Chapter One
Hilbert's 7th Problem: It's statement and origins

At the second International Congress of Mathematicians in Paris, 1900, the mathematician David Hilbert was invited to deliver a keynote address, just as Henri Poincaré had been invited to do at the first International Congress of Mathematicians in Zurich in 1896.

According to published versions of his lecture, which appeared in soon thereafter, Hilbert began his lecture with a bit of a motivation for offering a list of problems to inspire mathematical research:

...the close of a great epoch not only invites us to look back into the past but also directs our thoughts to the unknown future. The deep significance of certain problems for the advance of mathematical science in general and the important role which they play in the work of the individual investigator are not to be denied. As long as a branch of science offers an abundance of problems, so long is it alive; a lack of problems foreshadows extinction or the cessation of independent development. Just as every human undertaking pursues certain objects, so also mathematical research requires its problems.

In his lecture Hilbert posed 10 problems. In the published versions of his lecture Hilbert offered 23 problems (only 18 could really be considered to be problems rather than areas for further research). The distribution of his published problems is, roughly: 2 to Logic, 3 to Geometry, 7 to Number Theory, 10 to Analysis/Geometry, and 1 to Physics (and its foundations). To date 16 of these have been either solved or given counterexamples.

What will concern us in these notes is the seventh problem on Hilbert's list, which concerns the arithmetic nature of certain number–in particular Hilbert proposed that certain, specific, numbers are transcendental, i.e., not algebraic and so not the solutions of any integral polynomial equation $P(X) = 0$.

By the time Hilbert spoke in Paris transcendental numbers had already had a short yet fairly glorious history. The beginning of this history, so far as we can tell from what was written down, began with either Euler or Leibniz. What is important to take away from the possible exasperation these two mathematicians might have felt at feeling that there possibly were some numbers just beyond their grasps is that these *transcendental* numbers were not numbers that one would ordinarily encounter through algebraic methods. Rather than trace this history here we highlight research that influenced the early development of the study of transcendental numbers.

Some early developments that are relevant to these lectures

A natural place to begin with many historically oriented surveys is with Euler. In 1748 Euler published *Introductio in analysis infintiorum* (Introduction to Analysis of the Infinite) in which he did several things that are relevant to the development of transcendental number theory and to these lectures:

- Summed the series $1/1^k + 1/2^k + 1/3^k + ...$ for all k even, $2 \leq k \leq 26$, which we represent by $\zeta(k)$ (and which of course may extended to be a function of a complex variable z). We note for later reference that Euler showed that $\zeta(2) = \frac{\pi^2}{6}$.

• Derived the formula

$$e^{ix} = \cos(x) + i\sin(x)$$

• From the above equality, obtained the relationship

$$e^{i\pi} = -1.$$

• Found continued fraction expansions for e and e^2 (which were non-terminating thus showing that each of these numbers is irrational).

• Made an interesting conjecture concerning the nature of certain logarithms of rational numbers. As Euler's conjecture remarkably foreshadows part of Hilbert's seventh problem, it is worth stating. Euler wrote:

... the logarithms of [rational] numbers which are not the powers of the base are neither rational nor irrational ...

When Euler uses the terminology "irrational" he means what we would call "irrational and algebraic." Euler went on to say that *... it is with justice that they [the above logarithms] are called transcendental quantities.*

In this terminology, Euler's conjecture has the simple, more modern, formulation:

Conjecture (Euler) *For any two positive rational numbers $a/b \neq 1$ and c/d, the number*

$$log_{\frac{a}{b}} \frac{c}{d} \quad \left(= \frac{\log \frac{c}{d}}{\log \frac{a}{b}}\right)$$

is either rational or transcendental.

This conjecture applies to each of the number $\log_{\frac{1}{2}} 4 = -2, \log_{\frac{1}{2}} 3$ and $\log_3 5$, asserting that the latter two are transcendental. But it also asserts that a number like $2^{\sqrt{2}}$ is irrational, because if this conjecture is translated from one about logarithms into one about exponents, may be stated as:

Conjecture (Euler) Exponential Version *Suppose r and s are nonzero rational numbers and $r^\beta = s$. Then, if β is not rational it must be transcendental.*

In other words a rational number to an irrational, algebraic power cannot be rational. We will return to this formulation below.

Although Euler's discovery that e has a non-repeating continued fraction expansion implies that e is irrational, this discovery is not in the current of mathematical thought that includes Hilbert's seventh problem. That honor belong to a proof of the irrationality of e that was given by Fourier:

• Fourier (1815) Using the series representation

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}$$

gave an alternate proof that e is irrational.

We will look at this instructive proof below.

Although Euler had speculated that there might exist non-algebraic numbers, none were known, indeed they were not known to exist, until the work of Liouville in the middle of the nineteenth century.

•Liouville (1844/1851) Transcendental numbers exist.

This result is actually the corollary Liouville established concerning *how well* an algebraic number can be approximated by a rational number:

Liouville's Theorem. *Suppose α is an algebraic number of degree $d > 1$. Then there exists a positive constant $c(\alpha)$ such that for any rational number $\frac{a}{b}$*

$$\left|\alpha - \frac{a}{b}\right| > \frac{c(\alpha)}{b^d}. \tag{1}$$

The proof of Liouville's Theorem is entirely elementary, requiring only an application of the Mean Value Theorem from calculus and a bit of insight. Because this proof is so straightforward it is worth seeing it at least once (this proof is outlined in the exercises at the end of this chapter).

It is an elementary exercise to deduce that the number ℓ, below, does not satisfy the conclusion of Liouville's Theorem for any constant or for any d.

$$\begin{aligned}\ell = \sum_{n=1}^{\infty} 10^{-n!} &= \frac{1}{10} + \frac{1}{100} + \frac{1}{1000000} + \frac{1}{1000000000000000000000000} + \cdots \\ &= .1\ 1\ \underbrace{000}_{\text{3 zeros}}\ 1\ \underbrace{00000000000000000}_{\text{17 zeros}}\ 1\ \underbrace{000\ldots00}_{\text{95 zeros}}\ 1\ 000\ldots\end{aligned}$$

•Hermite (1873) Using the series representation for the numbers

$$e^n = \sum_{k=0}^{\infty} \frac{n}{k!}$$

proved that e is transcendental. (See next chapter.)

•Cantor (1874) There are only countably many algebraic numbers but there are uncountably many transcendental numbers

•Lindemann(1882) Using the above series representation e^z, Euler's Formula $e^{i\pi} = -1$, and results about algebraic numbers, proved that $i\pi$ is transcendental, and therefore π is transcendental.

•Lindemann actually proved that for any nonzero algebraic number α, the number e^α is transcendental. (See next chapter.)

•Weierstrass (1884) supplied the proof of something Lindemann had claimed, but not proved. This is now called the Lindemann-Weierstrass Theorem.

Theorem (Lindemann-Weierstrass). *Suppose $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ are distinct, nonzero algebraic numbers. Then any number of the form*

$$a_1 e^{\alpha_1} + a_2 e^{\alpha_2} + \cdots + a_\ell e^{\alpha_\ell},$$

with all a_j algebraic and not all zero, is transcendental.

Back to Hilbert's Lecture.

With the above survey as background we are now in a better position to understand the scope of Hilbert's proposed seventh problem. Hilbert began his statement of this problem with:

Hermite's arithmetical theorems on the exponential function and their extension by Lindemann are certain of the admiration of all generations of mathematicians.

Following his introductory words Hilbert continued,

I should like, therefore, to sketch a class of problems which, in my opinion, should be attacked as here next in order.

We state these problems as being separate parts of Hilbert's seventh problem.

Part 1.

... we expect transcendental functions to assume, in general, transcendental values for [...] algebraic arguments [...] we shall still consider it highly probable that the exponential function $e^{i\pi z}$ [...] will [...] always take transcendental values for irrational algebraic values of the argument z

We recall that a function $f(z)$ is transcendental function if there does not exist a nonzero polynomial $P(x, y)$ so that the function $P(z, f(z))$ is identically zero. For example e^z is a transcendental functions whereas $\sqrt{z}$ and $x^5 - x^2 + 1$ are not. So Hilbert speculated that if $f(z)$ is a transcendental function and α is a nonzero algebraic number, then $f(\alpha)$ is a transcendental number.

By what we have already seen, the only example Hilbert knew was the transcendental function $f(z) = e^z$, which by the Hermite-Lindemann Theorem is transcendental whenever $z = \alpha$ for a nonzero, algebraic number α.

After referring to a geometric version of his conjecture (see exercises) Hilbert continued that he believed that

the expression α^β, for an algebraic base and an irrational algebraic exponent , e. g., the number $2^{\sqrt{2}}$ or e^π, always represents a transcendental or at least an irrational number.

Hilbert's suggestion is that α^β should be transcendental whenever it has an algebraic base α (which implicitly requires $\alpha \neq 0, 1$) and an irrational algebraic exponent β. It is interesting to compare Hilbert's Conjecture with Euler's Conjecture from a century and a half earlier. If we slightly reformulate Euler's Conjecture we will see that Hilbert simply expanded the arithmetic nature of the numbers under consideration.

Euler's Conjecture (1748). If a is a nonzero rational number and β is an irrational algebraic number then a^β is irrational.

For example $2^{\sqrt{2}}$ is irrational.

Hilbert's Conjecture (1900). If a and β are algebraic, with $a \neq 0$ or 1, and β irrational then a^β is transcendental.

For example $2^{\sqrt{2}}, 2^{\sqrt{-2}}, i^{\sqrt{2}}$ and $e^\pi (= (-1)^{-i})$ are transcendental.

Just as Euler's Conjecture could be stated in terms of either exponents or logarithms, Hilbert's Conjecture concerning α^β has alternate formulations. These equivalent formulations have played important roles not only in the eventual solutions of Hilbert's seventh problem in the early 1930s, but to questions that have become central to transcendental number theory. Rather than return to these equivalencies later it seems best to give them now.

Second Version of Hilbert's Conjecture. Suppose ℓ and β are complex numbers, with $\ell \neq 0$ and β irrational. Then at least one of the numbers

$$\beta,\ e^\ell,\ e^{\beta\ell}$$

is transcendental.

Third Version of Hilbert's Conjecture. Suppose α and β are nonzero algebraic numbers. If $\dfrac{\log \alpha}{\log \beta}$ is irrational then it is transcendental.

It is an interesting exercise, and it is at the end of this chapter, to establish the equivalencies of these three versions of Hilbert's conjecture.

Important precursors to the solution of Hilbert's seventh problem

Hilbert's seventh problem, i.e., the transcendence of α^β, was solved independently by A. O. Gelfond and Th. Schneider, in 1934, using similar methods. In order to appreciate their solutions to this problem, and how their methods extend to other problems, it is useful to first understand earlier developments. The ones that will aid our understanding of twentieth-century transcendental number theory most are Fourier's proof for the irrationality of e (1815), Hermite's proof for the transcendence of e (1873), and Lindemann's proof for the transcendence of π (1882).

We begin with Fourier's simple proof of the following theorem

Theorem: *e is irrational.*

At the heart of Fourier's proof is the simplicity, and regularity, of the power series representation for e

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}$$

Proof. Suppose $e = \frac{B}{A}$ where A and B are positive integers. The rationality of e translates to the relationship $Ae - B = 0$. Fourier's idea is to replace e by its series representation and then truncate the series into a main term and a tail (knowing that the tail may be made as small as desired) This idea will eventually yield a positive integer that is strictly less than 1.

If we substitute the power series representation for e into the above equation we obtain the equation:

$$A\Big(\sum_{k=0}^{\infty} \frac{1}{k!}\Big) - B = 0. \tag{2}$$

For any integer $N \geq 1$ it is possible to separate the power series for e into a *main term*, M_N, and *tail*, T_N,

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} = \underbrace{\sum_{k=0}^{N} \frac{1}{k!}}_{M_N} + \underbrace{\sum_{k=N+1}^{\infty} \frac{1}{k!}}_{T_N}.$$

If we substitute this expression into (2) we have an equation:

$$A\Big(M_N + T_N\Big) - B = 0. \tag{3}$$

Fourier's idea is to rewrite this equation as $A \times M_N - B = -A \times T_N$ and, realizing that $|T_N|$ may be made to be a small quantity, obtain an inequality of the form

$$|\text{ nonzero integer }| < \text{a small quantity.}$$

It is easiest to follow this argument if we view (2) as:

$$A\Big(\underbrace{\sum_{k=0}^{N}\frac{1}{k!}}_{M_N}+\underbrace{\sum_{k=N+1}^{\infty}\frac{1}{k!}}_{T_N}\Big)-B=0.$$

which, after using $N!$ for a common denominator for the terms in the main term may be rewritten as:

$$A\Big(\sum_{k=0}^{N}\frac{\left(\frac{N!}{k!}\right)}{N!}+\sum_{k=N+1}^{\infty}\frac{1}{k!}\Big)-B=A\Big(\frac{1}{N!}\underbrace{(\sum_{k=0}^{N}\frac{N!}{k!})}_{M_N^*}+\underbrace{\sum_{k=N+1}^{\infty}\frac{1}{k!}}_{T_N}\Big)-B=0.$$

Note that for each k, $0\le k\le N$, the fraction $\dfrac{N!}{k!}$ in the modified main term, M_N^*, is a positive integer, thus so is their sum. For clarity we rewrite the above equation as:

$$A\Big(\frac{1}{N!}\times M_N^*+T_N\Big)-B=0.$$

If we multiply this equation by $N!$ and rearrange terms slightly, we obtain:

$$|A\times M_N^*-N!B|=N!A\times T_N$$

The rest of the proof requires two parts:

1. Show that the expression: $A\times M_N^*-N!B$ is a nonzero integer.

2. Show that for a suitably chosen N: $|N!A\times T_N|$ is less than 1.

If we establish **1** and **2** we will have the conclusion:

$$0<|A\times M_N^*-N!B|<1,$$

where the expression is an integer. This is, of course, a contradiction.

The reason **1** *holds.* Since the main term is a truncation of the series representation for e, and we are assuming e is rational and equals $\frac{B}{A}$, if $A\times M_N^*-N!B=0$ we obtain the contradictory inequalities:

$$e=\frac{B}{A}=\frac{M_N^*}{N!}<e.$$

Note that this holds for any N.

The reason **2** *holds.* We have:

$$
\begin{aligned}
N! \times A \times T_N &= N! \times A \sum_{k=N+1}^{\infty} \frac{1}{k!} = A \sum_{k=N+1}^{\infty} \frac{N!}{k!} \\
&= A\left(\frac{1}{N+1} + \frac{1}{(N+1)(N+2)} + \frac{1}{(N+1)(N+2)(N+3)} + \cdots\right)
\end{aligned}
$$

We are free to specify a value for N; taking $N+1 = 2A$ the above sum becomes

$$
\begin{aligned}
&= \frac{A}{2A} + \frac{A}{(2A)(2A+1)} + \frac{A}{(2A)(2A+1)(2A+2)} + \cdots \\
&< \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \\
&= 1\ .
\end{aligned}
$$

Thus we have deduced that the positive, nonzero integer $N! \times A \times T_N$ is less than 1. This contradiction establishes the irrationality of e.

Rather than leaving this proof behind its outline is so important that it deserves to be summarized. This proof consists of a sequence of easily understood steps. The proof begins with the assumption that e is a rational number. This assumption, followed by a simple argument using the power series for e, leads to a nonzero, positive integer that is less than 1. We will see that this basic structure holds in many, indeed almost all, transcendence proofs. And almost always, the most difficult part of the proof is to show that the integer derived in the proof is not equal to zero.

Before we explore these difficulties let's look at an instructive failed proof: an attempt to establish the transcendence of e through a direct application of Fourier's approach.

Sketch of the failed proof.

Assume e is algebraic, so we have an integral polynomial equation $P(e) = 0$, explicitly:

$$r_0 + r_1 e + r_2 e^2 + \cdots + r_d e^d = 0,\ \ r_d \neq 0$$

Following Fourier's method, for each $n, 1 \leq n \leq d$, use the series representation:

$$e^n = \sum_{k=0}^{\infty} \frac{n^k}{k!} = \underbrace{\sum_{k=0}^{N} \frac{n^k}{k!}}_{M_N(n)} + \underbrace{\sum_{k=N+1}^{\infty} \frac{n^k}{k!}}_{T_N(n)}$$

Substituting each of these expressions into the presumed vanishing algebraic relationship above, we obtain: $r_0 + r_1\big(M_N(1) + T_N(1)\big) + r_2\big(M_N(2) + T_N(2)\big) +$

$\cdots + r_d\big(M_N(d) + T_N(d)\big) = 0$ which yields:

$$\begin{aligned}\big|r_0 + r_1\big(M_N(1)\big) + r_2\big(M_N(2)\big) + \cdots + r_d\big(M_N(d)\big)\big| \\ = \big|r_1\big(T_N(1)\big) + r_2\big(T_N(2)\big) + \cdots + r_d\big(T_N(d)\big)\big| \quad (4)\end{aligned}$$

We rewrite each term $M_N(n)$ as

$$M_N(n) = \frac{1}{N!}\big(\underbrace{\sum_{k=0}^{N} \frac{N!}{k!} n^k}_{M_N^*(n)}\big),$$

where we note that $M_N^*(n)$ is an integer.

So if we multiply (4) by $N!$ we obtain:

$$\begin{aligned}\big|N! r_0 + r_1\big(M_N^*(1)\big) + r_2\big(M_N^*(2)\big) + \cdots + r_d\big(M_N^*(d)\big)\big| \\ = N!\big|r_1\big(T_N(1)\big) + r_2\big(T_N(2)\big) + \cdots + r_d\big(T_N(d)\big)\big|\end{aligned}$$

Each of the terms $T_N(n)$ is of the form

$$\frac{n^{N+1}}{(N+1)!} + \text{convergent series};$$

indeed

$$\big|T_N(n)\big| < \frac{n^{N+1}}{(N+1)!} \times e^n.$$

From these inequalities it is possible to obtain:

$$0 < |\text{complicated nonzero integer}| \quad \leq N!\frac{d^{N+1}}{(N+1)!} \times \underbrace{e^d \times \max\{|r_1|, \ldots, |r_d|\}}_{\text{a fixed quantity}}.$$

Unfortunately, as $N \to \infty$ the right-hand side of the above inequality grows without bound, so no contradiction is obtained. However a significant modification of this proof *could* succeed; we sketch this hopeful proof next.

An important first modification of the above, failed sketch.

The idea is to separate the power series for e^n, for each n, $1 \leq n \leq d$, into a main term, an intermediate term, and a tail, and hope to manipulate the intermediate terms so that a linear combination of them vanishes (and do this in such a way that the tails can be made arbitrarily small).

$$e^n = \sum_{k=0}^{\infty} \frac{n^k}{k!} = \underbrace{\sum_{k=0}^{N} \frac{n^k}{k!}}_{M_N(n)} + \underbrace{\sum_{k=N+1}^{N'} \frac{n^k}{k!}}_{I_{N,N'}(n)} + \underbrace{\sum_{k=N'+1}^{\infty} \frac{n^k}{k!}}_{T_{N'}(n)}$$

Therefore, assuming $r_0 + r_1 e + r_2 e^2 + \cdots + r_d e^d = 0,\ r_d \neq 0$ we obtain:

$$\begin{aligned} r_0 + r_1\big(M_N(1) + I_{N,N'}(1) + T_{N'}(1)\big) + r_2\big(M_N(2) + I_{N,N'}(2) + T_{N'}(2)\big) \\ + \cdots + r_d\big(M_N(d) + I_{N,N'}(d) + T_{N'}(d)\big) = 0, \end{aligned}$$

which leads to

$$\begin{aligned} &\underbrace{r_0 + r_1\big(M_N(1)\big) + r_2\big(M_N(2)\big) + \cdots + r_d\big(M_N(d)\big)}_{\text{sum of main terms}} \\ &= -\Big(\underbrace{r_1\big(I_{N,N'}(1)\big) + r_2\big(I_{N,N'}(2)\big) + \cdots + r_d\big(I_{N,N'}(d)\big)}_{\text{sum of intermediate terms}}\Big) \\ &- \Big(\underbrace{r_1\big(T_{N'}(1)\big) + r_2\big(T_{N'}(2)\big) + \cdots + r_d\big(T_{N'}(d)}_{\text{sum of tails}}\Big) \end{aligned}$$

IF it were possible to arrange things so that the sum of intermediate terms vanishes, then after multiplying through by $N!$, as before, and letting $M_N^*(n) = \sum_{k=0}^{N} \frac{N!}{k!} n^k$, we would be left with an equation:

$$\begin{aligned} &\big|N! r_0 + r_1\big(M_N^*(1)\big) + r_2\big(M_N^*(2)\big) + \cdots + r_d\big(M_N^*(d)\big)\big| \\ &\qquad = N!\big|\big(r_1\big(T_{N'}(1)\big) + r_2\big(T_{N'}(2)\big) + \cdots + r_d\big(T_{N'}(d)\big)\big|, \end{aligned}$$

where the expression on the left-hand side is now an integer.

If we look at the leading terms of the tails, as before, and *if* it were possible to show that the expression $N! r_0 + r_1\big(M_N^*(1)\big) + r_2\big(M_N^*(2)\big) + \cdots + r_d\big(M_N^*(d)\big)$ is nonzero we would have an inequality:

$$0 < |\text{complicated nonzero integer}| \quad < N! \frac{d^{N'+1}}{(N'+1)!} \times e^d \times \max\{|r_1|, \ldots, |r_d|\}.$$

Then keeping N fixed and letting $N' \to \infty$, the right-hand side approaches 0. Therefore we obtain:

$$0 < |\text{complicated nonzero integer}| \quad < 1,$$

which is, of course, a contradiction.

The above sketch of a possibly successful proof for the transcendence of e can be made into a formal proof (one which is not as closely aligned with Hermite's original proof as one due to Hurwitz in 1893). This proof is accomplished not by using approximations to the values $e^n, 1 \leq n \leq d$, obtained by simply dividing the power series representation for e^n into a main term, an intermediate term, and a tail. Rather it requires manipulating the power series for e^z so that the sum of the intermediate terms in the above sketch vanishes at the appropriate values of z.

Exercises.

1. a) Derive the following result from Liouville's Theorem: *Let α be a real number. Suppose that for each positive real number c and each positive integer d, there exists a rational number p/q satisfying the inequality*

$$\left|\alpha - \frac{p}{q}\right| < \frac{c}{q^d} \ .$$

Then α is transcendental.

b) Deduce from Liouville's Theorem that the number $\ell = \sum_{n=1}^{\infty} 10^{-n!}$ is transcendental.

2. Prove the three versions of the α^β conjecture are equivalent. (Hint. The easiest way to do this is to show the first implies the second, the second implies the third, and the negation of the first implies the negation of the third.)

3. Hilbert stated part of his seventh problem in a geometric form:
If, in an isosceles triangle, the ratio of the base angle to the angle at the vertex be algebraic but not rational, the ratio between base and side is always transcendental.

a) To which values of the standard trigonometric functions does this conjecture apply?

b) Does this version of Hilbert's conjecture follow from either of what we have called Part 1 and Part 2 of his conjecture?

4. Derive from one of the results stated in this chapter that if α is a nonzero algebraic number then both $\cos(\alpha)$ and $\sin(\alpha)$ are transcendental.

5. Prove that e^z is a transcendental function.

6. Can Fourier's proof of the irrationality of e be modified to establish the irrationality of e^m, where m is an integer?

7. This exercise outlines a proof of Liouville's Theorem. Fill in the details to justify each step.
a) We only have to consider the case in which α is a real number. (The result is easily seen to be true if α is not real.)

b) Our objective is to demonstrate the existence of a constant c that depends only on α for which the inequality in Liouville's Theorem is satisfied for *all* rational numbers p/q. We may as well restrict ourselves to the case where $\left|\alpha - \frac{p}{q}\right| \leq 1$.

c) Let

$$P(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0 \ ,$$

where $a_d, a_{d-1}, \ldots, a_0 \in \mathbf{Z}$ with $a_d > 0$ be the minimal polynomial for α. Then

$$P\left(\frac{p}{q}\right) = a_d \frac{p^d}{q^d} + a_{d-1} \frac{p^{d-1}}{q^{d-1}} + \cdots + a_1 \frac{p}{q} + a_0 \ = \frac{N}{q^d} \ ,$$

where N is a nonzero integer.

d) It follows from **c** that

$$\frac{1}{q^d} \leq \left| P\left(\frac{p}{q}\right) \right| \ = \left| P(\alpha) - P\left(\frac{p}{q}\right) \right| \ .$$

e) By the Mean Value Theorem these exists a real number φ between α and p/q such that

$$P(\alpha) - P\left(\frac{p}{q}\right) = P'(\varphi)\left(\alpha - \frac{p}{q}\right) \ .$$

f) Combine the inequalities from **d** and **e** to conclude the proof. (Remark. Make sure that your constant depends only on α and not on φ.)