MATH 8174: Assignment 13

1. Recall that in Exercise 4 of Assignment 12, we defined a 3-dimensional Lie algebra L over the field k with basis $\{a, b, c\}$ satisfying

$$[a, b] = c, \quad [b, c] = a, \text{ and } \quad [c, a] = b.$$

If $x = x_1a + x_2b + x_3c$ and $y = y_1a + y_2b + y_3c$ are arbitrary elements of L, show that the Killing form on L is given by

$$\langle x, y \rangle = -2(x_1y_1 + x_2y_2 + x_3y_3).$$

2. An ideal J of a Lie algebra L is said to be *characteristic* if $\delta J \subseteq J$ for all derivations $\delta \in \mathrm{Der}(L)$; we will write this as J char L.

(i) Show that if J char L, then the restriction of $\mathrm{ad}(x)$ to J lies in $\mathrm{Der}(J)$ for all $x \in L$.

(ii) Show that if M char N and $N \trianglelefteq L$, then $M \trianglelefteq L$.

(iii) Show that if M char N and N char L, then M char L.

(iv) Show that L^i, $L^{(i)}$ and $Z(L)$ are all characteristic ideals of L.

[Note that parts of (ii) and (iv) were used in our proof of Cartan's criterion for semisimplicity.]

3. Let k be a field and let L be a Lie algebra of dimension 2 over k. Prove that L is solvable.

4. Let k be a field of characteristic 2 and let L be the 5-dimensional subspace of $M_3(k)$ spanned by the matrices

$$\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \epsilon = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(i) By computing $[\alpha, \beta], \ldots, [\delta, \epsilon]$, show that L is a subalgebra of $\mathfrak{gl}_3(k)$ and that $L^2 (= L^{(1)})$ is spanned by α, β and γ.

(ii) Show that $L^{(1)}$ is simple and that $L^{(1)}$ is contained in every nonzero ideal of L.

(iii) Show that L is semisimple, but that L is not a direct sum of simple ideals.

(This shows that Theorem 8.11 fails in fields of characteristic 2.)