MATH 8174: Assignment 6

1. Prove that the representations $\mathrm{Hom}_{\mathbb{C}}(U \otimes V, W)$ and $\mathrm{Hom}_{\mathbb{C}}(U, V^* \otimes W)$ are equivalent.
2. (Extended form of Schur's lemma.) Suppose that A is a finite dimensional algebra over a field K that is not necessarily algebraically closed, and suppose that M is an irreducible A-module. Show that the A-module homomorphisms from M to itself (called the *endomorphisms* of M) form a division ring. Show that this division ring contains K in its center and is finite dimensional as a K-vector space.
3. Suppose that A and M are as in Question 2, and that M' is an A-module that is isomorphic to a direct sum of n copies of M. Show that the endomorphism ring of M' is isomorphic to the ring of $n \times n$ matrices over the endomorphism ring of M.
4. Let $G = SL(2,3)$ denote the group of 2×2 matrices of determinant 1 over the field with three elements. Show that G has 24 elements. Show that G has a central subgroup of order 2 with quotient isomorphic to A_4. Show that $-I$ (where I is the identity matrix) is the only element of order 2 in G, and use this to show that a Sylow 2-subgroup of G is isomorphic to Q_8, and normal in G with quotient of order 3. Show that G has seven conjugacy classes of elements, and that these elements have orders $1, 2, 3, 3, 4, 6$ and 6. Find the orders of the centralizers of these elements. Find the character table of G.
5. Let G be the group $GL(2,2)$ of invertible 2×2 matrices over the field $\mathbb{F}_2$ of two elements. Show that G is isomorphic to S_3, and deduce that G has an irreducible representation of dimension 2 over $\mathbb{F}_2$. Show also that G has a two-dimensional indecomposable representation over $\mathbb{F}_2$ that is not irreducible. Using these representations, show that we have

$$\mathbb{F}_2 G \cong M_2(\mathbb{F}_2) \oplus \mathbb{F}_2(\mathbb{Z}/2\mathbb{Z}).$$