Problem 1: Let $X$ be a set, and let $\mathcal{O}_{\text{coctbl}}$ be the set of all $U \subset X$ such that $X \setminus U$ is countable or the whole set $X$. Show that $\mathcal{O}_{\text{coctbl}}$ is a topology on $X$. Which topology on $X$ is finer, the cofinite topology $\mathcal{O}_{\text{cofin}}$ or the topology $\mathcal{O}_{\text{coctbl}}$? (6P)

Problem 2: Consider the following topologies on $\mathbb{R}$:

(a) $\mathcal{O}_1$, the standard topology,

(b) $\mathcal{O}_2$, the topology generated by all sets of the form $(a, b)$ or $(a, b) \setminus K$, where $a < b$ and

$$K = \left\{ \frac{1}{n} \mid n \in \mathbb{N}^* \right\},$$

(c) $\mathcal{O}_3$, the cofinite topology,

(d) $\mathcal{O}_4$, the upper limit topology, having all sets $(a, b]$, $a < b$ as basis,

(e) $\mathcal{O}_5$, the topology having all sets $(-\infty, a)$, $a \in \mathbb{R}$ as basis.

Determine, for each of these topologies, which of the others it contains. (6P)

Problem 3: Let $(\mathcal{O}_j)_{j \in J}$ be a family of topologies on a set $X$. Show that the intersection $\bigcap_{j \in J} \mathcal{O}_j$ is a topology on $X$. Is the union $\bigcup_{j \in J} \mathcal{O}_j$ a topology on $X$? (4P)