Problem 1: Let $A$ be a non-empty subset of a metric space $(X, d)$. Define the function $f : X \to \mathbb{R}$ by

$$f(x) := d(x, A) := \inf_{y \in A} d(x, y), \quad x \in X.$$ 

Prove that $f$ is continuous. (4P)

Problem 2: Prove that for every point $x \in \mathbb{R}^n$ the set of neighborhoods of $x$ with respect to the euclidean metric coincides with the set of neighborhoods with respect to the square metric. (3P)

Problem 3: Let $\ell^2$ denote the space of all sequences $x = (x_k)_{k \in \mathbb{N}}$ of real numbers $x_k \in \mathbb{R}$ such that $\sum_{k \in \mathbb{N}} x_k^2$ converges.

a) Prove that for $x, y \in \ell^2$ the sum $\sum_{k \in \mathbb{N}} |x_k y_k|$ converges. (3P)

b) Let $\lambda \in \mathbb{R}$ and $x, y \in \ell^2$. Show that then $x + y \in \ell^2$ and $\lambda x \in \ell^2$. (2P)

c) Show that

$$d(x, y) := \left(\sum_{k=0}^{\infty} (x_k - y_k)^2\right)^{1/2}$$

is a metric on $\ell^2$. (4P)