Problem 1: Show that \( \mathbb{Q} \) is not closed in \( \mathbb{R} \).

Problem 2: Prove the following inequality by Cauchy-Schwarz:

\[
(\sum_{i=1}^{n} x_i y_i)^2 \leq \left( \sum_{i=1}^{n} x_i^2 \right) \left( \sum_{i=1}^{n} y_i^2 \right)
\]
for vectors \((x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathbb{R}^n\).

Hint: Expand \( \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i y_j - x_j y_i)^2 \).

Problem 3: Use the Cauchy-Schwarz inequality to prove that

\[
d(x, y) := \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}
\]
for vectors \((x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathbb{R}^n\)

defines a metric on \( \mathbb{R}^n \). It is called the euclidean metric.

Problem 4: Let \( C \) be the set of all continuous functions \( f : [a, b] \to \mathbb{R} \). For \( f, g \in C \) define

\[
d(f, g) = \int_{a}^{b} |f(t) - g(t)| \, dt.
\]

Show that \( d \) is a metric on \( C \). Hint: You can use theorems from calculus resp. analysis to prove the claim.