Set theory following Jech

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These are largely self-contained notes developing set theory, following Jech. For Jech Chapters 1–8 these notes are sketchy; just hitting a few points in Jech that we found needed more discussion. Starting with Jech Chapter 9 we essentially rewrite Jech, supplying proofs for most exercises. Some chapters are missing or not fully developed.

Contents

1. Elementary set theory

A *partial ordering* of P is a binary relation $<$ on P such that

$\forall p \in P(p \not< p)$.
$\forall p, q, r \in P[p < q$ and $q < r$ imply that $p < r]$.

We say that (P, R) is a poset in the *second sense* iff R is transitive, reflexive on P, and anti-symmetric: $\forall p, q \in P[p \leq q \leq p \rightarrow p = q]$.

Proposition 1.1. *(i) If $(P, <)$ is a poset, define $p \leq q$ iff $p < q$ or $p = q$. Then $(P, \leq)$ is a poset in the second sense.*

(ii) If $(P, \leq)$ is a poset in the second sense, define $p < q$ iff $p \leq q$ and $p \neq q$. Then $(P, <)$ is a poset.

(iii) If $(P, <)$ is a poset, define $p \leq q$ iff $p < q$ or $p = q$. Then $(P, \leq)$ is a poset in the second sense, and if we define $p <' q$ iff $p \leq q$ and $p \neq q$, then $< = <'$.

(iv) If $(P, \leq)$ is a poset in the second sense, define $p < q$ iff $p \leq q$ and $p \neq q$. Then define $p \leq' q$ iff $p < q$ orp $= q$. Then $\leq = \leq'$.

Proof. (i): Clearly $\leq$ is transitive and reflexive on P. Suppose that $p \leq q \leq p$. Thus $(p < q \vee p = q) \wedge (q < p \vee q = p)$. Then logically we have the following cases.

Case 1. $p < q < p$. Then $p < p$, contradiction.

Case 2. $p < p$, contradiction.

Case 3. $p = q$.

(ii): Clearly $\forall p[p \not< p]$. Suppose that $p < q < r$. Then $p \leq q$, $p \neq q$, $q \leq r$, and $q \neq q$. Hence $p \leq r$. If $p = r$, then $p \leq q \leq p$, so $p = q$, contradiction. Thus $p \neq r$, so $p < r$.

(iii): $p <' q$ iff $(p \leq q \wedge p \neq q)$ iff $((p < q \vee p = q) \wedge p \neq q)$ iff $(p < q \wedge p \neq q)$ iff $p < q$.

(iv): $p \leq' q$ iff $(p < q \vee p = q)$ iff $((p \leq q \wedge p \neq q) \vee p = q)$ iff $p \leq q$. $\square$

We define

$$
\begin{array}{llllllll}
(\alpha, \beta) < (\gamma, \delta) & \text{iff} & \max\{\alpha, \beta\} < \max\{\gamma, \delta\} & \text{or} & & & & \\
& & \max\{\alpha, \beta\} = \max\{\gamma, \delta\} & \text{and} & \alpha < \gamma & \text{or} & & \\
& & \max\{\alpha, \beta\} = \max\{\gamma, \delta\} & \text{and} & \alpha = \gamma & \text{and} & \beta < \delta. &
\end{array}
$$

It is easily shown that $<$ is a well-order of $\mathbf{ON} \times \mathbf{ON}$. $\Gamma(\alpha, \beta)$ is the order type of$\Gamma(\alpha, \beta)$ $\{(\xi, \eta) : (\xi, \eta) < (\alpha, \beta)$. Clearly

Lemma 1.2. $\Gamma(\alpha + 1, \alpha + 1) = \Gamma(\alpha, \alpha) + \alpha \cdot 2 + 1$. $\square$

Lemma 1.3. $\Gamma(n, n)) = n^2$ *for all* $n \in \omega$.

Proof. Induction on n, using Lemma 1.2. $\square$

Lemma 1.4. $\Gamma(\omega, \omega) = \omega$. $\square$

Theorem 1.5. $\forall \alpha[\Gamma(\alpha, \alpha) \leq \omega^\alpha]$.

Proof. Induction on α. It is clear for $\alpha \leq \omega$. Now assume it for $\alpha \geq \omega$. Then

$$\begin{aligned}\omega^{\alpha+1} &= \omega^\alpha \cdot \omega > \omega^\alpha + \omega^\alpha + \omega^\alpha + \omega^\alpha \\ &\geq \omega^\alpha + \alpha + \alpha + 1 = \omega^\alpha + \alpha \cdot 2 + 1 \\ &\geq \Gamma(\alpha+1, \alpha+1).\end{aligned}$$

Thus our statement holds for $\alpha + 1$. Now suppose inductively that γ is a limit ordinal. Then

$$\Gamma(\gamma, \gamma) = \bigcup_{\alpha<\gamma} \Gamma(\alpha, \alpha) \leq \bigcup_{\alpha<\gamma} \omega^\alpha = \omega^\gamma. \qquad \square$$

Lemma 1.6. *If $\alpha < \beta$, then $(\alpha, \gamma) < (\beta, \gamma)$.*

Proof. *Case 1.* $\gamma \leq \alpha$. Then $\max(\alpha, \gamma) = \alpha < \beta = \max(\beta, \gamma)$, so $(\alpha, \gamma) < (\beta, \gamma)$.
Case 2. $\alpha < \gamma < \beta$. Then $\max(\alpha, \gamma) = \gamma < \beta = \max(\beta, \gamma)$, so $(\alpha, \gamma) < (\beta, \gamma)$.
Case 3. $\beta \leq \gamma$. Then $\max(\alpha, \gamma) = \gamma = \max(\beta, \gamma)$ and $\alpha < \beta$, so $(\alpha, \gamma) < (\beta, \gamma)$. $\square$

Lemma 1.7. *If $\alpha < \beta$, then $(\gamma, \alpha) < (\gamma, \beta)$.*

Proof. *Case 1.* $\gamma \leq \alpha$. Then $\max(\gamma, \alpha) = \alpha < \beta = \max(\gamma, \beta)$. So $(\gamma, \alpha) < (\gamma, \beta)$.
Case 2. $\alpha < \gamma < \beta$. Then $\max(\alpha, \gamma) = \gamma < \beta = \max(\gamma, \beta)$, so $(\gamma, \alpha) < (\gamma, \beta)$.
Case 3. $\beta \leq \gamma$. Then $\max(\gamma, \alpha) = \gamma = \max(\gamma, \beta)$ and $\alpha < \beta$, so $(\gamma, \alpha) < (\gamma, \beta)$. $\square$

Lemma 1.8. $\beta \leq \Gamma(\beta, 0)$.

Proof. In fact, $(0,0) < (1,0) < (2,0) < \cdots < (\xi, 0) \cdots$ for $\xi < \beta$. $\square$

Lemma 1.9. $\beta + \gamma \leq \Gamma(\beta, \gamma)$.

Proof. Induction on γ. It holds for $\gamma = 0$ by Lemma 1.8. Assume it for γ. Then $\Gamma(\beta, \gamma+1) > \Gamma(\beta, \gamma) \geq \beta + \gamma$, so $\Gamma(\beta, \gamma+1) \geq \beta + \gamma + 1$. Now assume that it holds for all $\delta < \gamma$, with γ limit. Then $\Gamma(\beta, \gamma) \geq \sup_{\delta<\gamma} \Gamma(\beta, \delta) \geq \sup_{\delta<\gamma}(\beta + \delta) = \beta + \gamma$. $\square$

Lemma 1.10. *If $\Gamma(\alpha, \alpha) = \alpha$ and $\beta, \gamma < \alpha$, then $\beta + \gamma < \alpha$.*

Proof. By Lemma 1.9, $\beta + \gamma \leq \Gamma(\beta, \gamma) < \Gamma(\alpha, \alpha) = \alpha$. $\square$

Lemma 1.11. *If $\Gamma(\alpha, \alpha) = \alpha$ and $\beta, \gamma < \alpha$, then $\beta \cdot \gamma \leq \Gamma(\beta + \gamma, \beta + \gamma)$.*

Proof. Induction on γ. It is clear for $\gamma = 0$. Assume that $\gamma + 1 < \alpha$ and $\beta \cdot \gamma \leq \Gamma(\beta + \gamma, \beta + \gamma)$. Then

$$\begin{aligned}\Gamma(\beta+\gamma+1, \beta+\gamma+1) &= \Gamma(\beta+\gamma, \beta+\gamma) + (\beta+\gamma) \cdot 2 + 1 \\ &\geq \beta \cdot \gamma + \beta = \beta \cdot (\gamma + 1).\end{aligned}$$

Now suppose that γ is limit and $\forall\delta < \gamma[\beta \cdot \delta \le \Gamma(\beta + \delta, \beta + \delta)]$. Then

$$\begin{aligned}\Gamma(\beta+\gamma,\beta+\gamma) &= \bigcup_{\delta<\gamma}\Gamma(\beta+\delta,\beta+\delta)\\ &\ge \bigcup_{\delta<\gamma}\beta\cdot\delta = \beta\cdot\gamma.\end{aligned} \qquad \square$$

Theorem 1.12. *If $\Gamma(\alpha, \alpha) = \alpha$, then $\alpha = 0$ or there is a β such that $\alpha = \omega^{\omega^\beta}$.* $\square$

Theorem 1.13. *If $\alpha = 0$ or $\alpha = \omega^{\omega^\xi}$ for some ξ, then $\Gamma(\alpha, \alpha) = \alpha$.*

Proof. We may assume that $\alpha = \omega^{\omega^\xi}$ with $\xi > 0$.
Case 1. $\xi = \eta + 1$ for some η. Note that

$$\omega^{\omega^{\eta+1}} = \omega^{\omega^\eta\cdot\omega} = \bigcup_{n\in\omega}\omega^{\omega^\eta\cdot n}.$$

(1) $\forall n \in \omega\backslash 1\forall\beta < \omega^{\omega^\eta\cdot n}[\Gamma(\beta,\beta) \le \omega^{\omega^{\eta\cdot n}} \cdot \beta]$.

For a fixed $n \in \omega\backslash 1$ we prove this by induction on β. It is clear for $\beta = 0$. Assume it for β. Then

$$\begin{aligned}\Gamma(\beta+1,\beta+1) = \Gamma(\beta,\beta)+\beta\cdot 2+1 &\le \omega^{\omega^{\eta\cdot n}}\cdot\beta+\beta\cdot 2+1\\ &< \omega^{\omega^{\eta\cdot n}}\cdot\beta+\omega^{\omega^{\eta\cdot n}} = \omega^{\omega^{\eta\cdot n}}\cdot(\beta+1).\end{aligned}$$

Now assume that $\beta < \omega^{\omega^\eta\cdot n}$ is limit, and for all $\delta < \beta$, $\Gamma(\delta, \delta) \le \omega^{\omega^{\eta\cdot n}} \cdot \delta$. Then

$$\Gamma(\beta,\beta) = \bigcup_{\delta<\beta}\Gamma(\delta,\delta) \le \bigcup_{\delta<\beta}(\omega^{\omega^{\eta\cdot n}}\cdot\delta) = \omega^{\eta\cdot n}\cdot\beta.$$

This proves (1). If follows that $\forall\beta < \alpha[\Gamma(\beta, \beta) < \alpha]$.
Case 2. ξ is limit.

(2) $\forall\eta \in \xi\backslash 1\forall\beta < \omega^{\omega^\eta}[\Gamma(\beta,\beta) \le \omega^{\omega^\eta} \cdot \beta]$.

The proof is like that for (1). $\square$

Theorem 1.14. *There is a well-ordering $<$ of ${}^{<\omega}$**On** such that for every ordinal α, ${}^{<\omega}\omega_\alpha$ is an initial segment under $<$, of order type ω_α.*

Proof. First we show by induction on n that there is a well-order $<_n$ of ^{n}**On** such that for every cardinal κ, $<_n \cap(\kappa\times\kappa)$ is a well-order of ${}^n\kappa$.
$n = 0$: ${}^0\mathbf{On} = \{\emptyset\}$
$n = 1$: Let $<_1=<$. Then

$$<_1 \cap(\kappa\times\kappa) = \{(\alpha,\beta) : \alpha < \beta < \kappa\},$$

which is a well-order of κ.

$n \geq 1$; define

$$\begin{aligned} f <_{n+1} g \quad \text{iff} \quad & f, g \in {}^{n+1}\mathbf{On} \text{ and } \max(\mathrm{rng}(f)) < \max(\mathrm{rng}(g)) \\ & \text{or } \max(\mathrm{rng}(f)) = \max(\mathrm{rng}(g)) \text{ and } \exists m \leq n \\ & [\forall p < m[f(p) = g(p)] \text{ and } f(m) < g(m)] \end{aligned}$$

Clearly $<_{n+1}$ is a well-order of ${}^{n+1}\mathbf{On}$. If κ is a cardinal, clearly $<_{n+1} \cap (\kappa \times \kappa)$ is a well-order of ${}^{n+1}\kappa$.

Now we define, for $f, g \in {}^{<\omega}\mathbf{On}$, $f < g$ iff

$$\begin{aligned} & f = \emptyset \neq g \quad \text{or} \\ & \exists n \in \omega \backslash \{0\}[f, g \in {}^{n}\mathbf{On} \text{ and } f <_n g \text{ or} \\ & \exists m, n \in \omega \backslash \{0\}[m < n \text{ and } f \in {}^{m}\mathbf{On} \text{ and } g \in {}^{n}\mathbf{On} \\ & \text{and } \max(\mathrm{rng}(f)) = \max(\mathrm{rng}(g))]]. \end{aligned}$$

Clearly $<$ is as desired. □

Lemma 1.15. *If P is perfect, then there exist real numbers $r < s$ such that $P \cap (-\infty, r]$ and $P \cap [s, \infty)$ are perfect.*

Proof. Let $a = \inf(P)$ and $b = \sup(P)$; perhaps $a = -\infty$ or $b = \infty$, or both. We consider two cases.

Case 1 $(a, b) \subseteq P$. Then choose r, s with $a < r < s < b$. To check that this works, note that

$$P \cap (-\infty, r] = \begin{cases} [a, r] & \text{if } a \in P, \\ (a, r] & \text{otherwise (so that } a = -\infty), \end{cases}$$

so clearly $(-\infty, r]$ is perfect. Similarly, $[s, \infty)$ is perfect.

Case 2 $(a, b) \not\subseteq P$. Choose $x \in (a, b) \backslash P$. Since P is closed and $x \notin P$, choose c, d with $c < x < d$ and $(c, d) \cap P = \emptyset$. Then

(1) $a < c$.

For, suppose that $c \leq a$. Since $a < x$, this implies that $(a, x) \subseteq (c, d)$. But $(a, x) \cap P \neq \emptyset$ by the definition of a; so $(c, d) \cap P \neq \emptyset$, contradiction. So (1) holds.

Similarly,

(2) $d < b$.

In fact, suppose that $b \leq d$. Since $x < b$, this implies that $(x, b) \subseteq (c, d)$. But $(x, b) \cap P \neq \emptyset$ by the definition of b; so $(c, d) \cap P \neq \emptyset$, contradiction. So (2) holds.

Now take r, s such that $c < r < x < s < d$. We claim that $(-\infty, r] \cap P$ is perfect. It is nonempty by (1). Since $(-\infty, r] \cap P = (-\infty, r) \cap P$, clearly it has no isolated points.

Similarly, $[s, \infty) \cap P$ is perfect. □

Lemma 1.16. *If P is perfect, then there exist $a < b$ such that $P \cap [a, b]$ is perfect.*

Proof. By Lemma 1.15, choose r such that $P \cap (-\infty, r]$ is perfect. Applying Lemma 1.15 to $P \cap (-\infty, r]$, we obtain s such that $P \cap [s, r] = P \cap (-\infty, r] \cap [s, \infty)$ is perfect. $\square$

Lemma 1.17. *If P is perfect, then there exist disjoint closed intervals $[a, b]$ and $[c, d]$ such that $P \cap [a, b]$ and $P \cap [c, d]$ are perfect.*

Proof. By Lemma 1.15, choose $r < s$ with $P \cap (-\infty, r]$ and $P \cap [s, \infty)$ perfect. By Lemma 1.16 choose a, b, c, d so that $P \cap (-\infty, r] \cap [a, b]$ and $P \cap [s, \infty) \cap [c, d]$ are perfect. Now $P \cap (-\infty, r] \cap [a, b] = P \cap [a, \min(r, b)]$ and $P \cap [s, \infty) \cap [c, d] = P \cap [\max(s, c), d]$. $\square$

Theorem 1.18. *Every perfect set has cardinality 2^ω.*

Proof. Let P be perfect. We define perfect sets Q_s for each $s \in {}^{<\omega}2$ by recursion on $\mathrm{dmn}(s)$. Let $Q_\emptyset$ be a bounded perfect subset of P, by Lemma 1.17. If Q_s has been defined, let Q_{s0} and Q_{s1} be disjoint perfect subsets of Q_s, by Lemma 1.18.

For each $t \in {}^\omega 2$ choose $r_t \in \bigcap_{n\in\omega} Q_{t\restriction n}$. This gives 2^ω elements of F. $\square$

Theorem 1.19. *If F is an uncountable closed set, then there exist a perfect set P and a countable set S such that $F = P \cup S$.*

Proof. Let F be an uncountable closed set. For every $A \subseteq \mathbb{R}$ let A' be the set of all limit points of A.

(1) A' is closed.

For, suppose that $a \in \overline{A'}$. Thus for any b, c, if $b < a < c$ then $(b, c) \cap A' \neq \emptyset$, and hence $(b, c) \cap A \neq \emptyset$. Thus $a \in A'$. So A' is closed.

Now let

$$\begin{aligned} F_0 &= F; \\ F_{\alpha+1} &= F'_\alpha; \\ F_\alpha &= \bigcap_{\gamma<\alpha} F_\gamma \quad \text{for } \alpha \text{ limit.} \end{aligned}$$

Thus $F_0 \supseteq F_1 \supseteq \cdots$, so there is an ordinal θ such that $\forall \alpha > \theta [F_\alpha = F_\eta]$. Let $P = F_\theta$.

(2) $|F \backslash P| \leq \omega$.

For, let $\langle J_k : k \in \omega \rangle$ enumerate all intervals (r, s) with $r < s$ and r, s rational. Now $F \backslash P = \bigcup_{\alpha<\theta}(F_\alpha \backslash F'_\alpha)$. Thus for any $a \in F \backslash P$ there is a unique $\alpha_a < \theta$ such that $a \in F_{\alpha_a} \backslash F'_{\alpha_a}$. a is an isolated point of $F_{\alpha_a} \backslash F'_{\alpha_a}$, so there is a $k_a \in \omega$ such that $J_{k_a} \cap (F_{\alpha_a} \backslash F'_{\alpha_a}) = \{a\}$. Clearly $k(a) \neq k(b)$ for $a \neq b$, so k is a bijection from ω onto $F \backslash P$. So (2) holds.

Now $F = P \cup (F \backslash P)$. Clearly P is closed with no isolated points. $\square$

Theorem 1.20. (Baire category theorem) *If D_n is a dense open set of reals for $n \in \omega$, then $D \overset{\text{def}}{=} \bigcap_{n\in\omega} D_n$ is dense.*

Proof. Let $\langle J_k : k \in \omega \rangle$ enumerate all intervals (r, s) with $r < s$ and r, s rational. Say $J_k = (r_k, q_k)$ for all $k \in \omega$. Let K be any nonempty open interval; we show that

$D \cap K \neq \emptyset$. Let $I_0 = K$. Suppose that I_n has been defined, so that it is a nonempty open interval. Then $I_n \cap D_n$ is nonempty and open. Let $k(n)$ be the smallest integer such that $[q_{k(n)}, r_{k(n)}] \subseteq I_n \cap D_n$ and $|r_{k(n)} - q_{k(n)}| < \frac{1}{n}$. Then set $I_{n+1} = (q_{k(n)}, r_{k(n)})$. Thus $I_{n+1} \subseteq I_n$. This finishes the construction.

(1) If $n < m$, then $[q_{k(m)}, r_{k(m)}] \subseteq (q_{k(n)}, r_{k(n)})$.

For, $[q_{k(m)}, r_{k(m)}] \subseteq I_m \subseteq I_{n+1} = (q_{k(n)}, r_{k(n)})$.

(2) $\langle q_{k(n)} : n \in \omega\rangle$ is Cauchy.

In fact, let $\varepsilon > 0$ be given. Choose a positive integer n such that $\frac{1}{n} < \varepsilon$. Then for any $m, p > n$, by (1) we have $q_{k(m)}, q_{k(p)} \in (q_{k(n)}, r_{k(n)})$, and hence $|q_{k(m)} - q_{k(p)}| < \frac{1}{n}$. So (2) holds.

By (2), let $a = \lim_{n\to\infty} q_{k(n)}$. Then $a \in K$, since for all $n > 0$ we have, by (1), $q_{k(n)} \in [q_{k(n)}, r_{k(n)}] \subseteq (q_{k(0)}, r_{k(0)}) \subseteq [q_{k(0)}, r_{k(0)}]$; so $a \in [q_{k(0)}, r_{k(0)}] \subseteq I_0 = K$. Also, for each n we have $a \in D_n$, since for any $m > n$ we have, by (1), $q_{k(m)} \in [q_{k(m)}, r_{k(m)}] \subseteq (q_{k(n)}, r_{k(n)}) \subseteq [q_{k(n)}, r_{k(n)}]$; so $a \in [q_{k(n)}, r_{k(n)}] \subseteq D_n$. □

The *Baire space* is ${}^\omega\omega$ with the product topology, in which the sets $O(s) \stackrel{\text{def}}{=} \{f \in {}^\omega\omega : s \subseteq f\}$ form a basis, where s is a finite sequence of natural numbers.

Let $d(f, f) = 0$ for any $f \in {}^\omega\omega$, andd $d(f, g) = \frac{1}{2^{n+1}}$ where n is minimum such that $f(n) \neq g(n)$ for $f \neq g$.

Proposition 1.21. *d is a metric, and the topology determined by d is the standard topology.*

Proof. Condition for the metric. Let f, g, h be given distinct functions. Let p, m, n be the least difference places for f, h; f, g; g, h. Then $f(p) \neq g(p)$ or $g(p) \neq h(p)$. So $m \leq p$ or $n \leq p$. Hence $2^{m+1} \leq 2^{p+1}$ or $2^{n+1} \leq 2^{p+1}$, hence $\frac{1}{2^{p+1}} \leq \frac{1}{2^{m+1}}$ or $\frac{1}{2^{p+1}} \leq \frac{1}{2^{n+1}}$; hence $d(f, h) \leq d(f, g)$ or $d(f, h) \leq d(g, h)$. Thus the triangle inequality holds.

The metric topology coincides with the indicated topology: first we show that $O(s)$ is open in the metric topology. Let $f \in O(s)$. Let $\varepsilon = \frac{1}{2^n}$. Suppose that $g \in S_\varepsilon(f)$. Then $d(f, g) < \varepsilon$, so g agrees with f on n, and so $g \in O(s)$, as desired. Conversely we show for any positive ε that S_ε is open in the indicated topology. Suppose that $f \in S_\varepsilon$. Choose n such that $\frac{1}{2^n} < \varepsilon$. Let $s = f \restriction n$. For any $g \in O(s)$, the least m such that $f(m) \neq g(m)$ is at least as big as n; so $d(f, g) \leq \frac{1}{2^{n+1}}$. Hence $g \in S_\varepsilon$, as desired.

Separability: let C be the collection of all eventually constant sequences, and suppose that $O(s)$ is given, with notation as on page 40. Let $f \supseteq s$ be eventually constant. Then $f \in O(s)$, as desired.

Completeness: Suppose that $\langle f_m : m \in \omega\rangle$ is a Cauchy sequence. For each $n \in \omega$ let $N(n)$ be smallest such that $\forall m, p \geq N(n)[d(f_m, f_p) \leq \frac{1}{2^{n+1}}]$. Define $g(n) = f_{N(n+1)}(n)$ for all $n \in \omega$. Now

(1) $N(n) \leq N(n+1)$ for all n.

In fact, for all $m, p \geq N(n+1)$ we have $d(f_m, f_p) \leq \frac{1}{2^{n+2}} < \frac{1}{2^{n+1}}$, so $N(n) \leq N(n+1)$.

(2) $d(g, f_{N(n+1)}) \leq \frac{1}{2^{n+1}}$ for all n.

We prove (2) by induction on n. It is trivial for $n = 0$. Assume it for n. Then $g \restriction n = f_{N(n+1)} \restriction n$ by the definition of d, and $g(n) = f_{N(n+1)}(n)$, so $g \restriction (n+1) = f_{N(n+1)} \restriction (n+1)$. We have $d(f_{N(n+1)}, f_{N(n+2)}) \leq \frac{1}{2^{n+2}}$ since $N(n+1), N(n+2) \geq N(n+1)$, so $f_{N(n+1)} \restriction (n+1) = f_{N(n+2)} \restriction (n+1)$. Hence $g \restriction (n+1) = f_{N(n+2)} \restriction (n+1)$. Thus (2) holds.

Now suppose that $\varepsilon > 0$. Choose n so that $\frac{1}{2^n} < \varepsilon$. Then for any $m \geq N(n+1)$ we have

$$d(g, f_m) \leq d(g, f_{N(n+1)}) + d(f_{N(n+1)}, f_m) \leq \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} = \frac{1}{2^n}. \qquad \square$$

We now give a proof that ${}^\omega\omega$ is homeomorphic to the irrationals. Let $a = \langle a_0, a_1, \ldots\rangle$ be an infinite sequence of integers such that $a_i > 0$ for all $i > 0$. We want to give a precise definition of the continued fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots}}}}$$

To start with, we assume that a is a sequence of positive real numbers with domain either ω or some positive integer. We define $[a_0, \ldots, a_l]$ for each $l < \mathrm{dmn}(a)$ by recursion:

$$[a_0] = a_0;$$
$$[a_0, \ldots, a_{k+1}] = a_0 + \frac{1}{[a_1, \ldots, a_{k+1}]}$$

We want to be very explicit as to how these approximations can be written as certain fractions. To this end we make the following recursive definitions:

$$p(a,0) = a_0; \quad q(a,0) = 1;$$
$$p(a,1) = a_0a_1 + 1; \quad q(a,1) = a_1.$$

For $k \geq 2$:

$$\text{(1)} \qquad \begin{aligned} p(a,k) &= a_kp(a,k-1) + p(a,k-2); \\ q(a,k) &= a_kq(a,k-1) + q(a,k-2). \end{aligned}$$

Also, let $a' = \langle a_1, a_2, \ldots\rangle$. Now we claim that for all $i \in \omega$,

$$\begin{aligned} p(a,i+1) &= a_0p(a',i) + q(a',i); \\ q(a,i+1) &= p(a',i). \end{aligned}$$

We prove these equations by induction on i. For $i = 0$ we have

$$\begin{aligned} p(a,1) &= a_0a_1 + 1 = a_0p(a',0) + q(a',0); \\ q(a,1) &= a_1 = p(a',0), \end{aligned}$$

as desired. For $i = 1$,

$$\begin{aligned}
p(a,2) &= a_2 p(a,1) + p(a,0) \\
&= a_0 a_1 a_2 + a_2 + a_0 \\
&= a_0(a_1 a_2 + 1) + a_2 \\
&= a_0 p(a',1) + q(a',1); \\
q(a,2) &= a_2 q(a,1) + q(a,0) \\
&= a_1 a_2 + 1 \\
&= p(a',1),
\end{aligned}$$

as desired. Now we do the inductive step for $i \geq 2$:

$$\begin{aligned}
p(a,i+1) &= a_{i+1} p(a,i) + p(a,i-1) \\
&= a_{i+1}(a_0 p(a',i-1) + q(a',i-1)) + a_0 p(a',i-2) + q(a',i-2) \\
&= a_0(a_{i+1} p(a',i-1) + p(a',i-2)) + a_{i+1} q(a',i-1) + q(a',i-2) \\
&= a_0 p(a',i) + q(a',i); \\
q(a,i+1) &= a_{i+1} q(a,i) + q(a,i-1) \\
&= a_{i+1} p(a',i-1) + p(a',i-2) \\
&= p(a',i),
\end{aligned}$$

as desired. So the above equations hold.

Note by an easy induction that $p(a,k), q(a,k) > 0$ for all k. Now we claim:

$$[a_0, \ldots, a_k] = \frac{p(a,k)}{q(a,k)} \tag{2}$$

for every $k \in \omega$. We prove (2) by induction on k. For $k = 0$, we have

$$[a_0] = a_0 = \frac{p(a,0)}{q(a,0)},$$

as desired. For $k = 1$, we have

$$[a_0, a_1] = a_0 + \frac{1}{a_1} = \frac{a_0 a_1 + 1}{a_1} = \frac{p(a,1)}{q(a,1)},$$

as desired. Inductively, for $k \geq 2$,

$$\begin{aligned}
[a_0, \ldots, a_k] &= a_0 + \frac{1}{[a_1, \ldots, a_k]} \\
&= a_0 + \frac{q(a',k-1)}{p(a',k-1)} \\
&= \frac{a_0 p(a',k-1) + q(a',k-1)}{p(a',k-1)} \\
&= \frac{p(a,k)}{q(a,k)},
\end{aligned}$$

as desired.

From now on we shall write p_k, q_k in place of $p(a,k), q(a,k)$ if a is understood. We also define $p_{-1} = 1$ and $q_{-1} = 0$. Then the equations (1) also hold for $k = 1$, since

$$\begin{aligned} a_1 p_0 + p_{-1} &= a_0 a_1 + 1 = p_1 \quad \text{and} \\ a_1 q_0 + q_{-1} &= a_1 = q_1. \end{aligned}$$

Next we claim that for $k \geq 1$,

(3) $$q_k p_{k-1} - p_k q_{k-1} = -(q_{k-1} p_{k-2} - p_{k-1} q_{k-2}).$$

In fact, multiply the equations (1) by q_{k-1} and p_{k-1} respectively:

$$\begin{aligned} p_k q_{k-1} &= a_k p_{k-1} q_{k-1} + p_{k-2} q_{k-1}; \\ q_k p_{k-1} &= a_k a_{k-1} p_{k-1} + q_{k-2} p_{k-1}. \end{aligned}$$

Subtracting the first of these equations from the second gives (3).

Now $q_0 p_{-1} - p_0 q_{-1} = 1$, so by (3) and induction we get, for $k \geq 0$,

(4) $$q_k p_{k-1} - p_k q_{k-1} = (-1)^k.$$

Hence for $k \geq 1$ we have

(5) $$\frac{p_{k-1}}{q_{k-1}} - \frac{p_k}{q_k} = \frac{(-1)^k}{q_k q_{k-1}} \ .$$

Next, for any $k \geq 1$,

(6) $$q_k p_{k-2} - p_k q_{k-2} = (-1)^{k-1} a_k \ .$$

To see this, multiply the equations (1) by q_{k-2} and p_{k-2} respectively:

$$\begin{aligned} p_k q_{k-2} &= a_k p_{k-1} q_{k-2} + p_{k-2} q_{k-2}; \\ q_k p_{k-2} &= a_k q_{k-1} p_{k-2} + q_{k-2} p_{k-2}. \end{aligned}$$

Now subtract the first from the second and use (4): (6) follows.

From (6):

(7) $$\frac{p_{k-2}}{q_{k-2}} - \frac{p_k}{q_k} = \frac{(-1)^{k-1} a_k}{q_k q_{k-2}} \ .$$

Hence:

(8) $$\left\langle \frac{p_{2k}}{q_{2k}} : k \in \omega \right\rangle \text{ is an increasing sequence;}$$

(9) $$\left\langle \frac{p_{2k+1}}{q_{2k+1}} : k \in \omega \right\rangle \text{ is an decreasing sequence;}$$

Next we claim

$$\frac{p_{2k}}{q_{2k}} < \frac{p_{2l+1}}{q_{2l+1}} \text{ for all } k, l \in \omega \tag{10}$$

In fact, let $m = \max(k, l)$. Then

$$\begin{aligned}\frac{p_{2k}}{q_{2k}} &\leq \frac{p_{2m}}{q_{2m}} \quad \text{by (8)}\\ &< \frac{p_{2m+1}}{q_{2m+1}} \quad \text{by (5)}\\ &\leq \frac{p_{2l+1}}{q_{2l+1}} \quad \text{by (9)}\end{aligned}$$

So (10) holds. Next we claim:

$$p_k < p_{k+1} \text{ and } q_{k+1} < q_{k+2} \quad \text{for all } k \in \omega. \tag{11}$$

In fact, this is clear from the recursive definitions.

Now we assume that our sequence a is infinite, and all a_i are positive integers. It follows from (8), (9), (10), (11), and (5) that the approximations $\frac{p_k}{q_k}$ converge, and by definition the limit is the value of the infinite continued fraction described at the beginning. For a_0 a negative integer but all a_i positive integers for $i > 0$, we define $a' = \langle 1, a_1, a_2, \ldots \rangle$ and define the continued fraction to be

$$a_0 - 1 + \lim_{k \to \infty} \frac{p(a', k)}{q(a', k)}$$

Now we want to see how to represent any real number as a finite or infinite continued fraction. We make a recursive definition for any real number $\alpha > 1$. Let $r(\alpha, 0) = \alpha$. Suppose that we have defined $r(\alpha, i) > 1$. Write $r(\alpha, i) = a(\alpha, i) + s(\alpha, i+1)$ with $a(\alpha, i)$ a positive integer and $s(\alpha, i+1)$ a nonnegative real < 1. If $s(\alpha, i+1) = 0$, the construction stops. Otherwise we define $r(\alpha, i+1) = \frac{1}{s(\alpha, i+1)}$. This finishes the construction. Let $l(\alpha)$ be the index i such that $s(\alpha, i+1) = 0$, or $l(\alpha) = \omega$ if there is no such index. We need the following technical fact.

(12) If $\alpha > 1$ and $l(\alpha) > 1$, then $l(r(\alpha, 1)) = l(\alpha) - 1$, and for each $j \leq l(\alpha) - 1$ we have $r(r(\alpha, 1), j) = r(\alpha, j+1)$ and $a(r(\alpha, 1), j) = a(\alpha, j+1)$.

The second equality of the conclusion follows from the first, so we do not worry about it. By induction on j we prove that $r(r(\alpha, 1), j)$ is defined and equals $r(\alpha, j+1)$ for each $j \leq l(\alpha) - 1$. For $j = 0$ we have $r(r(\alpha, 1), 0)$ defined and it equals $r(\alpha, 1)$, as desired. Now assume our result for j, with $j + 1 \leq l(\alpha) - 1$. Then

$$r(r(\alpha, 1), j) = r(\alpha, j+1) = a(\alpha, j+1) + s(\alpha, j+2).$$

Thus $a(r(\alpha, 1), j) = a(\alpha, j+1)$ and $s(r(\alpha, 1), j+1) = s(\alpha, j+2)$. Now $j + 2 \leq l(\alpha)$, so $s(\alpha, j+2) > 0$, and hence by definition, $r(\alpha, j+2) = \frac{1}{s(\alpha, j+2)}$. Hence $r(r(\alpha, 1), j+1) = r(\alpha, j+2)$, as desired.

Finally, if $j = l(\alpha)$, then $r(\alpha, j) = a(\alpha, j)$, and hence $r(r(\alpha, 1), j-1) = r(\alpha, j) = a(\alpha, j+1)$ and so $l(r(\alpha, 1)) = j-1$, as desired in (13).

(13) If $\alpha > 1$ and $n \leq l(\alpha)$, then $\alpha = [a(\alpha, 0), a(\alpha, 1), \ldots, a(\alpha, n-1), r(\alpha, n)]$.

We prove this by induction on n. For $n = 0$, $[r(\alpha, 0)] = \alpha$. Assume that our condition is true for n, and $n+1 \leq l(\alpha)$. Then

$$\begin{aligned}
&[a(\alpha,0), a(\alpha,1), \ldots, a(\alpha,n), r(\alpha,n+1)] \\
&\qquad = a(\alpha,0) + \frac{1}{[a(\alpha,1), a(\alpha,2), \ldots, a(\alpha,n), r(\alpha,n+1)]} \\
&\qquad = a(\alpha,0) + \frac{1}{[a(r(\alpha,1),0)a(r(\alpha,1),1), \ldots, a(r(\alpha,1),n-1), r(r(\alpha,1),n)]} \\
&\qquad = a(\alpha,0) + \frac{1}{r(\alpha,1)} \\
&\qquad = a(\alpha,0) + s(\alpha,1) \\
&\qquad = \alpha,
\end{aligned}$$

completing the inductive proof.

(14) If $\alpha > 1$ is rational, then the above definition of $r(\alpha, i)$'s terminates after finitely many steps.

In fact, it suffices to show that if $r(\alpha, i) = \frac{b}{c}$ with b, c positive integers and g.c.d$(b, c) = 1$, and $r(\alpha, i+1)$ is defined, then $r(\alpha, i+1)$ has the form $\frac{d}{e}$, with d and e positive integers with $e < c$. To prove this, recall that $r(\alpha, i) = a(\alpha, i) + s(\alpha, i+1)$, with $s(\alpha, i+1)$ a nonnegative real < 1, and $r(\alpha, i+1) = \frac{1}{s(\alpha, i+1)}$. Thus

$$\frac{b}{c} = r(\alpha,i) = a(\alpha,i) + s(\alpha,i+1) \quad \text{and hence}$$

$$b = ca(\alpha,i) + cs(\alpha,i+1); \tag{15}$$

Hence

$$\begin{aligned}
r(\alpha,i+1) &= \frac{1}{s(\alpha,i+1)} \\
&= \frac{1}{r(\alpha,i) - a(\alpha,i)} \\
&= \frac{1}{\frac{b}{c} - a(\alpha,i)} \\
&= \frac{c}{b - ca(\alpha,i)} \\
&= \frac{c}{cs(\alpha,i+1)} \quad \text{by (15)},
\end{aligned}$$

and $cs(\alpha, i+1)$ is a positive integer $< c$, as desired.

(16) If α is rational, then there exist integers $a_0, a_1, \ldots, a_n$ with $a_i > 0$ for all $i > 0$ such that $\alpha = [a_0, a_1, \ldots, a_n]$.

In fact, let m be an integer such that $\alpha + m > 1$; if $\alpha > 1$, let $m = 0$. By (14), $n \stackrel{\text{def}}{=} l(\alpha + m)$ is finite. We then have $r(\alpha + m, n) = a(\alpha + m, n)$. Hence by (13) we have $\alpha + m = [a(\alpha + m, 0), \ldots, a(\alpha + m, n)]$, and the desired conclusion follows.

(17) If $\langle a_0, a_1, \ldots\rangle$ is a sequence of rational numbers each greater than 0, then also $[a_0, a_1, \ldots, a_n]$ is rational for each n.

This is clear from the basic definition, by induction.

(18) Let $\alpha > 1$ be irrational. Then by (17), the sequence

$$b \stackrel{\text{def}}{=} \langle a(\alpha, 0), a(\alpha, 1), \ldots\rangle$$

never terminates. We claim that for each positive integer n,

$$\alpha = \frac{p(b, n-1)r(\alpha, n) + p(b, n-2)}{q(b, n-1)r(\alpha, n) + q(b, n-2)}.$$

We prove by induction that for every positive integer n, this holds for all irrationals $\alpha > 1$. First, the case $n = 1$:

$$\begin{aligned}
\frac{p(b,0)r(\alpha,1) + p(b,-1)}{q(b,0)r(\alpha,1) + q(b,-1)} &= \frac{a(\alpha,0)r(\alpha,1) + 1}{r(\alpha,1)} \\
&= a(\alpha,0) + \frac{1}{r(\alpha,1)} \\
&= a(\alpha,0) + s(\alpha,1) \\
&= r(\alpha,0) \\
&= \alpha,
\end{aligned}$$

as desired. Now we assume our statement for n. In fact, we apply it to $r(\alpha, 1)$ rather than α. Note that $r(\alpha, 1) > 1$, and it is irrational by (17) and (13). Let

$$\begin{aligned}
c &= \langle a(\alpha,1), a(\alpha,2), \ldots\rangle \\
&= \langle a(r(\alpha,1),0), a(r(\alpha,1),1), \ldots\rangle,
\end{aligned}$$

by (12). Hence, starting with the inductive hypothesis,

$$\begin{aligned}
r(\alpha,1) &= \frac{p(c,n-1)r(r(\alpha,1),n) + p(c,n-2)}{q(c,n-1)r(r(\alpha,1),n) + q(c,n-2)} \\
&= \frac{p(c,n-1)r(\alpha,n+1) + p(c,n-2)}{q(c,n-1)r(\alpha,n+1) + q(c,n-2)}.
\end{aligned}$$

Hence, using the equations following (1),

$$
\begin{aligned}
\alpha &= r(\alpha,0)\\
&= a(\alpha,0)+s(\alpha,1)\\
&= a(\alpha,0)+\frac{1}{r(\alpha,1)}\\
&= a(\alpha,0)+\frac{q(c,n-1)r(\alpha,n+1)+q(c,n-2)}{p(c,n-1)r(\alpha,n+1)+p(c,n-2)}\\
&= \frac{a(\alpha,0)p(c,n-1)r(\alpha,n+1)+a(\alpha,0)p(c,n-2)+q(c,n-1)r(\alpha,n+1)+q(c,n-2)}{p(c,n-1)r(\alpha,n+1)+p(c,n-2)}\\
&= \frac{p(b,n)r(\alpha,n+1)+p(b,n-1)}{q(b,n)r(\alpha,n+1)+q(b,n-1)},
\end{aligned}
$$

which finishes the inductive proof of (18).

We now omit the parameter b, as it is understood in what follows.

(19) Let $\alpha > 1$ be irrational. Then for every positive integer n,

$$\alpha - \frac{p_n}{q_n} = \frac{(p_{n-1}q_{n-2}-q_{n-1}p_{n-2})(r_n-a_n)}{(q_{n-1}r_n+q_{n-2})(q_{n-1}a_n+q_{n-2})}.$$

To prove this, first note by (18) and (1) that

$$\alpha - \frac{p_n}{q_n} = \frac{p_{n-1}r_n-p_{n-2}}{q_{n-1}r_n+q_{n-2}} - \frac{p_{n-1}a_n+p_{n-2}}{q_{n-1}a_n+q_{n-2}}. \tag{20}$$

Now we have

$$
\begin{aligned}
&(p_{n-1}r_n-p_{n-2})(q_{n-1}a_n+q_{n-2})-(p_{n-1}a_n+p_{n-2})(q_{n-1}r_n+q_{n-2})\\
&\quad= p_{n-1}q_{n-1}a_nr_n+p_{n-1}q_{n-2}r_n+p_{n-2}q_{n-1}a_n+p_{n-2}q_{n-2}\\
&\qquad - p_{n-1}q_{n-1}a_nr_n-p_{n-1}q_{n-2}a_n-p_{n-2}q_{n-1}r_n-p_{n-2}q_{n-2}\\
&\quad= (p_{n-1}q_{n-2}-q_{n-1}p_{n-2})(r_n-a_n).
\end{aligned}
$$

Hence from (20) we get (19).

(21) For irrational $\alpha > 1$ we have

$$\alpha = [a(\alpha,0), a(\alpha,1), \ldots].$$

In fact, note from (4) that $p_{n-1}q_{n-2}-q_{n-1}p_{n-2} = (-1)^{n-1}$, while by definition we have $r(\alpha,n)-a(\alpha,n) = s(\alpha,n+1) < 1$. Hence by (19),

$$\left|\alpha - \frac{p_n}{q_n}\right| \le \frac{1}{(q_{n-1}r_n+q_{n-2})(q_{n-1}a_n+q_{n-2})} < \frac{1}{q_{n-2}^2},$$

and hence (21) follows from (11).

Now for any irrational $\alpha > 1$, define

$$f(\alpha) = \langle a(\alpha,0), a(\alpha,1), \ldots\rangle.$$

Then by the above results, f is a one-to-one function mapping the set $\mathcal{N}$ of irrationals $>$ 1 onto the set ${}^{\omega}(\omega\backslash 1)$. The latter set is clearly homeomorphic to ${}^{\omega}\omega$.

(22) The set of irrationals > 1 is homeomorphic to the entire set of irrationals.

To see this, define g by setting, for each irrational $x > 1$,

$$g(x) = \begin{cases} x+m & \text{if } 0 < m < x < m+1 \text{ with } m \in \omega, \\ x+3m+1 & \text{if } -m < x < -m+1 \text{ with } m \in \omega. \end{cases}$$

Then g maps $(m, m+1)_{\text{irr}}$ one-one onto $(2m, 2m+1)_{\text{irr}}$ for each positive integer m, and $(-m, -m+1)_{\text{irr}}$ one-one onto $(2m+1, 2m+2)_{\text{irr}}$ for each $m \in \omega$. Clearly g is the desired homeomorphism.

Thus to finish this digression it suffices to show that f, defined above, is a homeomorphism. To do this, we need the following fact.

(23) Suppose that $a_0, \ldots, a_n, b_0, \ldots, b_{n-1}$ are positive integers and r is a real number > 1. Assume that

$$\begin{aligned} &[a_0, \ldots, a_{n-1}] < [b_0, \ldots, b_{n-1}, r] < [a_0, \ldots, a_n] \quad \text{if } n \text{ is odd} \\ &[a_0, \ldots, a_{n-1}] > [b_0, \ldots, b_{n-1}, r] > [a_0, \ldots, a_n] \quad \text{if } n \text{ is even} \end{aligned}$$

Then $a_i = b_i$ for all $i < n$. Cf here (2), (8), (9), (10).

We prove (23) by induction on n. For $n = 1$ the assumption is that $a_0 < b_0 + \frac{1}{r} < a_0 + \frac{1}{a_1}$. So clearly $a_0 = b_0$. Now assume (23) for an odd n; we prove it for $n+1$ and $n+2$. So, first suppose that

$$[a_0, \ldots, a_n] > [b_0, \ldots, b_n, r] > [a_0, \ldots, a_{n+1}].$$

Thus

$$a_0 + \frac{1}{[a_1, \ldots, a_n]} > b_0 + \frac{1}{[b_1, \ldots, b_n, r]} > a_0 + \frac{1}{[a_1, \ldots, a_{n+1}]},$$

and it follows that $a_0 = b_0$ and

$$[a_1, \ldots, a_n] < [b_1, \ldots, b_n, r] < [a_1, \ldots, a_{n+1}];$$

then the inductive hypothesis yields $a_i = b_i$ for all $i = 1, \ldots, n$, which proves our statement for $n+1$.

The inductive step to $n+2$ is clearly similar. So (23) holds.

Now to show that f is continuous, suppose that $s \in {}^{n}(\omega\backslash 1)$; we want to show that $f^{-1}[O(s)]$ is open. We may assume that $n = 2m+1$ for some natural number m. Let $\alpha \in f^{-1}[O_s]$. Define $a_i = a(\alpha, i)$ for all i. Thus $a_0 = s_0, \ldots, a_{2m} = s_{2m}$. By (2) and

(8)–(10) we have $[a_0, \ldots, a_{2m}] < \alpha < [a_0, \ldots, a_{2m+1}]$. Choose ε so that $[a_0, \ldots, a_{2m}] + \varepsilon < \alpha < \alpha + \varepsilon < [a_0, \ldots, a_{2m+1}]$. We claim:

(24) For every irrational $\beta > 1$, if $|\alpha - \beta| < \varepsilon$, then $\beta \in f^{-1}[O(s)]$.

This will prove continuity of f. To prove (24), assume its hypothesis, and let $b_i = b(\beta, i)$ for all i.

Case 1. $\beta < \alpha$. Thus $\alpha - \beta < \varepsilon$. Hence $[a_0, \ldots, a_{2m}] < [a_0, \ldots, a_{2m}] + \varepsilon < \alpha < \beta + \varepsilon$, so $[a_0, \ldots, a_{2m}] < \beta$. If $[a_0, \ldots, a_{2m+1}] \le \beta$, then by (8)–(10), $\alpha < \beta$, contradiction. So $\beta < [a_0, \ldots, a_{2m+1}]$. Now $\beta = [b_0, \ldots, b_{2m}, r_{2m+1}]$ by (13), so by (23), $a_i = b_i$ for all $i \le 2m$, as desired.

Case 2. $\alpha < \beta$. Thus $\beta - \alpha < \varepsilon$, so $\beta < \alpha + \varepsilon$. Hence

$$[a_0, \ldots, a_{2m}] < \alpha < \beta < \alpha + \varepsilon < [a_0, \ldots, a_{2m+1}],$$

and the argument is finished as in Case 1.

So (24) holds, and f is continuous.

(25) f is an open mapping.

For, suppose that $\alpha > 1$ is irrational, and ε is a positive real number; we want to show that $f[S_\varepsilon(\alpha)]$ is open. Let $b \in f[S_\varepsilon(\alpha)]$; we want to find a finite sequence s such that $b \in O(s) \subseteq f[S_\varepsilon(\alpha)]$. Say $b = f(\beta)$ with $\beta \in S_\varepsilon(\alpha)$. So $|\alpha - \beta| < \varepsilon$. Choose m such that

$$\frac{1}{q(b, 2m)q(b, 2m+1)} < \varepsilon - |\alpha - \beta|.$$

This is possible by (11). Let $s = \langle b_0, \ldots, b_{2m+1} \rangle$. So $b \in O(s)$. Now suppose that $c \in O(s)$. Then

$$[b_0, \ldots, b_{2m}] = [c_0, \ldots, c_{2m}] < [c] < [c_0, \ldots, c_{2m+1}] = [b_0, \ldots, b_{2m+1}]$$

by (8)–(10). Also,

$$[b_0, \ldots, b_{2m}] = [c_0, \ldots, c_{2m}] < \beta < [c_0, \ldots, c_{2m+1}] = [b_0, \ldots, b_{2m+1}]$$

by (8)–(10). Now

$$\begin{aligned} [b_0, \ldots, b_{2m+1}] - [b_0, \ldots, b_{2m}] &= \frac{p(b, 2m+1)}{q(b, 2m+1)} - \frac{p(b, 2m)}{q(b, 2m)} \quad \text{by (2)} \\ &= \frac{1}{q(b, 2m)q(b, 2m+1)} \\ &< \varepsilon - |\alpha - \beta|. \end{aligned}$$

Hence

$$|[c] - \alpha| \le |[c] - \beta| + |\beta - \alpha| < \varepsilon,$$

and so $c = f([c]) \in f[S_\varepsilon(\alpha)]$, as desired.

In summary:

Theorem 1.22. *${}^{\omega}\omega$ is homeomorphic to the irrationals.* $\square$

Seq is the set of all finite sequences of natural numbers. A *sequential tree* is a subset T of Seq such that $\forall t \in T \forall n \in \omega[(t \restriction n) \in T]$. If T is a sequential tree, then

$$[T] = \{f \in {}^{\omega}\omega : \forall n \in \omega[(f \restriction n) \in T\}.$$

Proposition 1.23. *$[T]$ is closed in ${}^{\omega}\omega$.*

Proof. Suppose that $f \in {}^{\omega}\omega \backslash [T]$. Choose $n \in \omega$ such that $(f \restriction n) \notin T$. Then $f \in O(f \restriction n) \subseteq {}^{\omega}\omega \backslash [T]$. $\square$

Proposition 1.24. *If F is a closed subset of ${}^{\omega}\omega$, then*

$$T_F \stackrel{\text{def}}{=} \{s \in \text{Seq} : \exists f \in F[s \subseteq f]\}$$

is a sequential tree, and $[T_F] = F$.

Proof. Clearly T_F is a sequential tree. Now suppose that $f \in [T_F]$ but $f \notin F$. Since F is closed, choose s such that $f \in O(s)$ and $O(s) \cap F = \emptyset$. Say s has length n. We have $s \subseteq f$. Since $f \in T_F$ we have $s = f \restriction n \in T_F$. Choose $g \in F$ such that $s \subseteq g$. Then $g \in O(s) \cap F$, contradiction.

If $f \in F$ and $n \in \omega$, then $(f \restriction n) \in T_F$. Hence $f \in [T_F]$. $\square$

A sequential tree T is *perfect* iff $\forall t \in T \exists s_1, s_2 \geq t[s_1 \not\leq s_2$ and $s_2 \not\leq s_1]$.

Proposition 1.25. *Let $F \subseteq {}^{\omega}\omega$ be closed. Then F is perfect iff T_F is perfect.*

Proof. Let $F \subseteq {}^{\omega}\omega$ be closed. First suppose that T_F is not perfect. So there is a $t \in T_F$ such that all elements above t are comparable. Let $f \in F$ with $t \subseteq f$. We claim that $\{f\}$ is open in F; hence f is isolated in F and so F is not perfect. Clearly $O(t) \cap F = \{f\}$.

Now suppose that F is not perfect. Choose $f \in {}^{\omega}\omega$ such that $O(f) \cap F = \{f\}$. So there is a $t \in T_F$ such that all elements above t are comparable. Hence T_F is not perfect. $\square$

Lemma 1.26. *Let T be a sequential tree. Define*

$$T' = \{t \in T : \exists s_1, s_2 \geq t[s_1 \textit{ and } s_2 \textit{ are incomparable.}]\}.$$

Then $|[T] \backslash [T']| \leq \omega$.

Proof. For each $f \in [T] \backslash [T']$ let $s_f = f \restriction n$, where n is minimum such that $(f \restriction n) \notin T$. Clearly $s_f = s_g$ implies that $f = g$. Hence $\langle s_f : f \in [T] \backslash [T'] \rangle$ is one-one. $\square$

Now let T be any sequential tree. Define

$$\begin{aligned} T_0 &= T; \\ T_{\alpha+1} &= T'_{\alpha}; \\ T_{\alpha} &= \bigcap_{\beta<\alpha} T_{\beta} \quad \text{for } \alpha \text{ limit.} \end{aligned}$$

Since T is countable, there is a $\theta < \omega_1$ such that $T_{\theta+1} = T_\theta$.

Proposition 1.27. $[T]\backslash[T_\theta]$ *is countable.*

Proof.

$$\forall\alpha\left[\left[\bigcap_{\beta<\alpha} T_\beta\right] = \bigcap_{\beta<\alpha}[T_\beta]\right]. \tag{1}$$

In fact,

$$\begin{aligned} f \in \left[\bigcap_{\beta<\alpha} T_\beta\right] \quad &\text{iff} \quad \forall n \in \omega\left[(f \restriction n) \in \bigcap_{\beta<\alpha} T_\beta\right] \\ &\text{iff} \quad \forall n \in \omega \forall \beta < \alpha[(f \restriction n) \in T_\beta] \\ &\text{iff } \forall \beta < \alpha \forall n \in \omega[(f \restriction n) \in T_\beta] \\ &\text{iff } \forall \beta < \alpha[f \in [T_\beta]] \\ &\text{iff } f \in \bigcap_{\beta<\alpha}[T_\beta]. \end{aligned}$$

Thus (1) holds.

$$[T]\backslash[T_\theta] = \bigcup_{\alpha<\theta}([T_\alpha]\backslash[T'_\alpha]). \tag{2}$$

For, suppose that $f \in [T]\backslash[T_\theta]$. Let β be minimum such that $f \notin [T_\beta]$. If β is limit, then

$$f \notin [T_\beta] = \left[\bigcap_{\alpha<\beta} T_\alpha\right] = \bigcap_{\alpha<\beta}[T_\alpha],$$

and so there is an $\alpha < \beta$ such that $f \notin [T_\alpha]$, contradiction.

So β is a successor $\alpha + 1$. Since clearly $\beta \le \theta$, we have $\alpha < \theta$. Now $[T_\beta] = [T'_\alpha]$, so $f \in [T_\alpha]\backslash[T'_\alpha]$. This proves $\subseteq$.

For $\supseteq$, note that if $\alpha < \theta$, then $[T'_\alpha] = [T_{\alpha+1}] \supseteq [T_\theta]$, and $\supseteq$ follows.

Hence (2) holds, and so $[T]\backslash[T_\theta]$ is countable. □

Clearly $T'_\theta = T_\theta$. Thus $[T] = ([T]\backslash[T_\theta]) \cup [T_\theta]$ is a decomposition into a countable set and a perfect set.

Theorem 1.28. ${}^\omega\omega \times {}^\omega\omega$ *is homeomorphic to* ${}^\omega\omega$.

Proof. Let f be a bijection from ω onto $\omega \times \omega$. Define $F : {}^\omega\omega \times {}^\omega\omega \to {}^\omega\omega$ by

$$(F(x, y))(n) = f^{-1}(x(n), y(n)).$$

F is one-one: suppose that $(x, y) \neq (x', y')$. Say wlog $x(n) \neq x'(n)$. Then $(F(x, y))(n) = f^{-1}(x(n), y(n)) \neq f^{-1}(x'(n), y'(n)) = (F(x', y'))(n)$.

F maps onto ${}^{\omega}\omega$: suppose that $z \in {}^{\omega}\omega$. Define $x(n) = 1^{st}(f(z(n))$ and $y(n) = 2^{nd}(f(z(n))$ for all n. Then for any $n \in \omega$,

$$(F(x, y))(n) = f^{-1}(x(n), y(n)) = f^{-1}(1^{st}(f(z(n)), 2^{nd}(f(z(n))) = f^{-1}(f(z(n)) = z(n).$$

F is continuous: suppose that $s \subseteq \omega \times \omega$ is a finite function, and $(x, y) \in F^{-1}[\{z \in {}^{\omega}\omega : s \subseteq z\}]$. Let $\text{dmn}(u) = \text{dmn}(s)$ with $u(n) = 1^{st}(f(s(n))$, and let $\text{dmn}(v) = \text{dmn}(s)$ with $v(n) = 2^{nd}(f(s(n))$. Then $s \subseteq F(x, y)$, and so for any $n \in \text{dmn}(s)$, $s(n) = (F(x, y))(n) = f^{-1}(x(n), y(n))$, and hence $u(n) = 1^{st}(f(s(n)) = x(n)$ and $v(n) = 2^{nd}(f(s(n)) = y(n)$. So $(x, y) \in U_u \times U_v$. Suppose that $(w, t) \in U_u \times U_v$. Then for all $n \in \text{dmn}(s)$,

$$(F(w, t))(n) = f^{-1}(w(n), t(n)) = f^{-1}(u(n), v(n)) = f^{-1}(1^{st}(f(s(n))), 2^{nd}f(s(n)) = s(n).$$

Thus $U_u \times U_v \subseteq F^{-1}[\{z \in {}^{\omega}\omega : s \subseteq z\}]$. So F is continuous.

F is open: Suppose that $u, v \subseteq \omega \times \omega$ are finite functions and $z \in F[U_u \times U_v]$. Choose $x \in U_u$ and $y \in U_v$ so that $z = F(x, y)$. Let n be greater than each member of $\text{dmn}(u) \cup \text{dmn}(v)$. Let $w(i) = z(i)$ for all $i < n$. Then $z \in U_w$. We claim that $U_w \subseteq F[U_u \times U_v]$. Suppose that $t \in U_w$. Let $a(i) = 1^{st}(f(t(i))$ for all i, and $b(i) = 2^{nd}(f(t(i))$ for all i. By "onto" above, $t = F(a, b)$. Now $a \in U_u$, since if $i \in \text{dmn}(u)$, then

$$a(i) = 1^{st}(f(t(i))) = 1^{st}(f(w(i))) = 1^{st}(f(z(i))) = 1^{st}(f((F(x, y))(i))) = x(i) = u(i).$$

Similarly, $b \in U_v$. $\square$

Theorem 1.29. *${}^{\omega}({}^{\omega}\omega)$ is homeomorphic to ${}^{\omega}\omega$.*

Proof. Let $B = \{U : \exists F \in [\omega]^{<\omega} \exists V[V$ is a function with domain F and $\forall m \in F[V_m$ is an open subset of ${}^{\omega}\omega$ and $U = \{x \in {}^{\omega}({}^{\omega}\omega) : \forall m \in F[x_m \in V_m]\}]]$. B is the standard base for the topology on ${}^{\omega}({}^{\omega}\omega)$. Let $C = \{U : \exists F \in [\omega]^{<\omega} \exists a, G[G : F \to [\omega]^{<\omega}$ and a is a function with domain F and for all $m \in F[a_m : G_m \to \omega]$ and $U = \{x \in {}^{\omega}({}^{\omega}\omega) : \forall m \in F[a_m \subseteq x_m]\}]\}$

Lemma 1. *C is a base for the topology on ${}^{\omega}({}^{\omega}\omega)$.*

Proof. First, every member of C is open. For, let $U \in C$, with associated F, a, G. For each $m \in F$ let $V_m = \{y \in {}^{\omega}\omega : a_m \subseteq y\}$. Then V_m is open in ${}^{\omega}\omega$. We claim that

$$U = \{x \in {}^{\omega}({}^{\omega}\omega) : \forall m \in F[x_m \in V_m]\},$$

so that U is open. In fact, if $x \in U$, then $\forall m \in F[a_m \subseteq x_m]$, and so $\forall m \in F[x_m \in V_m]$. Conversely, if $\forall m \in F[x_m \in V_m]$, then $\forall m \in F[a_m \subseteq x_m]$, and hence $x \in U$.

Second, every member of B is a union of members of C. For, suppose that $U \in B$, with associated F, V. Take any $x \in U$. For $m \in F$ we have $x_m \in V_m$, so there exist $G_m \in [\omega]^{<\omega}$ and $a_m : G_m \to \omega$ such that $x_m \in U_{a_m} \subseteq V_m$. So $a_m \subseteq x_m$. Thus $x \in V$ with $V = \{y \in {}^{\omega}({}^{\omega}\omega) : \forall m \in F[a_m \subseteq y_m]\}]\}$, and $V \subseteq U$. $\square$

Let $g : \omega \to \omega \times \omega$ be a bijection. Define $H : {}^\omega({}^\omega\omega) \to {}^\omega\omega$ by setting, for any $x \in {}^\omega({}^\omega\omega)$,

$$(H(x))_n = x_{1^{\rm st}(g(n))}(2^{\rm nd}(g(n))).$$

H is one-one: suppose that $H(x) = H(y)$. Take any $m, p \in \omega$. Then

$$x_m(p) = (H(x))_{g^{-1}(m,p)} = (H(y))_{g^{-1}(m,p)} = y_m(p).$$

H is onto: suppose that $y \in {}^\omega\omega$. Define $x \in {}^\omega({}^\omega\omega)$ by setting, for any $m, p \in \omega$, $x_m(p) = y(g^{-1}(m,p))$. Then for any $n \in \omega$,

$$(H(x))_n = x_{1^{\rm st}(g(n))}(2^{\rm nd}(g(n))) = y(n).$$

H is continuous: suppose that $b : K \to \omega$ with K finite, and $x \in H^{-1}[U_b]$, where $U_b = \{y \in {}^\omega\omega : b \subseteq y\}$. Thus $H(x) \in U_b$, so $b \subseteq H(x)$. Thus

$$\forall n \in K[b_n = x_{1^{\rm st}(g(n))}(2^{\rm nd}(g(n)))].$$

Let $F = \{m : \exists n \in K[1^{\rm st}(g(n)) = m]\}$. For each $m \in F$ let $G_m = \{p \in \omega : \exists n \in K[g(n) = (m,p)]\}$ and define $a_m(p) = b_{g^{-1}(m,p)}$ for all $m \in G_m$ and $p \in G_m$. Let $V = \{y \in {}^\omega({}^\omega\omega) : \forall m \in F[a_m \subseteq y_m]\}$. So $V \in C$. For all $m \in F$ and $p \in G_m$ we have $a_m(n) = x_m(n)$, so $x \in V$. For all $y \in V$ and $n \in K$ we have

$$(H(y))_n = y_{1^{\rm st}(g(n))}(2^{\rm nd}(g(n))) = a_{1^{\rm st}(g(n))}(2^{\rm nd}(g(n))) = b_n,$$

so $H(y) \in U_b$. So $x \in V \subseteq H^{-1}[U_b]$.

H^{-1} is continuous: suppose that $U \in C$ with associated F, a, G, and $x \in H[U]$. Say $x = H(y)$ with $y \in U$. Thus

$$\forall n \in \omega[x_n = y_{1^{\rm st}(g(n))}(2^{\rm nd}(g(n)))] \quad \text{and} \quad \forall m \in F \forall p \in G_m[a_m(p) = y_m(p).$$

Hence

$$\forall m \in F \forall p \in G_m[a_m(p) = x_{g^{-1}(m,p)}].$$

Now let $K = \{n \in \omega : \exists m \in F \exists p \in G_m[n = g^{-1}(m,p)]\}$. For each $n \in K$ let $s(n) = a_{1^{\rm st}(g(n))}(2^{\rm nd}(g(n)))$. Then for any $n \in K$, $s(n) = a_{1^{\rm st}(g(n))}(2^{\rm nd}(g(n))) = x_n$. So $x \in U_s$. Suppose that $y \in U_s$. Define $z_m(p) = y(g^{-1}(m,p))$ for all $m, p \in \omega$. Then for all $n \in \omega$, $y(n) = z_{1^{\rm st}(g(n))}(2^{\rm nd}(g(n)))$, so $y = H(z)$. For any $m \in F$ and $p \in G_m$,

$$a_m(p) = s(g(m,p)) = y(g^{-1}(m,p)) = z_m(p)),$$

and hence $z \in U$. Thus $x \in U_s \subseteq H[U]$. $\square$

Proposition 1.30. *If λ is an infinite cardinal, $\langle \kappa_\alpha : \alpha < \lambda \rangle$ is a sequence of nonzero cardinals, and $\forall \alpha, \beta < \lambda[\alpha < \beta \to \kappa_\alpha \leq \kappa_\beta]$, then*

$$\prod_{\alpha<\lambda} \kappa_\alpha = \left(\bigcup_{\alpha<\lambda} \kappa_\alpha \right)^\lambda.$$

Proof. Let $\mu = \bigcup_{\alpha<\lambda} \kappa_\alpha$.

(1) If $\Gamma \in [\lambda]^\lambda$, then $\mu \leq \prod_{\alpha\in\Gamma} \kappa_\alpha$.

In fact, Γ is cofinal in λ, so for any $\beta < \lambda$ there is an $\alpha \in \Gamma$ such that $\beta \leq \alpha$. Hence $\kappa_\beta \leq \prod_{\alpha\in\Gamma} \kappa_\alpha$, and (1) follows.

Now write $\lambda = \bigcup_{\alpha<\lambda} \Gamma_\alpha$, with each Γ_α of size λ and $\forall\alpha, \beta < \lambda[\alpha \neq \beta \to \Gamma_\alpha \cap \Gamma_\beta = \emptyset]$. Then

$$\prod_{\alpha<\lambda} \kappa_\alpha \leq \prod_{\alpha<\lambda} \mu = \mu^\lambda = \prod_{\alpha<\lambda} \mu \leq \prod_{\alpha<\lambda} \prod_{\beta\in\Gamma_\alpha} \kappa_\beta = \prod_{\alpha<\lambda} \kappa_\alpha. \qquad \square$$

Proposition 1.31. *If κ is a limit cardinal, then $2^\kappa = (2^{<\kappa})^{\mathrm{cf}(\kappa)}$.*

Proof. Write $\kappa = \sum_{i<\mathrm{cf}(\kappa)} \lambda_i$, with each $\lambda_i < \kappa$. Then

$$2^\kappa = 2^{\sum_{i<\mathrm{cf}(\kappa)} \lambda_i} = \prod_{i<\mathrm{cf}(\kappa)} 2^{\lambda_i} \leq \prod_{i<\mathrm{cf}(\kappa)} 2^{<\kappa} = (2^{<\kappa})^{\mathrm{cf}(\kappa)} \leq (2^\kappa)^{\mathrm{cf}(\kappa)} \leq 2^\kappa. \qquad \square$$

Proposition 1.32. *If κ is singular and $\exists\mu < \kappa\forall\nu \in [\mu, \kappa)[2^\nu = \lambda]$, then $2^\kappa = \lambda$.*

Proof. We may assume that $\mathrm{cf}(\kappa) \leq \mu$. Then $2^{<\kappa} = \lambda = 2^\mu$, and so

$$2^\kappa = (2^{<\kappa})^{\mathrm{cf}(\kappa)} = (2^\mu)^{\mathrm{cf}(\kappa)} = 2^\mu = \lambda. \qquad \square$$

Assume that κ is a limit cardinal and $\forall\mu < \kappa\exists\nu \in [\mu, \kappa)[2^\mu < 2^\nu]$. Let $\lambda = 2^{<\kappa}$. Then $\mathrm{cf}(\lambda) = \mathrm{cf}(\kappa)$ and $2^\kappa = \lambda^{\mathrm{cf}(\lambda)}$. $\square$

Proof. Let $\langle\theta_\xi : \xi < \mathrm{cf}(\kappa)\rangle$ be a strictly increasing sequence of cardinals with supremum κ. Now we define by recursion $\langle\rho_\xi : \xi < \mathrm{cf}(\kappa)\rangle$ with each $\rho_\xi < \kappa$. Assuming that ρ_η has been defined for all $\eta < \xi$, with $\xi < \mathrm{cf}(\kappa)$, let $\sigma = \bigcup_{\eta<\xi} \rho_\eta$. Then $\sigma < \kappa$, so by assumption there is a $\rho_\xi \in [\sigma \cup \theta_\xi^+, \kappa)$ such that $2^\sigma < 2^{\rho_\xi}$. Now $\forall\xi < \mathrm{cf}(\kappa)[\theta_\xi < \rho_\xi]$, so $\bigcup_{\xi<\mathrm{cf}(\kappa)} \rho_\xi = \kappa$. Also, $\langle 2^{\rho_\xi} : \xi < \mathrm{cf}(\kappa)\rangle$ is strictly increasing, so $\mathrm{cf}(\lambda) = \mathrm{cf}(\kappa)$. Then $2^\kappa = (2^{<\kappa})^{\mathrm{cf}(\kappa)} = \lambda^{\mathrm{cf}(\lambda)}$. $\square$

Proposition 1.33. *Define $\gimel(\kappa) = \kappa^{\mathrm{cf}(\kappa)}$.*

(i) If κ is a successor cardinal, then $2^\kappa = \gimel(\kappa)$.

(ii) If κ is a limit cardinal and $\exists\mu < \kappa\forall\nu \in [\mu, \kappa)[2^\nu = 2^\mu]$, then $2^\kappa = 2^{<\kappa} \cdot \gimel(\kappa)$.

(iii) If κ is a limit cardinal and $\forall\mu < \kappa\exists\nu \in (\mu, \kappa)[2^\mu < 2^\nu]$, then $2^\kappa = \gimel(2^{<\kappa})$.

Proof. (i) is clear. For (ii), $2^\kappa = 2^\mu = 2^{<\kappa}$. Also, $\gimel(\kappa) = \kappa^{\mathrm{cf}(\kappa)} \leq 2^\kappa$, so (ii) follows. For (iii), see above. $\square$

Proposition 1.34. *Let κ, λ be cardinals > 1, with at least one of them infinite. Then one of the following holds:*

(i) $\kappa^\lambda = 2^\lambda$.

(ii) $\kappa^\lambda = \kappa$.
(iii) There is a μ *such that* $\mathrm{cf}(\mu) \le \lambda < \mu$ *and* $\kappa^\lambda = \mu^{\mathrm{cf}(\mu)}$.

Proof. If κ is finite, then λ is infinite, and $\kappa^\lambda = 2^\lambda$. So assume that κ is infinite. If λ is finite, then $\kappa^\lambda = \kappa$, so assume that λ is infinite. If $\kappa \le \lambda$, then $\kappa^\lambda = 2^\lambda$. So assume that $\lambda < \kappa$. Let μ be minimum such that $\mu^\lambda \ge \kappa$; so $\mu \le \kappa$. Note that

(1) $\mu^\lambda = \kappa^\lambda$.

(2) $\forall \nu < \mu[\nu^\lambda < \mu]$.

In fact, otherwise $\mu^\lambda \le \nu^\lambda \le \mu^\lambda$, contradicting the minimality of μ.
Case 1. $\mu \le \lambda$. Then $\kappa^\lambda = \mu^\lambda = 2^\lambda$.
Case 2. $\lambda < \mu$.
Subcase 2.1. $\lambda < \mathrm{cf}(\mu)$. Then $\kappa^\lambda = \mu^\lambda = \mu \le \kappa$, so $\mu = \kappa$.
Subcase 2.2. $\mathrm{cf}(\mu) \le \lambda$. Then $\kappa^\lambda = \mu^\lambda = \mu^{\mathrm{cf}(\mu)}$. □

Proposition 1.35. *If* κ *is strong limit, then* $2^\kappa = \kappa^{\mathrm{cf}(\kappa)}$. □

Proposition 1.36. *If* $\kappa \le 2^{\mathrm{cf}(\kappa)}$*, then* $\kappa^{\mathrm{cf}(\kappa)} = 2^{\mathrm{cf}(\kappa)}$.

Proof. $\kappa^{\mathrm{cf}(\kappa)} \le (2^{\mathrm{cf}(\kappa)})^{\mathrm{cf}(\kappa)} = 2^{\mathrm{cf}(\kappa)})$. □

The *singular cardinal hypothesis,* SCH, is the statement

$$\forall \text{ singular } \kappa[2^{\mathrm{cf}(\kappa)} < \kappa \to \kappa^{\mathrm{cf}(\kappa)} = \kappa^+].$$

Proposition 1.37. (SCH) *If* κ *is singular and* $\forall \mu < \kappa \exists \nu \in [\mu, \kappa)[2^\mu < 2^\nu]$. *Then* $2^\kappa = (2^{<\kappa})^+$.

Proof. In fact, $\mathrm{cf}(2^{<\kappa}) = \mathrm{cf}(\kappa)$ and $2^\kappa = (2^{<\kappa})^{\mathrm{cf}(2^{<\kappa})}$, and $2^{\mathrm{cf}(2^{<\kappa})} = 2^{\mathrm{cf}(\kappa)} < 2^{<\kappa}$, so $2^\kappa = (2^{<\kappa)})^+$ by SCH. □

Proposition 1.38. *If* $2 \le \kappa \le 2^\lambda$*, then* $\kappa^\lambda = 2^\lambda$.

Proof. For, $\kappa^\lambda \le (2^\lambda)^\lambda = 2^\lambda \le \kappa^\lambda$. □

Proposition 1.39. (SCH) *If* $2^\lambda < \kappa$ *and* $\lambda < \mathrm{cf}(\kappa)$*, then* $\kappa^\lambda = \kappa$.

Proof. *Case 1.* $\exists \mu < \kappa[\kappa \le \mu^\lambda]$. Let μ be smallest with this property. Then $\kappa^\lambda \le \mu^\lambda \le \kappa^\lambda$, so $\kappa^\lambda = \mu^\lambda$. Now $\forall \nu < \mu[\nu^\lambda < \kappa]$. If $\nu < \mu$ and $\mu \le \nu^\lambda$, then $\kappa \le \mu^\lambda \le \nu^\lambda$, contradiction. So $\forall \nu < \mu[\nu^\lambda < \mu]$. In particular, $\lambda < 2^\lambda < \mu$.
Subcase 1.1. $\mathrm{cf}(\mu) > \lambda$. Then $\mu^\lambda = \mu$. Hence $\kappa^\lambda \le \mu^\lambda = \mu \le \kappa$. So $\kappa^\lambda = \kappa$.
Subcase 1.2. $\mathrm{cf}(\mu) \le \lambda$. Then $\mu^\lambda = \mu^{\mathrm{cf}(\mu)}$. Now if $\mu \le 2^\lambda$, then $\kappa^\lambda = \mu^\lambda \le 2^\lambda < \kappa$, contradiction. So $2^\lambda < \mu$, and hence $2^{\mathrm{cf}(\mu)} < \mu$. Hence by SCH, $\kappa^\lambda = \mu^\lambda = \mu^{\mathrm{cf}(\mu)} = \mu^+ \le \kappa$, as desired. □
Case 2. $\forall \mu < \kappa[\mu^\lambda < \kappa]$. Hence $\kappa^\lambda = \kappa$, since $\lambda < \mathrm{cf}(\kappa)$. □

Proposition 1.40. (SCH) *If* $2^\lambda < \kappa$ *and* $\mathrm{cf}(\kappa) \le \lambda$*, then* $\kappa^\lambda = \kappa^+$.

Proof. We have $2^{\text{cf}(\kappa)} < \kappa$, so by SCH, $\kappa^{\text{cf}(\kappa)} = \kappa^+$.

Case 1. $\exists\mu < \kappa[\kappa \leq \mu^\lambda]$. Let μ be smallest with this property. Then $\kappa^\lambda \leq \mu^\lambda \leq \kappa^\lambda$, so $\kappa^\lambda = \mu^\lambda$. Now $\forall\nu < \mu[\nu^\lambda < \kappa]$. If $\nu < \mu$ and $\mu \leq \nu^\lambda$, then $\kappa \leq \mu^\lambda \leq \nu^\lambda$, contradiction. So $\forall\nu < \mu[\nu^\lambda < \mu]$.

Subcase 1.1. $\text{cf}(\mu) > \lambda$. Then $\mu < \kappa$ since $\text{cf}(\kappa) \leq \lambda$. Now $\kappa^\lambda = \mu^\lambda = \mu < \kappa$, contradiction.

Subcase 1.2. $\text{cf}(\mu) \leq \lambda$. Then $\kappa^\lambda = \mu^\lambda = \mu^{\text{cf}(\mu)}$.

Subsubcase 1.2.1. $2^{\text{cf}(\mu)} < \mu$. Then by SCH, $\kappa < \kappa^\lambda = \mu^{\text{cf}(\mu)} = \mu^+ \leq \kappa$, contradiction.

Subsubcase 1.2.2. $\mu \leq 2^{\text{cf}(\mu)}$. Then $\kappa^\lambda = \mu^\lambda \leq 2^\lambda < \kappa$, contradiction.

Case 2. $\forall\mu < \kappa[\mu^\lambda < \kappa$. Hence $\kappa^\lambda = \kappa^{\text{cf}(\kappa)} = \kappa^+$. □

Proposition 1.41. (The Milner-Rado Paradox) *For every ordinal $\alpha < \kappa^+$ there are sets $X_n \subseteq \alpha$ for $n \in \omega$ such that $\alpha = \bigcup_{n\in\omega} X_n$ and for each $n \in \omega$ the order type of X_n is $\leq \kappa^n$.*

Proof. We proceed by induction on α. The conclusion is obvious if $\alpha < \kappa$, so suppose that $\kappa \leq \alpha$, inductively. If $\alpha = \beta + 1$, by the inductive hypothesis write $\beta = \bigcup_{n\in\omega} X_n$ with each X_n of order type $\leq \kappa^n$. Define $Y_0 = \{\beta\}$ and $Y_{n+1} = X_n$ for all $n \in \omega$. Then $\alpha = \bigcup_{n\in\omega} Y_n$ and each Y_n has order type $\leq \kappa^n$.

Now suppose that α is a limit ordinal. Let $\langle\beta_\xi : \xi < \text{cf}(\alpha)\rangle$ be a strictly increasing continuous sequence with supremum α. Note that $\text{cf}(\alpha) \leq \kappa$. By the inductive hypothesis, for each $\xi < \text{cf}(\alpha)$ write $\beta_\xi = \bigcup_{n\in\omega} X^\xi_n$, where each X^ξ_n has order type $\leq \kappa^n$. For each $n \in \omega$ and $\xi < \text{cf}(\alpha)$, let $Y^{\xi+1}_n = X^{\xi+1}_n \backslash \beta_\xi$. Let $Z_0 = \emptyset$, and for each $n \in \omega$ let $Z_{n+1} = \bigcup_{\xi<\text{cf}(\alpha)} Y^{\xi+1}_n$. For each $n \in \omega$ and $\xi < \text{cf}(\alpha)$ let f^ξ be an isomorphism of Y^ξ_n onto an ordinal $\leq \kappa^n$. For any $n \in \omega$ we define $g : Z_{n+1} \to \kappa^{n+1}$ as follows. Let $\gamma \in Z_{n+1}$. Choose $\xi < \text{cf}(\alpha)$ such that $\beta_\xi \leq \gamma < \beta_{\xi+1}$. Then $\gamma \in Y^{\xi+1}_n$, and we set

$$g(\gamma) = \kappa^n \cdot \xi + f^{\xi+1}_n(\gamma).$$

Clearly g is a strictly increasing function. So the order type of Z_{n+1} is at most κ^{n+1}. Now take any $\gamma < \alpha$. Choose $\xi < \text{cf}(\alpha)$ so that $\beta_\xi \leq \gamma < \beta_{\xi+1}$. Choose $n \in \omega$ such that $\gamma \in X^{\xi+1}_n$. Then $\gamma \in Y^{\xi+1}_n$, and so $\gamma \in Z_{n+1}$. This finishes the inductive proof. □

Proposition 1.42. $\prod_{n<\omega} \aleph_n = \aleph_\omega^\omega$.

Proof.

$$\begin{aligned}\prod_{n\in\omega} \aleph_n \leq \aleph_\omega^\omega = \left(\sum_{n<\omega} \aleph_n\right)^\omega &\leq \left(\prod_{n<\omega} \aleph_n\right)^\omega \\ &= \prod_{n<\omega} \aleph_n^\omega = \prod_{n<\omega} (2^\omega \cdot \aleph_n) \\ &= \prod_{n<\omega} 2^\omega \cdot \prod_{n<\omega} \aleph_n\end{aligned}$$

$$
\begin{aligned}
&= 2^\omega \cdot \prod_{n<\omega} \aleph_n \le \prod_{n<\omega} \aleph_n \cdot \prod_{n<\omega} \aleph_n \\
&= \prod_{n<\omega} \aleph_n. \qquad \square
\end{aligned}
$$

Proposition 1.43. $\prod_{\alpha<\omega+\omega} \aleph_\alpha = \aleph_{\omega+\omega}^\omega$.

Proof.

$$
\begin{aligned}
&\prod_{\alpha<\omega+\omega} \aleph_\alpha \le (\aleph_{\omega+\omega})^\omega = \left(\sum_{\alpha<\omega+\omega} \aleph_\alpha\right)^\omega \le \left(\prod_{\alpha<\omega+\omega} \aleph_\alpha\right)^\omega = \prod_{\alpha<\omega+\omega} \aleph_\alpha^\omega \\
&= \prod_{n<\omega} \aleph_n^\omega \cdot \prod_{n<\omega} \aleph_{\omega+n}^\omega = \prod_{n\in\omega} (2^\omega \cdot \aleph_n) \cdot \prod_{n\in\omega} (\aleph_\omega^\omega \cdot \aleph_{\omega+n}) \\
&= \aleph_\omega^\omega \cdot \prod_{\alpha<\omega+\omega} \aleph_\alpha = \prod_{n\in\omega} \aleph_n \cdot \prod_{\alpha<\omega+\omega} \aleph_\alpha = \prod_{\alpha<\omega+\omega} \aleph_\alpha. \qquad \square
\end{aligned}
$$

Proposition 1.44. *If* $\forall\alpha[2^{\aleph_\alpha} = \aleph_{\alpha+\beta}]$, *then* $\beta < \omega$.

Proof. Suppose that $\beta \ge \omega$. Let α be minimum such that $\alpha+\beta > \beta$. Then $0 < \alpha \le \beta$.

(1) α is limit.

For, suppose that $\alpha = \gamma + 1$. Then $\gamma + \beta \le \beta$, so $\gamma + 1 + \beta = \gamma + \beta \le \beta$, contradiction.

Let $\kappa = \aleph_{\alpha+\alpha}$. Then $\mathrm{cf}(\kappa) = \mathrm{cf}(\alpha) < \kappa$, so κ is singular. For each $\xi < \alpha$ we have $\xi + \beta = \beta$, and hence $2^{\aleph_{\alpha+\xi}} = \aleph_{\alpha+\xi+\beta} = \aleph_{\alpha+\beta}$. So $2^{\aleph_{\alpha+\alpha}} = \aleph_{\alpha+\beta}$. But $\alpha + \alpha + \beta > \alpha + \beta$, so $\aleph_{\alpha+\beta} < \aleph_{\alpha+\alpha+\beta}$, which contradicts the assumption of the proposition. $\square$

Proposition 1.45. $\prod_{\alpha<\omega_1+\omega} \aleph_\alpha = \aleph_{\omega_1+\omega}^{\omega_1}$.

Proof. First we prove

(1) For any infinite cardinal κ we have $\aleph_\kappa^\kappa = \prod_{\alpha<\kappa} \aleph_\alpha$.

In fact, write $\kappa = \bigcup_{\alpha<\kappa} \Gamma_\alpha$ with the Γ_α's pairwise disjoint and of size κ. Then

$$
\aleph_\kappa^\kappa = \prod_{\alpha<\kappa} \aleph_\kappa = \prod_{\alpha<\kappa} \sum_{\beta\in\Gamma_\alpha} \aleph_\beta \le \prod_{\alpha<\kappa} \prod_{\beta\in\Gamma_\alpha} \aleph_\beta = \prod_{\alpha<\kappa} \aleph_\alpha \le \aleph_\kappa^\kappa.
$$

Hence

$$
\begin{aligned}
\aleph_{\omega_1+\omega}^{\aleph_1} &= \left(\sum_{n\in\omega} \aleph_{\omega_1+n}\right)^{\aleph_1} \le \left(\prod_{n\in\omega} \aleph_{\omega_1+n}\right)^{\aleph_1} = \prod_{n\in\omega} \aleph_{\omega_1+n}^{\aleph_1} \\
&= \prod_{n\in\omega} (\aleph_{\omega_1}^{\aleph_1} \cdot \aleph_{\omega_1+n}) = \aleph_{\omega_1}^{\aleph_1} \cdot \prod_{n\in\omega} \aleph_{\omega_1+n} \\
&= \prod_{\alpha<\omega_1+\omega} \aleph_\alpha \quad \text{by (1)} \\
&\le \prod_{\alpha<\omega_1+\omega} \aleph_{\omega_1+\omega} = \aleph_{\omega_1+\omega}^{\aleph_1}. \qquad \square
\end{aligned}
$$

Proposition 1.46. $\aleph_\omega^{\omega_1} = \aleph_\omega^\omega \cdot 2^{\omega_1}$

Proof.

$$\aleph_\omega^{\aleph_1} \leq \prod_{n\in\omega} \aleph_n^{\aleph_1} = \prod_{n\in\omega} (\aleph_n \cdot 2^{\aleph_1}) = 2^{\aleph_1} \cdot \prod_{n\in\omega} \aleph_n \leq 2^{\aleph_1} \cdot \aleph_\omega^{\aleph_0} \leq \aleph_\omega^{\aleph_1}. \qquad \square$$

Proposition 1.47. *If* $\alpha < \omega_1$*, then* $\aleph_\alpha^{\omega_1} = \aleph_\alpha^\omega \cdot 2^{\omega_1}$*.*

Proof. Induction on α. For $\alpha < \omega$,

$$\aleph_\alpha^{\aleph_1} = \aleph_\alpha \cdot \aleph_0^{\aleph_1} = \aleph_\alpha \cdot 2^{\aleph_1} \leq \aleph_\alpha^{\aleph_0} \cdot 2^{\aleph_1} \leq \aleph_\alpha^{\aleph_1}.$$

Now suppose, inductively, that $\omega \leq \alpha < \omega_1$. If α is a limit ordinal, then

$$\aleph_\alpha^{\aleph_1} \leq \prod_{\beta<\alpha} \aleph_\beta^{\aleph_1} = \prod_{\beta<\alpha} (\aleph_\beta^{\aleph_0} \cdot 2^{\aleph_1}) = 2^{\aleph_1} \cdot \prod_{\beta<\alpha} \aleph_\beta^{\aleph_0} \leq 2^{\aleph_1} \cdot (\aleph_\alpha^{\aleph_0})^{|\alpha|} = 2^{\aleph_1} \cdot \aleph_\alpha^{\aleph_0} \leq \aleph_\alpha^{\aleph_1}.$$

Finally, if α is a successor ordinal, write $\alpha = \beta + n$ where β is a limit ordinal and $n > 0$. Then

$$\aleph_\alpha^{\aleph_1} = \aleph_\alpha \cdot \aleph_\beta^{\aleph_1} = \aleph_\alpha \cdot \aleph_\beta^{\aleph_0} \cdot 2^{\aleph_1} \leq \aleph_\alpha^{\aleph_0} \cdot 2^{\aleph_1} \leq \aleph_\alpha^{\aleph_1}. \qquad \square$$

Proposition 1.48. *If* $\alpha < \omega_2$*, then* $\aleph_\alpha^{\omega_2} = \aleph_\alpha^{\omega_1} \cdot 2^{\omega_2}$*.*

Proof. Induction on α. For $\alpha = 0$,

$$\aleph_0^{\aleph_2} = \aleph_0^{\aleph_1} \cdot 2^{\aleph_2}.$$

Now suppose inductively that α is a limit ordinal. Then

$$\aleph_\alpha^{\aleph_2} \leq \prod_{\beta<\alpha} \aleph_\beta^{\aleph_2} = \prod_{\beta<\alpha} (\aleph_\beta^{\aleph_1} \cdot 2^{\aleph_2}) \leq \prod_{\beta<\alpha} (\aleph_\alpha^{\aleph_1} \cdot 2^{\aleph_2}) = \aleph_\alpha^{\aleph_1} \cdot 2^{\aleph_2} \leq \aleph_\alpha^{\aleph_2}.$$

Finally, suppose inductively that $\alpha = \beta + 1$. Then

$$\aleph_\alpha^{\alpha_2} = \aleph_\beta^{\aleph_2} \cdot \aleph_\alpha = \aleph_\beta^{\aleph_1} \cdot 2^{\aleph_2} \cdot \aleph_\alpha = \aleph_\alpha^{\aleph_1} \cdot 2^{\aleph_2}. \qquad \square$$

Proposition 1.49. *If* κ *is regular limit, then* $\kappa^{<\kappa} = 2^{<\kappa}$*.*

Proof.

$$\kappa^{<\kappa} = \sum_{\lambda<\kappa} \kappa^\lambda = \sum_{\lambda<\kappa} \sum_{\alpha<\kappa} |{}^\lambda\alpha| \leq \sum_{\lambda<\kappa} \sum_{\lambda<\kappa} 2^\lambda = 2^{<\lambda} \leq \kappa^{<\kappa}. \qquad \square$$

Proposition 1.50. *If* κ *is regular and strong limit, then* $\kappa^{<\kappa} = \kappa$*.*

Proof.

$$\kappa^{<\kappa} = 2^{<\kappa} = \kappa. \qquad \square$$

Proposition 1.51. *If κ is singular and not strong limit, then $\kappa^{<\kappa} = 2^{<\kappa} > \kappa$.*

Proof. There is a $\lambda < \kappa$ such that $2^\lambda \geq \kappa$. Then $\kappa^\mu = 2^\mu$ for all $\mu \in [\lambda, \kappa)_{\mathrm{card}}$. So

$$\kappa < \kappa^{\mathrm{cf}(\kappa)} \leq \kappa^{<\kappa} = \sum_{\mu<\kappa} \kappa^\mu = \sum_{\lambda\leq\mu<\kappa} \kappa^\mu = \sum_{\lambda\leq\mu<\kappa} 2^\mu \leq 2^{<\kappa} \leq \kappa^{<\kappa}. \qquad \square$$

Proposition 1.52. *If κ is a limit cardinal and $\lambda \geq \mathrm{cf}(\kappa)$, then $\kappa^\lambda = (\bigcup_{\mu<\kappa} \mu^\lambda)^{\mathrm{cf}(\kappa)}$.*

Proof. Let $\kappa = \sum_{\alpha<\mathrm{cf}(\kappa)} \kappa_\alpha$ with each $\kappa_\alpha < \kappa$. Then

$$\kappa^\lambda \leq \left(\prod_{\alpha<\mathrm{cf}(\kappa)} \kappa_\alpha\right)^\lambda = \prod_{\alpha<\mathrm{cf}(\kappa)} \kappa_\alpha^\lambda \leq \left(\bigcup_{\alpha<\mathrm{cf}(\kappa)} \kappa_\alpha^\lambda\right)^{\mathrm{cf}(\kappa)} \leq (\kappa^\lambda)^{\mathrm{cf}(\kappa)} = \kappa^\lambda. \qquad \square$$

Proposition 1.53. *If κ is singular and strong limit, then $2^{<\kappa} = \kappa$ and $\kappa^{<\kappa} = \kappa^{\mathrm{cf}(\kappa)}$.*

Proof. Clearly $2^{<\kappa} = \kappa$. Now if $\lambda, \mu < \kappa$, then $\mu^\lambda < \kappa$. It follows that $\kappa^\lambda = \kappa^{\mathrm{cf}(\kappa)}$ for all λ such that $\mathrm{cf}(\kappa) \leq \lambda < \kappa$. Hence

$$\kappa^{<\kappa} = \sum_{\lambda<\kappa} \kappa^\lambda = \sum_{\mathrm{cf}(\kappa)\leq\lambda<\kappa} \kappa^\lambda = \sum_{\mathrm{cf}(\kappa)\leq\lambda<\kappa} \kappa^{\mathrm{cf}(\kappa)} = \kappa^{\mathrm{cf}(\kappa)}. \qquad \square$$

2. Advanced set theory

The *collection principle* is the statement

$$\forall X, p_1, \ldots, p_n \exists Y \forall u \in X[\exists v \varphi(u, v, p_1, \ldots, p_n) \rightarrow \exists v \in Y \varphi(u, v, p_1, \ldots, p_n)].$$

Proposition 2.1. *In ZFC the collection principle holds.*

Proof. Let $\psi(u, \alpha, p_1, \ldots, p_n)$ be the formula

$$\begin{aligned}
&[\neg\exists v[\varphi(u, v, p_1, \ldots, p_n)] \wedge \alpha = 0] \vee \\
&[\exists v[\varphi(u, v, p_1, \ldots, p_n)] \wedge \alpha \text{ is an ordinal } \wedge \exists v \in V_\alpha[\varphi(u, v, p_1, \ldots, p_n)] \wedge \\
&\forall \beta[\beta \text{ is an ordinal } \wedge \exists v \in V_\beta[\varphi(u, v, p_1, \ldots, p_n)] \rightarrow \alpha \leq \beta]].
\end{aligned}$$

Let $X, p_1, \ldots, p_n$ be given. Clearly $\forall u \in X \exists! \alpha[\psi(u, \alpha, p_1, \ldots, p_n)]$. Hence by the replacement axiom, choose Γ such that $\forall u \in X \exists \alpha \in \Gamma[\psi(u, \alpha, p_1, \ldots, p_n)]$. Now let $Y = \bigcup_{\alpha \in \Gamma}\{v \in V_\alpha : \varphi(u, v, p_1, \ldots, p_n)\}$. Suppose that $u \in X$ and $\exists v \varphi(u, v, p_1, \ldots, p_n)$. Choose $\alpha \in \Gamma$ such that $\psi(u, \alpha, p_1, \ldots, p_n)$. Then there is a $v \in V_\alpha$ such that $\varphi(u, v, p_1, \ldots, p_n)$. So $v \in Y$, as desired. $\square$

Proposition 2.2. *The replacement axioms are derivable from the other axioms plus the collection principle.*

Proof. Suppose that $X, p_1, \ldots, p_n$ are given, and $\forall u \in X \exists! v \varphi(u, v, p_1, \ldots, p_n)$. Choose Y so that

$$\forall u \in X[\exists v \varphi(u, v, p_1, \ldots, p_n) \rightarrow \exists v \in Y \varphi(u, v, p_1, \ldots, p_n)].$$

Then $\forall u \in X \exists v \in Y \varphi(u, v, p_1, \ldots, p_n)]$, as desired. $\square$

Proposition 2.3. *$\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ are in $V_{\omega+\omega}$.*

Proof. We have $\text{rank}(\omega) = \omega$. Thus $\omega \in V_{\omega+1}$. A definition of $\mathbb{Z}$ runs as follows. Define $(m, n)E(i, j)$ iff $m, n, i, j \in \omega$ and $m + j = i + n$. E is an equivalence relation. Let $\mathbb{Z}' = (\omega \times \omega)/E$, and $\omega' = \{[(m, 0)]_E : m \in \omega\}$. Then

(1) $\omega \cap (\mathbb{Z}' \backslash \omega') = \emptyset$.

For, suppose that $[(m, n)]_E \in \omega$. Then $(m, n) \in [(m, n)]_E$, so $(m, n) \in \omega$. Clearly $(m, n) \neq 0$, so $0 \in (m, n)$, contradiction.

Now we define $\mathbb{Z} = \omega \cup (\mathbb{Z}' \backslash \omega')$. To see that $\text{rank}(\mathbb{Z}) < \omega + \omega$, first note that if $i, j \in \omega$ then $\text{rank}((i, j)) < \omega$. Hence $\omega \times \omega \subseteq V_\omega$, and so $\omega \times \omega \in V_{\omega+1}$. For each $a \in \mathbb{Z}'$ we have $a \subseteq \omega \times \omega \backslash V_\omega$, so $a \in V_{\omega+1}$. So $\mathbb{Z}' \subseteq V_{\omega+1}$, and hence $\mathbb{Z}' \backslash \omega' \subseteq V_{\omega+1}$. Thus $\mathbb{Z} \subseteq V_{\omega+1}$, and so $\mathbb{Z} \in V_{\omega+2}$.

Now we consider $\mathbb{Q}$. Define $(a, b)F(c, d)$ iff $a, b, c, d \in \mathbb{Z}$, $b, d \neq 0$, and $ad = bc$. Then F is an equivalence relation on $\mathbb{Z} \times (\mathbb{Z} \backslash \{0\})$, and we define $\mathbb{Q}' = (\mathbb{Z} \times (\mathbb{Z} \backslash \{0\}))/F$. We also define $\mathbb{Z}'' = \{(a, 1) : a \in \mathbb{Z}\}$.

(2) $\mathbb{Z} \cap \mathbb{Q}' = 0$.

In fact, suppose that $x \in \mathbb{Z} \cap \mathbb{Q}'$. Say $x = [(a,b)]_F$.

Case 1. $x \in \omega$. Now $(a,b) \in x$, so $(a,b) \in \omega$, and this gives a contradiction, as above.

Case 2. $x \notin \omega$. Say $x = [(m,n)]_E$. Now $(m,n)E(m+1,n+1)$, so also $(m,n)F(m+1,n+1)$. Hence $mn+m = nm+n$, and so $m = n$. Now $(m,m)E(0,0)$. It follows that $(0,0)F(0,0)$, and so $(0,0) \in \mathbb{Z} \times (\mathbb{Z}\backslash\{0\})$, contradiction.

Thus (2) holds. We define $\mathbb{Q} = \mathbb{Z} \cup (\mathbb{Q}'\backslash\mathbb{Z}'')$. Now for any $(a,b) \in \mathbb{Q}'$ we have $a,b \in \mathbb{Z}$, and hence $a,b \in V_{\omega+2}$. So $\{a\},\{a,b\} \subseteq V_{\omega+2}$ and so $\{a\},\{a,b\} \in V_{\omega+3}$. Hence $(a,b) \subseteq V_{\omega+3}$, hence $(a,b) \in V_{\omega+4}$. Hence $\mathbb{Q}' \subseteq V_{\omega+4}$. Hence $\mathbb{Q} \subseteq V_{\omega+4}$ and so $\mathbb{Q} \in V_{\omega+5}$.

Next comes the real numbers. A subset A of $\mathbb{Q}$ is a *Dedekind cut of rationals* provided the following conditions hold:

(3) $\mathbb{Q} \neq A \neq 0$;
(4) For all $r,s \in \mathbb{Q}$, if $r < s$ and $s \in A$, then $r \in A$.
(5) A has no largest element.

Let $\mathbb{R}'$ be the set of all Dedekind cuts. For each rational number a, let $D_a = \{x \in \mathbb{Q} : x < a\}$. Clearly D_a is a Dedekind cut. Let $\mathbb{Q}'' = \{D_a : a \in \mathbb{Q}\}$.

(6) $\mathbb{Q} \cap \mathbb{R}' = 0$.

For, suppose that $x \in \mathbb{Q} \cap \mathbb{R}'$. If $x \in \omega$, a contradiction follows since each Dedekind cut is infinite. Suppose that $x \in \mathbb{Z}\backslash\omega$. Say $x = [(m,n)]_E$. Now $(m,n) \in x$, so $(m,n) \in \mathbb{Q}$. But each rational is clearly an infinite set, while (m,n) has at most two elements, contradiction. A similar contradiction is obtained if $x \notin \mathbb{Z}$. Hence (6) holds.

We define $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R}\backslash\mathbb{Q}'')$. For the rank of $\mathbb{R}$, note that each Dedekind cut is a subset of $\mathbb{Q}$, which is a subset of $V_{\omega+4}$. So $\mathbb{R}' \subseteq V_{\omega+5}$. Hence $\mathbb{R} \subseteq V_{\omega+5}$ and so $\mathbb{R} \in V_{\omega+6}$. □

Proposition 2.4. *Let A be an infinite set. For each $P \in [A]^{<\omega}$ let $\hat{P} = \{Q \in [A]^{<\omega} : P \subseteq Q\}$. Let F be the set of all $X \subseteq [A]^{<\omega}$ such that $\exists P \in [A]^{<\omega}[\hat{P} \subseteq X$. Then F is a nonprincipal filter on $[A]^{<\omega}$.*

Proof. This is clear, except possibly that F is nonprincipal. Suppose that F is principal; say $F = \{X \subseteq S : X_0 \subseteq X\}$. So $X_0 \in F$; choose $P \in S$ such that $\dot{P} \subseteq X_0$. Let $P \subset Q \in S$. Then $\dot{Q} \in F$, so $X_0 \subseteq \dot{Q}$. Hence $\dot{P} \subseteq \dot{Q}$. But $P \in \dot{P}$, so $P \in \dot{Q}$ and $Q \subseteq P$, contradiction. □

Proposition 2.5. *Let $X \in F$ iff $M_X \stackrel{\text{def}}{=} \{n : A_n\backslash X$ is infinite$\}$ is finite. If $X \subseteq Y$, then $\forall n[A_n\backslash Y \subseteq A_n\backslash X]$, and hence $M_Y \subseteq M_X$. So if $X \in F$ and $X \subseteq Y$, then $Y \in F$. Now $\forall n[A_n\backslash(X \cap Y) = (A_n\backslash X) \cup (A_n\backslash Y)]$; so $M_{X\cap Y} = M_X \cup M_Y$. Hence $X,Y \in F$ implies that $X \cap Y \in F$. Note that $M_\emptyset = \omega$; hence $\emptyset \notin F$. Thus F is a filter. We take D to be any ultrafilter containing F.*

Then D is not a p-point.

Proof. For any n we have

$$M_{\omega\backslash A_n} = \{m : A_m \cap A_n \text{ is infinite}\} = \{n\};$$

so $\omega\backslash A_n \in F \subseteq D$, and hence $A_n \notin D$. Now suppose that $X \in D$ and $X \cap A_n$ is finite for all n. Thus

$$M_{\omega\backslash X} = \{n : A_n \cap X \text{ is infinite}\} = \emptyset,$$

and so $\omega\backslash X \in F \subseteq D$, contradiction.

Proposition 2.6. *If $2^\omega = \omega_1$, then a p-point exists.*

Proof. First we note that there are exactly 2^ω partitions of ω. First, there are at least 2^ω partitions into exactly two sets. Namely, for each $A \subseteq \omega\backslash 1$ let $\mathscr{P}_A = \{A, \omega\backslash A\}$. Clearly this gives at least 2^ω partitions into exactly two sets. Now for each $\alpha \in (\omega+1)\backslash 1$ there at most $\prod_{\beta<\alpha} 2^\omega = 2^\omega$ partitions of ω into α sets. So there are exactly 2^ω partitions of ω.

Let $\langle \mathscr{A}_\alpha : \alpha < \omega_1\rangle$ be an enumeration of all partitions of ω. We construct by recursion a sequence $\langle X_\alpha : \alpha < \omega_1\rangle$ of infinite subsets of ω. Suppose that X_α has been defined. If there is a $Y \in \mathscr{A}_\alpha$ such that $X_\alpha \cap Y$ is infinite, we take the least such in some well-order, and define $X_{\alpha+1} = X_\alpha \cap Y$. Suppose that $X_\alpha \cap Y$ is finite for all $Y \in \mathscr{A}_\alpha$. Then we let $X_{\alpha+1} \subseteq X_\alpha$ be such that $|X_{\alpha+1} \cap Y| = 1$ for all $Y \in \mathscr{A}_\alpha$ such that $X_\alpha \cap Y \neq \emptyset$.

Now suppose that α is limit, and X_β has been defined for all $\beta < \alpha$ so that if $\beta < \gamma < \alpha$ then $X_\gamma \subseteq^* X_\beta$. Let $\langle \beta_n : n < \omega\rangle$ be a strictly increasing sequence of ordinals with supremum α. Define

$$i_n = \text{ least element of } \bigcap_{m\leq n} X_{\beta_m}\backslash\{i_m : m < n\};$$

then let $X_\alpha = \{i_n : n < \omega\}$. So $X_\alpha \subseteq^* X_\gamma$ for each $\gamma < \alpha$. This finishes our construction.

Now we define

$$D = \{Y \subseteq \omega : X_\alpha \subseteq^* Y \text{ for some } \alpha < \omega_1\}.$$

If $m \in \omega$, then $X_0 \subseteq^* \omega\backslash\{m\}$, so $\omega\backslash\{m\} \in D$. Clearly D is a filter on ω. Now suppose that $Y \subseteq \omega$. If $X_\alpha \subseteq^* Y$ for some $\alpha < \omega_1$, then $Y \in D$. Suppose that $X_\alpha \not\subseteq^* Y$ for all $\alpha < \omega_1$. If Y is finite, then $\omega\backslash Y \in D$ by the above. So, suppose that Y is infinite. Let $\mathscr{A}_\alpha$ be the partition $\{Y\} \cup \{\{i\} : i \in \omega\backslash Y\}$. Then $X_{\alpha+1}$ has exactly one element in common with Y, hence $\omega\backslash Y \in D$. Thus D is an ultrafilter.

Finally, suppose that $\mathscr{A}_\alpha$ is a partition of ω into infinitely many infinite parts, and $B \notin D$ for all $B \in \mathscr{A}_\alpha$. Then by construction we have $|B \cap X_{\alpha+1}| \leq 1$ for all $B \in \mathscr{A}_\alpha$, as desired. □

Proposition 2.7. *The completion of a BA is unique up to isomorphism.*

Proof. Let C and D be completions of B. Define $\pi : C \to D$ by

$$\pi(c) = \sum^B \{u \in B : u \leq c\}.$$

Clearly if $c \leq d$ then $\pi(c) \leq \pi(d)$. Now suppose that $c \not\leq d$. Choose $u \in B$ with $0 < u \leq c \cdot -d$. Then $u \leq \pi(c)$, and $u \cdot d = 0$, so $u \cdot \pi(d) = 0$; so $\pi(c) \not\leq \pi(d)$. In particular,

π is one-one. It remains only to show that π maps onto D. Suppose that $d \in D$. Let $c = \sum^C \{u \in B : u \leq d\}$. We claim that $\pi(c) = d$. If $u \in B$ and $0 \neq u \leq \pi(c) \cdot -d$, then $u \cdot d = 0$, so $u \cdot e = 0$ whenever $e \in B$ and $e \leq d$, so $u \cdot \pi(c) = 0$, contradiction. If $u \in B$ and $0 \neq u \leq d \cdot -\pi(c)$, then $u \leq d$ and $u \leq \pi(c)$, contradiction. $\square$

Theorem 2.8. *Every Boolean algebra has a completion.*

Proof. Let A be a BA. A set $U \subseteq A^+$ is a *cut* iff

$$\forall p \in A^+ \forall q \in U[p \leq q \rightarrow p \in U].$$

For every $p \in A^+$ let $U_p = \{x \in A^+ : x \leq p\}$. Note that U_p is a cut. A cut U is *regular* iff

$$\forall p \in A^+ \backslash U \exists q \in A^+[U_q \cap U = \emptyset].$$

$$\forall p \in A^+[U_p \text{ is regular}]$$

In fact, if $q \in A^+ \backslash U_p$, then $q \not\leq p$, so with $r = q \cdot -p$ we have $U_r \cap U_p = \emptyset$.

Now let B be the set of all regular cuts. We claim that under inclusion B is a complete BA, and A can be isomorphically embeded as a dense subalgebra in B.

A^+ is a regular cut.

(1) The intersection of a family of regular cuts is a regular cut.

For, let $\mathscr{A}$ be a family of regular cuts. Clearly $\bigcap \mathscr{A}$ is a cut. Now suppose that $p \in A^+ \backslash \bigcap \mathscr{A}$. Thus $p \in \bigcup_{U \in \mathscr{A}} (A^+ \backslash U)$. Choose $U \in \mathscr{A}$ such that $p \in (A^+ \backslash U)$. Then choose $q \in A^+$ such that $U_q \cap U = \emptyset$. Hence $U_q \cap \bigcap \mathscr{A} = \emptyset$. So $\bigcap \mathscr{A}$ is a regular cut.

For any cut U let $\overline{U}$ be the intersection of all regular cuts containing U.

(2) For any cut U, $\overline{U} = \{p : \forall q \leq p(U_q \cap U \neq \emptyset\}$.

For, let $W = \{p : \forall q \leq p(U_q \cap U \neq \emptyset\}$.

$\subseteq$: it suffices to show that $U \subseteq W$ and W is a regular cut. If $p \in U$ and $q \leq p$, then $U_q \subseteq U$, so $U_q \cap U \neq \emptyset$. Thus $U \subseteq W$. Clearly W is a cut. Suppose that $p \notin W$. So there is a $q \leq p$ such that $U_q \cap U = \emptyset$. Clearly then $U_q \cap W = \emptyset$.

$\supseteq$: it suffices to show that if $U \subseteq V$ and V is a regular cut, then $W \subseteq V$. Suppose that $p \in W \backslash V$. Since V is regular, choose $q \leq p$ such that $U_q \cap V = \emptyset$. By the definition of W we have $U_q \cap U \neq \emptyset$. Take any $r \in U_q \cap U$; then $r \in U_q \cap V$, contradiction.

This proves (2).

For $u, v \in B$, let $u \cdot v = u \cap v$ and $u + v = \overline{u \cup v}$. Then clearly $u \cdot v$ is the g.l.b. of u and v under $\subseteq$, and $u + v$ is the l.u.b. of u and v under $\subseteq$. So the commutative, associative, and absorption laws hold. Next note:

(2) $u + u = u$.

In fact, $u + u = u + u \cdot (u + u) = u$.

(3) $u \cdot u = u$.

In fact, $u \cdot u = u \cdot (u + u \cdot u) = u$.

Now note that the second distributive law can be proved once we prove the first one:

$$(u+v)\cdot(u+w) = u\cdot u + u\cdot w + v\cdot u + v\cdot w = u + v\cdot w.$$

To prove the first distributive law, note that the direction $\supseteq$ is clear. Now suppose that $p \in u\cdot(v+w)$. So $p \in u$ and $p \in v+w$, so by (1), $\forall q \leq p[(v\cup w)\cap U_q \neq \emptyset$. So if $q \leq p$, choose $r \in (v\cup w)\cap U_q$; so $r \in (v\cup w)$ and $r \in u$, so $r \in u\cap(v\cup w)\cap U_q$. So we have shown that $\forall q \leq p[u\cap(v\cup w)\cap U_q \neq \emptyset$. So by (1) again, $p \in \overline{u\cap(v\cup w)}$. Thus $u\cdot(v+w) \subseteq u\cdot v + u\cdot w$, and the distributive law holds.

Next, for any regular cut u let $-u = \{p : U_p \cap u = \emptyset\}$.

(4) $-u$ is a regular cut.

In fact, clearly it is a cut. Now suppose that $p \notin -u$. Then $U_p \cap u$ is nonempty, and we let q be a member of it. Clearly $U_q \cap -u = \emptyset$, as desired.

To show that $u\cdot -u = 0$, suppose that $p \in u\cap -u$. Then $p \in U_p \cap u = 0$, contradiction.

To show that $u + -u = 1$, suppose that v is a regular cut, $u \cup -u \subseteq v$, and $v \neq 1$. Say $p \notin v$. By the definition of regular cut, choose $q \leq p$ such that $U_q \cap v = 0$. Thus $U_q \cap (u \cup -u) = 0$. But then $q \in -u$ by definition, contradiction.

So we have checked that a BA is obtained. Since B is closed under arbitrary intersections, the meet of any set of elements exists. Thus B is complete.

It remains to show that A is isomorphic to a dense subalgebra of B. Note that we can define $U_0 = \emptyset$. For any $p \in A$ let $f(p) = U_p$. Clearly $U_p \cap U_q = U_{p\cdot q}$, so f preserves $\cdot$. If $p \in A$, then

$$\begin{aligned} -U_p &= \{q : U_q \cap U_p = \emptyset\} \\ &= \{q : p\cdot q = 0\} \\ &= \{q : q \leq -p\} \\ &= U_{-p}. \end{aligned}$$

Hence f preserves $-$. Clearly f is one-one. Clearly $f[A]$ is dense in B. □

Proposition 2.9. *Assume that U is an ultrafilter on S and $f : S \to T$. Let $f_*(U) = \{X \subseteq T : f^{-1}[X] \in U\}$. Then $f_*(U)$ is an ultrafilter on T.*

Proof. $f^{-1}[T] = S \in U$, so $T \in f_*(U)$. Suppose that $X \in f_*(U)$ and $X \subseteq Y \subseteq T$. Then $f^{-1}[X] \in U$ and $f^{-1}[X] \subseteq f^{-1}[Y]$, so $f^{-1}[Y] \in U$ and hence $Y \in f_*(U)$. Suppose that $X, Y \in f_*(U)$. Thus $f^{-1}[X], f^{-1}[Y] \in U$, hence $f^{-1}[X\cap Y] = f^{-1}[X] \cap f^{-1}[Y] \in U$, so $X \cap Y \in f_*(U)$. $f^{-1}[\emptyset] = \emptyset \notin U$, so $\emptyset \notin f_*(U)$. If $X \subseteq Y$, then either $f^{-1}[X] \in U$, hence $X \in f_*(U)$, or $S\backslash f^{-1}[X] = f^{-1}[Y\backslash X] \in U$, hence $Y\backslash X \in f_*(U)$. □

Proposition 2.10. *Let U be an ultrafilter on ω, and let $a \in {}^{\omega}\mathbb{R}$ be bounded. Then there is a unique U-limit c such that $\forall \varepsilon > 0[\{n \in \omega : |a_n - c| < \varepsilon\} \in U]$.*

Proof. Since $\{a_n : n \in \omega\}$ is bounded above, there is a b such that $A_b \stackrel{\text{def}}{=} \{n : a_n \leq b\}$ is infinite. Since $\{a_n : n \in \omega\}$ is bounded below, there is a d such that $d \leq b$ for each b such that A_b is infinite. Let c be the glb of all b such that A_b is infinite. For each $\varepsilon > 0$

let $C_\varepsilon = \{n : a_n \in (c-\varepsilon, c+\varepsilon)\}$. Note that if $\varepsilon < \delta$ then $C_\varepsilon \subseteq C_\delta$. Each C_ε is infinite. In fact, $\{n : a_n \leq c-\varepsilon\} = A_{c-\varepsilon}$, so $A_{c-\varepsilon}$ is finite, as otherwise $c < c-\varepsilon$. Now there is a b such that A_b is infinite and $b < c+\varepsilon$, since $c+\varepsilon$ is not a lower bound for all b with A_b infinite. So C_ε is infinite. The collection of all sets C_ε has fip, so let U be an ultrafilter on ω containing all sets C_ε. For any $\varepsilon > 0$, $\{n : |a_n - c| < \varepsilon\} = C_\varepsilon \in U$.

For uniqueness, suppose that $\{n : |a_n - e| < \varepsilon\} \in U$ for all $\varepsilon > 0$, with, say, $c < e$. Choose $m \in \{n : |a_n - c| < (e-c)/3\} \cap \{n : |a_n - d| < (e-c)/3\}$. Then $e - c \leq |c - a_m| + |e - a_m| < 2(e-c)/3$, contradiction. □

Proposition 2.11. *Let D be a nonprincipal ultrafilter on ω. Then the following are equivalent:*

(i) D is a p-point.

(ii) $\forall A \in \omega D[A_0 \supseteq A_1 \supseteq \cdots \to \exists X \in D \forall n \in \omega[X \backslash A_n$ is finite $]$.

Proof. First suppose that the definition of p-point holds. Assume that $A_0 \supseteq A_1 \supseteq \cdots$ are members of D. If this sequence is eventually constant, the desired conclusion is clear. So assume that it is not eventually constant. Let

$$P = \{\omega \backslash A_0, A_0 \backslash A_1, \ldots, \bigcap_{m\in\omega} A_m\} \backslash \{0\}.$$

So P is a partition of ω with ω members. None of them is in D, except possibly $\bigcap_{m\in\omega} A_m$; and if it is in D, the desired conclusion is clear. So assume that none of them is in D. Then by the p-point property let $X \in D$ such that $X \cap Y$ is finite for every $Y \in P$. Now the desired conclusion follows by induction.

Second, suppose that the condition of the propositiom holds, and we are given a partition $\langle A_n : n \in \omega\rangle$ of ω into infinitely many parts, each not in D. Then

$$(\omega\backslash A_0) \supseteq (\omega\backslash(A_0 \cup A_1)) \supseteq \cdots,$$

and each $\omega\backslash(A_0 \cup \ldots A_n)$ is in D, so we get $X \in D$ such that $X \cap (A_0 \cup \ldots A_n) = 0$ is finite for all n, as desired. □

Proposition 2.12. *If $(P, <)$ is a countable linearly ordered set and D is a p-point on P, then there is an $X \in D$ of order type ω or ω^*.*

Proof.

(1) If $\emptyset \neq X \in D$ and $(x, \infty) \cap X \in D$ for every $x \in X$, then D has a member of order type ω.

In fact, X does not have a largest element, so let $\langle x_i : i < \omega\rangle$ be a strictly increasing sequence of members of X cofinal in X. Then $(x_{i+1}, \infty) \cap X \subseteq (x_i, \infty) \cap X$ for all i. Choose $Y \in D$ such that $Y \subseteq^* (x_i, \infty) \cap X$ for all i. Then

$$\begin{aligned}
&Y \cap (-\infty, x_0) \text{ is finite;}\\
&Y \cap [x_i, x_{i+1}) \text{ is finite;}\\
&Y \cap \{y : x_i < y \text{ for all } i\} \text{ is finite.}
\end{aligned}$$

The last thing holds because $Y \cap \{y : x_i < y \text{ for all } i\} \subseteq Y \backslash X$. Thus Y is a member of D of order type ω, proving (1).

(2) If $\emptyset \neq X \in D$ and $(-\infty, x) \cap X \in D$ for every $x \in X$, then D has a member of order type ω^*.

The proof is very similar to that of (1): X does not have a smallest element, so let $\langle x_i : i < \omega \rangle$ be a strictly decreasing sequence of members of X coinitial in X. Then $(-\infty, x_{i+1}) \cap X \subseteq (-\infty, x_i) \cap X$ for all i. Choose $Y \in D$ such that $Y \subseteq^* (-\infty, x_i) \cap X$ for all i. Then

$$
\begin{aligned}
&Y \cap (x_0, \infty) \text{ is finite;}\\
&Y \cap [x_{i+1}, x_i) \text{ is finite;}\\
&Y \cap \{y : y < x_i \text{ for all } i\} \text{ is finite.}
\end{aligned}
$$

The last thing holds because $Y \cap \{y : y < x_i \text{ for all } i\} \subseteq Y \backslash X$. Thus Y is a member of D of order type ω^*, proving (2).

Now we use these two facts. Let $M = \{x : (x, \infty) \in D\}$.

Case 1. $M = \emptyset$. Thus $(-\infty, x] \in D$ for all x, and hence $(-\infty, x) \in D$ for all x. So we can apply (2) with $X = P$.

Case 2. $M \neq \emptyset$, and M does not have an upper bound. One can take $X = P$ and apply (1).

Case 3. $M \neq \emptyset$, and M has a least upper bound x, and x is the largest element of P. Let $X = P \backslash \{x\}$ and apply (1).

Case 4. $M \neq \emptyset$, M has a least upper bound $x \in M$, and x is not the largest element of P. Thus $(-\infty, y] \in D$ for all $y > x$, hence $(x, y] \in D$ for all $y > x$. We can apply (2) with $X = (x, \infty)$.

Case 5. $M \neq \emptyset$, and M has a least upper bound $x \notin M$, and x is not the largest element of P. Then $(y, x] \in D$ for all $y \in M$, and we can apply (1) with $X = (-\infty, x]$.

Case 6. $M \neq \emptyset$, and M has an upper bound, but no least upper bound. Let $N = P \backslash M$. If $M \in D$, apply (1), while if $N \in D$ apply (2). □

Proposition 2.13. *An ultrafilter D on ω is Ramsey iff every function $f : \omega \to \omega$ is either one-one on a set in D or constant on a set in D.*

Proof. $\Rightarrow$: Assume that D is Ramsey, $f : \omega \to \omega$, and f is not constant on any set in D. Then $\langle f^{-1}[\{i\}] : i \in \omega, f^{-1}[\{i\}] \neq \emptyset \rangle$ is a partition of ω with no entries in D. Hence there is an $X \in D$ such that $X \cap f^{-1}[\{i\}]$ has just one element, for every $i \in \omega$ for which $f^{-1}[\{i\}]$ is nonempty. Hence f is one-one on X.

$\Leftarrow$: assume the indicated condition, and suppose that $\langle A_n : n \in \omega \rangle$ is a partition of ω with no entries in D. For each $i \in \omega$ let $f(i)$ be the n such that $i \in A_n$. Then f is not constant on any element of D, so f is one-one on some element X of D. Let $X \subseteq Y$ be such that $|Y \cap A_n| = 1$ for every $n \in \omega$. So $Y \in D$ and it is as desired. □

Proposition 2.14. *If $f : \omega \to \omega$, D is an ultrafilter on ω, and $D = f_*(D)$, then $\{n : f(n) = n\} \in D$.*

Proof. Let $X = \{n : f(n) < n\}$ and $Y = \{n : n < f(n)\}$. We want to show that $X, Y \notin D$. For each $n \in X$ the sequence $\langle n > f(n) > f(f(n)) > \cdots \rangle$ is finite; let $l(n)$ be the length of this sequence. Let $X_0 = \{n \in X : l(n) \text{ is even}\}$, and $X_1 = \{n \in X : l(n) \text{ is odd}\}$. Then $X_0 \cap f^{-1}[X_0] = \emptyset$, since if $n \in X_0 \cap f^{-1}[X_0]$ then $f(n) \in X_0$, and so $f(n) > f(f(n)) > f(f(f(n))) > \cdots$ (of even length), and $n > f(n)$ so $n \notin X_0$, contradiction. It follows that $X_0 \notin D$, as otherwise also $f^{-1}[X_0] \in D$. Similarly, $X_1 \notin D$. Therefore $X \notin D$.

Let $Y' = \{n : n < f(n) < f(f(n)) < \cdots \text{ is a finite sequence}\}$. As above, $Y \notin D$

Let $Z = \{n : n < f(n) < f(f(n)) \cdots \text{ is infinite}\}$. Define $m \equiv n$ iff $m, n \in Z$ and $\exists i, j[f^i(m) = f^j(n)]$. Then $\equiv$ is an equivalence relation on Z. In fact, it is clearly reflexive on Z and symmetric. Suppose that $m \equiv n \equiv p$. Say $f^i(m) = f^j(n)$ and $f^k(n) = f^l(p)$. If $j < k$, then $f^{i+k-j}(m) = f^k(n) = f^l(p)$. If $k \leq j$, then $f^i(m) = f^j(n) = f^{l+j-k}(p)$. So $\equiv$ is transitive. For each $n \in Z$ let a_n be a representative of the equivalence class of n, and let $l(n)$ be the least integer m such that there exist i, j such that $m = i + j$ and $f^i(n) = f^j(a_n)$. Let $Z' = \{n \in Z : l(n) \text{ is even}\}$ and $Z'' = \{n \in Z : l(n) \text{ is odd}\}$. Then $f^{-1}[Z'] \subseteq Z''$. In fact, let $p \in f^{-1}[Z']$. So $f(p) \in Z'$, so $l(f(p))$ is even. Say $l(f(p)) = i + j$ with $f^i(f(p)) = f^j(a_{f(p)})$. Then $p \equiv a_{f(p)}$, so $a_p = a_{f(p)}$. Thus $f^{i+1}(p) = f^j(a_p)$.

Case 1. $p = f^u(a_p)$ for some u. Then $f^{i+1+u}(a_p) = f^j(a_p)$, so $i + 1 + u = j$. Now $f(p) = f^{u+1}(a_p)$, so $i + j \leq u + 1$. Thus $i + i + 1 + u \leq u + 1$, so $i = 0$. Hence $j = u + 1$. Suppose that $f^s(p) = f^t(a_p)$ with $s + t < u$.

Subcase 1.1. $s \neq 0$. Then $u + 1 \leq s - 1 + t < u - 1$, contradiction.

Subcase 1.2. $s = 0$. Then $t = u$ but also $t < u$, contradiction.

Thus $l(p) = u$. Now $j = u + 1$ is even, so $l(u)$ is odd, as desired.

Case 2. $p \neq f^u(a_p)$ for all u. Suppose that $f^k(p) = f^u(a_p)$ with $k + u < i + 1 + j$. Then $k \neq 0$. Now $f^{k-1}(f(p)) = f^u(a_p)$, so $i + j \leq k - 1 + u$. Now $k - 1 + u < i + j$, contradiction. It follows that $l(p) = i + 1 + j$, which is odd, as desired.

It follows that $Z' \notin D$. By symmetry, $Z'' \notin D$. Hence $Z \notin D$, so $\{n : f(n) = n\} \in D$. □

Proposition 2.15. *If $D \leq E$ and $E \leq D$, then $D \equiv E$.*

Proof. Note that $(f \circ g)_* = f_* \circ g_*$, since for any X,

$$\begin{aligned} X \in (f \circ g)_*(U) \quad &\text{iff} \quad (f \circ g)^{-1}[X] \in U \\ &\text{iff} \quad g^{-1}[f^{-1}[X]] \in U \\ &\text{iff} \quad f^{-1}[X] \in g_*(U) \\ &\text{iff} \quad X \in f_*(g_*(U)). \end{aligned}$$

Now suppose that $D \leq E \leq D$. Say $f, g : \omega \to \omega$ with $E = f_*(D)$ and $D = g_*(E)$. Then $E = f_*(g_*(E)) = (f \circ g)_*(E)$. By Proposition 2.14 it follows that $M \stackrel{\text{def}}{=} \{n : f(g(n)) = n\} \in E$. First suppose that M is finite. Then E is principal; say $\{m\} \in E$. Then for any $X \subseteq \omega$,

$$X \in D \text{ iff } g^{-1}[X] \in E \text{ iff } m \in g^{-1}[X] \text{ iff } g(m) \in X;$$

so D is principal with $\{g(m)\} \in D$. Let h be any permutation of ω such that $h(m) = g(m)$. Then

$$X \in D \text{ iff } g(m) \in X \text{ iff } h(m) \in X \text{ iff } m \in h^{-1}[X] \text{ iff } h^{-1}[X] \in E;$$

so $h_*(E) = D$, as desired.

Second, suppose that M is infinite. Note that $g \restriction M$ is one-one. Write $M = M_0 \cup M_1$, where M_0, M_1 are disjoint and infinite. Say $M_0 \in E$. Then there is a permutation h of ω such that $h[M_0] = g[M_0]$. For any $X \subseteq \omega$,

$$\begin{aligned} X \in D \quad &\text{iff} \quad g^{-1}[X] \in E \\ &\text{iff} \quad M_0 \cap g^{-1}[X] \in E \\ &\text{iff} \quad M_0 \cap h^{-1}[X] \in E \\ &\text{iff} \quad h^{-1}[X] \in E; \end{aligned}$$

so $D = h_*(E)$, as desired. □

Proposition 2.16. *An ultrafilter D on ω is minimal iff it is Ramsey*

Proof. First suppose that D is Ramsey and $E \leq D$, with both D and E nonprincipal. Let $f : \omega \to \omega$ be such that $E = f_*(D)$. If f is constant on $X \in D$, say with value m, then $X \subseteq f^{-1}[\{m\}]$, so $f^{-1}[\{m\}] \in D$ and hence $\{m\} \in E$, contradiction. Hence f is one-one on some member X of D. Let $g : \omega \to \omega$ be such that $g \circ f = \mathrm{Id} \restriction X$. Then for every $Y \subseteq \omega$,

$$Y \in g_*(E) \text{ iff } g^{-1}[Y] \in E \text{ iff } f^{-1}[g^{-1}[Y]] \in D \text{ iff } (g \circ f)^{-1}[Y] \cap X \in D \text{ iff } y \in D.$$

Thus $D \leq E$, so $D \equiv E$ by Proposition 2.15.

Second, suppose that D is minimal. Let $f : \omega \to \omega$, and suppose that f is not constant on any member of D; we show that f is one-one on some member of D. In fact, $f_*(D) \leq D$, so $f_*(D) \equiv D$. Hence there is a permutation g of ω such that $g_*(f_*(D)) = D$. Thus $(g \circ f)_*(D) = D$, so by Proposition 2.14 $\{n : g(f(n)) = n\} \in D$. Obviously f is one-one on this set, as desired. □

Proposition 2.17. *If ω_α is singular, then there is no nonprincipal ω_α-complete ultrafilter on ω_α*

Proof. Suppose that D is a nonprincipal ω_α-complete ultrafilter on ω_α, with ω_α singular. Say $\omega_\alpha = \bigcup_{\beta < \mathrm{cf}(\alpha)} \kappa_\beta$ with each κ_β a cardinal less than ω_α. For each $\beta < \mathrm{cf}(\alpha)$, and each $\xi \in \kappa_\beta$, $\{\xi\} \notin D$, so $\omega_\alpha \backslash \{\xi\} \in D$. Hence $\omega_\alpha \backslash \kappa_\beta = \bigcap_{\xi \in \kappa_\beta} \{\xi\} \in D$. So $\emptyset = \bigcap_{\beta < \mathrm{cf}(a)} (\omega_\alpha \backslash \kappa_\beta) \in D$, contradiction. □

Proposition 2.18. *Suppose that κ is a regular cardinal, $|A| \geq \kappa$, and*

$$F = \{X \subseteq [A]^{<\kappa} : \exists P \in [A]^{<\kappa} [\{Q \in [A]^{<\kappa} : P \subseteq Q\} \subseteq X]$$

Then F is a κ-complete filter on $[A]^{<\kappa}$.

Proof. If $P \in [A]^{<\kappa}$, then $\{Q \in [A]^{<\kappa} : P \subseteq Q\} \neq \emptyset$, and hence $\emptyset \notin F$. Now suppose that $X \in F$ and $X \subseteq Y \subseteq [A]^{<\omega}$. Choose $P \in [A]^{<\kappa}$ so that $\{Q \in [A]^{<\kappa} : P \subseteq Q\} \subseteq X$. Then $\{Q \in [A]^{<\kappa} : P \subseteq Q\} \subseteq Y$, so $Y \in F$. Now suppose that $\mathscr{A} \in F^{<\kappa}$. For each $X \in \mathscr{A}$ choose $P_X \in [A]^{<\kappa}$ so that $\{Q \in [A]^{<\kappa} : P_x \subseteq Q\} \subseteq X$. Let $P' = \bigcup_{X \in \mathscr{A}} P_X$. Then $P' \in [A]^{<\kappa}$, and $\{Q \in [A]^{<\kappa} : P' \subseteq Q\} \subseteq \bigcap \mathscr{A}$, so $\bigcap \mathscr{A} \in F$. □

Proposition 2.19. *Let A be a subalgebra of a BA B and let $u \in B \backslash A$. Then there are ultrafilters F, G on B such that $u \in F \backslash G$ and $F \cap A = G \cap A$.*

Proof. We claim that $\{a \in A : -a \leq u\} \cup \{u\}$ has fip. Otherwise let $a_0, \ldots, a_{m-1} \in A$ be such that $-a_i \leq u$ for all $i < m$ and $a_0 \cdot \ldots \cdot a_{m-1} \cdot u = 0$. So

$$u \leq -a_0 + \cdots + -a_{m-1} \leq u,$$

contradiction. So the claim holds. Let F be an ultrafilter containing $\{a \in A : -a \leq u\} \cup \{u\}$. So $u \in F$. Now we claim that $(F \cap A) \cup \{-u\}$ has fip. Otherwise we get $a \in F \cap A$ such that $a \cdot -u = 0$. So $a \leq u$, and hence $-a \in F$, contradiction. So this claim holds. If G is an ultrafilter such that $(F \cap A) \cup \{-u\} \subseteq G$, the desired conclusion of the Proposition holds. □

Proposition 2.20. *If B is κ-complete and κ-saturated, then B is complete.*

Proof. Suppose that $X \subseteq B$. Let $X' = \{a \in B : a \leq x \text{ for some } x \in X\}$. Let $Y \subseteq X'$ be maximal pairwise disjoint. Then $\sum Y$ exists. If $x \in X$ and $x \not\leq \sum Y$, then $x \cdot -\sum Y \neq 0$, so there is a nonzero $a \in X'$ such that $a \leq x \cdot -\sum Y$. Then $Y \cup \{a\} \subseteq X'$ is pairwise disjoint, so $a \in Y$, hence $-\sum Y \leq -a$ and so $a \leq -a$, contradiction. Hence $\sum Y$ is an upper bound for X; so $\sum Y = \sum X$. □

Proposition 2.21. *Let $B = \mathscr{P}(\kappa)/I_{\rm NS}$. Thus $X \equiv Y$ iff $X, Y \subseteq \kappa$ and $X \triangle Y \in I_{\rm NS}$ defines an equivalence relation, and B consists of all equivalence classe, with natural Boolean operations. The equivalence class of $Y \subseteq \kappa$ is denoted by $[Y]$. For a system $X \in {}^{\kappa}\mathscr{P}(\kappa)$ the g.l.b. of $\{[X_\alpha] : \alpha < \kappa\}$ is $[\triangle_{\alpha<\kappa} X_\alpha]$.*

Proof. First we show that it is a lower bound. Suppose that $\beta < \kappa$. Let $C = \{\xi : \beta < \xi\}$. So C is club, and if $\xi \in (\triangle_{\alpha<\kappa} X_\alpha \backslash X_\beta) \cap C$, then $\xi \in X_\beta$, contradiction. Now suppose that $[Y] \leq [X_\alpha]$ for all $\alpha < \kappa$. Let C_α be club such that $(Y \backslash X_\alpha) \cap C_\alpha = 0$, for each $\alpha < \kappa$. Then it is enough to show that

$$(Y \backslash \triangle_{\alpha<\kappa} X_\alpha) \cap \triangle_{\alpha<\kappa} C_\alpha = 0.$$

Suppose, to the contrary, that ξ is a member of this intersection. Choose $\alpha < \xi$ such that $\xi \notin X_\alpha$. But $\xi \in C_\alpha$, so this contradicts $(Y \backslash X_\alpha) \cap C_\alpha = 0$. □

Proposition 2.22. *Suppose that $\omega < \lambda < \kappa$. Then E^κ_λ is the union of κ pairwise disjoint stationary sets.*

Proof. Let $W \subseteq E^\kappa_\lambda$ be stationary. For every $\alpha \in W$ let $\langle a^\alpha_\xi : \xi < \lambda \rangle$ be strictly increasing with sup α.

(1) There is a $\nu < \lambda$ such that for all $\eta < \kappa$ the set $\{\alpha \in W : a^\alpha_\nu \geq \eta\}$ is stationary.

In fact, suppose not. Thus for all $\nu < \lambda$ there exist a $\eta_\nu < \kappa$ and a club C_ν in κ such that for all $\alpha \in W \cap C_\nu[a^\alpha_\nu < \eta_\nu]$. Let $\rho = \sup_{\nu<\lambda} \eta_\nu$. Then for all $\alpha \in \bigcap_{\nu<\lambda} C_\nu \cap W$ and all $\nu < \lambda$ we have $a^\alpha_\nu < \rho$. Hence for all $\alpha \in \bigcap_{\nu<\lambda} C_\nu \cap W$ we have $\alpha \leq \rho$. Since $|\bigcap_{\nu<\lambda} C_\nu \cap W| = \kappa$ and $\rho < \kappa$, this is a contradiction. So (1) holds.

Let ν be as in (1). For each $\alpha \in W$ let $f(\alpha) = a^\alpha_\nu$. So f is regressive. So by Fodor's theorem, for every $\eta < \kappa$ there exist a stationary set S_η contained in the set of (1) and a $\sigma_\eta \geq \eta$ such that $f(\alpha) = \sigma_\eta$ for all $\alpha \in S_\eta$. If $\sigma_\eta \neq \sigma_{\eta'}$, then $S_\eta \cap S_{\eta'} = \emptyset$. Since κ is regular, the range of σ has size κ, and the result follows. □

Theorem 2.23. *If κ is a regular uncountable cardinal, then every stationary subset W of $\{\alpha < \kappa : \mathrm{cf}(\alpha) < \alpha\}$ is a union of κ disjoint stationary sets.*

Proof. For each limit ordinal $\alpha \in W$ let $f(\alpha) = \mathrm{cf}(\alpha)$. By Fodor's theorem there is a $\lambda < \kappa$ and a stationary subset W' of W such that f takes the value λ on W'. Thus $W' \subseteq E^\lambda_\kappa$ and the result follows from the above generalization. □

Theorem 2.24. *Let κ be an uncountable regular cardinal. Then every stationary subset of κ is the disjoint union of κ stationary subsets.*

Proof. Let A be a stationary subset of κ. If $A \cap \{\alpha < \kappa : \mathrm{cf}(\alpha) < \alpha\}$ is stationary, then the above theorem applies, and so $A \cap \{\alpha < \kappa : \mathrm{cf}(\alpha) < \alpha\}$ is a disjoint union of κ stationary sets. We can take the union of one of them with $A \backslash \{\alpha < \kappa : \mathrm{cf}(\alpha) < \alpha\}$ to get the desired decomposition of A. So we can assume that $A \cap \{\alpha < \kappa : \mathrm{cf}(\alpha) < \alpha\}$ is not stationary; let C be club such that $A \cap \{\alpha < \kappa : \mathrm{cf}(\alpha) < \alpha\} \cap C = \emptyset$. Then $A \cap C$ is stationary with every element a regular cardinal. It suffices now, as above, to show that $A \cap C$ is a disjoint union of κ stationary subsets. We may assume that $\omega \notin A \cap C$. So $A \cap C$ is stationary set of regular uncountable cardinals. By Lemma 8.9 of Jech the set $W \stackrel{\text{def}}{=} \{\alpha \in A \cap C : A \cap C \cap \alpha \text{ is not a stationary subset of } \alpha\}$ is stationary. For each $\alpha \in W$ let C_α be a club in α such that $A \cap C \cap C_\alpha = \emptyset$. Now let $\langle a^\alpha_\xi : \xi < \alpha\rangle$ be the strictly increasing enumeraton of C_α.

Now we claim

(1) $\exists \xi < \kappa \forall \eta < \kappa[\{\alpha \in W : \xi < \alpha \text{ and } a^\alpha_\xi \geq \eta\} \text{ is stationary}]$.

For, assume not. Then for every $\xi < \kappa$ there is an $\eta(\xi) < \kappa$ and a club C_ξ such that $\forall \alpha \in W \cap C_\xi[\xi < \alpha \to a^\alpha_\xi < \eta(\xi)]$. Let $D = \triangle_{\xi<\kappa} C_\xi$. Then D is club, and $\forall \beta \in D \forall \xi < \beta[\beta \in C_\xi]$. Hence $\forall \beta \in D \cap W[\xi < \beta \to a^\beta_\xi < \eta(\xi)]$.

(∗) $E \stackrel{\text{def}}{=} \{\gamma \in D : \forall \xi < \gamma[\eta(\xi) < \gamma]\}$ is club.

For closure, suppose that $\alpha < \kappa$ is a limit ordinal and $E \cap \alpha$ is unbounded. Then $\alpha \in D$ since D is club. If $\xi < \alpha$, then there is a $\beta \in E \cap \alpha$ such that $\xi < \beta$. Hence $\eta(\xi) < \beta < \alpha$. So $\alpha \in E$.

For unbounded, suppose that $\alpha < \kappa$. Choose $\beta_0 \in D$ with $\alpha < \beta_0$. Having defined β_n, let $\gamma = \bigcup\{\eta(\xi) : \xi < \beta_n\}$ and choose $\beta_{n+1} > \beta_n, \gamma$ with $\beta_n \in D$. Let $\delta = \bigcup_{n\in\omega} \beta_n$. Then $\delta \in D$, and clearly also $\delta \in E$. This proves (∗).

Now let $\gamma < \alpha$ be two ordinals in $W \cap E$. Then $\forall \xi < \gamma[a^\alpha_\xi < \eta(\xi) < \gamma]$. Hence $a^\alpha_\gamma = \gamma$. But $\gamma \in W$ and $a^\alpha_\gamma \notin W$, contradiction. So (1) holds.

We take ξ as in (8.6). For each $\eta < \kappa$ let $W_\eta = \{\alpha \in W : \xi < \alpha \text{ and } a^\alpha_\xi \geq \eta\}$; so W_η is stationary. For each $\alpha < \kappa$ let $f(\alpha) = a^\alpha_\xi$. Then for each $\eta < \kappa$ the function $f \restriction W_\eta$ is regressive, so there is a stationary subset W'_η of W_η and an ordinal γ_η such that $f(\alpha) = \gamma_\eta$ for all $\alpha \in W'_\eta$. Thus for all $\eta < \kappa$, take any $\alpha \in W'_\eta$; then $\gamma_\eta = f(\alpha) = a^\alpha_\xi \geq \eta$. Now we define $\eta(\delta)$ for $\delta < \kappa$ by induction. Let $\eta(0) = 0$. If $\eta(\delta)$ has been defined for all δ, ε, let $\eta(\varepsilon) = \bigcup_{\delta<\varepsilon}(\gamma_{\eta(\delta)} + 1)$. Then for all $\delta, \varepsilon < \kappa$, if $\delta < \varepsilon$ then $\gamma_{\eta(\delta)} < \eta(\varepsilon) \leq \gamma_{\eta(\varepsilon)}$. Now suppose that $\delta < \varepsilon$ and $\alpha \in W'_{\gamma_{\eta(\delta)}} \cap W'_{\gamma_{\eta(\varepsilon)}}$. Then $\gamma_{\eta(\delta)} = f(\alpha) = \gamma_{\eta(\varepsilon)}$, contradiction. □

Theorem 2.25. *Suppose that $\langle \kappa_\alpha : \alpha < \beta \rangle$ is the strictly increasing enumeration of the first β inaccessibles. Then for each $\alpha < \beta$, if $\alpha < \kappa_\alpha$, then the set of all regular cardinals less than κ_α is nonstationary.*

Proof. Since $\beta \overset{\text{def}}{=} \mathrm{cf}(\bigcup_{\xi<\alpha} \kappa_\xi) \leq \alpha$, we have $\beta < \kappa_\alpha$. Let C be the set of all strong limit cardinals in the interval (β, κ_α). Clearly C is club and has no regular cardinals as members. □

Theorem 2.26. *If κ is a Mahlo cardinal, then the set I of all inaccessibles below κ is stationary.*

Proof. Suppose not, and let C be club in κ such that $C \cap I = \emptyset$. The set D of all strong limit cardinals is club in κ. Let B be the set of all regular cardinals below κ. Then $B \cap C \cap D = \emptyset$, contradiction. □

Theorem 2.27. *Let κ be Mahlo, and let $\langle \lambda_\xi : \xi < \alpha \rangle$ be the increasing emumeration of all inaccessibles, where α is an ordinal or $\alpha =$ **ON**. Say $\kappa = \lambda_\xi$. Then $\xi = \kappa$.*

Proof. By Theorem 2.25, $\lambda_\xi \leq \xi$. By induction, $\eta \leq \lambda_\eta$ for all η. □

Theorem 2.28. *If κ is weakly Mahlo, then the set I of weak inaccessibles below κ is stationary.*

Proof. Suppose not, and let C be club in κ such that $C \cap I = \emptyset$. The set D of all limit cardinals is club in κ. Let B be the set of all regular cardinals below κ. Then $B \cap C \cap D = \emptyset$, contradiction. □

Theorem 2.29. *If κ is uncountable and regular, then the club filter on κ is κ-complete, normal, and contains all complements of bounded subsets of κ.*

Proof. Let F be the club filter on κ. Suppose that $|I| < \kappa$ and $\langle c_i : i \in I \rangle$ is a system of members of F. For each $i \in I$ let C_i be club with $C_i \subseteq c_i$. Then $\bigcap_{i\in I} C_i$ is club, and $\bigcap_{i\in I} C_i \subseteq \bigcap_{i\in I} c_i$.

Next, suppose that $c_\alpha \in F$ for all $\alpha < \kappa$. For each $\alpha < \kappa$ let C_α be club with $C_\alpha \subseteq c_\alpha$. Then $\triangle_{\alpha<\kappa} C_\alpha \subseteq \triangle_{\alpha<\kappa} c_\alpha$. In fact, suppose that $\beta \in \triangle_{\alpha<\kappa} C_\alpha$. Then $\forall \xi < \beta[\beta \in C_\xi]$, so $\forall \xi < \beta[\beta \in c_\xi]$. Hence $\beta \in \triangle_{\alpha<\kappa} c_\alpha$. It follows that F is normal.

If $A \subseteq \kappa$ and $\forall\alpha \in A[\alpha < \beta]$ with $\beta \in \kappa$, then $[\beta, \kappa) \subseteq (\kappa\backslash A)$, and so $\kappa\backslash A \in F$. $\square$

Functions f, g on a regular cardinal μ are *almost disjoint* iff there is an $\alpha < \mu$ such that $f(\beta) \neq g(\beta)$ for all $\beta \in [\alpha, \mu)$.

Lemma 2.30. *Suppose that κ is singular and* $\mathrm{cf}(\kappa) > \omega$. *Assume that $\lambda^{\mathrm{cf}(\kappa)} < \kappa$ for all $\lambda < \kappa$. Let $\langle \nu_\alpha : \alpha < \kappa\rangle$ be a strictly increasing continuous sequence of infinite cardinals with supremum κ. Suppose that $F \subseteq \prod_{\alpha<\mathrm{cf}(\kappa)} A_\alpha$ is an almost disjoint family of functions such that the set*

$$\{\alpha < \mathrm{cf}(\kappa) : |A_\alpha| \leq \nu_\alpha\}$$

is stationary. Then $|F| \leq \kappa$.

Proof. We may assume that each A_α is a set of ordinals, and $A_\alpha \subseteq \nu_\alpha$ for all α in some stationary subset S of $\mathrm{cf}(\kappa)$, with each $\alpha \in S$ a limit ordinal. Now for all $f \in F$ and all $\alpha \in S$ we have $f(\alpha) < \nu_\alpha$, and so there is a $g(\alpha) < \alpha$ such that $f(\alpha) < \nu_{g(\alpha)}$. So g is regressive on S. Let S_f be a stationary subset of S and let $\beta_f < \mathrm{cf}(\alpha)$ such that $g(\alpha) = \beta_f$ for all $\alpha \in S_f$. Define $\mathscr{F}(f) = (S_f, f \restriction S_f)$. Now for any stationary T, and any distinct $f, h \in F$, if $f \restriction T = h \restriction T$, then $f = h$ since f and h are almost disjoint. Hence $\mathscr{F}$ is one-one. Hence it suffices to show that $|\mathrm{rng}(\mathscr{F})| \leq \kappa$. Suppose not. Now

$$\mathrm{rng}(\mathscr{F}) = \bigcup_{T\subseteq\mathrm{cf}(\kappa)} \{(S_f, f \restriction S_f) : S_f = T\},$$

and $|\mathscr{P}(\mathrm{cf}(\kappa))| = 2^{\mathrm{cf}(\kappa)} < \kappa$, so there is a $T \subseteq \mathrm{cf}(\kappa)$ such that $\kappa < |\{(S_f, f \restriction S_f) : S_f = T\}|$. Let $\mathscr{G} = \{f \in \mathscr{F} : S_f = T\}$; so $|\mathscr{G}| > \kappa$. Next, $\mathscr{G} = \bigcup_{\gamma<\mathrm{cf}(\kappa)}\{f \in \mathscr{G} : \beta_f = \gamma\}$, so there is a $\gamma < \mathrm{cf}(\kappa)$ and an $\mathscr{H} \subseteq \mathscr{G}$ with $|\mathscr{H}| > \kappa$ such that $\forall f \in \mathscr{H}[\beta_f = \gamma]$. Now $|\mathscr{H}| \leq \prod_{\alpha<\mathrm{cf}(\kappa)} \nu_\gamma = \nu_\gamma^{\mathrm{cf}(\kappa)} < \kappa$, contradiction. $\square$

Lemma 2.31. *Suppose that κ is singular and* $\mathrm{cf}(\kappa) > \omega$. *Assume that $\lambda^{\mathrm{cf}(\kappa)} < \kappa$ for all $\lambda < \kappa$. Let $\langle \nu_\alpha : \alpha < \mathrm{cf}(\kappa)\rangle$ be a strictly increasing continuous sequence of cardinals with supremum κ. Suppose that $F \subseteq \prod_{\alpha<\mathrm{cf}(\kappa)} A_\alpha$ is an almost disjoint family of functions such that the set*

$$\{\alpha < \mathrm{cf}(\kappa) : |A_\alpha| \leq \nu_\alpha^+\}$$

is stationary. Then $|F| \leq \kappa^+$.

Proof. Assume the hypotheses. Let U be an ultrafilter on $\mathrm{cf}(\kappa)$ extending the club filter. If $S \in U$, then S is stationary, as otherwise $C \cap S = \emptyset$ for some club C, hence $\mathrm{cf}(\kappa)\backslash S$ would be in the club filter and hence in U, contradiction. Wlog each $A_\alpha \subseteq \nu_\alpha^+$. Now we define

$$f < g \quad \text{iff} \quad f, g \in F \text{ and } \{\alpha < \mathrm{cf}(\kappa) : f(\alpha) < g(\alpha)\} \in U.$$

Then $<$ is a linear order of F, since $\{\alpha : f(\alpha) \neq g(\alpha)\} \in U$ for distinct $f, g \in F$, as they are almost disjoint. Now for each $f \in F$ let $G_f = \{g \in F :$ there is a stationary T such that $\forall\alpha \in T[g(\alpha) < f(\alpha)]\}$. For each $g \in G_f$ choose a stationary set T_g with the indicated property. We claim that $|G_f| \leq \kappa$. For, suppose that $|G_f| > \kappa$. Now $G_f = \bigcup_{S \subseteq \mathrm{cf}(\kappa)} \{g \in G_f : T_g = S\}$, so there is an $S \subseteq \mathrm{cf}(\kappa)$ such that $G'_f = \{g \in G : T_g = S\}$ has size $> \kappa$. Let $G''_f = \{g \restriction S : g \in G'_f\}$. Then G''_f is an almost disjoint set of functions, since F is. Now $\forall h \in G''_f[h \in \prod_{\alpha \in S} \nu_\alpha]$, so this contradicts Lemma 2.30. This proves the claim that $|G_f| \leq \kappa$.

If $\langle f_\alpha : \alpha < \sigma \rangle$ is a cofinal sequence in F with respect to $<$, then we must have $\sigma \leq \kappa^+$, as otherwise $|G_{f_{\kappa^+}}| \geq \kappa^+$. Hence $|F| \leq \sum_{\alpha<\sigma} |G_{f_\alpha}| \leq \kappa^+$, as desired. □

Lemma 2.32. *Suppose that κ is singular and $\mathrm{cf}(\kappa) > \omega$. Assume that $\lambda^{\mathrm{cf}(\kappa)} < \kappa$ for all $\lambda < \kappa$. Let $\langle \nu_\alpha : \alpha < \mathrm{cf}(\kappa) \rangle$ be a strictly increasing continuous sequence of cardinals with supremum κ. Suppose that $\{\alpha < \mathrm{cf}(\kappa) : \nu_\alpha^{\mathrm{cf}(\nu_\alpha)} = \nu_\alpha^+\}$ is stationary in $\mathrm{cf}(\kappa)$.*

Then $\kappa^{\mathrm{cf}(\kappa)} = \kappa^+$.

Proof. Assume the hypotheses. For every $h \in {}^{\mathrm{cf}(\kappa)}\kappa$ define $f_h = \langle k^h_\alpha : \alpha < \mathrm{cf}(\kappa) \rangle$, where $\mathrm{dmn}(k^h_\alpha) = \mathrm{cf}(\kappa)$ and

$$k^h_\alpha(\xi) = \begin{cases} h(\xi) & \text{if } h(\xi) < \nu_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Let $F = \{f_h : h \in {}^{\mathrm{cf}(\kappa)}\kappa\}$. Then

(1) If h and h' are different members of ${}^{\mathrm{cf}(\kappa)}\kappa$, then f_h and $f_{h'}$ are almost disjoint.

In fact, say $h(\xi) \neq h'(\xi)$. Choose $\alpha < \mathrm{cf}(\kappa)$ such that $h(\xi), h'(\xi) < \nu_\alpha$. Then for all $\beta \in [\alpha, \mathrm{cf}(\kappa))$ we have $k^h_\beta(\xi) = h(\xi) \neq h'(\xi) = k^{h'}_\beta(\xi)$, and so $h^h_\beta \neq k^{h'}_\beta$, as desired in (1).

Now note that $F \subseteq \prod_{\alpha<\mathrm{cf}(\kappa)} {}^{\mathrm{cf}(\kappa)}\nu_\alpha$.

(2) $\{\alpha < \mathrm{cf}(\kappa) : \alpha$ is a limit ordinal and $\forall \lambda < \nu_\alpha(\lambda^{\mathrm{cf}(\kappa)} < \nu_\alpha)\}$ is club in $\mathrm{cf}(\kappa)$.

In fact, it is clearly closed. Now suppose that $\alpha_0 < \mathrm{cf}(\kappa)$; we want to find an ordinal $\beta \in (\alpha_0, \mathrm{cf}(\kappa))$ such that β is in the above set. We may assume that α_0 is a limit ordinal. Suppose that α_m has been defined. Now for all $\beta < \alpha_m$ we have $\nu_\beta^{\mathrm{cf}(\kappa)} < \kappa$, so there is a $\gamma(\beta) < \mathrm{cf}(\kappa)$ such that $\nu_\beta^{\mathrm{cf}(\kappa)} < \nu_{\gamma(\beta)}$. Let α_{m+1} be a limit ordinal $< \mathrm{cf}(\kappa)$ such that $\alpha_m < \alpha_{m+1}$ and $\gamma(\beta) < \alpha_{m+1}$ for all $\beta < \alpha_m$. Then the ordinal $\delta \stackrel{\text{def}}{=} \bigcup_{m\in\omega} \alpha_m$ is in the set of (2). In fact, clearly δ is a limit ordinal. Now suppose that $\lambda < \nu_\delta$. Choose $m \in \omega$ such that $\lambda < \nu_{\alpha_m}$. Now $\alpha_m < \alpha_{m+1}$, so

$$\lambda^{\mathrm{cf}(\kappa)} \leq \nu_{\alpha_m}^{\mathrm{cf}(\kappa)} < \nu_{\gamma(\alpha_m)} < \nu_{\alpha_{m+1}} < \nu_\delta.$$

Thus (2) holds.

(3) $\{\alpha < \mathrm{cf}(\kappa) : \nu_\alpha^{\mathrm{cf}(\kappa)} \leq \nu_{\alpha+1}\}$ is stationary.

In fact, let C be the set of (2), and let $S = \{\alpha < \mathrm{cf}(\kappa) : \nu_\alpha^{\mathrm{cf}(\nu_\alpha)} = \nu_\alpha^+\}$. So S is stationary by assumption, and hence $S' \stackrel{\text{def}}{=} \{\alpha \in C \cap S : \mathrm{cf}(\kappa) < \nu_\alpha\}$ is stationary. Take any $\alpha \in S'$. Now we apply Jech Theorem 5.20 with κ, λ replaced by $\nu_\alpha, \mathrm{cf}(\kappa)$. By the definition of S',

$\mathrm{cf}(\kappa) < \nu_\alpha$. If $\mu < \nu_\alpha$, then $\mu^{\mathrm{cf}(\kappa)} < \nu_\alpha$ by (2). Hence by Jech Theorem 5.20, $\nu_\alpha^{\mathrm{cf}(\kappa)} = \nu_\alpha$ or $\nu_\alpha^{\mathrm{cf}(\nu_\alpha)} = \nu_\alpha^+$.

Now by Lemma 2.31 we get $|F| \le \kappa^+$. But clearly $|F| = \kappa^{\mathrm{cf}(\kappa)}$, so $\kappa^{\mathrm{cf}(\kappa)} = \kappa^+$. $\square$

Theorem 2.33. *If κ is a singular cardinal such that $\mathrm{cf}(\kappa) > \omega$ and $\forall$ cardinals $\lambda < \kappa[2^\lambda = \lambda^+]$, then $2^\kappa = \kappa^+$.*

Proof. For any $\lambda < \kappa$ we have $\lambda < \lambda^{\mathrm{cf}(\lambda)} \le 2^\lambda = \lambda^+$, so by Lemma 3, $\kappa^{\mathrm{cf}(\kappa)} = \kappa^+$. By Theore 5.16(iii), $\kappa^{\mathrm{cf}(\kappa)} = 2^\kappa$. $\square$

Theorem 2.34. *If the singular cardinals hypothesis holds for all singular cardinals of cofinality ω, then it holds for all singular cardinals.*

Proof. We prove by induction on κ that, under the indicated assumption, if $2^{\mathrm{cf}(\kappa)} < \kappa$ then $\kappa^{\mathrm{cf}(\kappa)} = \kappa^+$. This is given for $\mathrm{cf}(\kappa) = \omega$. Assume that $\mathrm{cf}(\kappa) > \omega$ and it holds for smaller cardinals. So we assume that $2^{\mathrm{cf}(\kappa)} < \kappa$. Then

(1) If $\lambda < \kappa$, then $\lambda^{\mathrm{cf}(\kappa)} < \kappa$.

We prove this by induction on λ. It is obviously true for $\lambda \le \mathrm{cf}(\kappa)$, so suppose that $\mathrm{cf}(\kappa) < \lambda$. If $\rho < \lambda$ and $\lambda \le \rho^{\mathrm{cf}(\kappa)}$, then $\lambda^{\mathrm{cf}(\kappa)} \le \rho^{\mathrm{cf}(\kappa)} < \kappa$ by the inductive assumption. So, suppose that $\rho^{\mathrm{cf}(\kappa)} < \lambda$ for all $\rho < \lambda$. Recall that $\mathrm{cf}(\kappa) < \lambda$. Hence Theorem 5.20(iii) applies.

Case 1. $\mathrm{cf}(\lambda) > \mathrm{cf}(\kappa)$. Then $\lambda^{\mathrm{cf}(\kappa)} = \lambda < \kappa$ by Theorem 5.20(iii)(a).

Case 2. $\mathrm{cf}(\lambda) \le \mathrm{cf}(\kappa)$. Then $\lambda^{\mathrm{cf}(\kappa)} = \lambda^{\mathrm{cf}(\lambda)}$ by Theorem 5.20(iii)(b). Now $2^{\mathrm{cf}(\lambda)} < \lambda$, so by the inductive hypothesis on κ, $\lambda^{\mathrm{cf}(\lambda)} = \lambda^+ < \kappa$.

This proves (1). Let $\langle \rho_\alpha : \alpha < \mathrm{cf}(\kappa)\rangle$ be a normal sequence of infinite cardinals with limit κ. Then the set $S \stackrel{\mathrm{def}}{=} \{\alpha < \mathrm{cf}(\kappa) : \mathrm{cf}(\rho_\alpha) = \omega \text{ and } 2^\omega < \rho_\alpha\}$ is stationary. In fact, if $C \subseteq \mathrm{cf}(\kappa)$ is club, let $\langle b_\xi : \xi < \mathrm{cf}(\kappa)\rangle$ enumerate the members c of C such that $2^\omega < b_0$. Then $b_\omega \in S$. By the hypothesis, $\rho_\alpha^{\mathrm{cf}(\rho_\alpha)} = \rho_\alpha^+$ for each $\alpha \in S$. Hence by (1) and Lemma 3 we have $\kappa^{\mathrm{cf}(\kappa)} = \kappa^+$, completing the inductive proof. $\square$

Proposition 2.35. *Suppose that $\mathrm{cf}(\alpha) > \omega$ and $S \subseteq \alpha$. Then the following are equivalent:*

(i) S is stationary in α.

(ii) For every normal function $f : \mathrm{cf}(\alpha) \to \alpha$ with $rng(f)$ unbounded in α, the set $f^{-1}[S]$ is stationary in $\mathrm{cf}(\alpha)$.

(iii) For some normal function $f : \mathrm{cf}(\alpha) \to \alpha$ with $rng(f)$ unbounded in α, the set $f^{-1}[S]$ is stationary in $\mathrm{cf}(\alpha)$.

Proof. (i)$\Rightarrow$(ii): Suppose that S is stationary in α and $f : \mathrm{cf}(\alpha) \to \alpha$ is a normal function with $\mathrm{rng}(f)$ unbounded in α. Let C be club in $\mathrm{cf}(\alpha)$. Then clearly $f[C]$ is unbounded in α. To show that it is closed in α, suppose that $\beta < \alpha$ is a limit ordinal and $f[C] \cap \beta$ is unbounded in β. Let $\gamma = \sup\{\delta \in C : f(\delta) < \beta\}$. Then clearly γ is a limit ordinal, and $C \cap \gamma$ is unbounded in γ. So $\gamma \in C$. Clearly $f(\gamma) = \beta$. So $\beta \in f[C]$, as desired. So $f[C]$ is club in α. Choose $\delta \in S \cap f[C]$. Write $\delta = f(\varepsilon)$ with $\varepsilon \in C$. Then $\varepsilon \in f^{-1}[S] \cap C$. This shows that $f^{-1}[S]$ is stationary in $\mathrm{cf}(\alpha)$.

(ii)⇒(iii): obvious.

(iii)⇒(i): Suppose that $f : \mathrm{cf}(\alpha) \to \alpha$ is a normal function with $\mathrm{rng}(f)$ unbounded in α such that $f^{-1}[S]$ is stationary in $\mathrm{cf}(\alpha)$. We want to show that S is stationary in α. So, let C be club in α. Clearly $f^{-1}[C]$ is unbounded in $\mathrm{cf}(\alpha)$. To show that it is closed in $\mathrm{cf}(\alpha)$, suppose that $\beta < \mathrm{cf}(\alpha)$ is a limit ordinal, and $f^{-1}[C] \cap \beta$ is unbounded in β. Then clearly $f(\beta)$ is a limit ordinal, and $C \cap f(\beta)$ is unbounded in $f(\beta)$. So $f(\beta) \in C$, and hence $\beta \in f^{-1}[C]$. So $f^{-1}[C]$ is club in $\mathrm{cf}(\alpha)$. Hence we can choose $\gamma \in f^{-1}[S] \cap f^{-1}[C]$. So $f(\gamma) \in S \cap C$, as desired. $\square$

Proposition 2.36. *If S and T are stationary and $S \triangle T$ is nonstationary, then also $Tr(S) \triangle Tr(T)$ is nonstationary.*

Proof. Let C be club in κ such that $(S \triangle T) \cap C = \emptyset$. Thus $S \cap C = T \cap C$. Let C' be the set of all limits of members of C. So C' is club in κ and $C' \subseteq C$. We claim that $(\mathrm{Tr}(S) \triangle \mathrm{Tr}(T)) \cap C' = \emptyset$, as desired. For, suppose that $\alpha \in \mathrm{Tr}(S) \cap C'$; we show that $\alpha \in \mathrm{Tr}(T)$, and by symmetry we are through. Thus $\mathrm{cf}(\alpha) > \omega$ and $S \cap \alpha$ is stationary in α. We show that $T \cap \alpha$ is stationary in α, completing the proof. For, let D be club in α. Then also $C \cap D$ is club in α, since $\alpha \in C'$. So $T \cap C \cap D \cap \alpha = S \cap C \cap D \cap \alpha \neq \emptyset$, so in particular $T \cap \alpha \cap D \neq \emptyset$, as desired. $\square$

$S < T$ iff the following two conditions hold:

(1) $\{\alpha \in T : \mathrm{cf}(\alpha) \leq \omega\}$ is nonstationary in κ.

(2) $\{\alpha \in T : S \cap \alpha$ is nonstationary in $\alpha\}$ is nonstationary in κ.

We do not assume that S and T are stationary.

Proposition 2.37. *If $\omega < \lambda < \mu < \kappa$ and λ, μ, κ are regular. Then $E^\kappa_\lambda < E^\kappa_\mu$.*

Proof. let $C = (\mu, \kappa)$. We claim that

$$\{\alpha \in E^\kappa_\mu : E^\kappa_\lambda \cap \alpha \text{ is nonstationary in } \alpha\} \cap C = \emptyset.$$

In fact, suppose that α is in the indicated intersection. Let D be club in α such that $E^\kappa_\lambda \cap D = \emptyset$. Now $\alpha \in E^\kappa_\mu$, so $\mathrm{cf}(\alpha) = \mu$. Hence by an easy construction, D has a member β with cofinality λ. So $\beta \in E^\kappa_\lambda \cap D$, contradiction. $\square$

Lemma 2.38. *(i) $A < \mathrm{Tr}(A)$.*

(ii) If $A < B < C$ then $A < C$.

(iii) If $A < B$, $A \equiv A'$ mod I_{NS}, and $B \equiv B'$ mod I_{NS}, then $A' < B'$.

Proof. (i): first we note that $\{\alpha \in \mathrm{Tr}(A) : \mathrm{cf}(\alpha) \leq \omega\} = \emptyset$, and so it is nonstationary in κ. Second,

$$\begin{aligned}\{\alpha \in \mathrm{Tr}(A) :& A \cap \alpha \text{ is nonstationary in } \kappa\} \\ &= \{\alpha < \kappa : \mathrm{cf}(\alpha) > \omega \text{ and } A \cap \alpha \text{ is stationary in } \kappa \\ &\quad \text{and } A \cap \alpha \text{ is nonstationary in } \kappa\} = \emptyset;\end{aligned}$$

Hence $\{\alpha \in \mathrm{Tr}(A) : A \cap \alpha$ is nonstationary in $\kappa\}$ is nonstationary.

(ii): The assumptions $A < B < C$ mean

$$\begin{aligned}
&\{\alpha \in B : \mathrm{cf}(\alpha) \le \omega\} \text{ is nonstationary in } \kappa;\\
&\{\alpha \in C : \mathrm{cf}(\alpha) \le \omega\} \text{ is nonstationary in } \kappa;\\
&\{\alpha \in B : \alpha \cap A \text{ is nonstationary}\} \text{ is nonstationary;}\\
&\{\alpha \in C : \alpha \cap B \text{ is nonstationary}\} \text{ is nonstationary.}
\end{aligned}$$

We want to show

$$\{\alpha \in C : \alpha \cap A \text{ is nonstationary}\} \text{ is nonstationary.}$$

Our assumptions give us a club M in κ such that

$$\begin{aligned}
&\{\alpha \in B : \mathrm{cf}(\alpha) \le \omega\} \cap M = \emptyset;\\
&\{\alpha \in C : \mathrm{cf}(\alpha) \le \omega\} \cap M = \emptyset;\\
&\{\alpha \in B : \alpha \cap A \text{ is nonstationary}\} \cap M = \emptyset \text{ and}\\
&\{\alpha \in C : \alpha \cap B \text{ is nonstationary}\} \cap M = \emptyset.
\end{aligned}$$

Let M' be the set of all limits of members of M; so also M' is club in κ. Now it suffices to show that

$$\{\alpha \in C : \alpha \cap A \text{ is nonstationary}\} \cap M' = \emptyset.$$

So, suppose that $\alpha \in C \cap M'$; we show that $\alpha \cap A$ is stationary in α. To this end, let P be club in α, and let P' be the set of all of its limit points. Now $\alpha \in C \cap M$, so $\alpha \cap B$ is stationary. Since $\alpha \in M'$, it follows that $\alpha \cap M$ is club in α. So $M \cap P'$ is club in α, and so we can choose $\beta \in \alpha \cap B \cap M \cap P'$. Now $\beta \in B \cap M$, so $\beta \cap A$ is stationary in β. Since $\beta \in P'$, it follows that $P \cap \beta$ is club in β. So $\beta \cap A \cap P \neq \emptyset$, hence $A \cap P \neq \emptyset$, as desired.

(iii): Assume that $A < B$, $A \sim A'$ mod I_{NS}, and $B \sim B'$ mod I_{NS}. Thus $A \triangle A'$ is nonstationary, and $B \triangle B'$ is nonstationary. We want to show that $A' < B'$. Choose stationary sets M, N, P such that

$$\begin{aligned}
\{\alpha \in B : A \cap \alpha \text{ is nonstationary}\} \cap M &= \emptyset;\\
(A \triangle A') \cap N &= \emptyset;\\
(B \triangle B') \cap P &= \emptyset.
\end{aligned}$$

It now suffices to take any $\alpha \in M \cap N' \cap P \cap B'$ and show that $A' \cap \alpha$ is stationary. Let Q be club in α. Now also $N \cap \alpha$ is club in α, so $Q \cap N$ is club in α. Since $\alpha \in P \cap B'$, we have $\alpha \in B$. Then since $\alpha \in M$, we see that $A \cap \alpha$ is stationary. Now since $Q \cap N$ is club in α, we get $\beta \in A \cap Q \cap N$. Since $\beta \in A \cap N$, we get $\beta \in A'$. Thus $\beta \in A' \cap Q$, as desired. □

Proposition 2.39. *If $A < B$, then there is a club C such that $B \cap C \subseteq Tr(A)$.*

Proof. Assume that $A < B$. Thus by definition, $\{\alpha \in B : \mathrm{cf}(\alpha) \le \omega\}$ is nonstationary in κ, and also $\{\alpha \in B : A \cap \alpha \text{ is non-stationary in } \alpha\}$ is non-stationary in κ. Hence there is a club C in κ such that $C \cap \{\alpha \in B : \mathrm{cf}(\alpha) \leq \omega\} = \emptyset$ and also $C \cap \{\alpha \in B : A \cap \alpha$ is non-stationary in $\alpha\} = \emptyset$. Hence the Proposition holds. $\square$

Theorem 2.40. *$<$ on stationary sets is well-founded.*

Proof. Suppose to the contrary that there are stationary sets A_n for $n \in \omega$ such that $A_0 > A_1 > \cdots$. By Proposition 2.39 there are clubs C_n such that $A_n \cap C_n \subseteq \mathrm{Tr}(A_{n+1})$ for $n \in \omega$. For each $n \in \omega$ let

$$B_n = A_n \cap C_n \cap \mathrm{Lim}(C_{n+1}) \cap \mathrm{Lim}(\mathrm{Lim}(C_{n+2})) \cap \cdots.$$

Clearly each B_n is stationary. Also,

$$\forall n \in \omega[B_n \subseteq \mathrm{Tr}(B_{n+1})].$$

For, suppose that $\alpha \in B_n$. Then $\alpha \in A_n \cap C_n$, so $\alpha \in \mathrm{Tr}(A_{n+1})$. So $\mathrm{cf}(\alpha) > \omega$ and $A_{n+1} \cap \alpha$ is stationary in α. Now $\alpha \in \mathrm{Lim}(C_{n+1}) \cap \mathrm{Lim}(\mathrm{Lim}(C_{n+2})) \ldots$, so $C_{n+1} \cap \mathrm{Lim}(C_{n+2}) \ldots$ is club in α, and hence $A_{n+1} \cap \alpha \cap C_{n+1} \cap \mathrm{Lim}(C_{n+2}) \ldots$ is stationary in α. Thus $\alpha \in \mathrm{Tr}(B_{n+1})$.

Now for each $n \in \omega$ let $\alpha_n = \min(B_n)$. Now $B_{n+1} \cap \alpha_n$ is stationary. Hence $\alpha_{n+1} < \alpha_n$. Thus $\alpha_0 > \alpha_1 > \cdots$, contradiction. $\square$

The rank of a stationary subset of κ under $<$ is called its *order*.

$$o(A) = \sup\{o(X) + 1\text{''} X < A\};$$
$$o(\kappa) = \sup\{o(A) + 1 : A \subseteq \kappa, A \text{ stationary}\}.$$

In addition we set $o(\omega) = 0$ and $o(\alpha) = o(\mathrm{cf}(\alpha))$ for every limit ordinal α.

Proposition 2.41. *Assume that κ is regular and uncountable. If $|A| = \kappa$, then the set $\kappa \subseteq [\kappa]^{<\kappa}$ is closed unbounded in $[\kappa]^{<\kappa}$.* $\square$

Proposition 2.42. *Suppose that κ is uncountable and regular. If F is the club filter on κ and K is the club filter on $[\kappa]^{<\kappa}$, then $F = K \cap \mathscr{P}(\kappa)$.*

Proof. Suppose first that $X \in F$. Let C be club on κ with $C \subseteq X$. Then C is also club on $[\kappa]^{<\kappa}$, so $X \in K$. Second suppose that $X \in K \cap \mathscr{P}(\kappa)$. Let D be club on $[\kappa]^{<\kappa}$ such that $D \subseteq X$. Let C be the set of all limits of members of $\bigcup D$. So C is club in κ. We claim that $C \subseteq X$; this will show that $X \in F$, as desired. For, let $\alpha \in C$. Since α is a limit of members of $\bigcup D$ and $D \subseteq X \subseteq \kappa$, α must actually be a member of D, as a union of members of D. Hence $\alpha \in X$, as desired. $\square$

Theorem 2.43. *Let κ be uncountable and regular. The closed unbounded filter on $[\kappa]^{<\kappa}$ is κ-complete.*

Proof. Clearly if C and D are club, then so is $C \cap D$. So the club filter consists exactly of those sets which include a club. So to show that this filter is κ-complete it suffices to take any sequence $\langle C_\xi : \xi < \alpha\rangle$ of clubs, with $\alpha < \kappa$, and show that $\bigcap_{\xi<\alpha} C_\xi$ is club. Clearly it is closed. Let $x \in \mathscr{P}_\kappa(A)$. We define $\langle y_{\xi,m} : \xi < \alpha, m \in \omega\rangle$ by recursion on m, and within that by recursion on ξ. Suppose that $y_{\eta,n}$ has been defined, a member of $\mathscr{P}_\kappa(A)$, for each $n < m$ and $\eta < \alpha$. Let $y_{0,m}$ be a member of C_0 such that $\bigcup_{n<m,\eta<\alpha} y_{\eta,n} \subseteq y_{0,m}$. Note that this works for $m = 0$. If $y_{\eta,m}$ has been defined for each $\eta < \xi$, where $\xi > 0$, let $y_{\xi,m}$ be a member of C_ξ such that $\bigcup_{\eta<\xi} y_{\eta,m} \subseteq y_{\xi,m}$. This finishes the construction. Finally, let $z = \bigcup_{\xi<\alpha,m<\omega} y_{\xi,m}$. Clearly $z \in \bigcap_{\xi<\alpha} C_\xi$. □

Theorem 2.44. *If f is a function defined on a stationary set $S \subseteq [\lambda]^{<\kappa}$ such that $f(x) \in x$ for every nonempty $x \in S$, then there exist a stationary $T \subseteq S$ and an $a \in \lambda$ such that $\forall x \in T[f(x) = a]$.*

Proof. Assume the hypotheses, but suppose that the theorem is false. So for every $a \in A$ the set $\{x \in S : f(x) = a\}$ is nonstationary. Hence there is a club C_a such that $C_a \cap \{x \in S : f(x) = a\} = \emptyset$. Let $D = \triangle_{a\in\lambda} C_a$. So D is club. Hence $S \cap D \neq \emptyset$. Choose $a \in S \cap D$. Suppose that x is a nonempty member of $P_\kappa(\lambda)$. If $a \in x$, then $x \in C_a$, and so $f(x) \neq a$. Thus $f(x) \notin x$, contradiction. □

Proposition 2.45. *If D is a closed subset of $[A]^{<\kappa}$, then for every directed set $D \in [C]^{<\kappa}$ we have $\bigcup D \in C$.*

Proof. We proceed by induction on $|D|$. It is obvious if D is finite, and an easy inductive construction works if $|D| = \omega$. So suppose that $\omega < |D|$.

(1) If $E \subseteq D$, then there is a directed subset F of D such that $|F| \leq |E| + \omega$.

For, define $F_0 = E$. If F_m has been defined, for each pair $a, b \in F_m$ adjoin a set $c \in D$ such that $a, b \subseteq c$, forming F_{m+1}. Then $\bigcup_{m\in\omega} F_m$ is as desired.

Let $D = \{a_\alpha : \alpha < |D|\}$. Now we define a sequence $\langle E_\alpha : \alpha < |D|\rangle$. If E_β has been defined for all $\beta < \alpha$, by (1) let E_α be a directed subset of D such that $\bigcup_{\beta<\alpha} E_\beta \cup \{a_\alpha\} \subseteq E_\alpha$ and $|E_\alpha| \leq |\alpha| + \omega$. By the inductive hypothesis, $\bigcup E_\alpha \in D$ for every $\alpha < |D|$. Clearly $\bigcup E_\alpha \subseteq \bigcup E_\beta$ if $\alpha < \beta$, so $\bigcup D = \bigcup_{\alpha<|D|} \bigcup E_\alpha \in D$. □

Let $f : [A]^{<\omega} \to [A]^{<\kappa}$. $x \in [A]^{<\kappa}$ is a *closure point* of f iff $\forall e \in [x]^{<\omega}[f(e) \subseteq x]$. C_f is the set of all closure points of f.

Proposition 2.46. *C_f is a club of $[A]^{<\kappa}$.*

Proof. Obviously C_f is closed. To show that it is unbounded, suppose that $x \in \mathscr{P}_\kappa(A)$. We define $y_n \in \mathscr{P}_\kappa(A)$ by recursion. Let $y_0 = x$ and

$$y_{n+1} = y_n \cup \bigcup_{e\in[y_n]^{<\omega}} f(e).$$

Clearly $x \subseteq \bigcup_{n\in\omega} y_n \in C_f$. □

Note that $\bigcup_{n\in\omega} y_n$ is the smallest closure point of f that contains x. This will be used below.

Proposition 2.47. *For every club $C \subseteq [A]^{<\kappa}$ there is a function $f : [A]^{<\omega} \to [A]^{<\kappa}$ such that $C_f \subseteq C$.*

Proof. Let $f(\emptyset)$ be any member of C. If $f(e)$ has been defined whenever $|e| = n$, let $|e| = n + 1$. For each $x \in e$ choose $f(e) \in C$ so that $\bigcup_{x \in e} f(e \backslash \{x\}) \cup e \subseteq f(e)$. Clearly $C_f \subseteq C$. $\square$

Suppose that $|A| \geq \kappa$ and $A \subseteq B$. For $X \subseteq [B]^{<\kappa}$ the *projection* of X to A is

$$X \restriction A \stackrel{\text{def}}{=} \{x \cap A : x \in X\}.$$

For $Y \subseteq [A]^{<\kappa}$ the *lifting* of Y to B is

$$Y^B \stackrel{\text{def}}{=} \{x \in [B]^{<\kappa} : x \cap A \in Y\}.$$

Theorem 2.48. *Suppose that $|A| \geq \kappa$ and $A \subseteq B$. Then*
(i) If S is stationary in $[B]^{<\kappa}$, then $S \restriction A$ is stationary in $[A]^{<\kappa}$.
(ii) If S is stationary in $[A]^{<\kappa}$, then S^B is stationary in $[B]^{<\kappa}$.

Proof. (i): Suppose that C is club in $\mathscr{P}_\kappa(A)$. We show that C^B is club in $\mathscr{P}_\kappa(B)$. Suppose that $x_0 \subseteq \cdots \subseteq x_\xi \subseteq \cdots$ are elements of C^B for $\xi < \alpha < \kappa$. Thus $x_\xi \cap A \in C$ for all $\xi < \alpha$, so $\bigcup_{\xi<\alpha} x_\xi \cap A \in C$ and so $\bigcup_{\xi<\alpha} x_\xi \in C^B$. For unboundedness, suppose that $x \in \mathscr{P}_\kappa(B)$. Then $x \cap A \in \mathscr{P}_\kappa(A)$, so there is a $y \in C$ such that $x \cap A \subseteq y$. Then $x \subseteq x \cup y$ and $(x \cup y) \cap A = y$ and so $x \cup y \in C^B$. This shows that C^B is club in $\mathscr{P}_\kappa(B)$.

Now for (i), suppose that S is stationary in $\mathscr{P}_\kappa(B)$. Let C be club in $\mathscr{P}_\kappa(A)$. Then by the preceding paragraph, C^B is club in $\mathscr{P}_\kappa(B)$. Hence there is an $x \in S \cap C^B$. So $x \cap A \in (S|A) \cap C$, as desired.

(ii): We prove:

(1) If C is club in $\mathscr{P}_\kappa(B)$, then $C|A$ contains a club in $\mathscr{P}_\kappa(A)$.

For, by Proposition 2.47 we get $C_f \subseteq C$ for some $f : [B]^{<\omega} \to \mathscr{P}_\kappa(B)$. We now define $g : [A]^{<\omega} \to \mathscr{P}_\kappa(A)$. To do this, first let cl be the function above such that for any $x \in \mathscr{P}_\kappa(B)$, $\mathrm{cl}(x)$ is the smallest closure point under f containing x. Note:

(2) $\mathrm{cl}(x) = \bigcup\{\mathrm{cl}(e) : e \in [x]^{<\omega}\}$.

In fact, it is clear that the set on the right is the smallest closure point under f containing x.

Now for each $e \in [A]^{<\omega}$ we let $g(e) = \mathrm{cl}(e) \cap A$.

(3) $C_f \restriction A = C_g$.

In fact, let $x \in C_f \restriction A$. Then there is a $y \in C_f$ such that $x = y \cap A$. If $e \in [x]^{<\omega}$, then $e \in [y]^{<\omega}$, and hence $e \subseteq y \in C_f$, so $e \subseteq \mathrm{cl}(e) \subseteq y$. Hence $e \subseteq \mathrm{cl}(e) \cap A = g(e)$. This shows that $x \in C_g$.

Conversely, suppose that $x \in C_g$. Then $\bigcup\{g(e) : e \in [x]^{<\omega}\} \subseteq x$ since $x \in C_g$. Now

$$\begin{aligned}\bigcup\{g(e) : e \in [x]^{<\omega}\} &= \bigcup\{\mathrm{cl}(e) \cap A : e \in [x]^{<\omega}\} \\ &= A \cap \bigcup\{\mathrm{cl}(e) : e \in [x]^{<\omega}\} \\ &= A \cap \mathrm{cl}(x) \quad \text{by (2)}\end{aligned}$$

Thus $g(x) = A \cap \mathrm{cl}(x) = x$, and so $x \in C_f|A$. This proves (3).

Now by (3) we have $C_g = C_f \restriction A \subseteq C|A$, proving (1).

Now to prove (ii), suppose that S is stationary in $\mathscr{P}_\kappa(A)$ and C is club in $\mathscr{P}_\kappa(B)$. By (1), $C \restriction A$ contains a club D. Choose $a \in S \cap D$. So $a \in C \restriction A$; choose $x \in C$ such that $a = x \cap A$. Thus $x \in C \cap S^B$, as desired. $\square$

Proposition 2.49. *If $X \subseteq \kappa$ is nonstationary, then there is a regressive function f on $X\backslash\{0\}$ such that $\forall\gamma < \kappa[\{\alpha \in X\backslash\{0\} : f(\alpha) \leq \gamma\}$ is bounded$]$.*

Proof. Let C be club such that $C \cap X = \emptyset$. For each $\alpha \in X$ let $f(\alpha) = \sup(C \cap \alpha)$. Here $\sup(\emptyset) = 0$.

Let $\gamma < \kappa$; we want to show that $\{\alpha \in X\backslash\{0\} : \sup(C \cap \alpha) \leq \gamma\}$ is bounded. Choose $\delta \in C$ with $\gamma < \delta$. We claim that $\forall\alpha \in X[\delta < \alpha \to \sup(C \cap \alpha) > \gamma]$. In fact, if $\alpha \in X$ and $\delta < \alpha$, then $\gamma < \delta \leq \sup(C \cap \alpha)$. $\square$

Proposition 2.50. *Let S be a stationary subset of ω_1. For all $\alpha, \gamma < \omega_1$ with $\alpha \neq 0$ there is a closed subset A of S such that $\gamma < \min(A)$, A has order type α, and $\sup(A) \in S$.*

Proof. Induction on α. The case $\alpha = 1$ and the successor step are easy. So suppose our statement is true for all $\beta < \alpha$, where α is a limit ordinal. Let $\langle\beta_n : n \in \omega\rangle$ be a strictly increasing sequence of successor ordinals with supremum α, and let γ_n be such that $\beta_{n+1} = \beta_n + \gamma_n$ for all n.

(1) For every δ there is a closed subset A of S of order type α such that $\delta < \min(A)$.

In fact, by the inductive hypothesis define B_n inductively so that B_0 is a closed subset of S of order type β_0 with $\delta < \min(B_0)$, and B_{n+1} is a closed subset of S of order type γ_n with $\sup(B_n) < \min(B_{n+1})$. Then let $A = \bigcup_{n\in\omega} B_n$. Clearly A is as desired in (1).

Now by induction using (1) there is a sequence $\langle A_\xi : \xi < \omega_1\rangle$ of closed subsets of S, each of order type α, such that $\gamma < \min(A_0)$ and $\sup\left(\bigcup_{\eta<\xi} A_\eta\right) < \min(A_\xi)$ for each $\xi < \omega_1$. Let $\lambda_\xi = \sup\left(\bigcup_{\eta<\xi} A_\eta\right)$ for each $\xi < \omega_1$. Then λ is a normal function, and so there is a ξ such that $\lambda_\xi \in S$. Let $\langle\eta_n : n \in \omega\rangle$ be a strictly increasing sequence of ordinals with supremum ξ. Let C_0 be an initial segment of A_{η_0} of order type β_0, and let C_{n+1} be an initial segment of $A_{\eta_{n+1}}$ of order type γ_n. Then $D \stackrel{\text{def}}{=} \bigcup_{n\in\omega} C_n$ is a closed subset of C with order type α such that $\gamma < \min(D)$ and $\sup(D) = \lambda_\xi \in S$, finishing the inductive proof. $\square$

Lemma 2.51. *If κ is Mahlo, then $\{\alpha < \kappa : \alpha$ is inaccessible$\}$ is stationary in κ.*

Proof. Let C be club in κ. Define $f : \kappa \to \kappa$ by: $f(0) = 0$; $f(\alpha+1) = 2^{f(\alpha)}$, and $f(\alpha) = \bigcup_{\beta<\alpha} f(\beta)$ for α limit. Note that f maps into κ because κ is inaccessible. Since f is a normal function, its range is club in κ. Let C be any club in κ. Then $C \cap \mathrm{rng}(f) \cap \{\alpha < \kappa : \alpha \text{ is regular}\}$ is nonempty, and any member of it is inaccessible, as desired. □

Proposition 2.52. *Suppose that κ is the first inaccessible such that there are κ inaccessibles below κ. Then κ is not Mahlo.*

Proof. Suppose that κ is Mahlo. Let $S = \{\alpha < \kappa : \alpha \text{ is inaccessible}\}$. Let $\langle i_\xi : \xi < \kappa\rangle$ enumerate in increasing order all of the inaccessibles below κ. For each $\alpha \in S$ there is a unique $\xi < \kappa$ such that $i_\xi = \alpha$, and we set $f(\alpha) = \xi$. By the choice of κ, f is regressive. Since it is one-one, Fodor's theorem gives a contradiction. □

Proposition 2.53. *If κ is (limit, regular limit, weakly Mahlo) and $\{\lambda < \kappa : \lambda$ is strong limit$\}$ is unbounded in κ, then κ is (strong limit, inaccessible, Mahlo).*

Proof. Limit: Choose $\lambda < \kappa$ strong limit with $\mu < \lambda$. Then $2^\mu < \lambda < \kappa$.

Regular limit: By the above, κ is strong limit. Since κ is regular, κ is inaccessible.

Mahlo: $\{\lambda < \kappa : \lambda \text{ regular}\}$ is stationary in κ. By the above, κ is inaccessible. Hence κ is Mahlo. □

Recall that

$$\begin{aligned}\triangle_{\alpha<\kappa} X_\alpha &= \{\alpha < \kappa : \alpha \in \bigcap_{\beta<\alpha} X_\beta\}\\ &= \{\alpha < \kappa : \forall \beta < \alpha(\alpha \in X_\beta)\}.\end{aligned}$$

Now we define the diagonal union:

$$\nabla_{\alpha<\kappa} X_\alpha = \{\alpha < \kappa : \alpha \in \bigcup_{\beta<\alpha} X_\beta\}.$$

Then clearly $\kappa \backslash \triangle_{\alpha<\kappa} X_\alpha = \nabla_{\alpha<\kappa}(\kappa\backslash X_\alpha)$.

Propositioon 2.54. *An ideal I on κ is normal iff it is closed under diagonal unions.*

Proof. In fact, suppose that I is normal and $\langle X_\alpha : \alpha < \kappa\rangle$ is a system of members of I. Then $\triangle_{\alpha<\kappa}(\kappa\backslash X_\alpha) \in I'$, and so $\kappa\backslash\triangle_{\alpha<\kappa}(\kappa\backslash X_\alpha) \in I$. Clearly $\kappa\backslash\triangle_{\alpha<\kappa}(\kappa\backslash X_\alpha) = \nabla_{\alpha<\kappa} X_\alpha$.

The converse is proved similarly. □

Theorem 2.55. *A κ-complete ideal I on κ is normal iff for every $S_0 \notin I$ and any regressive function f defined on S_0 there is an $S \subseteq S_0$ with $S \notin I$ such that f is constant on S.*

Proof. $\Rightarrow$: Suppose that I is a κ-complete ideal on κ, it is normal, $S_0 \notin I$, and f is regressive on S_0. Suppose that the conclusion fails. Then for every $\gamma < \kappa$, the set $f^{-1}[\{\gamma\}]$ is in I, and hence $\nabla_{\gamma<\kappa} f^{-1}[\{\gamma\}] \in I$. Thus $S_0 \not\subseteq \nabla_{\gamma<\kappa} f^{-1}[\{\gamma\}]$; choose

$\alpha \in S_0 \backslash \nabla_{\gamma<\kappa} f^{-1}[\{\gamma\}]$. Then for every $\gamma < \alpha$ we have $\alpha \notin f^{-1}[\{\gamma\}$, contradicting f being regressive.

$\Leftarrow$: Assume the indicated condition. Suppose that $X_\alpha \in I$ for all $\alpha < \kappa$, but $\nabla_{\alpha<\kappa} X_\alpha \notin I$. Now for each $\beta \in \nabla_{\alpha<\kappa} X_\alpha$ there is an $f(\beta) < \beta$ such that $\beta \in X_{f(\beta)}$. By the condition, let S be a subset of $\nabla_{\alpha<\kappa} X_\alpha$ on which f takes a constant value γ, with $S \notin I$. But if $\beta \in S$, then $f(\beta) = \gamma$, and so $S \subseteq X_\gamma \in I$, contradiction. □

Proposition 2.56. *There is no normal nonprincipal filter on ω.*

Proof. Suppose that F is a normal nonprincipal filter on ω. Thus F is closed under diagonal intersections.

(1) If $X \in F$, then there is a $Y \in F$ such that $Y \subset X$.

In fact, X does not generate F, so there is some $Z \in F$ such that $X \not\subseteq Z$. So $X \cap Z \subset X$ and $X \cap Z \in F$, as desired in (1).

(2) There is a sequence $\langle X_i : i \in \omega \rangle$ of members of F such that $X_i \supset X_{i+1}$ for all i, and $\bigcap_{i\in\omega} X_i \notin F$.

In fact, we can construct by induction a strictly decreasing sequence of members of F, taking intersections at limit steps if the result is in F. The sequence eventually stops, and by (1) it stops at a limit ordinal because some intersection is in F. Then a cofinal subsequence gives what is desired.

Next, we define a sequence $\langle i_m : m \in \omega \rangle$ of integers by recursion so that

$(*)$ $m \notin \bigcap_{k\in\omega} X_k$ implies that $m \notin X_{i_m}$.
$(**)$ $n < m \to i_n < i_m$.

Case 1. $0 \in \bigcap_{k\in\omega} X_k$. Let $i_0 = 0$. So $(*)$ and $(**)$ hold.
Case 2. $0 \notin \bigcap_{k\in\omega} X_k$. Let i_0 be such that $0 \notin X_{i_0}$. So $(*)$ and $(**)$ hold.
Now suppose that i_m has been defined so that $(*)$ and $(**)$ hold.
Case 1. $m+1 \in \bigcap_{k\in\omega} X_k$. Choose $i_{m+1} \in (\omega \backslash (i_m + 1))$. So $(*)$ and $(**)$ hold.
Case 2. $m+1 \notin \bigcap_{k\in\omega} X_k$. Say $m+1 \notin X_s$. Then $\forall t \geq s[m+1 \notin X_t]$. Let i_{m+1} be such that $i_{m+1} > s, i_m$. So $(*)$ and $(**)$ hold.

Now we define $Y_m = X_{i_{m+1}}$ for all $m \in \omega$.

(3) $\triangle_{m\in\omega} Y_m \subseteq \bigcap_{i\in\omega} X_i \cup \{0\}$.

In fact, suppose that $n \in \triangle_{m\in\omega} Y_m$, $n \neq 0$, and $n \notin \bigcap_{i\in\omega} X_i$. Then $n \in Y_{n-1} = X_{i_n}$, contradiction. So (3) holds.

Now because $\bigcap_{i\in\omega} X_i$ is not in F, it follows from (3) that $0 \notin \bigcap_{i\in\omega} X_i$. Choose X_j such that $0 \notin X_j$. Then $\bigcap_{i\in\omega} X_i = X_j \cap (\bigcap_{i\in\omega} X_i \cup \{0\}) \in F$, contradiction. □

Proposition 2.57. *If κ is singular, then there is no normal ideal on κ that contains all bounded subsets of κ.*

Proof. Suppose that F is a proper normal filter on κ, where κ is singular. Say that $\langle \mu_\alpha : \alpha < \mathrm{cf}(\kappa)\rangle$ is strictly increasing with supremum κ. For $\alpha < \kappa$ let

$$X_\alpha = \begin{cases} \kappa\backslash\mu_\alpha & \text{if } \alpha < \mathrm{cf}(\kappa), \\ \kappa\backslash\alpha & \text{otherwise.} \end{cases}$$

If $\mathrm{cf}(\kappa) \leq \beta < \kappa$ and $\beta \in \triangle_{\alpha<\kappa} X_\alpha$, then $\beta \in \kappa\backslash\mu_\alpha$ for all $\alpha < \mathrm{cf}(\kappa)$, contradiction. Hence $\triangle_{\alpha<\kappa} X_\alpha$ is bounded and so F is improper, contradiction. □

For S, T stationary, we define $S < T$ iff ($\{\alpha \in T : S \cap \alpha$ is non-stationary in $\alpha\}$ is non-stationary in κ) and ($\{\alpha \in T : \mathrm{cf}(\alpha) \leq \omega\}$ is non-stationary). Also, we define for any set S, $\mathrm{Tr}(S) = \{\alpha < \kappa : \mathrm{cf}(\alpha) > \omega$ and $S \cap \alpha$ is stationary$\}$. For λ regular $< \kappa$, $E^\kappa_\lambda = \{\alpha < \kappa : \mathrm{cf}(\alpha) = \lambda\}$.

Proposition 2.58. *If $S, T \subseteq \kappa$ are stationary and $S \subseteq T$, then* $\mathrm{Tr}(S) \subseteq \mathrm{Tr}(T)$. □

Proposition 2.59. *If $S, T \subseteq \kappa$ are stationary, then* $\mathrm{Tr}(S \cup T) = \mathrm{Tr}(S) \cup \mathrm{Tr}(T)$.

Proof. $\supseteq$ follows from Proposition 2.58. For $\subseteq$, suppose that $\alpha \in \mathrm{Tr}(S \cup T)\backslash\mathrm{Tr}(S)$. Since $\alpha \notin \mathrm{Tr}(S)$, there is some club C in α such that $S \cap \alpha \cap C = \emptyset$. Now take any club D in α. Then also $C \cap D$ is club in α, so $\emptyset \neq (C \cap D) \cap (S \cup T) = C \cap D \cap T$. so $\alpha \in \mathrm{Tr}(T)$, as desired. □

Proposition 2.60. *If $S \subseteq \kappa$ is stationary, then* $\mathrm{Tr}(\mathrm{Tr}(S)) = \mathrm{Tr}(S)$.

Proof. Suppose that $\alpha \in \mathrm{Tr}(\mathrm{Tr}(S))$. Hence $\mathrm{cf}(\alpha) > \omega$ and $\mathrm{Tr}(S) \cap \alpha$ is stationary in α. We want to show that $S \cap \alpha$ is stationary in α. To this end, suppose that C is club in α. So also C' is club in α, so $\mathrm{Tr}(S) \cap \alpha \cap C' \neq \emptyset$; choose β in this set. So $S \cap \beta$ is stationary in β, and $\beta \in C'$ and hence $C \cap \beta$ is club in β, so $S \cap C \neq \emptyset$, as desired. □

Proposition 2.61. *If $S, T \subseteq \kappa$ are stationary and $S \simeq T$ (mod I_{NS}) then* $\mathrm{Tr}(S) \simeq \mathrm{T}$.

Proof. Suppose that $S \simeq T$ (mod I_{NS}). Let C be club in κ with $(S \triangle T) \cap C = \emptyset$. Let C' be the set of all limits of members of C. So C' is also club in κ. Suppose that $\alpha \in (Tr(S)\backslash Tr(T)) \cap C')$. Then $\mathrm{cf}(\alpha) > \omega$, $S \cap \alpha$ is stationary in α, and $T \cap \alpha$ is not stationary in α. Say D is club in α with $T \cap D = \emptyset$. Then $C' \cap D$ is club in α, so $S \cap C' \cap D \neq \emptyset$. But $(S\backslash T) \cap C' = \emptyset$, so $S \cap C' \subseteq T$. Hence $T \cap D \neq \emptyset$, contradiction.

Hence $(Tr(S)\backslash Tr(T)) \cap C') = \emptyset$. By symmetry, $(Tr(S) \triangle Tr(T)) \cap C' = \emptyset$. So $Tr(S) \simeq Tr(T)$. □

Proposition 2.62. $\mathrm{Tr}(E^\kappa_\lambda) = \{\alpha < \kappa : \mathrm{cf}(\alpha) \geq \lambda^+\}$.

Proof. First suppose that $\alpha \in \mathrm{Tr}(E^\kappa_\lambda)$. Suppose also that $\mathrm{cf}(\alpha) \leq \lambda$. Now $\mathrm{cf}(\alpha) > \omega$. Let $\langle \beta_\xi : \xi < \mathrm{cf}(\alpha)\rangle$ be strictly increasing and continuous with supremum α. Then $\mathrm{rng}(\beta)$ is club in α and $E^\kappa_\lambda \cap \mathrm{rng}(\beta) = \emptyset$, contradiction.

Second, suppose that $\alpha < \kappa$ and $\mathrm{cf}(\alpha) \geq \lambda^+$. Let C be club in α. Let $\langle \beta_\alpha : \xi < \mathrm{cf}(\alpha)\rangle$ be the strictly increasing enumeration of C. Then $\beta_\lambda \in E^\kappa_\lambda \cap C$. □

Theorem 2.63. $\mathrm{Tr}(E^\kappa_\lambda) = \{\alpha < \kappa : \mathrm{cf}(\alpha) \geq \lambda^+\}$.

Proof. First suppose that $\alpha \in \mathrm{Tr}(E^\kappa_\lambda)$. Suppose also that $\mathrm{cf}(\alpha) \leq \lambda$. Now $\mathrm{cf}(\alpha) > \omega$. Let $\langle \beta_\xi : \xi < \mathrm{cf}(\alpha)\rangle$ be strictly increasing and continuous with supremum α. Then $\mathrm{rng}(\beta)$ is club in α and $E^\kappa_\lambda \cap \mathrm{rng}(\beta) = \emptyset$, contradiction.

Second, suppose that $\alpha < \kappa$ and $\mathrm{cf}(\alpha) \geq \lambda^+$. Let C be club in α. Let $\langle \beta_\alpha : \xi < \mathrm{cf}(\alpha)\rangle$ be the strictly increasing enumeration of C. Then $\beta_\lambda \in E^\kappa_\lambda \cap C$. □

Theorem 2.64. *$S < T$ iff there is a club C such that $T \cap C \subseteq \mathrm{Tr}(S)$.*

Proof. $\Rightarrow$: Assume that $S < T$. Let C be club in κ such that $\{\alpha \in T : S \cap \alpha$ is non-stationary in $\alpha\} \cap C = \emptyset$ and $\{\alpha \in T : \mathrm{cf}(\alpha) \leq \omega\} \cap C = \emptyset$. Suppose that $\alpha \in T \cap C$. Then by definition, $S \cap \alpha$ is stationary in α and $\mathrm{cf}(\alpha) \geq \omega_1$. Hence $\alpha \in \mathrm{Tr}(S)$.

Conversely, suppose that C is club and $T \cap C \subseteq \mathrm{Tr}(S)$. Then $\forall \alpha \in T \cap C[\mathrm{cf}(\alpha) \geq \omega_1]$, so $\{\alpha \in T : \mathrm{cf}(\alpha) \leq \omega_1\} \cap C = \emptyset$. Suppose that $\beta \in \{\alpha \in T : S \cap \alpha$ is non-stationary in $\alpha\} \cap C$. Thus $\beta \in T$ and $S \cap \beta$ is non-stationary in β. Since $\beta \in T \cap C$, we have $\beta \in \mathrm{Tr}(S)$. Hence $S \cap \beta$ is stationary in β, contradiction. □

Corollary 2.65. *If $S < T$ and X is a stationary subset of T, then $S < X$.*

Proof. By Theorem 2.64. □

Theorem 2.66. *If $\mathrm{Tr}(A)$ is stationary, then so is A.*

Proof. Let C be club. Let $\overline{C}$ be the set of all limits of members of C; it is club too. Choose $\alpha \in \overline{C} \cap \mathrm{Tr}(A)$. Then $\alpha \cap A$ is stationary in α. Now $C \cap \alpha$ is unbounded in α, so $\alpha \cap A \cap C \neq \emptyset$. □

Theorem 2.67. *If S is stationary of order ν and $\mu < \nu$, then there is a stationary T of order μ such that $T < S$.*

Proof. Induction on ν. Suppose that it is true for all $\nu' < \nu$, and now assume that $\mu < \nu$ and S is stationary of order ν. By definition there is a stationary T of some order $\rho < \nu$ such that $\mu \leq \rho$ and $T < S$. By the inductive hypothesis there is a stationary U of order μ such that $U < T$ or $U = T$ (if $\mu = \rho$). Thus $U < S$. □

A stationary set E is *weakly canonical* of order ν iff the following condition holds:

($*$) If X is stationary of order ν, then $E \cap X \neq \emptyset$.

Theorem 2.68. *If E is weakly canonical of order ν and S is stationary of order $\nu + 1$, then $S \cap \mathrm{Tr}(E) \neq \emptyset$.*

Proof. Suppose that $S \cap \mathrm{Tr}(E) = \emptyset$. Let T be stationary of order ν such that $T < S$. Let F be club such that

(1) $S \cap F \subseteq \mathrm{Tr}(T)$.

Now we define $A = T \cap E$, $B = T \cap \mathrm{Tr}(E)$, and $C = T \backslash (A \cup B)$. Now $A \subseteq E$, so $\mathrm{Tr}(A) \subseteq \mathrm{Tr}(E)$, and hence $S \cap \mathrm{Tr}(A) = \emptyset$. Also, $B \subseteq \mathrm{Tr}(E)$, hence $\mathrm{Tr}(B) \subseteq \mathrm{Tr}(\mathrm{Tr}(E) \subseteq \mathrm{Tr}(E)$, so also $S \cap \mathrm{Tr}(B) = \emptyset$.

(2) $S \cap F \subseteq \mathrm{Tr}(C)$.

In fact,

$$\begin{aligned} S \cap F &= S \cap F \cap \mathrm{Tr}(T) = S \cap F \cap \mathrm{Tr}(A \cup B \cup C) \\ &= S \cap F \cap (\mathrm{Tr}(A) \cup \mathrm{Tr}(B) \cup \mathrm{Tr}(C)) = S \cap F \cap \mathrm{Tr}(C). \end{aligned}$$

Now if D is any club, then $\emptyset \neq S \cap F \cap D \subseteq \mathrm{Tr}(C) \cap D$. So $\mathrm{Tr}(C)$ is stationary, and hence by Theorem 4, C is stationary.

It follows that $C < S$. Since $C \subseteq T$, by Corollary 3, $\forall X[X$ stationary and $X < T$ imply $X < C]$. Hence $o(C) \geq o(T) = \nu$. Since $C \cap E = \emptyset$ and E is weakly canonical, C must have order $> \nu$. Since $C < S$ and S has order $\nu + 1$, this is a contradiction. $\square$

Corollary 2.69. *If E is weakly canoniical of order ν and S is stationary of order $\nu + 1$, then $E < S$.*

Proof. Suppose not: so $T \stackrel{\text{def}}{=} \{\alpha \in S : \alpha \cap E$ is nonstationary$\}$ is stationary. Thus $T \cap \mathrm{Tr}(E) = \emptyset$, contradicting Theorem 2.68. $\square$

Theorem 2.70. *If $\lambda < \mu$, then $E^\kappa_\lambda < E^\kappa_\mu$.*

Proof. By Theorem 2.63 $\mathrm{Tr}(E^\kappa_\lambda) = \{\alpha < \kappa : \mathrm{cf}(\alpha) \geq \lambda^+\} \supseteq E^\kappa_\mu$, so $E^\kappa_\lambda < E^\kappa_\mu$ by Theorem 2.64. $\square$

Theorem 2.71. *If E^κ_λ is weakly canonical of order ν, then $E^\kappa_{\lambda^+}$ is of order $\nu + 1$.*

Proof. Since $E^\kappa_\lambda < E^\kappa_{\lambda^+}$ by Theorem 2.70, we have $o(E^\kappa_{\lambda^+}) \geq \nu + 1$. Suppose that $o(E^\kappa_{\lambda^+}) > \nu + 1$. Then there is an $S < E^\kappa_{\lambda^+}$ such that $o(S) = \nu + 1$. By Corollary 2.69, $E^\kappa_\lambda < S$. Then there is a club M in κ such that

$$\{\alpha \in E^\kappa_{\lambda^+} : S \cap \alpha \text{ is non-stationary }\} \cap M = \emptyset;$$
$$\{\alpha \in S : E^\kappa_\lambda \cap \alpha \text{ is non-stationary }\} \cap M = \emptyset.$$

Let M' be the set of limit points of M. Choose $\alpha \in M' \cap E^\kappa_{\lambda^+}$. Then $S \cap \alpha$ is stationary. Let $D \subseteq \alpha$ be club with order type λ^+ consisting of limit ordinals. Then $M' \cap D$ is club in α. Choose $\beta \in S \cap \alpha \cap M' \cap D$. Then $\mathrm{cf}(\beta) \leq \lambda$. Since $\beta \in S \cap M'$, it follows that $E^\kappa_\lambda \cap \beta$ is stationary. Let U be club in β of order type $\mathrm{cf}(\beta)$. Choose $\gamma \in E^\kappa_\lambda \cap \beta \cap U$. Then $\mathrm{cf}(\gamma) = \lambda$. But $\mathrm{cf}(\gamma) < \lambda$ since $\mathrm{cf}(\beta) \leq \lambda$. This is a contradiction. $\square$

Proposition 2.72. *If E^κ_λ is weakly canonical of order ν and X is stationary of order $\nu + 1$, then $E^\kappa_{\lambda^+} \cap X \neq \emptyset$.*

Proof. By Corollary 2.70, $E^\kappa_\lambda < X$. Hence by Theorem 2.64 there is a club C such that $C \cap X \subseteq \mathrm{Tr}(E^\kappa_\lambda)$. So by Theorem 2.63 $\forall \alpha \in C \cap X[\mathrm{cf}(\alpha) \geq \lambda^+]$, Suppose that $\forall \alpha \in C \cap X[\mathrm{cf}(\alpha) > \lambda^+]$. Then $C \cap X \subseteq \mathrm{Tr}(E^\kappa_{\lambda^+})$ by Theorem 2.63. Thus by Theorem 2,64 $E^\kappa_{\lambda^+} < X$. Since $o(E^\kappa_{\lambda^+}) = \nu + 1$ by Theorem 2.71, this is a contradiction. Hence $E^\kappa_{\lambda^+} \cap X \neq \emptyset$. $\square$

Corollary 2.73. *If E^κ_λ is weakly canonical of order ν, then $E^\kappa_{\lambda^+}$ is weakly canonical of order $\nu+1$.* □

For the next few results we assume:

($\star$) λ is singular, λ^+ is the ν-th regular cardinal, and for each regular $\mu<\lambda$ which is the η-th regular cardinal, E^κ_μ is weakly canonical of order η.

Proposition 2.74. *Assume ($\star$). Suppose that S is stationary of order ν. Then there is a club U such that $\forall\alpha\in S\cap U[\mathrm{cf}(\alpha)\geq\lambda^+]$.*

Proof. For each regular $\mu<\lambda$, μ the η-th regular cardinal, let T_μ be stationary of order η^+ such that $T_\mu<S$. By Corollary 7, $E^\kappa_\mu<T_\mu$; so $E^\kappa_\mu<S$. Let U be club such that $S\cap U\subseteq\mathrm{Tr}(E^\kappa_\mu)$ for all regular $\mu<\lambda$. Then by Theorem 1. $\forall\alpha\in S\cap U[\mathrm{cf}(\alpha)\geq\lambda^+]$. □

Proposition 2.75. *Assume ($\star$). Then $E^\kappa_{\lambda^+}$ has order ν.*

Proof. Suppose not; so $o(E^\kappa_{\lambda^+})>\nu$. Let S be stationary of order ν with $S<E^\kappa_{\lambda^+}$. Let U be a club as in Proposition 2.74, and let D be a club such that $E^\kappa_{\lambda^+}\cap D\subseteq\mathrm{Tr}(S)$. Thus $\forall\alpha\in E^\kappa_{\lambda^+}\cap D[\alpha\cap S$ is stationary in $\alpha]$. Take any $\alpha\in E^\kappa_{\lambda^+}\cap D$ and let V be club in α of order type λ^+. Take any $\beta\in V\cap U'\cap S$. Then $\mathrm{cf}(\beta)<\lambda$ but also $\mathrm{cf}(\beta)\geq\lambda^+$, contradiction. □

Proposition 2.76. *Assume ($\star$). Suppose that X is stationary of order ν. Then for all regular $\mu<\lambda$, $E^\kappa_\mu<X$.*

Proof. Choose $Y<X$ stationary of order $\eta+1$, where μ is the η-th regular cardinal. By Corollary 2.69, $E^\kappa_\mu<Y$. □

Proposition 2.77. *Assume ($\star$). Suppose that X is stationary of order ν, then $E^\kappa_{\lambda^+}\cap X\neq\emptyset$.*

Proof. By Proposition 2.76, Theorem 2.63, and Theorem 2.64, there is a club C such that $\forall\alpha\in X\cap C\forall\mu<\lambda[\mu$ regular $\rightarrow\mathrm{cf}(\alpha)\geq\mu]$. So $\forall\alpha\in X\cap C[\mathrm{cf}(\alpha)\geq\lambda^+]$. If $\forall\alpha\in X\cap C[\mathrm{cf}(\alpha)>\lambda^+]$, then by Theorem 2.63, $X\cap C\subseteq\mathrm{Tr}(E^\kappa_{\lambda^+})$, hence $E^\kappa_{\lambda^+}<X$. Since X has order ν, this contradicts Proposition 2.75. □

Corollary 2.78. *Assume ($\star$). Then $E^\kappa_{\lambda^+}$ is weakly canonical of order ν.* □

For the next few results we assume:

($\star\star$) λ is regular limit, is the ν-th regular cardinal, and for each regular $\mu<\lambda$ which is the η-th regular cardinal, E^κ_μ is weakly canonical of order η.

Proposition 2.79. *Assume ($\star\star$). Suppose that S is stationary of order ν. Then there is a club U such that $\forall\alpha\in S\cap U[\mathrm{cf}(\alpha)\geq\lambda]$.*

Proof. For each regular $\mu<\lambda$, μ the η-th regular cardinal, let T_μ be stationary of order η^+ such that $T_\mu<S$. By Corollary 2.70, $E^\kappa_\mu<T_\mu$; so $E^\kappa_\mu<S$. Let U be club such

that $S \cap U \subseteq \mathrm{Tr}(E^\kappa_\mu)$ for all regular $\mu < \lambda$. Then by Theorem 2.63. $\forall \alpha \in S \cap U[\mathrm{cf}(\alpha) \geq \lambda]$. $\square$

Proposition 2.80. *Assume* $(\star\star)$. *Then* E^κ_λ *has order* ν.

Proof. Suppose not; so $o(E^\kappa_\lambda) > \nu$. Let S be stationary of order ν with $S < E^\kappa_\lambda$. Let U be a club as in Proposition 2.74, and let D be a club such that $E^\kappa_\lambda \cap D \subseteq \mathrm{Tr}(S)$. Thus $\forall \alpha \in E^\kappa_\lambda \cap D[\alpha \cap S$ is stationary in $\alpha]$. Take any $\alpha \in E^\kappa_\lambda \cap D$ and let V be club in α of order type λ. Take any $\beta \in V \cap U' \cap S$. Then $\mathrm{cf}(\beta) < \lambda$ but also $\mathrm{cf}(\beta) \geq \lambda$, contradiction. $\square$

Proposition 2.81. *Assume* $(\star\star)$. *Suppose that* X *is stationary of order* ν. *Then for all regular* $\mu < \lambda$, $E^\kappa_\mu < X$.

Proof. Choose $Y < X$ stationary of order $\eta + 1$, where μ is the η-th regular cardinal. By Corollary 2.69, $E^\kappa_\mu < Y$. $\square$

Proposition 2.82. *Assume* $(\star\star)$. *Suppose that* X *is stationary of order* ν, *then* $E^\kappa_\lambda \cap X \neq \emptyset$.

Proof. By Proposition 2.76, Theorem 2.63, and Theorem 2.64, there is a club C such that $\forall \alpha \in X \cap C \forall \mu < \lambda[\mu$ regular $\rightarrow \mathrm{cf}(\alpha) \geq \mu]$. So $\forall \alpha \in X \cap C[\mathrm{cf}(\alpha) \geq \lambda]$. If $\forall \alpha \in X \cap C[\mathrm{cf}(\alpha) > \lambda]$, then by Theorem 2.63, $X \cap C \subseteq \mathrm{Tr}(E^\kappa_\lambda)$, hence $E^\kappa_\lambda < X$. Since X has order ν, this contradicts Proposition 2.75. $\square$

Corollary 2.83. *Assume* $(\star\star)$. *Then* E^κ_λ *is weakly canonical of order* ν. $\square$

Lemma 2.84. *If* α *is a limit ordinal with* $\mathrm{cf}(\alpha) = \omega$, *and* $X \subseteq \alpha$, *then* X *is stationary in* α *iff* $\exists \beta < \alpha[(\beta, \alpha) \subseteq X]$.

Proof. $\Rightarrow$: Suppose $\neg \exists \beta < \alpha[(\beta, \alpha) \subseteq X]$. Then there is a cofinal subset of α of order type ω such that $C \cap X = \emptyset$. Note that C is club in α.

$\Leftarrow$: obvious. $\square$

Lemma 2.85. *There is no stationary set* S *such that* $S < E^\kappa_\omega$.

Proof. $\{\alpha \in E^\kappa_\omega : \mathrm{cf}(\alpha) \leq \omega\} = E^\kappa_\omega$, which is stationary. So the second condition in $S < E^\kappa_\omega$ fails. $\square$

Proposition 2.86. *If* S, T *are stationary and* $S \sim T$ *mod* I_{NS}, *then* $o(S) = o(T)$.

Proof. Suppose that $X < S$. Let C be club with $S \cap C \subseteq \mathrm{Tr}(X)$. Let D be club with $(T \backslash S) \cap D = \emptyset$. Then $T \cap C \cap D = T \cap S \cap C \cap D \subseteq \mathrm{Tr}(X)$. Hence by symmetry $o(S) = o(T)$. $\square$

Proposition 2.87. E^κ_ω *is canonical of order 0.*

Proof. For (1), suppose that $X \subseteq E^\kappa_\omega$, X stationary. If Y is stationary and $Y < X$, let C be club with $X \cap C \subseteq \mathrm{Tr}(Y)$. Then $\forall \alpha \in X \cap C[\mathrm{cf}(\alpha) > \omega$ and $Y \cap \alpha$ is stationary in $\alpha]$. Since $\mathrm{cf}(\alpha) = \omega$, this is a contradiction.

For (2), suppose that X is stationary of order 0 and $E^\kappa_\omega \cap X = \emptyset$. Then $\forall \alpha \in X[\mathrm{cf}(\alpha) > \omega]$. We claim that $E^\kappa_\omega < X$ (contradiction). For, let $\alpha \in X \cap \kappa$. Then $\mathrm{cf}\alpha) > \omega$ and $E^\kappa_\omega \cap \alpha$ is stationary in α, as desired. ☐

Proposition 2.88. *$o(E^\kappa_{\omega_1}) = 1$.*

Proof. By Lemma 2.69, $o(E^\kappa_{\omega_1}) \geq 1$. Now suppose that $A < B < E^\kappa_{\omega_1}$ with A, B stationary. Then

$$\{\alpha \in B : \mathrm{cf}(\alpha) \leq \omega\} \text{ is nonstationary in } \kappa;$$
$$\{\alpha \in B : \alpha \cap A \text{ is nonstationary}\} \text{ is nonstationary;}$$
$$\{\alpha \in E^\kappa_{\omega_1} : \alpha \cap B \text{ is nonstationary}\} \text{ is nonstationary.}$$

Thus there is a club M in κ such that

$$\{\alpha \in B : \mathrm{cf}(\alpha) \leq \omega\} \cap M = \emptyset; \tag{1}$$
$$\{\alpha \in B : \alpha \cap A \text{ is nonstationary}\} \cap M = \emptyset \text{ and}$$
$$\{\alpha \in E^\kappa_{\omega_1} : \alpha \cap B \text{ is nonstationary}\} \cap M = \emptyset.$$

Let M' be the set of all limits of members of M; so also M' is club in κ. Suppose that $\alpha \in E^\kappa_{\omega_1} \cap M'$. Thus $\mathrm{cf}(\alpha) = \omega_1$. Then $\alpha \cap B$ is stationary. Let U be a club in α of order type ω_1. Since $\alpha \in M'$, $M \cap \alpha$ is club in α. Choose $\beta \in \alpha \cap B \cap U \cap M'$. Then $\mathrm{cf}(\beta) > \omega$ by (1), but $\mathrm{cf}(\beta) = \omega$ since $\beta \in U$, contradiction. ☐

Lemma 2.89. *$o(E^\kappa_{\omega_2}) = 2$.*

Proof. Clearly $o(E^\kappa_{\omega_2}) \geq 2$. Now suppose that $A < B < C < E^\kappa_{\omega_2}$. Then there is a club M in κ such that

$$\{\alpha \in C : \mathrm{cf}(\alpha) \leq \omega\} \cap M = \emptyset; \tag{1}$$
$$\{\alpha \in B : \mathrm{cf}(\alpha) \leq \omega\} \cap M = \emptyset; \tag{2}$$
$$\{\alpha \in E^\kappa_{\omega_2} : \alpha \cap C \text{ is nonstationary}\} \cap M = \emptyset \text{ and}$$
$$\{\alpha \in C : \alpha \cap B \text{ is nonstationary}\} \cap M = \emptyset \text{ and}$$
$$\{\alpha \in B : \alpha \cap A \text{ is nonstationary}\} \cap M = \emptyset.$$

Let M' be the set of all limit points of members of M'. Take any $\alpha \in M' \cap E^\kappa_{\omega_2}$. So $\mathrm{cf}(\alpha) = \omega_2$. Let U be club in α with order type ω_2. Now $\alpha \cap C$ is stationary. Let $\beta \in \alpha \cap C \cap M' \cap U$. Then β has cofinality $\leq \omega_1$.

Case 1. $\mathrm{cf}(\beta) = \omega$. This contradicts (1).

Case 2. $\mathrm{cf}(\beta) = \omega_1$. Let V be club in β with order type ω_1. Then $M' \cap U' \cap V'$ is club in β. Now $\beta \cap B$ is stationary, so choose $\gamma \in B \cap U' \cap V'$. Since $\gamma \in V'$, we have $\mathrm{cf}(\gamma) = \omega$. Since $\gamma \in M'$, this contradicts (2). ☐

Theorem 2.90. *$o(\kappa) \geq \kappa$ iff κ is weakly inaccessible.*

Proof. First suppose that κ is not weakly inaccessible. Since κ is assumed to be regular, this means that $\kappa = \mu^+$ for some μ. Suppose that $o(\kappa) \geq \kappa$. Choose $A \subseteq \kappa$ be stationary such that $o(A) > \mu$. For each regular $\nu < \kappa$ we have $o(E^\kappa_\nu) \leq \nu < o(A)$. Say $o(E^\kappa_\nu) = \psi$. Let $B < A$ be stationary such that $o(B) = \psi + 1$. By Corollary 2.70. $E^\kappa_\nu < B$, hence $E^\kappa_\nu < A$. So, choose C_ν club in κ such that $A \cap C_\nu \subseteq \mathrm{Tr}(E^\kappa_\nu)$. By Theorem 2.63, this implies that each member of $A \cap C_\nu$ has cofinality greater than ν. Now the intersection D of all C_ν's is still club, since there are at most $\mu < \kappa$ of them. So each member of $A \cap D$ has cofinality greater than each regular cardinal less than κ, contradiction.

Suppose that κ is weakly inaccessible. Then $\kappa = \bigcup\{\lambda : \lambda < \kappa, \lambda \text{ regular}\}$, since κ is regular limit. So κ is the κ-th regular cardinal. For each $\nu < \kappa$ choose E stationary of order ν. Hence $o(\kappa) \geq \kappa$. □

Theorem 2.91. *$o(\kappa) \geq \kappa + 1$ iff κ is weakly Mahlo.*

Proof. Suppose that κ is weakly Mahlo. So $A \stackrel{\text{def}}{=} \{\lambda < \kappa : \lambda \text{ is regular}\}$ is stationary. Now by definition, κ is weakly inaccessible. If $\lambda < \kappa$ is regular, then $A \cap (\kappa \backslash \lambda^+)$ is a subset of $\mathrm{Tr}(E^\kappa_\lambda)$. For, if $\mu \in A \cap (\kappa \backslash \lambda^+)$, then μ is regular and greater than λ, hence $\mu \in \mathrm{Tr}(E^\kappa_\lambda)$. This shows that $E^\kappa_\lambda < A$. So $o(A) \geq \kappa$. Hence $o(\kappa) \geq \kappa + 1$.

Finally, suppose that $o(\kappa) \geq \kappa + 1$. Then κ is weakly inaccessible by the above. There is a stationary A with $o(A) = \kappa$, and hence by Corollary 2.70 we have $E^\kappa_\lambda < A$ for every regular $\lambda < \kappa$. Let $\langle \mu_\xi : \xi < \kappa \rangle$ enumerate in increasing order all of the regular cardinals less than κ. For each $\xi < \kappa$ there is a club C_ξ such that $A \cap C_\xi \subseteq \mathrm{Tr}(E^\kappa_{\mu_\xi})$. The set D of all cardinals less than κ is a club. Let $E = \triangle_{\xi < \kappa}(D \cap C_\xi)$. Take any $\nu \in E \cap A$. Then for all $\xi < \nu$ we have $\nu \in D \cap C_\xi$, so $\nu \cap E^\kappa_{\mu_\xi}$ is stationary in ν, and hence $\mathrm{cf}(\nu) > \mu_\xi$. If $\mathrm{cf}(\nu) < \nu$, then $\mathrm{cf}(\nu) \leq \mu_{\mathrm{cf}(\nu)} < \mathrm{cf}(\nu)$, contradiction. So $\mathrm{cf}(\nu) = \nu$. That is, ν is regular. Thus $E \cap A$ is a stationary set of regular cardinals, so κ is weakly Mahlo. □

Suppose that F is a normal κ-complete filter on $[A]^{<\kappa}$, A set $X \subseteq [A]^{<\kappa}$ is *F-positive* iff $[A]^{<\kappa} \backslash X \notin F$.

Proposition 2.92. *Suppose that F is a normal κ-complete filter on $[A]^{<\kappa}$, and $X \subseteq \mathscr{P}_\kappa(A)$ is F-positive. Also suppose that g is a function with domain X and $\forall x \in F[g(x) \in [x]^{<\omega}]$. Then g is constant on some F-positve $Y \subseteq X$.*

Proof.

(1) There is an $m \in \omega$ such that $\{x \in X : |g(x)| = m\}$ is F-positive.

In fact, otherwise the set $Y_m \stackrel{\text{def}}{=} \{x \in X : |g(x)| \neq m\}$ is in F for all $m \in \omega$, and hence by κ-completeness and because (implicitly) κ is uncountable and regular, we get $\emptyset = \bigcap_{m \in \omega} Y_m \in F$, contradiction.

So we can assume that $|g(x)| = m$ for all $x \in X$, and proceed by induction on m. The case $m = 0$ is trivial. Now assume inductively that $m > 0$. For each $x \in X$ let $h(x)$ be the least member of $g(x)$. We claim that there is an $a \in A$ such that $\{x \in X : h(x) = a\}$ is F-positive. Otherwise, the set $Y_a \stackrel{\text{def}}{=} \{x \in X : h(x) \neq a\}$ is in F for all $a \in A$. Then also $Z \stackrel{\text{def}}{=} \triangle_{a \in A} Y_a \in F$. Take any $x \in Z$. Then for all $a \in x$ we have $x \in Y_a$, hence $h(x) \neq a$, contradiction. Hence our claim is true. Choose such an a, and let $W = \{x \in X : h(x) = a\}$. Define $g'(x) = g(x)\backslash\{a\}$ for all $x \in W$. Then the inductive hypothesis applies and gives the desired result. $\square$

Proposition 2.93. *If F is a normal κ-complete filter on $[A]^{<\kappa}$, then F contains all clubs.*

Proof. Let $C \subseteq [A]^{<\kappa}$ be club. By Lemma 8.26 there is an $f : [A]^{<\omega} \to [A]^{<\kappa}$ such that $C_f \subseteq C$, so it suffices to show that $C_f \in F$. Suppose not. Then the set $[A]^{<\kappa}\backslash C_f$ is F-positive. For all $x \in [A]^{<\kappa}\backslash C_f$ there is a $g(x) \in [x]^{<\omega}$ such that $f(g(x)) \not\subseteq x$. Hence by exercise 8.16 there is an F-positive set $P \subseteq [A]^{<\kappa}\backslash C_f$ such that g has constant value, say u, on P. Thus for all $x \in P$ we have $u \in [x]^{<\omega}$ and $f(u) \not\subseteq x$. Now $f(u) \in [A]^{<\kappa}$. For each $a \in f(u)$ the set $t_a = \{x \in [A]^{<\kappa} : a \in x\}$ is in F, and so $\bigcap_{a\in f(u)} t_a \in F$. Now

$$\bigcap_{a\in f(u)} t_a \in F = \{x \in [A]^{<\kappa} : f(u) \subseteq x\} \subseteq [A]^{<\kappa}\backslash P,$$

contradicting P being F-positive. $\square$

Proposition 2.94. *For every $F : [A]^{<\omega} \to A$ there is a countable x closed under F.*

Proof. Define $\langle y_i : i < \omega\rangle$ by recursion. Let $y_0 = \{a\}$ for any $a \in A$. If y_i has been defined, let

$$y_{i+1} = y_i \cup \bigcup\{f(z) : z \in [y_i]^{<\omega}\}.$$

Clearly each y_i is countable, and $\bigcup_{i\in\omega} y_i$ is as desired. $\square$

Proposition 2.95. *If D is a p-point and $X_0, X_1, \ldots \in D$ with $X_0 \supseteq X_1 \supseteq \cdots$, then there is a $Y \in D$ such that $\forall n \in \omega[Y\backslash X_n$ is finite].*

Case 1. $\bigcap_{i\in\omega} X_i \in D$. Then apply the P-point condition to $\{X_i\backslash X_{i+1} : i \in \omega\} \cup \{\omega\backslash \bigcap_{i\in\omega} X_i\}$.

Case 2. $\bigcap_{i\in\omega} X_i \notin D$. Then apply the P-point condition to $\{X_i\backslash X_{i+1} : i \in \omega\} \cup \{\bigcap_{i\in\omega} X_i\} \cup \{\omega\backslash X_0\}$. $\square$

9. Combinatorial set theory

Suppose that ρ is a nonzero cardinal number, $\langle \lambda_\alpha : \alpha < \rho \rangle$ is a sequence of cardinals, and σ, κ are cardinals. We also assume that $1 \le \sigma \le \lambda_\alpha \le \kappa$ for all $\alpha < \rho$. Then we write

$$\kappa \to (\langle \lambda_\alpha : \alpha < \rho \rangle)^\sigma$$

provided that the following holds:

For every $f : [\kappa]^\sigma \to \rho$ *there exist* $\alpha < \rho$ *and* $\Gamma \in [\kappa]^{\lambda_\alpha}$ *such that* $f[[\Gamma]^\sigma]] \subseteq \{\alpha\}$.

In this case we say that Γ is *homogeneous for* f. The following colorful terminology is standard. We imagine that α is a color for each $\alpha < \rho$, and we color all of the σ-element subsets of κ. To say that Γ is homogeneous for f is to say that all of the σ-element subsets of Γ get the same color. Usually we will take σ and ρ to be a positive integers. If $\rho = 2$, we have only two colors, which are conventionally taken to be red (for 0) and blue (for 1). If $\sigma = 2$ we are dealing with ordinary graphs.

Note that if $\rho = 1$ then we are using only one color, and so the arrow relation obviously holds by taking $\Gamma = \kappa$. If κ is infinite and $\sigma = 1$ and ρ is a positive integer, then the relation holds no matter what σ is, since

$$\kappa = \bigcup_{i<\rho} \{\alpha < \kappa : f(\{\alpha\}) = i\},$$

and so there is some $i < \rho$ such that $|\{\alpha < \kappa : f(\{\alpha\}) = i\}| = \kappa \ge \lambda_i$, as desired.

In case $\lambda_\alpha = \mu$ for all $\alpha < \rho$ we write $\kappa \to (\mu)^\sigma_\rho$

Proposition 9.1. *If* $\kappa \to (\lambda)^n_m$ *and* $\kappa < \kappa'$. *then* $\kappa' \to (\lambda)^n_m$.

Proof. Let $F : [\kappa']^n \to m$. Let $F' = F \restriction [\kappa]^n$. Choose $H \in [\kappa]^\lambda$ such that $F' \restriction [H]^n$ is constant. Then $F \restriction [H]^n = F' \restriction [H]^n$ is constant. □

Proposition 9.2. *If* $\kappa \to (\lambda)^n_m$ *and* $0 < n' < n$. *then* $\kappa' \to (\lambda)^n_m$.

Proof. Let $F : [\kappa']^n \to m$. Let $F' = F \restriction [\kappa]^n$. Choose $H \in [\kappa]^\lambda$ such that $F' \restriction [H]^n$ is constant. Then $F \restriction [H]^n = F' \restriction [H]^n$ is constant. □

Theorem 9.3. (Ramsey) *If* n *and* r *are positive integers, then* $\omega \to (\omega)^n_r$.

Proof. We proceed by induction on n. The case $n = 1$ is trivial, as observed above. So assume that the theorem holds for $n \ge 1$, and now suppose that $f : [\omega]^{n+1} \to r$. For each $m \in \omega$ define $g_m : [\omega \backslash \{m\}]^n \to r$ by:

$$g_m(X) = f(X \cup \{m\}).$$

Then by the inductive hypothesis, for each $m \in \omega$ and each infinite $S \subseteq \omega$ there is an infinite $H^S_m \subseteq S \backslash \{m\}$ such that g_m is constant on $[H^S_m]^n$. We now construct by recursion two sequences $\langle S_i : i \in \omega \rangle$ and $\langle m_i : i \in \omega \rangle$. Each m_i will be in ω, and we will have

$S_0 \supseteq S_1 \supseteq \cdots$. Let $S_0 = \omega$ and $m_0 = 0$. Suppose that S_i and m_i have been defined, with S_i an infinite subset of ω. We define

$$\begin{aligned} S_{i+1} &= H^{S_i}_{m_i} \quad \text{and} \\ m_{i+1} &= \text{the least element of } S_{i+1} \text{ greater than } m_i. \end{aligned}$$

Clearly $S_0 \supseteq S_1 \supseteq \cdots$ and $m_0 < m_1 < \cdots$. Moreover, $m_i \in S_i$ for all $i \in \omega$.

(1) For each $i \in \omega$, the function g_{m_i} is constant on $[\{m_j : j > i\}]^n$.

In fact, $\{m_j : j > i\} \subseteq S_{i+1}$ by the above, and so (1) is clear by the definition.

Let $p_i < r$ be the constant value of $g_{m_i} \restriction [\{m_j : j > i\}]^n$, for each $i \in \omega$. Hence

$$\omega = \bigcup_{j<r} \{i \in \omega : p_i = j\};$$

so there is a $j < r$ such that $K \stackrel{\text{def}}{=} \{i \in \omega : p_i = j\}$ is infinite. Let $L = \{m_i : i \in K\}$. We claim that $f[[L]^{n+1}] \subseteq \{j\}$, completing the inductive proof. For, take any $X \in [L]^{n+1}$; say $X = \{m_{i_0}, \ldots, m_{i_n}\}$ with $i_0 < \cdots < i_n$. Then

$$f(X) = g_{m_{i_0}}(\{m_{i_1}, \ldots, m_{i_n}\}) = p_{i_0} = j. \qquad \square$$

Theorem 9.4. (Ramsey) *Suppose that $n, r, l_0, \ldots, l_{r-1}$ are positive integers, with $n \le l_i$ for each $i < r$. Then there is a $k \ge l_i$ for each $i < r$ and $k \ge n$ such that*

$$k \to (l_0, \ldots, l_{r-1})^n.$$

Proof. Assume the hypothesis, but suppose that the conclusion fails. Thus for every k such that $k \ge l_i$ for each $i < r$ with $k \ge n$ also, we have $k \not\to (l_0, \ldots, l_{r-1})^n$, which means that there is a function $f_k : [k]^n \to r$ such that for each $i < r$, there is no set $S \in [k]^{l_i}$ such that $f_k[[S]^n] \subseteq \{i\}$. We use these functions to define a certain $g : [\omega]^n \to r$ which will contradict the infinite version of Ramsey's theorem. Let $M = \{k \in \omega : k \ge l_i$ for each $i < r$ and $k \ge n\}$.

To define g, we define functions $h_i : [i]^n \to r$ by recursion. h_0 has to be the empty function. Now suppose that we have defined h_i so that $S_i \stackrel{\text{def}}{=} \{s \in M : f_s \restriction [i]^n = h_i\}$ is infinite. This is obviously true for $i = 0$. Then

$$S_i = \bigcup_{s:[i+1]^n \to r} \{k \in S_i : f_k \restriction [i+1]^n = s\},$$

and so there is a $h_{i+1} : [i+1]^n \to r$ such that $S_{i+1} \stackrel{\text{def}}{=} \{k \in S_i : f_k \restriction [i+1]^n = h_{i+1}\}$ is infinite, finishing the construction.

Clearly $h_i \subseteq h_{i+1}$ for all $i \in \omega$. Hence $g = \bigcup_{i\in\omega} h_i$ is a function mapping $[\omega]^n$ into r. By the infinite version of Ramsey's theorem choose $v < r$ and $Y \in [\omega]^\omega$ such that

$g[[Y]^n] \subseteq \{v\}$. Take any $Z \in [Y]^{l_v}$. Choose i so that $Z \subseteq i$, and choose $k \in S_i$. Then for any $X \in [Z]^n$ we have

$$f_k(X) = h_i(X) = g(X) = v,$$

so Z is homogeneous for f_k, contradiction. $\square$

Theorem 9.5. *Let D be a nonprincipal ultrafilter on ω. Then the following are equivalent:*

(i) D is Ramsey.

(ii) For all positive integers n and k and all $F : [\omega]^n \to k$ there exist $X \in D$ and $i < k$ such that $F([X]^n) \subseteq \{i\}$.

Proof. First assume (ii). Let $\mathscr{A}$ be a partition of ω such that $\forall X \in \mathscr{A}[X \notin D]$. We will find $X \in D$ such that $\forall A \in \mathscr{A}[|X \cap A| \leq 1]$. Define $F : [\omega]^2 \to 2$ by

$$F(\{x,y\}) = \begin{cases} 1 & \text{if } \exists A, B \in \mathscr{A}[A \neq B \text{ and } x \in A \text{ and } y \in B] \\ 0 & \text{otherwise.} \end{cases}$$

Choose $X \in D$ and $i < 2$ such that $F([X]^2) \subseteq \{i\}$. If $A \in \mathscr{A}$ and x, y are distinct members of $X \cap A$ then clearly $F(\{x,y\}) = 0$. Hence $F([X]^2) \subseteq \{0\}$. Hence $X \subseteq A$ for some $A \in \mathscr{A}$. Hence $A \in D$, contradiction.

Now suppose that D is Ramsey.

(1) If $X_0 \supseteq X_1 \supseteq$ with each $X_i \in D$, then there exist $a_0 < a_1 < \cdots$ each in ω such that $\{a_0, a_1, \ldots\} \in D$, $a_0 \in X_0$ and $\forall n \in \omega[a_{n+1} \in X_{a_n}]$.

For, suppose that $X_0 \supseteq X_1 \supseteq$ with each $X_i \in D$. Since D is a p-point, there is a $Y \in D$ such that $\forall n[Y \subseteq X_n$ is finite$]$. Now define

$$\begin{aligned}
&y_0 = \text{least } z \in Y \text{ such that } \forall y > z[y \in X_0];\\
&y_1 = \text{least } z \in Y \text{ such that } z > y_0 \text{ and } \forall y > z[y \in X_{y_1}];\\
&\ldots\ldots\\
&y_n = \text{least } z \in Y \text{ such that } z > y_{n-1} \text{ and } \forall y > z[y \in X_{y_{n-1}}];\\
&\ldots\ldots
\end{aligned}$$

For each $n \in \omega$ let $A_n = \{z \in Y : y_n < z \leq y_{n+1}\}$. Each A_n is finite and hence is not in D. $\{z : z \leq y_0\}$ along with the A_n's is a partition with each piece not in D. Since D is Ramsey, there is a $z \in {}^{\omega}\omega$ such that $\mathrm{rng}(z) \in D$ and $\forall n \in \omega[z_n \in A_n]$.

(2) $\forall n \in \omega[z_{n+2} \in X_{z_n}]$.

For, $z_{n+2} \in A_{n+2}$ and hence $z_{n+2} > y_{n+2}$. It follows that $z_{n+2} \in X_{y_{n+1}}$. Now $y_{n+1} \geq z_n$, so $X_{y_{n+1}} \subseteq X_{z_n}$. Hence $z_{n+2} \in X_{z_n}$, and (2) holds.

Now for all $n \in \omega$ let $a_n = z_{2n}$ and $b_n = z_{2n+1}$. Then either $\{a_n : n \in \omega\} \in D$ or $\{b_n : n \in \omega\} \in D$.

Case 1. $\{a_n : n \in \omega\} \in D$. Choose $a'_0 \in X_0$ and let $a'_{n+1} = a_{n+1}$. Then $a'_{n+1} = a_{n+1} = z_{2n+2} \in X_{z_{2n}} = X_n$, as desired in (1).

Case 2. $\{b_n : n \in \omega\} \in D$. Choose $b'_0 \in X_0$ and let $b'_{n+1} = b_{n+1}$. Then $b'_{n+1} = b_{n+1} = z_{2n+3} \in X_{z_{2n+1}} = X_{b_n}$, as desired in (1).

Thus (1) holds.

Now to prove (ii), we proceed by induction on n. For $n = 1$, suppose that $F : \omega \to k$. Then $\omega = \bigcup_{i<k} F^{-1}[\{i\}]$, so choose $i < k$ such that $F^{-1}[\{i\}] \in D$. Then $F[F^{-1}[\{i\}]] = \{i\}$, as desired.

Now assume the result for n, and suppose that $F : [\omega]^{n+1} \to k$. For each $a \in \omega$ define $F_a : [\omega\backslash\{a\}]^n \to k$ by $F_a(b) = F(b \cup \{a\})$. By the inductive hypothesis choose $H_a \in D$ and $i_a < k$ such that $F_a[[H_a]^n] \subseteq \{i_a\}$. Then $\omega = \bigcup_{j<k}\{a \in \omega : i_a = j\}$. Choose $j < k$ so that $K \stackrel{\text{def}}{=} \{a \in \omega : i_a = j\} \in D$. Fix $a \in K$. Then $H_a \cap K \in D$ and for all $b \in H_a \cap K$ we have $F_b[[H_a \cap K]^n] \subseteq \{j\}$.

Now for each $n \in \omega$ let $X_n = H_a \cap K \cap H_0 \cap H_1 \cap \cdots \cap H_n$. By (1) choose $a_0 > a_1 < \cdots$ such that $a_0 \in X_0$ and $\forall n \in \omega[a_{n+1} \in X_{a_n}$, and $L \stackrel{\text{def}}{=} \{a_0.a_1, \ldots\} \in D$.

(3) $\forall i \in \omega[a_i \in L$ and $\{a_m : m > i\} \subseteq H_{a_i}]$.

For, obviously $a_i \in L$. If $m > i$, then $m-1 \geq i$, hence $a_{m-1} \geq a_i$, and $a_m \in X_{a_{m-1}} \subseteq H_{a_i}$.

(4) $\forall i \in \omega \forall x \in [\{a_m : m > i\}]^n[F_{a_i}(x) = j]$.

In fact, suppose that $i \in \omega$ and $x \in [\{a_m : m > i\}]^n$. Now for $m > i$ we have $a_m \in X_{a_{m-1}} \subseteq H_a \cap K$. Hence $x \in [H_a \cap K]^n$. Clearly $a_i \in H_a \cap K$. Hence $F_{a_i}(x) = j$. $\square$

Theorem 9.6. *For any infinite cardinal κ we have $2^\kappa \not\to (3)^2_\kappa$.*

Proof. Define $F : [{}^\kappa 2]^2 \to \kappa$ by setting $F(\{f, g\}) = \chi(f, g)$ for any two distinct $f, g \in {}^\kappa 2$. If $\{f, g, h\}$ is homogeneous for F with f, g, h distinct, let $\alpha = \chi(f, g)$. Then $f(\alpha), g(\alpha), h(\alpha)$ are distinct members of 2, contradiction. $\square$

Theorem 9.7. *For any infinite cardinal κ, the linear order ${}^\kappa 2$ does not contain a subset order isomorphic to κ^+ or to $(\kappa^+, >)$.*

Proof. The two assertions are proved in a very similar way, so we give details only for the first assertion. In fact, we assume that $\langle f_\alpha : \alpha < \kappa^+\rangle$ is a strictly increasing sequence of members of ${}^\kappa 2$, and try to get a contradiction. The contradiction will follow rather easily from the following statement:

(1) If $\gamma \leq \kappa$, $\Gamma \in [\kappa^+]^{\kappa^+}$, and $f_\alpha \restriction \gamma < f_\beta \restriction \gamma$ for any $\alpha, \beta \in \Gamma$ such that $\alpha < \beta$, then there exist $\delta < \gamma$ and $\Delta \in [\Gamma]^{\kappa^+}$ such that $f_\alpha \restriction \delta < f_\beta \restriction \delta$ for any $\alpha, \beta \in \Delta$ such that $\alpha < \beta$.

To prove this, assume the hypothesis. For each $\alpha \in \Gamma$ let $f'_\alpha = f_\alpha \restriction \gamma$. Clearly Γ does not have a largest element. For each $\alpha \in \Gamma$ let α' be the least member of Γ which is greater than α. Then

$$\Gamma = \bigcup_{\xi<\gamma} \{\alpha \in \Gamma : \chi(f'_\alpha, f'_{\alpha'}) = \xi\}.$$

Since $|\Gamma| = \kappa^+$, it follows that there are $\delta < \gamma$ and $\Delta \in [\Gamma]^{\kappa^+}$ such that $\chi(f'_\alpha, f'_{\alpha'}) = \delta$ for all $\alpha \in \Delta$. We claim now that $f'_\alpha \restriction \delta < f'_\beta \restriction \delta$ for any two $\alpha, \beta \in \Delta$ such that $\alpha < \beta$, as desired in (1). For, take any such α, β. Suppose that $f'_\alpha \restriction \delta = f'_\beta \restriction \delta$. (Note that we must have $f'_\alpha \restriction \delta \leq f'_\beta \restriction \delta$.) Now from $\chi(f'_\alpha, f'_{\alpha'}) = \delta$ we get $f'_{\alpha'}(\delta) = 1$, and from $\chi(f'_\beta, f'_{\beta'}) = \delta$

we get $f'_\beta(\delta) = 0$. Now $f'_{\alpha'} \upharpoonright \delta = f'_\alpha \upharpoonright \delta = f'_\beta \upharpoonright \delta$, so we get $f'_\beta < f'_{\alpha'} \leq f'_\beta$, contradiction. This proves (1).

Clearly from (1) we can construct an infinite decreasing sequence $\kappa > \gamma_1 > \gamma_2 > \cdots$ of ordinals, contradiction. $\square$

Theorem 9.8. *For any infinite cardinal κ we have $2^\kappa \not\to (\kappa^+, \kappa^+)^2$.*

Proof. We consider ${}^\kappa 2$ under the lexicographic order. Let $\langle f_\alpha : \alpha < 2^\kappa \rangle$ be a one-one enumeration of ${}^\kappa 2$. Define $F : 2^\kappa \to 2$ by setting, for any $\alpha < \beta < \kappa$,

$$F(\{\alpha, \beta\}) = \begin{cases} 0 & \text{if } f_\alpha < f_\beta, \\ 1 & \text{if } f_\beta < f_\alpha. \end{cases}$$

If $2^\kappa \to (\kappa^+, \kappa^+)^2$ holds, then there is a set $\Gamma \in [2^\kappa]^{\kappa^+}$ which is homogeneous for F. If $F(\{\alpha, \beta\}) = 0$ for all distinct $\alpha < \beta$ in Γ, then $\langle f_\alpha : \alpha \in \Gamma \rangle$ is a strictly increasing sequence of length o.t.(Γ), contradicting Theorem 9.7. A similar contradiction is reached if $F(\{\alpha, \beta\}) = 1$ for all distinct $\alpha < \beta$ in Γ. $\square$

Theorem 9.9. (Erdös-Rado) *For every positive integer n, $\beth_n^+ \to (\aleph_1)^{n+1}_\omega$.*

Proof. First assume that $n = 1$. Let $\kappa = (2^\omega)^+$ and $F : [\kappa]^2 \to \omega$. For each $a \in \kappa$ let F_a be the function with domain $\kappa \backslash \{a\}$ defined by $F_a(x) = F(\{a, x\})$.

Claim. $\exists A \in [\kappa]^{2^\omega} \forall C \in [A]^\omega \forall u \in \kappa \backslash C \exists v \in A \backslash C [F_v \upharpoonright C = F_u \upharpoonright C]$.

Proof of claim. We construct an ω_1-sequence $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_\alpha \subseteq \cdots$ of members of $[\kappa]^{2^\omega}$. Let $A_0 = 2^\omega$. For limit α let $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$. To construct $A_{\alpha+1}$, for each countable $C \subseteq A_\alpha$ define $u \equiv v$ iff $u, v \in \kappa \backslash C$ and $F_u \upharpoonright C = F_v \upharpoonright C$. Since there are at most 2^ω functions from C into ω, there are at most 2^ω equivalence classes. Let K_C have one member from each equivalence class. Thus $\forall u \in \kappa \backslash C \exists v \in K_C [F_u \upharpoonright C = F_v \upharpoonright C]$. Let $A_{\alpha+1}$ be A_α union the union of all such sets K_C. Since $|A_\alpha| \leq 2^\omega$, there are at most 2^ω sets C, so $A_{\alpha+1}$ is as desired. Let $A = \bigcup_{\alpha<\omega_1} A_\alpha$. $\square$

Now fix $a \in \kappa \backslash A$. Fix $x_0 \in A$. Given $\{x_\beta : \beta < \alpha\} = C$ with $\alpha < \omega_1$, choose $x_\alpha \in A \backslash C$ so that $F_{x_\alpha} \upharpoonright C = F_a \upharpoonright C$. Let $X = \{x_\alpha : \alpha < \omega_1\}$. Now define $G : X \to \omega$ by $G(x) = F_a(x)$.

(1) If $\alpha < \beta < \omega_1$, then $F(\{x_\alpha, x_\beta\}) = G(x)$.

For, $F(\{x_\alpha, x_\beta\}) = F_{x_\beta}(x_\alpha) = F_a(x_\beta) = G(x)$. Now rng$(G)$ is countable, so there is an $H \in [X]^{\omega_1}$ such that G is constant on H. Hence F is constant on $[H]^2$. This finishes the case $n = 1$.

Now assume that $n \geq 2$ and the result holds for $n - 1$. Thus $\beth_{n-1}^+ \to (\omega_1)^n_\omega$. Let $\kappa = \beth_n^+$ and assume that $F : [\kappa]^{n+1} \to \omega$. For each $a \in \kappa$ let $F_a : [\kappa \backslash \{a\}]^n \omega$ be defined by $F_a(x) = F(x \cup \{a\})$. We claim:

$$\exists A \in [\beth_n^+]^{\beth_n} \forall C \in [A]^{\beth_{n-1}} \forall u \in \beth_n^+ \backslash C \exists v \in A \backslash C (F_u \upharpoonright [C]^n = F_v \upharpoonright [C]^n).$$

In fact, we construct a sequence $\langle A_\alpha : \alpha < \beth_{n-1}^+ \rangle$ of subsets of $\beth_n^+$, each of size $\beth_n$. Let $A_0 = \beth_n$, and for α limit let $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$. Now suppose that A_α has been defined, and $C \in [A_\alpha]^{\beth_{n-1}}$. Define $u \equiv v$ iff $u, v \in \beth_n^+ \backslash C$ and $F_u \restriction [C]^n = F_v \restriction [C]^n$). Now each $\left|{}^{[C]^n}\omega\right| = \beth_n$, so there are at most $\beth_n$ equivalence classes. Let K_C have exactly one member from each equivalence class. Let $A_{\alpha+1} = A_\alpha \cup \bigcup \left\{K_C : C \in [A_\alpha]^{\beth_{n-1}}\right\}$. Since $\beth_n^{\beth_{n-1}} = \beth_n$, we still have $|A_{\alpha+1}| = \beth_n$. Finally, let $A = \bigcup_{\alpha < \beth_{n-1}^+} A_\alpha$. Clearly A is as desired.

Now choose $a \in \kappa \backslash A$. Construct $X = \{x_\alpha : \alpha < \beth_{n-1}^+\} \subseteq A$ so that $\forall \alpha < \beth_{n-1}^+ [F_{x_\alpha} \restriction [\{x_\beta : \beta < \alpha\}]^n = F_a \restriction [\{x_\beta : \beta < \alpha\}]^n$. Define $G : [X]^n \to \omega$ so that $G(x) = F_a(x)$. By the inductive hypothesis there is an $H \in [X]^{\omega_1}$ such that G is constant on H with value b. Then if $\alpha_0 < \cdots < \alpha_n$, all in H, then

$$\begin{aligned} F(\{x_{\alpha_0}, \ldots, x_{\alpha_n}\}) &= F_{x_{\alpha_n}}(\{x_{\alpha_0}, \ldots, x_{\alpha_{n-1}}\}) \\ &= F_a(\{x_{\alpha_0}, \ldots, x_{\alpha_{n-1}}\}) = G(\{x_{\alpha_0}, \ldots, x_{\alpha_{n-1}}\}) = b. \end{aligned} \qquad \square$$

Theorem 9.10. (Dushnik, Miller) *For any infinite cardinal κ we have $\kappa \to (\kappa, \omega)^2$.*

Proof. Suppose that $f : [\kappa]^2 \to 2$; we want to find a set $X \in [\kappa]^\kappa$ such that $f[[X]^2] = \{0\}$, or a set $X \in [\kappa]^\omega$ such that $f[[X]^2] = \{1\}$.

For each $x \in \kappa$ let $B(x) = \{y \in \kappa \backslash \{x\} : f(\{x, y\}) = 1\}$. Now we claim:

Claim. Suppose that for every $X \in [\kappa]^\kappa$ there is an $x \in X$ such that $|B(x) \cap X| = \kappa]$. Then there is an infinite $X \subseteq \kappa$ such that $f[[X]^2] \subseteq \{1\}$.

Proof of claim. Assume the hypothesis. We define x_n, Y_n for $n \in \omega$ by recursion. Let $Y_0 = \kappa$. Assume that $Y_n \in [\kappa]^\kappa$ has been defined. Then by supposition there is an $x_n \in Y_n$ such that $|B(x_n) \cap Y_n| = \kappa$. Let $Y_{n+1} = B(x_n) \cap Y_n$. Now if $n < m < \omega$, then $x_m \in Y_{n+1} \subseteq B(x_n)$, and hence $f(\{x_n, x_m\}) = 1$. Thus $\{x_n : n \in \omega\}$ is an infinite subset of κ such that $f[[\{x_n : n \in \omega\}]^2] \subseteq \{1\}$, as desired. $\square$

First suppose that κ is regular, and assume that there is no $X \in [\kappa]^\kappa$ such that $f[[X]^2] \subseteq \{0\}$. We will verify the hypothesis of the claim; this gives the desired conclusion. So, suppose that $X \in [\kappa]^\kappa$. By Zorn's lemma let $Y \subseteq X$ be maximal such that $f[[Y]^2] \subseteq \{0\}$. Thus $|Y| < \kappa$ by assumption. Now

$$X \backslash Y \subseteq \bigcup_{y \in Y} \{x \in X \backslash Y : f(\{x, y\}) = 1\} = \bigcup_{y \in Y} [B(y) \cap (X \backslash Y)].$$

Since $|Y| < \kappa$ and κ is regular, there is a $y \in Y$ such that $|B(y) \cap X| = \kappa$. This verifies the hypothesis of the claim.

Second suppose that κ is singular, and suppose that there is no infinite $X \subseteq \kappa$ such that $f[[X]^2] \subseteq \{1\}$. Then by the claim,

$$(*) \qquad \exists X \in [\kappa]^\kappa \forall x \in X [|B(x) \cap X| < \kappa].$$

Let $\langle \lambda_\xi : \xi < \mathrm{cf}(\kappa) \rangle$ be a strictly increasing sequence of regular cardinals with supremum κ and with $\mathrm{cf}(\kappa) < \lambda_0$, and let $\langle Y_\xi : \xi < \mathrm{cf}(\kappa) \rangle$ be a system of pairwise disjoint subsets of

X such that $\forall\xi < \mathrm{cf}(\kappa)[|Y_\xi| = \lambda_\xi]$. By the regular case, $\lambda_\xi \to (\lambda_\xi, \omega)^2$ for each $\xi < \mathrm{cf}(\kappa)$. It follows that for each $\xi < \mathrm{cf}(\kappa)$ there is a $Z_\xi \in [Y_\xi]^{\lambda_\xi}$ such that $f[[Z_\xi]^2] \subseteq \{0\}$. Now for each $\xi < \mathrm{cf}(\kappa)$, by $(*)$,

$$Z_\xi = \bigcup_{\alpha<\mathrm{cf}(\kappa)} \{x \in Z_\xi : |B(x) \cap X| < \lambda_\alpha\}.$$

Since $|Z_\xi| = \lambda_\xi > \mathrm{cf}(\kappa)$ and λ_ξ is regular, there is an $h(\xi) < \mathrm{cf}(\kappa)$ such that

$$W_\xi \stackrel{\text{def}}{=} \{x \in Z_\xi : |B(x) \cap X| < \lambda_{h(\xi)}\}$$

has size λ_ξ.

Now we define a sequence $\langle\alpha_\xi : \xi < \mathrm{cf}(\kappa)\rangle$ of ordinals less than κ by recursion. If α_η has been defined for all $\eta < \xi$, with $\xi < \mathrm{cf}(\kappa)$, then the set $\{\alpha_\eta : \eta < \xi\} \cup \{\lambda_{h(\eta)} : \eta < \xi\}$ is bounded below κ and so there is an $\alpha_\xi < \kappa$ greater than each member of this set. Thus if $\eta < \xi$ then $\alpha_\eta < \alpha_\xi$ and $\lambda_{h(\eta)} < \alpha_\xi$. Now for any $\xi < \mathrm{cf}(\kappa)$ let

$$S_\xi = W_{\alpha_\xi} \backslash \bigcup \left\{ B(x) \cap X : x \in \bigcup_{\eta<\xi} W_{\alpha_\eta} \right\}.$$

Note that if $\eta < \xi < \mathrm{cf}(\kappa)$ then $|W_{\alpha_\eta}| = \lambda_{\alpha_\eta} < \lambda_{\alpha_\xi}$ and $\xi < \mathrm{cf}(k) < \lambda_0 < \lambda_{\alpha_\xi}$, so $\left|\bigcup_{\eta<\xi} W_{\alpha_\eta}\right| < \lambda_{\alpha_\xi}$. Moreover, if $\eta < \xi$ and $x \in W_{\alpha_\eta}$, then $|B(x) \cap X| < \lambda_{h(\alpha_\eta)} < \lambda_\xi$. Hence for each $x \in \bigcup_{\eta<\xi} W_{\alpha_\eta}$ we have $|B(x) \cap X| < \lambda_{\alpha_\xi}$. Hence $|S_\xi| = \lambda_{\alpha_\xi}$. Let $T = \bigcup_{\xi<\mathrm{cf}(\kappa)} S_\xi$. So $|T| = \kappa$. We claim that $f[[T]^2] \subseteq \{0\}$. For, suppose that $x, y \in T$ with $x \neq y$.

Case 1. There is a $\xi < \mathrm{cf}(\kappa)$ such that $x, y \in S_\xi$. Now $S_\xi \subseteq W_{\alpha_\xi} \subseteq Z_{\alpha_\xi}$, so $f(\{x, y\}) = 0$.

Case 2. There exist $\eta < \xi < \mathrm{cf}(\kappa)$ such that $x \in S_\eta$ and $y \in S_\xi$. (The case $x \in S_\xi$ and $y \in S_\eta$ is treated similarly.) Then $x \in W_{\alpha_\eta}$, so $y \notin B(x)$, i.e. $f(\{x, y\}) = 0$. □

A cardinal κ is *weakly compact* iff it is uncountable and $\kappa \to (\kappa)^2$.

Proposion 9.11. *Every weakly compact cardinal is inaccessible.*

Proof. To show that κ is regular, suppose to the contrary that $\kappa = \sum_{\alpha<\lambda} \mu_\alpha$, where $\lambda < \kappa$ and $\mu_\alpha < \kappa$ for each $\alpha < \lambda$. By the definition of infinite sum of cardinals, it follows that we can write $\kappa = \bigcup_{\alpha<\lambda} M_\alpha$, where $|M_\alpha| = \mu_\alpha$ for each $\alpha < \lambda$ and the M_α's are pairwise disjoint. Define $f : [\kappa]^2 \to 2$ by setting, for any distinct $\alpha, \beta < \kappa$,

$$f(\{\alpha, \beta\}) = \begin{cases} 0 & \text{if } \alpha, \beta \in M_\xi \text{ for some } \xi < \lambda, \\ 1 & \text{otherwise.} \end{cases}$$

Let H be homogeneous for f of size κ. First suppose that $f[[H]^2] = \{0\}$. Fix $\alpha_0 \in H$, and say $\alpha_0 \in M_\xi$. For any $\beta \in H$ we then have $\beta \in M_\xi$ also, by the homogeneity of H. So $H \subseteq M_\xi$, which is impossible since $|M_\xi| < \kappa$. Second, suppose that $f[[H]^2] = \{1\}$. Then any two distinct members of H lie in distinct M_ξ's. Hence if we define $g(\alpha)$ to be the

$\xi < \lambda$ such that $\alpha \in M_\xi$ for each $\alpha \in H$, we get a one-one function from H into λ, which is impossible since $\lambda < \kappa$.

To show that κ is strong limit, suppose that $\lambda < \kappa$ but $\kappa \leq 2^\lambda$. Now by Theorem 9.8 we have $2^\lambda \not\to (\lambda^+, \lambda^+)^2$. So choose $f : [2^\lambda]^2 \to 2$ such that there does not exist an $X \in [2^\lambda]^{\lambda^+}$ with $f \restriction [X]^2$ constant. Define $g : [\kappa]^2 \to 2$ by setting $g(A) = f(A)$ for any $A \in [\kappa]^2$. Choose $Y \in [\kappa]^\kappa$ such that $g \restriction [Y]^2$ is constant. Take any $Z \in [Y]^{\lambda^+}$. Then $f \restriction [Z]^2$ is constant, contradiction.

A *tree* is a partially ordered set $(T, <)$ such that for each $t \in T$, the set $\{s \in T : s < t\}$ is well-ordered by the relation $<$. Thus every ordinal is a tree, but that is not so interesting in the present context. We introduce some standard terminology concerning trees.

- $(t\downarrow) = \{s \in T : s < t\}$; $(t\uparrow) = \{s \in T : t < s\}$.

- For each $t \in T$, the order type of $\{s \in T : s < t\}$ is called the *height* of t, and is denoted by $\mathrm{ht}(t, T)$ or simply $\mathrm{ht}(t)$ if T is understood.

- A *root* of a tree T is an element of T of height 0, i.e., it is an element of T with no elements of T below it. Frequently we will assume that there is only one root.

- For each ordinal α, the α-th *level* of T, denoted by $\mathrm{Lev}_\alpha(T)$, is the set of all elements of T of height α.

- The *height* of T itself is the least ordinal greater than the height of each element of T; it is denoted by $\mathrm{ht}(T)$.

- A *chain* in T is a subset of T linearly ordered by $<$.

- A *branch* of T is a maximal chain of T.

- For each $\alpha \leq \mathrm{ht}(T)$ let $T_\alpha = \bigcup_{\beta<\alpha} \mathrm{Lev}_\beta(T)$.

- An *antichain* is a collection of pairwise incomparable elements.

- T is a *Suslin tree* iff the height of T is ω_1, every branch is countable, and every antichain is countable.

- For $\alpha \leq \omega_1$, a *normal α-tree* is a tree T satisfying the following conditions:
  (i) The height of T is ω_1.
  (ii) T has only one root.
  (iii) Each level of T is countable.
  (iv) If $x \in T$ is not maximal, then there are infinitely many elements greater than x at the next level.
  (v) For each $x \in T$ and each level l greater than the level of x there is an element $>$ x at that level.
  (vi) If $\beta < \alpha$ is a limit ordinal, x and y have level β, and $\{z : z < x\} = \{z : z < y\}$, then $x = y$.

Note that chains and branches of T are actually well-ordered, and so we may talk about their *lengths*.

Lemma 9.12. *If there is a Suslin tree, then there is a normal Suslin tree.*

Proof. Let T be a Suslin tree. For any tree S and any $s \in S$ let $S \uparrow s = \{t \in S : s \leq t\}$. Now define

$$T_1 = \{x \in T : T \uparrow x \text{ is uncountable}\}.$$

Obviously there is a root of T which is in T_1, so T_1 is nonempty. We let T_1 have the order from T. So T_1 is a tree. Clearly if $s < t \in T_1$, then also $s \in T_1$. Hence the level of an element of T_1 in T_1 is the same as its level in T. Now we claim that (v) holds for T_1. For, suppose that $t \in T_1$, and take an α greater than the level of t. Then

$$\begin{aligned} T \uparrow t = \{s \in T_1 \uparrow t &: s \text{ has level less than } \alpha\} \\ &\cup \bigcup \{T \uparrow s : s \in T_1 \uparrow t \text{ and } s \text{ has level } \alpha\}. \end{aligned}$$

Since the first set here is countable, it follows that there is an $s \in T \uparrow t$ at level α such that $T \uparrow s$ is uncountable. So $s \in T_1$, as desired. Thus T_1 is a Suslin tree satisfying (v).

An element $s \in T_1$ is a *branching point* iff it has at least two immediate successors.

(1) There are uncountably many branching points above each member of T_1.

In fact, suppose that $s \in T_1$ and the set B of branching points above s is countable. Let α be a level above the level of all members of $B \cup \{s\}$. Then for each t above s at level α, $T_1 \uparrow t$ is a chain, and hence is countable. So $\bigcup \{T_1 \uparrow t : t$ is above s at level $\alpha\}$ is countable, and (v) is contradicted for s. So (1) holds.

Let T_2 be the set of all branching points of T_1. So we still have a Suslin tree, and (v) continues to hold. Moreover, every element of T_2 is a branching point in T_2, for if $x \in T_2$ and y, z are distinct immediate successors of x in T, let y', z' be the least branching points of T_1 above y, z respectively. Then y', z' are distinct immediate successors of x in T_2.

Next, let $\mathscr{C}$ be the collection of all $C \subseteq T_2$ satisfying the following conditions.

(2) C is a chain, and $\forall t \in T_2 \forall x \in C[t \leq x \to t \in C]$.

(3) C has limit length α_C, and there are at least two elements $x \in T_2$ of level α_C such that $\forall y \in C[y < x]$.

For each $C \in \mathscr{C}$ we introduce a new element l_C producing T_3, and extend the order on T_2 by defining for any $x \in T_2$ and $C_1, C_2 \in \mathscr{C}$

$$\begin{aligned} x <' l_{C_1} \quad &\text{iff} \quad x \in C_1; \\ l_{C_1} <' x \quad &\text{iff} \quad \forall y \in C_1[y < x]; \\ l_{C_1} <' l_{C_2} \quad &\text{iff} \quad C_1 \subset C_9. \end{aligned}$$

Clearly $<'$ is irreflexive. To see that it is transitive, suppose that $x, y, z \in T_3$ and $x <' y <' z$.

Case 1. $x, y, z \in T_2$. Clearly $x <' z$.
Case 2. $x, y \in T_2$, $z = l_C$. Then $y \in C$, so $x \in C$; hence $x <' z$.
Case 3. $x, z \in T_2$, $y = l_C$. Then $x \in C$, hence $x < z$.

Case 4. $x \in T_2$, $y = l_{C_1}$, $z = l_{C_2}$. Then $x \in C_1 \subset C_2$, so $x \in C_2$ and hence $x <' z$.

Case 5. $x = l_C$, $y, z \in T_2$. Then $\forall w \in C[w < y]$, hence $\forall w \in C[w < z]$, so $x <' z$.

Case 6. $x = l_{C_1}$, $y \in T_2$, $z = l_{C_2}$. Then $\forall w \in C_1[w < y]$, and $y \in C_2$, so $C_1 \subset C_2$. So $x <' z$.

Case 7. $x = l_{C_1}$, $y = l_{C_2}$, $z \in T_2$. Then $C_1 \subset C_2$ and $\forall w \in C_2[w < z]$. hence $x <' z$.

Case 8. $x = l_{C_1}$, $y = l_{C_2}$, $z = l_{C_3}$. Then $C_1 \subset C_2 \subset C_3$, so $x <' z$.

Next, T_3 is a tree. For, suppose that $x_0 >' x_1 >' \cdots$ with each $x_i \in T_3$. If $\forall m \in \omega \exists n \geq m[x'_n \in T_2]$, a contradiction follows. Suppose that $\exists m \in \omega \forall n \geq m[x'_n \notin T_2]$. Say for $n \geq m$ that $x'_n = l_{C_n}$. Then $C_n \supset C_{n+1} \supset \cdots$. Choose $y_n \in C_n \backslash C_{n+1}$ for all $n \in \omega$. Then $\forall n \in \omega \forall z \in C_{n+1}[z < y_n]$, so $y_n > y_{n+1} > \cdots$, contradiction.

Next, suppose that $\langle x_\alpha : \alpha < \omega_1 \rangle$ is strictly increasing in T_3. If $\forall \alpha < \omega_1 \exists \beta \in [\alpha, \omega_1)[x_\beta \in T_2]$, we get a contradiction because T_2 is Suslin. Suppose $\exists \alpha < \omega_1 \forall \beta \in [\alpha, \omega_1)[x_\beta \notin T_2]$. For each $\beta \in [\alpha, \omega_1)$ say $x_\beta = l_{C_\beta}$. Then $C_\beta \subset C_{\beta+1} \subset \cdots$. Choose $y_\beta \in C_{\beta+1} \backslash C_\beta$ for all $\beta \in [\alpha, \omega_1)$. Then $y_\beta < y_{\beta+1} < \cdots$, again a contradiction.

Now suppose that $\langle x_\alpha : \alpha < \omega_1 \rangle$ is incomparable in T_3. If $\forall \alpha < \omega_1 \exists \beta \in [\alpha, \omega_1)[x_\beta \in T_2]$, we get a contradiction because T_2 is Suslin. Suppose $\exists \alpha < \omega_1 \forall \beta \in [\alpha, \omega_1)[x_\beta \notin T_2]$. For each $\beta \in [\alpha, \omega_1)$ say $x_\beta = l_{C_\beta}$. For each $\beta \in [\alpha, \omega_1)$ let y_β be at level α_{C_β} such that $\forall z \in C_\beta[z < y_\beta]$. Then y_β and y_γ are incomparable for $\beta \neq \gamma$, contradiction.

Thus T_3 is Suslin. Clearly (v) holds for T_3, and each element of T_3 has at least two immediate successors. Suppose that $\langle w_\gamma : \gamma < \beta \rangle$ is a chain in T_3 of limit length β, and $u \neq v$ are at level β such that $\forall \gamma < \beta[w_\gamma < u$ and $w_\gamma < v]$. If $\forall \gamma < \beta \exists \delta \in (\gamma, \beta)[w_\delta \in T_2]$ then the chain is in $\mathscr{C}$ and we get a contradiction. Suppose $\exists \gamma < \beta \forall \delta \in (\gamma, \beta)[w_\delta \notin T_2]$. Say $w_\delta = l_{C_\delta}$ for $\delta \in (\gamma, \beta)$. Then $\bigcup_{\delta \in (\gamma, \beta)} C_\delta \in \mathscr{C}$ and again we get a contradiction. Thus (vi) holds for T_3.

Now let T_4 be the set of elements at limit levels in T_3. Then all conditions except (ii) hold. Finally, let T_5 be all elements of T_3 above a fixed root. Then all conditions hold. □

• A subset U of a linear order L is *open* iff U is a union of open intervals (a, b) or $(-\infty, a)$ or (a, ∞). Here $(-\infty, a) = \{b \in L : b < a\}$ and $(a, \infty) = \{b \in L : a < b\}$. L itself is also counted as open. (If L has at least two elements, this follows from the other parts of this definition.) Note that if L has a largest element a, then $(a, \infty) = \emptyset$; similarly for smallest elements.

• An *antichain* in a linear order L is a collection of pairwise disjoint nonempty open sets.

• A linear order L has the *countable chain condition*, abbreviated ccc, iff every antichain in L is countable.

• A subset D of a linear order L is *topologically dense* in L iff $D \cap U \neq \emptyset$ for every nonempty open subset U of L. Then dense in the sense at the beginning of the chapter implies topologically dense. In fact, if D is dense in the original sense and U is a nonempty open set, take some non-empty open interval (a, b) contained in U. There is a $d \in D$ with $a < d < b$, so $D \cap U \neq \emptyset$. If $\emptyset \neq (a, \infty) \subseteq U$ for some a, choose $b \in (a, \infty)$, and then choose $d \in D$ such that $a < d < b$. Then again $D \cap U \neq \emptyset$. Similarly if $(-\infty, a) \subseteq U$ for some a.

Conversely, if L itself is dense, then topological denseness implies dense in the order sense; this is clear. On the other hand, take for example the ordered set ω; ω itself is topologically dense in ω, but ω is not dense in ω in the order sense.

• A linear order L is *separable* iff there is a countable subset C of L which is topologically dense in L. Note that if L is separable and (a, b) is a nonempty open interval of L, then (a, b), with the order induced by L ($x < y$ for $x, y \in (a, b)$ iff $x < y$ in L) is separable. In fact, if C is countable and topologically dense in L clearly $C \cap (a, b)$ is countable and topologically dense in (a, b). Similarly, $[a, b]$ is separable, taking $(C \cap [a, b]) \cup \{a, b\}$. This remark will be used shortly.

• A *Suslin line* is a linear ordered set $(S, <)$ satisfying the following conditions:
(i) S has ccc.
(ii) S is not separable.

Theorem 9.13. *If there is a Suslin tree then there is a Suslin line.*

Proof. By Theorem 9.12 we may assume that T is normal. Take any linear order $\prec$ of T. To show that $\mathscr{B}(T, \prec)$ is ccc, suppose that $\mathscr{A}$ is an uncountable collection of nonempty pairwise disjoint open intervals in $\mathscr{B}(T, \prec)$. For each $(B, C) \in \mathscr{A}$ choose $E_{(B,C)} \in (B, C)$. Remembering that each branch has limit length, we can also select an ordinal $\alpha_{(B,C)}$ such that

$$d(B, E_{(B,C)}), d(E_{(B,C)}, C) < \alpha_{(B,C)} < \mathrm{len}(E_{(B,C)})$$

We claim that $\langle b^{E(B,C)}(\alpha_{(B,C)}) : (B, C) \in \mathscr{A}\rangle$ is a system of pairwise incomparable elements of T, which contradicts the definition of a Suslin tree. In fact, suppose that (B, C) and (B', C') are distinct elements of $\mathscr{A}$ and $b^{E(B,C)}(\alpha_{(B,C)}) \le b^{E(B',C')}(\alpha_{(B',C')})$. It follows that $\alpha_{(B,C)} \le \alpha_{(B',C')}$ and

(1) $b^{E(B,C)}(\beta) = b^{E(B',C')}(\beta)$ for all $\beta \le \alpha_{(B,C)}$.

Hence

(2) If $\beta < d(B, E_{(B,C)})$, then $\beta < \alpha_{(B,C)}$, and so $b^B(\beta) = b^{E_{(B,C)}}(\beta) = b^{E_{(B',C')}}(\beta)$.

Now recall that $d(B, E_{(B,C)}) < \alpha_{(B,C)}$. Hence

$$b^B(d(B, E_{(B,C)})) \prec b^{E_{(B,C)}}(d(B, E_{(B,C)})) = b^{E_{(B',C')}}(d(B, E_{(B,C)})),$$

and so $B < E_{(B',C')}$. Similarly, $E_{(B',C')} < C$, as follows:

(3) If $\beta < d(C, E_{(B,C)})$, then $\beta < \alpha_{(B,C)}$, and so $b^C(\beta) = b^{E_{(B,C)}}(\beta) = b^{E_{(B',C')}}(\beta)$.

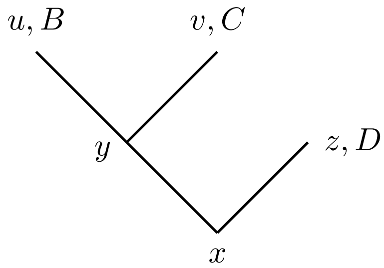
Now recall that $d(C, E_{(B,C)}) < \alpha_{(B,C)}$. Hence

$$b^C(d(C, E_{(B,C)})) \succ b^{E_{(B,C)}}(d(C, E_{(B,C)})) = b^{E_{(B',C')}}(d(C, E_{(B,C)})),$$

and so $C > E_{(B',C')}$. Hence $E_{(B',C')} \in (B, C)$. But also $E_{(B',C')} \in (B', C')$, contradiction.

To show that $\mathscr{B}(T, \prec)$ is not separable, it suffices to show that for each $\delta < \omega_1$ the set $\{B \in \mathscr{B}(T, \prec) : \mathrm{len}(B) < \delta\}$ is not dense in $\mathscr{B}(T, \prec)$. Take any $x \in T$ of height δ.

Since $\{y : y > x\}$ has elements of every level greater than δ, it cannot be a chain, as this would give a chain of size ω_1. So there exist incomparable $y, z > x$. Similarly, there exist incomparable $u, v > y$. Let B, C, D be branches containing u, v, z respectively. By symmetry say $B < C$. Illustration:



(4) $\mathrm{ht}(y) < d(B, C)$

This holds since $y \in B \cap C$.

(5) $d(B, D) \leq \mathrm{ht}(y)$ and $d(C, D) \leq \mathrm{ht}(y)$; hence $d(B, D) < d(B, C)$ and $d(C, D) < d(B, C)$.

In fact, $y \in B \backslash D$, so $d(B, D) \leq \mathrm{ht}(y)$ follows. Similarly $d(C, D) \leq \mathrm{ht}(y)$. Now the rest follows by (4).

(6) $d(B, D) = d(C, D)$.

For, if $d(B, D) < d(C, D)$, then $b^C(d(B, D)) = b^D(d(B, D)) \neq b^B(d(B, D))$, contradicting $d(B, D) < d(B, C)$, part of (5). If $d(C, D) < d(B, D)$, then $b^B(d(C, D)) = b^D(d(C, D)) \neq b^C(d(C, D))$, contradicting $d(C, D) < d(B, C)$, part of (5).

By (6) we have $B, C < D$, or $D < B, C$. Since we are assuming that $B < C$, it follows that

(7) $B < C < D$ or $D < B < C$.

Case 1. $B < C < D$. Thus (B, D) is a nonempty open interval. Suppose that there is some branch E with $\mathrm{len}(E) < \delta$ and $B < E < D$. Then $d(B, E), d(E, D) < \delta$. By Lemma 22.9 one of the following holds: $d(B, D) = d(B, E) < d(E, D)$; $d(B, D) = d(B, E) = d(E, D)$; $d(B, D) = d(E, D) < d(B, E)$. Hence $d(B, D) < \delta$. Since $x \in B \cap D$ and x has height δ, this is a contradiction.

Case 2. $D < B < C$. Thus (D, C) is a nonempty open interval. Suppose that there is some branch E with $\mathrm{len}(E) < \delta$ and $D < E < C$. Then $d(D, E), d(E, C) < \delta$. By Lemma 22.9 one of the following holds: $d(D, C) = d(D, E) < d(E, C)$; $d(D, C) = d(D, E) = d(E, C)$; $d(D, C) = d(E, D) < d(D, E)$; hence $d(D, C) < \delta$. Since $x \in C \cap D$ and x is of height δ, this is a contradiction. $\square$

Let $(L, <)$ be a linear order. We say that a linear order $(M, \prec)$ is a *completion* of L iff the following conditions hold:

(C1) $L \subseteq M$, and for any $a, b \in L$, $a < b$ iff $a \prec b$.

(C2) M is complete.

(C3) Every element of M is the lub of a set of elements of L.

(C4) If $a \in L$ is the lub in L of a subset X of L, then a is the lub of X in M.

Theorem 9.14. *Any linear order has a completion.*

Proof. Let $(L, <)$ be a linear order. We let M' be the collection of all $X \subseteq L$ such that the following conditions hold:

(1) For all $a, b \in L$, if $a < b \in X$ then $a \in X$.

(2) If X has a lub a in L, then $a \in X$.

We consider the structure $(M', \subset)$. It is clearly a partial order; we claim that it is a linear order. (Up to isomorphism it is the completion that we are after.) Suppose that $X, Y \in M'$ and $X \neq Y$; we want to show that $X \subset Y$ or $Y \subset X$. By symmetry take $a \in X \backslash Y$. Then we claim that $Y \subseteq X$ (hence $Y \subset X$). For, take any $b \in Y$. If $a < b$, then $a \in Y$ by (1), contradiction. Hence $b \leq a$, and so $b \in X$ by (1), as desired. This proves the claim.

Next we claim that $(M', \subset)$ is complete. For, suppose that $\mathscr{X} \subseteq M'$. Then $\bigcup \mathscr{X}$ satisfies (1). In fact, suppose that $c < d \in \bigcup \mathscr{X}$. Choose $X \in \mathscr{X}$ such that $d \in X$. Then $c \in X$ by (1) for X, and so $c \in \bigcup \mathscr{X}$. Now we consider two cases.

Case 1. $\bigcup \mathscr{X}$ does not have a lub in L. Then $\bigcup \mathscr{X} \in M'$, and it is clearly the lub of $\mathscr{X}$.

Case 2. $\bigcup \mathscr{X}$ has a lub in L; say a is its lub. Then

(3) $\bigcup \mathscr{X} \cup \{a\} = (-\infty, a]$.

In fact, $\subseteq$ is clear. Suppose that $b < a$. Then b is not an upper bound for $\bigcup \mathscr{X}$, so we can choose $c \in \bigcup \mathscr{X}$ such that $b < c$. Then $b \in \bigcup \mathscr{X}$ since $\bigcup \mathscr{X}$ satisfies (1). This proves (3).

Clearly $(-\infty, a] \in M'$. We claim that it is the lub of $\mathscr{X}$. Clearly it is an upper bound. Now suppose that Z is any upper bound. Then $\bigcup \mathscr{X} \subseteq Z$. If $a \notin Z$, then $\bigcup \mathscr{X} = Z$, contradicting (2) for Z. So $a \in Z$ and hence $(-\infty, a] \subseteq Z$.

Hence we have shown that $(M', \subset)$ is complete.

Now for each $a \in L$ let $f(a) = \{b \in L : b \leq a\}$. Clearly $f(a) \in M'$.

(4) For any $a, b \in L$ we have $a < b$ iff $f(a) \subset f(b)$.

For, suppose that $a, b \in L$. If $a < b$, clearly $f(a) \subseteq f(b)$, and even $f(a) \subset f(b)$ since $b \in f(b) \backslash f(a)$. The other implication in (4) follows easily from this implication by assuming that $b \leq a$.

(5) Every element of M' is a lub of elements of $f[L]$.

For, suppose that $X \in M'$, and let $\mathscr{X} = \{f(a) : a \in X\}$; we claim that X is the lub of $\mathscr{X}$. Clearly $f(a) \subseteq X$ for all $a \in X$, so X is an upper bound of $\mathscr{X}$. Suppose that $Y \in M'$ is any upper bound for $\mathscr{X}$. If $a \in X$, then $a \in f(a) \subseteq Y$, so $a \in Y$. Thus $X \subseteq Y$, as desired. So (5) holds.

(6) If $a \in L$ is the lub in L of $X \subseteq L$, then $f(a)$ is the lub in M' of $f[X]$.

For, assume that $a \in L$ is the lub in L of $X \subseteq L$. If $x \in X$, then $x \leq a$, so $f(x) \subseteq f(a)$. Thus $f(a)$ is an upper bound for $f[X]$ in M'. Now suppose that $Y \in M'$ and Y is an upper bound for $f[X]$. If $b \in L$ and $b < a$, then since a is the lub of X, there is a $d \in X$

such that $b < d \leq a$. So $f(d) \subseteq Y$, and hence $d \in Y$. Since $b < d$, we also have $b \in Y$. This shows that $f(a)\backslash\{a\} \subseteq Y$. If $a \in X$, then $f(a) \in f[X]$ and so $f(a) \subseteq Y$, as desired. Assume that $a \notin X$. Since a is the lub of X in L, there is no largest member of L which is less than a. Now suppose that $a \notin Y$. If $u \in Y$, then $u < a$, as otherwise $a \leq u$ and so $a \in Y$, contradiction. It follows that $Y = \{u \in L : u < a\}$. Clearly then a is the lub of Y. This contradicts (2). It follows that $a \in Y$. Hence $f(a) \subseteq Y$. So (6) holds.

Thus M' is as desired, up to isomorphism.

Finally, we need to take care of the "up to isomorphism" business. Non-rigorously, we just identify a with $f(a)$ for each $a \in L$. This is the way things are done in similar contexts in mathematics. Rigorously we proceed as follows; and a similar method can be used in other contexts. Let A be a set disjoint from L such that $|A| = |M'\backslash f[L]|$. For example, we could take $A = \{(L, X) : X \in M'\backslash f[L]\}$; this set is clearly of the same size as $M'\backslash f[L]$, and it is disjoint from L by the foundation axiom. Let g be a bijection from A onto $M'\backslash f[L]$. Now let $N = L \cup A$, and define $h : N \to M'$ by setting, for any $x \in N$,

$$h(x) = \begin{cases} f(x) & \text{if } x \in L, \\ g(x) & \text{if } x \in A. \end{cases}$$

Thus h is a bijection from N to M', and it extends f. We now define $x \ll y$ iff $x, y \in N$ and $h(x) \subset h(y)$. We claim that $(N, \ll)$ really is a completion of L. (Not just up to isomorphism.) We check the conditions for this. Obviously $L \subseteq N$. Suppose that $a, b \in L$. Then $a < b$ iff $f(a) \subset f(b)$ iff $h(a) \subset h(b)$ iff $a \ll b$. Now h is obviously an order-isomorphism from $(N \subset)$ onto $(M' \subset)$, so N is complete. Now take any element a of N. Then by (5), $h(a)$ is the lub of a set $f[X]$ with $X \subseteq L$. By the isomorphism property, a is the lub of X. Finally, suppose that $a \in L$ is the lub of $X \subseteq L$. Then by (6), $f(a)$ is the lub of $f[X]$ in M', i.e., $h(a)$ is the lub of $h[X]$ in M'. By the isomorphism property, a is the lub of X in N. $\square$

Theorem 9.15. *If L is a linear order and M, N are completions of L, then there is an isomorphism f of M onto N such that $f \restriction L$ is the identity.*

Proof. It suffices to show that if P is a completion of L and M', f, g, h, N are as in the proof of Theorem 9.14, then there is an isomorphism g from P onto N such that $g \restriction L$ is the identity.

For any $x \in P$ let $g'(x) = \{a \in L : a \leq_P x\}$. We claim that $g'(x) \in M'$. Clearly condition (1) holds. Now suppose that $g'(x)$ has a lub b in L. By (C4) for P, b is the lub of $g'(x)$ in P. But obviously x is the lub of $g'(x)$ in P, so $b = x \in g'(x)$. So (2) holds for $g'(x)$, and hence $g'(x) \in M'$.

Now we let $g(x) = h^{-1}(g'(x))$ for any $x \in P$. If $x \in L$, then $g'(x) = f(x) = h(x)$, and hence $g(x) = x$.

If $x <_P y$, clearly $g'(x) \subseteq g'(y)$, and hence $g(x) \leq_N g(y)$. By (C3) for P and y, there is an $a \in L$ such that $x <_P a \leq_P y$. So $a \in g'(y)\backslash g'(x)$. Hence $g'(x) \subset g'(y)$ and so $g(x) <_N g(y)$. Thus $\forall x, y \in P[x <_P y \to g(x) <_N g(y)]$. Hence $x \not<_P y$ iff $y \leq_P x$ iff $g(y) \leq_N g(x)$ iff $g(x) \not<_N g(y)$. So $\forall x, y \in P[x <_P y \leftrightarrow g(x) <_N g(y)]$.

It remains only to show that g is a surjection. Let $x \in N$. Set $y = \sup_P h(x)$. If $a \in h(x)$, then $a \leq_P y$ and so $a \in g'(y)$. Thus $h(x) \subseteq g'(y)$. Now suppose that $a \in g'(y)$.

So $a \leq_P y$. If $a <_P y$, then there is a $z \in h(y)$ such that $a <_P z \leq_P y$. It follows that $a \in h(y)$. If $a = y$, then $a \in h(x)$ by (2). So $g'(y) \subseteq h(y)$, showing that $g'(y) = h(x)$. Hence $g(y) = h^{-1}(g'(y)) = x$. $\square$

Corollary 9.16. *Suppose that L is a dense linear order, and M is a linear order. Then the following conditions are equivalent:*

(i) M is a completion of L.

(ii) (a) $L \subseteq M$

(b) M is complete.

(c) For any $a, b \in L$, $a <_L b$ iff $a <_M b$.

(d) For any $x, y \in M$, if $x <_M y$ then there is an $a \in L$ such that $x <_M a <_M y$.

Proof. (i)⇒(ii): Assume that M is a completion of L. then (a)–(c) are clear. Suppose that $x, y \in M$ and $x <_M y$. By (C3), choose $b \in L$ such that $x <_M b \leq_M y$. If $x \in L$, then choose $a \in L$ such that $x <_L a <_L b$; so $x <_M a <_M y$, as desired. Assume that $x \notin L$. Then by (C4), b is not the lub in L of $\{u \in L : u <_M x\}$, so there is some $a \in L$ such that $a <_L b$ and a is an upper bound of $\{u \in L : u <_M x\}$. Since by (C3) x is the lub of $\{u \in L : u <_M x\}$, it follows that $x <_M a <_M b \leq_M y$, as desired.

(ii)⇒(i): Assume (ii). Then (C1) and (C2) are clear. For (C3), let $x \in M$, and let $X = \{a \in L : a < x\}$. Then x is an upper bound for X, and (ii)(d) clearly implies that it is the lub of X. For (C4), suppose that $a \in L$ is the lub in L of a set X of elements of L. Suppose that $x \in M$ is an upper bound for X and $x < a$. Then by (ii)(d) there is an element $b \in L$ such that $x < b < a$. Then there is an element $c \in X$ such that $b < c \leq a$. It follows that $c \leq x$, contradiction. $\square$

Theorem 9.17. *The following conditions are equivalent:*

(i) There is a Suslin line.

(ii) There is a linearly ordered set $(L, <)$ satisfying the following conditions:

(a) L has no first or last elements.

(b) L is dense.

(c) Every nonempty subset of L which is bounded above has a least upper bound.

(d) No nonempty open subset of L is separable.

(e) L is ccc.

Proof. Obviously (ii) implies (i). Now suppose that (i) holds, and let S be a Suslin line. We obtain (ii) in two steps: first taking care of denseness, and then taking the completion to finish up.

We define a relation $\sim$ on S as follows: for any $a, b \in S$,

$$\begin{aligned} a \sim b \quad \text{iff} \quad & a = b, \\ & \text{or } a < b \text{ and } [a, b] \text{ is separable}, \\ & \text{or } b < a \text{ and } [b, a] \text{ is separable}. \end{aligned}$$

Clearly $\sim$ is an equivalence relation on S. Let L be the collection of all equivalence classes under $\sim$.

(1) If $I \in L$, then I is convex, i.e., if $a < c < b$ with $a, b \in I$, then also $c \in I$.

For, $[a, b]$ is separable, so $[a, c]$ is separable too, and hence $a \sim c$; so $c \in I$.

(2) If $I \in L$, then I is separable.

For, this is clear if I has only one or two elements. Suppose that I has at least three elements. Then there exist $a, b \in I$ with $a < b$ and $(a, b) \neq \emptyset$. Let $\mathscr{M}$ be a maximal pairwise disjoint set of such intervals. Then $\mathscr{M}$ is countable. Say $\mathscr{M} = \{(x_n, y_n) : n \in \omega\}$. Since $x_n \sim y_n$, the interval $[x_n, y_n]$ is separable, so we can let D_n be a countable dense subset of it. We claim that the following countable set E is dense in I:

$$\begin{aligned} E = \bigcup_{n\in\omega} D_n &\cup \{e : e \text{ is the largest element of } I\} \\ &\cup \{a : a \text{ is the smallest element of } I\}. \end{aligned}$$

Thus e and a are added only if they exist. To show that E is dense in I, first suppose that $a, b \in I$, $a < b$, and $(a, b) \neq \emptyset$. Then by the maximality of $\mathscr{M}$, there is an $n \in \omega$ such that $(a, b) \cap (x_n, y_n) \neq \emptyset$. Choose $c \in (a, b) \cap (x_n, y_n)$. Then $\max(a, x_n) < c < \min(b, y_n)$, so there is a $d \in D_n \cap (\max(a, x_n), \min(b, y_n)) \subseteq (a, b)$, as desired. Second, suppose that $a \in I$ and $(a, \infty) \neq \emptyset$; here $(a, \infty) = \{x \in I : a < x\}$. We want to find $d \in E$ with $a < d$. If I has a largest element e, then e is as desired. Otherwise, there are $b, c \in I$ with $a < b < c$, and then an element of $(a, c) \cap E$, already shown to exist, is as desired. Similarly one deals with $-\infty$. Thus we have proved (2).

Now we define a relation $<$ on L by setting $I < J$ iff $I \neq J$ and $a < b$ for some $a \in I$ and $b \in J$. By (1) this is equivalent to saying that $I < J$ iff $I \neq J$ and $a < b$ for all $a \in I$ and $b \in J$. In fact, suppose that $a \in I$ and $b \in J$ and $a < b$, and also $c \in I$ and $d \in J$, while $d \leq c$. If $d \leq a$, then $d \leq a < b$ with $d, b \in J$ implies that $a \in J$, contradiction. Hence $a < d$. Since also $d \leq c$ this gives $d \in I$, contradiction.

Clearly $<$ makes L into a simply ordered set. Except for not being complete in the sense of (c), L is close to the linear order we want.

To see that L is dense, suppose that $I < J$ but $(I, J) = \emptyset$. Take any $a \in I$ and $b \in J$. Then $(a, b) \subseteq I \cup J$, and $I \cup J$ is separable by (2), so $a \sim b$, contradiction.

For (d), by a remark in the definition of separable it suffices to show that no open interval (I, J) is separable. Suppose to the contrary that (I, J) is separable. Let $\mathscr{A}$ be a countable dense subset of (I, J). Also, let $\mathscr{B} = \{K \in L : I < K < J \text{ and } |K| > 2\}$. Any two distinct members of $\mathscr{B}$ are disjoint, and hence by ccc $\mathscr{B}$ is countable. In fact, each $K \in \mathscr{B}$ has the form (a, b), $[a, b)$, $(a, b]$, or $[a, b]$. since $|K| > 2$, and in each case the open interval (a, b) is nonempty. So ccc applies.

Define $\mathscr{C} = \mathscr{A} \cup \mathscr{B} \cup \{I, J\}$. By (2), each member of $\mathscr{C}$ is separable, so for each $K \in \mathscr{C}$ we can let D_K be a countable dense subset of K. Let $E = \bigcup_{K\in\mathscr{C}} D_K$. So E is a countable set. Fix $a \in I$ and $b \in J$. We claim that $E \cap (a, b)$ is dense in (a, b). (Hence $a \sim b$ and so $I = J$, contradiction.) For, suppose that $a \leq c < d \leq b$ with $(c, d) \neq \emptyset$.

Case 1. $[c]_\sim = [d]_\sim = I$. Then $D_I \cap (c, d) \neq \emptyset$, so $E \cap (c, d) \neq \emptyset$, as desired.

Case 2. $[c]_\sim = [d]_\sim = J$. Similarly.

Case 3. $I < [c]_\sim = [d]_\sim < J$. Then $[c]_\sim \in \mathscr{B} \subseteq \mathscr{C}$, so the desired result follows again.

Case 4. $[c]_\sim < [d]_\sim$. Choose $K \in \mathscr{A}$ such that $[c]_\sim < K < [d]_\sim$. Hence $c < e < d$ for any $e \in D_K$, as desired.

Thus we have obtained a contradiction, which proves that (I, J) is not separable.

Next, we claim that L has ccc. In fact, suppose that $\mathscr{A}$ is an uncountable family of pairwise disjoint open intervals. Let $\mathscr{B}$ be the collection of all endpoints of members of $\mathscr{A}$, and for each $I \in \mathscr{B}$ choose $a_I \in I$. Then

$$\{(a_I, a_J) : (I, J) \in \mathscr{A}\}$$

is an uncountable collection of pairwise disjoint nonempty open intervals in S, contradiction. In fact, given $(I, J) \in \mathscr{A}$, choose K with $I < K < J$. then $a_K \in (a_I, a_J)$. So $(a_I, a_J) \neq \emptyset$. Suppose that $(I, J), (I', J')$ are distinct members of $\mathscr{A}$. Wlog $J \leq I'$. Then $a_J \leq a_{I'}$, and it follows that $(a_I, a_J) \cap (a_{I'}, a_{J'}) = \emptyset$.

This finishes the first part of the proof. We have verified that L satisfies (b), (d), and (e). Now let M be the completion of L, and let N be M without its first and last elements. We claim that N finally satisfies all of the conditions in (ii). Clearly N is dense, it has no first or last elements, and every nonempty subset of it bounded above has a least upper bound. Next, suppose that $a < b$ in N and C is a countable subset of (a, b) which is dense in (a, b). Choose $c, d \in L$ such that $a < c < d < b$. For any $u, v \in C$ with $c < u < v < d$ choose $e_{uv} \in L$ such that $u < e_{uv} < v$; such an element exists by Corollary 21.15. We claim that $\{e_{uv} : u, v \in C,\ u < v\}$ is dense in (c, d) in L, which is a contradiction. For, given x, y such that $c < x < y < d$ in L, by the definition of denseness we can find $u, v \in C$ such that $x < u < v < y$; and then $x < e_{uv} < y$, as desired.

It remains only to prove that N has ccc. Suppose that $\mathscr{A}$ is an uncountable collection of nonempty open intervals of N. By Corollary 21.15, for each $(a, b) \in \mathscr{A}$ we can find $c, d \in L$ such that $a < c < d < b$. So this gives an uncountable collection of nonempty open intervals in L, contradiction. $\square$

Theorem 9.18. *If there is a Suslin line, then there is a Suslin tree.*

Proof. Assume that there is a Suslin line. Then by Theorem 9.17 we may assume that we have a linear order L satisfying the following conditions:

(1) L is dense, with no first or last elements.

(2) No nonempty open subset of L is separable.

(3) L is ccc.

Now we define by recursion elements a_α, b_α of L, for $\alpha < \omega_1$. If these have already been defined for all $\beta < \alpha$, then the set $\{a_\beta : \beta < \alpha\} \cup \{b_\beta : \beta < \alpha\}$ is countable, and hence by (2) it is not dense in L. Let (c, d) be an open interval disjoint from this set, and pick a_α, b_α so that $c < a_\alpha < b_\alpha < d$ Thus for any $\xi < \alpha$ one of these conditions holds:
$a_\xi < a_\alpha < b_\alpha < b_\xi$;
$a_\alpha < b_\alpha < a_\xi < b_\xi$;
$a_\xi < b_\xi < a_\alpha < b_\alpha$.

Hence

$$(1) \qquad \forall \xi, \alpha < \omega_1 [\xi < \alpha \to [[a_\alpha, b_\alpha] \subseteq (a_\xi, b_\xi) \text{ or } [a_\alpha, b_\alpha] \cap (a_\xi, b_\xi) = \emptyset].$$

Now we define a relation $\prec$ on ω_1 as follows: for any $\xi, \alpha < \omega_1$,

$$\xi \prec \alpha \text{ iff } \xi < \alpha \text{ and } [a_\alpha, b_\alpha] \subseteq (a_\xi, b_\xi).$$

If $\xi \prec \eta \prec \alpha$, then $\xi < \eta < \alpha$, hence $\xi < \alpha$, $[a_\eta, b_\eta] \subseteq (a_\xi, b_\xi)$, and $[a_\alpha, b_\beta] \subseteq (a_\eta, b_\eta)$, hence $[a_\alpha, b_\beta] \subseteq (a_\xi, b_\xi)$; so $\xi \prec \alpha$. Thus $\prec$ is transitive. Clearly it is irreflexive. So $\prec$ is a partial order on ω_1.

Now suppose that $\xi \prec \alpha$, $\eta \prec \alpha$, and $\xi \neq \eta$. We show that $\xi \prec \eta$ or $\eta \prec \xi$; hence $(\omega_1, \prec)$ is a tree. Wlog $\xi < \eta$. Now $[a_\alpha, b_\alpha] \subseteq (a_\xi, b_\xi) \cap (a_\eta, b_\eta)$, so $(a_\xi, b_\xi) \cap (a_\eta, b_\eta) \neq \emptyset$, so by (1) $[a_\eta, b_\eta] \subseteq (a_\xi, b_\xi)$. Thus $\xi \prec \eta$, as desired.

Now suppose that $\langle \alpha(\xi) < \xi < \omega_1 \rangle$ is $\prec$-increasing. Thus $\langle \alpha(\xi) < \xi < \omega_1 \rangle$ is $<$-increasing and $\forall \xi, \eta < \omega_1 [\xi < \eta \to [[a_{\alpha(\eta)}, b_{\alpha(\eta)}] \subseteq (a_{\alpha(\xi)}, b_{\alpha(\xi)})$. Then

$$\langle (a_{\alpha(\xi)}, b_{\alpha(\xi)}) \backslash [a_{\alpha(\xi+1)}, b_{\alpha(\xi+1)}] : \xi < \omega_1 \rangle$$

is a system of ω_1 pairwise disjoint open sets in L, contradiction.

Finally, if $\langle \alpha(\xi) : \xi < \omega_1 \rangle$ is a system of pairwise incomparable elements under $\prec$, then by (1), $\langle (a_{\alpha(\xi)}, b_{\alpha(\xi)}) : \sigma < \omega_1 \rangle$ is a system of pairwise disjoint open intervals in L, contradiction. $\square$

An *Aronszajn tree* is a tree of height ω_1 with all levels countable and no uncountable branches.

Theorem 9.19. *There is an Aronszajn tree.*

Proof. We start with the tree

$$T = \{s \in {}^{<\omega_1}\omega : s \text{ is one-one}\}.$$

under $\subset$. This tree clearly does not have a chain of size ω_1. But all of its infinite levels are uncountable, so it is not an ω_1-Aronszajn tree. We will define a subset of it that is the desired tree. We define a system $\langle S_\alpha : \alpha < \omega_1 \rangle$ of subsets of T by recursion; these will be the levels in the new tree.

Let $S_0 = \{\emptyset\}$. Now suppose that $\alpha > 0$ and S_β has been constructed for all $\beta < \alpha$ so that the following conditions hold for all $\beta < \alpha$:

(1_β) $S_\beta \subseteq {}^\beta\omega \cap T$.

(2_β) $\omega \backslash \text{rng}(s)$ is infinite, for every $s \in S_\beta$.

(3_β) For all $\gamma < \beta$, if $s \in S_\gamma$, then there is a $t \in S_\beta$ such that $s \subset t$.

(4_β) $|S_\beta| \leq \omega$.

(5_β) If $s \in S_\beta$, $t \in T$, and $\{\gamma < \beta : s(\gamma) \neq t(\gamma)\}$ is finite, then $t \in S_\beta$.

(6_β) If $s \in S_\beta$ and $\gamma < \beta$, then $s \restriction \gamma \in S_\gamma$.

(Vacuously these conditions hold for all $\beta < 0$.) If α is a successor ordinal $\varepsilon + 1$, we simply take

$$S_\alpha = \{s \cup \{(\varepsilon, n)\} : s \in S_\varepsilon \text{ and } n \notin \mathrm{rng}(s)\}.$$

Clearly (1_β)–(6_β) hold for all $\beta < \alpha + 1$.

Now suppose that α is a limit ordinal less than ω_1 and (1_β)–(6_β) hold for all $\beta < \alpha$. Since α is a countable limit ordinal, it follows that $\mathrm{cf}(\alpha) = \omega$. Let $\langle \delta_n : n \in \omega \rangle$ be a strictly increasing sequence of ordinals with supremum α. Now let $U = \bigcup_{\beta < \alpha} S_\beta$. Take any $s \in U$; we want to define an element $t_s \in {}^\alpha\omega \cap T$ which extends s. Let $\beta = \mathrm{dmn}(s)$.

Choose n minimum such that $\beta \leq \delta_n$. Now we define a sequence $\langle u_i : i \in \omega \rangle$ of members of U; u_i will be a member of $S_{\delta_{n+i}}$. By (3_{δ_n}), let u_0 be a member of S_{δ_n} such that $s \subseteq u_0$. Having defined a member u_i of $S_{\delta_{n+i}}$, use $(3_{\delta_{n+i+1}})$ to get a member u_{i+1} of $S_{\delta_{n+i+1}}$ such that $u_i \subseteq u_{i+1}$. This finishes the construction. Let $v = \bigcup_{i\in\omega} u_i$. Thus $s \subseteq v \in {}^\alpha\omega \cap T$. Unfortunately, condition (2) may not hold for v, so this is not quite the element t_s that we are after. We define $t_s \in {}^\alpha\omega$ as follows. Let $\gamma < \alpha$. Then

$$t_s(\gamma) = \begin{cases} v(\delta_{2n+2i}) & \text{if } \gamma = \delta_{n+i} \text{ for some } i \in \omega, \\ v(\gamma) & \text{if } \gamma \notin \{\delta_{n+i} : i \in \omega\}. \end{cases}$$

Clearly $t_s \in {}^\alpha\omega \cap T$. Since $v(\delta_{2n+2i+1}) \notin \mathrm{rng}(t_s)$ for all $i \in \omega$, it follows that $\omega \backslash \mathrm{rng}(t_s)$ is infinite.

We now define

$$S_\alpha = \bigcup_{s \in U} \{w \in {}^\alpha\omega \cap T : \{\varepsilon < \alpha : w(\varepsilon) \neq t_s(\varepsilon)\} \text{ is finite}\}.$$

Now we want to check that (1_α)–(6_α) hold. Conditions (1_α) and (3_α) are very clear. For (2_α), suppose that $w \in S_\alpha$. Then $w \in {}^\alpha\omega \cap T$ and there is an $s \in U$ such that $\{\varepsilon < \alpha : w(\varepsilon) \neq t_s(\varepsilon)\}$ is finite. Since $\omega\backslash\mathrm{rng}(t_s)$ is infinite, clearly $\omega\backslash\mathrm{rng}(w)$ is infinite. For (4_α), note that U is countable by the assumption that (4_β) holds for every $\beta < \alpha$, while for each $s \in U$ the set

$$\{w \in {}^\alpha\omega \cap T : \{\varepsilon < \alpha : w(\varepsilon) \neq t_s(\varepsilon)\} \text{ is finite}\}$$

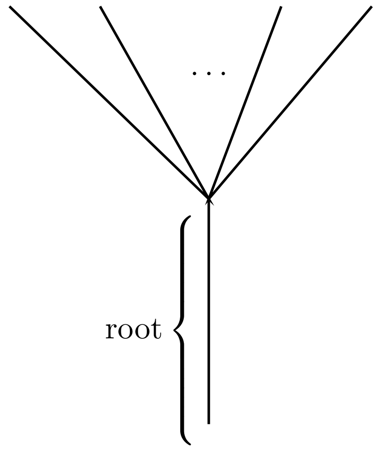
is also countable. So (4_α) holds. For (5_α), suppose that $w \in S_\alpha$, $x \in T$, and $\{\gamma < \alpha : w(\gamma) \neq x(\gamma)\}$ is finite. Choose $s \in U$ such that $\{\varepsilon < \alpha : w(\varepsilon) \neq t_s(\varepsilon)\}$ is finite. Then of course also $\{\varepsilon < \alpha : x(\varepsilon) \neq t_s(\varepsilon)\}$ is finite. So $x \in S_\alpha$, and (5_α) holds. Finally, for (6_α), suppose that $w \in S_\alpha$ and $\gamma < \alpha$; we want to show that $w \restriction \gamma \in S_\gamma$. Choose $s \in U$ such that $\{\varepsilon < \alpha : w(\varepsilon) \neq t_s(\varepsilon)\}$ is finite. Assume the notation introduced above when defining t_s. Choose $i \in \omega$ such that $\gamma \leq \delta_{n+i}$. Then

$$\begin{aligned} \{\varepsilon < \delta_{n+i} : w(\varepsilon) \neq u_i(\varepsilon)\} &= \{\varepsilon < \delta_{n+i} : w(\varepsilon) \neq v(\varepsilon)\} \\ &\subseteq \{\varepsilon < \delta_{n+i} : w(\varepsilon) \neq t_s(\varepsilon)\} \cup \{\delta_{n+j} : j < i\}, \end{aligned}$$

and the last union is clearly finite. It follows from $(5_{\delta_{n+1}})$ that $w \in S_\gamma$. So (6_α) holds.

This finishes the construction. Clearly $\bigcup_{\alpha<\omega_1} S_\alpha$ is the desired Aronszajn tree. $\square$

A collection $\mathscr{A}$ of sets forms a Δ-*system* iff there is a set r (called the *root* or *kernel* of the Δ-system) such that $A \cap B = r$ for any two distinct $A, B \in \mathscr{A}$. This is illustrated as follows:



Theorem 9.20. (Δ-system theorem) *If κ is an uncountable regular cardinal and $\mathscr{A}$ is a collection of finite sets with $|\mathscr{A}| \geq \kappa$, then there is a $\mathscr{B} \in [\mathscr{A}]^\kappa$ such that $\mathscr{B}$ is a Δ-system.*

Proof. First we prove the following special case of the theorem.

$(*)$ If $\mathscr{A}$ is a collection of finite sets each of size $m \in \omega$, with $|\mathscr{A}| = \kappa$, then there is a $\mathscr{B} \in [\mathscr{A}]^\kappa$ such that $\mathscr{B}$ is a Δ-system.

We prove this by induction on m. The hypothesis implies that $m > 0$. If $m = 1$, then each member of $\mathscr{A}$ is a singleton, and so $\mathscr{A}$ is a collection of pairwise disjoint sets; hence it is a Δ-system with root $\emptyset$. Now assume that $(*)$ holds for m, and suppose that $\mathscr{A}$ is a collection of finite sets each of size $m+1$, with $|\mathscr{A}| = \kappa$, and with $m > 0$. We consider two cases.

Case 1. There is an element x such that $\mathscr{C} \stackrel{\text{def}}{=} \{A \in \mathscr{A} : x \in A\}$ has size κ. Let $\mathscr{D} = \{A \backslash \{x\} : A \in \mathscr{C}\}$. Then $\mathscr{D}$ is a collection of finite sets each of size m, and $|\mathscr{D}| = \kappa$. Hence by the inductive assumption there is an $\mathscr{E} \in [\mathscr{D}]^\kappa$ which is a Δ-system, say with kernel r. Then $\{A \cup \{x\} : A \in E\} \in [\mathscr{A}]^\kappa$ and it is a Δ-system with kernel $r \cup \{x\}$.

Case 2. Case 1 does not hold. Let $\langle A_\alpha : \alpha < \kappa \rangle$ be a one-one enumeration of $\mathscr{A}$. Then from the assumption that Case 1 does not hold we get:

$(**)$ For every x, the set $\{\alpha < \kappa : x \in A_\alpha\}$ has size less than κ.

We now define a sequence $\langle \alpha(\beta) : \beta < \kappa \rangle$ of ordinals less than κ by recursion. Suppose that $\alpha(\beta)$ has been defined for all $\beta < \gamma$, where $\gamma < \kappa$. Then $\Gamma \stackrel{\text{def}}{=} \bigcup_{\beta<\gamma} A_{\alpha(\beta)}$ has size less than κ, and so by $(**)$, so does the set

$$\bigcup_{x \in \Gamma} \{\delta < \kappa : x \in A_\delta\}.$$

Thus we can choose $\alpha(\gamma) < \kappa$ such that for all $x \in \Gamma$ we have $x \notin A_{\alpha(\gamma)}$. This implies that $A_{\alpha(\gamma)} \cap A_{\alpha(\beta)} = \emptyset$ for all $\beta < \gamma$. Thus we have produced a pairwise disjoint system $\langle A_{\alpha(\beta)} : \beta < \kappa \rangle$, as desired. (The root is $\emptyset$ again.)

This finishes the inductive proof of $(*)$

Now the theorem itself is proved as follows. Let $\mathscr{A}'$ be a subset of $\mathscr{A}$ of size κ. Then

$$\mathscr{A}' = \bigcup_{m \in \omega} \{A \in \mathscr{A}' : |A| = m\}.$$

Hence there is an $m \in \omega$ such that $\{A \in \mathscr{A}' : |A| = m\}$ has size κ. So $(*)$ applies to give the desired conclusion. $\square$

Theorem 9.21. (general Δ-system theorem) *Suppose that κ and λ are cardinals, $\omega \leq \kappa < \lambda$, λ is regular, and for all $\alpha < \lambda$, $|[\alpha]^{<\kappa}| < \lambda$. Suppose that $\mathscr{A}$ is a collection of sets, with each $A \in \mathscr{A}$ of size less than κ, and with $|\mathscr{A}| \geq \lambda$. Then there is a $\mathscr{B} \in [\mathscr{A}]^\lambda$ which is a Δ-system.*

Proof.

(1) There is a regular cardinal μ such that $\kappa \leq \mu < \lambda$.

In fact, if κ is regular, we may take $\mu = \kappa$. If κ is singular, then $\kappa^+ \leq |[\kappa]^{<\kappa}| < \lambda$, so we may take $\mu = \kappa^+$.

We take μ as in (1). Let $S = \{\alpha < \lambda : \alpha \text{ is a limit ordinal and } \mathrm{cf}(\alpha) = \mu\}$. Then S is a stationary subset of λ.

Let $\mathscr{A}_0$ be a subset of $\mathscr{A}$ of size λ. Now $\left|\bigcup_{A \in \mathscr{A}_0} A\right| \leq \lambda$ since $\kappa < \lambda$. Let a be an injection of $\bigcup_{A \in \mathscr{A}_0} A$ into λ, and let A be a bijection of λ onto $\mathscr{A}_0$. Set $b_\alpha = a[A_\alpha]$ for each $\alpha < \lambda$. Now if $\alpha \in S$, then $|b_\alpha \cap \alpha| \leq |b_\alpha| = |A_\alpha| < \kappa \leq \mu = \mathrm{cf}(\alpha)$, so there is an ordinal $g(\alpha)$ such that $\sup(b_\alpha \cap \alpha) < g(\alpha) < \alpha$. Thus g is a regressive function on S. By Jech Theorem 8,7, there exist a stationary $S' \subseteq S$ and a $\beta < \lambda$ such that $g[S'] = \{\beta\}$. For each $\alpha \in S'$ let $F(\alpha) = b_\alpha \cap \alpha$. Thus $F(\alpha) \in [\beta]^{<\kappa}$, and $|[\beta]^{<\kappa}| < \lambda$, so there exist an $S'' \in [S']^\lambda$ and a $B \in [\beta]^{<\kappa}$ such that $b_\alpha \cap \alpha = B$ for all $\alpha \in S''$.

Now we define $\langle \alpha_\xi : \xi < \lambda \rangle$ by recursion. For any $\xi < \lambda$, α_ξ is a member of S'' such that

(2) $\alpha_\eta < \alpha_\xi$ for all $\eta < \xi$, and
(3) $\delta < \alpha_\xi$ for all $\delta \in \bigcup_{\eta < \xi} b_{\alpha_\eta}$.

Since $\left|\bigcup_{\eta < \xi} b_{\alpha_\eta}\right| < \lambda$, this is possible by the regularity of λ.

Now let $\mathscr{A}_1 = A[\{\alpha_\xi : \xi < \lambda\}]$ and $r = a^{-1}[B]$. We claim that $C \cap D = r$ for distinct $C, D \in \mathscr{A}_1$. For, write $C = A_{\alpha_\xi}$ and $D = A_{\alpha_\eta}$. Without loss of generality, $\eta < \xi$. Suppose that $x \in r$. Thus $a(x) \in B \subseteq b_{\alpha_\xi}$, so by the definition of b_{α_ξ} we have $x \in A_{\alpha_\xi} = C$. Similarly $x \in D$. Conversely, suppose that $x \in C \cap D$. Thus $x \in A_{\alpha_\xi} \cap A_{\alpha_\eta}$, and hence $a(x) \in b_{\alpha_\xi} \cap b_{\alpha_\eta}$. By the definition of α_ξ, since $a(x) \in b_{\alpha_\eta}$ we have $a(x) < \alpha_\xi$. So $a(x) \in b_{\alpha_\xi} \cap \alpha_\xi = B$, and hence $x \in r$.

Clearly $|\mathscr{A}_1| = \lambda$. $\square$

$A, B \in [\omega]^\omega$ are *almost disjoint* iff $A \cap B$ is finite.

Theorem 9.22. *There is a family of 2^ω pairwise almost disjoint infinite sets of natural numbers.*

Proof. Let $X = \bigcup_{n\in\omega} {}^n2$. Then $|X| = \omega$, since X is clearly infinite, while

$$|X| \le \sum_{n\in\omega} 2^n \le \omega \cdot \omega = \omega.$$

Let f be a bijection from ω onto X. Then for each $g \in {}^\omega 2$ let $x_g = \{g \restriction n : n \in \omega\}$. So x_g is an infinite subset of X. If $g, h \in {}^\omega 2$ and $g \ne h$, choose n so that $g(n) \ne h(n)$. Then clearly $x_g \cap x_h \subseteq \{g \restriction i : i \le n\}$, and so this intersection is finite. Thus we have produced 2^ω pairwise almost disjoint infinite subsets of X. That carries over to ω. Namely, $\{f^{-1}[x_g] : g \in {}^\omega 2\}$ is a family of 2^ω pairwise almost disjoint infinite subsets of ω, as is easily checked. $\square$

For κ a regular cardinal, functions $f, g \in {}^\kappa\kappa$ are *almost disjoint* iff $|\{\alpha < \kappa : f(\alpha) = g(\alpha)\}| < \kappa$.

Proposition 9.23. *If κ is a regular cardinal, there is an almost disjoint family of κ^+ members of ${}^\kappa\kappa$.*

Proof. For each $\xi < \kappa$ define $f^\xi \in {}^k\kappa$ by $f^\xi(\alpha) = \xi$ for all $\alpha < \kappa$. Clearly $\{f^\xi : \xi < \kappa\}$ is an almost disjoint family.

Now suppose that $\{g_\nu : n < \kappa\}$ is an almost disjoint family. It suffices to find a function h almost disjoint with each g_ν. For each $\alpha < \kappa$ choose $h(\alpha) \in \kappa\backslash\{g_\nu(\alpha) : \nu < \alpha\}$. $\square$

A tree T is a *Kurepa tree* iff T has height ω_1, each level is countable, and T has more than ω_1 branches.

Lemma 9.24. *There is a Kurepa tree iff there is a set $\mathscr{F} \subseteq \mathscr{P}(\omega_1)$ such that $|\mathscr{F}| \ge \omega_1^+$ and $\forall\alpha < \omega_1[|\{X \cap \alpha : X \in \mathscr{F}\}| < \omega_1]$.*

Proof. $\Leftarrow$: suppose that $\mathscr{F}$ is as indicated. Let

$$T = \bigcup_{\alpha<\omega_1} \{\chi_{X\cap\alpha} : X \in \mathscr{F}\},$$

where $\chi_{X\cap\alpha}$ is the characteristic function of $X \cap \alpha$ within α. Clearly this gives a Kurepa tree.

$\Rightarrow$: Assume that T is a Kurepa tree. Now $|T| = \omega_1$, so we may assume that $T = \omega_1$. Let $\mathscr{F}$ be the set of all branches of length ω_1 in T; $|\mathscr{F}| \ge \omega_1^+$. For each $\alpha < \omega_1$ fix $\delta < \omega_1$ so that $\alpha \subseteq T_\delta$. Now for each $X \in \mathscr{F}$ let $\langle x^X_\xi : \xi < \omega_1\rangle$ enumerate X in increasing order. Define $f(x^X_\delta) = \{x^X_\beta : \beta < \delta\} = X \cap T_\delta$. Since $\{x^X_\delta : X \in \mathscr{F}\}$ is a subset of Lev_δ, which has size less than ω_1, it follows that $\{X \cap T_\delta : X \in \mathscr{F}\}$ has size less than ω_1. For each $Y \in \{X \cap T_\delta : X \in \mathscr{F}\}$ let $g(y) = Y \cap \alpha$. Then g maps $\{X \cap T_\delta : X \in \mathscr{F}\}$ onto $\{X \cap \alpha : X \in \mathscr{F}\}$. Hence $|\{X \cap \alpha : X \in \mathscr{F}\}| \le |\{X \cap T_\delta : X \in \mathscr{F}\}| < \omega_1$. $\square$

- A cardinal κ has the *tree property* iff every κ-tree has a chain of size κ.

Equivalently, κ has the tree property iff there is no κ-Aronszajn tree.

• A cardinal κ has the *linear order property* iff every linear order $(L, <)$ of size κ has a subset with order type κ or κ^* under $<$.

Lemma 9.25. *For any regular cardinal κ, the linear order property implies the tree property.*

Proof. Assume the linear order property, and let $(T, <)$ be a κ-tree. For each $x \in T$ and each $\alpha \le \mathrm{ht}(x, T)$ let x^α be the element of height α below x. Thus x^0 is the root which is below x, and $x^{\mathrm{ht}(x)} = x$. For each $x \in T$, let $T \restriction x = \{y \in T : y < x\}$. If x, y are incomparable elements of T, then let $\chi(x, y)$ be the smallest ordinal $\alpha \le \min(\mathrm{ht}(x), \mathrm{ht}(y))$ such that $x^\alpha \ne y^\alpha$. Let $<'$ be a well-order of T. Then we define, for any distinct $x, y \in T$,

$$x <'' y \quad \text{iff} \quad x < y, \text{ or } x \text{ and } y \text{ are incomparable and } x^{\chi(x,y)} <' y^{\chi(x,y)}.$$

We claim that this gives a linear order of T. To prove transitivity, suppose that $x <'' y <'' z$. Then there are several possibilities. These are illustrated in diagrams below.

Case 1. $x < y < z$. Then $x < z$, so $x <'' z$.

Case 2. $x < y$, while y and z are incomparable, with $y^{\chi(y,z)} <' z^{\chi(y,z)}$.

Subcase 2.1. $\mathrm{ht}(x) < \chi(y, z)$. Then $x = x^{\mathrm{ht}(x)} = y^{\mathrm{ht}(x)} = z^{\mathrm{ht}(x)}$ so that $x < z$, hence $x <'' z$.

Subcase 2.2. $\chi(y, z) \le \mathrm{ht}(x)$. Then x and z are incomparable. In fact, if $z < x$ then $z < y$, contradicting the assumption that y and z are incomparable; if $x \le z$, then $y^{\mathrm{ht}(x)} = x = x^{\mathrm{ht}(x)} = z^{\mathrm{ht}(x)}$, contradiction. Now if $\alpha < \chi(x, z)$ then $y^\alpha = x^\alpha = z^\alpha$; it follows that $\chi(x, z) \le \chi(y, z)$. If $\alpha < \chi(y, z)$ then $\alpha \le \mathrm{ht}(x)$, and hence $x^\alpha = y^\alpha = z^\alpha$; this shows that $\chi(y, z) \le \chi(x, z)$. So $\chi(y, z) = \chi(x, z)$. Hence $x^{\chi(x,z)} = y^{\chi(x,z)} = y^{\chi(y,z)} <' z^{\chi(y,z)} = z^{\chi(x,z)}$, and hence $x <'' z$.

Case 3. x and y are incomparable, and $y < z$. Then x and z are incomparable. Now if $\alpha < \chi(x, y)$, then $x^\alpha = y^\alpha = z^\alpha$; this shows that $\chi(x, y) \le \chi(x, z)$. Also, $x^{\chi(x,y)} <' y^{\chi(x,y)} = z^{\chi(x,y)}$, and this implies that $\chi(x, z) \le \chi(x, y)$. So $\chi(x, y) = \chi(x, z)$. It follows that $x^{\chi(x,z)} = x^{\chi(x,y)} <' y^{\chi(x,y)} = z^{\chi(x,z)}$, and hence $x <'' z$.

Case 4. x and y are incomparable, and also y and z are incomparable. We consider subcases.

Subcase 4.1. $\chi(y, z) < \chi(x, y)$. Now if $\alpha < \chi(y, z)$, then $x^\alpha = y^\alpha = z^\alpha$; so $\chi(y, z) \le \chi(x, z)$. Also, $x^{\chi(y,z)} = y^{\chi(y,z)} <' z^{\chi(y,z)}$, so that $\chi(x, z) \le \chi(y, z)$. Hence $\chi(x, z) = \chi(y, z)$, and $x^{\chi(x,z)} = y^{\chi(y,z)} <' z^{\chi(y,z)}$, and hence $x <'' z$.

Subcase 4.2. $\chi(y, z) = \chi(x, y)$. Now $x^{\chi(x,y)} <' y^{\chi(x,y)} = y^{\chi(y,z)} <' z^{\chi(y,z)} = z^{\chi(x,y)}$. It follows that $\chi(x, z) \le \chi(x, y)$. For any $\alpha < \chi(x, y)$ we have $x^\alpha = y^\alpha = z^\alpha$ since $\chi(y, z) = \chi(x, y)$. So $\chi(x, y) = \chi(x, z)$. Hence $x^{\chi(x,z)} = x^{\chi(x,y)} <' y^{\chi(x,y)} = y^{\chi(y,z)} <' z^{\chi(y,z)} = z^{\chi(x,z)}$, so $x <'' z$.

Subcase 4.3. $\chi(x, y) < \chi(y, z)$. Then $x^{\chi(x,y)} <' y^{\chi(x,y)} = z^{\chi(x,y)}$, and if $\alpha < \chi(x, y)$ then $x^\alpha = y^\alpha = z^\alpha$. It follows that $x <'' z$

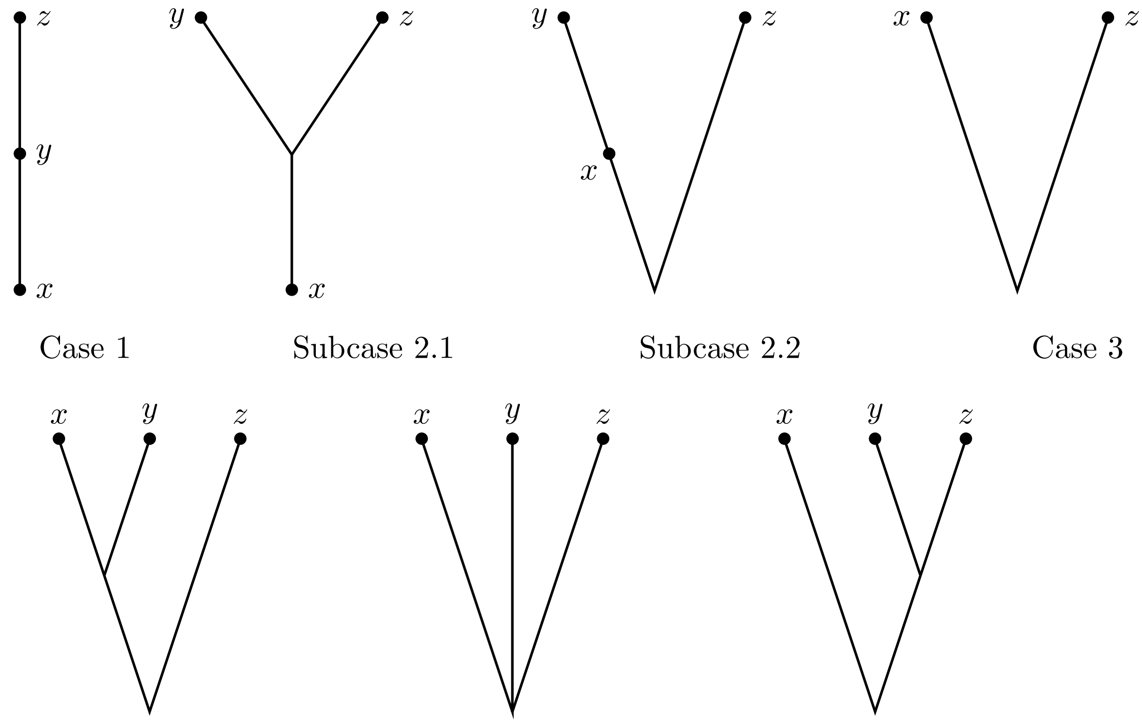
Clearly any two elements of T are comparable under $<''$, so we have a linear order. The following property is also needed.

(*) If $t < x, y$ and $x <'' a <'' y$, then $t < a$.

In fact, suppose not. If $a \leq t$, then $a < x$, hence $a <'' x$, contradiction. So a and t are incomparable. Then $\chi(a,t) \leq \mathrm{ht}(t)$, and hence $x <'' y <'' a$ or $a <'' x <'' y$, contradiction.

Now by the linear order property, $(T, <'')$ has a subset L of order type κ or κ^*. First suppose that L is of order type κ. Define

$$B = \{t \in T : \exists x \in L \forall a \in L[x \leq'' a \to t \leq a]\}.$$



Case 1 Subcase 2.1 Subcase 2.2 Case 3

Subcase 4.1 Subcase 4.2 Subcase 4.3

We claim that B is a chain in T of size κ. Suppose that $t_0, t_1 \in B$ with $t_0 \neq t_1$, and choose $x_0, x_1 \in L$ correspondingly. Say wlog $x_0 <'' x_1$. Now $t_0 \in B$ and $x_0 \leq'' x_1$, so $t_0 \leq x_1$. And $t_1 \in B$ and $x_1 \leq x_1$, so $t_1 \leq x_1$. So t_0 and t_1 are comparable.

Now let $\alpha < \kappa$; we show that B has an element of height α. For each t of height α let $V_t = \{x \in L : t \leq x\}$. Then

$$\{x \in L : \mathrm{ht}(x) \geq \alpha\} = \bigcup_{\mathrm{ht}(t)=\alpha} V_t;$$

since there are fewer than κ elements of height less than κ, this set has size κ, and so there is a t such that $\mathrm{ht}(t) = \alpha$ and $|V_t| = \kappa$. We claim that $t \in B$. To prove this, take any $x \in V_t$ such that $t < x$. Suppose that $a \in L$ and $x \leq'' a$. Choose $y \in V_t$ with $a <'' y$ and $t < y$. Then $t < x$, $t < y$, and $x \leq'' a <'' y$. If $x = a$, then $t \leq a$, as desired. If $x <'' a$, then $t < a$ by (*).

This finishes the case in which L has a subset of order type κ. The case of order type κ^* is similar, but we give it. So, suppose that L has order type κ^*. Define

$$B = \{t \in T : \exists x \in L \forall a \in L[a \leq'' x \to t \leq a]\}.$$

We claim that B is a chain in T of size κ. Suppose that $t_0, t_1 \in B$ with $t_0 \neq t_1$, and choose $x_0, x_1 \in L$ correspondingly. Say wlog $x_0 <'' x_1$. Now $t_0 \in B$ and $x_0 \leq x_0$, so $t_0 \leq x_0$. and $t_1 \in B$ and $x_0 \leq'' x_1$, so $t_1 \leq x_0$. So t_0 and t_1 are comparable.

Now let $\alpha < \kappa$; we show that B has an element of height α. For each t of height α let $V_t = \{x \in L : t \leq x\}$. Then

$$\{x \in L : \mathrm{ht}(x) \geq \alpha\} = \bigcup_{\mathrm{ht}(t)=\alpha} V_t;$$

since there are fewer than κ elements of height less than κ, this set has size κ, and so there is a t such that $\mathrm{ht}(t) = \alpha$ and $|V_t| = \kappa$. We claim that $t \in B$. To prove this, take any $x \in V_t$ such that $t < x$. Suppose that $a \in L$ and $a \leq'' x$. Choose $y \in V_t$ with $y <'' a$ and $t < y$. Then $t < x$, $t < y$, and $y <'' a \leq'' x$. If $a = x$, then $t < a$, as desired. If $a <'' x$, then $t < a$ by (*). □

Theorem 9.26. *For any uncountable cardinal κ the following conditions are equivalent:*

(i) κ is weakly compact.

(ii) κ is inaccessible, and it has the linear order property.

(iii) κ is inaccessible, and it has the tree property.

(iv) For any cardinal λ such that $1 < \lambda < \kappa$ we have $\kappa \to (\kappa)^2_\lambda$.

Proof. (i)⇒(ii): Assume that κ is weakly compact. First we need to show that κ is inaccessible.

To show that κ is regular, suppose to the contrary that $\kappa = \sum_{\alpha<\lambda} \mu_\alpha$, where $\lambda < \kappa$ and $\mu_\alpha < \kappa$ for each $\alpha < \lambda$. By the definition of infinite sum of cardinals, it follows that we can write $\kappa = \bigcup_{\alpha<\lambda} M_\alpha$, where $|M_\alpha| = \mu_\alpha$ for each $\alpha < \lambda$ and the M_α's are pairwise disjoint. Define $f : [\kappa]^2 \to 2$ by setting, for any distinct $\alpha, \beta < \kappa$,

$$f(\{\alpha, \beta\}) = \begin{cases} 0 & \text{if } \alpha, \beta \in M_\xi \text{ for some } \xi < \lambda, \\ 1 & \text{otherwise.} \end{cases}$$

Let H be homogeneous for f of size κ. First suppose that $f[[H]^2] = \{0\}$. Fix $\alpha_0 \in H$, and say $\alpha_0 \in M_\xi$. For any $\beta \in H$ we then have $\beta \in M_\xi$ also, by the homogeneity of H. So $H \subseteq M_\xi$, which is impossible since $|M_\xi| < \kappa$. Second, suppose that $f[[H]^2] = \{1\}$. Then any two distinct members of H lie in distinct M_ξ's. Hence if we define $g(\alpha)$ to be the $\xi < \lambda$ such that $\alpha \in M_\xi$ for each $\alpha \in H$, we get a one-one function from H into λ, which is impossible since $\lambda < \kappa$.

To show that κ is strong limit, suppose that $\lambda < \kappa$ but $\kappa \leq 2^\lambda$. Now by Theorem 20.7 we have $2^\lambda \not\to (\lambda^+, \lambda^+)^2$. So choose $f : [2^\lambda]^2 \to 2$ such that there does not exist an $X \in [2^\lambda]^{\lambda^+}$ with $f \restriction [X]^2$ constant. Define $g : [\kappa]^2 \to 2$ by setting $g(A) = f(A)$ for any

$A \in [\kappa]^2$. Choose $Y \in [\kappa]^\kappa$ such that $g \restriction [Y]^2$ is constant. Take any $Z \in [Y]^{\lambda^+}$. Then $f \restriction [Z]^2$ is constant, contradiction.

So, κ is inaccessible. Now let $(L, <)$ be a linear order of size κ. Let $\prec$ be a well order of L. Now we define $f : [L]^2 \to 2$; suppose that $a, b \in L$ with $a \prec b$. Then

$$f(\{a,b\}) = \begin{cases} 0 & \text{if } a < b, \\ 1 & \text{if } b > a. \end{cases}$$

Let H be homogeneous for f and of size κ. If $f[[H]^2] = \{0\}$, then H is well-ordered by $<$. If $f[[H]^2] = \{1\}$, then H is well-ordered by $>$.

(ii)⇒(iii): By Lemma 9.25.

(iii)⇒(iv): Assume (iii). Suppose that $F : [\kappa]^2 \to \lambda$, where $1 < \lambda < \kappa$; we want to find a homogeneous set for F of size κ. We construct by recursion a sequence $\langle t_\alpha : \alpha < \kappa \rangle$ of members of ${}^{<\kappa}\kappa$; these will be the members of a tree T. Let $t_0 = \emptyset$. Now suppose that $0 < \alpha < \kappa$ and $t_\beta \in {}^{<\kappa}\kappa$ has been constructed for all $\beta < \alpha$. We now define t_α by recursion; its domain will also be determined by the recursive definition, and for this purpose it is convenient to actually define an auxiliary function $s : \kappa \to \kappa+1$ by recursion. If $s(\eta)$ has been defined for all $\eta < \xi$, we define

$$s(\xi) = \begin{cases} F(\{\beta, \alpha\}) & \text{where } \beta < \alpha \text{ is minimum such that } s \restriction \xi = t_\beta, \text{ if there is such a } \beta, \\ \\ \kappa & \text{if there is no such } \beta. \end{cases}$$

Now eventually the second condition here must hold, as otherwise $\langle s \restriction \xi : \xi < \kappa \rangle$ would be a one-one function from κ into $\{t_\beta : \beta < \alpha\}$, which is impossible. Take the least ξ such that $s(\xi) = \kappa$, and let $t_\alpha = s \restriction \xi$. This finishes the construction of the t_α's. Let $T = \{t_\alpha : \alpha < \kappa\}$, with the partial order $\subseteq$. Clearly this gives a tree.

By construction, if $\alpha < \kappa$ and $\xi < \mathrm{dmn}(t_\alpha)$, then $t_\alpha \restriction \xi \in T$. Thus the height of an element t_α is $\mathrm{dmn}(t_\alpha)$.

(2) The sequence $\langle t_\alpha : \alpha < \kappa \rangle$ is one-one.

In fact, suppose that $\beta < \alpha$ and $t_\alpha = t_\beta$. Say that $\mathrm{dmn}(t_\alpha) = \xi$. Then $t_\alpha = t_\alpha \restriction \xi = t_\beta$, and the construction of t_α gives something with domain greater than ξ, contradiction. Thus (2) holds, and hence $|T| = \kappa$.

(3) The set of all elements of T of level $\xi < \kappa$ has size less than κ.

In fact, let U be this set. Then

$$|U| \le \prod_{\eta < \xi} \lambda = \lambda^\xi < \kappa$$

since κ is inaccessible. So (3) holds, and hence, since $|T| = \kappa$, T has height κ and is a κ-tree.

(4) If $t_\beta \subset t_\alpha$, then $\beta < \alpha$ and $F(\{\beta, \alpha\}) = t_\alpha(\mathrm{dmn}(t_\beta))$.

This is clear from the definition.

Now by the tree property, there is a branch B of size κ. For each $\xi < \lambda$ let

$$H_\xi = \{\alpha < \kappa : t_\alpha \in B \text{ and } t_\alpha^\frown\langle\xi\rangle \in B\}.$$

We claim that each H_ξ is homogeneous for F. In fact, take any distinct $\alpha, \beta \in H_\xi$. Then $t_\alpha, t_\beta \in B$. Say $t_\beta \subset t_\alpha$. Then $\beta < \alpha$, and by construction $t_\alpha(\mathrm{dmn}(t_\beta)) = F(\{\alpha, \beta\})$. So $F(\{\alpha, \beta\}) = \xi$ by the definition of H_ξ, as desired. Now

$$\{\alpha < \kappa : t_\alpha \in B\} = \bigcup_{\xi<\lambda} \{\alpha < \kappa : t_\alpha \in H_\xi\},$$

so since $|B| = \kappa$ it follows that $|H_\xi| = \kappa$ for some $\xi < \lambda$, as desired.

(iv)⇒(i): obvious. □

For κ an infinite cardinal, $\alpha < \kappa$ a limit ordinal, and $2 \le m < \kappa$, we write

$$\kappa \to (\alpha)^{<\omega}_m$$

to mean that for every $F : [\kappa]^{<\omega} \to m$ there is a set $H \subseteq \kappa$ of order type α such that for each $n \in \omega$, F is constant on $[H]^n$. A cardinal κ is *Ramsey* iff $\kappa \to (\kappa)^{<\omega}$.

Proposition 9.27. *Every Ramsey cardinal is weakly compact.*

Proof. Suppose that $F : [\kappa]^n \to 2$. Extend F in any way to a function mapping $[\kappa]^{<\omega}$ into 2. □

Proposition 9.28. *Every infinite poset either has an infinite chain or an infinite set of pairwise incomparable elements.*

Proof. Let P be an infinite poset. Let $\prec$ be a well-order of P. Define $F : [P]^2 \to 2$ by

$$F(\{x, y\}) = \begin{cases} 2 & \text{if } x < y \text{ and } x \prec y, \\ 1 & \text{if } x < y \text{ and } y \prec x, \\ 0 & \text{otherwise.} \end{cases}$$ □

For any infinite cardinal κ we define

$$2^\kappa_0 = \kappa;$$
$$2^\kappa_{n+1} = 2^{(2^\kappa_n)} \quad \text{for all } n \in \omega.$$

Theorem 9.29. *For every infinite cardinal κ and every positive integer n, $(2^\kappa_{n-1})^+ \to (\kappa^+)^n_\kappa$.*

Proof. Induction on n. For $n = 1$ we want to show that $\kappa^+ \to (\kappa^+)^1_\kappa$, and this is obvious. Now assume the statement for $n \ge 1$, and suppose that $f : [(2^\kappa_n)^+]^{n+1} \to \kappa$. For each $\alpha \in (2^\kappa_n)^+$ define $F_\alpha : [(2^\kappa_n)^+ \backslash \{\alpha\}]^n \to \kappa$ by setting $F_\alpha(x) = f(x \cup \{\alpha\})$.

(1) There is an $A \in [(2^\kappa_n)^+]^{2^\kappa_n}$ such that for all $C \in [A]^{2^\kappa_{n-1}}$ and all $u \in (2^\kappa_n)^+ \backslash C$ there is a $v \in A\backslash C$ such that $F_u \restriction [C]^n = F_v \restriction [C]^n$.

To prove this, we define a sequence $\langle A_\alpha : \alpha < (2^\kappa_{n-1})^+\rangle$ of subsets of $(2^\kappa_n)^+$, each of size 2^κ_n. Let $A_0 = 2^\kappa_n$, and for α limit let $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$. Now suppose that A_α has been defined, and $C \in [A_\alpha]^{2^\kappa_{n-1}}$. Define $u \equiv v$ iff $u, v \in (2^\kappa_n)^+\backslash C$ and $F_u \restriction [C]^n = F_v \restriction [C]^n$. Now $|^{[C]^n}\kappa| = 2^\kappa_n$, so there are at most 2^κ_n equivalence classes. Let K_C have exactly one element in common with each equivalence class. Let $A_{\alpha+1} = A_\alpha \cup \{K_C : C \in [A_\alpha]^{2^\kappa_{n-1}}\}$. Since $(2^\kappa_n)^{2^\kappa_{n-1}} = 2^\kappa_n$, we still have $|A_{\alpha+1}| = 2^\kappa_n$. This finishes the construction. Clearly $A \stackrel{\text{def}}{=} \bigcup_{\alpha\le(2^\kappa_{n-1})^+} A_\alpha$ is as desired in (1).

Take A as in (1), and fix $a \in (2^\kappa_n)^+\backslash A$. We now define a sequence $\langle x_\alpha : \alpha < (2^\kappa_{n-1})^+\rangle$ of elements of A. Given $C \stackrel{\text{def}}{=} \{x_\beta : \beta < \alpha\}$, by (1) let $x_\alpha \in A\backslash C$ be such that $F_{x_\alpha} \restriction [C]^n = F_a \restriction [C]^n$. This defines our sequence. Let $X = \{x_\alpha : \alpha < (2^\kappa_{n-1})^+\}$.

Now define $G : [X]^n \to \kappa$ by $G(x) = F_a(x)$. Suppose that $\alpha_0 < \cdots < \alpha_n < (2^\kappa_{n-1})^+$. Then

$$\begin{aligned} f(\{x_{\alpha_0}, \ldots, x_{\alpha_n}\}) &= F_{x_{\alpha_n}}(\{x_{\alpha_0}, \ldots, x_{\alpha_{n-1}}\}) \\ &= F_a(\{x_{\alpha_0}, \ldots, x_{\alpha_{n-1}}\}) \\ &= G(\{x_{\alpha_0}, \ldots, x_{\alpha_{n-1}}\}). \end{aligned}$$

Now by the inductive hypothesis there is an $H \in [X]^{\kappa^+}$ such that G is constant on $[H]^n$. By the above, f is constant on $[H]^{n+1}$. ☐

Proposition 9.30. $\omega_1 \to (\omega_1, \omega+1)^2$

Proof. Let $\{A, B\}$ be a partition of $[\omega_1]^2$. For each limit ordinal α let K_α be a maximal subset of α such that $[K_\alpha \cup \{\alpha\}]^2 \subseteq B$. If some K_α is infinite, this is as desired. So suppose that each K_α is finite. For each $m \in \omega$ let $T_m = \{\alpha < \omega_1 : \alpha$ is a limit ordinal and $|K_\alpha| = m\}$. The set $S \stackrel{\text{def}}{=} \{\alpha < \omega_1 : \alpha$ is a limit ordinal$\}$ is stationary, and $S = \bigcup_{m\in\omega} T_m$, so there is an $m \in \omega$ such that T_m is stationary.

First suppose that $m = 0$. Then for any $\alpha < \beta$, both in T_m, we must have $\{\alpha, \beta\} \in A$, since K_β is empty. Thus $[T_m]^2 \subseteq A$, as desired.

Now suppose that $m > 0$. For each $\alpha \in T_m$, let $f(\alpha)$ be the largest element of K_α. Then there is a stationary subset U of T_m such that f takes on a constant value, say γ, on U. We can repeat this argument with the next largest elements of the K_α's, etc., until finally we reach a stationary set V such that K_α has a constant value, say L, on V. Let γ be the largest element of L. We claim that $[V\backslash(\gamma+1)]^2 \subseteq A$, as desired.

For, suppose that $\alpha < \beta$, with $\alpha, \beta \in V\backslash(\gamma+1)$. Then $[L \cup \{\alpha, \beta\}]^2 \not\subseteq B$ since $L = K_\beta$, while $[L \cup \{\beta\}]^2 \subseteq B$ since $L = K_\beta$ and $[L \cup \{\alpha\}]^2 \subseteq B$ since $L = K_\alpha$. Hence $\{\alpha, \beta\} \in A$. ☐

Proposition 9.31. *For all infinite cardinals κ, $\kappa \not\to (\omega)^\omega$.*

Proof. Define $s \equiv t$ iff $s, t \in [\kappa]^\omega$ and $\{n : s(n) \neq t(n)\}$ is finite. Clearly $\equiv$ is an equivalencd relation on $[\omega]^\omega$. Pick a representative from each equivalence class. Now define

$F : [\omega]^\omega \to 2$ by setting $F(s) = 0$ iff $|s \triangle t|$ is even, where t is the representative in $[s]$. Suppose that H is a homogeneous set, with $|H| = \omega$. Let t be the representative in $[H]$. Then $H \cap t$ is infinite, since $H \backslash (H \cap t) = H \backslash t \subseteq H \triangle t$ is finite. Choose $m \in H \cap t$. Then $|(H \backslash \{m\}) \triangle t|$ is even iff $|H \triangle t|$ is odd, contradicting homogeneity. In fact, $(H \backslash \{m\}) \backslash t = H \backslash t$ while $t \backslash (H \backslash \{m\}) = (t \backslash H) \cup \{m\}$. □

Theorem 9.32. (König) *If T is a tree of height ω such that each level is finite, then T has an infinite branch.*

Proof. We define $t_) < t_1 < \cdots$ by recursion. T has finitely many roots. Let t_0 be a root such that $t_0 \uparrow$ is infinite. Having constructed t_n so that $t_n \uparrow$ is infinite, let t_{n+1} be an immediate successor of T_n such that $t_{n+1} \uparrow$ is infinite. □

Proposition 9.33. *If T is a normal α-tree, then T is isomorphic to a tree $\overline{T}$ whose elements are sequences with domain $\beta < \alpha$, ordered by $\subseteq$.*

Proof. We define the isomorphism f by defining $f \restriction \mathrm{lev}_\xi(T)$ by induction on ξ. The definition is done so that if a has level ξ, then $f(a) \in {}^\xi\omega$, and if $a < b$ then $f(a) \subset f(b)$. Let $f \restriction \mathrm{lev}_0(T) = \emptyset$. If $f \restriction \mathrm{lev}_\xi(T)$ has been defined and $a \in \mathrm{lev}_\xi(T)$, let g_a be a one-one function from the set of immediate successors of a onto ω, and for each immediate successor b of a let $f(b) = f(a)^\frown\langle g_a(b)\rangle$. If ξ is a limit ordinal and $f \restriction \mathrm{lev}_\eta(T)$ has been defined for every $\eta < \xi$, take any element a of level ξ. Let b_η be the predecessor of a at level η, for every $\eta < \xi$. Then define $f(a) = \bigcup_{\eta<\xi} f(b_\eta)$. □

Proposition 9.34. *If T is a normal tree of height ω_1 and T has an uncountable branch, then T has an uncountable antichain.*

Proof. Let $\langle x_\alpha : \alpha < \omega_1\rangle$ be such that if $\alpha < \beta < \omega_1$ then $\alpha <_T \beta$. For each $\alpha < \omega_1$ choose an immediate successor $y_{\alpha'} >_T x_\alpha$ with $y_{\alpha'} \not<_T x_{\alpha+1}$. Then $\{y_{\alpha'} : \alpha < \omega_1\}$ is an antichain. In fact, if $\alpha < \beta$ and $y_{\alpha'} <_Y y_{\beta'}$, then $y_{\alpha'}$ and x_β are comparable, hence $y_{\alpha'} \leq_T x_{\alpha+1}$, contradiction. □

Theorem 9.35. *There is an Aronszajn tree T such that there is a function $\varphi : T \to \mathbb{Q}$ with $\forall s, t \in Y[s <_T t \to \varphi(s) < \varphi(t)]$.*

Proof. Let $T = \omega_1$. The levels are as follows:

$$\begin{aligned} Lev_0 &= \{0\}; \\ Lev_1 &= \omega\backslash\{0\}; \\ Lev_{n+1} &= \{\omega \cdot n + k : k \in \omega\} \quad \text{for } 0 < n < \omega; \\ Lev_\alpha &= \{\omega \cdot \alpha + k\} \quad \text{for } \omega \leq \alpha < \omega_1. \end{aligned}$$

Now we define the tree order $<_T$ and the function φ by induction on the level, so that

$$\begin{aligned} &\forall \alpha < \omega_1 \forall x \in \omega_1 \forall q \in \mathbb{Q}[ht(x) < \alpha \text{ and } \varphi(x) < q \to \\ &\exists y \in Lev(\alpha)[x <_T y \text{ and } \varphi(y) = q]]. \end{aligned} \tag{$*$}$$

Let $\varphi(0) = 0$. The immediate successors of 0 are the members of L_1. Given L_α with $0 < \alpha < \omega_1$, let $\langle E^\xi_{\alpha+1} : \xi \in L_\alpha \rangle$ be a partition of $L_{\alpha+1}$, let the members of $E^\xi_{\alpha+1}$ be the immediate successors of ξ, and let φ map E^ξ_{a+1} one-one onto $\mathbb{Q} \cap (\varphi(\xi), \infty)$. Clearly if $(*)$ holds for α then it also holds for $\alpha + 1$.

Now suppose that γ is a limit ordinal and the construction works for smaller ordinals. Let $\langle (x_k, q_k) : k \in \omega \rangle$ enumerate all pairs (x, q) with $x \in T_\gamma$ and $q \in \mathbb{Q} \cap (\varphi(x), \infty)$.

(1) For each $k \in \omega$ there is a path P_k through T_γ such that $x_k \in P_k$ and $\sup\{\varphi(y) : y \in P_k\} = q_k$.

To prove (1), suppose that $k \in \omega$. Let $\langle \alpha_n : n \in \omega \rangle$ be strictly increasing with $ht(x_k) < \alpha_0$ and $\sup_{n\in\omega} \alpha_n = \gamma$. Also, let $\langle r_n : n \in \omega \rangle$ be a strictly increasing system of rationals with $\varphi(x_k) < r_0$ and $\sup_{n\in\omega} r_n = q_k$. Now we apply $(*)$ with α, x, q replaced by α_0, x_k, r_0. This gives $z_0 \in Lev(\alpha_0)$ such that $x_k <_T z_0$ and $\varphi(z_0) = \alpha_0$. Now suppose that z_n has been defined so that $z_n \in Lev(\alpha_n)$ and $\varphi(z_n) = \alpha_n$. Apply $(*)$ with α, x, q replaced by $\alpha_{n+1}, z_n, r_{n+1}$. This gives $z_{n+1} \in Lev(\alpha_{n+1})$ such that $z_n <_T z_{n+1}$ and $\varphi(z_{n+1}) = \alpha_{n+1}$. Let $P_k = \bigcup_{n\in\omega}(z_n \downarrow)$.

We put $\omega \cdot \gamma + k$ directly above P_k and let $\varphi(\omega \cdot \gamma + k) = q_k$. Clearly $(*)$ continues to hold. $\square$

A *special Aronszajn tree* is an Aronszajn tree T such that there is a function $f : T \to \mathbb{Q}$ such that $\forall s, t \in T[s <_T t \to f(s), f(t)]$.

Lemma 9.36. *Any countable linear order can be isomorphically embedded in* $\mathbb{Q}$.

Proof. Let L be a countable linear order. Say $L = \{a_m : n \in \omega\}$. We define $f(a_0) = 0$. Suppose that $f(a_m)$ has been defined for all $m < n$ so that it is an isomorphism into $\mathbb{Q}$.

Case 1. $a_n <_L a_m$ for all $m < n$. Let $f(a_n)$ be a rational less than each $f(a_m)$ for $m < n$.

Case 2. $a_m <_L a_n$ for all $m < n$. Similar to Case 1.

it Case 3. Otherwise. Let $A = \{a_m : m < n, a_m < a_n\}$ and $B = \{a_m : m < n, a_n < a_m\}$. Let $f(a_n)$ be a rational q such that $\forall x \in A[x < f(a_m)]$ and $\forall x \in B[f(a_m) < x]$. $\square$

Proposition 9.37. *An Aronszajn tree T is special iff T is the union of ω antichains.*

Proof. First suppose that T is special. Let $f : T \to \mathbb{Q}$ be such that $\forall s, t \in T[s <_T t \to f(s) < f(t)]$. For each rational number r let $X_r = \{s \in T : f(s) = r\}$. Clearly each X_r is an antichain (possibly empty), and $T = \bigcup_{r\in\mathbb{Q}} X_r$. $\square$

Now suppose that $T = \bigcup_{n\in\omega} A_n$ with each A_n an antichain. Let $B_n = A_n \setminus \bigcup_{m<n} A_m$. Then each B_n is an antichain, and $T = \bigcup_{n\in\omega} B_n$. Let $f[B_n] = n$ for each $n \in \omega$. For any $t \in T$ and $n \in \omega$ let

$$g_t(n) = \begin{cases} 1 & \text{if } n \leq f(t) \text{ and } \{s \in T : s \leq_T t\} \cap B_n \neq \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $s, t \in T$ and $s <_T t$. Say $s \in B_m$ and $t \in B_n$. Then $m \neq n$.

Case 1. $m < n$. Then $g_t(n) = 1$ and $g_s(n) = 0$. So $g_t \neq g_s$. Let p be minimum such that $g_t(p) \neq g_s(p)$. Thus $p \leq n$. Suppose that $g_t(p) = 0$ and $g_s(p) = 1$. Then $p \leq m$

and $\{u \in T : u \leq_T s\} \cap B_p \neq \emptyset$. Say $u \leq_T s$ and $u \in B_p$. Now $p \leq n$ and $g_t(p) = 0$, so $\{v \in T : v \leq_T t\} \cap B_p = \emptyset$. But $u \leq_T t$, contradiction. So $g_t(p) = 1$ and $g_s(p) = 0$, as desired.

Case 2. $n < m$. Then $g_s(m) = 1$ and $g_t(m) = 0$. So $g_t \neq g_s$. Let p be minimum such that $g_t(p) \neq g_s(p)$. Thus $p \leq m$. Suppose that $g_t(p) = 0$ and $g_s(p) = 1$. Then $\{v \in T : v \leq_T s\} \cap B_p \neq 0$. Say $u \leq_T s$ and $u \in B_p$. Since $g_t(p) = 0$ and $p \leq n$, we have $\{s \in T : s \leq_T t\} \cap B_p = \emptyset$. But $u \leq_T t$, contradiction.

(1) $\{g_t : t \in T\}$ is countable.

In fact, it suffices to show that $\{g_t : t \in T, f(t) = m\}$ is countable, for any $m \in \omega$,

$$\{g_t \in {}^{\omega}2 : f(t) = m\} \subseteq \{x \in {}^{\omega}\omega : \forall n > m[x(n) = 0],$$

and hence $\{g_t \in {}^{\omega}2 : f(t) = m\}$ is finite. Hence (1) holds.

Now by Proposition 9.36, the desired conclusion follows. $\square$

Proposition 9.38. *If $2^{<\kappa} = \kappa$, then there is an $\mathscr{A} \subseteq [\kappa]^{\kappa}$ such that $|\mathscr{A}| = 2^{\kappa}$ and $\forall A, B \in \mathscr{A}[A \neq B \to |A \cap B| < \kappa]$.*

Proof. Let $F : {}^{<\kappa}\kappa \to \kappa$ be a bijection. For each $f \in {}^{\kappa}2$ let $X_f = \{f \restriction x : x \in {}^{<\kappa}\kappa\}$. Let $\mathscr{A} = \{F[X_f]; f \in \kappa 2\}$. $\square$

Proposition 9.39. *If there is a family $\mathscr{F}$ of almost disjoint functions from ω_1 into ω with $|\mathscr{F}| = \omega_2$, then there is a family $\mathscr{S}$ of pairwise disjoint stationary subsets of ω_1 with $|\mathscr{S}| = \omega_2$.*

Proof. If $f \in \mathscr{F}$, then there is an $n_f \in \omega$ such that $S_f \stackrel{\text{def}}{=} \{\alpha < \omega_1 : f(\alpha) = n\}$ is stationary. Otherwise, for each $n \in \omega$ let C_n be club such that $\{\alpha < \omega_1 : f(\alpha) = n\} \cap C_n = \emptyset$. Then $\bigcap_{n\in\omega} C_n$ is still club; but it is empty, contradiction. Now $\mathscr{F} = \bigcup_{m\in\omega}\{f \in \mathscr{F} : n_f = m\}$, so there is an $m \in \omega$ such that $|\{f \in \mathscr{F} : n_f = m\}| = \aleph_2$. Say $\mathscr{G} = \{f \in \mathscr{F} : n_f = m\}$. If $f, g \in \mathscr{G}$ with $f \neq g$, then $S_f \cap S_g = \emptyset$. In fact, by almost disjointness choose $\beta < \omega_1$ such that $\forall\alpha \in [\beta, \omega_1)[f(\alpha) \neq g(\alpha)]$. Then choose $\alpha \in [\beta, \omega_1)$ with $\alpha \in S_f \cap S_g$. Then $f(\alpha) = m = g(\alpha)$. Thus $S_f \cap S_g$ is countable.

Write $\mathscr{G} = \{f_\alpha : \alpha < \omega_2\}$ with f one-one. For each $\alpha < \omega_2$ let

$$S'_\alpha = S_{f_\alpha} \setminus \bigcup_{\beta<\alpha} S_{f_\beta}.$$

Now for any $\alpha < \omega_2$, $S_{f_\alpha} \cap \bigcup_{\beta<\alpha} S_{f_\beta}$ is countable, so S'_α is stationary. Clearly $S'_\alpha \cap S'_\beta = \emptyset$ for $\alpha \neq \beta$. $\square$

Proposition 9.40. $\omega \not\to (\omega)^{<\omega}$.

Proof. Assume otherwise. Define $F : [\omega]^{<\omega} \to 2$ by

$$F(x) = \begin{cases} 1 & \text{if } |x| \in x, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $H \in [\omega]^{\omega}$. Let $n \in H$ with $n \neq 0$. Suppose that F is constant on $[H]^n$. If $n \in x \in [H]^n$ then $F(x) = 1$. If $x \in [H]^n$ with $n \notin x$, then $F(x) = 0$. So F is not constant on $[H]^n$. $\square$

10. Measurable cardinals

A *measure space* is a triple (X, Σ, μ) such that:

(1) X is a set

(2) Σ is a σ-algebra of subsets of X.

(3) μ is a measure on Σ.

Given a measure space as above, a subset A of X is a *μ-null* set iff there is an $E \in \Sigma$ such that $A \subseteq E$ and $\mu(E) = 0$.

Theorem 10.1. *If (X, Σ, μ) is a measure space, then the collection of μ-null sets is a σ-ideal of subsets of X.*

Proof. Let I be the collection of all μ-null sets. Clearly $\emptyset \in I$, and $B \subseteq A \in I$ implies that $B \in I$. Now suppose that $\langle A_i : i \in \omega \rangle$ is a system of members of I. For each $i \in \omega$ choose $E_i \in \Sigma$ such that $A_i \subseteq E_i$ and $\mu(E_i) = 0$. Then $\bigcup_{i \in I} A_i \subseteq \bigcup_{i \in I} E_i$, and

$$\mu\left(\bigcup_{i\in\omega} E_i\right) \leq \sum_{i\in\omega} \mu(E_i) = 0. \qquad \square$$

An *outer measure* on a set X is a function $\mu : \mathscr{P}(X) \to [0, \infty]$ satisfying the following conditions:

(1) $\mu(\emptyset) = 0$.

(2) If $A \subseteq B \subseteq X$, then $\mu(A) \leq \mu(B)$.

(3) For every $A \in {}^{\omega}\mathscr{P}(X)$, $\mu(\bigcup_{n\in\omega} A_n) \leq \sum_{n\in\omega} \mu(A_n)$.

If θ is an outer measure on a set X, then a subset E of X is *θ-measurable* iff for every $A \subseteq X$,

$$\theta(A) = \theta(A \cap E) + \theta(A \backslash E).$$

Note that every subset $E \subseteq X$ such that $\theta(E) = 0$ is automatically θ-measurable.

Theorem 10.2. *Let θ be an outer measure on a set X. Let Σ be the collection of all θ-measurable subsets of X. Then $(X, \Sigma, \theta \restriction \Sigma)$ is a measure space. Moreover, if $E \subseteq X$ and $\theta(E) = 0$, then $E \in \Sigma$.*

Proof. Note that Σ is obviously closed under complementation. Obviously

(1) If $A, E \subseteq X$, then $\theta(A) \leq \theta(A \cap E) + \theta(A \backslash E)$.

Clearly $\emptyset \in \Sigma$ and Σ is closed under complements. Next we show that Σ is closed under $\cup$. Suppose that $E, F \in \Sigma$ and $A \subseteq X$. Then

$$\begin{aligned}
\theta(A \cap (E \cup F)) + \theta(A \backslash (E \cup F)) &\leq \theta((A \cap (E \cup F) \cap E)) + \theta(A \cap (E \cup F) \backslash E))) \\
&\qquad + \theta(A \backslash (E \cup F)) \\
&= \theta(A \cap E) + \theta((A \backslash E) \cap F) + \theta((A \backslash E) \backslash F) \\
&= \theta(A \cap E) + \theta(A \backslash E) \\
&= \theta(A) \\
&\leq \theta(A \cap (E \cup F)) + \theta(A \backslash (E \cup F)) \quad \text{by (1).}
\end{aligned}$$

This proves that $E \cup F \in \Sigma$. Thus we have shown that Σ is a field of subsets of X.

Next we show that Σ is closed under countable unions. So, suppose that $E \in {}^{\omega}\Sigma$, and let $K = \bigcup_{n\in\omega} E_n$. For every $m \in \omega$ let

$$G_m = \bigcup_{n\leq m} E_n.$$

Then clearly each G_m is in Σ. Now we define $F_0 = G_0$, and for $m > 0$, $F_m = G_m \backslash G_{m-1}$. Then also each F_m is in Σ. By induction, $\bigcup_{n\leq m} F_n = G_m$. Hence $\bigcup_{n\in\omega} F_n = \bigcup_{n\in\omega} E_n$. Now temporarily fix a positive integer n and an $A \subseteq X$. Then

$$\theta(A \cap G_n) = \theta(A \cap G_n \cap G_{n-1}) + \theta(A \cap G_n \backslash G_{n-1}) = \theta(A \cap G_{n-1}) + \theta(A \cap F_n);$$

hence by induction $\theta(A \cap G_n) = \sum_{m\leq n} \theta(A \cap F_m)$.

Now we unfix n. Now $A \cap K = \bigcup_{n\in\omega}(A \cap F_n)$, so

$$\theta(A \cap K) \leq \sum_{n\in\omega} \theta(A \cap F_n) = \lim_{n\to\infty} \sum_{m\leq n} \theta(A \cap F_m) = \lim_{n\to\infty} \theta(A \cap G_m).$$

Also, note that if $m < n$ then $G_m \subseteq G_n$, hence $X \backslash G_n \subseteq X \backslash G_m$, and so

$$\theta(A \backslash K) = \theta\left(A \backslash \bigcup_{n\in\omega} G_n\right) = \theta\left(\bigcap_{n\in\omega} (A \backslash G_n)\right) \leq \inf_{n\in\omega} \theta(A \backslash G_n) = \lim_{n\to\infty} \theta(A \backslash G_n).$$

Hence

$$\begin{aligned}
\theta(A \cap K) + \theta(A \backslash K) &\leq \lim_{n\to\infty} \theta(A \cap G_n) + \lim_{n\to\infty} \theta(A \backslash G_n) \\
&= \lim_{n\to\infty} (\theta(A \cap G_n) + \theta(A \backslash G_n)) \\
&= \theta(A) \\
&\leq \theta(A \cap K) + \theta(A \backslash K).
\end{aligned}$$

This proves that $K \in \Sigma$, so that Σ is closed under countable unions.

Finally, suppose that $\langle E_n : n \in \omega \rangle$ is a system of pairwise disjoint members of Σ. Let $K = \bigcup_{n\in\omega} E_n$. Hence $\theta(K) \leq \sum_{n\in\omega} \theta(E_n)$. Conversely, for each $n \in \omega$ let $G_n = \bigcup_{m\leq n} E_m$. Then

$$\theta(G_{n+1}) = \theta(G_{n+1} \cap E_{n+1}) + \theta(G_{n+1} \backslash E_{n+1}) = \theta(E_{n+1}) + \theta(G_n).$$

Hence by induction, $\theta(G_n) = \sum_{m\leq n} \theta(E_m)$ for every n, and hence

$$\theta(K) \geq \theta(G_n) = \sum_{m\leq n} \theta(E_m),$$

and so $\theta(K) \geq \sum_{n\in\omega} \theta(E_n)$.

For the "moreover" statement, suppose that $E \subseteq X$ and $\theta(E) = 0$, Then for any $A \subseteq X$, $\theta(A) \leq \theta(A \cap E) + \theta(A \backslash E) = \theta(A \backslash E) \leq \theta(A)$. $\square$

For any $a, b \in \mathbb{R}$ let $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$. Note that if $a \geq b$, then $[a, b) = \emptyset$. Note that if $[a, b) = [c, d)$, $a < b$, and $c < d$, then $a = c$ and $b = d$. For any $a, b \in \mathbb{R}$ we define

$$\lambda([a, b)) = \begin{cases} 0 & \text{if } a \geq b, \\ b - a & \text{if } a < b. \end{cases}$$

A set of the form $[a, b)$ is called a *half-open interval.*

Lemma 10.3. *Suppose that I is a half-open interval, $\langle J_i : i \in \omega \rangle$ is a system of half-open intervals, and $I \subseteq \bigcup_{i \in \omega} J_i$. Then*

$$\lambda(I) \leq \sum_{j \in \omega} \lambda(J_i).$$

Proof. If $I = \emptyset$ this is obvious. So suppose that $I \neq \emptyset$. Then there exist real numbers $a < b$ such that $I = [a, b)$. Let

$$A = \left\{ x \in [a, b] : x - a \leq \sum_{j \in \omega} \lambda(J_j \cap (-\infty, x)) \right\}.$$

Obviously $a \in A$, and A is bounded above by b, so $c \stackrel{\text{def}}{=} \sup(A)$ exists. Now

$$\begin{aligned} c - a &= \sup_{x \in A}(x - a) \\ &\leq \sup_{x \in A} \sum_{j \in \omega} \lambda(J_j \cap (-\infty, x)) \\ &\leq \sum_{j \in \omega} \lambda(J_j \cap (-\infty, c)). \end{aligned}$$

Hence $c \in A$. Now suppose that $c < b$. Thus $c \in [a, b)$, so there is a $k \in \omega$ such that $c \in J_k$. Say $J_k = [u, v)$. Then $x \stackrel{\text{def}}{=} \min(v, b) > c$. Then $\lambda(J_j \cap (-\infty, c)) \leq \lambda(J_j \cap (-\infty, x))$ for each j, and $\lambda(J_k \cap (-\infty, x)) = \lambda(J_k \cap (-\infty, c)) + x - c$. Hence

$$\begin{aligned} &\sum_{j \in \omega} \lambda(J_j \cap (-\infty, x)) \geq \sum_{j \in \omega} \lambda(J_j \cap (-\infty, c)) + x - c \\ &\geq c - a + x - c = x - a. \end{aligned}$$

Here we used the above inequality on $c - a$. Thus we have shown that $x \in A$. But $x > c = \sup(A)$, contradiction.

Hence $c = b$, so $b \in A$. $\square$

Now for any $A \subseteq \mathbb{R}$ let

$$\theta'(A) = \inf \left\{ \sum_{j \in \omega} \lambda(I_j) : \langle I_j : j \in \omega \rangle \text{ is a sequence of half-open intervals} \right.$$
$$\left. \text{such that } A \subseteq \bigcup_{j \in \omega} I_j \right\}.$$

Lemma 10.4. *(i) θ' is an outer measure on $\mathbb{R}$.*
(ii) $\theta'(I) = \lambda(I)$ for every half-open interval I.

Proof. (i): Clearly (1) and (2) hold. Now for (3), suppose that $\langle A_i : i \in \omega \rangle$ is a sequence of subsets of X. Let $B = \bigcup_{i \in \omega} A_i$. For each $i \in \omega$ let $\langle I_{ij} : j \in \omega \rangle$ be a sequence of half-open intervals such that $A_i \subseteq \bigcup_{j \in \omega} I_{ij}$ and

$$\sum_{j \in \omega} \lambda(I_{ij}) \leq \theta'(A_i) + \frac{\varepsilon}{2^i}.$$

Note that this holds even if $\theta'(A_i) = \infty$. Let $p : \omega \to \omega \times \omega$ be a bijection.

$$B \subseteq \bigcup_{m \in \omega} I_{1^{st}(p(m)), 2^{nd}(p(m))}. \tag{1}$$

In fact, if $b \in B$, choose $i \in I$ such that $b \in A_i$, and then choose $j \in \omega$ such that $b \in I_{ij}$. Let $m = p^{-1}(i, j)$. Then

$$b \in I_{1^{st}(p(m)), 2^{nd}(p(m))},$$

as desired in (1).

$$\sum_{m \in \omega} \lambda(I_{1^{st}(p(m)), 2^{nd}(p(m))}) \leq \sum_{i \in \omega} \sum_{j \in \omega} \lambda(I_{ij}). \tag{2}$$

In fact, let $m \in \omega$, and set

$$n = \max(\{1^{st}(p(i)) : i \leq m\} \cup \{2^{nd}(p(i)) : i \leq m\}).$$

Then

$$\sum_{i=0}^{m} \lambda(I_{1^{st}(p(m)), 2^{nd}(p(m))}) \leq \sum_{i=0}^{n} \sum_{j=0}^{n} \lambda(I_{ij}) \leq \sum_{i \in \omega} \sum_{j \in \omega} \lambda(I_{ij}),$$

and (2) follows.

Hence using (1) we have

$$\begin{aligned}
\theta'\left(\bigcup_{i\in\omega} A_i\right) &= \theta'(B) \\
&\leq \sum_{m\in\omega} \lambda(I_{1^{st}(p(m)),2^{nd}(p(m))}) \\
&\leq \sum_{i\in\omega}\sum_{j\in\omega} \lambda(I_{ij}) \\
&\leq \sum_{i\in\omega}\left(\theta'(A_i) + \frac{\varepsilon}{2^i}\right) \\
&= \sum_{i\in\omega}\theta'(A_i) + \sum_{i\in\omega}\frac{\varepsilon}{2^i} \\
&= \sum_{i\in\omega}\theta'(A_i) + 2\varepsilon.
\end{aligned}$$

Hence (3) in the definition of outer measure holds.

Clearly $\theta'(I) \leq \lambda(I)$. The other inequality follows from a Lemma above. □

Corollary 10.5. *For θ' the explicit outer measure defined above on $\mathbb{R}$, and with*

$$\begin{aligned}
\Sigma_1 = \{E \subseteq \mathbb{R} : &\textit{for every } A \subseteq X, \\
&\theta'(A) = \theta'(A\cap E) + \theta'(A\backslash E)\},
\end{aligned}$$

the system $(\mathbb{R}, \Sigma_1, \theta' \restriction \Sigma_1)$ is a measure space. □

The measure space of this corollary is *Lebesgue measure.*

Lemma 10.6. *$(-\infty, x)$ is measurable for every $x \in \mathbb{R}$.*

Proof. First we show

(1) $\lambda(I) = \lambda(I \cap (-\infty, x)) + \lambda(I\backslash(-\infty, x))$ for every half-open interval I.

This is obvious if $I \subseteq (-\infty, x)$ or $I \subseteq [x, \infty)$. So assume that neither of these cases hold. Then with $I = [a, b)$ we must have $a < x < b$. Then

$$\begin{aligned}
\lambda(I \cap (-\infty, x)) + \lambda(I\backslash(-\infty, x)) &= \lambda([a, x)) + \lambda([x, b)) \\
&= \lambda([a, x)) + \lambda([x, b)) \\
&= x - a + b - x \\
&= b - a \\
&= \lambda([a, b)) \\
&= \lambda(I).
\end{aligned}$$

So (1) holds.

Now for the proof of the lemma, let $A \subseteq \mathbb{R}$ and let $\varepsilon > 0$. We show that $\theta'(A \cap (-\infty, x)) + \theta'(A \backslash (-\infty, x)) \leq \theta'(A) + \varepsilon$, which will prove the lemma. By the definition of θ', there is a sequence $\langle I_j : j \in \omega \rangle$ of half-open intervals such that $A \subseteq \bigcup_{j\in\omega} I_j$ and $\sum_{j\in\omega} \lambda(I_j) \leq \theta'(A) + \varepsilon$. Now $\langle I_j \cap (-\infty, x) : j \in \omega \rangle$ and $\langle I_j \backslash (-\infty, x) : j \in \omega \rangle$ are sequences of half-open intervals, $A \cap (-\infty, x) \subseteq \bigcup_{j\in\omega}(I_j \cap (-\infty, x))$, and $A \backslash (-\infty, x) \subseteq \bigcup_{j\in\omega}(I_j \backslash (-\infty, x))$, so

$$\begin{aligned}
\theta'(A \cap (-\infty, x)) + \theta'(A \backslash (-\infty, x)) &\leq \sum_{j=0}^{\infty} \lambda(I_j \cap (-\infty, x)) + \sum_{j=0}^{\infty} \lambda(I_j \backslash (-\infty, x)) \\
&= \sum_{j=0}^{\infty} \lambda(I_j) \leq \theta'(A) + \varepsilon. \qquad \square
\end{aligned}$$

Theorem 10.7. *Every Borel subset of $\mathbb{R}$ is Lebesgue measurable.*

Proof. It suffices to show that every open set is Lebesgue measurable. It then suffices to prove the following:

(1) If U is a nonempty open subset of $\mathbb{R}$, then there is a family $\mathscr{A}$ of half-open intervals with rational coefficients such that $U = \bigcup \mathscr{A}$.

To prove (1), let $\mathscr{A}$ be the set of all half-open intervals contained in U. Now take any $x \in U$. Since U is open, there are real numbers $y < z$ such that $x \in (y, z) \subseteq U$. Choose rational numbers r, s such that $y < r < x < s < z$. Then $x \in [r, s) \subseteq U$, as desired. □

Corollary 10.8. *Every Lebesgue null set is Lebesgue measurable. Every singleton is a null set, and every countable set is a null set.* □

Lemma 10.9. *Suppose that μ is a measure and E, F, G are μ-measurable. Then*

$$\mu(E \triangle F) \leq \mu(E \triangle G) + \mu(G \triangle F).$$

Proof.

$$\begin{aligned}
\mu(E \triangle F) &= \mu(E \backslash F) + \mu(F \backslash E) \\
&= \mu((E \backslash F) \cap G) + \mu((E \backslash F) \backslash G) + \mu(F \backslash E) \cap G) + \mu((F \backslash E) \backslash G) \\
&\leq \mu(G \backslash F) + \mu(E \backslash G) + \mu(G \backslash E) + \mu(F \backslash G) \\
&= \mu(E \triangle G) + \mu(G \triangle F). \qquad \square
\end{aligned}$$

Lemma 10.10. *If E is Lebesgue measurable with finite measure, then for any $\varepsilon > 0$ there is an open set $U \supseteq E$ such that $\theta'(E) \leq \theta'(U) \leq \theta'(E) + \varepsilon$. Moreover, there is a system $\langle K_j : j < \omega \rangle$ of open intervals such that $U = \bigcup_{j<\omega} K_j$ and $\theta'(U) \leq \sum_{j<\omega} \theta'(K_j) \leq \theta'(E) + \varepsilon$.*

Proof. By the basic definition of Lebesgue measure,

$$0 = \theta'(E) = \inf\left\{ \sum_{j\in\omega} \theta'(I_j) : \langle I_j : j \in \omega\rangle \text{ is a sequence of half-open intervals} \right.$$
$$\left. \text{such that } A \subseteq \bigcup_{j\in\omega} I_j \right\}.$$

Hence we can choose a sequence $\langle I_j : j \in \omega\rangle$ of half-open intervals such that $E \subseteq \bigcup_{j\in\omega} I_j$ and

$$\theta'\left(\bigcup_{j\in\omega} I_j\right) \le \sum_{j\in\omega} \theta'(I_j) \le \theta'(E) + \frac{\varepsilon}{2}.$$

Write $I_j = [a_j, b_j)$ with $a_j < b_j$. Define

$$\begin{aligned} K_j &= \left(a_j - \frac{\varepsilon}{2^{j+2}}, b_j\right); \quad \text{then} \\ E &\subseteq \bigcup_{j\in\omega} K_j \quad \text{and} \\ \theta'\left(\bigcup_{j\in\omega} K_j\right) &\le \sum_{j\in\omega} \theta'(K_j) \\ &= \sum_{j\in\omega} \left(\frac{\varepsilon}{2^{j+2}} + \theta'(I_j)\right) \\ &= \sum_{j\in\omega} \frac{\varepsilon}{2^{j+2}} + \sum_{j\in\omega} \theta'(I_j) \\ &\le \frac{\varepsilon}{2} + \theta'(E) + \frac{\varepsilon}{2} = \theta'(E) + \varepsilon. \end{aligned}$$ □

Corollary 10.11. *(i) If A is Lebesgue measurable and $\theta'(A)$ is finite, then $\theta'(A) = \inf\{\theta'(U) : U$ open, $A \subseteq U\}$.*

(ii) If A is Lebesgue measurable with finite measure, then $\theta'(A) = \sup\{\theta'(C) : C$ closed, $C \subseteq A\}$.

(iii) If A is measurable and $\theta'(A) = \infty$, then $\sup\{\theta'(C) : C$ closed, $C \subseteq A\} = \infty$.

Proof. Only (iii) needs a proof. Let $\varepsilon > 0$. For each $n \in \omega$ let

$$\begin{aligned} a_{2n} &= n; \\ b_{2n} &= n + 1; \\ a_{2n+1} &= -n - 1; \\ b_{2n+1} &= -n. \end{aligned}$$

For each $n \in \omega$ let C_n be a closed subset of $[a_n, b_n) \cap A$ such that

$$\theta'([a_n, b_n) \cap A \backslash C_n) < \frac{\varepsilon}{2^n}.$$

Then

$$\begin{aligned}
\theta'(A) &= \sum_{n \in \omega} \theta'([a_n, b_n) \cap A) \\
&= \lim_{n=0}^{\infty} \theta'([[a_0, b_0) \cap A] \cup \ldots \cup [[a_n, b_n) \cap A]) \\
&= \lim_{n=0}^{\infty} \theta'([[a_0, b_0) \cap A \backslash C_0] \cup \ldots \cup [[a_n, b_n) \cap A \backslash C_n]) \\
&\qquad + \theta'(C_0 \cup \ldots \cup C_n) \\
&= \lim_{n=0}^{\infty} \theta'([[a_0, b_0) \cap A \backslash C_0] \cup \ldots \cup [[a_n, b_n) \cap A \backslash C_n]) \\
&\qquad + \lim_{n \to \infty} \theta'(C_0 \cup \ldots \cup C_n) \\
&= \varepsilon + \lim_{n \to \infty} \theta'(C_0 \cup \ldots \cup C_n),
\end{aligned}$$

as desired.

Theorem 10.12. *Every set of Lebesgue measure 0 is included in a G_δ set of measure 0.*

Proof. Let X be Lebesgue measurable of measure 0. By the above, for every positive integer n there is an open set U_n such that $X \subseteq U_n$ and $\mu(U_n) \leq \frac{1}{n}$. Then $V = \bigcap_{n \in \omega} U_n$ is as desired. □

Theorem 10.13. *If X is Lebesgue measurable, then one can write $X = U \cup N$ with U a G_δ and N of measure 0.* □

Lemma 10.14. *Let θ' be as in the definition preceding Lemma 10.4. For any $A \subseteq \mathbb{R}$ and $c \in \mathbb{R}$ let $A + c = \{x + c : x \in A\}$. Then $\theta'(A) = \theta'(A + c)$.*

Proof. Let $\langle I_j : j \in \omega \rangle$ be a sequence of half-open intervals such that $A \subseteq \bigcup_{j \in \omega} I_j$. For each $j \in \omega$, $I_j + c$ is a half-open interval, and $A + c \subseteq \bigcup_{j \in \omega} (I_j + c)$. For each $\varepsilon > 0$ choose $\langle I_j : j \in \omega \rangle$ such that $\theta'(A) + \varepsilon > \sum_{j \in \omega} \lambda(I_j)$. Note that $\lambda(I_j) = \lambda(I_j + c)$. Hence

$$\theta'(A) + \varepsilon > \sum_{j \in \omega} \lambda(I_j) = \sum_{j \in \omega} \lambda(I_j + c) \geq \theta'(A + c).$$

Hence $\theta'(A) \geq \theta'(A + c)$. Similarly, $\theta'(A + c) \geq \theta'(A + c - c) = \theta'(A)$. □

Theorem 10.15. *If X is Lebesgue measurable then so is $X + c$, and $\mu(X) = \mu(X + c)$.*

Proof. Since X is measurable, for each $Y \subseteq \mathbb{R}$ we have

$$\mu(Y) = \mu(X \cap Y + \mu(Y \backslash X).$$

Then for all $c, x \in \mathbb{R}$ we have

$$\begin{aligned} x \in Y \cap (X+c) \quad &\text{iff} \quad x \in Y \text{ and } x \in X + c \\ &\text{iff} \quad x \in Y \text{ and } x - c \in X \\ &\text{iff} \quad x - c \in Y - c \text{ and } x - c \in X \\ &\text{iff} \quad x - c \in (Y - c) \cap X \\ &\text{iff} \quad x \in ((Y - c) \cap X) + c. \end{aligned}$$

Thus $Y \cap (X + c) = ((Y - c) \cap X) + c$. Hence by Theorem 10.14,

$$\mu(Y \cap (X + c)) = \mu(((Y - c) \cap X) + c) = \mu((Y - c) \cap X).$$

Similarly,

$$\begin{aligned} x \in Y \backslash (X+c) \quad &\text{iff} \quad x \in Y \text{ and } x \notin X + c \\ &\text{iff} \quad x \in Y \text{ and } x - c \notin X \\ &\text{iff} \quad x - c \in Y - c \text{ and } x - c \notin X \\ &\text{iff} \quad x - c \in (Y - c) \backslash X \\ &\text{iff} \quad x \in ((Y - c) \backslash X) + c. \end{aligned}$$

Thus $Y \backslash (X + c) = ((Y - c) \backslash X) + c$. Hence by Theorem 10.14,

$$\mu(Y \backslash (X + c)) = \mu(((Y - c) \backslash X) + c) = \mu((Y - c) \backslash X).$$

Hence

$$\begin{aligned} \mu(Y \cap (X + c)) + \mu(Y \backslash (X + c)) &= \mu(((Y - c) \cap X)) + \mu((Y - c) \backslash X) \\ &= \mu(Y - c) = \mu(Y). \end{aligned}$$ $\square$

Theorem 10.16. *There is a subset of $\mathbb{R}$ which is not Lebesgue measurable.*

Proof. Let X have exactly one element in common with each element of $\mathbb{R}/\mathbb{Q}$. Then

(1) $\forall u, v \in X[u \neq v \rightarrow v - u$ is irrational].

Let $q_0, q_1, \ldots$ enumerate the rational numbers in $[-1, 1]$. Let $X' = X \cap [0, 1]$. For each $m \in \omega$ let $Y_m = X' + q_m$. Then

(2) $$[0, 1] \subseteq \bigcup_{k \in \omega} Y_k \subseteq [-1, 2].$$

In fact, for the first inclusion let $r \in [0, 1]$. Let $v \in X'$ be the representative of $[r]$. Then $r - v$ is rational, and clearly $r - v \in [-1, 1]$. Say $r - v = q_k$. Hence $r = q_k + v \in Y_k$, as desired. The second inclusion is clear. If $m \neq n$ and $x \in Y_m \cap Y_n$, then there exist

$x, y \in X'$ such that $x + q_m = y + q_n$. Then $x - y$ is rational, contradicting (1). So the Y_ms are pairwise disjoint. Hence

$$1 \leq \sum_{m\in\omega} \mu(Y_m) = \sum_{m\in\omega} \mu(X'),$$

contradiction. □

Proposition 10.17. *An ultrafilter U on S is σ-complete iff there is no partition of S into countably many disjoint parts $S = \bigcup_{n\in\omega} X_n$ such that $X_n \notin U$ for all $n \in \omega$.*

Proof. $\Rightarrow$: obvious.

$\Leftarrow$: Suppose that U is not σ-complete. Then there is a system $\langle X_n : n \in \omega\rangle$ of members of U such that $\bigcap_{n\in\omega} X_n \notin U$. For each $i < \omega$ let $Y_i = \bigcap_{j<i} X_j$. Let $Z_i = Y_i \backslash Y_{i+1}$. We claim that $\langle Z_i : i < \omega\rangle^\frown\langle\bigcap_{i<\omega} X_i\rangle$ is a partition of S; since clearly none of these sets is in U, this will show that the indicated condition fails. Clearly the sets are pairwise disjoint. Now suppose that $s \in S$. We may assume that $s \notin \bigcap_{n\in\omega} X_n$. Let i be minimum such that $s \notin X_i$. Then $s \in Y_i$, as desired. □

For any set S, a it full measure on S is a measure on $\mathscr{P}(S)$ which gives singletons measure 0. A *two-valued* measure is a measure which takes only the values 0 and 1.

Theorem 10.18. *If μ is a full two-valued measure on S, and $U = \{X \subseteq S : \mu(X) = 1\}$, then U is a σ-complete ultrafilter on S.*

Proof. Clearly $S \in U$, and if $X \in U$ and $X \subseteq Y$, then $Y \in U$. Now suppose that $X, Y \in U$. Now $X \cup Y = (X\backslash Y) \cup (Y\backslash X) \cup (X \cap Y)$. If $\mu(X \cap Y) = 0$, then $1 = \mu(X) = \mu(X \cap Y) + \mu(X \cap Y) = \mu(X\backslash Y)$; similarly $\mu(Y\backslash X) = 0$ and hence $\mu(X \cup Y) = 2$, contradiction. Hence $\mu(X\cap Y) = 1$ and $X\cap Y \in U$. For any $X \subseteq S$ we have $S = X\cup(S\backslash X)$, and it follows that $X \in U$ or $(S\backslash X) \in U$. Hence U is an ultrafilter on S. U is σ-complete: we prove this by using Theorem 10.17. In fact, suppose that $\langle X_n : n \in \omega\rangle$ is a partition of S such that $X_n \notin U$ for all $n \in \omega$. Then $\mu(\bigcup_{n\in\omega} X_n) = \sum_{n\in\omega} \mu(X_n) = 0$, contradiction. □

Theorem 10.19. *If U is a nonprincipal σ-complete ultrafilter on S, define for any $X \subseteq S$,*

$$\mu(X) = \begin{cases} 1 & \textit{if } X \in U, \\ 0 & \textit{otherwise.} \end{cases}$$

Then μ is a two valued measure on S.

Proof. Clearly μ takes only the values 0 and 1. Clearly $\mu(\emptyset) = 0$. Now suppose that $X \subseteq Y \subseteq S$. If $\mu(X) = 1$, clearly $\mu(Y) = 1$. So $\mu(X) \leq \mu(Y)$. Since U is nonprincipal, clearly $\mu(\{a\}) = 0$. If $\langle X_n : n \in \omega\rangle$ is a pairwise disjoint system of subsets of S, then $\mu(\bigcup_{n\in\omega} X_n = 1$ iff $\bigcup_{n\in\omega} X_n \in U$ iff exactly one X_n is in U iff $\sum_{n\in\omega} \mu(X_n) = 1$. □

If μ is a measure on a set S, a subset $A \subseteq S$ is a μ-atom iff $\mu(A) = 0$ and for every $B \subseteq A$, either $\mu(B) = 0$ or $\mu(B) = \mu(A)$.

Proposition 10.20. *If μ is a measure on S and μ has an atom A, then*

$$U \stackrel{\text{def}}{=} \{X \subseteq S : \mu(X \cap A) = \mu(A)\}$$

is a σ-complete ultrafilter on S.

Proof. Obviously $\emptyset \notin U$, and $S \in U$. If $X \in U$ and $X \subseteq Y \subseteq S$, clearly $Y \in U$. If $X \subseteq S$, then $\mu(A) = \mu(X \cap A) + \mu((S \backslash X) \cap A)$; hence $X \in U$ or $(S \backslash X) \in U$, but not both. If $X, Y \in U$, then $\mu(A) = \mu(X \cap A) = \mu((X \backslash Y) \cap A) + \mu(X \cap Y \cap A)$; since $\mu((X \backslash Y) \cap A) \leq \mu((S \backslash Y) \cap A) = 0$, it follows that $\mu(A) = \mu(X \cap Y \cap A)$, so that $X \cap Y \in U$.

Finally, suppose that $X_i \in U$ for each $i < \omega$. Define $Y_0 = X_0$ and $Y_{i+1} = X_{i+1} \cap Y_i$. Thus $\forall i \in \omega[Y_i \in U]$. Since $Y_{i+1} \cap (Y_i \backslash Y_{i+1}) = \emptyset$, it follows that $\mu(Y_i \backslash Y_{i+1}) = 0$, since A is an atom. Hence

$$\begin{aligned}
\mu(A) &= \mu(Y_0 \cap A) \\
&= \mu\left(\left(\bigcap_{i\in\omega} X_i \cap A\right) \cup \bigcup_{i\in\omega}((Y_i \backslash Y_{i+1}) \cap A)\right) \\
&= \mu\left(\bigcap_{i\in\omega} X_i \cap A\right) + \sum_{i\in\omega} \mu((Y_i \backslash Y_{i+1}) \cap A) \\
&= \mu\left(\bigcap_{i\in\omega} X_i \cap A\right),
\end{aligned}$$

as desired. □

Lemma 10.21. *If κ is an infinite cardinal and U is a nonprincipal ultrafilter on κ, then the following conditions are equivalent:*

(i) U is not κ-complete.

(ii) There is a $\gamma < \kappa$ and a partition $\langle X_\alpha : a < \gamma\rangle$ of κ such that $\forall \alpha < \gamma[X_\alpha \notin U]$.

Proof. (i)⇒(ii): Choose $\gamma < \kappa$ and a system $\langle X_\alpha : \alpha < \gamma\rangle$ of elements of U such that $\bigcap_{\alpha<\gamma} X_\alpha \notin U$. For each $\alpha < \gamma$ let $Y_\alpha = (\kappa \backslash X_\alpha) \cap \bigcap_{\beta<\alpha} X_\beta$. Then $\{Y_\alpha : \alpha < \gamma\} \cup \{\bigcap_{\alpha<\gamma} X_\alpha\}$ is a partition of κ with all members not in U.

(ii)⇒(i): obvious. □

Lemma 10.22. *Let κ be the least cardinal such that there is a σ-complete ultrafilter U on κ. Then U is κ-complete.*

Proof. Suppose that U is not κ-complete. By Lemma 10.21, there is a $\gamma < \kappa$ and a partition $\langle X_\alpha : \alpha < \gamma\rangle$ of κ such that $\forall \alpha < \gamma[X_\alpha \notin U]$. Define $F : \kappa \to \gamma$ by: $F(\beta) =$ the α such that $\beta \in X_\alpha$. Clearly F maps onto γ. Define $D \subseteq \mathscr{P}(\gamma)$ by

$$Z \in D \quad \text{iff} \quad f^{-1}[Z] \in U.$$

Since $f^{-1}[\gamma] = \kappa$, we have $\gamma \in D$. Clearly if $Z \subseteq Z'$ and $Z \in D$, then $Z' \in D$. If $Z, Z" \in D$, then $f^{-1}[Z \cap Z'] = f^{-1}[Z] \cap f^{-1}[Z'] \in U$, so $Z \cap Z' \in D$. For any $Z \subseteq \gamma$ we have $\kappa = f^{-1}[\gamma] = f^{-1}[Z] \cup f^{-1}[\gamma \backslash Z]$, and it follows that $Z \in D$ or $(\gamma \backslash Z) \in D$. Thus D is an ultrafilter on γ. Clearly it is σ-complete. Suppose that $\alpha < \gamma$ and $\{\alpha\} \in D$. Thus $X_\alpha = F^{-1}[\{\alpha\}] \in U$, contradiction. This contradicts the minimality of κ. □

A cardinal κ is *measurable* iff κ is uncountable and there is a κ-complete nonprincipal ultrafilter on κ.

Corollary 10.23. *If κ is the least cardinal such that there is a nontrivial two-valued measure on κ, then κ is measurable.*

Theorem 10.24. *Every measurable cardinal is inaccessible.*

Proof. Let κ be measurable. Say that D is a nontrivial κ-complete ultrafilter on κ. If $X \in [\kappa]^{<\kappa}$ then $X \notin D$. Clearly κ is regular. Now suppose that $\lambda < \kappa \leq 2^\lambda$. Let S be a set of functions mapping λ into 2 such that $|S| = \kappa$. Let U be a nonprincipal ultrafilter on S. For each $\alpha < \lambda$ let

$$X_\alpha = \begin{cases} \{f \in S : f(\alpha) = 0\} & \text{if this set is in } U \\ \{\{f \in S : f(\alpha) = 1\} & \text{otherwise.} \end{cases}$$

Since U is κ-complete, $Y \stackrel{\text{def}}{=} \bigcap_{\alpha<\lambda} X_\alpha$. But this set has at most one member, namely the function g such that for each $\alpha < \lambda$,

$$g(\alpha) = \begin{cases} 0 & \text{if } X_\alpha = \{f \in S : f(\alpha) = 0\} \in U, \\ 1 & \text{if } \{f \in S : f(\alpha) = 1\} \in U. \end{cases}$$

This is a contradiction. □

Let μ be a nontrivial measure on a set S. We define

$$I_\mu = \{X \subseteq S : \mu(X) = 0\}.$$

A σ-complete ideal I on a set S is σ-saturated provided that
(i) $\forall x \in S[\{x\} \in I]$;
(ii) Every family of pairwise disjoint sets each not in I is countable.

Proposition 10.25. *Let μ be a nontrivial measure on a set S. Then I_μ is σ-saturated.* □

Proposition 10.26. *If κ is the least cardinal having a nontrivial σ-additive measure μ, then I_μ is κ-complete.*

Proof. Suppose that I_μ is not κ-complete. Let $\langle X_\alpha : \alpha < \gamma \rangle$ be a system of null sets such that $\bigcup_{\alpha<\gamma} X_\alpha$ has positive measure. For each $\alpha < \gamma$ let $Y_\alpha = X_\alpha \backslash \bigcup_{\beta<\alpha} X_\beta$. Then the Y_αs are pairwise disjoint and $Z \stackrel{\text{def}}{=} \bigcup_{\alpha<\gamma} X_\alpha = \bigcup_{\alpha<\gamma} Y_\alpha$. Let $m = \mu(Z)$. Let $f : Z \to \gamma$ be defined by

$$f(x) = \text{ the } \alpha < \gamma \text{ such that } x \in Y_\alpha.$$

Define $\nu : \mathscr{P}(\gamma) \to \mathbb{R}^+$ by

$$\nu(Z) = \frac{1}{m}\mu(f^{-1}[Z]).$$

Then ν is clearly a measure on γ. It is nontrivial, since $\nu(\{\alpha\}) = \mu(Y_\alpha) = 0$. This contradicts the minimality of κ. $\square$

Proposition 10.27. *If κ is the least cardinal such that there is a σ-complete σ-saturated ideal I on κ, then I is κ-complete.*

Proof. Suppose not. Say $\gamma < \kappa$ and $\langle X_\alpha : \alpha < \gamma\rangle$ is a system of elements of I such that $\bigcup_{\alpha<\gamma} X_\alpha \notin I$. Let f be as in the proof of Proposition 10.26. Define an ideal J on γ by

$$X \in J \quad \text{iff} \quad f^{-1}[X] \in I.$$

Clearly J is a σ-saturated σ-complete ideal on γ, contradicting the minimality of I. $\square$

If $\langle r_i : i \in I\rangle$ is a system of real numbers, then

$$\sum_{i\in I} r_i = \sup\left\{\sum_{\in J} r_i : J \in [I]^{<\omega}\right\}.$$

Let κ be an uncountable cardinal. A measure μ on a set S is κ-additive iff for every $\gamma < \kappa$ and every system $\langle X_\alpha : \alpha < \gamma\rangle$ of subsets of S,

$$\mu\left(\bigcup_{\alpha<\gamma} X_\alpha\right) = \sum_{\alpha<\gamma} \mu(X_\alpha).$$

Corollary 10.28. *If μ is a κ-additive measure, then I_μ is κ-complete.*

Proposition 10.29. *If μ is a measure on S and I_μ is κ-complete, then μ is κ-additive.*

Proof. Let $\gamma < \kappa$ and let $\langle X_\alpha : \alpha < \gamma\rangle$ be a system of subsets of S. For all $\alpha < \gamma$ let $Y_\alpha = X_\alpha \backslash \bigcup_{\beta<\alpha} X_\beta$. Then $\bigcup_{\alpha<\gamma} X_\alpha = \bigcup_{\alpha<\gamma} Y_\alpha$. Write

$$\{Y_\alpha : \alpha < \gamma\} = \{Z_n : n \in \omega\} \cup \{W_\alpha : \alpha < \gamma\}$$

where each W_α has measure 0. Since I_μ is κ-complete we have

$$\begin{aligned}
\mu\left(\bigcup_{\alpha<\gamma} Y_\alpha\right) &= \mu\left(\bigcup_{n\in\omega} Z_n\right) + \mu\left(\bigcup_{\alpha<\gamma} W_\alpha\right)\\
&= \sum_{n\in\omega} \mu(Z_n)\\
&= \sum_{n\in\omega} \mu(Z_n) + \sum_{\alpha<\gamma} \mu(W_\alpha)\\
&= \sum_{\alpha<\gamma} \mu(Y_\alpha)
\end{aligned}$$

$\square$

Corollary 10.30. *Let κ be the least cardinal such that there is a nontrivial σ-additive measure μ on κ. Then μ is κ-additive.* □

An uncountable cardinal κ is *real-valued measurable* iff there is a nontrivial κ-additive measure on κ.

Proposition 10.31. *If κ is real-valued measurable, then κ is regular.*

Proof. Let μ be a κ-additive measure on κ. Since μ is nontrivial and κ-additive, every subset of κ of size less than κ has measure 0. So κ is not the union of fewer than κ sets each of size less than κ. □

Proposition 10.32. *If there exists an atomless non-trivial σ-additive measure, then there is a non-trivial σ-additive measure on some $\kappa \leq 2^\omega$.*

Proof. Let μ be an atomless non-trivial σ-additive measure on a set S. We construct a tree of subsets of S, ordered by $\supseteq$. Let the 0th level be S. Now suppose that $X \in T$ has been defined so that X has positive measure Then there is a $Y \subset X$ such that $0 < \mu(Y) < \mu(X)$. Let $Z = X \backslash Y$. We add Y and Z to T. If α is a limit ordinal, then the α-th level consists of all intersections $\bigcap_{\xi<\alpha} X_\xi$ where each X_ξ is at level ξ and the intersection has positive measure.

The levels consist of pairwise disjoint positive measure sets, and so each level is countable.

If b is a branch, then $\bigcap_{X\in b} X$ has measure 0. Clearly every branch is countable.

Hence T has height at most ω_1.

(1) There are at most 2^ω branches.

In fact, it suffices to show that for each $\gamma < \omega_1$ there are at most 2^ω branches of height γ. By the above there are countably many elements of T of height less than γ. Let A be the set of elements of T of height less than γ. Then the number of branches of height γ is at most $|{}^\gamma A|$. (1) follows.

Let $\langle b_\alpha : \alpha < \kappa$ enumerate all branches $b = \langle X_\xi : \xi < \gamma\rangle$ such that $\bigcap\{X_\xi : \xi < \gamma\}$ is nonempty. For each $\alpha < \kappa$ let $Z_\alpha = \bigcap\{X \in b_\alpha\}$. Then $\{Z_\alpha : \alpha < \kappa\}$ is a partition of S into κ nonempty sets, each of measure 0. Now we define $f : S \to \kappa$ and ν with domain $\mathscr{P}(S)$:

$$\begin{aligned} f(x) = \alpha \quad &\text{iff} \quad x \in Z_\alpha; \\ \nu(W) &= \mu(f^{-1}[W]). \end{aligned}$$

We claim that ν is a non-trivial σ-additive measure on κ. Clearly $\mu(\emptyset) = 0$ and $\mu(\kappa) = 1$. If $W \subseteq V$, clearly $\nu(W) \leq \nu(V)$. For any$\alpha < \kappa$ we have $\nu(\{\alpha\}) = \mu(f^{-1}[\{\alpha\}] = \mu(Z_\alpha) = 0$. Now suppose that $W_0, W_1, \ldots$ are pairwise disjoint subsets of κ. Then

$$\nu\left(\bigcup_{n\in\omega} W_n\right) = \mu\left(f^{-1}\left[\bigcup_{n\in\omega} W_n\right]\right) = \mu\left(\bigcup_{n\in\omega} f^{-1}[W_n]\right) = \sum_{n\in\omega} \mu(f^{-1}[W_n]) = \sum_{n\in\omega} \nu(W_n).$$

□

Corollary 10.33. it If μ is an atomless non-trivial σ-additive measure on a set S, then there is a partition of S into at most 2^ω null sets. $\square$

Proposition 10.34. *If there is an atomless non-trivial σ-additive measure on some set S, then there is an atomless non-trivial σ-additive measure on $\mathbb{R}$.*

Proof. Let μ be an atomless non-trivial σ-additive measure on $Y \subseteq \mathbb{R}$; see Proposition 10.31. For any $Z \subseteq \mathbb{R}$ define $\mu'(Z) = \mu(Y \cap Z)$. $\square$

Lemma 10.35. *If I is a σ-complete, σ-saturated ideal on S, then either there exists $Z \subseteq S$ such that $I \restriction Z \stackrel{\text{def}}{=} \{X \subseteq Z : X \in I\}$ is a prime ideal or there exists a σ-complete, σ-saturated ideal on some $\kappa \leq 2^\omega$.*

Proof. Suppose I is as indicated. Assume

(1) For every $Z \subseteq S$, the set $\mathscr{P}(Z) \cap I$ is not a maximal ideal on Z.

We construct a tree T of subsets of S by recursion. Each member of T is not in I. The 0th level of T is $\{S\}$. If $X \in T$, then by (1) there is a subset Y of X such that $Y, X\backslash Y \notin I$; we let Y and $X\backslash Y$ be the successors of X in the tree. If α is a limit ordinal, then the α-th level of T consists of all intersections $\bigcap_{\xi<\alpha} X_\xi$ such that each X_ξ is at level ξ and $\bigcap_{\xi<\alpha} X_\xi \notin I$.

Each branch of T has countable length, since if $\langle X_\xi : \xi < \alpha\rangle$ is a branch, with each X_ξ at level ξ, then $\langle X_\xi\backslash X_{\xi+1} : \xi < \alpha\rangle$ is a pairwise disjoint system of elements of $\mathscr{P}(S)\backslash I$ so α is countable by the σ-saturation of I.

It follows that T has at most 2^ω branches. Let $\langle b_\xi : \xi < \kappa\rangle$ enumerate all of the branches of T such that $\bigcap b_\xi \neq \emptyset$, and let $Z_\xi = \bigcap b_\xi$. Clearly $\langle Z_\xi : \xi < \kappa\rangle$ is a partition of S into κ nonempty sets, all in I. Now let

$$J = \left\{ W \subseteq \kappa : \bigcup_{\xi\in W} Z_\xi \in I \right\}.$$

Clearly J is a σ-complete ideal on κ. Since $Z_\alpha \in I$, we have $\{\alpha\} \in J$ for each $\alpha < \kappa$. J is proper since I is proper. Suppose that $\mathscr{A}$ is a family of pairwise disjoint subsets of κ each not in J. Then $\{\bigcup A : A \in \mathscr{A}\}$ is a family of pairwise disjoint subsets of S each not in I, so $\mathscr{A}$ is countable. Thus J is σ-saturated. $\square$

Lemma 10.36. *Let μ be an atomless measure on S. If $\mu(Y) > 0$, then there is a $W \subseteq Y$ such that $0 < \mu(W) \leq \frac{1}{2}\mu(Y)$.*

Proof. Since μ is atomless, there is a $V \subseteq Y$ such that $0 < \mu(V) < \mu(Y)$. Then $\mu(V) \leq \frac{1}{2}\mu(Y)$ or $\mu(Y\backslash V) \leq \frac{1}{2}\mu(Y)$, and $0 \neq \mu(V), \mu(Y\backslash V)$. $\square$

By repeated applications of this lemma we get

Lemma 10.37. *If μ is an atomless measure on S, $\mu(Y) > 0$, and $\delta > 0$, then there is a $W \subseteq Y$ such that $0 < \mu(W) < \delta$.* $\square$

Lemma 10.38. *If μ is an atomless measure on S and $\mu(Y) > \varepsilon$, then there is a $V \subseteq Y$ such that $\varepsilon \leq \mu(V) < \mu(Y)$.*

Proof. By Lemma 10.37, choose $W \subseteq Y$ such that $0 < \mu(W) < \mu(Y) - \varepsilon$. Then $\mu(Y) = \mu(W) + \mu(Y\backslash W)$, hence

$$\mu(Y) > \mu(Y\backslash W) = \mu(Y) - \mu(W) > \varepsilon. \qquad \square$$

Lemma 10.39. *If μ is a measure on S, β is a countable limit ordinal, and $Y_0, Y_1, \ldots \in$* dmn(μ) *and $Y_0 \subseteq Y_1 \cdots \subseteq Y_\alpha \subseteq \cdots$ for $\alpha < \beta$, then $\mu(\bigcup_{\alpha<\beta} Y_\alpha) = \sup\{\mu(Y_\alpha) : \alpha < \beta\}$.*

Proof. Define $W_\alpha = Y_\alpha \backslash \bigcup_{\gamma<\alpha} Y_\gamma$ for all $\alpha < \beta$. Then $\langle W_\alpha : \alpha < \beta\rangle$ is disjointed, $\forall \alpha < \beta[\bigcup_{\gamma\leq\alpha} W_\alpha = Y_\alpha]$, and $\bigcup_{\alpha<\beta} W_\alpha = \bigcup_{\alpha<\beta} Y_\alpha$. Hence

$$\mu\left(\bigcup_{\alpha<\beta} Y_\alpha\right) = \mu\left(\bigcup_{\alpha<\beta} W_\alpha\right) = \sum_{\alpha<\beta} \mu(W_\alpha) = \sup_{\alpha<\beta} \sum_{\gamma\leq\alpha} \mu(W_\gamma) = \sup_{\alpha<\beta} \mu(Y_\alpha). \qquad \square$$

Lemma 10.40. *If μ is a measure on S, $Y_0, Y_1, \ldots \in$* dmn(μ) *for $\alpha < \beta$, with β a countable limit ordinal; and $Y_0 \supseteq Y_1 \cdots \supseteq Y_\alpha \supseteq \cdots$ for $\alpha < \beta$, then $\mu(\bigcap_{\alpha<\beta} Y_\alpha) = \inf\{\mu(Y_\alpha) : \alpha < \beta\}$.*

Proof. We have $S\backslash Y_0 \subseteq S\backslash Y_1 \subseteq \cdots$; hence $\mu(\bigcup_{\alpha<\beta}(S\backslash Y_\alpha) = \sup\{\mu(S\backslash Y_\alpha) : \alpha < \beta\}$. Hence

$$\begin{aligned}\mu\left(\bigcap_{\alpha<\beta} Y_\alpha\right) &= \mu(S) - \mu\left(\bigcup_{\alpha<\beta} (S\backslash Y_\alpha\right) \\ &= \mu(S) - \sup\{\mu(S\backslash Y_\alpha) : \alpha < \beta\} = \inf\{\mu(Y_\alpha) : \alpha < \beta\}. \qquad \square\end{aligned}$$

Lemma 10.41. *Suppose that μ is an atomless measure on S and $Z_0 \subseteq S$. Then there is a $Z \subseteq Z_0$ such that $\mu(Z) = (1/2)\mu(Z_0$.*

Proof. If $\mu(Z_0) = 0$, we are through, so suppose that $\mu(Z_0) > 0$. We construct Z_α for $\alpha < \omega_1$ by recursion, so that always $\mu(Z_\alpha) \geq \frac{1}{2}\mu(Z_0)$. If Z_α has been constructed, let $Z_{\alpha+1} = Z_\alpha$ if $\mu(Z_\alpha) = \frac{1}{2}\mu(Z_0)$. Suppose that $\mu(Z_\alpha) > \frac{1}{2}\mu(Z_0)$. By Lemma 9.131, choose $Z_{\alpha+1} \subseteq Z_\alpha$ such that $\frac{1}{2}\mu(Z_0) \leq \mu(Z_{\alpha+1}) < \mu(Z_\alpha)$.

If β is limit and Z_α has been constructed for all $\alpha < \beta$, let $Z_\beta = \bigcap_{\alpha<\beta} Z_\alpha$. By Lemma 9.133 $\frac{1}{2}\mu(Z_0) \leq \mu(Z_\beta)$.

The sets $Z_\alpha\backslash Z_{\alpha+1}$ are pairwise disjoint and of positive measure. Hence the construction stops at some $\alpha < \omega_1$. Then $\mu(Z_\alpha) = \frac{1}{2}\mu(Z_0)$. $\square$

Theorem 10.42. *If there is an atomless σ-additive measure on a set S, then there is a σ-additive measure on $\mathbb{R}$ which extends Lebesgue measure.*

Proof. Let μ be an atomless σ-additive measure on $\mathbb{R}$. Now we define

$$X_0 = \mathbb{R};$$
$$\mu(X_{s\frown\langle 0\rangle}) = \mu(X_{s\frown\langle 1\rangle)}) = \frac{1}{2}\mu(X_s);$$

So X_s is defined for all $s \in {}^{<\omega}2$. Now define $\nu_1 \in {}^{\omega}2$ by

$$\nu_1(Y) = \mu\left(\bigcup\left(\left\{\bigcap_{n\in\omega} X_{f\restriction n} : f \in Y\right\}\right)\right)$$

(1) ν_1 is an atomless σ-additive measure on ${}^{\omega}2$.

For, clearly $\nu_1(\emptyset) = 0$. For any $r \in \mathbb{R}$ let $f \in {}^{\omega}2$ be such that $r \in X_{f\restriction m}$ for all $m \in \omega$. Then $r \in X_f$. Hence $\bigcup\{X_f : f \in {}^{\omega}2\} = \mathbb{R}$ and so $\nu_1({}^{\omega}2) = 1$. Clearly $Z_1 \subseteq Z_2$ implies that $\nu_1(Z_1) \leq \nu_1(Z_2)$. For each $f \in {}^{\omega}2$ it is clear that $\mu(X_f) = 0$, so $\nu_1(\{f\}) = 0$. Now suppose that $\langle Z_n : n \in \omega\rangle$ is a pairwise disjoint system of subsets of ${}^{\omega}2$. Then $\langle\bigcup\{X_f : f \in Z_n\} : n \in \omega\rangle$ is a pairwise disjoint system of subsets of $\mathbb{R}$, and so

$$\begin{aligned}
\nu_1\left(\bigcup_{n\in\omega} Z_n\right) &= \mu\left(\bigcup\left\{X_f : f \in \bigcup_{n\in\omega} Z_n\right\}\right) \\
&= \mu\left(\bigcup_{n\in\omega}\bigcup\{X_f : f \in Z_n\}\right) \\
&= \sum_{n\in\omega}\mu\left(\bigcup\{X_f : f \in Z_n\}\right) \\
&= \sum_{n\in\omega}\nu_1(Z_n).
\end{aligned}$$

Clearly ν_1 is atomless.

(2) Clearly there is a bijection F from ${}^{\omega}2$ onto $[0,1]$. Now for each $X \subseteq [0,1]$ define $\nu_2(X) = \nu_1(F^{-1}[X])$. We check that ν_2 is a non-trivial atomless measure on $[0,1]$. Clearly $\nu_2(\emptyset) = 0$. If $X \subseteq Y$, then $\nu_2(X) \leq \nu_2(Y)$. For each $x \in [0,1]$ we have $|F^{-1}[\{x\}]| \leq 2$; so $\nu_2(\{x\}) = 0$. If $\langle X_n : n \in \omega\rangle$ is a pairwise disjoint system of subsets of $[0,1]$, then $\langle F^{-1}[X_n] : n \in \omega\rangle$ is a pairwise disjoint system of subsets of ${}^{\omega}2$. Hence

$$\begin{aligned}
\nu_2\left(\bigcup_{n\in\omega} X_n\right) &= \nu_1\left(F^{-1}\left[\bigcup_{n\in\omega} X_n\right]\right) = \nu_1\left(\bigcup_{n\in\omega} F^{-1}[X_n]\right) \\
&= \sum_{n\in\omega}\nu_1(F^{-1}[X_n]) = \sum_{n\in\omega}\nu_2(X_n).
\end{aligned}$$

So ν_2 is a nontrivial σ-additive measure on $[0,1]$. Clearly it is atomless.

(3) If $k+1 \leq 2^n$, then $\nu_2([k/2^n,(k+1)/2^n)) = 1/2^n$.

For, let $(k/2^n) = .f_0 f_1 \cdots f_n \cdots$. Thus $f_m = 0$ for all $m \geq n$. Let $Z = \{h \in {}^\omega 2 : (h \restriction (n+1)) = f\}$. Then $F[Z] = [k/2^n,(k+1)/2^n)$ Hence $\nu_2([k/2^n,(k+1)/2^n)) = \nu_1(F^{-1}[k/2^n,(k+1)/2^n)) = \nu_1(X_f) = 1/2^n$.

(4) The collection of Borel subsets of $[0,1]$ is the σ-algebra of subsets of $[0,1]$ generated by $\{[a,b) : 0 \leq a < b \leq 1\}$.

In fact, each $[a,b)$ as here is Borel, since $[a,b) = \{a\} \cup (a,b)$. If U is open, then $U = \bigcup\{[a,b) : [a,b) \subseteq U, a,b \in \mathbb{Q}\}$. So (4) holds.

(5) It follows that ν_2 coincides with Lebesgue measure on the collection of Borel sets. $\square$

An *Ulam* (λ^+,λ)-matrix is a function $\langle A_{\alpha,\eta} : \alpha < \lambda^+, \eta < \lambda\rangle$ with the following properties:

(1) $\forall\alpha < \lambda^+ \forall\eta < \lambda[A_{\alpha,\eta} \subseteq \lambda^+]$.

(2) $\forall\alpha,\beta < \lambda^+ \forall\eta < \lambda[\alpha \neq \beta \rightarrow [A_{\alpha,\eta} \cap A_{\beta,\eta} = \emptyset]$.

(3) $\forall\alpha < \lambda^+ \left|\lambda^+ \backslash \bigcup_{\eta<\lambda} A_{\alpha,\eta}\right| \leq \lambda$.

Theorem 10.43. *An Ulam (λ^+,λ)-matrices exist.*

Proof. For each $\xi < \lambda^+$ with $\xi \neq 0$ let f_ξ be a function mapping ω onto ξ. For $\alpha < \lambda^+$ and $\eta < \lambda$ let

$$A_{\alpha,\eta} = \{\xi < \lambda^+ : \xi \neq 0 \text{ and } f_\xi(n) = \alpha\}$$

Clearly (1) and (2) hold. For (3), note that if $\alpha < \lambda^+$ then $\forall\xi > \alpha\exists\eta < \lambda[f_\xi(\eta) = \alpha]$; hence $(\lambda^+ \backslash \bigcup_{\eta<\lambda} A_{\xi,\eta}) \subseteq \alpha + 1$, proving (3). $\square$

Theorem 10.44. *There is no λ^+-complete σ-saturated ideal on λ^+.*

Proof. By Theorem 10.43, let $\langle A_{\alpha,\eta} : \alpha < \lambda^+, \eta < \lambda\rangle$ be an Ulam (λ^+,λ)-matrix. Suppose that I is a λ^+-complete σ-saturated ideal on λ^+. For each $\alpha < \lambda^+$ there is an $\eta < \lambda$ such that $A_{\alpha\eta} \notin I$; otherwise $\bigcup_{\eta<\lambda} A_{\alpha\eta} \in I$. It follows that there exist a $W \in [\lambda^+]^{\lambda^+}$ and an $\eta < \lambda$ such that $\forall\alpha \in W[A_{\alpha\eta} \notin I]$. Now for distinct $\alpha,\beta \in W$ we have $A_{\alpha\eta} \cap A_{\beta\eta} = \emptyset$, contradiction, since W is uncountable. $\square$

Theorem 10.45. (Ulam) *Suppose that there is a nontrivial σ-additive measure on a set S. Then one of the following holds:*

(i) There is a two-valued measure on S, and $|S|$ is $\geq$ the first inaccessible cardinal.

(ii) There is an atomless nontrivial measure on 2^ω, and 2^ω is $\geq$ the first regular limit cardinal.

Proof. Let μ be a nontrivial σ-additive measure on S.

Case 1. μ has an atom. By Proposition 10.20, there is a σ-complete ultrafilter on S. Then by Theorem 10.19 there is a two-valued measure on S. By Corollary 10.23, $|S|$ is $\geq$ than the first measurable cardinal, and by Theorem 10.24 that measurable cardinal is inaccessible.

Case 2. μ is atomless. By Proposition 10.34 there is an atomless nontrivial measure on 2^ω. Let κ be minimum such that there is an atomless nontrivial measure μ on κ. By Proposition 10.30 μ is κ-additive. By the argument in the proof of Proposition 10.31, κ is regular. By Theorem 10.44, κ is a limit cardinal. $\square$

For $f.g \in {}^\omega\omega$, we write $f <^* g$ iff $\exists n \in \omega \forall m \geq n[f(m) < g(m)]$. A sequence $\langle f_\alpha : \alpha < \kappa \rangle$ is a κ-scale iff the following hold:

(i) $\forall \alpha < \kappa[f_\alpha \in {}^\omega\omega]$.
(ii) $\forall \alpha, \beta < \kappa[\alpha < \beta \to f_\alpha <^* f_\beta]$.
(iii) $\forall g \in {}^\omega\omega \exists \alpha < \kappa[g <^* f_\alpha]$.

Lemma 10.46. *If there is a κ-scale, then κ is not real-valued measurable.*

Proof. Let $\langle f_\alpha : \alpha < \kappa \rangle$ be a scale. We define $A : \omega \times \omega \to \mathscr{P}(\kappa)$ as follows:

$$\alpha \in A_{nk} \quad \text{iff} \quad f_\alpha(n) = k.$$

Now $\forall n \in \omega \forall \alpha < \kappa \exists k \in \omega[\alpha \in A_{nk}]$, so $\forall n \in \omega[\bigcup_{k\in\omega} A_{nk} = \kappa]$.

Now suppose that μ is a nontrivial κ-additive measure on κ. For each $n \in \omega$ let $k_n \in \omega$ be such that

$$\mu(A_{n0} \cup A_{n1} \cup \cdots \cup A_{nk_n}) \geq 1 - \frac{1}{2^{n+1}}.$$

Let $B_n = A_{n0} \cup A_{n1} \cup \cdots \cup A_{nk_n}$ and $C = \bigcap_{n\in\omega} B_n$.

Now note that

(1) For any measure μ on a set X we have $\mu(\bigcup_{n\in\omega} C_n) \leq \sum_{n\in\omega} \mu(C_n)$ (where the sum may be infinite). In fact, define $D_n = C_n \backslash \bigcup_{m<n} C_m$. Then $\bigcup_{n\in\omega} C_n = \bigcup_{n\in\omega} D_n$, and so

$$\begin{aligned}
\mu\left(\bigcup_{n\in\omega} C_n\right) &= \mu\left(\bigcup_{n\in\omega} D_n\right) \\
&= \sum_{n\in\omega} \mu(D_n) \\
&\leq \sum_{n\in\omega} \mu(C_n),
\end{aligned}$$

as desired.

(2) $\mu(C) \geq 1/2$.

In fact,

$$\begin{aligned}
\mu(\kappa\backslash C) &= \mu\left(\bigcup_{n\in\omega} (\kappa\backslash B_n)\right) \\
&\leq \sum_{n\in\omega} \mu(\kappa\backslash B_n) \\
&\leq \sum_{n\in\omega} \frac{1}{2^{n+2}} \\
&= \frac{1}{2}.
\end{aligned}$$

This proves (2).

(3) $\forall\alpha \in C \forall n \in \omega[f_\alpha(n) \leq k_n]$.

In fact, suppose that $\alpha \in C$ and $n \in \omega$. Let $k = f_\alpha(n)$, Now $\alpha \in B_n$, so there is an $i \leq k_n$ such that $\alpha \in A_{ni}$. Hence $f_\alpha(n) = i$, proving (3).

By (3), for each $\alpha \in C$ we have $k_n \not<^* f_\alpha$. Since $\mu(C) > 0$, C has size κ. This contradicts $\langle f_\alpha : \alpha < \kappa\rangle$ being a scale. □

Theorem 10.47. *Assuming CH, there is an ω_1-scale.*

Proof. Suppose that f_β has been constructed for all $\beta < \alpha$. If $\alpha = 0$, just let $f_0(m) = g_0(m) + 1$ for all $m \in \omega$. If $\alpha = \beta + 1$, let $f_\alpha(m) = \max(g_\beta(m), f_\beta(m)) + 1$ for all $m \in \omega$. If α is a limit ordinal, let $\langle \beta^\alpha_n : n \in \omega\rangle$ be a strictly increasing sequence of ordinals with supremum α, and for each $m \in \omega$ let $f_\alpha(m) = \max(\{f_{\beta^\alpha_n}(m) + 1 : n \leq m\} \cup \{g_\alpha(m) + 1\})$. Then

(∗) For all $\beta < \alpha \in \omega_1$, $f_\beta <^* f_\alpha$ and $g_\alpha < f_\alpha$.

In fact, the second condition in (∗) is obvious. We prove the first condition by induction on α. Suppose that it is true for all $\alpha' < \alpha$, and now suppose that $\beta < \alpha$. So $\alpha \neq 0$. Suppose that α is a successor ordinal $\gamma + 1$. If $\beta < \gamma$, then $f_\beta <^* f_\gamma$ by the inductive hypothesis, and $f_\gamma < f_\alpha$ by definition, so $f_\beta <^* f_\alpha$; and $\beta = \gamma$ is clear. Finally, suppose that α is a limit ordinal. Then there is an $n \in \omega$ such that $\beta < \beta^\alpha_n$; we have $f_\beta <^* f_{\beta^\alpha_n}$ by the inductive hypothesis, and clearly $f_{\beta^\alpha_n} <^* f_\alpha$, so $f_\beta <^* f_\alpha$.

Corollary 10.48. *If there is measure on 2^ω, then $2^\omega > \aleph_1$.* □

Theorem 10.49. *If κ is measurable, then it is weakly compact.*

Proof. Let κ be a measurable cardinal, and let U be a nonprincipal κ-complete ultrafilter on κ.

Since U is nonprincipal, $\kappa\backslash\{\alpha\} \in U$ for every $\alpha < \kappa$. Then κ-completeness implies that $\kappa\backslash F \in U$ for every $F \in [\kappa]^{<\kappa}$.

Now we show that κ is regular. For, suppose it is singular. Then we can write $\kappa = \bigcup_{\alpha<\lambda} \Gamma_\alpha$, where $\lambda < \kappa$ and each Γ_α has size less than κ. So by the previous paragraph, $\kappa\backslash\Gamma_\alpha \in U$ for every $\alpha < \kappa$, and hence

$$\emptyset = \bigcap_{\alpha<\lambda} (\kappa\backslash\Gamma_\alpha) \in U,$$

contradiction.

Next, κ is strong limit. For, suppose that $\lambda < \kappa$ and $2^\lambda \geq \kappa$. Let $S \in [{}^\lambda 2]^\kappa$. Let $\langle f_\alpha : \alpha < \kappa\rangle$ be a one-one enumeration of S. Now for each $\beta < \lambda$, one of the sets $\{\alpha < \kappa : f_\alpha(\beta) = 0\}$ and $\{\alpha < \kappa : f_\alpha(\beta) = 1\}$ is in U, so we can let $\varepsilon(\beta) \in 2$ be such that $\{\alpha < \kappa : f_\alpha(\beta) = \varepsilon(\beta)\} \in U$. Then

$$\bigcap_{\beta<\lambda} \{\alpha < \kappa : f_\alpha(\beta) = \varepsilon(\beta)\} \in U;$$

this set clearly has only one element, contradiction.

Thus we now know that κ is inaccessible. Finally, we check the tree property. Let $(T, \prec)$ be a tree of height κ such that every level has size less than κ. Then $|T| = \kappa$, and we may assume that actually $T = \kappa$. Let $B = \{\alpha < \kappa : \{t \in T : \alpha \preceq t\} \in U\}$. Clearly any two elements of B are comparable under $\prec$. Now take any $\alpha < \kappa$; we claim that $\mathrm{Lev}_\alpha(T) \cap B \neq \emptyset$. In fact,

$$\kappa = \{t \in T : \mathrm{ht}(t, T) < \alpha\} \cup \bigcup_{t \in \mathrm{Lev}_\alpha(T)} \{s \in T : t \preceq s\}. \tag{1}$$

Now by regularity of κ we have $|\{t \in T : \mathrm{ht}(t, T) < \alpha\}| < \kappa$, and so the complement of this set is in U, and then (1) yields

$$\bigcup_{t \in \mathrm{Lev}_\alpha(T)} \{s \in T : t \preceq s\} \in U. \tag{2}$$

Now $|\mathrm{Lev}_\alpha(T)| < \kappa$, so from (2) our claim easily follows.

Thus B is a branch of size κ, as desired. □

A *normal measure on* κ is a normal κ-complete nonprincipal ultrafilter on κ.

Lemma 10.50. *If D is a normal measure on κ, then every set in D is stationary.*

Proof. By Lemma 8.11, every club is in D. □

Theorem 10.51. *If U is a nonprincipal κ-complete ultrafilter on κ, then there is a function $f : \kappa \to \kappa$ such that $f_*(U) = \{X \subseteq \kappa : f^{-1}[X] \in U\}$ is a normal measure on κ.*

Proof. For $f, g \in {}^\kappa\kappa$ we define $f \equiv g$ iff $\{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in U$. Clearly $\equiv$ is an equivalence relation on ${}^\kappa\kappa$. We define $f <^* g$ iff $\{\alpha < \kappa : f(\alpha) < g(\alpha)\} \in U$. Clearly $<^*$ is irreflexive and transitive. Moreover,

(1) If $f, g \in {}^\kappa\kappa$ and $f \not\equiv g$, then $f <^* g$ or $g <^* f$.

In fact, $\{\alpha < \kappa : f(\alpha) = g(\alpha)\} \notin U$, so $\{\alpha < \kappa : f(\alpha) \neq g(\alpha)\} \in U$. Since

$$\{\alpha < \kappa : f(\alpha) \neq g(\alpha)\} = \{\alpha < \kappa : f(\alpha) < g(\alpha)\} \cup \{\alpha < \kappa : g(\alpha) < f(\alpha)\},$$

(1) follows.

(2) There is no infinite sequence $f_0 > f_1 > \cdots$.

For, suppose there is, and for each $n \in \omega$ let $X_n = \{\alpha < \kappa : f_{n+1}(\alpha) < f_n(\alpha)\}$. Choose $\alpha \in \bigcap_{n\in\omega} X_n$. Then $f_{n+1}(\alpha) < f_n(\alpha)$ for all n, contradiction.

Now let ${}^\kappa\kappa/\equiv = \{[f] : f \in {}^\kappa\kappa\}$. For $x, y \in {}^\kappa\kappa/\equiv$ define $x < y$ iff $\exists f \in x \exists g \in y[f <^* g]$.

(3) $[f] < [g]$ iff $f <^* g$.

In fact, $\Leftarrow$ is clear. Now suppose that $[f] < [g]$. Say $f \equiv f'$, $g \equiv g'$, and $f' <^* g'$. Then

$$\begin{aligned}&\{\alpha < \kappa : f(\alpha) = f'(\alpha)\} \cap \{\alpha < \kappa : f'(\alpha) < g'(\alpha)\} \cap \{\alpha < \kappa : g(\alpha) = g'(\alpha)\}\\ &\subseteq \{\alpha < \kappa : f(\alpha) < g(\alpha)\},\end{aligned}$$

and $\Rightarrow$ follows.

(3) $<$ is a well-order of ${}^\kappa\kappa/\equiv$.

This is clear, using (3).

Let $d(\alpha) = \alpha$ for all $\alpha < \kappa$. Then for all $\gamma < \kappa$, $\{\alpha < \kappa : d(\alpha) > \gamma\} \in U$. Now let x be minimum in ${}^\kappa\kappa/\equiv$ such that there is an $f \in x$ such that for all $\gamma < \kappa[\{\alpha < \kappa : f(\alpha) > \gamma\} \in U]$. This is possible because $[d]$ satisfies this condition. Fix such an $f \in x$. We claim that f is as desired in the theorem.

Clearly $f_*(U)$ is closed upwards, is κ-complete and is an ultrafilter on κ. Suppose that $\{\beta\} \in f_*(U)$. Thus $f^{-1}[\{\beta\}] = \{\alpha < \kappa : f(\alpha) = \beta\} \in U$. But also $\{\alpha < \kappa : f(\alpha) > \beta\} \in U$, so $\emptyset \in U$, contradiction. So $f_*(U)$ is a nonprincipal κ-complete ultrafilter on κ.

Now to show that it is normal, by exercise 8.8 (page 104) it suffices to show that if $X \in D$ with $0 \notin X$ and h is a function which is regressive on X, then there is a $Y \subseteq X$ such that $Y \in D$ and h is constant on Y. Define $g : \kappa \to \kappa$ by setting, for any $\alpha \in \kappa$,

$$g(\alpha) = \begin{cases} \alpha & \text{if } \alpha \notin f^{-1}[X], \\ h(f(\alpha)) & \text{if } \alpha \in f^{-1}[X]. \end{cases}$$

If $\alpha \in f^{-1}[X]$, then $g(\alpha) = h(f(\alpha)) < f(\alpha)$, so $f^{-1}[X] \subseteq \{\alpha < \kappa : g(\alpha) < f(\alpha)\}$. Now $f^{-1}[X] \in U$, so $\{\alpha < \kappa : g(\alpha) < f(\alpha)\} \in U$. Thus $g < f$. By the minimality of f it follows that $\{\alpha < \kappa : g(\alpha) > \gamma\} \notin U$, so $\{\alpha < \kappa : g(\alpha) \leq \gamma\} \in U$. By the κ-completeness of U it follows that there is a $\delta \leq \gamma$ such that $Y \overset{\text{def}}{=} \{\alpha < \kappa : g(\alpha) = \delta\} \in U$. Also, $Y\backslash\{\delta\} \in U$. For $\alpha \in Y\backslash\{\delta\}$ we have $g(\alpha) = \delta \neq \alpha$, so $\delta = g(\alpha) = h(f(\alpha))$. Thus h is constant on $f[Y\backslash\{\delta\}]$. Now $f^{-1}[f[Y\backslash\{\delta\}]] \supseteq Y\backslash\{\delta\}$, so $f^{-1}[f[Y\backslash\{\delta\}]] \in U$, and hence $f[Y\backslash\{\delta\}] \in D$. Finally, note that $Y\backslash\{\delta\} \subseteq f^{-1}[X]$, and so $f[Y\backslash\{\delta\}] \subseteq X$. $\square$

Corollary 10.52. *If κ is a measurable cardinal, then there is a normal measure on κ.* $\square$

Lemma 10.53. *Let A be a set of infinite cardinals such that for every regular cardinal κ, the set $A \cap \kappa$ is non-stationary in κ. Then there is a one-one regressive function with domain A.*

Proof. We proceed by induction on $\gamma \overset{\text{def}}{=} \bigcup A$. Note that γ is a cardinal; it is 0 if $A = \emptyset$. The cases $\gamma = 0$ and $\gamma = \omega$ are trivial, since then $A = \emptyset$ or $A = \{\omega\}$ respectively.

Next, suppose that γ is a successor cardinal κ^+. Then $A = A' \cup \{\kappa^+\}$ for some set A' of infinite cardinals less than κ^+. Then $\bigcup A' < \kappa^+$, so by the inductive hypothesis there is a one-one regressive function f on A'. We can extend f to A by setting $f(\kappa^+) = \kappa$, and so we get a one-one regressive function defined on A.

Suppose that γ is singular. Let $\langle \mu_\xi : \xi < \text{cf}(\gamma)\rangle$ be a strictly increasing continuous sequence of infinite cardinals with supremum γ, with $\text{cf}(\gamma) < \mu_0$. Note then that for every cardinal $\lambda < \gamma$, either $\lambda < \mu_0$ or else there is a unique $\xi < \mathrm{cf}(\gamma)$ such that $\mu_\xi \leq \lambda < \mu_{\xi+1}$. For every $\xi < \mathrm{cf}(\gamma)$ we can apply the inductive hypothesis to $A \cap \mu_\xi$ to get a one-one regressive function g_ξ with domain $A \cap \mu_\xi$. We now define f with domain A. In case $\mathrm{cf}(\gamma) = \omega$ we define, for each $\lambda \in A$,

$$f(\lambda) = \begin{cases} g_0(\lambda) + 2 & \text{if } \lambda < \mu_0, \\ \mu_\xi + g_{\xi+1}(\lambda) + 1 & \text{if } \mu_\xi < \lambda < \mu_{\xi+1}, \\ \mu_\xi & \text{if } \lambda = \mu_{\xi+1}, \\ 1 & \text{if } \lambda = \mu_0, \\ 0 & \text{if } \lambda = \gamma \in A. \end{cases}$$

Here the addition is ordinal addition. Clearly f is as desired in this case. If $\mathrm{cf}(\gamma) > \omega$, let $\langle \nu_\xi : \xi < \mathrm{cf}(\gamma) \rangle$ be a strictly increasing sequence of limit ordinals with supremum $\mathrm{cf}(\gamma)$. Then we define, for each $\lambda \in A$,

$$f(\lambda) = \begin{cases} g_0(\lambda) + 1 & \text{if } \lambda < \mu_0, \\ \mu_\xi + g_{\xi+1}(\lambda) + 1 & \text{if } \mu_\xi < \lambda < \mu_{\xi+1}, \\ \nu_\xi & \text{if } \lambda = \mu_\xi, \\ 0 & \text{if } \lambda = \gamma \in A. \end{cases}$$

Clearly f works in this case too.

Finally, suppose that γ is a regular limit cardinal. By assumption, there is a club C in γ such that $C \cap \gamma \cap A = \emptyset$. We may assume that $C \cap \omega = \emptyset$. Let $\langle \mu_\xi : \xi < \gamma \rangle$ be the strictly increasing enumeration of C. Then we define, for each $\lambda \in A$,

$$f(\lambda) = \begin{cases} g_0(\lambda) + 1 & \text{if } \lambda < \mu_0, \\ \mu_\xi + g_{\xi+1}(\lambda) + 1 & \text{if } \mu_\xi < \lambda < \mu_{\xi+1}, \\ 0 & \text{if } \lambda = \gamma \in A. \end{cases}$$

Clearly f works in this case too. □

Lemma 10.54. *Suppose that κ is weakly compact, and S is a stationary subset of κ. Then there is a regular $\lambda < \kappa$ such that $S \cap \lambda$ is stationary in λ.*

Proof. Suppose not. Thus for all regular $\lambda < \kappa$, the set $S \cap \lambda$ is non-stationary in λ. Let C be the collection of all infinite cardinals less than κ. Clearly C is club in κ, so $S \cap C$ is stationary in κ. Clearly still $S \cap C \cap \lambda$ is non-stationary in λ for every regular $\lambda < \kappa$. So we may assume from the beginning that S is a set of infinite cardinals.

Let $\langle \lambda_\xi : \xi < \kappa \rangle$ be the strictly increasing enumeration of S. Let

$$T = \left\{ s : \exists \xi < \kappa \left[s \in \prod_{\eta < \xi} \lambda_\eta \text{ and } s \text{ is one-one} \right] \right\}.$$

For every $\xi < \kappa$ the set $S \cap \lambda_\xi$ is non-stationary in every regular cardinal, and hence by Lemma 10.53 there is a one-one regressive function s with domain $S \cap \lambda_\xi$. Now $S \cap \lambda_\xi = \{\lambda_\eta : \eta < \xi\}$. Hence $s \in T$.

Clearly T forms a tree of height κ under $\subseteq$. Now for any $\alpha < \kappa$,

$$\prod_{\beta<\alpha} \lambda_\beta \leq \left(\sup_{\beta<\alpha} \lambda_\beta\right)^{|\alpha|} < \kappa.$$

Hence by the tree property there is a branch B in T of size κ. Thus $\bigcup B$ is a one-one regressive function with domain S, contradicting Fodor's theorem. $\square$

Theorem 10.55. *Every weakly compact cardinal is Mahlo, hyper-Mahlo, hyper-hyper-Mahlo, etc.*

Proof. Let κ be weakly compact. Let $S = \{\lambda < \kappa : \lambda \text{ is regular}\}$. Suppose that C is club in κ. Then C is stationary in κ, so by Lemma 10.54 there is a regular $\lambda < \kappa$ such that $C \cap \lambda$ is stationary in λ; in particular, $C \cap \lambda$ is unbounded in λ, so $\lambda \in C$ since C is closed in κ. Thus we have shown that $S \cap C \neq \emptyset$. So κ is Mahlo.

Let $S' = \{\lambda < \kappa : \lambda \text{ is a Mahlo cardinal}\}$. Suppose that C is club in κ. Let $S'' = \{\lambda < \kappa : \lambda \text{ is regular}\}$. Since κ is Mahlo, S'' is stationary in κ. Then $C \cap S''$ is stationary in κ, so by Lemma 10.54 there is a regular $\lambda < \kappa$ such that $C \cap S'' \cap \lambda$ is stationary in λ. Hence λ is Mahlo, and also $C \cap \lambda$ is unbounded in λ, so $\lambda \in C$ since C is closed in κ. Thus we have shown that $S' \cap C \neq \emptyset$. So κ is hyper-Mahlo.

Let $S''' = \{\lambda < \kappa : \lambda \text{ is a hyper-Mahlo cardinal}\}$. Suppose that C is club in κ. Let $S^{iv} = \{\lambda < \kappa : \lambda \text{ is Mahlo}\}$. Since κ is hyper-Mahlo, S^{iv} is stationary in κ. Then $C \cap S^{iv}$ is stationary in κ, so by Lemma 10.54 there is a regular $\lambda < \kappa$ such that $C \cap S^{iv} \cap \lambda$ is stationary in λ. Hence λ is hyper-Mahlo, and also $C \cap \lambda$ is unbounded in λ, so $\lambda \in C$ since C is closed in κ. Thus we have shown that $S''' \cap C \neq \emptyset$. So κ is hyper-hyper-Mahlo.

Etc. $\square$

Theorem 10.56. *Let κ be a measurable cardinal, and let D be a normal measure on κ. Suppose that $\lambda < \kappa$ and $F : [\kappa]^{<\omega} \to \lambda$. Then there is an $H \in D$ homogeneous for F.*

Hence every measurable cardinal is Ramsey.

Proof. It suffices to prove that for each $n \in \omega\backslash\{0\}$ there is a set $H_n \in D$ such that F is constant on $[H_n]^n$; then $\bigcap_{n\in\omega\backslash\{0\}} H_n$ is as desired. We prove this by induction on n. It is clear for $n = 1$. Now assume it for n. For each $\alpha < \kappa$ define F_α with domain $[\kappa\backslash\{\alpha\}]^n$ by $F_\alpha(x) = F(x \cup \{\alpha\})$. By the inductive hypothesis, for each $\alpha < \kappa$ there is a set $X_\alpha \in D$ such that F_α is constant on $[X_\alpha]^n$, say with constant value i_α. Define

$$Y = \left\{\alpha < \kappa : \alpha \in \bigcap_{\gamma<\alpha} X_\gamma\right\}.$$

Since D is normal, we have $Y \in D$. Now if $\gamma < \alpha_1 < \cdots < \alpha_n$ are in Y, then $\{\alpha_1, \ldots, \alpha_n\} \in [X_\gamma]^n$; hence

$$F(\{\gamma, \alpha_1, \ldots, \alpha_n\}) = F_\gamma(\{\alpha_1, \ldots, \alpha_n\}) = i_\gamma.$$

Now there exist a $j \in \lambda$ and a $H \in D$ with $H \subseteq Y$ such that $\forall\gamma \in H[i_\gamma = j]$. Hence $\forall x \in [H]^{n+1}[F(x) = j]$. $\square$

A cardinal κ is *strongly compact* iff κ is uncountable, regular, and for any set S, every κ-complete filter on S can be extended to a κ-complete ultrafilter on S.

Theorem 10.57. *Every strongly compact cardinal is measurable.*

Proof. Let κ be strongly compact. Define $F = \{X \subseteq \kappa : |\kappa \backslash X| < \kappa$. Clearly F is a κ-complete filter. Its extension to a κ-complete ultrafilter shows that κ is measurable. □

Now assume that $|A| \geq \kappa$. Let $\mathscr{F}_{A\kappa}$ be the filter on $[A]^\kappa$ generated by the sets

$$\hat{P} \stackrel{\text{def}}{=} \{Q \in [A]^\kappa : P \subseteq Q\}.$$

for $P \subseteq [A]^\kappa$. Clearly $\mathscr{F}_{A\kappa}$ is a proper κ-complete filter on $[A]^\kappa$. An ultrafilter on $[A]^\kappa$ extending $\mathscr{F}_{A\kappa}$ is a *fine measure for* κ, A.

Proposition 10.58. *If κ is strongly compact and $|A| \geq \kappa$, then there is a fine measure for κ, A.* □

A fine measure U for κ, A is *normal for* κ, A iff $\forall f : [A]^\kappa \to A[\forall P \in U[f(P) \in P] \to \exists P :\in U[f$ is constant on $\{Q \in U : Q \subseteq P\}]]$.

Proposition 10.59. *Let U be a fine measure for κ, A. Then the following are equivalent:*
(i) U is normal for κ, A.
(ii) $\forall X \in {}^AU[\triangle_{a\in A} X_a \stackrel{\text{def}}{=} \{x \in [A]^\kappa : x \in \bigcap_{a \in X} X_a\} \in U]$.

Proof. $\Rightarrow$: Assume that U is normal for κ, A, $X_a \in U$ for all $a \in A$, and $\triangle_{a\in A} X_a \notin U$. Thus $M \stackrel{\text{def}}{=} \{x \in [A]^{<\kappa} : x \notin \bigcap_{a\in x} X_a\} \in U$. Fix $a \in A$. For each $x \in M$, choose $f(x) \in x$ such that $x \notin X_{f(x)}$; let $f(x) = a$ for $x \in [A]^{<\kappa} \backslash M$. Then f is constant, say with value a, on some $N \in U$. Thus $\forall x \in M \cap N(x \notin X_a)$. So $\emptyset = M \cap N \cap X_a \in U$, contradiction.

$\Leftarrow$: Assume closure under diagonal intersections, and suppose that $f : [A]^{<\kappa} \to A$ such that $f(x) \in x$ for all $x \in M$, where $M \in U$. For all $a \in A$ let $X_a = \{x \in [A]^{<\kappa} : f(x) \neq a\}$. It suffices to get a contradiction from the assumption that $X_a \in U$ for all $a \in A$. Choose $x \in \triangle_{a\in A} X_a \cap M$. Then $x \in \bigcap_{a\in x} X_a \subseteq X_{f(x)}$, so $f(x) \neq f(x)$, contradiction. □

A cardinal κ is *supercompact* iff κ is unbounded, regular, and for every A with $|A| \geq \kappa$ there is a normal measure for A, κ.

Lemma 10.60. *If κ is an uncountable cardinal and μ is a κ-additive measure on S, and if $\langle X_\alpha : \alpha < \gamma\rangle$ is a system of subsets of S with $\gamma < \kappa$, then*

$$\mu\left(\bigcup_{\alpha<\gamma} X_\alpha\right) \leq \sum_{\alpha<\gamma} \mu(X_\alpha).$$

Proof. Define $Y_\alpha = X_\alpha \backslash \bigcup_{\beta<\alpha} X_\beta$ for all $\alpha < \gamma$. Then $\bigcup_{\alpha<\gamma} X_\alpha = \bigcup_{\alpha<\gamma} Y_\alpha$, and

$$\mu\left(\bigcup_{\alpha<\gamma} X_\alpha\right) = \mu\left(\bigcup_{\alpha<\gamma} Y_\alpha\right) = \sum_{\alpha<\gamma} \mu(Y_\alpha) \leq \sum_{\alpha<\gamma} \mu(X_\alpha).$$ □

Proposition 10.61. *Suppose that μ is a two-valued measure and U is the ultrafilter of all sets of measure 1. Then μ is κ-additive iff U is κ-complete.*

Proof. Assume that μ is a 2-valued measure on S. Note that since μ is 2-valued, for any disjoint $X, Y \subseteq S$ either $\mu(X) = 0$ or $\mu(Y) = 0$.

$\Rightarrow$: Suppose that $X_\alpha \in U$ for all $\alpha < \gamma$, where $\gamma < \kappa$. Then $\mu(S \backslash X_\alpha) = 0$ for all $\alpha < \gamma$, and hence $\mu(\bigcup_{\alpha<\gamma}(S\backslash X_\alpha)) \leq \sum_{\alpha<\gamma}\mu(S\backslash X_\alpha) = 0$, and so $\bigcap_{\alpha<\gamma} X_\alpha \in U$.

$\Leftarrow$: Suppose that $\langle X_\alpha : \alpha < \gamma\rangle$ is a system of pairwise disjoint subsets of S, with $\gamma < \kappa$. If $\mu(X_\alpha) = 0$ for all $\alpha < \gamma$, then $(S\backslash X_\alpha) \in U$ for all $\alpha < \gamma$, hence $\bigcap_{\alpha<\gamma}(S\backslash X_\alpha) \in U$, and so $\mu(\bigcap_{\alpha<\gamma}(S\backslash X_\alpha)) = 1$ and hence $\mu(\bigcup_{\alpha<\gamma} X_\alpha) = 0$, as desired. Otherwise, $\mu(X_\alpha) = 1$ for exactly one $\alpha < \gamma$. So, using what has just been shown,

$$\mu\left(\bigcup_{\beta<\gamma} X_\beta\right) = \mu(X_\alpha) + \mu\left(\bigcup_{\substack{\beta<\gamma\\ \beta\neq\alpha}} X_\beta\right) = \mu(X_\alpha) + \sum_{\substack{\beta<\gamma\\ \beta\neq\alpha}} \mu(X_\beta) = \sum_{\beta<\gamma}\mu(X_\beta). \qquad \square$$

Proposition 10.62. *A measure U on κ is normal iff the diagonal function is the least function f such that $\forall\gamma < \kappa[\{\alpha : f(\alpha) < \gamma\} \in U]$.*

Recall that by definition that U is normal iff it is a κ-complete nonprincipal ultrafilter on κ. The diagonal function d is defined by $d(\alpha) = \alpha$ for all $\alpha < \kappa$. $f < g$ means that $\{\alpha < \kappa : f(\alpha) < g(\alpha)\} \in U$. By exercise 8.8, U is normal iff every regressive function on a set in U is constant on a subset which is in U.

Proof. $\Rightarrow$: Suppose that U is normal. Then for each $\gamma < \kappa$, the set $\{\alpha : \alpha > \gamma\}$ is in U, because U is nonprincipal and κ-complete. So d is a function of the sort mentioned. Now suppose that $f : \kappa \to \kappa$ and $\{\alpha : f(\alpha) > \gamma\} \in U$; we want to show that $d \leq f$. Suppose not. Then $\{\alpha : f(\alpha) < \alpha\} \in U$, so by the equivalent of normality, there is a $\gamma < \kappa$ such that $\{\alpha : f(\alpha) = \gamma\} \in U$. But also $\{\alpha : f(\alpha) > \gamma\} \in U$, contradiction.

$\Leftarrow$: Suppose that $S \in U$ and f is regressive on S; we want to find $S_0 \subseteq S$ with $S_0 \in U$ such that f is constant on S_0. Suppose that there is no such S_0. Then for every γ, $\{\alpha : f(\alpha) > \gamma\} \in U$. Hence by assumption, $\{\alpha : \alpha \leq f(\alpha)\} \in U$. The intersection of this set with S is empty, contradiction. $\square$

Proposition 10.63. *Let D be a normal measure on κ and let $f : [\kappa]^{<\omega}$ be such that*

$$f(x) = \begin{cases} 0 & \text{if } x = \emptyset \text{ or } (x \neq \emptyset \text{ and } \min(x) = 0) \text{ or} \\ f(x) < \min(x) & \text{if } x \neq \emptyset \text{ and } 0 < \min(x). \end{cases}$$

Then there is an $H \in D$ such that $\forall n \in \omega[f$ is constant on $[H]^n]$.

Proof. It sufices to show that

$(*)$ For all $n \in \omega\backslash 1$ and all $f : [\kappa]^n \to \kappa$, if $f(s) = 0$ or $f(s) < \min(s)$ for all $s \in [\kappa]^n$, then there is a $H \in D$ such that f is constant on $[H]^n$.

In fact, suppose that $(*)$ holds, and suppose now that $f : [\kappa]^{<\omega} \to \kappa$ and $f(s) = 0$ or $f(s) < \min(s)$ for all $s \in [\kappa]^{<\omega}$. For each positive integer n, apply $(*)$ to $f \restriction [\kappa]^n$ to get $H_n \in D$ such that f is constant on $[H_n]^n$. Then $\bigcap_{n\in\omega\backslash 1} H_n$ is as desired.

Now we prove $(*)$ by induction on n. For $n = 1$, f is regressive on $\kappa\backslash 1$, and we can apply the comment before 10.19.

Assume that $(*)$ holds for $n \geq 1$, and suppose now that $f : [\kappa]^{n+1} \to \kappa$ such that $f(s) = 0$ or $f(s) < \min(s)$ for all $s \in [\kappa]^{n+1}$. For each $\alpha < \kappa$ define $f_\alpha : [\kappa]^n \to \kappa$ by

$$f_\alpha(s) = \begin{cases} 0 & \text{if } \alpha \not< \min(s), \\ f(\{\alpha\} \cup s) & \text{if } \alpha < \min(s). \end{cases}$$

Clearly $f_\alpha(s) = 0$ or $f_\alpha(s) < \min(s)$ for all $s \in [\kappa]^n$. Hence by the inductive hypothesis there is an $X_\alpha \in D$ such that f_α is constant on $[H_\alpha]^n$, say with value γ_α. Let $X = \triangle_{\alpha<\kappa} X_\alpha$. So $X \in D$. Now $\gamma_\alpha < \alpha$ for all nonzero $\alpha < \kappa$. In fact, choose $s \in [H_\alpha]^n$ such that $\alpha < \min(s)$. Then $\gamma_\alpha = f_\alpha(s) = f(\{\alpha\} \cup s) < \alpha$. Thus γ is regressive on $X\backslash\{0\} \in D$, so there is a $K \subseteq X\backslash\{0\}$ such that $K \in D$ and $\gamma_\alpha = \gamma$ for all $\alpha \in K$. We claim that $f \restriction [K]^{n+1}$ is constant, with value γ. For, take any $t \in [K]^{n+1}$, and write $t = \{\alpha\} \cup s$ with α the least element of t and $\alpha < \min(s)$. If $\beta \in S$, then $\alpha < \beta$, so by the definition of diagonal intersection, $\beta \in X_\alpha$. Thus $s \in [X_\alpha]^n$. Hence $f(t) = f_\alpha(s) = \gamma_\alpha = \gamma$, as desired. $\square$

Proposition 10.63. *If κ is measurable, then there is a normal measure on $[\kappa]^{<\kappa}$.*

Proof. Let U be a normal κ-additive nonprincipal ultrafilter on κ. (See Corollary 10.52.) We define

$$D = \{X \subseteq [\kappa]^{<\kappa} : X \cap \kappa \in U\}.$$

Clearly D is an ultrafilter on $[\kappa]^{<\kappa}$. It is nonprincipal, since for any $a \in [\kappa]^{<\kappa}$ we have

$$([\kappa]^{<\kappa}\backslash\{a\}) \cap \kappa = \{\alpha < \kappa : \alpha \neq a\} \supseteq \{\alpha < \kappa : \sup(a) < \alpha \in U.$$

Clearly D is κ-complete. To show that D is fine, suppose that $a \in [\kappa]^{<\kappa}$. Then

$$\hat{a} \cap \kappa = \{\alpha \in \kappa : \alpha \in \hat{a}\} = \{\alpha < \kappa : a \subseteq \alpha\} = \{\alpha \in \kappa : \sup(a) \leq \alpha\} \in U,$$

and hence $\hat{a} \in D$. Finally, suppose that $f : [\kappa]^{<\kappa} \to \kappa$ and $f(P) \in P$ for all $P \in X$, where $X \in D$. Then $X \cap \kappa \in U$, $f \restriction (X \cap \kappa) : X \cap \kappa \to \kappa$, and $f(\alpha) < \alpha$ for all $\alpha \in X \cap \kappa$. Hence there is a $Y \in U$ with $Y \subseteq X \cap \kappa$ such that f is constant on Y. Then $Y \in D$ is as desired. $\square$

11. Borel and analytic sets

A *Polish space* is a space which is homeomorphic to a complete separable metric space. By the proof of Proposition 1.21, ${}^\omega\omega$ is a Polish space.

Lemma 11.1. *For any Polish space X there is a continuous mapping from ${}^\omega\omega$ onto X.*

Proof. Let X be a complete separable metric space. We construct a mapping f of ${}^\omega\omega$ onto X as follows. Let Seq$= {}^{<\omega}\omega$. For each $s \in$Seq we define a closed ball C_s so that:

(i) $C_\emptyset = X$.
(ii) $\mathrm{diam}(C_s) \leq 1/n$, where n is the length of s.
(iii) $\mathrm{int}(C_s) \subseteq \bigcup_{k\in\omega} C_{s\frown\langle k\rangle}$
(iv) If $s \subseteq t$ then $\mathrm{center}(C_t) \subseteq C_s$.

Suppose that $s \in$ Seq and C_s have been constructed, where s has length n. Let D be a countable dense subset of X, and let $E = D \cap C_s$. Say $E = \{e_k : k \in \omega\}$. For each $k \in \omega$ let $C_{s\frown\langle k\rangle}$ be the closed ball about e_k with radius $\frac{1}{2(n+1)}$. Clearly (ii) and (iv) hold. For (iii), suppose that $x \in \mathrm{int}(C_s)$. Choose k so that $e_k \in S_{1/2(n+1)}(x)$. Thus $d(x, e_k) < \frac{1}{2(n+1)}$, so $x \in C_{s\frown\langle k\rangle}$, as desired. This completes the construction.

Now let $a \in {}^\omega\omega$. Then by condition (iv), for every $k \in \omega$ we can choose $x_k \in \bigcap_{m\leq k} C_{a\restriction m}$. We claim that $\langle x_k : k \in \omega\rangle$ is Cauchy. For, let $\varepsilon > 0$ be given. Choose n so that $\frac{1}{n+1} < \varepsilon$. Suppose that $k, l \geq n$. Then $x_k, x_l \in C_{a\restriction n}$, so $d(x_k, x_l) \leq \frac{1}{n+1} < \varepsilon$, as desired. Let y be the limit of $\langle x_k : k \in \omega\rangle$. We claim that $y \in \bigcap_{n\in\omega} C_{a\restriction n}$. Suppose that $n \in \omega$ and $y \notin C_{a\restriction n}$. Since $C_{a\restriction n}$ is closed, there is a positive integer m such that $S_{1/m}(y) \cap C_{a\restriction n} = \emptyset$. Choose M such that for all $k \geq M$, $d(y, x_k) < \frac{1}{m}$. Then if $k \geq M, n$, we have $d(y, x_k) < \frac{1}{m}$, but also $x_k \in C_{a\restriction n}$, contradiction. Thus our claim holds. We also claim that $\bigcap_{n\in\omega} C_{a\restriction n}$ does not contain any other elements. This is clear from (ii). For each $a \in {}^\omega\omega$ we define $f(a)$ to be the unique point in $\bigcap\{C_s : s \subseteq a\}$.

To show that f is continuous, suppose that $a \in f^{-1}[S_{1/n}(x)]$ we want to find a positive ε such that $a \in S_\varepsilon(a) \subseteq f^{-1}[S_{1/n}(x)]$. Since $f(a) \in S_{1/n}(x)$, choose m such that $S_{1/m}(f(a)) \subseteq S_{1/n}(x)$. Now

(1) $C_{a\restriction(m+1)} \subseteq S_{1/n}(x)$.

For, we have $f(a) \in C_{a\restriction(m+1)}$, and for any $y \in C_{a\restriction(m+1)}$ we have $d(f(a), y) \leq \frac{1}{m+1} < \frac{1}{m}$ by (i), so $y \in S_{1/n}(x)$. Thus (1) holds.

Now we claim that $S_{1/(m+2)}(a) \subseteq f^{-1}[S_{1/n}(x)]$, as desired. For, let $b \in S_{1/(m+2)}(a)$. Thus $a \restriction (m+1) = b \restriction (m+1)$, so $f(b) \in C_{b\restriction(m+1)} = C_{a\restriction(m+1)} \subseteq S_{1/n}(x)$, as desired.

So f is continuous.

To show that f maps onto X, take any $x \in X$. Suppose that s has been defined so that $x \in \mathrm{int}(C_s)$. Take $\varepsilon > 0$ so that $S_\varepsilon(x) \subseteq \mathrm{int}(C_s)$, and $\varepsilon < \frac{1}{2(n+1)}$. Take $e_k \in E \cap (S_\varepsilon(x))$. Then $x \in \mathrm{int}(C_{s\frown\langle k\rangle})$, as desired. $\square$

Let X be a Polish space. $A \subseteq X$ is a *Borel* set in X iff it belongs to the smallest σ-field of subsets of X containing all closed subsets. Now we define

$${}^X\mathbf{\Sigma}^0_1 = \text{ the collection of all open sets}$$

$$
\begin{aligned}
& {}^X\mathbf{\Pi}^0_1 = \text{ the collection of all closed sets} \\
\text{for } \alpha > 0\text{:} \quad & {}^X\mathbf{\Sigma}^0_\alpha = \left\{ \bigcup_{n\in\omega} A_n : \forall n \in \omega \left[A_n \in \bigcup_{\beta<\alpha} {}^X\mathbf{\Pi}^0_\beta \right] \right\} \\
\text{for } \alpha > 0\text{:} \quad & {}^X\mathbf{\Pi}^0_\alpha = \text{ the collection of all complements of sets in } {}^X\mathbf{\Sigma}^0_\alpha
\end{aligned}
$$

We omit the superscript ^{X} in what follows.

Proposition 11.2. *In any metric space, every open set is the union of countably many closed sets.*

Proof. Let U be open. For each positive integer n and each $x \notin U$ let $V_{x,n}$ be an open ball around x of radius $1/n$. Let $W_n = \bigcup_{x\notin U} V_{x,n}$. So $X\backslash U \subseteq W_n$ for each n. Let $F_n = X\backslash W_n$. So F_n is a closed set contained in U. We claim that $U = \bigcup_{n\in\omega\backslash 1} F_n$ (as desired). For, let $y \in U$. Choose a positive integer n and an open ball W about y of radius $1/n$ such that $W \subseteq U$. We claim that $y \in F_n$. For, suppose not. So $y \in W_n$, and so we can choose $x \in X\backslash U$ such that $y \in V_{x,n}$. Thus $d(x, y) < n$, so $x \in W \subseteq U$, contradiction. $\square$

Proposition 11.3. *For all α, β, if $1 \le \alpha < \beta$ then*
(1) $\Sigma^0_\alpha \subseteq \Sigma^0_\beta$,
(2) $\Sigma^0_\alpha \subseteq \Pi^0_\beta$,
(3) $\Pi^0_\alpha \subseteq \Sigma^0_\beta$,
(4) $\Pi^0_\alpha \subseteq \Pi^0_\beta$,
(5) $\Sigma^0_\beta = \{\bigcup_{n\in\omega} B_n : \forall n \in \omega[B_n \in \bigcup_{\alpha<\beta} \Pi^0_\alpha]\}$.
(6) $\Pi^0_\beta = \{\bigcap_{n\in\omega} B_n : \forall n \in \omega[B_n \in \bigcup_{\alpha<\beta} \Sigma^0_\alpha]\}$.

Proof. For $\beta = 1$, (1)–(6) hold vacuously.

Now assume (1)–(6) hold for β; we prove them for $\beta + 1$. (5) holds by definition. For (6),

$$
\begin{aligned}
Y \in \Pi^0_{\beta+1} \quad & \text{iff} \quad (X\backslash Y) \in \Sigma^0_{\beta+1} \\
& \text{iff} \quad \exists B \in {}^\omega\bigcup_{\alpha<\beta+1} \Pi^0_\alpha \left[(X\backslash Y) = \bigcup_{n\in\omega} B_n \right] \\
& \text{iff} \quad \exists B \in {}^\omega\bigcup_{\alpha<\beta+1} \Pi^0_\alpha \left[Y = \bigcap_{n\in\omega} (X\backslash B_n) \right] \\
& \text{iff} \quad \exists B \in {}^\omega\bigcup_{\alpha<\beta+1} \Sigma^0_\alpha \left[Y = \bigcap_{n\in\omega} B_n \right]
\end{aligned}
$$

This gives (6) for $\beta + 1$. (2) follows. For (1) we take two cases.

Case 1. $\alpha = 1$. Then $\Sigma^0_1 \subseteq \Sigma^0_{\beta+1}$ by Lemma 11.2.

Case 2. $\alpha > 1$. Suppose that $A \in \mathbf{\Sigma}^0_\alpha$. Hence $A \in \mathbf{\Sigma}^0_{\beta+1}$ by definition. So (1) holds.

(4) is clear by (1). For (3), if $A \in \mathbf{\Pi}^0_\alpha$ then $A \in \Sigma^0_{\beta+1}$ by definition.

Now assume inductively that β is limit. (5) and (3) are true for β by definition. For (2), if $A \in \Sigma^0_\alpha$ with $\alpha < \beta$, then $X\backslash A \in \Pi^0_\alpha$, hence $X\backslash A \in \Sigma^0_\beta$ by definition, and so $A \in \Pi^0_\beta$. For (1), suppose that $A \in \Sigma^0_\alpha$ with $\alpha < \beta$. Then $A \in \Pi^0_{\alpha+1}$ by (2), hence $A \in \Sigma^0_\beta$ by definition. For (4), if $A \in \Pi^0_\alpha$ with $\alpha < \beta$, then $X\backslash A \in \Sigma^0_\alpha$, hence $X\backslash A \in \Sigma^0_\beta$ by (1), so $A \in \Pi^0_\beta$. For (6),

$$\begin{aligned}
Y \in \Pi^0_\beta \quad &\text{iff} \quad (X\backslash Y) \in \Sigma^0_\beta \\
&\text{iff} \quad \exists B \in {}^\omega \bigcup_{\alpha<\beta} \Pi^0_\alpha \left[(X\backslash Y) = \bigcup_{n\in\omega} B_n\right] \\
&\text{iff} \quad \exists B \in {}^\omega \bigcup_{\alpha<\beta} \Pi^0_\alpha \left[Y = \bigcap_{n\in\omega} (X\backslash B_n)\right] \\
&\text{iff} \quad \exists B \in {}^\omega \bigcup_{\alpha<\beta} \Sigma^0_\alpha \left[Y = \bigcap_{n\in\omega} B_n\right] \qquad \square
\end{aligned}$$

Theorem 11.4. *For any Polish space X the union of all sets $\mathbf{\Sigma}^0_n$ and $\mathbf{\Pi}^0_n$ is the set of Borel sets.*

Proof. By induction, each $\mathbf{\Sigma}^0_n$ and $\mathbf{\Pi}^0_n$ consists of Borel sets. Clearly the union of all these sets is a σ field of sets. □

Proposition 11.5. *Each $\mathbf{\Sigma}^0_\alpha$ is closed under finite unions.* □

Proposition 11.6. *Each $\mathbf{\Pi}^0_\alpha$ is closed under finite intersections.*

Proof. This is clear for $\alpha = 1$. Suppose that $\alpha > 1$ and $A, B \in \mathbf{\Pi}^0_\alpha$. Then $X\backslash A, X\backslash B \in \mathbf{\Sigma}^0_\alpha$, so $(X\backslash A) \cup (X\backslash B) \in \mathbf{\Sigma}^0_\alpha$, so $A \cap B = X\backslash((X\backslash A) \cup (X\backslash B)) \in \mathbf{\Pi}^0_\alpha$. □

Proposition 11.7. *Each $\mathbf{\Sigma}^0_\alpha$ is closed under finite intersectins.*

Proof. This is clear by Propositions 11.3 and 11.6. □

Proposition 11.8. *Each $\mathbf{\Pi}^0_\alpha$ is closed under finite unions.*

Proof. This is clear by Proposition 11.7. □

Proposition 11.9. *Let X, Y be Polish spaces and $f : X \to Y$ be continuous.*

(i) If $W \in {}^Y\mathbf{\Sigma}_\alpha$, then $f^{-1}[W] \in {}^X\mathbf{\Sigma}^0_\alpha$.

(i) If $W \in {}^Y\mathbf{\Pi}_\alpha$, then $f^{-1}[W] \in {}^X\mathbf{\Pi}^0_\alpha$.

Proof. (i) and (ii) are clear for $\alpha = 0$. Assume that they are true for α. Suppose that $W \in {}^Y\mathbf{\Sigma}^0_{\alpha+1}$. Write $W = \bigcup_{n\in\omega} A_n$, where each $A_n \in {}^Y\mathbf{\Pi}^0_\alpha(Y)$. Then $f^{-1}[W] = \bigcup_{n\in\omega} f^{-1}[A_n]$, and by the inductive hypothesis each $f^{-1}[A_n]$ is in ${}^X\mathbf{\Pi}^0_\alpha(X)$, so $f^{-1}[W] \in {}^X\mathbf{\Sigma}^0_{\alpha+1}(X)$. The other inductive steps are similar. □

Proposition 11.10. *Let d be a metric on a set X. Define*

$$\hat{d}(x,y) = \frac{d(x,y)}{1+d(x,y)}.$$

Then:

(i) $\hat{d}$ is a metric on X.
(ii) d and $\hat{d}$ induce the same topology on X.
(iii) $\hat{d}(x,y) < 1$ for all x, y.
(iv) $\forall x \in {}^{\omega}X[x$ Cauchy under d implies that x is Cauchy under $\hat{d}]$.

Proof. (i): Clearly $\hat{d}(x,y) = 0$ iff $x = y$, and $\hat{d}(x,y) = \hat{d}(y,x)$. Next,

$$\begin{aligned}
&\hat{d}(x,y) + \hat{d}(y,z) - \hat{d}(x,z) \geq 0 \quad \text{iff} \\
&\frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)} - \frac{d(x,z)}{1+d(x,z)} \geq 0 \quad \text{iff} \\
&d(x,y) + d(x,y)d(y,z) + d(x,y)d(x,z) + d(x,y)d(y,z)d(x,z) \\
&\ + d(y,z) + d(y,z)d(x,y) + d(y,z)d(x,z) + d(y,z)d(x,y)d(x,z) \\
&\ - d(x,z) - d(x,z)d(x,y) - d(x,z)d(y,z) - d(x,z)d(x,y)d(y,z) \geq 0 \quad \text{iff} \\
&d(x,y) + d(x,y)d(y,z) + d(y,z) + d(y,z)d(x,y) + d(x,y)d(y,z)d(x,z) - d(x,z) \geq 0,
\end{aligned}$$

and the last statement is true.

(ii): $B_d(x,\varepsilon)$ is open in the topology determined by $\hat{d}$: First note that $B_d(x,\varepsilon) \subseteq B_{\hat{d}}(x,\varepsilon)$, since for all $y \in B_d(x,\varepsilon)$ we have

$$\hat{d}(x,y) = \frac{d(x,y)}{1+d(x,y)} \leq d(x,y) < \varepsilon.$$

It follows that $B_{\hat{d}}(x,\varepsilon)$ is open in the d-topology. Now suppose that $y \in B_d(x,\varepsilon)$. We want to find δ such that $B_{\hat{d}}(y,\delta) \subseteq B_d(x,\varepsilon)$. Let $\varepsilon' = \varepsilon - d(x,y)$ and $\delta = \frac{\varepsilon'}{1+\varepsilon'}$. So $\delta < 1$. Hence $\delta + \delta\varepsilon' = \varepsilon'$, hence $\varepsilon'(1-\delta) = \delta$, hence $\varepsilon' = \frac{\delta}{1-\delta}$. Suppose that $z \in B_{\hat{d}}(y,\delta)$. Thus

$$\begin{aligned}
&\hat{d}(y,z) < \delta \text{ hence } \frac{d(y,z)}{1+d(y,z)} < \delta \text{ hence } d(y,z) < \delta + \delta d(y,z) \text{ hence} \\
&d(y,z)(1-\delta) < \delta \text{ hence } d(y,z) < \frac{\delta}{1-\delta} \text{ hence } d(y,z) < \varepsilon' \text{ hence} \\
&d(y,z) < \varepsilon - d(x,y) \text{ hence } d(x,z) \leq d(y,z) + d(x,y) < \varepsilon.
\end{aligned}$$

(iii): clear. (iv): finally, suppose that $x \in {}^{\omega}X$ is Cauchy under d but not under $\hat{d}$. Say $\varepsilon > 0$, and for all $N \in \omega$, there exist $m, n > N$ such that $\hat{d}(x^m, x^n) > \varepsilon$. Choose $N \in \omega$ such that $\forall m, n \in \omega[d(x^m, x_n) < \varepsilon$. Choose $m, n > N$ such that $\hat{d}(x^m, x^n) > \varepsilon$. Then

$$d(x^m, x_n) < \varepsilon < \varepsilon < \hat{d}(x^m, , x^n) = \frac{d(x^m, x^n)}{1+d(x^m, x^n)} < d(x^m, x^n),$$

contradiction. □

Theorem 11.11. *If $X_0, X_1, \ldots$ are Polish spaces, then $\prod_{n\in\omega} X_n$ is Polish.*

Proof. Suppose that d_n is a complete metric on X_n such that $\forall x, y \in X_n[d(x, y) < 1]$. Define $\hat{d}$ on $\prod_{n\in\omega} X_n$ by

$$\hat{d}(x, y) = \sum_{n\in\omega} \frac{1}{2^{n+1}} d_n(x(n), y(n)).$$

Clearly $\hat{d}(x, y) = 0$ iff $x = y$, and $\hat{d}(x, y) = \hat{d}(y, x)$. Next,

$$\begin{aligned}
\hat{d}(x, y) + \hat{d}(y, z) &= \sum_{n\in\omega} \frac{1}{2^{n+1}} d_n(x(n), y(n)) + \sum_{n\in\omega} \frac{1}{2^{n+1}} d_n(y(n), z(n)) \\
&= \sum_{n\in\omega} \frac{1}{2^{n+1}} (d_n(x(n), y(n)) + d_n(y(n), z(n)) \\
&\geq \sum_{n\in\omega} \frac{1}{2^{n+1}} d_n(x(n), z(n)) = \hat{d}(x, z).
\end{aligned}$$

Next, suppose that $\langle x^n : n \in \omega\rangle$ is a Cauchy sequence. Take any $m \in \omega$. We claim that $\langle x^n(m) : n \in \omega\rangle$ is a Cauchy sequence. For, take any $\varepsilon > 0$, let $\varepsilon' = \frac{\varepsilon}{\frac{1}{2^{m+1}}}$ and choose N so that $\forall n \geq N[\hat{d}(x^N, x^n) < \varepsilon']$. Thus for all $n \geq N$,

$$\sum_{p\in\omega} \frac{1}{2^{p+1}} d_p(x^N(p), x^n(p)) < \varepsilon'.$$

Then $\frac{1}{2^{m+1}} d_m(x^N(m), x^n(m)) < \varepsilon'$, and hence $d_m(x^N(m)x^n(m)) < \varepsilon$.

This proves the claim. For each $m \in \omega$ let $y(m) = \lim_{n\in\omega} x^n(m)$. Then $\lim_{n\in\omega} x^n = y$. For, let $\varepsilon > 0$. Choose M so that $\frac{1}{2^M} < \frac{\varepsilon}{2}$. Note that

$$\sum_{m\geq M} \frac{1}{2^{m+1}} = \frac{1}{2^M} < \frac{\varepsilon}{2}.$$

Choose $N \geq M$ so that for all $i < M$ and all $m \geq N$, $d_i(x^i(m), y(m)) < \frac{\varepsilon}{2M}$. Then for any $n \geq N$,

$$\begin{aligned}
\hat{d}(x^n, y) &= \sum_{m\in\omega} \frac{1}{2^{m+1}} d_m(x^n(m), y(m)) \\
&\leq \sum_{i<M} d_i(x^i(m), y(m)) + \sum_{i\geq M} \frac{1}{2^{i+1}} \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Thus $\hat{d}$ is a complete metric on $\prod_{n\in\omega} X_n$. Next, $\hat{d}$ determines the usual topology on $\prod_{n\in\omega} X_n$. For, let U be basic open in $\prod_{n\in\omega} X_n$. Then we can write $U = \prod_{n\in\omega} Y_n$, where each Y_n is open in X_n and there is a finite $F \subseteq \omega$ such that $Y_n = X_n$ for all $n \notin F$. Let $x \in U$. We want to find an $\varepsilon > 0$ such that $B_{\hat{d}}(x, \varepsilon) \subseteq U$. Choose $\varepsilon > 0$ so that for all $n \in F$, $B_{d_n}(x_n, \varepsilon) \subseteq Y_n$. Let

$$\varepsilon' = \frac{\varepsilon}{\prod_{n\in F} 2^{n+1}}.$$

Suppose that $y \in B_{\hat{d}}(x, \varepsilon')$. Thus $\hat{d}(x, y) < \varepsilon'$, i.e.,

$$\sum_{n\in\omega} \frac{1}{2^{n+1}} d_n(x(n), y(n)) < \varepsilon'.$$

If $n \in F$, then $\frac{1}{2^{n+1}} d_n(x(n), y(n)) < \varepsilon'$, and hence $d_n(x(n), y(n)) < \varepsilon$. It follows that $B_{\hat{d}}(x, \varepsilon) \subseteq U$.

Conversely, given $\varepsilon > 0$ and $x \in \prod_{n\in\omega} X_n$, we want to find a basic open subset V of $\prod_{n\in\omega} X_n$ such that $x \in V \subseteq B_{\hat{d}}(x, \varepsilon)$. Choose N so that

$$\sum_{n\geq N} \frac{1}{2^{n+1}} < \frac{\varepsilon}{2},$$

and then define

$$\varepsilon' = \frac{\varepsilon}{2N \prod_{n<N} 2^{n+1}}.$$

For each $n < N$ let $Y_n = B_{d_n}(x(n), \varepsilon')$ and $Y_n = X_n$ for all $n \geq N$. Then $x \in \prod_{n\in\omega} Y_n \subseteq B_{\hat{d}}(x, \varepsilon)$.

It remains only to show that $\prod_{n\in\omega} X_n$ is separable. For each $n \in \omega$ let D_n be a countable dense subset of X_n. For each $n \in \omega$ let $a_n \in X_n$. Let $E = \{x \in \prod_{n\in\omega} X_n :$ there is a finite $F \subseteq \omega$ such that $\forall n \in F[x_n \in D_n]$ and $\forall n \in \omega\backslash F[x_n = a_n]$. Thus E is countable. Let U be a basic open subset of $\prod_{n\in\omega} X_n$. Say $F \subseteq \omega$ is finite and $U = \prod_{n\in\omega} V_n$ with each V_n open in X_n and $V_n = X_n$ for all $n \notin F$. Clearly there is an $x \in E \cap U$. □

Corollary 11.12. *${}^{\omega}2$ is a Polish space.* □

Lemma 11.13. *${}^{\omega}2$ has a subspace homeomorphic to ${}^{\omega}\omega$.*

Proof. Clearly ${}^{\omega}\omega$ is homeomorphic to ${}^{\omega}(\omega\backslash\{0\})$. Now let $M = \{x \in {}^{\omega}2 : x$ has infinitely many 0's and infinitely many 1's, starting with 1, and with no two 0's in a row$\}$. For each $x \in M$ write $x = 1^{n(x,0)}01^{n(x,1)}\cdots$. Let $f(x) = \langle n(x,0), n(x,1), \ldots\rangle$. Clearly f is a bijection of M onto ${}^{\omega}(\omega\backslash\{0\})$. Now suppose that U is basic open in ${}^{\omega}(\omega\backslash\{0\})$. Say $U = \{x \in^{\omega} (\omega\backslash\{0\}) : s \subseteq x$ where for some $m \in \omega$, $s \in {}^{m}(\omega\backslash\{0\})$. Then

$$f^{-1}[U] = \{x \in M : \langle s_0, 0, s_1, \ldots, 0, s_{m-1}\rangle \subseteq x\},$$

an open set in M.

Finally, suppose that U is basic open in ${}^\omega 2$. Say $s \in {}^{<\omega}2$ and $U = \{x \in {}^\omega 2 : s \subseteq x\}$. Let $t = \langle s_0 0 s_1 \ldots 0 s_{m-1}\rangle$, where $m = \mathrm{dmn}(s)$. Let $V = \{x \in M : t \subseteq x$. Then V is basic open in M, and $f[V] = U$. □

Lemma 11.14. *If X is an uncountable Polish space, then for every decreasing sequence $F_0 \supseteq F_1 \supseteq \cdots$ of nonempty closed subsets of X with $\mathrm{diam}(F_n) \to 0$ the intersection $\bigcap_{n\in\omega} F_n$ is a singleton.*

Proof. For each $n \in \omega$ choose $x_n \in F_n$.

(1) x is Cauchy.

For, suppose that $\varepsilon > 0$. Choose $N \in \omega$ such that $\forall n \geq N[\mathrm{diam}(F_n) < \varepsilon]$. Suppose that $m, n \geq N$. Then $x_m, x_n \in F_N$, so $d(x_m, x_n) < \varepsilon$.

By (1), let $y = \lim_{n\to\omega} x_n$.

(2) $y \in \bigcap_{n\in\omega} F_n$.

In fact, take any $n \in \omega$. Let $\varepsilon = \mathrm{diam}(F_n)$. Choose $N \in \omega$ so that $\forall m \geq N[d(x_m, y) < \varepsilon]$. Then $y \in F_n$. This proves (2).

If $z \neq y$, choose n so that $\mathrm{diam}(F_n) < \frac{1}{2}d(y,z)$. Then choose $m \geq n$ so that $d(y, x_m) < \frac{1}{2}d(y,z)$. Suppose that $z \in F_m$. Then

$$d(y,z) \leq d(y,x_m) + d(x_m,z) < \frac{1}{2}d(y,z) + \frac{1}{2}d(y,z) = d(y,z),$$

contradiction. □

Lemma 11.15. *If X is a dense-in-itself Polish space, then X has a subset homeomorphic to ${}^\omega 2$.*

Proof. We define $U_s \subseteq X$ for $s \in {}^{<\omega}2$ by induction on $\mathrm{dmn}(s)$. Let $U_\emptyset = X$. If U_s has been defined and is a nonempty open set, choose distinct x_0, x_1 in U_s; this is possible since X is dense-in-itself. Let $\varepsilon > 0$ be such that $\varepsilon < \frac{1}{3}d(x_0,x_1)$ and $S_\varepsilon(x_i) \subseteq U_s$ for $\varepsilon \in 2$. Let $U_{s\frown\langle\varepsilon\rangle} = S_\varepsilon(x_\varepsilon)$ for $\varepsilon \in 2$. If $y \in B_\varepsilon(x_0 \cap B_\varepsilon(x_1)$, then $d(x_0,x_1) \leq d(x_0,y) + d(y,x_1) \leq \frac{2}{3}d(x_0,x_1)$, contradiction.

Now for each $f \in {}^\omega 2$ let $\{y_f\} = \bigcap\{\mathrm{cl}(U_s) : s \in {}^{<\omega}2, s \subseteq f\}$. This is possible by Lemma 11.14 and our construction. We claim that $g \stackrel{\text{def}}{=} \langle y_f : f \in {}^\omega 2\rangle$ is a homeomorphism from ${}^\omega 2$ onto $\mathrm{rng}(y)$. Clearly g is a bijection. Suppose that $f \in g^{-1}[S_\varepsilon(z)]$. So $f \in {}^\omega 2$ and $z \in X$. Choose $s \in {}^{<\omega}2$ with $s \subseteq f$ so that $\mathrm{diam}(U_s) < \varepsilon - d(y_f, z)$. Thus $f \in \{k \in {}^\omega 2 : s \subseteq k\}$. If $s \subseteq k \in {}^\omega 2$, then $d(y_k, z) \leq d(y_k, y_f) + d(y_f, z) < \varepsilon$. Thus g is continuous. Suppose that $w \in g[\{f : s \subseteq f\}]$ with $s \in {}^\omega 2$. Say $w = y_f$ with $s \subseteq f$. Say $\mathrm{diam}(U_s) = \varepsilon$ and $d(x_s, y_f) = \delta < \varepsilon$. If $d(y_k, y_f) < \varepsilon - \delta$, then $d(y_k, x_s) \leq d(y_k, y_f) + d(y_f, x_s) < \varepsilon$. Thus also g^{-1} is continuous. □

Lemma 11.16. *If X is an uncountable Polish space, then there exist disjoint Y, Z with $X = Y \cup Z$, Y countable, and Z closed with no isolated points.*

Proof. Let $\{U_n : n \in \omega\}$ be a countable base for X. Define $Y = \bigcup\{U_n : U_n$ countable$\}$, and $Z = X\backslash Y$. If $z \in Z$ is isolated, then there is a U_n with $Z \cap U_n = \{z\}$. Then $U_n \subseteq Y \cup \{z\}$, hence U_n is countable, so $U_n \subseteq Y$, contradiction. $\square$

Lemma 11.17. *If X is an uncountable Polish space, then X has a subset homeomorphic to ${}^{\omega}\omega$.*

Proof. By Lemmas 11.13, 11.15. $\square$

Lemma 11.18. *If X, Y are Polish spaces, $x \in X$, and $A \subseteq X \times Y$, let $A_x = \{y \in Y : (x, y) \in A\}$. Let $\alpha \geq 1$.*

(i) If $A \subseteq X \times Y$ is Σ^0_α and $x \in X$, then A_x is Σ^0_α.

(ii) If $A \subseteq X \times Y$ is Π^0_α and $x \in X$, then A_x is Π^0_α.

Proof. Induction on α. For $\alpha = 1$, suppose that $A \subseteq X \times Y$ is open. Take any $x \in X$. Suppose that $y \in Y$ and $(x, y) \in A$. Let $U \times V$ be an open ball with $(x, y) \in U \times V \subseteq A$. Then $y \in V \subseteq A_x$. So A_x is open.

Still with $\alpha = 1$, suppose that $A \subseteq X \times Y$ is closed. Then $(X \times Y)\backslash A$ is open. For any $x \in X$, $((X \times Y)\backslash A)_x$ is open in Y. Now

$$Y\backslash((X \times Y)\backslash A)_x = \{y \in Y : (x, y) \notin ((X \times Y)\backslash A)\} = \{y \in Y : (x, y) \in A\} = A_x,$$

and so A_x is closed.

Now suppose inductively that $\alpha > 1$. Suppose that $A \subseteq X \times Y$ is Σ^0_α. Say $A = \bigcup_{n\in\omega} B_n$ with each B_n Π^0_β for some $\beta < \alpha$. Take any $x \in X$. Then for any $y \in Y$,

$$y \in A_x \quad \text{iff} \quad (x, y) \in A \quad \text{iff} \quad \exists n \in \omega[(x, y) \in B_n] \quad \text{iff} \quad \exists n \in \omega[y \in (B_n)_x],$$

and by the inductive hypothesis each $(B_n)_x$ is Π^0_β in Y. So A_x is Σ^0_α in Y.

Now suppose that $A \subseteq X \times Y$ is Π^0_α. Thus $(X \times Y)\backslash A$ is Σ^0_α, so also $((X \times Y)\backslash A)_x$ is Σ^0_α. Now

$$y \in ((X \times Y)\backslash A)_x \quad \text{iff} \quad (x, y) \in (X \times Y)\backslash A \quad \text{iff} \quad (x, y) \notin A \quad \text{iff} \quad y \notin A_x.$$

Hence A_x is Π^0_α. $\square$

Lemma 11.19. *Suppose that X is an uncountable Polish space and $\alpha \geq 1$. Then there exist $U, V \subseteq {}^{\omega}\omega \times X$ such that:*

(i) $U \in \Sigma^0_\alpha$, and $\forall A \subseteq X[A \in \Sigma^0_\alpha \leftrightarrow \exists x \in {}^{\omega}\omega[A = U_x]]$.

(ii) $V \in \Pi^0_\alpha$, and $\forall A \subseteq X[A \in \Pi^0_\alpha \leftrightarrow \exists x \in {}^{\omega}\omega[A = V_x]]$.

Proof. Induction on α. First we take $\alpha = 1$. Let $\langle W_n : n \in \omega\rangle$ enumerate a base for X. Then we set

$$U = \left\{(x, y) \in {}^{\omega}\omega \times X : y \in \bigcup_{n\in\omega} W_{x(n)}\right\}.$$

Clearly U_x is open for any $x \in {}^\omega\omega$. If $A \subseteq X$ is open, write $A = \bigcup_{n\in\omega} W_{x(n)}$. So $A = U_x$. Next we show that U is open in ${}^\omega\omega \times X$. Take any $(x, y) \in U$. Choose $n \in \omega$ such that $y \in W_{x(n)}$. Then

$$(x, y) \in \{w \in {}^\omega\omega : w(n) = x(n)\} \times W_{x(n)} \subseteq U.$$

This shows that U is open.

Now let $V = ({}^\omega\omega \times X)\backslash U$. Then V is closed. If $A \subseteq X$ is closed, choose $x \in {}^\omega\omega$ such that $(X\backslash A) = U_x$. Then for any $y \in X$,

$$y \in V_x \quad \text{iff} \quad (x, y) \in V \quad \text{iff} \quad (x, y) \notin U \quad \text{iff} \quad y \notin U_x \quad \text{iff} \quad y \in A.$$

This takes care of $\alpha = 1$.

Now assume inductively that $\alpha > 1$.

Case 1. α is a limit ordinal less than ω_1. Let $\langle \beta_n < n \in \omega\rangle$ be a sequence of ordinals ≥ 1, each less than α, with supremum α. For each $n \in \omega$ let $V_n \subseteq {}^\omega\omega \times X$ be a $\Pi^0_{\beta_n}$ set universal for $\Pi^0_{\beta_n}$. For each $x \in {}^\omega\omega$ and $n \in \omega$ define $x^n \in {}^\omega\omega$ by

$$x^n(m) = x(2^n(2m + 1) - 1).$$

(1) For each $n \in \omega$ the function $f_n : {}^\omega\omega \to {}^\omega\omega$ defined by $f_n(x) = x^n$ is continuous.

In fact, suppose that $n \in \omega$, $s \in {}^q\omega$, and $x \in f_n^{-1}[\{y \in {}^\omega\omega : s \subseteq y\}]$. Thus $s \subseteq f_n(x) = x^n$, so $\forall m < q[s(m) = x^n(m) = x(2^n(2m + 1) - 1)]$. Let $t = x \restriction (2^n(2q + 1) - 1)$. Thus $x \in \{z \in {}^\omega\omega : t \subseteq z\}$. If $z \in {}^\omega\omega$ and $t \subseteq z$, then $s \subseteq f_n(z)$, since for any $m < q$ we have $s(m) = x^n(m) = x(2^n(2m + 1) - 1)$. Thus (1) holds.

Now define

$$U = \{(x, y) \in {}^\omega\omega \times X : \exists n[(x^n, y) \in V_n].$$

Now fix $n \in \omega$. Define $g : {}^\omega\omega \times X \to {}^\omega\omega \times X$ by $g(x, y) = (x^n, y) = (f_n(x), y)$. Clearly g is continuous. Now $U''_n \stackrel{\text{def}}{=} \{(x, y) \in {}^\omega\omega \times X : (x^n, y) \in V_n\}$ is in $\Pi^0_{\alpha_n}$, since $U''_n = g^{-1}[V_n]$. Hence $U = \bigcup_{n\in\omega} U''_n \in \Sigma^0_\alpha$.

Now to see that U is universal, suppose that $A \subseteq X$ is Σ^0_α. Then we can write $A = \bigcup_{n\in\omega} B_n$ where each B_n is $\Pi^0_{\beta_n}$. Choose $z_n \in {}^\omega\omega$ such that $B_n = (V_n)_{z_n}$. Define $x(2^n(2m + 1) - 1) = z_n(m)$. Then for any $y \in X$,

$$\begin{aligned} y \in A \quad &\text{iff} \quad \exists n \in \omega[y \in B_n] \quad \text{iff} \quad \exists n \in \omega[y \in (V_n)_{z_n}] \quad \text{iff} \quad \exists n \in \omega[(z_n, y) \in V_n] \\ &\text{iff} \quad [(x^n, y) \in V_n] \quad \text{iff} \quad [(x, y) \in U] \quad \text{iff} \quad [y \in U_x]. \end{aligned}$$

This takes care of Σ^0_α. Suppose that $B \in \Pi^0_\alpha$. Then we choose U for $({}^\omega\omega \times X)\backslash B$. Then $({}^\omega\omega \times X)\backslash U$ is as desired.

The non-limit case is similar. □

Lemma 11.20. *Suppose that Y is a subspace of X and $\alpha \geq 1$.*

(i) ${}^Y\Sigma^0_\alpha = \{Y \cap U : U \in {}^X\Sigma^0_\alpha\}$.

(ii) ${}^Y\Pi^0_\alpha = \{Y \cap U : U \in {}^X\Pi^0_\alpha\}$.

Proof. Induction on α. First, $\alpha = 1$: The conditions (i) and (ii) hold by the definition of subspace. Second, suppose inductively that $\alpha > 1$. Then for any $A \subseteq Y$,

$$\begin{aligned}
A \in {}^Y\Sigma^0_\alpha \quad &\text{iff} \quad \exists B \in {}^\omega \bigcup_{\beta<\alpha} {}^Y\Pi^0_\beta \left[A = \bigcup_{n\in\omega} B_n\right] \\
&\text{iff} \quad \exists B \in {}^\omega \bigcup_{\beta<\alpha} {}^X\Pi^0_\beta \left[A = \bigcup_{n\in\omega} (B_n \cap Y)\right] \\
&\text{iff} \quad \exists B \in {}^\omega \bigcup_{\beta<\alpha} {}^X\Pi^0_\beta \left[A = Y \cap \bigcup_{n\in\omega} B_n\right] \\
&\text{iff} \quad \exists C \in {}^X\Sigma^0_\alpha[A = C \cap Y] \\
A \in {}^Y\Pi^0_\alpha \quad &\text{iff} \quad (Y\backslash A) \in {}^Y\Sigma^0_\alpha \\
&\text{iff} \quad \exists V \in {}^X\Sigma^0_\alpha[Y\backslash A = V \cap Y] \\
&\text{iff} \quad \exists V \in {}^X\Sigma^0_\alpha[A = (X\backslash V) \cap Y] \\
&\text{iff} \quad \exists W \in {}^X\Pi^0_\alpha[A = Y \cap W]
\end{aligned}$$

□

Theorem 11.21. *Suppose that X is an uncountable Polish space and $\alpha \geq 1$. Then:*

(i) There is a Σ^0_α subset U of $X \times X$ such that for every Σ^0_α subset A of X there is an $x \in X$ such that $A = U_x$.

(ii) There is a Π^0_α subset V of $X \times X$ such that for every π^0_α subset A of X there is an $x \in X$ such that $A = V_x$.

Proof. (i): By Lemma 11.17, X has a subset Y homeomorphic to ${}^\omega\omega$. By Lemma 11.19 let $U \subseteq Y \times X$ be Σ^0_α such that $\forall A \subseteq X \exists y \in Y[U_y = A]$. By Lemma 1.18 let $V \subseteq X \times X$ be Σ^0_α such that $V \cap (Y \times X) = U$. Clearly V is as desired.. (ii): Similarly. □

Proposition 11.22. *If U is Σ^0_α in $X \times X$, then $\{x \in X : (x, x) \in U\}$ is Σ^0_α in X.*

Proof. Let $f(x) = (x, x)$ for all $x \in X$. Then f is continuous. In fact, suppose that U and V are open in X and $x \in f^{-1}[U \times V]$. Then $x \in U \cap V \subseteq f^{-1}[U \times V]$. □

Corollary 11.23. *For every $\alpha \geq 1$ there is a set $A \subseteq {}^\omega\omega$ which is Σ^0_α but not Π^0_α.*

Proof. Let U be as in Theorem 11.21(i) with $X = {}^\omega\omega$. Let

$$A = \{x \in {}^\omega\omega : (x, x) \in U\}.$$

By Proposition 11.22, A is Σ^0_α in X. Suppose that also $A \in \Pi^0_\alpha$. Then $X\backslash A$ is Σ^0_α. By Theorem 11.21(i) choose $x \in X$ such that $X\backslash A = U_x$. Thus $\forall a \in X[a \notin A \text{ iff } (a, x) \in U]$. Hence $x \in A$ iff $(x, x) \in U$ iff $x \notin A$, contradiction. □

Corollary 11.24. *For every $\alpha \geq 1$ there is a set $A \subseteq {}^\omega\omega$ which is Π^0_α but not Σ^0_α.*

Proof. Let U be as in Theorem 11.21(ii) with $X = {}^\omega\omega$. Let

$$A = \{x \in {}^\omega\omega : (x,x) \in U\}.$$

By the proof of Proposition 11.22, A is Π^0_α in X. Suppose that also $A \in \Sigma^0_\alpha$. Then $X\backslash A$ is Π^0_α. By Theorem 11.21(ii) choose $x \in X$ such that $X\backslash A = U_x$. Thus $\forall a \in X[a \notin A$ iff $(a,x) \in U]$. Hence $x \in A$ iff $(x,x) \in U$ iff $x \notin A$, contradiction. $\square$

A subset A of a Polish space X is *analytic* iff there is a continuous function $f : {}^\omega\omega \to X$ such that $A = \mathrm{rng}(f)$. A *projection onto* X of set $S \subseteq X \times Y$ is $P = \{x \in X : \exists y \in Y[(x,y) \in S]\}$.

Proposition 11.25. *Every closed set in a Polish space is analytic.*

Proof. Let C be a closed subset of a Polish space X. Clearly C is a Polish space. Hence C is analytic by Theore 11.1. $\square$

Theorem 11.26. *For any Polish space X and any $A \subseteq X$ the following are equivalent:*

(i) A is analytic.

(ii) There is a Polish space Y, a Borel set B in Y, and a continuous function $f : Y \to X$ such that $A = f[B]$.

(iii) There is a Polish space Y and a Borel set B in $X \times Y$ such that A is the projection of B onto X.

(iv) There is a closed set C in $X \times {}^\omega\omega$ such that A is the projection of C onto X.

Proof.

(*) In any Polish space X, every Borel set is the projection of a closed subset of $X \times {}^\omega\omega$.

In fact, let P be the set of all subsets of X that are the projection of a closed subset of $X \times {}^\omega\omega$.

(1) Every closed subset of X is in P.

For, let C be a closed subset of X, and take any $f \in {}^\omega\omega$. Then $C \times \{f\}$ is a closed subset of $X \times {}^\omega\omega$ with projection C.

Now we recall from the proof of Theorem 1.29 the implicit definition of a homeomorphism from ${}^\omega\omega$ onto ${}^\omega({}^\omega\omega)$. Let $g : \omega \to \omega \times \omega$, and for each $a \in {}^\omega\omega$ and $n \in \omega$ define $a_{(n)} \in {}^\omega\omega$ by $a_{(n)}(k) = a(g^{-1}(n,k))$.

(2) P is closed under countable unions and countable intersections.

For, suppose that $A_n \in P$ for all $n \in \omega$. For each $n \in \omega$ let F_n be a closed subset of $X \times {}^\omega\omega$ such that $A_n = \{x \in X : \exists y[(x,y) \in F_n]\}$. Then for all $x \in X$,

$$\begin{aligned} x \in \bigcup_{n\in\omega} A_n \quad &\text{iff} \quad \exists n \in \omega \exists y \in {}^\omega\omega[(x,y) \in F_n] \\ &\text{iff} \quad \exists a, b \in {}^\omega\omega[(x,a) \in F_{b(0)}] \\ &\text{iff} \quad \exists c \in {}^\omega({}^\omega\omega)[(x, c(0)) \in F_{(c(1))(0)}]; \end{aligned}$$

$$
\begin{aligned}
x \in \bigcap_{n\in\omega} A_n \quad &\text{iff} \quad \forall n \in \omega \exists a \in {}^{\omega}\omega[(x,a) \in F_n] \\
&\text{iff} \quad \exists c \in {}^{\omega}({}^{\omega}\omega)\forall n \in \omega[(x,c(n)) \in F_n] \\
&\text{iff} \quad \exists c \in {}^{\omega}({}^{\omega}\omega)[(x,c) \in \bigcap_{n\in\omega}\{(x,d) : d \in {}^{\omega}({}^{\omega}\omega), (x,d(n)) \in F_n\}.
\end{aligned}
$$

(3) $Y \stackrel{\text{def}}{=} \{(x,c) \in X \times {}^{\omega}({}^{\omega}\omega) : (x,c(0)) \in F_{(c(1))(0)}\}$ is a closed subset of $X \times {}^{\omega}({}^{\omega}\omega)$.

To show this, suppose that $(y,d) \notin \{(x,c) : (x,c(0)) \in F_{(c(1))(0)}\}$. So $(y,d(0)) \notin F_{(d(1))(0)}$. Let U,V be open so that $(y,d(0)) \in U \times V$ and $(U \times V) \cap F_{(d(1))(0)} = \emptyset$. Let

$$
W = \{h \in {}^{\omega}({}^{\omega}\omega) : h(0) = d(0)\}
$$

Then $(y,d) \in U \times W$. Suppose that $(z,e) \in U \times W$. Then $(z,e(0)) = (z,d(0)) \in U \times V$, and hence $(z,e(0)) \notin F_{(d(1))(0)}$. So $(z,e) \notin Y$. This proves (3).

It follows that $\bigcup_{n\in\omega} A_n$ is the projection of a closed subset of $X \times {}^{\omega}({}^{\omega}\omega)$. Since ${}^{\omega}({}^{\omega}\omega)$ is homeomorphic to ${}^{\omega}\omega$, $\bigcup_{n\in\omega} A_n$ is the projection of a closed subset of $X \times {}^{\omega}\omega$. Thus $\bigcup_{n\in\omega} A_n \in P$.

Also, $\bigcap_{n\in\omega} A_n$ is the projection of a closed subset of $X \times {}^{\omega}\omega$. Thus $\bigcap_{n\in\omega} A_n \in P$. Hence (2) follows.

(4) Every Borel set is the projection of a closed subset of $X \times {}^{\omega}\omega$.

(5) Every Borel set is analytic.

(6) Every continuous image of a Borel set is analytic.

Proof of Lemma 11.26: (i)$\Rightarrow$(ii): obvious

(ii)$\Rightarrow$(i): Suppose that A is a continuous image of a Borel set B. By (6), A is analytic.

(i)$\Rightarrow$(iv): If A is analytic, let f be a continous mapping of ${}^{\omega}\omega$ onto A. The set $\{(f(x),x) : x \in {}^{\omega}\omega\}$ is a closed set in $X \times {}^{\omega}\omega$. In fact, if $(u,v) \in (X \times {}^{\omega}\omega)\backslash\{(f(x),x) : x \in {}^{\omega}\omega\}$, then $u \neq f(v)$. Let U,V be disjoint open sets with $u \in U$ and $f(v) \in V$. Then $(u,v) \in U \times f^{-1}[V] \subseteq (X \times {}^{\omega}\omega)\backslash\{(f(x),x) : x \in {}^{\omega}\omega\}$. In fact, if $(u',v') \in U \times f^{-1}[V]$, then $u' \in U$ and $v' \in f^{-1}[V]$, hence $f(v') \in V$, and so $u' \neq f(v')$. So $\{(f(x),x) : x \in {}^{\omega}\omega\}$ is a closed set in $X \times {}^{\omega}\omega$. Now $A = \{u \in X : \exists v \in {}^{\omega}\omega[(u,v) \in \{(f(x),x) : x \in {}^{\omega}\omega\}$.

(iv)$\Rightarrow$(iii): obvious.

(iii)$\Rightarrow$(ii): obvious. □

Recall the following definition: SEQ is the set of all finite sequences of members of ω. For each $s \in$SEQ, $U_s = \{f \in {}^{\omega}\omega : s \subseteq f\}$. Suppose that $\langle A_s : s \in\text{SEQ}\rangle$ is given. We define

$$
\mathscr{A}(\langle A_s : s \in \text{SEQ}\rangle) = \bigcup_{f\in{}^{\omega}\omega}\bigcap_{n\in\omega} A_{f\restriction n}.
$$

Proposition 11.27. *For any system $\langle B_s : s \in \text{SEQ}\rangle$ we have*

$$
\mathscr{A}(\langle B_s : s \in \text{SEQ}\rangle) = \bigcup_{f\in{}^{\omega}\omega}\bigcap_{n\in\omega}(B_{f\restriction 0} \cap B_{f\restriction 1} \cap \ldots \cap B_{f\langle n}). \qquad \square
$$

A system $\langle B_s : s \in \mathrm{SEQ}\rangle$ is *special* iff $\forall s, t \in \mathrm{SEQ}[B_t \subseteq B_s]$.

Corollary 11.28. *For any system $\langle A_s : s \in \mathrm{SEQ}\rangle$ there is a special system $\langle B_s : s \in \mathrm{SEQ}\rangle$ such that*

$$\mathscr{A}(\langle A_s : s \in \mathrm{SEQ}\rangle) = \mathscr{A}(\langle B_s : s \in \mathrm{SEQ}\rangle). \qquad \square$$

Theorem 11.29. *A set A in a Polish space X is analytic iff A is the result of the operation $\mathscr{A}$ applied to a family of closed sets.*

Proof. First we show that if F_s, $s \in \mathrm{Seq}$ are closed sets then $A \stackrel{\mathrm{def}}{=} \mathscr{A}(\langle F_s : s \in \mathrm{Seq}\rangle)$ is analytic. For any $x \in X$ we have

$$\begin{aligned} x \in A \quad &\text{iff} \quad \exists a \in {}^\omega\omega \left[x \in \bigcap_{n\in\omega} F_{a\restriction n}\right] \\ &\text{iff} \quad \exists a \left[(x,a) \in \bigcap_{n\in\omega} B_n\right], \end{aligned}$$

where for each $n \in \omega$. $B_n = \{(x, a) : x \in F_{a\restriction n}\}$.

(1) $\forall n \in \omega[B_n$ is closed].

In fact, for each $s \in {}^n\omega$, the set $F_s \times U_s$ is closed. Then observe that $B_n = \bigcup_{s\in {}^n\omega}(F_s \times U_s)$.

From (1) and the above equation it follows that A is analytic.

Conversely, suppose that A is analytic. Then there is a continuous function $f : {}^\omega\omega \to X$ such that $A = \mathrm{rng}(f)$.

$$\forall a \in {}^\omega\omega \left[\bigcap_{n\in\omega} f[U_{a\restriction n}] = \bigcap_{n\in\omega} \overline{f[U_{a\restriction n}]} = \{f(a)\}\right]. \tag{2}$$

In fact, clearly $f(a) \in \bigcap_{n\in\omega} f[U_{a\restriction n}]$, Suppose that $b \in \bigcap_{n=0}^\infty f[U_{a\restriction n}]$, and $b \neq f(a)$. Let U be open such that $f(a) \in U$ and $b \notin U$. Thus $a \in f^{-1}[U]$; choose n such that $U_{a\restriction n)} \subseteq f^{-1}[U]$. Now $b \in f[U_{a\restriction n}]$; choose $x \in U_{a\restriction n}$ such that $b = f(x)$. Then $x \in f^{-1}[U]$, so $b = f(x) \in U$, contradiction. Thus $\bigcap_{n\in\omega} f[U_{a\restriction n}] = \{f(a)\}$.

Now suppose that $f(c) \in \bigcap_{n\in\omega} \overline{f[\mathscr{O}(a \restriction n)]}$ and $f(c) \neq f(a)$. Let U, V be disjoint open sets with $f(a) \in U$ and $f(c) \in V$. Choose n so that $\mathscr{O}(a \restriction n) \subseteq f^{-1}[U]$. Thus $f[\mathscr{O}(a \restriction n)] \subseteq U$. Choose $w \in V \cap f[\mathscr{O}(a \restriction n)]$. Then $w \in V \cap U$, contradiction. Hence $\bigcap_{n\in\omega} \overline{f[\mathscr{O}(a \restriction n)]} = \{f(a)\}$.

Thus (2) holds. Clearly

$$A = \bigcup_{a\in{}^\omega\omega} \bigcap_{n\in\omega} \overline{a[U_{a\restriction n}]} \qquad \square$$

If $\mathcal{F}$ is a collection of subsets of X, then

$$\mathcal{A}(\mathcal{F}) = \{\mathcal{A}(A) : A \text{ maps } {}^{<\omega}\omega \text{ into } \mathcal{F}\}.$$

Lemma 11.30. $\mathcal{F} \subseteq \mathcal{A}(\mathcal{F})$.

Proof. For any $A \in \mathcal{F}$ and any $s \in {}^{<\omega}\omega$ let $B_s = A$. Then

$$\bigcup_{a \in {}^\omega\omega} \bigcap_{n \in \omega} B_{a \restriction n} = A. \qquad \square$$

Lemma 11.31. *There are bijections $u : \omega \times \omega \to \omega$, $v : {}^\omega\omega \times {}^\omega({}^\omega\omega) \to {}^\omega\omega$ and functions $\varphi, \psi : {}^{<\omega}\omega \to {}^{<\omega}\omega$ such that for all $(\alpha, \gamma) \in {}^\omega\omega \times {}^\omega({}^\omega\omega)$ and all β, s, m, n, if $v(\alpha, \gamma) = \beta$ and $s = \beta \restriction u(m, n)$, then $\varphi(s) = \alpha \restriction m$ and $\psi(s) = \gamma_m \restriction n$.*

Proof. Define $u(m, n) = 2^m(2n + 1) - 1$. Clearly u is a bijection. Note that

(1) $m \leq 2^m - 1$

In fact, this is clear for $m = 0$. Assuming it is true for m, then $m+1 \leq 2^m - 1 + 1 = 2^m < 2^m \cdot 2 = 2^{m+1}$ and so $m + 1 \leq 2^{m+1} - 1$.

(2) $m \leq u(m, n)$.

This is clear by (1).

(3) If $n < p$, then $u(m, n) < u(m, p)$.

For, $u(m, n) = 2^m(2n + 1) - 1 < 2^m(2p + 1) - 1 = u(m, p$.

For each $k \in \omega$ let $l(k), r(k)$ be such that $k = u(l(k), r(k))$.

Define $(v(\alpha, \gamma))(k) = u(\alpha(k), \gamma_{l(k)}(r(k)))$. Thus $v : {}^\omega\omega \times {}^\omega({}^\omega\omega) \to {}^\omega\omega$.

(4) v is one-one.

For, suppose that (α, γ) and (α', γ') are distinct members of ${}^\omega\omega \times {}^\omega({}^\omega\omega)$. If $\alpha \neq \alpha'$, clearly $v(\alpha, \gamma) \neq v(\alpha', \gamma')$. If $\gamma \neq \gamma'$, choose $s \in \omega$ such that $\gamma_s \neq \gamma'_s$. Say $\gamma_s(t) \neq \gamma'_s(t)$. Let $k = u(s, t)$. Then $\gamma_{l(k)}(r(k)) = \gamma_s(t) \neq \gamma'_s(t) = \gamma'_{l(k)}(r(k))$. So $v(\alpha, \gamma) \neq v(\alpha', \gamma')$. Thus v is one-one.

(5) v maps onto ${}^\omega\omega$.

In fact, let $\beta \in {}^\omega\omega$. Define $\alpha \in {}^\omega\omega$ and $\gamma \in {}^\omega({}^\omega\omega)$ by

$$\begin{aligned} \alpha(k) &= l(\beta(k)); \\ \gamma_n(m) &= r(\beta(u(n, m))). \end{aligned}$$

Then for any $\kappa \in \omega$,

$$\begin{aligned} (v(\alpha, \gamma))(k) &= u(\alpha(k), \gamma_{l(k)}(r(k))) \\ &= u(l(\beta(k)), r(\beta(u(l(k), r(k))))) \\ &= u(l(\beta(k)), r(\beta(k))) = \beta(k). \end{aligned}$$

This proves (5).

So v is a bijection.

Now we define φ. For $s \in {}^{<\omega}\omega$, by (2) we have $l(\mathrm{dmn}(s)) \le u(l(\mathrm{dmn}(s)), r(\mathrm{dmn}(s))) = \mathrm{dmn}(s)$, and we let $\varphi(s) = l \circ (s \restriction l(\mathrm{dmn}(s)))$.

To define ψ, let $s \in {}^{<\omega}\omega$. For $i < r(\mathrm{dmn}(s))$ by (3) we have $u(l(\mathrm{dmn}(s)), i) < u(l(\mathrm{dmn}(s)), r(\mathrm{dmn}(s))) = \mathrm{dmn}(s)$, and we define

$$\psi(s) = r \circ \langle s(u(l(\mathrm{dmn}(s)), i)) : i < r(\mathrm{dmn}(s))\rangle.$$

Now suppose that $\alpha, \gamma, \beta, s, m, n$ are given with $(\alpha, \gamma) \in {}^{\omega}\omega \times {}^{\omega}({}^{\omega}\omega)$, $v(\alpha, \gamma) = \beta$, and $s = \beta \restriction u(m, n)$. Then $\mathrm{dmn}(s) = u(m, n)$, $l(\mathrm{dmn}(s)) = m$, and

$$\begin{aligned}
\varphi(s) &= l \circ (s \restriction l(\mathrm{dmn}(s))) = l \circ (s \restriction m) \\
&= l \circ (\beta \restriction m) = l \circ ((v(\alpha, \gamma)) \restriction m) \\
&= \langle l((v(\alpha, \gamma))(i) : i < m\rangle \\
&= \langle l(u(\alpha(i), \gamma_{l(i)}(r(i)))) : i < m\rangle \\
&= \alpha \restriction m.
\end{aligned}$$

Finally,

$$\begin{aligned}
\psi(s) &= r \circ \langle s(u(l(\mathrm{dmn}(s)), i)) : i < r(\mathrm{dmn}(s))\rangle \\
&= \langle r(s(u(m, i)) : i < n\rangle \\
&= \langle r(\beta(u(m, i))) : i < n\rangle \\
&= \langle r((v(\alpha, \gamma))(u(m, i))) : i < n\rangle \\
&= \langle r((u(\alpha(u(m, i)), \gamma_{l(u(m,i))}(r(u(m, i))) : i < n\rangle \\
&= \langle \gamma_m(i) : i < n\rangle \\
&= \gamma_m \restriction n.
\end{aligned}$$

□

Theorem 11.32. $\mathcal{A}(\mathcal{A}(\mathcal{F})) = \mathcal{A}(\mathcal{F})$.

Proof. By Lemma 1, $\mathcal{A}(\mathcal{F}) \subseteq \mathcal{A}(\mathcal{A}(\mathcal{F}))$. Now suppose that $B \in \mathcal{A}(\mathcal{A}(\mathcal{F}))$. Say $B = \mathcal{A}(C)$, where C maps ${}^{<\omega}\omega$ into $\mathcal{A}(\mathcal{F})$. Thus $B = \bigcup_{a \in {}^{\omega}\omega} \bigcap_{n \in \omega} C_{a \restriction n}$. Now for each $s \in {}^{<\omega}\omega$ let D_s map ${}^{<\omega}\omega$ into $\mathcal{A}(\mathcal{F})$ such that $C_s = \mathcal{A}(D_s)$. Thus $C_s = \bigcup_{a \in {}^{\omega}\omega} \bigcap_{n \in \omega} D_{s, a \restriction n}$. Hence

$$\begin{aligned}
x \in B \quad &\text{iff} \quad \exists a \in {}^{\omega}\omega \forall m \in \omega [x \in C_{a \restriction m}] \\
&\text{iff} \quad \exists a \in {}^{\omega}\omega \forall m \in \omega \exists b_m \in {}^{\omega}\omega \forall n \in \omega [x \in D_{a \restriction m, b_m \restriction n}].
\end{aligned}$$

Now for each $s \in {}^{<\omega}\omega$ let $E_s = D_{\varphi(s), \psi(s)}$. note that

(2) $\mathcal{A}(E) = \bigcup_{c \in {}^{\omega}\omega} \bigcap_{p \in \omega} E_{c \restriction p} = \bigcup_{c \in {}^{\omega}\omega} \bigcap_{p \in \omega} D_{\varphi(c \restriction p), \psi(c \restriction p)}$.

We claim:

(3) $B = \mathcal{A}(E)$.

In fact, first suppose that $x \in B$. Choose $a \in {}^\omega\omega$ such that $x \in \bigcap_{m\in\omega} C_{a\restriction m}$. Then for any $m \in \omega$, $x \in C_{a\restriction m}$; so there is a $b_m \in {}^\omega\omega$ such that $x \in \bigcap_{n\in\omega} D_{a\restriction m, b_m\restriction n}$. Let $\beta = v(a, b)$. Take any $k \in \omega$ and let m, n be such that $k = u(m, n)$. Then $\varphi(\beta \restriction k) = a \restriction m$ and $\psi(\beta \restriction k) = b_m \restriction n$. So $x \in E_{\varphi(b\restriction k),\psi(b\restriction k)}$. This shows that $x \in \mathcal{A}(E)$.

Second, suppose that $x \in \mathcal{A}(E)$. Choose $c \in {}^\omega\omega$ such that $x \in \bigcap_{p\in\omega} E_{c\restriction p}$, using (2). Choose (α, γ) so that $v(\alpha, \gamma) = c$. Take any $m, n \in \omega$ and let $k = u(m, n)$. Then $E_{c\restriction p} = D_{\varphi(c\restriction p),\psi(c\restriction p)}$. Now $\varphi(c \restriction p) = \alpha \restriction m$ and $\psi(c \restriction p) = \gamma_m \restriction n$. So $x \in D_{\alpha\restriction m,\gamma_m\restriction n}$. By the equivalents for $x \in B$ at the beginning of this proof, this shows that $x \in B$. □

Theorem 11.33. *The collection of all analytic sets in a Polish space is closed under countable unions and intersections, continuous images, inverse images, and $\mathscr{A}$.*

Proof. Closure under countable unions and intersections was proved in the proof of Lemma 11.26. Clearly the collection of analytic sets is closed under continuous images. To show that it is closed under inverse images, suppose that $f : X \to Y$ is a continuous function from a Polish space X to a Polish space Y, and $A \subseteq Y$ is analytic. Then A is the projection of a closed set C in $Y \times {}^\omega\omega$. Thus $A = \{a \in Y :$there is an $x \in {}^\omega\omega$ such that $(a, x) \in C\}$. Define $g : X \times {}^\omega\omega \to Y \times {}^\omega\omega$ by setting $g(a, z) = (f(a), z)$. Then g is clearly continuous. Hence $g^{-1}[C]$ is closed in $X \times {}^\omega\omega$. We claim that $f^{-1}[A]$ is the projection of $g^{-1}[C]$ (and hence $f^{-1}[A]$ is analytic). For,

$$\begin{aligned} b \in f^{-1}[A] \quad & \text{iff} \quad f(b) \in A \\ & \text{iff} \quad \text{there is an } x \in {}^\omega\omega \text{ such that } (f(b), x) \in C \\ & \text{iff} \quad (a, x) \in g^{-1}[C]. \end{aligned}$$

Closure under $\mathscr{A}$: Theorem 11.32. □

Now for any Polish space X we define

$$\begin{aligned} \mathbf{\Sigma}^1_1 &= \text{the collection of all analytic sets;} \\ \mathbf{\Pi}^1_1 &= \text{the collection of all complements of analytic sets;} \\ \mathbf{\Sigma}^1_{n+1} &= \text{the collection of all projections of } \mathbf{\Pi}^1_n\text{-sets in } X \times {}^\omega\omega; \\ \mathbf{\Pi}^1_{n+1} &= \text{the collection of all complements of } \mathbf{\Sigma}^1_{n+1}\text{-sets in } X; \\ \mathbf{\Delta}^1_n &= \mathbf{\Sigma}^1_n \cap \mathbf{\Pi}^1_n. \end{aligned}$$

Proposition 11.34. *In any Polish space,*

(1) $\Sigma^1_n \subseteq \Sigma^1_{n+1}$*;*

(2) $\Pi^1_n \subseteq \Pi^1_{n+1}$*;*

(3) If $A \in \Sigma^1_n$*, then* $A \times {}^\omega\omega$ *is in* Σ^1_n *in* $X \times {}^\omega\omega$*.*

(4) If $A \in \Pi^1_n$*, then* $A \times {}^\omega\omega$ *is in* Π^1_n *in* $X \times {}^\omega\omega$*.*

(5) $\Sigma^1_n \subseteq \Pi^1_{n+1}$*;*

(6) $\Pi^1_n \subseteq \Sigma^1_{n+1}$.

We prove this by induction on n First we consider $n = 1$. For (1), suppose that A is analytic. Then by Lemma 11.26, A is the projection of some closed set C in $X \times {}^\omega\omega$. Since open sets are analytic, C is the complement of an analytic set, so that $C \in \Pi^1_1$. Hence $A \in \Sigma^1_2$ by definition.

For (2), let $B \in \Pi^1_1$. Then $X\backslash B$ is analytic, so $X\backslash B \in \Sigma^1_2$ by (1). Hence $B \in \Pi^1_2$ by definition.

For (3), suppose again that A is analytic. By Lemma 11.26, let B be a closed set in $X \times {}^\omega\omega$ such that $A = \{x \in X : \text{there is a } y \in {}^\omega\omega \text{ such that } (x, y) \in B\}$. Then $B \times {}^\omega\omega$ is closed in $X \times {}^\omega\omega \times {}^\omega\omega$, and $A \times {}^\omega\omega = \{(x, y) \in X \times {}^\omega\omega : \text{there is a } z \in {}^\omega\omega \text{ such that } (x, y, z) \in B \times {}^\omega\omega\}$. So $A \times {}^\omega\omega$ is analytic in $X \times {}^\omega\omega$.

For (4), suppose that $B \in \Pi^1_1$. Then ${}^\omega\omega\backslash B$ is analytic, and so $({}^\omega\omega\backslash B) \times {}^\omega\omega$ is analytic in $X \times {}^\omega\omega$ by (3). Now note that

$$({}^\omega\omega\backslash B) \times {}^\omega\omega = (X \times {}^\omega\omega)\backslash(B \times {}^\omega\omega);$$

So $B \times {}^\omega\omega$ is in Π^1_1 for the space $X \times {}^\omega\omega$.

Next we take (6). Suppose that $B \in \Pi^1_1$. Then by (4), $B \times {}^\omega\omega$ is in Π^1_1 too. Clearly B is its projection, so $B \in \Sigma^1_2$.

For (5), suppose again that A is analytic. Then $X\backslash A \in \Pi^1_1$, so by (6), $X\backslash A \in \Sigma^1_2$. Hence $A \in \Pi^1_2$.

This finishes the case $n = 1$. Now assume (1)–(6) for n; we prove them for $n + 1$.

For (1), suppose that $A \in \Sigma^1_{n+1}$. Then A is the projection of some $B \in \Pi^1_n$. By (2) for n, $B \in \Pi^1_{n+1}$. So $A \in \Sigma^1_{n+2}$.

For (2), suppose that $B \in \Pi^1_{n+1}$. Thus $X\backslash B \in \Sigma^1_{n+1}$. By (1), $X\backslash B \in \Sigma^1_{n+2}$, so $B \in \Pi^1_{n+2}$.

For (3), suppose that $A \in \Sigma^1_{n+1}$. Say A is the projection of B, where B is in Π^1_n in $X \times {}^\omega\omega$. By (4) for n, $B \times {}^\omega\omega$ is in Π^1_n for $X \times {}^\omega\omega \times {}^\omega\omega$. Clearly $A \times {}^\omega\omega$ is the projection of $B \times {}^\omega\omega$, so $A \times {}^\omega\omega$ is in Σ^1_{n+1}.

For (4), suppose that $B \in \Pi^1_{n+1}$. Thus $X\backslash B \in \Sigma^1_{n+1}$, so by (3), $(X\backslash B) \times {}^\omega\omega$ is in Σ^1_{n+1} for $X \times {}^\omega\omega$. Etc., as in (4) for $n = 1$.

For (6), suppose that $B \in \Pi^1_{n+1}$. Then $B \times {}^\omega\omega$ is in Π^1_{n+1} too by (4), and B is clearly its projection, so $B \in \Sigma^1_{n+2}$.

Finally, for (5), suppose that $A \in \Sigma^1_{n+1}$. Then $X\backslash A \in \Pi^1_{n+1}$, so by (6), $X\backslash A \in \Sigma^1_{n+2}$. Hence $A \in \Pi^1_{n+2}$.

Theorem 11.35. *For all $n \in \omega\backslash\{0\}$ there is a $U \subseteq {}^\omega\omega \times {}^\omega\omega$ such that $U \in \boldsymbol{\Sigma}^1_n$ and for every $A \in \boldsymbol{\Sigma}^1_n$ there is a $v \in {}^\omega\omega$ such that*

$$A = \{x \in {}^\omega\omega : (x, v) \in U\}.$$

Proof. We proceed by induction on n. By Theorem 1.28 let h be a homeomorphism of ${}^\omega\omega \times {}^\omega\omega$ onto ${}^\omega\omega$. For $n = 1$ let V be a $\boldsymbol{\Sigma}^0_1$-set for ${}^\omega\omega \times {}^\omega\omega$ such that for every $\boldsymbol{\Sigma}^0_1$-set A in ${}^\omega\omega$ there is an $x \in {}^\omega\omega$ such that $A = V_x$; V exists by Theorem 11.21(i). For $n > 1$ let V

be a $\mathbf{\Sigma}^0_{n-1}$-set for ${}^\omega\omega \times {}^\omega\omega$ such that for every $\mathbf{\Sigma}^0_{n-1}$-set A in ${}^\omega\omega$ there is an $x \in {}^\omega\omega$ such that $A = V_x$; again V exists by Theorem 11.21(i). Define

$$U = \{(x, y) \in {}^\omega\omega \times {}^\omega\omega : \exists a \in {}^\omega\omega[(h(x, a), y) \notin V]\} \tag{1}$$

For $n = 1$ the set $\{(x, y, a) : (h(x, a), y) \in V\}$ is open. In fact, define $k(x, y, a) = (h(x, a), y)$. Clearly k is a homeomorphism from ${}^\omega\omega \times {}^\omega\omega \times {}^\omega\omega$ onto ${}^\omega\omega \times {}^\omega\omega$. Now $\{(x, y, a) : (h(x, a), y) \in V\} = k^{-1}[V]$. It follows that $\{(x, y, a) : (h(x, a), y) \in V\}$ is open, so $\{(x, y, a) : (h(x, a), y) \notin V\}$ is closed. Therefore U is analytic, i.e. $U \in \mathbf{\Sigma}^1_1$.

For $n > 1$, U is Σ_{n-1} by the same argument.

(2) If A is $\mathbf{\Sigma}^1_1$ then there is a closed set B such that

$$x \in A \quad \text{iff} \quad \exists a \in {}^\omega\omega[(x, a) \in B] \tag{3}$$

(4) If $n > 1$ and A is $\mathbf{\Sigma}^1_n$ then there is a $\mathbf{\Pi}^1_{n-1}$ set B such that (3) holds.

Now let A be $\mathbf{\Sigma}^1_n$. Then $C \overset{\text{def}}{=} {}^\omega\omega \backslash h(B)$ is open (if $n = 1$) or $\mathbf{\Sigma}^1_{n-1}$ (if $n > 1$). Choose v so that $C = \{u : (u, v) \in V\}$. Then

$$\begin{aligned} x \in A \quad &\text{iff} \quad \exists a \in {}^\omega\omega[(x, a) \in B] \\ &\text{iff} \quad \exists a \in {}^\omega\omega[h(x, a) \notin C] \\ &\text{iff} \quad \exists a \in {}^\omega\omega[(h(x, a), v) \notin V \\ &\text{iff} \quad (x, v) \in U. \end{aligned}$$

□

Lemma 11.36. *If $f : Y \to X$ is a continuous function between Polish spaces, then for any $n \geq 1$, the inverse image under f of a Σ^1_n set is Σ^1_n; similarly for Π^1_n.*

Proof. By induction on n. First suppose that $n = 1$. For Σ^1_1, we want to show that if A is analytic in X, then $f^{-1}[A]$ is analytic in Y. By Lemma 11.26, there is an closed set C in $X \times {}^\omega\omega$ such that $A = \{x \in X : \text{there is a } y \in {}^\omega\omega \text{ such that } (x, y) \in C\}$. Let $g : Y \times {}^\omega\omega \to X \times {}^\omega\omega$ be defined by $g(y, z) = (f(y), z)$. Clearly g is continuous, and so $g^{-1}[C]$ is closed. Now

$$\begin{aligned} f^{-1}[A] &= \{y \in Y : f(y) \in A\} \\ &= \{y \in Y : \text{ there is a } z \in {}^\omega\omega \text{ such that } (f(y), z) \in C\} \\ &= \{y \in Y : \text{ there is a } z \in {}^\omega\omega \text{ such that } g(y, z) \in C\} \\ &= \{y \in Y : \text{ there is a } z \in {}^\omega\omega \text{ such that } (y, z) \in g^{-1}[C]\}, \end{aligned}$$

and so $f^{-1}[A]$ is analytic.

If B is Π^1_1, then $X \backslash B$ is analytic, and hence

$$f^{-1}[B] = f^{-1}[X \backslash (X \backslash B)] = Y \backslash f^{-1}[X \backslash B]$$

shows that $f^{-1}[B] \in \Pi^1_1$.

This takes care of $n = 1$. Now take $A \in \Sigma^1_{n+1}$, inductively. Then there is a Π^1_n set C such that A is the projection of C. The argument clearly now proceeds as in the case $n = 1$. $\square$

Corollary 11.37. *For each $n \in \omega\backslash\{0\}$ there is a set $A \subseteq {}^\omega\omega$ which is $\mathbf{\Sigma}^1_n$ but not $\mathbf{\Pi}^1_n$.*

Proof. Let $f(x) = (x, x)$ for all $x \in {}^\omega\omega$. Then $f : {}^\omega\omega \to {}^\omega\omega \times {}^\omega\omega$ is continuous, and $A = f^{-1}[U]$, so by Lemma 11.36, A is Σ^1_n. Suppose that A is Π^1_n. By Lemma 11.35, choose U and $v \in {}^\omega\omega$ such that ${}^\omega\omega\backslash A = \{x \in {}^\omega\omega : (x, v) \in U\}$. Then $v \in A$ iff $(v, v) \in U$ iff $v \notin A$, contradiction. $\square$

Let X be a Polish space and A, B two disjoint analytic subsets of X. We say that A and B are it separated by a Borel set iff there is a Borel set D such that $A \subseteq D$ and $B \subseteq X\backslash D$.

Theorem 11.38. *Let X be any Polish space. Any two disjoint analytic sets in X are separated by a Borel set.*

Proof.

(1) If $A = \bigcup_{n\in\omega} A_n$ and $B = \bigcup_{n\in\omega} B_n$ are such that for all m and n, A_m and B_n are separated by a Borel set, than A and B are separated by a Borel set.

For each m and n let D_{nm} be a Borel set such that $A_n \subseteq D_{nm}$ and $B_m \subseteq X\backslash D_{nm}$. Then let

$$D = \bigcup_{n\in\omega} \bigcap_{m\in\omega} D_{nm}.$$

Now first suppose that $x \in A$. Say $x \in A_n$. Then $x \in \bigcap_{m=0}^\infty D_{n,m} \subseteq D$. So $A \subseteq D$. Next we want to show that $D \subseteq X\backslash B$. So take any $n \in \omega$. Now $X\backslash B = \bigcap_{m=0}^\infty (X\backslash B_m)$. Each $D_{n,m}$ is a subset of $X\backslash B_m$, so $\bigcap_{m=0}^\infty D_{n,m} \subseteq \bigcap_{m=0}^\infty (X\backslash B_m) = X\backslash B$. Hence $D \subseteq X\backslash B$.

Now let A and B be disjoint analytic sets. Let f and g be continuous functions with domain ${}^\omega\omega$ such that $A = \mathrm{rng}(f)$ and $B = \mathrm{rng}(g)$. For each $s \in \mathrm{Seq}$ let $A_s = f[U_s]$ and $B_s = g[U_s]$. Clearly each A_s and B_s are analytic. For each $s \in \mathrm{Seq}$ we have $A_s = \bigcup_{n\in\omega} A_{s\frown\langle n\rangle}$ and $B_s = \bigcup_{n\in\omega} B_{s\frown\langle n\rangle}$. For each $a \in {}^\omega\omega$ we have

$$\{f(a)\} = \bigcap_{n\in\omega} f[U_{a\restriction n}] = \bigcap_{n\in\omega} A_{s\restriction n},$$

and similarly for g and B.

Now let $a, b \in {}^\omega\omega$. Since $\mathrm{rng}(f) \cap \mathrm{rng}(g) = \emptyset$, it folows that $f(a) \neq g(b)$. Let G_a and G_b be disjoint open neighborhoods of $f(a)$ and $g(b)$ respectively. Since $a \in f^{-1}[G_a]$, there is an $s \in Seq$ such that $a \in U_s \subseteq f^{-1}[G_a]$. There is an n such that $s = a \restriction n$. so $A_{a\restriction n} = f[U_s] \subseteq G_a$. Similarly for b; so we may aasume that $B_{a\restriction n} = g[U_s] \subseteq G_b$. Hence $A_{a\restriction n}$ and $B_{b\restriction n}$ are separated by the Borel set G_a.

Now suppose that A and B are not separated. Since $A = \bigcup_{\in\omega} A_{\langle n\rangle}$ and $B = \bigcup_{\in\omega} B_{\langle n\rangle}$, by (1) there exist $n_0, m_0 \in \omega$ such that A_{n_0} and B_{m_0} are not separated. Similarly there are n_1, m_1 such that $A_{n_0 n_1}$ and $B_{m_0 m_1}$ are not separated. Continuing, we get $a \stackrel{\text{def}}{=} \langle n_0, n_1, \ldots\rangle$

and $b \stackrel{\text{def}}{=} \langle m_0, m_1, \ldots \rangle$ such that for every k, $A_{n_0 n_1 \ldots n_k}$ and $B_{m_0 m_1, \ldots m_k}$ are not separated. This contradicts the above. $\square$

Theorem 11.39. (Suslin) *If both A and $X \backslash A$ are analytic, then A is Borel.*

Proof. Assume that both A and $X \backslash A$ are analytic. By Theorem 11.38 let B be a Borel set such that $A \subseteq B$ and $X \backslash A \subseteq X \backslash B$. Then $B \subseteq A$, so $A = B$. $\square$

Proposition 11.40. *Lebesgue measure is σ-finite. That is, if A is measurable, then there exist measurable sets A_n for $n \in \omega$ such that $\forall n \in \omega[\mu(A_n) < \infty]$ and $A = \bigcup_{n \in \omega} A_n$.*

Proof. Recall from Theorem 10.2 that the collection of Lebesgue measurable sets forms a field of sets. Hence for A measurable we have

$$A = \bigcup_{n \in \omega} (A \cap [-n, n]) \qquad \square$$

Proposition 11.41. *If $E \subseteq [0.1]$ is measurable, then there exist a G_δ U and an F_σ F such that $F \subseteq E \subseteq U$ and $\mu(U \backslash F) = 0$.*

Proof. For each $n \in \omega$ let U_n be an open set such that $E \subseteq U_n$ and $\mu(U_n) \leq \mu(E) + \frac{1}{1+n}$. U_n exists by Lemma 11.11(i). Let $U = \bigcap_{n \in \omega} U_n$. Then $\mu(U) = \mu(E)$. Since $\mu(U) = \mu(E) + \mu(U \backslash E)$, it follows that $\mu(U \backslash E) = 0$.

Applying this argument to $[0, 1] \backslash E$ we get a G_δ H such that $[0, 1] \backslash E \subseteq H$ and $\mu(H \backslash ([0, 1] \backslash E) = 0$. Then $[0, 1] \backslash H$ is an F_σ, $[0, 1] \backslash H \subseteq E$, and $\mu(E \backslash ([0, 1] \backslash H) = 0$. $\square$

Corollary 11.42. *$A \subseteq \mathbb{R}$ is measurable iff there is an F_σ F and a G_δ G such that $F \subseteq A \subseteq G$ with $G \backslash F$ a nullset.*

Proof. $\Rightarrow$: Suppose that A is measurable. Choose U as in Proposition 11.41. Then

$$\mu(A \triangle U) = \mu((A \backslash U) \cup (U \backslash A)) = \mu(U \backslash A) = 0.$$

$\Leftarrow$: Let $\mathscr{A}$ be the set of all $A \subseteq \mathbb{R}$ such that there exist an F_σ F and a G_δ G such that $F \subseteq A \subseteq G$ with $G \backslash F$ a nullset. It suffices to show that $\mathscr{A}$ contains all closed sets and is closed under complementation and countable unions.

Every closed set is an F_σ and a G_δ, so clearly every closed set is in $\mathscr{A}$.

Now suppose that $A \in \mathscr{A}$; we show that $\mathbb{R} \backslash A \in \mathscr{A}$. Let F be an F_σ and G a G_δ such that $F \subseteq A \subseteq G$ and $\mu(G \backslash F)$. Then $\mathbb{R} \backslash F$ is a G_δ, $\mathbb{R} \backslash G$ is an F_σ, $\mathbb{R} \backslash G \subseteq \mathbb{R} \backslash A \subseteq \mathbb{R} \backslash F$, and

$$\mu((\mathbb{R} \backslash F) \backslash (\mathbb{R} \backslash G) = \mu(G \backslash F) = 0.$$

Finally, suppose that $A \in {}^\omega\mathscr{A}$. Then there exist an ω-sequence G of G_δs and an ω-sequence F of F_σs such that $\forall n \in \omega[F_n \subseteq A_n \subseteq G_n]$ and $\mu(G_n \backslash F_n) = 0$. Then

$$\bigcup_{n \in \omega} F_n \subseteq \bigcup_{n \in \omega} A_n \subseteq \bigcap_{n \in \omega} G_n$$

and

$$\mu\left(\bigcup_{n\in\omega} G_n \setminus \bigcup_{n\in\omega} F_n\right) \le \sum_{n\in\omega} \mu(G_n\setminus F_n) = 0 \qquad \square$$

Lemma 11.43. *Let B be the σ-algebra of Borel sets, M the σ-algebra of measurable sets, I_μ the ideal in B of null sets, and I'_μ the ideal in M of null sets. Then $B/I_\mu \cong M/I'_\mu$.*

Proof. For each $A \in B$ let $f([A]_{I_\mu}) = [A]_{I'_\mu}$. Then f is well-defined and one-one. It is onto by Lemma 11.42. Clearly then it is the desired isomorphism. $\square$

Proposition 11.44. *Let B be the BA of Borel sets, and I the ideal of measure 0 Borel sets. Then B/I is σ-complete.*

Proof. Let $\langle [A_n] : n \in \omega\rangle$ be given. Clearly $[\bigcup_{n\in\omega} A_n]$ is an upper bound for $\langle [A_n] : n \in \omega\rangle$. Let $[C]$ be any upper bound. Then $\forall n \in \omega[A_n\setminus C$ has measure 0]. Hence $(\bigcup_{n\in\omega} A_n)\setminus C$ has measure 0. $\square$

Proposition 11.45. *Let B be the BA of Borel sets, and I the ideal of measure 0 Borel sets. Then I is σ-saturated.*

Proof. Clearly every singleton is in I. Suppose that S is an uncountable collection of pairwise disjoint subsets of $\mathbb{R}$ each not in I. Then

$$S = \sum_{n\in\omega}\left\{s \in S : \mu(s) > \frac{1}{n+1}\right\},$$

so there exist an uncountable $S' \subseteq S$ and a $n \in \omega$ such that $\forall s \in S'[\mu(s) > \frac{1}{n+1}]$. Let $S'' \subseteq S'$ have more than $\mu(1)(n+1)$ elements. Clearly this is a contradiction. $\square$

Proposition 11.46. *Let B be the BA of Borel sets, and I the ideal of measure 0 Borel sets. Then B/I is complete.*

Proof. Suppose that $X \subseteq B$. Let Y be maximal such that Y is pairwise disjoint and $\forall y \in Y \exists x \in X[y \le x]$. By Proposition 11.45 Y is countable. By Proposition 11.44 ΣY exists. We claim that ΣY is the supremum of X. Suppose that $x \in X$ and $x \cdot -\Sigma Y \neq 0$, Then $Y \cup \{x \cdot -\Sigma Y\}$ contadicts the maximality of Y. Thus ΣY is an upper bound for X. Suppose that z is an upper bound for X, but $\Sigma Y \cdot -z \neq 0$. Then choose $y \in Y$ such that $y \cdot -z \neq 0$, and choose $x \in X$ such that $y \le x$. Then $x \dot{-} z \neq 0$, contradiction. $\square$

Proposition 11.47. $\forall n \in \omega\setminus\{0\}\forall X \subseteq {}^n\mathbb{R}\exists A \supseteq X[A$ *is measurable and* $\forall Z \subseteq A\setminus X[Z$ *measurable implies that Z is null*]].

Proof. For $X \subseteq Y \subseteq {}^n\mathbb{R}$ we define $\mu^*(X,Y) = \inf\{\mu(A) : A$ is measurable and $Y \supseteq A \supseteq X\}$.

Case 1. $\mu^*(X,Y) < \infty$. Let A be such that $Y \supseteq A \supseteq X$, A is measurable, and $\mu(A)$ is minimum among all such A. Suppose that $Z \subseteq A\setminus X$ and Z is measurable. If $\mu(Z) > 0$ then $A\setminus Z$ contradicts the minimality of $\mu(A)$.

Case 2. $\mu^*(X,Y)=\infty$. Now let

$$Z_0=\{x\in {}^n\mathbb{R}:\forall i<n[-1<x_i<1] \quad \text{and, for } m>0,$$
$$Z_m=\{x\in {}^n\mathbb{R}:\forall i<n[-m-1<x_i\le m \text{ or } m\le x_i<m+1].$$

Thus $\langle Z_i : i<\omega\rangle$ is a system of pairwise disjoint sets, and ${}^n\mathbb{R}=\bigcup_{m\in\omega}Z_m$. Note that $\forall m\in\omega[|Z_m|=2^n]$. Hence $\mu^*(X\cap Z_m,Y\cap Z_m)<\infty$. We apply Case 1 to $X\cap Z_m$ and $Y\cap Z_m$: there is a measurable $A_m\supseteq X\cap Z_m$ such that $A_m\subseteq Y\cap Z_m$ and $\mu(A_m)$ is minimum. Let $A=\bigcup_{m\in\omega}A_m$. Then $Y\supseteq A\supseteq X$ and A is measurable. Suppose that $W\subseteq A\backslash X$. Then for each $m\in\omega$ we have $W\cap Z_m\subseteq(A_m\backslash X)\cap Z_m$, so $W\cap Z_m$ has measure 0. Hence so does W. □

For any topological space X, $Y\subseteq X$ is *nowhere dense* iff $X\backslash\overline{Y}$ is dense.

Proposition 11.48. *A is nowhere dense iff $X\backslash A$ contains a dense open set.*

Proof. ⇒: Assume that A is nowhere dense. Now $A\subseteq\overline{A}$, so $X\backslash\overline{A}\subseteq X\backslash A$, as desired.

⇐: Assume that D is dense open and $D\subseteq X\backslash A$. Then $A\subseteq X\backslash D$, and $X\backslash D$ is closed, so $\overline{A}\subseteq X\backslash D$, hence $D\subseteq X\backslash\overline{A}$. Since D is dense, so is $X\backslash\overline{A}$. So A is nowhere dense. □

Proposition 11.49. *A is nowhere dense iff for every nonempty open set G there is a nonempty open set $H\subseteq G$ such that $A\cap H=\emptyset$.*

Proof. ⇒: Assume that A is nowhere dense and G is a nonempty open set. Since $X\backslash\overline{A}$ is dense, we have $G\backslash\overline{A}\neq\emptyset$, and $G\backslash\overline{A}$ is open, as desired.

⇐: Assume the indicated condition. To show that $X\backslash\overline{A}$ is dense, let G be a nonempty open set, and suppose that $(X\backslash\overline{A})\cap G=\emptyset$. Thus $G\subseteq\overline{A}$. Choose H open, $H\subseteq G$, with $H\cap A=\emptyset$. This contradicts $G\subseteq\overline{A}$. □

Proposition 11.50. *If F is closed, then $F\backslash\mathrm{int}(F)$ is nowhere dense.*

Proof. We use Proposition 11.48. In fact, $X\backslash(F\backslash\mathrm{int}(F))=(X\backslash F)\cup\mathrm{int}(F)$ is clearly open dense. □

Proposition 11.51. *If G is open, then $\overline{G}\backslash G$ is nowhere dense.*

Proof. We use Proposition 11.48.

$$X\backslash(\overline{G}\backslash G)=(X\backslash\overline{G})\cup G,$$

and this set is open dense, since if a nonempty open set H is such that $G\cap H=\emptyset$, then $H\cap\overline{G}=\emptyset$ too, and so $H\subseteq X\backslash\overline{G}$.

Proposition 11.52. *The collection of all nowhere dense sets is an ideal in $\mathscr{P}(X)$.*

Proof. Clearly $\emptyset$ is nowhere dense.

Suppose that A and B are nowhere dense. Choose dense open sets C, D such that $C \subseteq X\backslash A$ and $D \subseteq X\backslash B$. Then $C \cap D$ is clearly dense open, and $C \cap D \subseteq (X\backslash(A \cup B)$. Thus $A \cup B$ is nowhere dense.

Suppose that A is nowhere dense and $B \subseteq A$. Then $X\backslash A \subseteq X\backslash B$. Hence by Proposition 11.48 B is nowhere dense. $\square$

Proposition 11.52. *For $x \in {}^n\mathbb{R} \cup {}^\omega\omega \cup {}^\omega 2$, $\{x\}$ is nowhere dense.*

Proof. We use Proposition 11.48.

${}^n\mathbb{R}$: Clearly ${}^n\mathbb{R}\backslash\{x\}$ is open. It is also dense. For, let $U \subseteq {}^n\mathbb{R}$ be nonempty and open. If $x \in U$, then there is a $S_\varepsilon(x) \subseteq U$. Let $y_i = x_i + \frac{\varepsilon}{2n}$ for each $i < n$. Then

$$d(x, y) = \sqrt{\sum_{i<n}(x_i - y_i)^2} = \sqrt{\sum_{i<n}\frac{\varepsilon^2}{4n^2}} < \varepsilon.$$

So ${}^n\mathbb{R}\backslash\{x\}$ is dense.

${}^\omega\omega$: For, again ${}^\omega\omega\backslash\{x\}$ is open. If U is basic open, say $U = \{y : s \subseteq y\}$ with $s \in {}^{<\omega}\omega$. Choose $y \in U\backslash\{x\}$. So ${}^\omega\omega\backslash\{x\}$ is dense.

$\omega 2$: similarly $\square$

A set is *meager* iff it is the countable union of nowhere dense sets. *A has the Baire property* iff there is an open set G such that $A \triangle G$ is meager.

Theorem 11.53. *The sets having the Baire property form a σ-algebra.*

Proof. First,

$$(*) \qquad A \triangle \overline{G} \subseteq (A \triangle G) \cup (\overline{G}\backslash G).$$

In fact, if $x \in A\backslash\overline{G}$, obviously $x \in A\backslash G \subseteq$ rhs. If $x \in \overline{G}\backslash A$, then either $x \in G$, hence $x \in G\backslash A \subseteq$ rhs, or $x \notin G$, hence $x \in \overline{G}\backslash G \subseteq$ rhs. So $(*)$ holds.

It follows that if $A \triangle G$ is meager, then so is $A \triangle \overline{G}$. Hence

$$(X\backslash A)\triangle(X\backslash\overline{G}) = ((X\backslash A) \cap \overline{G}) \cup ((X\backslash\overline{G}) \cap A = (\overline{G}\backslash A) \cup (A\backslash\overline{G}) = A \triangle \overline{G}.$$

Thus $X\backslash A$ has the Baire property.

For unions, note that

$$\left(\bigcup_{i\in\omega} A_i\right) \triangle \left(\bigcup_{i\in\omega} G_i\right) \subseteq \bigcup_{i\in\omega}(A_i \triangle G_i). \qquad \square$$

Corollary 11.54. *Every Borel set has the Baire property.* $\square$

Proposition 11.55. *The following are equivalent:*

(i) A has the Baire property.

(ii) There is an open set G and a meager set P such that $A = G\triangle P$.
(iii) There is a closed set F and a meager set P such that $A = F\triangle P$.

Proof. (i)⇒(ii): Assume that A has the Baire property. So there is an open set G such that $A\triangle G$ is meager. Now $A = G\triangle(A\triangle G)$.

(ii)⇒(i): Assume (ii). Let G be open and P meager such that $A = G\triangle P$. Then $A\triangle G = (G\triangle P)\triangle G = P$.

(ii)⇒(iii): Assume (ii) with G, P as indicated. Now $\overline{G}\backslash G$ is nowhere dense, so $Q \stackrel{\text{def}}{=} (\overline{G}\backslash G)\triangle P$ is meager, using Proposition 11.53. Now $G = (\overline{G}\backslash G)\triangle\overline{G}$ and hence

$$A = G\triangle P = ((\overline{G}\backslash G)\triangle\overline{G})\triangle P = \overline{G}\triangle Q.$$

(iii)⇒(ii): Assume (iii) and let F, P be as indicated. Then $N \stackrel{\text{def}}{=} F\backslash\mathrm{int}(F)$ is nowhere dense. So $N\triangle P$ is meager, and

$$A = F\triangle P = (N\triangle\mathrm{int}(F))\triangle P = \mathrm{int}(F)\triangle(N\triangle P). \qquad \square$$

Proposition 11.56. *The collection of all sets having the Baire property is the σ-field of subsets of X generated by the open sets and meager sets.*

Proof. Call this collection $\mathscr{A}$. Clearly $\mathscr{A}$ contains the open sets and meager sets. Each A with the Baire property is clearly in the indicated σ-field. $\square$

Proposition 11.57. *Let B be the σ-algebra of Borel sets, I the ideal of meager Borel sets, C the σ-algebra of sets with the Baire property, J the ideal of meager sets. For each $a \in B$ let $f([a]_I) = [a]_J$. Then f is a well-defined isomorphism of B/I onto C/J.*

Proof. Everything is clear except f being onto. Take any a with the Baire property. By Proposition 11.55 there exist an open set b and a meager set c such that $a = b\triangle c$. Then $a\triangle b = c$, so $[a] = [b]$. $\square$

Lemma 11.58. *Suppose that $A \subseteq {}^2\mathbb{R}$ is meager. Then $\{x \in \mathbb{R} : A_x$ is not meager$\}$ is meager.*

Proof. It suffices to prove

(∗) If A is nowhere dense, then $\{x \in \mathbb{R} : A_x$ is not nowhere dense$\}$ is meager.

In fact, suppose that (∗) holds. Let A be meager. Write $A = \bigcup_{n\in\omega} B_n$ with each B_n nowhere dense. Then for each $n \in \omega$, $\{x \in \mathbb{R} : B_{nx}$ is not nowhere dense$\}$ is meager. Now for any $x \in \mathbb{R}$,

$$\begin{aligned} A_x &= \{y \in \mathbb{R} : (x,y) \in A\} = \{y \in \mathbb{R} : (x,y) \in \bigcup_{n\in\omega} B_n\} \\ &= \bigcup_{n\in\omega}\{y \in \mathbb{R} : (x,y) \in B_n\} = \bigcup_{n\in\omega} B_{nx}. \end{aligned}$$

Hence $\forall n \in \omega[B_{nx}$ is nowhere dense$] \to A_x$ is meager, so $\{x \in \mathbb{R} : A_x$ is not meager$\} \subseteq \bigcup_{n\in\omega}\{x \in \mathbb{R} : B_{nx}$ is not nowhere dense$\}$. This big union is meager, so $\{x \in \mathbb{R} : A_x$ is not meager$\}$ is meager, as desired.

Now to prove $(*)$, suppose that A is nowhere dense. Let $\{V_n : n \in \omega\}$ be an open basis for the topology on ${}^2\mathbb{R}$. Let $G = {}^2\mathbb{R}\backslash A$. So G is open dense. For each $n \in \omega$ let

$$H_n = \{x \in \mathbb{R} : \exists y \in V_n[(x, y) \in G]\}.$$

$(**)$ H_n is an open subset of $\mathbb{R}$.

In fact, suppose that $x \in H_n$. Choose $y \in V_n$ so that $(x, y) \in G$. Since G is open, there are intervals U, W such that $x \in U$, $y \in W \subseteq V_n$, and $(x, y) \in U \times W \subseteq G$. Then $U \subseteq H_n$, since for any $u \in U$ we can choose $w \in W$ and then $w \in V_n$ and $(u, w) \in G$. so $u \in H_n$. So $(**)$ holds.

$(***)$ H_n is dense.

In fact, let $U \subseteq \mathbb{R}$ be a nonempty open set. Then $G \cap (U \times V_n) \neq \emptyset$ since G is dense. For $(x, y) \in G \cap (U \times V_n)$ we have $x \in H_n \cap U$. So $(***)$ holds.

From $(**)$, $(***)$ and the Baire category theorem it follows that $\bigcap_{n\in\omega} H_n$ is dense.

$(****)$ $\forall x \in \bigcap_{n\in\omega} H_n[G_x$ is dense open in $\mathbb{R}]$.

In fact, $G_x = \{y \in \mathbb{R} : (x, y) \in G\}$, so clearly G_x is open. Now let $W \subseteq \mathbb{R}$ be a nonempty open set. Choose $n \in \omega$ such that $V_n \subseteq W$. Since $x \in H_n$, there is a $y \in V_n$ such that $(x, y) \in G$. Hence $y \in G_x \cap V_n \subseteq B_x \cap W$. So $(****)$ holds.

Now for all $x \in \bigcap_{n\in\omega} H_n[\mathbb{R}\backslash G_x$ is nowhere dense$]$. Hence

$$\{x : \mathbb{R}\backslash G_x \text{ is not nowhere dense}\} \subseteq \bigcup_{n\in\omega}(\mathbb{R}\backslash H_n),$$

and this big union is meager by $(**)$ and $(***)$.

Note that $\mathbb{R}\backslash G_x = \{y \in \mathbb{R} : (x, y) \notin G\} = \{y \in \mathbb{R} : (x, y) \in A\} = A_x$. Hence $(*)$ holds. □

Lemma 11.59. *Let $A, B \subseteq \mathbb{R}$, and suppose that A is meager in $\mathbb{R}$ or B is meager in $\mathbb{R}$. Then $A \times B$ is meager.*

Proof. By symmetry suppose that A is meager in $\mathbb{R}$. Say $A = \bigcup_{n\in\omega} A'_n$ with each A'_n nowhere dense. Then for each $n \in \omega$, $\mathbb{R}\backslash A'_n$ is dense open in $\mathbb{R}$. Then $(\mathbb{R}\backslash A'_n) \times \mathbb{R}$ is dense open in ${}^2\mathbb{R}$. Hence $A'_n \times B$ is nowhere dense in ${}^2\mathbb{R}$. So $A \times B$ is meager. □

Lemma 11.60. *If $A \subseteq {}^2\mathbb{R}$ has the property of Baire and $\{x \in \mathbb{R} : A_x$ is not meager$\}$ is meager, then A is meager.*

Proof. Assume the hypotheses, but suppose that A is not meager. Write $A = G \triangle P$ with U open and P meager. Write $G = \bigcup_{n\in\omega}(U_n \times V_n)$, with U_n, V_n open. Then there is an n_0 such that $U_{n_0} \times V_{n_0}$ is not meager. Then by Lemma B, U_{n_0} and V_{n_0} are not meager.

$(*)\ \forall x \in U_{n_0}[V_{n_0}\backslash P_x \subseteq A_x]$.

In fact, suppose that $x \in U_{n_0}$ and $y \in V_{n_0}\backslash P_x$. Then $y \in V_{n_0}$ and $(x, y) \notin P$, so $(x, y) \in U_{n_0} \times V_{n_0} \subseteq G$. Thus $(x, y) \in G\backslash P \subseteq G \triangle P = A$, and hence $y \in A_x$. So $(*)$ holds.

Now if $V_{n_0}\backslash P_x$ is meager, then $V_{n_0} = (V_{n_0} \cap P_x) \cup (V_{n_0}\backslash P_x)$ is meager, contradiction. Thus $\forall x \in U_{n_0}[A_x$ is not meager]. So $U_{n_0} \subseteq \{x \in \mathbb{R} : A_x$ is not meager$\}$; but this last set is meager by hypothesis, and U_{n_0} is not meager, contradiction. $\square$

Lemma 11.61. *Let $A \subseteq \mathbb{R}\times\mathbb{R}$ have the Baire property. Then A is meager iff $\{x \in \mathbb{R} : A_x$ is not meager$\}$ is meager,*

Proof. By Lemmas 11.58 and 11.60. $\square$

Lemma 11.62. *For any set S in a Polish space X there exists a set $A \supseteq S$ which has the Baire property and is such that whenever $Z \subseteq A\backslash S$ then Z is meager.*

Proof. For $S \subseteq X$ let

$$D(S) = \{x \in X : \forall U \in \mathscr{O}[x \in U \to U \cap S \text{ is not meager}]\}.$$

Then

$$\begin{aligned} X\backslash D(S) &= \{x \in X : \exists U \in \mathscr{O}[x \in U \text{ and } U \cap S \text{ is meager}]\} \\ &= \bigcup\{U \in \mathscr{O} : U \cap S \text{ is meager}\}. \end{aligned}$$

Thus $X\backslash D(S)$ is open, so $D(S)$ is closed. Also note that

$$S\backslash D(S) = \bigcup\{S \cap U : U \in \mathscr{O} \text{ and } U \cap S \text{ is meager}\}.$$

Since $\mathscr{O}$ is countable, it follows that $S\backslash D(S)$ is meager. Let

$$A = S \cup D(S).$$

So $S \subseteq A$. Now $A = (S\backslash D(S)) \cup D(S)$, so A is the union of a meager set and a closed set. Hence A has the Baire property by Lemma 11.63. Now suppose that $Z \subseteq A\backslash S$ has the Baire property; we want to show that Z is meager. Suppose not. Choose G open such that $Z \triangle G$ is meager. Since Z is not meager, $G \neq \emptyset$. Choose $\emptyset \neq U \in \mathscr{O}$ with $U \subseteq G$. Thus $U\backslash Z$ is meager. Now $Z \subseteq D(S)\backslash S$, so $S \subseteq X\backslash Z$, hence $U \cap S \subseteq U\backslash Z$. So $U \cap S$ is meager.

Now U is not meager, by the Baire category theorem. Since $U\backslash Z$ is meager, it follows that $U \cap Z$ is not meager, and hence $U \cap Z \neq \emptyset$. Take any $x \in U \cap Z$. Now $Z \subseteq D(S)$, so it follows that $U \cap S$ is not meager, contradiction. $\square$

Theorem 11.65. *Every analytic set of reals is Lebesgue measurable.*

Proof. Let A be an analytic set of reals. Let $f : {}^\omega\omega \to \mathbb{R}$ be continuous with range A. For each $s \in \mathrm{Seq}$ let $A_s = f[U_s]$. Then

$$A = \mathscr{A}(\langle A_s : s \in \mathrm{Seq}\rangle) = \mathscr{A}(\langle \overline{A_s} : s \in \mathrm{Seq}\rangle). \tag{1}$$

In fact, suppose that $a \in A$. Say $f(x) = a$. Now

$$\mathscr{A}(\langle A_s : s \in \mathrm{Seq}\rangle) = \bigcup_{y \in {}^\omega\omega} \bigcap_{n\in\omega} A_{y\restriction n}.$$

Now $\forall n \in \omega[x \in U_{x\restriction n}]$, so $\forall n \in \omega[a \in f[U_{x\restriction n}]]$, so $\forall n \in \omega[x \in A_{x\restriction n}]$. This shows that $x \in \mathscr{A}(\langle A_s : s \in \mathrm{Seq}\rangle)$. Clearly $\mathscr{A}(\langle A_s : s \in \mathrm{Seq}\rangle) \subseteq \mathscr{A}(\langle \overline{A_s} : s \in \mathrm{Seq}\rangle)$.

Suppose that $a \in \mathscr{A}(\langle \overline{A_s} : s \in \mathrm{Seq}\rangle)$. Choose $x \in {}^\omega\omega$ such that $a \in \bigcap_{n\in\omega} \overline{A_{x\restriction n}}$. Thus $a \in \bigcap_{n\in\omega} \overline{f[U_{x\restriction n}]}$. Suppose that $a \neq f(x)$. Let V, W be disjoint open sets such that $a \in V$ and $f(x) \in W$. Now $x \in f^{-1}[W]$, so there is an open Z such that $x \in Z \subseteq f^{-1}[W]$. There is an $n \in \omega$ such that $x \in U_{x\restriction n} \subseteq Z$. Now $f(x) = a \in V$, so $x \in f^{-1}[V]$. Hence there is an $m \in \omega$ such that $x \in U_{x\restriction m} \subseteq f^{-1}[V]$. Let $p = \max\{m, n\}$. Then $U_{x\restriction p} \subseteq f^{-1}[W] \cap f^{-1}[V] = \emptyset$, contradiction.

This proves (1).

(2) $\forall s \in \mathrm{Seq}\left[A_s = \bigcup_{n\in\omega} A_{s\frown\langle n\rangle}\right]$.

For, $U_s = \bigcup_{n\in\omega} U_{s\frown\langle n\rangle}$, so

$$A_s = f[U_s] = f\left[\bigcup_{n\in\omega} U_{s\frown\langle n\rangle}\right] = \bigcup_{n\in\omega} f[U_{s\frown\langle n\rangle}] = \bigcup_{n\in\omega} A_{s\frown\langle n\rangle}$$

Now by Proposition 11.47, for each $s \in \mathrm{Seq}$ let $B_s \supseteq A_s$ be measurable such that every measurable $Z \subseteq B_s\backslash A_s$ is null. For each $s \in \mathrm{Seq}$, $\overline{A_s}$ is measurable. Let $B'_s = B_s \cap \overline{A_s}$ then $A_s \subseteq B'_s \subseteq \overline{A_s}$ and every measurable $Z \subseteq B'_s\backslash A_s$ is null. Now by (1) we have

$$A = \mathscr{A}(\langle B'_s : s \in \mathrm{Seq}\rangle).$$

Hence

$$B'_\emptyset\backslash A = B'_\emptyset\backslash \bigcup_{a\in{}^\omega\omega}\bigcap_{n\in\omega} B'_{a\restriction n}.$$

Now we claim that

$$B'_\emptyset\backslash \bigcup_{a\in{}^\omega\omega}\bigcap_{n\in\omega} B'_{a\restriction n} \subseteq \bigcup_{s\in\mathrm{Seq}}\left(B'_s\backslash\bigcup_{k\in\omega} B'_{s\frown\langle k\rangle}\right). \tag{3}$$

For, suppose that $x \in B'_\emptyset$ is in the right side but not in the left side. Then for every $x \in {}^\omega\omega$ and every $s \in \mathrm{Seq}$, if $x \in B'_s$ then $s \in B'_{s\frown\langle k\rangle}$ for some $k \in \omega$. Hence there is a k_0 such that $x \in B'_{\langle k_0\rangle}$, then there is a k_1 such that $x \in B'_{\langle k_0,k_1\rangle}$, etc., producing $a \overset{\text{def}}{=} \langle k_0, k_1, \ldots\rangle$ such that $x \in \bigcap_{n\in\omega} B'_{a\restriction n}$, so that x is not in the left side of (3). Thus (3) holds.

It follows that

$$B'_0\backslash A \subseteq \bigcup_{s\in\mathrm{Seq}} \left(B'_s\backslash \bigcup_{k\in\omega} B'_{s\frown\langle k\rangle}\right) \tag{4}$$

$$\forall s \in \mathrm{Seq}\left[B'_s\backslash \bigcup_{k\in\omega} B'_{s\frown\langle k\rangle} \text{ is null}\right]. \tag{5}$$

For, let $s\in\mathrm{Seq}$ and let $Z = B'_s\backslash\bigcup_{k\in\omega} B'_{s\frown\langle k\rangle}$. Then

$$Z = B'_s\backslash \bigcup_{k\in\omega} B'_{s\frown\langle k\rangle} \subseteq B'_s\backslash \bigcup_{k\in\omega} A_{s\frown\langle k\rangle} = B'_s\backslash A_s.$$

Since Z is measurable and $Z\subseteq B'_s\backslash A_s$, Z is null. This proves (5). Hence by (4), $B'_0\backslash A$ is null. Since $A\subseteq B'_0$ and B'_0 is measurable, also A is measurable. $\square$

Theorem 11.66. *Every analytic set has the Baire property.*

Proof. Let A be an analytic set of reals. Let $f : {}^\omega\omega \to \mathbb{R}$ be continuous with range A. For each $s\in\mathrm{Seq}$ let $A_s = f[U_s]$. Then (1) and (2) in the proof of Theorem 11.65 hold. By Lemma 11.64, there exists for each $s\in Seq$ a set $B_s\supseteq A_s$ with the Baire property such that every set $Z\subseteq B_s\backslash A_s$ with the Baire property is meager. Now $\overline{A_s}$ is closed, and hence by 11.56 has the Baire property. For each $s\in\mathrm{Seq}$ let $B'_s = B_s\cap\overline{A_s}$. Since $B'_\emptyset$ has the Baire property, it suffices to show that $B'_\emptyset\backslash A$ is meager, for then $A = B'_\emptyset\backslash(B'_\emptyset\backslash A)$ has the Baire property.

Now by Lemma 11.62, for each $s\in\mathrm{Seq}$ let B_s have the Baire property such that $B_s\supset A_s$ and whenever $Z\subseteq B_s\backslash A_s$ then Z is meager. Let $B'_s = B_s\cap\overline{A_s}$. Then B'_s has the Baire property and whenever $Z\subseteq B'_s\backslash A_s$ then Z is meager. Now by (1) we have

$$A = \mathscr{A}(\langle B'_s : s\in\mathrm{Seq}\rangle).$$

Hence

$$B'_\emptyset\backslash A = B'_\emptyset\backslash \bigcup_{a\in{}^\omega\omega}\bigcap_{n\in\omega} B'_{a\restriction n}.$$

Now we claim that

$$B'_\emptyset\backslash \bigcup_{a\in{}^\omega\omega}\bigcap_{n\in\omega} B'_{a\restriction n} \subseteq \bigcup_{s\in\mathrm{Seq}}\left(B'_s\backslash\bigcup_{k\in\omega} B'_{s\frown\langle k\rangle}\right). \tag{3}$$

For, suppose that $x\in B'_\emptyset$ is in the right side but not in the left side. Then for every $x\in{}^\omega\omega$ and every $s\in\mathrm{Seq}$, if $x\in B'_s$ then $s\in B'_{s\frown\langle k\rangle}$ for some $k\in\omega$. Hence there is a k_0 such that $x\in B'_{\langle k_0\rangle}$, then there is a k_1 such that $x\in B'_{\langle k_0,k_1\rangle}$, etc., producing $a \stackrel{\text{def}}{=} \langle k_0,k_1,\ldots\rangle$ such that $x\in\bigcap_{n\in\omega} B'_{a\restriction n}$, so that x is not in the left side of (3). Thus (3) holds.

It follows that

$$B_0'\backslash A \subseteq \bigcup_{s\in\mathrm{Seq}}\left(B_s'\backslash\bigcup_{k\in\omega}B'_{s\frown\langle k\rangle}\right) \tag{4}$$

$$\forall s\in\mathrm{Seq}\left[B_s'\backslash\bigcup_{k\in\omega}B'_{s\frown\langle k\rangle}\text{ is meager}\right]. \tag{5}$$

For, let $s\in\mathrm{Seq}$ and let $Z=B_s'\backslash\bigcup_{k\in\omega}B'_{s\frown\langle k\rangle}$. Then

$$Z=B_s'\backslash\bigcup_{k\in\omega}B'_{s\frown\langle k\rangle}\subseteq B_s'\backslash\bigcup_{k\in\omega}A_{s\frown\langle k\rangle}=B_s'\backslash A_s.$$

Since $Z\subseteq B_s'\backslash A_s$, Z is meager. This proves (5). Hence by (4), $B_0'\backslash A$ has the property of Baire. Since $A\subseteq B_0'$ and B_0' has the property of Baire, also A has the property of Baire. $\square$

Proposition 11.67. *Every separable metric space has a countable base for its topology.*

Proof. Let D be dense, Define

$$\mathscr{U}=\left\{U_{\frac{1}{n}}(x):x\in D,n\geq 1\right\}.$$

Now suppose that V is an open set and $x\in V$. Choose $m>0$ such that $B_{1/m}(x)\subseteq V$. Choose $y\in D\cap B_{1/2m}(x)$. Then $x\in B_{1/2m}(y)\subseteq B_{1/m}(x)\subseteq V$. For, let $z\in B_{1/2m}(y)$. Thus $d(z,x)\leq d(z,y)+d(y,x)<1/2m+1/2m=1/m$. $\square$

Suppose that X is a Polish space and $F\subseteq X$ is closed. Then we define

$$\begin{aligned}\Gamma^0(F)&=F\\ \Gamma^{\alpha+1}(F)&=\{x\in\Gamma^\alpha(F):x\text{ is not an isolated point of }\Gamma^\alpha(F)\}\\ \Gamma^\alpha(F)&=\bigcap_{\beta<\alpha}\Gamma^\beta(F)\quad\text{for }\alpha\text{ limit.}\end{aligned}$$

Proposition 11.68. *Suppose that X is a Polish space and $F\subseteq X$ is closed. Then $\forall\alpha[\Gamma^\alpha(F)$ is closed.* $\square$

Proposition 11.69. *Suppose that X is a Polish space and $F\subseteq X$ is closed. Then $\forall\alpha<\omega_1[\Gamma^\alpha(F)\backslash\Gamma^{\alpha+11}(F)$ is countable$]$.*

Proof. Let $\mathscr{A}$ be a countable base for X. $\Gamma^\alpha(F)\backslash\Gamma^{\alpha+1}(F)$ consists of the isolated points of $\Gamma^\alpha(F)$. For each $x\in\Gamma^\alpha(F)\backslash\Gamma^{\alpha+1}(F)$ let $U_x\in\mathscr{A}$ be such that $U_x\cap\Gamma^\alpha(F)=\{x\}$. Since $\mathscr{A}$ is countable, so is $\Gamma^\alpha(F)\backslash\Gamma^{\alpha+1}(F)$. $\square$

Proposition 11.70. *If $\Gamma^1(F) = F$, then F is perfect, and $\Gamma^\alpha(F) = F$ for all $\alpha < \omega_1$.*

Proof. $\Gamma^1(F) = F$ implies that F does not have any isolated points; hence it is perfect. $\Gamma^\alpha(F) = F$ for all $\alpha < \omega_1$ by an easy induction. $\square$

Proposition 11.71. *There is an ordinal $\alpha < \omega_1$ such that $\Gamma^\alpha(F) = \Gamma^{\alpha+1}(F)$.*

Proof. Suppose not. For each $\alpha < \omega_1$ let U_{n_α} be such that $U_{n_\alpha} \cap \Gamma^\alpha(F)\backslash\Gamma^{\alpha+1}(F)$ is a singleton. Clearly $n : \omega_1 \to \omega$ is one-one, contradiction. $\square$

Theorem 11.72. *Let X be a Polish space and $F \subseteq X$ closed. Then there exist a perfect, or empty, P and a countable A such that $F = P \cup A$ and $P \cap A = \emptyset$.*

Proof. Let α be minimum such that $\Gamma^\alpha(F) = \Gamma^{\alpha+1}(F)$. Then

$$F = \Gamma^\alpha(F) \cup \bigcup_{\beta<\alpha} (\Gamma^\beta(F)\backslash\Gamma^{\beta+1}(F)). \qquad \square$$

Lemma 11.73. *Let X be a Polish space and $A \subseteq X$ uncountable. Then there are disjoint open sets V_1 and V_2 such that $A \cap V_1$ and $A \cap V_2$ are uncountable.*

Proof. Suppose not.

(1) For each $n > 1$ there is an open cover $U_{n0}, U_{n1}, \ldots$ of X, each U_{ni} an open ball of radius $\frac{1}{n}$.

In fact, with $D \subseteq X$ countable and dense, for each $x \in D$ let V_x have radius $\frac{1}{n}$ with $x \in V_x$. Now given any $y \in X$, let W be an open ball with center y and radius $\frac{1}{2n}$. There is an $x \in D \cap W$. Then $d(x, y) < \frac{1}{2n}$, and so $y \in V_x$. This proves (1).

Now for each $n > 1$ choose $x(n) \in \omega$ such that $U_{nx(n)} \cap A$ is uncountable. For each $n > 1$ let $A_n = A\backslash\overline{U_{nx(n)}}$. For each n, if A_n is uncountable then "Suppose not" is contradicted. So each A_n is countable. Now

$$A\backslash \bigcup_{n>1} A_n = \bigcap_{n>1} \overline{U_{nx(n)}}.$$

But clearly $\bigcap_{n>1} \overline{U_{nx(n)}}$ has at most one element. Hence A is countable, contradiction. $\square$

Theorem 11.74. *For any Polish space X and any uncountable analytic subset $A \subseteq X$, there is a perfect set $P \subseteq A$.*

Proof. Let $f : {}^\omega\omega \to X$ be continuous with range A.

(1) If $V \subseteq {}^\omega\omega$ is open and $f[V]$ is uncountable, then there are disjoint open subsets W_1 and W_2 of V such that $f[W_1]$ and $f[W_2]$ are uncountable.

For, by Lemma 11.73 there are disjoint open subsets U_0 and U_1 of X such that $f[V] \cap U_0$ and $f[V] \cap U_1$ are uncountable. Now let $W_1 = f^{-1}[U_0] \cap V$ and $W_2 = f^{-1}[U_1] \cap V$. Clearly (1) holds.

(2) There is a function $\tau : {}^{<\omega}2 \to {}^{<\omega}\omega$ such that

(a) $\tau_\emptyset = \emptyset$.

(b) If $\sigma_1 \subseteq \sigma_2$, then $\tau_{\sigma_1} \subseteq \tau_{\sigma_2}$.

(c) $\forall \sigma \in {}^{<\omega}2[f[\{x \in {}^\omega\omega : \tau_\sigma \subseteq x\}]$ is uncountable.

(d) $\forall \sigma \in {}^{<\omega}2[f[\{x \in {}^\omega\omega : \tau_{\sigma\frown\langle 0\rangle} \subseteq x\}] \cap f[\{x \in {}^\omega\omega : \tau_{\sigma\frown\langle 1\rangle} \subseteq x\}] = \emptyset$.

We construct τ by recursion. $\tau_\emptyset = \emptyset$ clearly satisfies (a)–(c). If τ_σ has been defined, we apply (1) with $V = \{x \in {}^\omega\omega : \tau_\sigma \subseteq x\}$; this gives disjoint open W'_1, W'_2 contained in V. Both W'_1 and W'_2 are the union of countably many basic open subsets of ${}^\omega\omega$, giving the desired extension of V.

Now define $g : {}^\omega 2 \to {}^\omega\omega$ by $g(x) = \bigcup_{n\in\omega} \tau_{x\restriction n}$. To see that g is continuous, take any basic open set U_y of ${}^\omega\omega$ and suppose that $x \in g^{-1}[U_y]$. Then there is an $n \in \omega$ such that $y \subseteq \tau_{x\restriction n}$. Then $x \in U_{x\restriction n} \subseteq g^{-1}[U_y]$.

Now $\mathrm{rng}(f \circ g)$ is closed, by Theorem 3.1.12 of Engelking. Hence by Theorem 11.72, X has a perfect subset. □

Proposition 11.75. *The operations $\bigcup_{n\in\omega}$ and $\bigcap_{n\in\omega}$ are special cases of $\mathscr{A}$.*

Proof. (i) Define $A_s = B_{\mathrm{dmn}(s)}$ for every $s \in \mathrm{seq}$. Then

$$\bigcap_{n\in\omega} B_n = \bigcup_{\alpha\in{}^\omega\omega} \bigcap_{n\in\omega} A_{\alpha\restriction n}.$$

(ii) Let $A_\emptyset = \bigcup_{n\in\omega} B_n$, and for each nonempty $s \in \mathrm{seq}$ let $A_s = B_{s(0)}$. Then

$$\bigcup_{n\in\omega} B_n = \bigcup_{\alpha\in{}^\omega\omega} \bigcap_{n\in\omega} A_{\alpha\restriction n}.$$

Proposition 11.76. *Suppose that $\langle A_s : s \in \mathrm{Seq}\rangle$ is a system of Borel sets such that*

(i) $\forall s, t \in \mathrm{Seq}[s \subseteq t$ implies that $A_t \subseteq A_s]$,

(ii) $\forall s \in \mathrm{Seq}\forall m, n \in \omega[m \neq n$ implies that $A_{s\frown\langle m\rangle} \cap A_{s\frown\langle n\rangle} = \emptyset]$.

Then $\mathscr{A}(\langle A_s < s \in \mathrm{Seq}\rangle)$ is a Borel set.

Proof. We claim that

$$\bigcup_{a\in{}^\omega\omega} \bigcap_{n\in\omega} A_{a\restriction n} = \bigcap_{n\in\omega} \bigcup \{A_s : \mathrm{dmn}(s) = n\}.$$

For, first suppose that x is in the left side. Choose $a \in {}^\omega\omega$ such that $x \in \bigcap_{n\in\omega} A_{a\langle n}$. Suppose that $n \in \omega$. Then $x \in A_{a\restriction n}$. Thus x is in the right side.

Now suppose that x is in the right side. For each $n \in \omega$ choose s_n with domain n such that $x \in A_{s_n}$. Clearly $m < n$ implies that $s_m < s_n$. Hence x is in the left side. □

Proposition 11.77. *Suppose that A_n for $n = 0, \ldots$ are pairwise disjoint analytic sets. Then there exist pairwise disjoint Borel sets D_n for $n = 0, \ldots$ such that $\forall n \in \omega[A_n \subseteq D_n]$.*

Proof. For each n, A_n and $\bigcup_{m\neq n} A_m$ are disjoint Analytic sets. Hence by Lemma 11.38 there is a Borel set B_n such that $A_n \subseteq B_n$ and $B_n \cap \bigcup_{m\neq n} A_m = \emptyset$. Let $C_n = B_n \cap \bigcap_{m\neq n}(X\backslash B_m)$. Then C_n is Borel. If $x \in A_n$, then $x \in B_n$ and for all $m \neq n$, $x \in (X\backslash B_m)$. So $A_n \subseteq C_n$. Clearly $C_m \cap C_n = \emptyset$ for $m \neq n$.

Proposition 11.78. *If A has the Baire property then there exist a G_δ set G and an F_σ set F such that $G \subseteq A \subseteq F$ and $F\backslash G$ is meager.*

Proof.

(1) Any meager set is contained in a meager F_σ set.

For, let A be meager. Say $A = \bigcup_{n\in\omega} B_n$, each B_n nowhere dense. Then also each $\overline{B_n}$ is nowhere dense, and $A \subseteq \bigcup_{n\in\omega} \overline{B_n}$.

Now suppose that A has the Baire property. Let G be open such that $A\triangle G$ is meager. By (1) let Q be an F_σ such that Q is meager and $A\triangle G \subseteq Q$.

(2) $A = (G\backslash Q)\triangle(A \cap Q)$.

In fact, $G\backslash A \subseteq Q$, so $G\backslash Q \subseteq A$; hence $G\backslash Q \subseteq A\backslash Q$. Similarly, $A\backslash G \subseteq Q$, so $A\backslash Q \subseteq G$; hence $A\backslash Q \subseteq G\backslash Q$. So $A\backslash Q = G\backslash Q$. Note that $G\backslash Q$ is a G_δ. So

$$(G\backslash Q)\triangle(A \cap Q) = (A\backslash Q)\triangle(A \cap Q) = (A\backslash Q) \cup (A \cap Q) = A.$$

Since $A\backslash Q$ and $A \cap Q$ are disjoint, we have A equal to the disjoint union of a G_δ and a meager set.

Applying this to the complement of A, we get $X\backslash A = H \cup K$ with H a G_δ and K meager. Hence $A = (X\backslash H) \cap (X\backslash K)$. Now $X\backslash H \supseteq A$ is an F_σ, and $(X\backslash H)\backslash A = (X\backslash H) \cap (H \cup K) = K\backslash H$ is meager.

Proposition 11.79. *If A has the Baire property then there exists a unique regular open set U such that $A\triangle U$ is meager.*

Proof.

$(*)$ If H is open, then $\mathrm{int}(\mathrm{cl}(H))\backslash H$ is nowhere dense.

For, let H be any open set. Let $G = \mathrm{int}(\mathrm{cl}(H))$. Then G is regular open. Now suppose that U is a nonempty open set and $U \cap (X\backslash \mathrm{cl}(G\backslash H)) = \emptyset$. Then $U \subseteq \mathrm{cl}(G\backslash H)$, so $U \subseteq \mathrm{cl}(G) = G$, and also $U \subseteq X\backslash H$; so $U \subseteq G\backslash H \subseteq \mathrm{cl}(H)\backslash H$. But $\mathrm{cl}(H)\backslash H$ is nowhere dense by Fact 1 on page 93, so this is a contradiction. Therefore $(*)$ holds.

$(**)$ Any open set H has the form $H = G\backslash \mathrm{cl}(N)$ where G is regular open and N is nowhere dense.

For, let $G = \mathrm{int}(\mathrm{cl}(H))$ and $N = G\backslash H$. Now $G\backslash H \subseteq \mathrm{cl}(G\backslash H)$, so $G\backslash \mathrm{cl}(N) = G\backslash \mathrm{cl}(G\backslash H) \subseteq G\backslash(G\backslash H) = H$. If $H \cap \mathrm{cl}(N) \neq \emptyset$, then $H \cap N \neq \emptyset$, contradiction. So $H \cap \mathrm{cl}(N) = \emptyset$ and so $H \subseteq G\backslash \mathrm{cl}(N)$. Thus $H = G\backslash \mathrm{cl}(N)$. By $(*)$, N is nowhere dense.

Now for the proposition, let A have the Baire property. So let G be open with $P \stackrel{\text{def}}{=} A\triangle G$ meager. Let $N = \mathrm{int}(\mathrm{cl}(G))\backslash G$. By $(*)$, N is nowhere dense. Clearly $N\triangle\mathrm{int}(\mathrm{cl}(G)) = G$.

$P\triangle N$ is meager. Now $A\triangle\text{int}(\text{cl}(G))\triangle N = P$. So $A\triangle\text{int}(\text{cl}(G)) = N\triangle P$ is meager. This proves existence.

For uniqueness, we prove

$(***)$ If $G\triangle P = H\triangle Q$ with P, Q meager, G regular open, and H open, then $H \subseteq G$. (Uniqueness follows.)

We have $H\backslash\text{cl}(G) \subseteq H\triangle G = P\triangle Q$. So $H\backslash\text{cl}(G)$ is open and meager, hence is empty. So $H \subseteq \text{cl}(G)$. Hence $H \subseteq \text{int}(\text{cl}(G)) = G$, as desired. □

Now suppose that $k, l \in \omega$. Let $X = {}^k\omega \times {}^l({}^\omega\omega)$. Define

$$S_X \stackrel{\text{def}}{=} \{(m, \sigma) : m \in {}^k\omega. \text{ and } \sigma \in {}^l({}^{<\omega}\omega)\}.$$

Now a *Borel code* for a subset of X is a pair (T, l) such that

(1) $T \subseteq {}^{<\omega}\omega, \emptyset \in T$, and $\forall x \in T \forall y \in {}^{<\omega}\omega[y \subseteq x \to y \in T]$.

(2) $l : T \to (\{0\} \times \{0, 1\}) \cup (\{1\} \times S_X)$ and for all $\sigma \in T$,
  (i) If $l(\emptyset) = (0, 0)$, then $\sigma^\frown\{0\} \in T$ and $\forall n \geq 1[\sigma^\frown\{n\} \notin T]$.
  (ii) If $l(\emptyset) = (1, (m, \sigma))$ for some $(m, \sigma) \in S_X$, then $\forall n \in \omega[\sigma^\frown\{n\} \notin T]$.

12. Models of set theory

We assume a basic knowledge of model theory. The *language of set theory* is the first-order language $\mathscr{L}_{set}$ with just one non-logical constant, the binary relation symbol $\in$. If M is a class, E is a binary relation, and $\varphi(\overline{x})$ is a formula of $\mathscr{L}_{set}$ then we define the *relativization* φ^{ME} of φ to M and E as follows:

$$\begin{array}{rcl}
(x \in y)^{ME} & \text{is} & xEy; \\
(x = y)^{ME} & \text{is} & x = y \\
(\exists x\varphi)^{ME} & \text{is} & \exists x \in M\varphi^{ME}; \\
(\forall x\varphi)^{ME} & & \text{similarly}; \\
(\neg\varphi)^{ME} & \text{is} & \neg\varphi^{ME}; \\
\wedge, \vee, \rightarrow, \leftrightarrow & & \text{similarly}.
\end{array}$$

The incompletenss theorems of Gödel are roughly as follows:

First incompleteness theorem. If Γ is a computable set of sentences proving a sufficient amount of number theory, then there is a sentence φ such that neither φ nor $\neg\varphi$ is provable from Γ.

Second incompleteness theorem. If Γ is a computable set of sentences proving a sufficient amount of number theory, then the consistency of Γ, formulated in number theoretic form is not provable from Γ.

Tarski's theorem about truth does not involve the notion of proof. We formulate it and give a complete proof modulo some background in recursion theory. We need a definitional expansion of ZFC having an individual constant 0 and a one-place function symbol S; 0 is defined as $\emptyset$, and S is the function assigning $x \cup \{x\}$ to each set x. For simplicity we assume that the symbols are certain natural numbers. Then we define, for any formula φ,

$$\#(\varphi) = \prod_{i<\mathrm{dmn}(\varphi)} p_i^{\varphi_i},$$

where p is the sequence of primes; $p_0 = 2, p_1 = 3, p_2 = 5, \ldots$. Terms $\overline{n}$ are defined by recursion for each $n \in \omega$:

$$\begin{aligned}
\overline{0} &= 0; \\
\overline{n+1} &= S^n 0 \quad \text{that is, } n\ S\text{'s followed by } 0
\end{aligned}$$

We say that a formula $T(x)$ with one free variable x is a *truth definition* iff the following hold:

$$\begin{array}{rl}
 & \mathrm{ZFC} \vdash \forall x[T(x) \rightarrow x \in \omega]; \\
\text{if } \sigma \text{ is a sentence, then} & \mathrm{ZFC} \vdash \sigma \leftrightarrow T(\overline{\#\sigma})
\end{array}$$

Theorem 12.1. (Tarski) *A truth definition does not exist.*

Proof. Suppose it does. Let $\varphi_0, \varphi_1, \ldots$ enumerate all formulas with one free variable x. For each $m \in \omega$ let $f(m) = \#(\varphi_m(\overline{m}))$. Then f is a recursive function, and so is represented by a formula $\chi(x,y)$ in ZFC. This means that

$$\text{if } f(m)=n, \text{ then } \begin{cases} \text{ZFC} \vdash \chi(\overline{m},\overline{n}) & \text{and} \\ \\ \text{ZFC} \vdash \forall y[\chi(\overline{m},y) \rightarrow y=\overline{n}]. \end{cases}$$

Let $\psi(x)$ be the formula $\exists y[\chi(x,y) \wedge \neg T(y)]$. Say that ψ is φ_m. Let σ be the sentence $\varphi_m(\overline{m})$. Thus $f(m) = \#(\sigma)$. Hence $\text{ZFC} \vdash \chi(\overline{m}, \overline{\#(\sigma)})$, so $\text{ZFC} \vdash \neg T(\overline{\#(\sigma)}) \rightarrow \psi(\overline{m})$, i.e.,

(1) $$\text{ZFC} \vdash \neg T(\overline{\#(\sigma)}) \rightarrow \sigma.$$

On the other hand, $\text{ZFC} \vdash \chi(\overline{m},y) \rightarrow y = \overline{\#(\sigma)}$, so $\text{ZFC} \vdash \neg T(y) \wedge \chi(\overline{m},y) \rightarrow \neg T(\overline{\#\sigma})$, and hence

$$\text{ZFC} \vdash \sigma \rightarrow \neg T(\overline{\#\sigma}),$$

which together with (1) gives

$$\text{ZFC} \vdash \neg T(\overline{\#(\sigma)}) \leftrightarrow \sigma.$$

This contradicts our assumption that

$$\text{ZFC} \vdash T(\overline{\#(\sigma)}) \leftrightarrow \sigma,$$

since we assume that ZFC is consistent. □

We take Łoś's theorem as follows.

Theorem 12.2. (Łoś) *Suppose that $\mathscr{L}$ is a first-order language, $\overline{A} = \langle \overline{A_i} : i \in I\rangle$ is a system of $\mathscr{L}$-structures, F is an ultrafilter on I, and $a \in {}^{\omega}\prod_{i\in I} a_i$. The values of a will be denoted by $a^0, a^1, \ldots$. Let $\pi : \prod_{i\in I} A_i \rightarrow \prod_{i\in I} A_i/F$ be the natural mapping, taking each element of $\prod_{i\in I} A_i$ to its equivalence class under $\equiv^A_F$. For each $i \in I$ let $pr_i : \prod_{j\in I} A_j \rightarrow A_i$ be defined by setting $pr_i(x) = x_i$ for all $x \in \prod_{i\in I} A_i$. Suppose that φ is any formula of $\mathscr{L}$. Then*

$$\prod_{i\in I} \overline{A_i}/F \models \varphi[\pi \circ a] \text{ iff } \{i \in I : \overline{A_i} \models \varphi[\mathrm{pr}_i \circ a]\} \in F. \qquad \square$$

Now let $\overline{A}$ be any structure, S any set, and U an ultrafilter on S. For each $a \in A$ let c^S_a be the function with domain S such that $c^S_a(x) = a$ for all $x \in S$. Then define $j^S(a) = [c_a]$ for all $a \in A$, where $[c_a]$ is the equivalence class of c_a in the ultrapower ${}^S\overline{A}/U$.

Theorem 12.3. *j^S is an elementary embedding of $\overline{A}$ into ${}^S\overline{A}/U$.*

Proof. By Łoś's theorem, ${}^S\overline{A}/U \models \varphi[j^S(a_0), \ldots, j^S(a_{n-1})]$ iff $\{x \in S : \overline{A} \models \varphi[a_0, \ldots, a_{n-1}]\} \in U$ iff $\overline{A} \models \varphi[a_0, \ldots, a_{n-1}]$. □

• Suppose that $\mathbf{M} \subseteq \mathbf{N}$ are classes and $\varphi(x_1, \ldots, x_n)$ is a formula of our set-theoretical language. We say that φ is *absolute for* $\mathbf{M}, \mathbf{N}$ iff

$$\forall x_1, \ldots, x_n \in \mathbf{M}[\varphi^{\mathbf{M}}(x_1, \ldots, x_n) \text{ iff } \varphi^{\mathbf{N}}(x_1, \ldots, x_n)].$$

An important special case of this notion occurs when $\mathbf{N} = \mathbf{V}$. Then we just say that φ is *absolute for* $\mathbf{M}$.

More formally, we associate with three formulas $\mu(y, w_1, \ldots, w_m)$, $\nu(y, w_1, \ldots, w_m)$, $\varphi(x_1, \ldots, x_n)$ another formula "φ is absolute for μ, ν", namely the following formula:

$$\forall x_1, \ldots, x_n \left[\bigwedge_{1 \leq i \leq n} \mu(x_i) \to [\varphi^\mu(x_1, \ldots, x_n) \leftrightarrow \varphi^\nu(x_1, \ldots, x_n)] \right].$$

In full generality, very few formulas are absolute, as we will see later. Usually we need to assume that the sets are transitive. Then there is an important set of formulas all of which are absolute; this class is defined as follows.

• The set of Δ_0-formulas is the smallest set Γ of formulas satisfying the following conditions:
  (a) Each atomic formula is in Γ.
  (b) If φ and ψ are in Γ, then so are $\neg\varphi$ and $\varphi \wedge \psi$.
  (c) If φ is in Γ, then so are $\exists x \in y\varphi$ and $\forall x \in y\varphi$.

Recall here that $\exists x \in y\varphi$ and $\forall x \in y\varphi$ are abbreviations for $\exists x(x \in y \wedge \varphi)$ and $\forall x(x \in y \to \varphi)$ respectively.

Theorem 12.4. *If* $\mathbf{M}$ *is transitive and* φ *is* Δ_0*, then* φ *is absolute for* $\mathbf{M}$*.*

Proof. We show that the collection of formulas absolute for $\mathbf{M}$ satisfies the conditions defining the set Δ_0. Absoluteness is clear for atomic formulas. It is also clear that if φ and ψ are absolute for $\mathbf{M}$, then so are $\neg\varphi$ and $\varphi \wedge \psi$. Now suppose that φ is absolute for $\mathbf{M}$; we show that $\exists x \in y\varphi$ is absolute for $\mathbf{M}$. Implicitly, φ can involve additional parameters $w_1, \ldots, w_n$. Assume that $y, w_1, \ldots, w_n \in \mathbf{M}$. First suppose that $\exists x \in y\varphi(x, y, w_1, \ldots, w_n)$. Choose $x \in y$ so that $\varphi(x, y, w_1, \ldots, w_n)$. Since $\mathbf{M}$ is transitive, $x \in \mathbf{M}$. Hence by the "inductive assumption", $\varphi^{\mathbf{M}}(x, y, w_1, \ldots, w_n)$ holds. This shows that $(\exists x \in y\varphi(x, y, w_1, \ldots, w_n))^{\mathbf{M}}$. Conversely suppose that $(\exists x \in y\varphi(x, y, w_1, \ldots, w_n))^{\mathbf{M}}$. Thus $\exists x \in \mathbf{M}[x \in y \wedge \varphi^{\mathbf{M}}(x, y, w_1, \ldots, w_n)$. By the inductive assumption, $\varphi(x, y, w_1, \ldots, w_n)$. So this shows that $\exists x \in y\varphi(x, y, w_1, \ldots, w_n)$. The case $\forall x \in y\varphi$ is treated similarly. $\square$

Ordinals and special kinds of ordinals are absolute since they could have been defined using Δ_0 formulas:

Theorem 12.5. *The following are absolute for any transitive class:*

(i) x is an ordinal	*(iii) x is a successor ordinal*	*(v) x is ω*
(ii) x is a limit ordinal	*(iv) x is a finite ordinal*	*(vi) x is i (each $i < 10$)*

Proof.

$$\begin{aligned}
x \text{ is an ordinal} &\leftrightarrow \forall y \in x \forall z \in y[z \in x] \wedge \forall y \in x \forall z \in y \forall w \in z[w \in y];\\
x \text{ is a limit ordinal} &\leftrightarrow \exists y \in x[y = y] \wedge x \text{ is an ordinal} \wedge \forall y \in x \exists z \in x(y \in z);\\
x \text{ is a successor ordinal} &\leftrightarrow x \text{ is an ordinal} \wedge x \neq \emptyset \wedge x \text{ is not a limit ordinal};\\
x \text{ is a finite ordinal} &\leftrightarrow \forall y[y \notin x] \vee (x \text{ is a successor ordinal}\\
&\quad \wedge \forall y \in x(\forall z[z \notin y] \vee y \text{ is a successor ordinal}));\\
x = \omega &\leftrightarrow x \text{ is a limit ordinal} \wedge \forall y \in x(y \text{ is a finite ordinal});
\end{aligned}$$

finally, we do (vi) by induction on i. The case $i = 0$ is clear. Then

$$y = i + 1 \leftrightarrow \exists x \in y[x = i \wedge \forall z \in y[z \in x \vee z = x] \wedge \forall z \in x[z \in y] \wedge x \in y]. \qquad \square$$

The following theorem, while obvious, will be very useful in what follows.

Theorem 12.6. *Suppose that S is a set of sentences in our set-theoretic language, and $\mathbf{M}$ and $\mathbf{N}$ are classes which are models of S. Suppose that*

$$S \models \forall x_1, \ldots, x_n[\varphi(x_1, \ldots, x_n) \leftrightarrow \psi(x_1, \ldots, x_n)].$$

Then φ is absolute for $\mathbf{M}, \mathbf{N}$ iff ψ is. $\square$

Of course we will usually apply this when S is a subset of ZFC.

We let ZF be our axioms without the axiom of choice, and ZF − Inf the axioms ZF without the axiom of infinity. The status of the functions that we have defined requires some explanation. Whenever we defined a function $\mathbf{F}$ of n arguments, we have implicitly assumed that there is an associated formula φ whose free variables are among the first $n + 1$ variables, so that the following is derivable from the axioms assumed at the time of defining the function:

$$\forall v_0, \ldots, v_{n-1} \exists! v_n \varphi(v_0, \ldots, v_n).$$

Recall that "$\exists! v_n$" means "there is exactly one v_n". Now if we have a class model $\mathbf{M}$ in which this sentence holds, then we can define $\mathbf{F}^{\mathbf{M}}$ by setting, for any $x_0, \ldots, x_{n-1} \in \mathbf{M}$,

$$\mathbf{F}^{\mathbf{M}}(x_0, \ldots, x_{n-1}) = \text{ the unique } y \text{ such that } \varphi^{\mathbf{M}}(x_0, \ldots, x_{n-1}, y).$$

In case $\mathbf{M}$ satisfies the indicated sentence, we say that $\mathbf{F}$ *is defined in* $\mathbf{M}$. Given two class models $\mathbf{M} \subseteq \mathbf{N}$ in which $\mathbf{F}$ is defined, we say that $\mathbf{F}$ is *absolute for* $\mathbf{M}, \mathbf{N}$ provide that φ is. Note that for $\mathbf{F}$ to be absolute for $\mathbf{M}, \mathbf{N}$ it must be defined in both of them.

Proposition 12.7. *Suppose that $\mathbf{M} \subseteq \mathbf{N}$ are models in which $\mathbf{F}$ is defined. Then the following are equivalent:*

(i) $\mathbf{F}$ is absolute for $\mathbf{M}, \mathbf{N}$.

(ii) For all $x_0, \ldots, x_{n-1} \in \mathbf{M}$ we have $\mathbf{F}^{\mathbf{M}}(x_0, \ldots, x_{n-1}) = \mathbf{F}^{\mathbf{N}}(x_0, \ldots, x_{n-1})$.

Proof. Let φ be as above.

Assume (i), and suppose that $x_0, \ldots, x_{n-1} \in \mathbf{M}$. Let $y = \mathbf{F}^{\mathbf{M}}(x_0, \ldots, x_{n-1})$. Then $y \in \mathbf{M}$, and $\varphi^{\mathbf{M}}(x_0, \ldots, x_{n-1}, y)$, so by (i), $\varphi^{\mathbf{N}}(x_0, \ldots, x_{n-1}, y)$. Hence $\mathbf{F}^{\mathbf{N}}(x_0, \ldots, x_{n-1}) = y$.

Assume (ii), and suppose that $x_0, \ldots, x_{n-1}, y \in \mathbf{M}$. Then

$$\begin{aligned}
\varphi^{\mathbf{M}}(x_0, \ldots, x_{n-1}, y) \quad &\text{iff} \quad \mathbf{F}^{\mathbf{M}}(x_0, \ldots, x_{n-1}) = y \quad \text{(definition of } \mathbf{F}\text{)} \\
&\text{iff} \quad \mathbf{F}^{\mathbf{N}}(x_0, \ldots, x_{n-1}) = y \quad \text{(by (ii))} \\
&\text{iff} \quad \varphi^{\mathbf{N}}(x_0, \ldots, x_{n-1}, y) \quad \text{(definition of } \mathbf{F}\text{)}. \qquad \square
\end{aligned}$$

The following theorem gives many explicit absoluteness results, and will be used frequently along with some similar results below. Note that we do not need to be explicit about how the relations and functions were really defined.

Theorem 12.8. *The following relations and functions were defined by formulas equivalent to Δ_0-formulas on the basis of* ZF−Inf*, and hence are absolute for all transitive class models of* ZF − Inf*:*

(i) $x \in y$
(ii) $x = y$
(iii) $x \subseteq y$
(iv) $\{x, y\}$
(v) $\{x\}$
(vi) (x, y)
(vii) $\emptyset$
(viii) $x \cup y$
(ix) $x \cap y$
(x) $x \backslash y$
(xi) $x \cup \{x\}$
(xii) x is transitive
(xiii) $\bigcup x$
(xiv) $\bigcap x$ *(with* $\bigcap \emptyset = \emptyset$*)*

Note here, for example, that in (iv) we really mean the 2-place function assigning to sets x, y the unordered pair $\{x, y\}$.

Proof. (i) and (ii) are already Δ_0 formulas. (iii):

$$x \subseteq y \leftrightarrow \forall z \in x(z \in y).$$

(iv):

$$z = \{x, y\} \leftrightarrow \forall w \in z(w = x \vee w = y) \wedge x \in z \wedge y \in z.$$

(v): Similarly. (vi):

$$z = (x, y) \leftrightarrow \forall w \in z[w = \{x, y\} \vee w = \{x\}] \wedge \exists w \in z[w = \{x, y\}] \wedge \exists w \in z[w = \{x\}].$$

(vii):

$$x = \emptyset \leftrightarrow \forall y \in x(y \neq y).$$

(viii):

$$z = x \cup y \leftrightarrow \forall w \in z(w \in x \vee w \in y) \wedge \forall w \in x(w \in z) \wedge \forall w \in y(w \in z).$$

(ix):

$$z = x \cap y \leftrightarrow \forall w \in z(w \in x \wedge w \in y) \wedge \forall w \in x(w \in y \rightarrow w \in z).$$

(x):

$$z = x \backslash y \leftrightarrow \forall w \in z(w \in x \wedge w \notin y) \wedge \forall w \in x(x \notin y \rightarrow w \in z).$$

(xi):

$$y = x \cup \{x\} \leftrightarrow \forall w \in y(w \in x \vee w = x) \wedge \forall w \in x(w \in y) \wedge x \in y.$$

(xii):

$$x \text{ is transitive} \leftrightarrow \forall y \in x(y \subseteq x).$$

(xiii):

$$y = \bigcup x \leftrightarrow \forall w \in y \exists z \in x(w \in z) \wedge \forall w \in x(w \subseteq y).$$

(xiv):

$$\begin{aligned} y = \bigcap x \leftrightarrow &[x \neq \emptyset \wedge \forall w \in y \forall z \in x(w \in z) \\ &\wedge \forall w \in x \forall t \in w[\forall z \in x(t \in z) \rightarrow t \in y] \vee [x = \emptyset \wedge y = \emptyset]. \end{aligned}$$ □

A stronger form of Theorem 12.8. For each of the indicated relations and functions, we do not need the model to be all of ZF − Inf. In fact, we need only finitely many of the axioms of ZF − Inf: enough to prove the uniqueness condition for any functions involved, and enough to prove the equivalence of the formula with a Δ_0-formula, since Δ_0 formulas are absolute for any transitive class model. To be absolutely rigorous here, one would need an explicit definition for each relation and function symbol involved, and then an explicit proof of equivalence to a Δ_0 formula; given these, a finite set of axioms becomes clear. And since any of the relations and functions of Theorem 12.8 require only finitely many basic relations and functcions, this can always be done. For Theorem 12.8 it is easy enough to work this all out in detail. We will be interested, however, in using this fact for more complicated absoluteness results to come.

As an illustration, however, we do some details for the function $\{x, y\}$. The definition involved is naturally taken to be the following:

$$\forall x, y, z[z = \{x, y\} \leftrightarrow \forall w[w \in z \leftrightarrow w = x \vee x = y]].$$

The axioms involved are the pairing axiom and one instance of the comprehension axiom:

$$\begin{aligned} &\forall x, y \exists w[x \in w \wedge y \in w]; \\ &\forall x, y, w \exists z \forall u(u \in z \leftrightarrow u \in w \wedge (u = x \vee u = y)). \end{aligned}$$

$\{x, y\}$ is then absolute for any transitive class model of these three sentences, by the proof of (iv) in Theorem 12.8, for which they are sufficient.

For further absoluteness results we will not reduce to Δ_0 formulas. We need the following extensions of the absoluteness notion.

• Suppose that $\mathbf{M} \subseteq \mathbf{N}$ are classes, and $\varphi(w_1, \ldots, w_n)$ is a formula. Then we say that φ is *absolute upwards* for $\mathbf{M}, \mathbf{N}$ iff for all $w_1, \ldots, w_n \in \mathbf{M}$, if $\varphi^{\mathbf{M}}(w_1, \ldots, w_n)$, then $\varphi^{\mathbf{N}}(w_1, \ldots, w_n)$. It is *absolute downwards* for $\mathbf{M}, \mathbf{N}$ iff for all $w_1, \ldots, w_n \in \mathbf{M}$, if $\varphi^{\mathbf{N}}(w_1, \ldots, w_n)$, then $\varphi^{\mathbf{M}}(w_1, \ldots, w_n)$. Thus φ is absolute for $\mathbf{M}, \mathbf{N}$ iff it it is both absolute upwards for $\mathbf{M}, \mathbf{N}$ and absolute downwards for $\mathbf{M}, \mathbf{N}$.

Theorem 12.9. *Suppose that $\varphi(x_1,\ldots,x_n,w_1,\ldots,w_m)$ is absolute for $\mathbf{M},\mathbf{N}$. Then*

(i) $\exists x_1,\ldots\exists x_n\varphi(x_1,\ldots,x_n,w_1,\ldots,w_m)$ is absolute upwards for $\mathbf{M},\mathbf{N}$.

(ii) $\forall x_1,\ldots\forall x_n\varphi(x_1,\ldots,x_n,w_1,\ldots,w_m)$ is absolute downwards for $\mathbf{M},\mathbf{N}$. □

Theorem 12.10. *Absoluteness is preserved under composition. In detail: suppose that $\mathbf{M}\subseteq\mathbf{N}$ are classes, and the following are absolute for $\mathbf{M},\mathbf{N}$:*

$\varphi(x_1,\ldots,x_n)$;
$\mathbf{F}$, an n-ary function ;
For each $i=1,\ldots,n$, an m-ary function $\mathbf{G}_i$.

Then the following are absolute:

(i) $\varphi(\mathbf{G}_1(x_1,\ldots,x_m),\ldots,\mathbf{G}_n(x_1,\ldots,x_m))$.

(ii) The m-ary function assigning to $x_1,\ldots,x_m$ the value

$$\mathbf{F}(\mathbf{G}_1(x_1,\ldots,x_m),\ldots,\mathbf{G}_n(x_1,\ldots,x_m)).$$

Proof. We use Theorem 12.9:

$$\begin{aligned}
\varphi(\mathbf{G}_1(x_1,\ldots,x_m),\ldots,\mathbf{G}_n(x_1,\ldots,x_m)) \leftrightarrow &\exists z_1,\ldots\exists z_n\Bigg[\varphi(z_1,\ldots,z_n)\\
&\wedge\bigwedge_{i=1}^n(z_i=\mathbf{G}_i(x_1,\ldots,x_m))\Bigg];\\
\varphi(\mathbf{G}_1(x_1,\ldots,x_m),\ldots,\mathbf{G}_n(x_1,\ldots,x_m)) \leftrightarrow &\forall z_1,\ldots\forall z_n\Bigg[\bigwedge_{i=1}^n(z_i=\mathbf{G}_i(x_1,\ldots,x_m))\\
&\to\varphi(z_1,\ldots,z_n)\Bigg];\\
y=\mathbf{F}(\mathbf{G}_1(x_1,\ldots,x_m),\ldots,\mathbf{G}_n(x_1,\ldots,x_m)) \leftrightarrow &\exists z_1,\ldots\exists z_n\Bigg[(y=\mathbf{F}(z_1,\ldots,z_n))\\
&\wedge\bigwedge_{i=1}^n(z_i=\mathbf{G}_i(x_1,\ldots,x_m))\Bigg];\\
y=\mathbf{F}(\mathbf{G}_1(x_1,\ldots,x_m),\ldots,\mathbf{G}_n(x_1,\ldots,x_m)) \leftrightarrow &\forall z_1,\ldots\forall z_n\Bigg[\bigwedge_{i=1}^n(z_i=\mathbf{G}_i(x_1,\ldots,x_m))\\
&\to(y=\mathbf{F}(z_1,\ldots,z_n))\Bigg].
\end{aligned}$$

□

Theorem 12.11. *Suppose that $\mathbf{M}\subseteq\mathbf{N}$ are classes, $\varphi(y,x_1,\ldots,x_m,w_1,\ldots,w_n)$ is absolute for $\mathbf{M},\mathbf{N}$, and $\mathbf{F}$ and $\mathbf{G}$ are n-ary functions absolute for $\mathbf{M},\mathbf{N}$. Then the following are also absolute for $\mathbf{M},\mathbf{N}$:*

(i) $z\in\mathbf{F}(x_1,\ldots,x_m)$.

(ii) $\mathbf{F}(x_1,\ldots,x_m)\in z$.

(iii) $\exists y\in\mathbf{F}(x_1,\ldots,x_m)\varphi(y,x_1,\ldots,x_m,w_1,\ldots,w_n)$.

(iv) $\forall y \in \mathbf{F}(x_1, \ldots, x_m)\varphi(y, x_1, \ldots, x_m, w_1, \ldots, w_n)$.
(v) $\mathbf{F}(x_1, \ldots, x_m) = \mathbf{G}(x_1, \ldots, x_m)$.
(vi) $\mathbf{F}(x_1, \ldots, x_m) \in \mathbf{G}(x_1, \ldots, x_m)$.

Proof.

$$\begin{aligned}
z \in \mathbf{F}(x_1, \ldots, x_m) &\leftrightarrow \exists w[z \in w \wedge w = \mathbf{F}(x_1, \ldots, x_m)];\\
&\leftrightarrow \forall w[w = \mathbf{F}(x_1, \ldots, x_m) \to z \in w];\\
\mathbf{F}(x_1, \ldots, x_m) \in z &\leftrightarrow \exists w \in z[w = \mathbf{F}(x_1, \ldots, x_m)];
\end{aligned}$$

$$\begin{aligned}
&\exists y \in \mathbf{F}(x_1, \ldots, x_m)\varphi(y, x_1, \ldots, x_m, w_1, \ldots, w_n)\\
&\leftrightarrow \exists w \exists y \in w[w = \mathbf{F}(x_1, \ldots, x_m) \wedge \varphi(y, x_1, \ldots, x_m, w_1, \ldots, w_n)];\\
&\leftrightarrow \forall w[w = \mathbf{F}(x_1, \ldots, x_m) \to \exists y \in w\varphi(y, x_1, \ldots, x_m, w_1, \ldots, w_n)];
\end{aligned}$$

(iv)–(vi) are proved similarly. □

We now give some more specific absoluteness results.

Theorem 12.12. *The following relations and functions are absolute for all transitive class models of* ZF − Inf*:*

(i) x is an ordered pair	*(iv)* $\mathrm{dmn}(R)$	*(vii)* $R(x)$
(ii) $A \times B$	*(v)* $\mathrm{rng}(R)$	*(viii) R is a one-one function*
(iii) R is a relation	*(vi) R is a function*	*(ix) x is an ordinal*

Note concerning (vii): This is supposed to have its natural meaning if R is a function and x is in its domain; otherwise, $R(x) = \emptyset$.

Proof.

$$\begin{aligned}
x \text{ is an ordered pair} &\leftrightarrow \left(\exists y \in \bigcup x\right)\left(\exists z \in \bigcup x\right)[x = (y, z)];\\
y = A \times B &\leftrightarrow (\forall a \in A)(\forall b \in B)[(a, b) \in y] \wedge\\
&\qquad (\forall z \in y)(\exists a \in A)(\exists b \in B)[z = (a, b)];\\
R \text{ is a relation} &\leftrightarrow \forall x \in R[x \text{ is an ordered pair}];\\
x = \mathrm{dmn}(R) &\leftrightarrow (\forall y \in x)\left(\exists z \in \bigcup\bigcup R\right)[(x, z) \in R] \wedge\\
&\qquad \left(\forall y \in \bigcup\bigcup R\right)\left(\forall z \in \bigcup\bigcup R\right)[(y, z) \in R \to y \in x];\\
x = \mathrm{rng}(R) &\leftrightarrow (\forall y \in x)\left(\exists z \in \bigcup\bigcup R\right)[(z, x) \in R] \wedge\\
&\qquad \left(\forall y \in \bigcup\bigcup R\right)\left(\forall z \in \bigcup\bigcup R\right)[(y, z) \in R \to z \in x];\\
R \text{ is a function} &\leftrightarrow R \text{ is a relation } \wedge \left(\forall x \in \bigcup\bigcup R\right)\left(\forall y \in \bigcup\bigcup R\right)\\
&\qquad \left(\forall z \in \bigcup\bigcup R\right)[(x, y) \in R \wedge (x, z) \in R \to y = z];
\end{aligned}$$

$$
\begin{aligned}
y = R(x) \leftrightarrow & [R \text{ is a function} \wedge (x,y) \in R] \vee \\
& [R \text{ is not a function} \wedge (\forall z \in y)(z \neq z)] \vee \\
& [x \notin \text{dmn}(R) \wedge (\forall z \in y)(z \neq z)];
\end{aligned}
$$

$$
\begin{aligned}
R \text{ is a one-one function} \leftrightarrow & R \text{ is a function} \wedge \\
& \forall x \in \text{dmn}(R) \forall y \in \text{dmn}(R)[R(x) = R(y) \rightarrow x = y];
\end{aligned}
$$

$$x \text{ is an ordinal} \leftrightarrow x \text{ is transitive} \wedge (\forall y \in x)(y \text{ is transitive}). \qquad \square$$

Theorem 12.13. *If* $\mathbf{M}$ *is a transitive class model of* ZF*, then* $\mathbf{M}$ *is closed under the following set-theoretic operations:*

(i) $\cup$
(ii) $\cap$
(iii) $(a, b) \mapsto a \backslash b$
(iv) $(a, b) \mapsto \{a, b\}$
(v) $(a, b) \mapsto (a, b)$
(vi) $x \mapsto x \cup \{x\}$
(vii) $\bigcup$
(viii) $\bigcap$

Moreover, $[\mathbf{M}]^{<\omega} \subseteq \mathbf{M}$.

Proof. (i)–(viii) are all very similar, so we only treat (i). Let $a, b \in \mathbf{M}$. Then because $\mathbf{M} \models$ ZF, there is a $c \in \mathbf{M}$ such that $(c = a \cup b)^{\mathbf{M}}$. By absoluteness, $c = a \cup b$.

Now we prove that $x \in \mathbf{M}$ for all $x \in [\mathbf{M}]^{<\omega}$ by induction on $|x|$. If $|x| = 0$, then $x = \emptyset$. Now $\mathbf{M} \models \exists v \forall w[w \notin v]$. So choose $s \in \mathbf{M}$ such that $\mathbf{M} \models \forall w[w \notin s]$. By transitivity, $s = \emptyset$. Thus $\emptyset \in \mathbf{M}$. If $a \in \mathbf{M}$, then $\mathbf{M} \models \exists v \forall w[w \in v \leftrightarrow w = a]$. Choose $s \in \mathbf{M}$ such that $\mathbf{M} \models \forall w[w \in s \leftrightarrow w = a]$. By absoluteness, $s = \{a\}$. So $\{a\} \in \mathbf{M}$. So our statement holds for all x with $|x| = 1$. Now suppose that $x \in \mathbf{M}$ for all $x \subseteq \mathbf{M}$ such that $|x| = n$. Suppose that $y \subseteq \mathbf{M}$ and $|y| = n + 1$. Take any $a \in y$. Then $|y \backslash \{a\}| = n$, so $y \backslash \{a\} \in M$. Hence by (i), $y = (y \backslash \{a\}) \cup \{a\} \in \mathbf{M}$. $\square$

The hierarchy of sets is defined recursively as follows:

Theorem 12.14. *There is a class function* $V : \mathbf{On} \rightarrow \mathbf{V}$ *satisfying the following conditions:*

(i) $V_0 = \emptyset$.
(ii) $V_{\alpha+1} = \mathscr{P}(V_\alpha)$.
(iii) $V_\gamma = \bigcup_{\alpha<\gamma} V_\alpha$ *for* γ *limit.*

Theorem 12.15. *For every ordinal* α *the following hold:*

(i) V_α *is transitive.*
(ii) $V_\beta \subseteq V_\alpha$ *for all* $\beta < \alpha$.

Proof. We prove these statements simultaneously by induction on α. They are clear for $\alpha = 0$. Assume that both statements hold for α; we prove them for $\alpha + 1$. First we prove

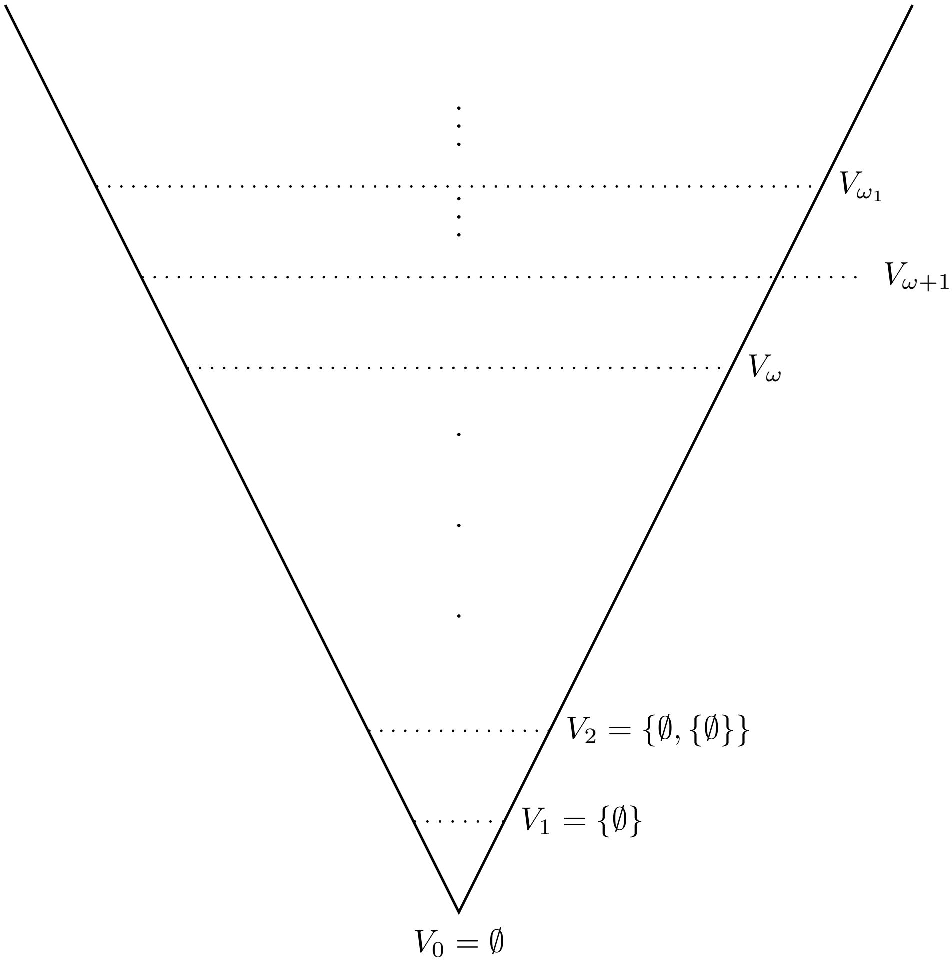
(1) $V_\alpha \subseteq V_{\alpha+1}$.

In fact, suppose that $x \in V_\alpha$. By (i) for α, the set V_α is transitive. Hence $x \subseteq V_\alpha$, so $x \in \mathscr{P}(V_\alpha) = V_{\alpha+1}$. So (1) holds.

Now (ii) follows. For, suppose that $\beta < \alpha + 1$. Then $\beta \leq \alpha$, so $V_\beta \subseteq V_\alpha$ by (ii) for α (or trivially if $\beta = \alpha$). Hence by (1), $V_\beta \subseteq V_{\alpha+1}$.

To prove (i) for $\alpha+1$, suppose that $x \in y \in V_{\alpha+1}$. Then $y \in \mathscr{P}(V_\alpha)$, so $y \subseteq V_\alpha$, hence $x \in V_\alpha$. By (1), $x \in V_{\alpha+1}$, as desired.

For the final inductive step, suppose that γ is a limit ordinal and (i) and (ii) hold for all $\alpha < \gamma$. To prove (i) for γ, suppose that $x \in y \in V_\gamma$. Then by definition of V_γ, there is an $\alpha < \gamma$ such that $y \in V_\alpha$. By (i) for α we get $x \in V_\alpha$. So $x \in V_\gamma$ by the definition of V_γ. Condition (ii) for γ is obvious. $\square$



A very important fact about this hierarchy is that every set is a member of some V_α. To prove this, we need the notion of transitive closure. We introduced and used this notion in Chapter 8, but we will prove the following independent of this.

Theorem 12.16. *For any set a there is a transitive set b with the following properties:*
(i) $a \subseteq b$.
(ii) For every transitive set c such that $a \subseteq c$ we have $b \subseteq c$.

Proof. We first make a definition by recursion. Define $\mathbf{G} : \mathbf{On} \times \mathbf{V} \to \mathbf{V}$ by setting, for an $\alpha \in \mathbf{On}$ and any $x \in \mathbf{V}$

$$\mathbf{G}(\alpha, x) = \begin{cases} a & \text{if } x = \emptyset, \\ x(m) \cup \bigcup x(m) & \text{if } x \text{ is a function with domain } m+1 \text{ with } m \in \omega, . \\ 0 & \text{otherwise} \end{cases}$$

By Theorem 9.7 let $\mathbf{F} : \mathbf{On} \to \mathbf{V}$ be such that $\mathbf{F}(\alpha) = \mathbf{G}(\alpha, \mathbf{F} \restriction \alpha)$ for any $\alpha \in \mathbf{On}$. Let $d = \mathbf{F} \restriction \omega$. Then $d_0 = \mathbf{F}(0) = \mathbf{G}(0, \mathbf{F} \restriction 0) = \mathbf{G}(0, \emptyset) = a$. For any $m \in \omega$ we have $d_{m+1} = \mathbf{F}(m+1) = \mathbf{G}(m+1, \mathbf{F} \restriction (m+1)) = \mathbf{F}(m) \cup \bigcup \mathbf{F}(m) = d_m \cup \bigcup d(m)$. Let $b = \bigcup_{m\in\omega} d_m$. Then $a = d_0 \subseteq b$. Suppose that $x \in y \in b$. Choose $m \in \omega$ such that $y \in d_m$. Then $x \in \bigcup d_m \subseteq d_{m+1} \subseteq b$. Thus b is transitive. Now suppose that c is a transitive set such that $a \subseteq c$. We show by induction that $d_m \subseteq c$ for every $m \in \omega$. First, $d_0 = a \subseteq c$, so this is true for $m = 0$. Now assume that it is true for m. Then $d_{m+1} = d_m \cup \bigcup d_m \subseteq c \cup \bigcup c = c$, completing the inductive proof.

Hence $b = \bigcup_{m\in\omega} d_m \subseteq c$. □

The set shown to exist in Theorem 12.16 is called the *transitive closure* of a, and is denoted by $\mathrm{trcl}(a)$.

Theorem 12.17. *Every set is a member of some V_α.*

Proof. Suppose that this is not true, and let a be a set which is not a member of any V_α. Let $A = \{x \in \mathrm{trcl}(a \cup \{a\}) : x \text{ is not in any of the sets } V_\alpha\}$. Then $a \in A$, so A is nonempty. By the foundation axiom, choose $x \in A$ such that $x \cap A = 0$. Suppose that $y \in x$. Then $y \in \mathrm{trcl}(a \cup \{a\})$, so y is a member of some V_α. Let α_y be the least such α. Let $\beta = \bigcup_{y\in x} \alpha_y$. Then by 12.1(ii), $x \subseteq V_\beta$. So $x \in V_{\beta+1}$, contradiction. □

An important technical consequence of Theorem 12.17 is the following definition, known as *Scott's trick*:

• Let R be a class equivalence relation on a class A. For each $a \in A$, let α be the smallest ordinal such that there is a $b \in V_\alpha$ with $(a, b) \in R$, and define

$$\mathrm{type}_R(a) = \{b \in V_\alpha : (a, b) \in R\}.$$

This is the "reduced" equivalence class of a. It could be that the collection of b such that $(a, b) \in R$ is a proper class, but $\mathrm{type}_R(a)$ is always a set.

On the basis of our hierarchy we can define the important notion of *rank* of sets:

• For any set x, the rank of x, denoted by $\mathrm{rank}(x)$, is the smallest ordinal α such that $x \in V_{\alpha+1}$.

We take $\alpha + 1$ here instead of α just for technical reasons. Some of the most important properties of ranks are given in the following theorem.

Theorem 12.18. *Let x be a set and α an ordinal. Then*

(i) $V_\alpha = \{y : \mathrm{rank}(y) < \alpha\}$.

(ii) For all $y \in x$ we have $\mathrm{rank}(y) < \mathrm{rank}(x)$.

(iii) $\mathrm{rank}(y) \leq \mathrm{rank}(x)$ *for every $y \subseteq x$.*

(iv) $\mathrm{rank}(x) = \sup_{y\in x}(\mathrm{rank}(y) + 1)$.

(v) $\mathrm{rank}(\alpha) = \alpha$.

(vi) $V_\alpha \cap \mathbf{On} = \alpha$.

Proof. (i): Suppose that $y \in V_\alpha$. Then $\alpha \neq 0$. If α is a successor ordinal $\beta + 1$, then $\mathrm{rank}(y) \leq \beta < \alpha$. If α is a limit ordinal, then $y \in V_\beta$ for some $\beta < \alpha$, hence $y \in V_{\beta+1}$ also, so $\mathrm{rank}(y) \leq \beta < \alpha$. This proves $\subseteq$.

For $\supseteq$, suppose that $\beta \stackrel{\text{def}}{=} \text{rank}(y) < \alpha$. Then $y \in V_{\beta+1} \subseteq V_\alpha$, as desired.

(ii): Suppose that $x \in y$. Let $\text{rank}(y) = \alpha$. Thus $y \in V_{\alpha+1} = \mathscr{P}(V_\alpha)$, so $y \subseteq V_\alpha$ and hence $x \in V_\alpha$. Then by (i), $\text{rank}(x) < \alpha$.

(iii): Let $\text{rank}(x) = \alpha$. Then $x \in V_{\alpha+1}$, so $x \subseteq V_\alpha$. Let $y \subseteq x$. Then $y \subseteq V_\alpha$, and so $y \in V_{\alpha+1}$. Thus $\text{rank}(y) \leq \alpha$.

(iv): Let α be the indicated sup. Then $\geq$ holds by (ii). Now if $y \in x$, then $\text{rank}(y) < \alpha$, and hence $y \in V_{\text{rank}(y)+1} \subseteq V_\alpha$. This shows that $x \subseteq V_\alpha$, hence $x \in V_{\alpha+1}$, hence $\text{rank}(x) \leq \alpha$, finishing the proof of (iv).

(v): We prove this by transfinite induction. Suppose that it is true for all $\beta < \alpha$. Then by (iv),

$$\text{rank}(\alpha) = \sup_{\beta<\alpha}(\text{rank}(\beta) + 1) = \sup_{\beta<\alpha}(\beta + 1) = \alpha.$$

Finally, for (vi), using (i) and (v),

$$V_\alpha \cap \mathbf{On} = \{\beta \in \mathbf{On} : \beta \in V_\alpha\} = \{\beta \in \mathbf{On} : \text{rank}(\beta) < \alpha\} = \{\beta \in \mathbf{On} : \beta < \alpha\} = \alpha. \quad \square$$

Theorem 12.19. *(i) $n \leq |V_n| \in \omega$ for any $n \in \omega$.*

(ii) For any ordinal α, $|V_{\omega+\alpha}| = \beth_\alpha$.

Proof. (i) is clear by ordinary induction on n. We prove (ii) by the three-step transfinite induction (where γ is a limit ordinal below):

$$\begin{aligned}
|V_\omega| &= \left|\bigcup_{n\in\omega} V_n\right| = \omega = \beth_0 \quad \text{by (i);}\\
|V_{\omega+\alpha+1}| &= |\mathscr{P}(V_{\omega+\alpha})|\\
&= 2^{|V_{\omega+\alpha}|}\\
&= 2^{\beth_\alpha} \quad \text{(inductive hypothesis)}\\
&= \beth_{\alpha+1};\\
|V_{\omega+\gamma}| &= \left|\bigcup_{\beta<\gamma} V_{\omega+\beta}\right|\\
&\leq \sum_{\beta<\gamma} |V_{\omega+\beta}|\\
&= \sum_{\beta<\gamma} \beth_\beta \quad \text{(inductive hypothesis)}\\
&\leq \sum_{\beta<\gamma} \beth_\gamma\\
&= |\gamma| \cdot \beth_\gamma\\
&= \beth_\gamma \quad .
\end{aligned}$$

To finish this last inductive step, note that for each $\beta < \gamma$ we have $\beth_\beta = |V_{\omega+\beta}| \leq |V_{\omega+\gamma}|$, and hence $\beth_\gamma \leq |V_{\omega+\gamma}|$. $\square$

Many of the results and proofs below are taken from Kunen 2011, and we indicate the exact place.

Lemma 12.20. *If $\alpha \geq \omega^2$, then $|V_\alpha| = \beth_\alpha$.*

Proof. We have $|V_{\omega+\alpha}| = \beth_\alpha$ for all α. If $\alpha \geq \omega^2$, write $\alpha = \omega^2 + \beta$. Then $|V_\alpha| = |V_{\omega^2+\beta}| = |V_{\omega+\omega^2+\beta}| = \beth_{\omega^2+\beta} = \beth_\alpha$. □

Theorem 12.21. *If κ is inaccessible, then $(V_\kappa, \in)$ is a model of ZFC.*

Proof. For the axioms, see pp. 5ff.

(1) Extensionality. Relativized to V_κ, this is

$$\forall X, Y \in V_\kappa[\forall u \in V_\kappa[u \in X \leftrightarrow u \in Y] \rightarrow X = Y].$$

Since V_κ is transitive. $\forall u[u \in X \leftrightarrow u \in Y]$. Hence $X = Y$.

(2) Pairing. We want to show that

$$\forall a, b \in V_\kappa \exists c \in V_\kappa \forall x \in V_\kappa[x \in c \leftrightarrow x = a \text{ or } x = b].$$

Suppose that $a, b \in V_\kappa$. Choose $\alpha, \beta < \kappa$ such that $a \in V_\alpha$ and $b \in V_\beta$. Say $\alpha \leq \beta$. Then $\{a, b\} \subseteq V_\beta$, so $\{a, b\} \in V_{\beta+1} \subseteq V_\kappa$.

(3) Separation, We want to show that

$$\forall p_0, \ldots, p_{n-1}, X \in V_\kappa \exists Y \in V_\kappa \forall u \in V_\kappa[u \in Y \leftrightarrow u \in X \text{ and } \varphi^{V_\kappa}(u, p_0, \ldots, p_{n-1})].$$

Given $p_0, \ldots, p_{n-1}, X \in V_\kappa$ let

$$Y = \{u \in X : \varphi^{V_\kappa}(u, p_0, \ldots, p_{n-1})\}.$$

Since $Y \subseteq X \in V_\kappa$, it follows that $Y \in V_\kappa$.

(4) Union. We want to show that

$$\forall X \in V_\kappa \exists Y \in V_\kappa \forall u \in V_\kappa[u \in Y \leftrightarrow \exists z \in V_\kappa[z \in X \text{ and } u \in z]].$$

Given $X \in V_\kappa$ let $Y = \bigcup X$. Clearly $Y \in V_\kappa$. Now suppose that $u \in V_\kappa$.

Case 1. $u \in Y$. Then there is an $z \in X$ such that $u \in z$. Clearly $z \in V_\kappa$.

Case 2. There is a $z \in V_\kappa$ such that $z \in X$ and $u \in z$. Clearly then $u \in Y$.

(5) Power set. We want to show that

$$\forall X \in V_\kappa \exists Y \in V_\kappa \forall u \in V_\kappa[u \in Y \leftrightarrow \forall v \in V_\kappa[v \in u \rightarrow v \in X]].$$

Suppose that $X \in V_\kappa$. Let $Y = \mathscr{P}(X)$. Clearly $Y \in V_\kappa$. Now take any $u \in V_\kappa$.

Case 1. $u \in Y$. Thus $u \subseteq X$. Suppose that $v \in V_\kappa$ and $v \in u$. Then $v \in X$.

Case 2. $\forall v \in V_\kappa[v \in u \rightarrow v \in X]$, Clearly then $u \subseteq X$, so $u \in Y$.

(6) Infinity. This axiom is

$$\exists S[\exists x \in S \forall y[y \notin x] \wedge \forall x \in S \exists y \in S \forall z[z \in y \leftrightarrow z \in x \vee z = x].$$

So we want to prove that

$$\begin{aligned}&\exists S \in V_\kappa[\exists x \in V_\kappa[x \in S \wedge \forall y \in V_\kappa[y \notin x] \wedge \forall x \in V_\kappa[x \in S \rightarrow \\ &\exists y \in V_\kappa[y \in S \wedge \forall z \in V_\kappa[z \in y \leftrightarrow z \in x \vee z = x]]]]\end{aligned}$$

Let $S = \omega$; clearly $S \in V_\kappa$. Now $\emptyset \in \omega$ and for all y, $y \notin \emptyset$. Now suppose that $x \in V_\kappa \cap S$. Then also $x \cup \{x\} \in S$. So the above is clear.

(7) Replacement. An instance of this axiom is

$$\forall \overline{p}[\forall x, y, z[\varphi(x, y, \overline{p}) \wedge \varphi(x, z, \overline{p}) \rightarrow y = z] \rightarrow \forall X \exists Y \forall y(y \in Y \leftrightarrow \exists x \in X \varphi(x, y, \overline{p}))]$$

Thus we want to prove that

$$\begin{aligned}&\forall \overline{p} \in V_\kappa[\forall x, y, z, \overline{p} \in V_\kappa[\varphi^{V_\kappa}(x, y, \overline{p}) \wedge \varphi^{V_\kappa}(x, z, \overline{p}) \rightarrow y = z] \\ &\rightarrow \forall X \in V_\kappa \exists Y \in V_\kappa \forall y \in V_\kappa[y \in Y \leftrightarrow \exists x \in V_\kappa[x \in X \wedge \varphi^{V_\kappa}(x, y, \overline{p}))]]\end{aligned}$$

Thus assume that $\overline{p} \in V_\kappa$ and $\forall x, y, z, \overline{p} \in V_\kappa[\varphi^{V_\kappa}(x, y, \overline{p}) \wedge \varphi^{V_\kappa}(x, z, \overline{p}) \rightarrow y = z]$. For all $x \in V_\kappa$ let $f(x, \overline{p})$ be such that $\varphi^{V_\kappa}(x, f(x, \overline{p}), \overline{p})$. Now take any $X \in V_\kappa$. Let $Y = \{f(x, \overline{p}) : x \in X$. Clearly $Y \in V_\kappa$ is as desired.

(8) Regularity. This axiom is

$$\forall x[\exists y \in x \rightarrow \exists y \in x \forall z \in x[z \notin y].$$

So we want to show that

$$\forall x \in V_\kappa[\exists y[y \in V_\kappa \wedge y \in x] \rightarrow \exists y \in V_\kappa[y \in x \wedge \forall z \in V_\kappa[z \in x \rightarrow z \notin y]]].$$

So assume that $x \in V_\kappa$ and $\exists y[y \in V_\kappa \wedge y \in x$. Let y be a member of x of smallest rank.

(9) Choice. This axiom is

$$\begin{aligned}&\forall \mathscr{A}[\forall x \in \mathscr{A} \exists a[a \in x] \wedge \forall x, y \in \mathscr{A}[x \neq y \rightarrow \neg \exists a[a \in x \wedge a \in y]] \rightarrow \\ &\exists A \forall x \in \mathscr{A}[\exists a[a \in x \wedge a \in A \wedge \forall b[b \in x \wedge b \in A \rightarrow a = b]]]].\end{aligned}$$

Thus we want to show that

$$\begin{aligned}&\forall \mathscr{A} \in V_\kappa[\forall x \in V_\kappa[x \in \mathscr{A} \rightarrow \exists a[a \in V_\kappa \wedge a \in x]] \\ &\wedge \forall x, y \in V_\kappa[x \neq y \rightarrow [x, y \in \mathscr{A} \rightarrow \neg \exists a \in V_\kappa[a \in x \wedge a \in y]] \rightarrow \\ &\exists A \in V_\kappa \forall x \in V_\kappa[x \in \mathscr{A} \rightarrow [\exists a \in V_\kappa \wedge a \in x \wedge a \in A \wedge \forall b \in V_\kappa[b \in x \wedge b \in A \rightarrow a = b]]]].\end{aligned}$$

So, assume that $\mathscr{A} \in V_\kappa$ and

$$\begin{aligned}&\forall x \in V_\kappa[x \in \mathscr{A} \to \exists a[a \in V_\kappa \wedge a \in x]]\\ &\wedge \forall x, y \in V_\kappa[x \neq y \to [x, y \in \mathscr{A} \to \neg\exists a \in V_\kappa[a \in x \wedge a \in y]].\end{aligned}$$

This simplifies to

$$\begin{aligned}&\forall x[x \in \mathscr{A} \to \exists a[a \in x]]\\ &\wedge \forall x, y \in \mathscr{A}[x \neq y \to \neg\exists a[a \in x \wedge a \in y]].\end{aligned}$$

By the axiom of choice let A have exactly one element in common with each member of $\mathscr{A}$. Let $A' = A \cap \bigcup \mathscr{A}$. So A' is as desired. $\square$

Theorem 12.22. *If* ZFC *is consistent, then so is* ZFC+ *"there do not exist uncountable inaccessible cardinals".*

Proof. Let

$$\mathbf{M} = \{x : \forall\alpha[\alpha \text{ inaccessible } \to x \in V_\alpha]\}$$

Thus $\mathbf{M}$ is a class, and $\mathbf{M} \subseteq V_\alpha$ for every inaccessible α (if there are such). We claim that $\mathbf{M}$ is a model of ZFC+"there do not exist uncountable inaccessible cardinals". To prove this, we consider two possibilities.

Case 1. $\mathbf{M} = \mathbf{V}$. Then of course $\mathbf{M}$ is a model of ZFC. Suppose that α is inaccessible. Then since $\mathbf{M} = \mathbf{V}$ we have $\mathbf{V} \subseteq V_\alpha$, which is not possible, since V_α is a set. Thus $\mathbf{M}$ is a model of ZFC + "there do not exist uncountable inaccessible cardinals".

Case 2. $\mathbf{M} \neq \mathbf{V}$. Let x be a set which is not in $\mathbf{M}$. Then there is an ordinal α such that α is inaccessible and $x \notin V_\alpha$. In particular, there is an inaccessible α, and we let κ be the least such.

(1) $\mathbf{M} = V_\kappa$.

In fact, if $x \in \mathbf{M}$, then $x \in V_\alpha$ for every inaccessible α, so in particular $x \in V_\kappa$. On the other hand, if $x \in V_\kappa$, then $x \in V_\alpha$ for every $\alpha \geq \kappa$, so $x \in V_\alpha$ for every inaccessible α, and so $x \in \mathbf{M}$. So (1) holds.

Now we show that V_κ is as desired. By Theorem 12.21, V_κ is a model of ZFC. Suppose that $x \in V_\kappa$ and $(x \text{ is an inaccessible cardinal})^{V_\kappa}$; we want to get a contradiction. In particular, $(x \text{ is an ordinal})^{V_\kappa}$, so by absoluteness, x is an ordinal. Absoluteness clearly implies that x is infinite. We claim that x is a cardinal. For, if $f : y \to x$ is a bijection with $y < x$, then clearly $f \in V_\kappa$, and hence by absoluteness $(f : y \to x \text{ is a bijection and } y < x)^{V_\alpha}$, contradiction. Similarly, x is regular; otherwise there is an injection $f : y \to x$ with $\text{rng}(f)$ unbounded in x, so clearly $f \in V_\kappa$, and absolutenss again yields a contradiction. Thus x is a regular cardinal. Hence, since κ is the smallest inaccessible, there is a $y \in x$ such that there is a one-one function g from x into $\mathscr{P}(y)$. Again, $g \in V_\kappa$, and easy absoluteness results contradicts $(x \text{ is an inaccessible cardinal})^{V_\kappa}$. $\square$

Theorem 12.23. *Let Γ be a set of sentences, φ a sentence, and $\mathbf{M}$ a class. Let $\Gamma^{\mathbf{M}} = \{\chi^{\mathbf{M}} : \chi \in \Gamma\}$. Suppose that $\Gamma \models \varphi$. Then*

$$\Gamma^{\mathbf{M}} \models \mathbf{M} \neq \emptyset \to \varphi^{\mathbf{M}}.$$

Proof. Assume the hypothesis of the theorem, let $\overline{A} = (A, E)$ be any set theory structure, assume that $\overline{A}$ is a model of $\Gamma^{\mathbf{M}}$, and suppose that $A \cap \mathbf{M} \neq \emptyset$. We want to show that $\overline{A}$ is a model of $\varphi^{\mathbf{M}}$. To do this, we define another structure $\overline{B} = (B, F)$ for our language. Let $B = A \cap \mathbf{M}$, and let $F = E \cap (B \times B)$. Now we claim:

(*) For any formula χ and any $c \in {}^{\omega}B$, $\overline{A} \models \chi^{\mathbf{M}}[c]$ iff $\overline{B} \models \chi[c]$.

We prove (*) by induction on χ:

$$
\begin{array}{rll}
\overline{A} \models (v_i = v_j)^{\mathbf{M}}[c] & \text{iff} & c_i = c_j \\
& \text{iff} & \overline{B} \models (v_i = v_j)[c]; \\
\overline{A} \models (v_i \in v_j)^{\mathbf{M}}[c] & \text{iff} & c_i E c_j \\
& \text{iff} & c_i F c_j \\
& \text{iff} & \overline{B} \models (v_i \in v_j)[c]; \\
\overline{A} \models (\neg\chi)^{\mathbf{M}}[c] & \text{iff} & \text{not}[\overline{A} \models \chi^{\mathbf{M}}[c]] \\
& \text{iff} & \text{not}[\overline{B} \models \chi[c]] \quad \text{(induction hypothesis)} \\
& \text{iff} & \overline{B} \models \neg\chi[c]; \\
\overline{A} \models (\chi \to \theta)^{\mathbf{M}}[c] & \text{iff} & [\overline{A} \models \chi^{\mathbf{M}}[c] \text{ implies that } \overline{A} \models \theta^{\mathbf{M}}[c]] \\
& \text{iff} & [\overline{B} \models \chi[c] \text{ implies that } \overline{B} \models \theta[c] \\
& & \qquad\qquad \text{(induction hypothesis)} \\
& \text{iff} & \overline{B} \models (\chi \to \theta)[c].
\end{array}
$$

We do the quantifier step in each direction separately. First suppose that $\overline{A} \models (\forall v_i \chi)^{\mathbf{M}}[c]$. Thus $\overline{A} \models [\forall v_i [v_i \in \mathbf{M} \to \chi^{\mathbf{M}}][c]$. Take any $b \in B$. Then $b \in \mathbf{M}$, so $\overline{A} \models \chi^{\mathbf{M}}[c^i_b]$. By the inductive hypothesis, $\overline{B} \models \chi[c^i_b]$. This proves that $\overline{B} \models \forall v_i \chi[c]$.

Conversely, suppose that $\overline{B} \models \forall v_i \chi[c]$. Suppose that $a \in A$ and $\overline{A} \models (v_i \in \mathbf{M})[c^i_a]$. Then $a \in B$, so $\overline{B} \models \chi[c^i_a]$. By the inductive hypothesis, $\overline{A} \models \chi^{\mathbf{M}}[c^i_a]$. So we have shown that $\overline{A} \models \forall v_i [v_i \in \mathbf{M} \to \chi^{M}][c]$. That is, $\overline{A} \models (\forall v_i \chi)^{\mathbf{M}}[c]$.

This finishes the proof of (*).

Now $\overline{A}$ is a model of $\Gamma^{\mathbf{M}}$, so by (*), $\overline{B}$ is a model of Γ. Hence by assumption, $\overline{B}$ is a model of φ. So by (*) again, $\overline{A}$ is a model of $\varphi^{\mathbf{M}}$. □

The following theorem gives the basic idea of consistency proofs in set theory; we express this as follows. Remember by the completeness theorem that a set Γ of sentences is consistent iff it has a model.

Corollary 12.24. *Suppose that Γ and Δ are collections of sentences in our language of set theory. Suppose that $\mathbf{M}$ is a class, and $\Gamma \models [\mathbf{M} \neq \emptyset$ and $\varphi^{\mathbf{M}}]$ for each $\varphi \in \Delta$. Then Γ consistent implies that Δ is consistent.*

Proof. Suppose to the contrary that Δ does not have a model. Then trivially $\Delta \models \neg(x = x)$. By Proposition 12.23, $\Delta^{\mathbf{M}} \models \mathbf{M} \neq \emptyset \to \neg(x = x)$. Hence by hypothesis we get $\Gamma \models \neg(x = x)$, contradiction. □

We now want to consider to what extent sentences can reflect to proper subclasses of $\mathbf{V}$; this is a natural extension of our considerations for absoluteness.

Actually we are dealing here with a set-theoretic version of the model theoretic notion of elementary substructure. The model theoretic notion will be important later on, so we describe the basic definition and give an important lemma about the notion.

If $\overline{A} = (A, R)$ and $\overline{B} = (B, S)$ are set theory structures, then we say that $\overline{A}$ is an *elementary substructure* of $\overline{B}$, in symbols $\overline{A} \preceq \overline{B}$, iff $A \subseteq B$, $R = S \cap (A \times A)$, and for every formula $\varphi(x_0, \ldots, x_{n-1})$ and all $a_0, \ldots, a_{n-1} \in A$, $\overline{A} \models \varphi[a_0, \ldots, a_n]$ iff $\overline{B} \models \varphi[a_0, \ldots, a_n]$. (See Chapter 2.)

Lemma 12.25. (Tarski) *Let $\overline{A} = (A, R)$ and $\overline{B} = (B, S)$ be set theory structures, and suppose that $A \subseteq B$ and $R = S \cap (A \times A)$. Then the following conditions are equivalent:*

(i) $\overline{A} \preceq \overline{B}$.

(ii) For every formula $\forall x\varphi(x, y_0, \ldots, y_{n-1})$ and all $a_0, \ldots, a_{n-1} \in A$, if $\forall b \in A[\overline{B} \models \varphi(b, a_0, \ldots, a_{n-1})]$ then $\forall b \in B[\overline{B} \models \varphi(b, a_0, \ldots, a_{n-1})]$.

Proof. (i)⇒(ii): Assume (i), and suppose that $\forall x\varphi(x, y_0, \ldots, y_{n-1})$ is a formula, $a_0, \ldots, a_{n-1} \in A$, and $\forall b \in A[\overline{B} \models \varphi(b, a_0, \ldots, a_{n-1})]$. Since $\overline{A} \preceq \overline{B}$, it follows that $\forall b \in A[\overline{A} \models \varphi(b, a_0, \ldots, a_{n-1})]$. Thus $\overline{A} \models \forall x\varphi(x, a_0, \ldots, a_{n-1})$. Then again by $\overline{A} \preceq \overline{B}$, $\overline{B} \models \forall x\varphi(x, a_0, \ldots, a_{n-1})$. So $\forall b \in B[\overline{B} \models \varphi(b, a_0, \ldots, a_{n-1})]$.

(ii)⇒(i): Assume (ii). We prove for $a_0, \ldots, a_{n-1} \in A$

$$\overline{A} \models \varphi[a_0, \ldots, a_n] \quad \text{iff } \overline{B} \models \varphi[a_0, \ldots, a_n]$$

by induction on φ. The atomic cases are clear, as are the induction steps involving $\neg$ and $\rightarrow$. Now suppose that $\overline{A} \models \forall x\varphi(x, a_0, \ldots, a_n)$. Thus $\forall b \in A[\overline{A} \models \varphi(b, a_0, \ldots, a_{n-1})]$. Hence by the inductive hypothesis, $\forall b \in A[\overline{B} \models \varphi(b, a_0, \ldots, a_{n-1})]$. Hence by (ii), $\forall b \in B[\overline{B} \models \varphi(b, a_0, \ldots, a_{n-1})$, i.e., $\overline{B} \models \forall x\varphi(x, a_0, \ldots, a_n)$.

Conversely, suppose that $\overline{B} \models \forall x\varphi(x, a_0, \ldots, a_n)$. Thus $\forall b \in B[\overline{B} \models \varphi(b, a_0, \ldots, a_n)]$. Hence $\forall b \in A[\overline{B} \models \varphi(b, a_0, \ldots, a_n)]$; then $\forall b \in A[\overline{A} \models \varphi(b, a_0, \ldots, a_n)]$ by the inductive hypothesis, that is, $\overline{A} \models \forall x\varphi(x, a_0, \ldots, a_n)$. □

Lemma 12.26. *Suppose that $\mathbf{M}$ and $\mathbf{N}$ are classes with $\mathbf{M} \subseteq \mathbf{N}$. Let $\varphi_0, \ldots, \varphi_n$ be a list of formulas such that if $i \leq n$ and ψ is a subformula of φ_i, then there is a $j \leq n$ such that φ_j is ψ. Then the following conditions are equivalent:*

(i) Each φ_i is absolute for $\mathbf{M}, \mathbf{N}$.

(ii) If $i \leq n$ and φ_i has the form $\forall x\varphi_j(x, y_1, \ldots, y_t)$ with $x, y_1, \ldots, y_t$ exactly all the free variables of φ_j, then

$$\forall y_1, \ldots, y_t \in \mathbf{M}[\forall x \in \mathbf{M}\varphi_j^{\mathbf{N}}(x, y_1, \ldots, y_t) \rightarrow \forall x \in \mathbf{N}\varphi_j^{\mathbf{N}}(x, y_1, \ldots, y_t)].$$

Proof. (i)⇒(ii): Assume (i) and the hypothesis of (ii). Suppose that $y_1, \ldots, y_t \in \mathbf{M}$ and $\forall x \in \mathbf{M}\varphi_j^{\mathbf{N}}(x, y_1, \ldots, y_t)$. Thus by absoluteness $\forall x \in \mathbf{M}\varphi_j^{\mathbf{M}}(x, y_1, \ldots, y_t)$. Hence by absoluteness again, $\forall x \in \mathbf{N}\varphi_j^{\mathbf{N}}(x, y_1, \ldots, y_t))$.

(ii)⇒(i): Assume (ii). We prove that φ_i is absolute for $\mathbf{M}, \mathbf{N}$ by induction on the length of φ_i. This is clear if φ_i is atomic, and it easily follows inductively if φ_i has the form $\neg\varphi_j$ or $\varphi_j \to \varphi_k$. Now suppose that φ_i is $\forall x\varphi_j(x, y_1, \ldots, y_t)$, and $y_1, \ldots, y_t \in \mathbf{M}$. then

$$\begin{aligned}
\varphi_i^{\mathbf{M}}(y_1, \ldots, y_t) &\leftrightarrow \forall x \in \mathbf{M}\varphi_j^{\mathbf{M}}(x, y_1, \ldots, y_t) \quad \text{(definition of relativization)}\\
&\leftrightarrow \forall x \in \mathbf{M}\varphi_j^{\mathbf{N}}(x, y_1, \ldots, y_t) \quad \text{(induction hypothesis)}\\
&\leftrightarrow \forall x \in \mathbf{N}\varphi_j^{\mathbf{N}}(x, y_1, \ldots, y_t) \quad \text{(by (ii)}\\
&\leftrightarrow \varphi_i^{\mathbf{N}}(y_1, \ldots, y_t) \quad \text{(definition of relativization)}
\end{aligned}$$ □

Theorem 12.27. *Suppose that $Z(\alpha)$ is a set for every ordinal α, and the following conditions hold:*

(i) If $\alpha < \beta$, then $Z(\alpha) \subseteq Z(\beta)$.

(ii) If γ is a limit ordinal, then $Z(\gamma) = \bigcup_{\alpha<\gamma} Z(\alpha)$.

Let $\mathbf{Z} = \bigcup_{\alpha\in\mathbf{On}} Z(\alpha)$. Then for any formulas $\varphi_0, \ldots, \varphi_{n-1}$,

$$\forall\alpha\exists\beta > \alpha[\varphi_0, \ldots, \varphi_{n-1} \text{ are absolute for } Z(\beta), \mathbf{Z}].$$

Proof. Assume the hypothesis, and let an ordinal α be given. We are going to apply Lemma 15.4 with $\mathbf{N} = \mathbf{Z}$, and we need to find an appropriate $\beta > \alpha$ so that we can take $\mathbf{M} = Z(\beta)$ in 15.4.

We may assume that $\varphi_0, \ldots, \varphi_{n-1}$ is subformula-closed; i.e., if $i < n$, then every subformula of φ_i is in the list. Let A be the set of all $i < n$ such that φ_i begins with a universal quantifier. Suppose that $i \in A$ and φ_i is the formula $\forall x\varphi_j(x, y_1, \ldots, y_t)$, where $x, y_1, \ldots, y_t$ are exactly all the free variables of φ_j. We now define a class function $\mathbf{G}_i$ as follows. For any sets $y_1, \ldots, y_t$,

$$\mathbf{G}_i(y_1, \ldots, y_t) = \begin{cases} \text{the least } \eta \text{ such that } \exists x \in Z(\eta)\neg\varphi_j^{\mathbf{Z}}(x, y_1, \ldots, y_t) & \text{if there is such,}\\ 0 & \text{otherwise.}\end{cases}$$

Then for each ordinal ξ we define

$$\mathbf{F}_i(\xi) = \sup\{\mathbf{G}_i(y_1, \ldots, y_t) : y_1, \ldots, y_t \in Z(\xi)\};$$

note that this supremum exists by the replacement axiom.

Now we define a sequence $\gamma_0, \ldots, \gamma_p, \ldots$ of ordinals by induction on $n \in \omega$. Let $\gamma_0 = \alpha + 1$. Having defined γ_p, let

$$\gamma_{p+1} = \max(\gamma_{p+1}, \sup\{\mathbf{F}_i(\xi) : i \in A, \xi \leq \gamma_p\} + 1).$$

Finally, let $\beta = \sup_{p\in\omega} \gamma_p$. Clearly $\alpha < \beta$ and β is a limit ordinal.

(1) If $i \in A$, $y_1, \ldots, y_t \in Z(\beta)$, and $\exists x \in \mathbf{Z}\neg\varphi_i^{\mathbf{Z}}(x, y_1, \ldots, y_t)$, then there is an $x \in Z(\beta)$ such that $\neg\varphi_i^{\mathbf{Z}}(x, y_1, \ldots, y_t)$.

In fact, choose p such that $y_1, \ldots, y_t \in Z(\gamma_p)$. Then $\mathbf{G}_i(y_1, \ldots, y_t) \leq \mathbf{F}_i(\gamma_p) < \gamma_{p+1}$. Hence an x as in (1) exists, with $x \in Z(\gamma_{p+1})$.

(1) clearly gives the desired conclusion. $\square$

Corollary 12.28. (The reflection theorem) *For any formulas $\varphi_1, \ldots, \varphi_n$,*

$$\text{ZF} \models \forall\alpha\exists\beta > \alpha[\varphi_1, \ldots, \varphi_n \text{ are absolute for } V_\beta]. \qquad \square$$

Theorem 12.29. *Suppose that $\mathbf{Z}$ is a class and $\varphi_1, \ldots, \varphi_n$ are formulas. Then*

$$\forall X \subseteq \mathbf{Z}\exists A[X \subseteq A \subseteq \mathbf{Z} \text{ and } \varphi_1, \ldots, \varphi_n \text{ are absolute}$$
$$\text{for } A, \mathbf{Z} \text{ and } |A| \leq \max(\omega, |X|)].$$

Proof. We may assume that $\varphi_1, \ldots, \varphi_n$ is subformula closed. For each ordinal α let $Z(\alpha) = \mathbf{Z} \cap V_\alpha$. Clearly there is an ordinal α such that $X \subseteq V_\alpha$, and hence $X \subseteq Z(\alpha)$. Now we apply Theorem 12.27 to obtain an ordinal $\beta > \alpha$ such that

$$\text{(1)} \qquad \varphi_1, \ldots, \varphi_n \text{ are absolute for } Z(\beta), \mathbf{Z}.$$

Let $\prec$ be a well-order of $Z(\beta)$. Let B be the set of all $i < n$ such that φ_i begins with a universal quantifier. Suppose that $i \in B$ and φ_i is the formula $\forall x\varphi_j(x, y_1, \ldots, y_t)$, where $x, y_1, \ldots, y_t$ are exactly all the free variables of φ_j. We now define a function H_i for each $i \in B$ as follows. For any sets $y_1, \ldots, y_t \in Z(\beta)$,

$$H_i(y_1, \ldots, y_t) = \begin{cases} \text{the } \prec\text{-least } x \in Z(\beta) \text{ such that } \neg\varphi_i^{Z(\beta)}(x, y_1, \ldots, y_t) & \text{if there is such,} \\ \text{the } \prec\text{-least element of } Z(\beta) & \text{otherwise.} \end{cases}$$

Let $A \subseteq Z(\beta)$ be closed under each function H_i, with $X \subseteq A$. We claim that A is as desired. To prove the absoluteness, it suffices to take any formula φ_i with $i \in A$, with notation as above, assume that $y_1, \ldots, y_t \in A$ and $\exists x \in \mathbf{Z}\neg\varphi_j^{\mathbf{Z}}(x, y_1, \ldots, y_t)$, and find $x \in A$ such that $\neg\varphi_j^{\mathbf{Z}}(x, y_1, \ldots, y_t)$. Now there is an $x \in Z(\beta)$ such that $\neg\varphi_j^{\mathbf{Z}}(x, y_1, \ldots, y_t)$. Hence $H_i(y_1, \ldots, y_t)$ is an element of A such that $\neg\varphi_j^{\mathbf{Z}}(H_i(y_1, \ldots, y_t), y_1, \ldots, y_t)$, as desired.

It remains only to check the cardinality estimate. This is elementary. $\square$

Lemma 12.30. *Suppose that $\mathbf{F}$ is a bijection from A onto $\mathbf{M}$, and for any $a, b \in A$ we have $a \in b$ iff $\mathbf{F}(a) \in \mathbf{F}(b)$. Then for any formula $\varphi(x_1, \ldots, x_n)$ and any $x_1, \ldots, x_n \in A$,*

$$\varphi^A(x_1, \ldots, x_n) \leftrightarrow \varphi^{\mathbf{M}}(\mathbf{F}(x_1), \ldots, \mathbf{F}(x_n)).$$

Proof. An easy induction on φ. $\square$

Let $\mathbf{A}$ be a class and $\mathbf{R}$ a class relation with $\mathbf{R} \subseteq \mathbf{A} \times \mathbf{A}$. For any $x \in \mathbf{A}$ we define $\text{pred}_{\mathbf{AR}}(x) = \{y \in \mathbf{A} : (y, x) \in \mathbf{R}\}$. We say that $\mathbf{R}$ is *set-like* on $\mathbf{A}$ iff $\mathbf{R} \subseteq \mathbf{A} \times \mathbf{A}$ and $\text{pred}_{\mathbf{AR}}(x)$ is a set for all $x \in \mathbf{A}$.

Suppose that R is well-founded and set-like on A. For each $y \in A$ we define

$$\text{mos}_{AR}(y) = \{\text{mos}_{AR}(x) : x \in A \text{ and } xRy\}.$$

Lemma 12.31. *If R is well-founded and set-like on A, then* $\text{mos}_{AR}[A]$ *is transitive.*

Proof. Assume that R is well-founded and set-like on A, and $u \in v \in \text{mos}_{AR}[A]$. Say $v = \text{mos}_{AR}(y)$ with $y \in A$. Since $u \in v$, choose $x \in A$ with xRy and $u = \text{mos}_{AR}(x)$. Thus $u \in \text{mos}_{AR}[A]$. $\square$

A relation R is *extensional on A* iff $\forall x, y \in A[\{z \in A : zRx\} = \{z \in A : zRy\} \to x = y]$.

Lemma 12.32. (I.9.34) *If A is transitive, then $\in$ is extensional on A.* $\square$

Lemma 12.33. (I.9.35) *Suppose that R is well-founded and set-like on A. Then* mos_{AR} *is one-one iff R is extensional on A.*

Proof. If R is not extensional on A, then there exist $x, y \in A$ such that $\{z \in A : zRx\} = \{z \in A : zRy\}$ but $x \neq y$. Hence $\text{mos}_{AR}(x) = \text{mos}_{AR}(y)$, so mos is not one-one.

Now suppose that R is extensional on A; we show that mos_{AR} is one-one. Suppose that mos_{AR} is not one-one. So there exist distinct $a, b \in A$ such that $\text{mos}_{AR}(a) = \text{mos}_{AR}(b)$. Let $X = \{c \in A : \exists d \in A[c \neq d \text{ and } \text{mos}_{AR}(c) = \text{mos}_{AR}(d)]\}$. Thus $a \in X$, so $X \neq \emptyset$. Let c be an R-minimal element of X. By definition of X, let $d \in A$ be such that $c \neq d$ and $\text{mos}_{AR}(c) = \text{mos}_{AR}(d)$. Since R is extensional on A and $c \neq d$, there are two cases.

Case 1. There is a $z \in A$ such that zRc but $\text{not}(zRd)$. Then $\text{mos}_{AR}(z) \in \text{mos}_{AR}(c) = \text{mos}_{AR}(d) = \{\text{mos}_{AR}(x) : x \in A \text{ and } xRd\}$. Say $\text{mos}_{AR}(z) = \text{mos}_{AR}(x)$ with xRd and $x \in A$. Since c is an R-minimal element of X and zRc, it follows that $z \notin X$. Hence $\forall y \in A[\text{mos}_{AR}(z) = \text{mos}_{AR}(y) \to z = y]$. Since $\text{mos}_{AR}(z) = \text{mos}_{AR}(x)$, we thus have $z = x$. But $\text{not}(zRd)$ while xRd, contradiction.

Case 2. There is a $z \in A$ such that $\text{not}(zRc)$ but zRd. Then $\text{mos}_{AR}(z) \in \text{mos}_{AR}(d) = \text{mos}_{AR}(c) = \{\text{mos}_{AR}(x) : x \in A \text{ and } xRc\}$. Say $\text{mos}_{AR}(z) = \text{mos}_{AR}(x)$ with xRc and $x \in A$. Since c is an R-minimal element of X and xRc, it follows that $x \notin X$. Hence $\forall y \in A[\text{mos}_{AR}(x) = \text{mos}_{AR}(y) \to x = y]$. Since $\text{mos}_{AR}(z) = \text{mos}_{AR}(x)$, we thus have $z = x$. But $\text{not}(zRc)$ while xRc, contradiction. $\square$

Theorem 12.34. *Suppose that $\mathbf{Z}$ is a transitive class and $\varphi_0, \ldots, \varphi_{m-1}$ are sentences. Suppose that X is a transitive subset of $\mathbf{Z}$. Then there is a transitive set M such that $X \subseteq M$, $|M| \leq \max(\omega, |X|)$, and for every $i < m$, $\varphi_i^M \leftrightarrow \varphi_i^{\mathbf{Z}}$.*

Proof. We may assume that the extensionality axiom is one of the φ_i's. Now we apply Theorem 12.29 to get a set A as indicated there. By Proposition 12.31, there is a transitive set M and a bijection mos from A onto M such that for any $a, b \in A$, $a \in b$ iff $\text{mos}_{AR}(a) \in \text{mos}_{AR}(b)$. Hence all of the desired conditions are clear, except possibly $X \subseteq M$. Now $\text{mos}_{AR}(x) = x$ for all $x \in X$. Hence $X \subseteq M$. $\square$

Corollary 12.35. *Suppose that S is a set of sentences containing ZFC. Suppose also that $\varphi_0, \ldots, \varphi_{n-1} \in S$. Then*

$$S \models \exists M \left(M \text{ is transitive, } |M| = \omega, \text{ and } \bigwedge_{i<n} \varphi_i^M \right).$$

Proof. Take $\mathbf{Z} = \mathbf{V}$ and $X = \omega$ in Theorem 12.34. $\square$

The following corollary can be taken as a basis for working with countable transitive models of ZFC.

Theorem 12.36. *Suppose that S is a consistent set of sentences containing ZFC. Expand the basic set-theoretic language by adding an individual constant $\mathbf{M}$. Then the following set of sentences is consistent:*

$$S \cup \{\mathbf{M} \text{ is transitive}\} \cup \{|\mathbf{M}| = \omega\} \cup \{\varphi^{\mathbf{M}} : \varphi \in S\}.$$

Proof. Suppose that the indicated set is not consistent. Then there are $\varphi_0, \ldots, \varphi_{m-1}$ in S such that

$$S \models \mathbf{M} \text{ is transitive and } |\mathbf{M}| = \omega \to \neg \bigwedge_{i<n} \varphi_i^{\mathbf{M}};$$

it follows that

$$S \models \neg\exists \mathbf{M} \left(\mathbf{M} \text{ is transitive, } |\mathbf{M}| = \omega, \text{ and } \bigwedge_{i<n} \varphi_i^{\mathbf{M}} \right),$$

contradicting Corollary 12.35 $\square$

Theorem 12.37. *Suppose that κ is an uncountable regular cardinal and $\langle A(\xi) : \xi \leq \kappa \rangle$ satisfies the following conditions:*

(i) $\forall \xi < \eta \leq \kappa [A(\xi) \subseteq A(\eta)]$.

(ii) $\forall$ *limit* $\eta \leq \kappa \left[A(\eta) = \bigcup_{\xi<\eta} A(\xi) \right]$.

(iii) $\forall \xi < \kappa [|A(\xi)| < \kappa$.

(iv) $|A(\kappa)| = \kappa$.

Then $\forall \xi < \kappa \exists \eta < \kappa [\xi < \eta$ *and* $A(\eta) \preceq A(\kappa)$ *and* η *is a limit ordinal].*

Proof. Let A be the set of all formulas in the $\in$-language of set theory which begin with a universal quantifier. For each $\varphi \in A$ we define a function G_φ as follows. Say φ is $\forall x \psi_\varphi(x, y_1, \ldots, y_n)$. Then G_φ is a function with domain ${}^n(A(\kappa))$ such that for any $a_1, \ldots, a_n \in A(\kappa)$,

$$G_\varphi(a_1, \ldots, a_n) = \begin{cases} \text{least } \alpha < \kappa : \exists x \in A(\alpha) \neg\psi_\varphi(x, a_1, \ldots, a_n) & \text{if there is such an } \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Then for each ordinal $\alpha < \kappa$ we define

$$F_\varphi(\alpha) = \sup\{G_\varphi(a_1, \ldots, a_n) : a_1, \ldots, a_n \in A(\alpha)\}.$$

Note that $F_\varphi(\alpha) < \kappa$ since $|A(\alpha)| = |\alpha| < \kappa$.

Now we define a sequence γ_i of ordinals less than κ by recursion on $i < \omega$. Now by (iv) we can let γ_0 be greater than ξ such that $A(\gamma_0) \neq \emptyset$. Having defined γ_i, let

$$\gamma_{i+1} = \max(\gamma_i + 1, \sup\{F_\varphi(\eta) : \varphi \text{ a formula}, \eta \leq \gamma_i\}).$$

Let $\eta = \bigcup_{i\in\omega} \gamma_i$. Note that $\eta < \kappa$. Clearly $\xi < \eta$ and $A(\eta) \neq \emptyset$. Now we claim

(*) If $\varphi \in A$, say $\varphi = \forall x \psi_\varphi(x, y_1, \ldots, y_n)$, then

$$\forall a_1, \ldots, a_n \in A(\eta)[\forall x \in A(\eta)\psi_\varphi^{A(\kappa)}(x, a_1, \ldots, a_n) \to \forall x \in A(\kappa)\psi_\varphi^{A(\kappa)}(x, a_1, \ldots, a_n)].$$

In fact, suppose that $\varphi \in A$, $\varphi = \forall x\psi_\varphi(x, y_1, \ldots, y_n)$, $a_1, \ldots, a_n \in A(\eta)$, and $\exists x \in A(\kappa)\neg\psi_\varphi^{A(\kappa)}(x, a_1, \ldots, a_n)$. Say $a_1, \ldots, a_n \in A(\gamma_i)$. Then $G_\varphi(a_1, \ldots, a_n) < F_\varphi(\gamma_i) < \gamma_{i+1} < \eta$, so $\exists x \in A(\eta)\neg\psi_\varphi(x, a_1, \ldots, a_n)$. This proves (*).

Now we prove by induction on φ that for any $a_1, \ldots, a_n \in A(\eta)$, $\varphi^{A(\eta)}(a_1, \ldots, a_n)$ iff $\varphi^{A(\kappa)}(a_1, \ldots, a_n)$. This is clear for atomic formulas, and the inductive steps for $\neg$ and $\to$ are clear. Now suppose inductively that φ is $\forall x\psi_\varphi(x, y_1, \ldots, y_n)$. First suppose that $\varphi^{A(\eta)}(a_1, \ldots, a_n)$. Thus $\forall x \in A(\eta)\psi_\varphi^{A(\eta)}(x, a_1, \ldots, a_n)$. Hence by the induction hypothesis, $\forall x \in A(\eta)\psi_\varphi^{A(\kappa)}(x, a_1, \ldots, a_n)$, so by (*), $\forall x \in A(\kappa)\psi_\varphi^{A(\kappa)}(x, a_1, \ldots, a_n)$, i.e., $\varphi^{A(\kappa)}(a_1, \ldots, a_n)$.

Second, suppose that $\varphi^{A(\kappa)}(a_1, \ldots, a_n)$, i.e., $\forall x \in A(\kappa)\psi_\varphi^{A(\kappa)}(x, a_1, \ldots, a_n)$. So $\forall x \in A(\eta)\psi_\varphi^{A(\kappa)}(x, a_1, \ldots, a_n)$, hence by the inductive hypothesis, $\forall x \in A(\eta)\psi_\varphi^{A(\eta)}(x, a_1, \ldots, a_n)$. This finishes the induction. $\square$

Proposition 12.38. *If $\{a\} \in U$, then $\prod_{i\in I} \overline{A}_i/U \cong \overline{A}_a$.*

Proof. Let $b_i \in A_i$ for all $i \in I$. For each $c \in A_a$ define

$$x_i^c = \begin{cases} b_i & \text{if } i \in I\backslash\{a\}, \\ c & \text{if } i = a. \end{cases}$$

Then for each $c \in A_a$ let $f(c) = [x^c]$, the equivalence class of x^c in $\prod_{i\in I} \overline{A_i}/U$. We claim that f is an isomorphism from $\overline{A}_a$ onto $\prod_{i\in I} \overline{A_i}/U$.

one-one: Suppose that $c, d \in A_a$ and $c \neq d$. Then

$$\begin{aligned} f(c) = f(d) \quad &\text{iff} \quad [x^c] = [x^d] \quad \text{iff} \quad \{i \in I : x^c(i) = x^d(i)\} \in U \\ &\text{iff} \quad x^c(a) = x^d(a) \quad \text{iff} \quad c = d; \end{aligned}$$

hence $f(c) \neq f(d)$.

onto: Suppose $[y]$ is given. Let $c = y(a)$. Then $f(c) = [x^c]$, and

$$f(c) = [y] \quad \text{iff} \quad \{i \in I : y(i) = x^c(i)\} \in U \quad \text{iff} \quad y(a) = x^c(a) \quad \text{iff} \quad y(a) = c;$$

so $f(c) = [y]$.

fundamental operation g: Say g is m-ary. Let $c_0, \ldots, c_{m-1} \in A_a$. Then

$$f(g(c_0, \ldots, c_{m-1})) = [x^{g(c_0,\ldots,c_{m-1}))}]$$

and

$$g(f(c_0), \ldots, f(c_{m-1})) = g([x^{c_0}], \ldots, [x^{c_{m-1}}]) = [\langle g(x^{c_0}(i), \ldots, x^c_{m-1}(i)) : i \in I\rangle]$$

Now for any $i \in I$,

$$\begin{aligned} &[[x^{g(c_0,\ldots,c_{m-1})}(i)] = [\langle g(x^{c_0}(i), \ldots, x^c_{m-1}(i)) : i \in I\rangle] \\ \text{iff}\quad & x^{(g(c_0,\ldots,c_{m-1}}(a) = \langle g(x^{c_0}(i), \ldots, x^c_{m-1}(a)) \\ \text{iff}\quad & g(c_0, \ldots, c_{m-1} = g(c_0, \ldots, c_{m-1}) \end{aligned}$$

fundamental relation R: say R is m-ary. Suppose that $c_0, \ldots, c_{m-1} \in A_a$.

$$\begin{aligned} \langle f(c_0), \ldots, f(c_{m-1})\rangle \in R \quad \text{iff}\quad & \langle [x^{c_0}], \ldots, [x^{cm-1}]\rangle \in R \\ \text{iff}\quad & \{i \in I : \langle x_i^{c_0}, \ldots, x^{c_{m-1}}\rangle \in R\} \in U \\ \text{iff}\quad & \langle x_a^{c_0}, \ldots x_a^{c_{m-1}}\rangle \in R \quad \text{iff} \quad \langle c_0, \ldots, c_{m-1}\rangle \in R. \end{aligned}$$ □

Proposition 12.39. *If U is a principal ideal, then j is an isomorphism from $\overline{A}$ onto ${}^I\overline{A}/U$.*

Proof. Say $\{a\} \in U$. It suffices to show that j maps onto ${}^I\overline{A}/U$. Let $[b] \in {}^I\overline{A}/U$. We claim that $j(b_a) = [b]$. Thus we want to show that $[c_{b_a}] = [b]$. We have $c_{b_a}(a) = b_a$, so $a \in \{i \in I; c_{b_a}(i) = b_i\} \in U$. Hence $[c_{b_a}] = [b]$. □

Theorem 12.40. *Suppose that κ is a measurable cardinal, and U is an ultrafilter. Let $(A, <^*) = {}^\kappa(\kappa, <)/U$, and let $j : \kappa \to A$ be the canonical embedding. (Recall that $j(\alpha) = [c_\alpha]$ for all $\alpha < \kappa$.) Then:*

(i) $(A, <^*)$ is a linear order.

(ii) If U is σ-complete, then $(A, <^*)$ is a well-ordering. Hence there is an order isomorphism h from $(A, <^*)$ onto some ordinal λ.

(iii) If U is κ-complete, then $\forall \alpha < \kappa[h(j(\alpha)) = \alpha]$

(iv) Let U be κ-complete and nonprincipal. Define $d(\alpha) = \alpha$ for all $\alpha < \kappa$. Then $h([d]) \geq \kappa$.

(v) Let U be κ-complete and nonprincipal. Then U is normal iff $h([d]) = \kappa$.

Proof. (i): $\{\xi < \kappa : \xi \not< \xi\} = \kappa \in U$ so $\forall x \in A[x \not<^* x]$ by Łoś's theorem. If $[x] <^* [y] <^* [z]$ then

$$\{\xi < \kappa : x_\xi < y_\xi\} \cap \{\xi < \kappa : y_\xi < z_\xi\} \subseteq \{\xi < \kappa : x_\xi < z_\xi\},$$

and it follows that $[x] <^* [z]$. Given distinct $[x], [y]$ in A, we have $\{\xi < \kappa : x_\xi < y_\xi\} \cup \{\xi < \kappa : y_\xi < x_\xi\} = \kappa \in U$. so $\{\xi < \kappa : x_\xi < y_\xi\} \in U$ or $\{\xi < \kappa : y_\xi < x_\xi\} \in U$, hence $[x] <^* [y]$ or $[y] <^* [x]$.

(ii): suppose to the contrary that $\cdots <^* [x_2] <^* [x_1] <^* [x_0]$. For each $n \in \omega$ let $A_n = \{\xi < \kappa : x_{n+1}(\xi) < x_n(\xi)\}$. Thus each A_n is in U. Let $B = \bigcap_{n\in\omega} A_n$. So $B \in U$. For any $\xi \in B$ we have $\cdots < x_2(\xi) < x_1(\xi) < x_0(\xi$, contradiction.

(iii): By induction on $\alpha < \kappa$. $h(j(0)) = h([c_0]) = 0$. Assume it is true for β with $\beta < \kappa$. Now $\beta + 1$ is the successor of β, so $\{\xi < \kappa : c_{\beta+1}(\xi)$ is the successor of $c_\beta(\xi)\} = \kappa \in U$, so $[c_{\beta+1}]$ is the succssor of c_β. Hence $j(\beta + 1) = \beta + 1$. Now suppose that $\beta < \kappa$ is limit and the result holds for all $\gamma < \beta$. Then for all $\alpha < \beta$

$$\{\xi < \kappa : c_\alpha(\xi) < c_\beta(\xi)\} = \kappa \in U,$$

so $[c_\alpha] < [c_\beta]$, hence $\alpha < h([c_\beta])$. Hence $\beta \le h([c_\beta])$. Suppose that $\beta < h([c_\beta])$. Say $\beta = h(c_\delta)$. Then $\delta < \beta$ so $h(c_\delta) = \delta$, contradiction.

(iv): If $\beta < \kappa$, clearly $[c_\beta] \le [d]$ Hence $\kappa \le h([d])$.

(v): $\Rightarrow$: Suppose that U is normal. Suppose that $a \in h([d])$. Say $a = h([f])$, Then $[f] < [d]$, so $\{\alpha < \kappa[f(\alpha) < \alpha]\} \in U$, hence since U is normal, by Exercise 8.8 there is a $\gamma < \kappa$ such that $\{\alpha < \kappa : f(\alpha) = \gamma\} \in U$, hence $[f] = [c_\gamma]$ and so $h([f]) = \gamma < \kappa$. Thus $h([d]) \le \kappa$, and by (iv), $h([d]) = \kappa$.

$\Leftarrow$: Assume that $h([d]) = \kappa$, and suppose that $f \in {}^\kappa\kappa$ is such that $\{\alpha < \kappa : f(\alpha) < \alpha\} \in U$. Thus $[f] < [d]$, so $h([f]) < h([d]) = \kappa$. Say $h([f]) = \alpha < \kappa$. Hence $h([f]) < h([c_\alpha])$, so $[f] < [c_\alpha]$. Hence $\{\beta < \kappa : f(\beta) \in \alpha\} \in U$. Now $\{\beta < \kappa : f(\beta) \in \alpha\} = \bigcup_{\gamma<\alpha}\{\beta < \kappa : f(\beta) = \gamma\}$ It follows that there is a $\gamma < \alpha$ such that $\{\beta < \kappa : f(\beta) = \gamma\} \in U$. So U is normal. $\square$

Proposition 12.41. *If M is a transitive class, $X, Y \in M$, and $M \models |X| \le |Y|$, then $|X| \le |Y|$.*

Proof. Assume that M is a transitive class, $X, Y \in M$, and $M \models |X| \le |Y|$. Let $f \in M$ and in M $f : X \to Y$ with f one-one. By absoluteness, f is a one-one function from X into Y. $\square$

Proposition 12.42. *If M is a transitive class, $\alpha \in M$, and α is a cardinal, then $M \models \alpha$ is a cardinal.*

Proof. Assume the hypotheses, but suppose that $M \not\models \alpha$ is a cardinal. Then there exist in M a $\beta < \alpha$ and a one-one function f mapping β onto α. By absoluteness, f is really such a function, contradiction. $\square$

Proposition 12.43. *If κ is inaccessible, then there is an $\alpha < \kappa$ such that $(V_\alpha, \in) \preceq (V_\kappa, \in)$.*

Proof. Let $\mathscr{H}$ be a set of Skolem functions. Define $\alpha_0 = 0$. If α_m has been defined, let α_{m+1} be such that $V_{\alpha_{m+1}}$ contains V_{α_m} and $h[V_{\alpha_m}] \subseteq V_{\alpha_{m+1}}$. Let $\beta = \sup_{m\in\omega} \alpha_m$. Then V_β is as desired. $\square$

13. Constructible sets

A set $X \subseteq M$ is *definable over* $\overline{M}$ iff there is a formula $\varphi(x, \overline{y})$ and a tuple $\overline{b}$ of elements of M, such that $X = \{a \in M : \overline{M} \models \varphi(a, \overline{b})\}$. We define

$$\begin{aligned}
\operatorname{def}(\overline{M}) &= \{X \subseteq M : X \text{ is definable over } \overline{M}\};\\
L_0 &= \emptyset;\\
L_{\alpha+1} &= \operatorname{def}(L_\alpha);\\
L_\gamma &= \bigcup_{\alpha<\gamma} L_\alpha \quad \text{for } \gamma \text{ limit};\\
L &= \bigcup_{\alpha\in\mathbf{ON}} L_\alpha.
\end{aligned}$$

Proposition 13.1. *If F is a finite subset of M, then F is definable over $\overline{M}$, and hence $F \in \operatorname{def}(\overline{M})$.*

Proof. $F = \{a \in M : (M, \in) \models \bigvee_{b\in F} a = b\}$. □

Proposition 13.2. *If F is a finite subset of L, then $F \in L$.*

Proof. Choose α so that $F \subseteq L_\alpha$. Then $F \in \operatorname{def}_A(L_\alpha)$ by Proposition 1. □

Proposition 13.3. $L_\alpha \subseteq V_\alpha$. □

Proposition 13.4. *For any ordinal α,*
(i) L_α is transitive.
(ii) $L_\beta \subseteq L_\alpha$ for all $\beta < \alpha$.

Proof. We prove both statements simultaneously by induction on α. Both statements are clear for $\alpha = 0$. Now assume them for α. For (ii), suppose that $\beta < \alpha + 1$.

Case 1. $\beta = \alpha$. If $a \in L_\beta$, then $(L_\beta, \in) \models a = a$, so $a \in \operatorname{def}(L_\beta, \in)$, and hence $a \in L_\alpha$.

Case 9. $\beta < \alpha$. Then $L_\beta \subseteq L_\alpha$ by the inductive hypothesis, and $L_\alpha \subseteq L_{\alpha+1}$ by Case 1.

Hence (ii) holds for $\alpha + 1$. Now suppose that $X \in L_{\alpha+1}$. Then $X \subseteq L_\alpha \subseteq L_{\alpha+1}$ by (i). So (i) holds for $\alpha + 1$.

Clearly the induction step to a limit ordinal works. □

Proposition 13.5. (Lemma 13.2) $\alpha = L_\alpha \cap \mathbf{ON}$.

Proof. We prove this by induction on α. It is obvious for $\alpha = 0$, and the inductive step when α is limit is clear. So, suppose the statement holds for β and we want to prove it for $\beta + 1$. If $\gamma \in L_{\beta+1} \cap \mathbf{ON}$, then $\gamma \in \operatorname{def}(L_\beta)$, so $\gamma \subseteq L_\beta \cap \mathbf{ON} = \beta$; hence $\gamma \leq \beta$. This shows that $L_{\beta+1} \cap \mathbf{ON} \subseteq \beta + 1$.

If $\gamma < \beta$, then by the inductive hypothesis, $\gamma \in L_\beta \cap \mathbf{ON} \subseteq L_{\beta+1} \cap \mathbf{ON}$. Thus it remains only to show that $\beta \in L_{\beta+1}$. Now there is a natural Δ_0 formula $\varphi(x)$ which expresses that x is an ordinal:

$$\forall y \in x \forall z \in y(z \in x) \wedge \forall y \in x \forall z \in y \forall w \in z(w \in y);$$

this just says that x is transitive and every member of x is transitive. Now $\varphi(x)$ is absolute, so

$$\beta = L_\beta \cap \mathbf{ON} = \{x \in L_\beta : (L_\beta, \in) \models \varphi(x)\} \in \mathrm{def}(L_\beta) = L_{\beta+1}. \qquad \square$$

Proposition 13.6. $\alpha \in L_{\alpha+1}$.

Proof. $\alpha = \{x \in L_\alpha : (L_\alpha, \in) \models [\alpha \text{ is an ordinal}]$, using absoluteness. $\square$

Proposition 13.7. $L_\alpha \in L_{\alpha+1}$.

Proof. $L_\alpha = \{x \in L_\alpha : (L_\alpha, \in) \models x = x\} \in \mathrm{def}(L_\alpha, \in) = L_{\alpha+1}$. $\square$

Theorem 13.8. (Theorem 13.3) *L is a model of ZFC.*

Proof. We take the axioms in order.

Extensionality. $\forall x, y[\forall w[w \in x \leftrightarrow w \in y] \to x = y]$. Assume that $x, y \in L$ and for all $w \in L[w \in x \leftrightarrow w \in y]$. Since L is transitive, $\forall w[w \in x \leftrightarrow w \in y]$. Hence $x = y$.

Comprehension. Given a formula φ with free variables among $x, z, w_1, \ldots, w_n$, an instance of comprehension is

$$\forall z \forall \overline{w} \exists y \forall x[x \in y \leftrightarrow x \in z \wedge \varphi].$$

So, let $b, \overline{c} \in L$. Say $b, \overline{c} \in L_\alpha$. Choose $\beta > \alpha$ so that φ is absolute for L_β, L. Say $\varphi = \varphi(x, z, \overline{w})$. Let

$$y = \{d \in L_\beta : (L_\beta, \in) \models \varphi(d, b, \overline{c})\}.$$

Then $y \in L_{\beta+1} \subseteq L$, and

$$\forall d \in L_\beta[d \in y \leftrightarrow \varphi(d, b, \overline{c})];$$

hence by absoluteness,

$$\forall d \in L[d \in y \leftrightarrow \varphi(d, b, \overline{c})].$$

Pairing. $\forall x, y \exists z[x \in z \wedge y \in z]$. Given $x, y \in L$, choose α so that $x, y \in L_\alpha$. Now $L_\alpha \in L_{\alpha+1} \subseteq L$.

Union. $\forall \mathscr{A} \exists A \forall Y \forall x[x \in Y \wedge Y \in \mathscr{A} \to x \in A]$. Let $\mathscr{A} \in L$. Say $\mathscr{A} \in L_\alpha$. Suppose that $Y \in L$, $x \in Y$, and $Y \in \mathscr{A}$. Then $Y \in L_\alpha$ and $x \in L_\alpha$.

Power set. $\forall x \exists y \forall z[\forall w \in z[w \in x] \to z \in y]$. Let $x \in L$; say $x \in L_\alpha$. Let $y = \mathscr{P}(x) \cap L$. Say $y \subseteq L_\beta$. Wlog $\alpha < \beta$. We claim that $y = \{a \in L_\beta : (L_\beta, \in) \models \forall w[w \in a \to w \in x]\}$. In fact, suppose that $a \in y$. Then $a \subseteq x$ and $a \in L_\beta$. Hence $\forall w \in L_\beta[w \in a \to w \in x]$. Thus $y \subseteq \{a \in L_\beta : (L_\beta, \in) \models \forall w[w \in a \to w \in x]\}$. Conversely, suppose that $a \in L_\beta$ and $(L_\beta, \in) \models \forall w[w \in a \to w \in x]\}$. If $w \in a$, then $w \in L_\beta$ and hence $w \in x$. So $a \subseteq x$. Hence $a \in y$. Thus $y = \{a \in L_\beta : (L_\beta, \in) \models \forall w[w \in a \to w \in x]\}$. It follows that $y \in L_{\beta+1} \subseteq L$.

Now suppose that $z \in L$ and $\forall w \in L[w \in z \to w \in x]$. Then $\forall w[w \in z \to w \in x]$, i.e., $z \subseteq x$. Hence $z \in y$.

Infinity.

$$\exists x[\exists y \in x \forall w[w \notin y] \wedge \forall y \in x \exists z \forall w[w \in z \leftrightarrow w \in y \vee w = y]].$$

We take $x = \omega$. $\emptyset \in \omega$ satisfies the first part of the formula. Now suppose that $y \in L$ and $y \in \omega$. Then $y + 1 = y \cup \{y\} \in \omega$ and it is constructible by transitivity. Clearly then the second part of the formula holds.

Replacement. Given $\varphi(x, y, A, \overline{w})$ with free variables among $x, y, A, w_1, \ldots, w_n$, the following is an axiom.

$$\forall A \forall w_1 \ldots \forall w_n [\forall x \in A \exists y[\varphi \wedge \forall y'[\varphi(x, y', A, \overline{w}) \to y = y'] \to \exists Y \forall x \in A \exists y \in Y \varphi].$$

Suppose that $A, \overline{w} \in L$; say $A, \overline{w} \in L_\alpha$. We assume that

$$\forall x \in A \exists y \in L[\varphi \wedge \forall y' \in L[\varphi(x, y', A, \overline{w}).$$

Hence for all $x \in A$ choose α_x so that there is a $y \in L_{\alpha_x}$ such that $\forall y' \in L[\varphi(x, y', A, \overline{w}) \to y = y']$. Let β be greater than α_x for each $x \in A$. Then for all $x \in A$ there is a $y \in L_\beta$ such that $\varphi(x, y, A, \overline{w})$, as desired.

Foundation. $\forall x[\exists y \in x \to \exists y \in x \forall z \in y[z \notin x]]$. Take any $x \in L$ with $x \neq \emptyset$. Choose $y \in x$ such that $x \cap y = \emptyset$. Then $y \in L$ is as desired.

Choice. It suffices to define a well-order of L. In fact, we define a well-order $<_\alpha \in L$ of L_α for each α. Using the considerations about the corner notation we can define a sequence $\langle \varphi_n : n \in \omega \rangle$ in L enumerating all formulas of the form $\varphi(x, \overline{y})$. Then we define $<_0 = \emptyset$. If $<_\alpha$ has been defined, let $\langle \overline{b}_\xi : \xi < \beta \rangle$ enumerate all finite sequences of members of L_α, which we well-order lexicographically. This enables us to well-order the members of $\mathrm{def}(L_\alpha)$. We define $<_{\alpha+1}$ to be L_α with all the new members of $\mathrm{def}(L_\alpha)$ adjoined at the end. For γ limit we let $<_\gamma = \bigcup_{\alpha<\gamma} <_\alpha$. Finally, $< = \bigcup_{\alpha \in \mathbf{ON}} <_\alpha$. □

A set-theoretic formula φ is Δ_0-*special* iff (i) the only logical symbols in φ are $\neg, \wedge, \exists$. (In particular, $=$ does not occur.)

(ii) the only occurrence of $\in$ has the form $v_i \in v_j$ with $i \neq j$.

(iii) every occurrence of $\exists$ has the form $\exists v_i \in v_j \varphi$ with $i \neq j$.

We assume that a Δ_0 formula has quantifier parts of the form $\exists v_i \in v_j \varphi$ or $\forall v_i \in v_j \varphi$ with $i \neq j$.

Lemma 13.9. *Every Δ_0 formula φ is equivalent in ZF to a Δ_0-special formula ψ.*

Proof. We prove this by induction on φ.

(1) φ is $v_i \in v_i$. Let ψ be $\exists v_{i+1} \in v_i[v_{i+1} = v_i]$,

(2) φ is $v_i = v_j$. Let ψ be

$$\neg \exists v_k \in v_i[v_k \notin v_j] \wedge \neg \exists v_k \in v_j[v_k \notin v_i]$$

where $k = \max(i, j) + 1$.

(3) φ is $v_i \in v_j$ with $i \neq j$. Let ψ be φ.

(4) Inductively, the cases of $\wedge, \neg, \vee, \rightarrow, \leftrightarrow, \exists$ are clear.

(5) φ is $\forall v_i \in v_j \varphi'$. Say φ' is equivalent to the Δ_0-special formula ψ'. Then ψ is $\neg\exists v_i \in v_j \neg\psi'$. □

The following are the Gödel operations.

$$\begin{aligned}
G_1(X, Y) &= \{X, Y\};\\
G_2(X, Y) &= X \times Y;\\
G_3(X, Y) &= \{(u, v) : u \in X \wedge v \in Y \wedge u \in v\};\\
G_4(X, Y) &= X \backslash Y;\\
G_5(X, Y) &= X \cap Y;\\
G_6(X) &= \bigcup X;\\
G_7(X) &= \mathrm{dmn}(X);\\
G_8(X) &= \{(u, v) : (v, u) \in X\};\\
G_9(X) &= \{(u, v, w) : (u, w, v) \in X\};\\
G_{10}(X) &= \{(u, v, w) : (v, w, u) \in X\}.
\end{aligned}$$

Now we indicate precisely the notion of composition. For $0 \leq i < n$ define P^n_i by $P^n_i(X_0, \ldots, X_{n-1}) = X_i$. If F is m-ary and $G_0, \ldots, G_{m-1}$ are n-ary, then

$$\begin{aligned}
&C^m_n(F, G_0, \ldots, G_{m-1}) \text{ is the } n\text{-ary function } H \text{ such that}\\
&H(X_0, \ldots, X_{n-1}) = F(G_0(X_0, \ldots, X_{n-1}), \ldots, G_{m-1}(X_0, \ldots, X_{n-1})).
\end{aligned}$$

A *composition* of $G_0, \ldots, G_{10}$ is any function which is in the closure of $\{G_0, \ldots, G_{10}\} \cup \{P^n_i : 0 \leq i < n\}$ under the composition functions C^m_n. Gfcn is the set of all such compositions.

Lemma 13.10. *For each $n \geq 1$ define $H_n(X_0, \ldots, X_{n-1}) = X_0 \times \cdots \times X_{n-1}$. Then H_n is a composition of $G_1, \ldots, G_{10}$.*

Proof. We prove this by induction on n. For $n = 1$ we take $H_1 = P^1_0$; for $n = 2$ we take $H_2 = G_2$. Now suppose that $n \geq 2$ and H_n is a composition of $G_1, \ldots, G_n$. Then

$$\begin{aligned}
H_{n+1}(X_0, \ldots, X_n) &= H_n(X_0, \ldots, X_{n-1}) \times X_n = G_2(H_n(X_0, \ldots, X_{n-1}), X_n)\\
&= G_2(H_n(P^{n+1}_0(X_0, \ldots, X_n), \ldots, P^{n+1}_{n-1}(X_0, \ldots, X_n)), P^{n+1}_n(X_0, \ldots, X_n)).
\end{aligned}$$

Hence

$$H_{n+1} = C^2_{n+1}(G_2, C^n_{n+1}(H_n, P^{n+1}_0, \ldots, P^{n+1}_{n-1}), P^{n+1}_n).$$ □

Theorem 13.11. (Theorem 13.4) *If $\varphi(v_0, \ldots, v_{n-1})$ is a Δ_0 formula with free variables among those shown, then there is an n-ary composition G of Gödel functions such that for all $X_0, \ldots, X_{n-1}$,*

$$G(X_0, \ldots, X_{n-1}) = \{(u_0, \ldots, u_{n-1}) : u_0 \in X_0, \ldots, u_{n-1} \in X_{n-1} \text{ and } \varphi(u_0, \ldots, u_{n-1})\}.$$

Proof. We may assume that $\varphi(v_0,\ldots,v_{n-1})$ is Δ_0-special. We proceed by induction on φ.

Case 1. $\varphi(v_0,\ldots,v_{n-1})$ is an atomic formula $v_i\in v_j$ with $i,j<n$ and $i\neq j$. We treat this case by induction on $n\geq 2$.

Subcase 1a. $n=2$. Then

$$\begin{aligned}\{(u_0,u_1):u_0\in X_0\wedge u_1\in X_1\wedge u_0\in u_1\}&=G_3(X_0,X_1);\\ \{(u_0,u_1):u_0\in X_0\wedge u_1\in X_1\wedge u_1\in u_0\}&=\\ \{(u_0,u_1):(u_1,u_0)\in G_3(X_2,X_1)\}&=\\ \{(u_0,u_1):(u_1,u_0)\in G_3(P^2_1(X_1,X_2),P^2_0(X_1,X_2))\}&=\\ \{(u_0,u_1):(u_1,u_0)\in (C^2_2(G_3,P^2_1,P^2_0))(X_1,X_2)\}&=\\ (G_8(C^2_2(G_3,P^2_1,P^2_0)))(X_1,X_2))&=\\ (C^1_2(G_8,C^2_2(G_3,P^2_1,P^2_0)))(X_1,X_2).&\end{aligned}$$

Subcase 1b. Assume the result for n. Now we are given $\varphi(v_0,\ldots,v_n)$. Let $\overline{X}=(X_0,\ldots,X_n)$.

Subsubcase 1b1. $i,j\leq n-1$. By the inductive hypothesis let K be an n-ary composition such that

$$K(X_0,\ldots,X_{n-1})=\{(u_0,\ldots,u_{n-1}):u_0\in X_0,\ldots,u_{n-1}\in X_{n-1},u_i\in u_j\}.$$

Then

$$\begin{aligned}&\{(u_0,\ldots,u_n):u_0\in X_0,\ldots,u_n\in X_n,u_i\in u_j\}\\ &\quad=\{(u_0,\ldots,u_n):(u_0,\ldots,u_{n-1})\in K(X_0,\ldots,X_{n-1}),u_n\in X_n\}\\ &\quad=K(X_0,\ldots,X_{n-1})\times X_n\\ &\quad=G_2(K(X_0,\ldots,X_{n-1}),X_n)\\ &\quad=(C^2_{n+1}(G_2,C^n_{n+1}(K,P^{n+1}_0,\ldots,P^{n+1}_{n-1}),P^{n+1}_n))(\overline{X}).\end{aligned}$$

Subsubcase 1b9. $i=n$ or $j=n$, and $i,j\neq n-1$. Then by Subcase 1b1 there is an $(n+1)$-ary composition K such that

$$K(X_0,\ldots,X_n)=\{(u_0,\ldots,u_{n-2},u_n,u_{n-1}):u_0\in X_0,\ldots,u_n\in X_n,u_i\in u_j\}.$$

Now note that $(u_0,\ldots,u_{n-2},u_n,u_{n-1})=((u_0,\ldots,u_{n-2}),u_n,u_{n-1})$. Hence

$$\begin{aligned}\{(u_0,\ldots,u_n):u_0\in X_0,\ldots,u_n\in X_n,u_i\in u_j\}&=G_9(K(X_0,\ldots,X_n))\\ &=(C^1_{n+1}(G_9,C^{n+1}_{n+1}(K,P^{n+1}_0,\ldots,P^{n+1}_n)))(\overline{X}).\end{aligned}$$

Subsubcase 1b3. $i=n-1$, $j=n$. Let

$$\begin{aligned}K_0&=C^2_{n+1}(G_3,P^{n+1}_{n-1},P^{n+1}_n);\\ K_1&=C^{n-1}_{n+1}(H_{n-1},P^{n+1}_0,\ldots,P^{n+1}_{n-2});\\ K_2&=C^2_{n+1}(G_2,K_0,K_1);\\ K_3&=C^1_{n+1}(G_{10},K_2).\end{aligned}$$

Then for any $X_0, \ldots, X_n$ we have

$$\begin{aligned}
K_0(X_0, \ldots, X_n) &= G_3(X_{n-1}, X_n);\\
K_1(X_0, \ldots, X_n) &= X_0 \times \cdots \times X_{n-2};\\
K_2(X_0, \ldots, X_n) &= G_3(K_{n-1}, K_n) \times (X_0 \times \cdots \times X_{n-2});\\
K_3(X_0, \ldots, X_n) &= \{(u_0, \ldots, u_n) : u_0 \in X_0, \ldots, u_n \in X_n, u_{n-1} \in u_n\}.
\end{aligned}$$

Subsubcase 1b4. $i = n$, $j = n-1$. Let

$$\begin{aligned}
K_0 &= C^2_{n+1}(G_3, P^{n+1}_{n-1}, P^{n+1}_n);\\
K_1 &= C^{n-1}_{n+1}(H_{n-1}, P^{n+1}_0, \ldots, P^{n+1}_{n-2});\\
K_2 &= C^2_{n+1}(G_2, K_0, K_1);\\
K_3 &= C^1_1(G_9, G_{10});\\
K_4 &= C^1_{n+1}(K_3, K_2).
\end{aligned}$$

Then for any $X_0, \ldots, X_n$ we have

$$\begin{aligned}
K_0(X_0, \ldots, X_n) &= G_3(X_{n-1}, X_n);\\
K_1(X_0, \ldots, X_n) &= X_0 \times \cdots \times X_{n-2};\\
K_2(X_0, \ldots, X_n) &= G_3(K_{n-1}, K_n) \times X_0 \times \cdots \times X_{n-2};\\
K_3(X) &= \{(u, v, w) : (u, w, v) \in G_{10}(X)\} = \{(u, v, w) : (w, v, u) \in X\}.
\end{aligned}$$

Hence

$$\begin{aligned}
&(u_0, \ldots, u_n) \in K_4(X_0, \ldots, X_n) \quad \text{iff}\\
&(u_0, \ldots, u_n) \in G_9(G_{10}(G_3(X_{n-1}, X_n) \times X_0 \times \cdots \times X_{n-2})) \quad \text{iff}\\
&(u_n, u_{n-1}, (u_0, \ldots, u_{n-2})) \in (G_3(X_n, X_{n-1}) \times X_0 \times \cdots \times X_{n-2}) \quad \text{iff}\\
&u_0 \in X_0, \ldots, u_n \in X_n, u_n \in u_{n-1}.
\end{aligned}$$

Case 9. φ is $\neg\psi$. Choose G for ψ. Then

$$\begin{aligned}
&\{(u_0, \ldots, u_{n-1}) : u_0 \in X_0, \ldots, u_{n-1} \in X_{n-1} \text{ and } \varphi(u_0, \ldots, u_{n-1})\}\\
&\quad = X_0 \times \cdots \times X_{n-1} \backslash G(X_0, \ldots, X_{n-1})\\
&\quad = H_n(X_0, \ldots, X_{n-1}) \backslash G(X_0, \ldots, X_{n-1})\\
&\quad = C^2_m(G_4, H_n, G)(X_0, \ldots, X_{n-1}).
\end{aligned}$$

Case 3. φ is $\psi \wedge \chi$. Let K_1, K_2 work for ψ, χ respectively. Then

$$\begin{aligned}
&\{(u_0, \ldots, u_{n-1}) : u_0 \in X_0, \ldots, u_{n-1} \in X_{n-1} \text{ and } \varphi(u_0, \ldots, u_{n-1})\}\\
&\quad = K_1(X_0, \ldots, X_{n-1}) \cap K_2(X_0, \ldots, X_{n-1})\\
&\quad = C^2_n(G_5, K_1, K_2)(X_0, \ldots, X_{n-1}).
\end{aligned}$$

Case 4. φ is $\exists v_j \in v_i\psi(v_0,\ldots,v_{n-1})$. Let ψ' be obtained from ψ by replacing all free occurrences of v_j by v_n. Then φ is logically equivalent to $\exists v_n \in v_i\psi'(v_0,\ldots,v_n)$, which we denote by φ'.

By one of the initial cases, there is a composition K such that

$$K(X_0,\ldots,X_n) = \{(u_0,\ldots,u_n) : u_0 \in X_0 \wedge \ldots \wedge u_n \in X_n \wedge u_n \in u_i\}.$$

By the inductive hypothesis we get a composition L such that

$$L(X_0,\ldots,X_n) = \{(u_0,\ldots,u_n) : u_0 \in X_0 \wedge \ldots \wedge u_n \in X_n \wedge \psi'(u_0,\ldots,u_n)\}.$$

Let $G = C^2_{n+1}(G_5, K, L)$. Thus

$$\begin{aligned} G(X_0,\ldots,X_n) = \{(u_0,\ldots,u_n) : u_0 \in X_0 \wedge \ldots \wedge u_n \in X_n \\ \wedge u_n \in u_i \wedge \psi'(u_0,\ldots,u_n)\}. \end{aligned}$$

Now we claim

$$(*) \qquad \begin{aligned} \{(u_0,\ldots,u_{n-1}) : & u_0 \in X_0 \wedge \ldots \wedge u_{n-1} \in X_{n-1} \wedge \varphi'(u_0,\ldots,u_{n-1})\} \\ & = \operatorname{dmn}(G(X_0,\ldots,X_{n-1},\bigcup X_i)). \end{aligned}$$

In fact, with $v \neq u_i$,

$$\begin{aligned} & u_0 \in X_0 \wedge \ldots \wedge u_{n-1} \in X_{n-1} \wedge \varphi'(u_0,\ldots,u_{n-1}) \\ & \leftrightarrow u_0 \in X_0 \wedge \ldots \wedge u_{n-1} \in X_{n-1} \wedge \exists v \in u_i\psi'(u_0,\ldots,u_n,v) \\ & \leftrightarrow u_0 \in X_0 \wedge \ldots \wedge u_{n-1} \in X_{n-1} \wedge \exists v \left[v \in u_i \wedge \psi'(u_0,\ldots,u_{n-1},v) \wedge v \in \bigcup X_i\right] \\ & \leftrightarrow u_0 \in X_0 \wedge \ldots \wedge u_{n-1} \in X_{n-1} \wedge (u_0,\ldots,u_{n-1}) \in \operatorname{dmn}(\{(u_0,\ldots,u_{n-1},v) : \\ & \quad u_0 \in X_0 \wedge \ldots \wedge u_{n-1} \in X_{n-1} \wedge v \in u_i \wedge v \in \bigcup X_i \wedge \psi'(u_0,\ldots,u_{n-1},v)\}) \\ & \leftrightarrow (u_0,\ldots,u_{n-1}) \in \operatorname{dmn}(G(X_0,\ldots,X_{n-1},\bigcup X_i)). \end{aligned}$$

So $(*)$ holds. Now let

$$\begin{aligned} M &= C^{n+1}_n(G, P^n_0,\ldots,P^n_{n-1}, C^1_n(G_6, P^n_{i-1})); \\ N &= C^1_n(G_7, M). \end{aligned}$$

Then

$$\{(u_0,\ldots,u_{n-1}) : u_0 \in X_1 \wedge \ldots \wedge u_{n-1} \in X_{n-1} \wedge \varphi'(u_0,\ldots,u_{n-1})\} = N(X_0,\ldots,X_{n-1}).$$

Since φ and φ' are logically equivalent, this completes the proof. $\square$

For the following theorem, note that $(x,y) = \{\{x\},\{x,y\}\}$ and $\bigcup(x,y) = \{x\} \cup \{x,y\} = \{x,y\}$. Define $(x,y,z) = ((x,y),z)$. Then $\bigcup(x,y,z) = \{(x,y),z\}$ and $\bigcup\bigcup(x,y,z) = (x,y) \cup z$ and $\bigcup\bigcup\bigcup(x,y,z) = \{x,y\} \cup \bigcup z$.

We treat $\exists u \in \bigcup x$ as an abbreviation for $\exists a \in x \exists u \in a$, with a a new variable; and $\exists u \in \bigcup\bigcup x$ as an abbreviation for $\exists a \in x \exists b \in a \exists u \in b$ with a and b new variables; etc. Similarly, $\forall a \in \bigcup x$ abbreviates $\forall b \in x \forall a \in b$ and $\forall a \in \bigcup\bigcup x$ abbreviates $\forall b \in x \forall c \in b \forall a \in c$; etc.

Proposition 13.12. *Define*

$$M_0 = \{G_i : 1 \le i \le n\} \cup \{P_i^n : i < n \in \omega\};$$
$$M_{k+1} = M_k \cup \{C_n^m(G_i, \overline{H}) : 1 \le i \le 10, \overline{H} \in M_k\}.$$

Here in the definition of M_{m+1}, m and n are 1 or 2, depending on G_i, and $\overline{H}$ is a sequence of length 1 or 9.

Then Gfcn $= \bigcup_{m\in\omega} M_m$.

Proof. By induction, $M_m \subseteq$ Gfcn for all $m \in \omega$, so $\bigcup_{m\in\omega} M_m \subseteq$ Gfcn. Let $N = \bigcup_{m\in\omega} M_m$. Now let $\mathscr{F} = \{F :$for all $\overline{H}$ in N, $C_l^k(F, \overline{H}) \in N\}$. We claim that $N \subseteq \mathscr{F}$. Clearly $M_0 \subseteq \mathscr{F}$. Suppose that $M_m \subseteq \mathscr{F}$, and $F \in M_{m+1}$. Say $F = C_t^s(G_i, \overline{L})$ with $L \in M_m$. Suppose that $\overline{H} \in N$. Then for any $\overline{X}$,

$$\begin{aligned}(C_l^k(F, \overline{H}))(\overline{X}) &= F(H_0(\overline{X}), \ldots, H_l(\overline{X}))\\ &= C_l^k(G_i, L_0(\overline{H}(\overline{X})), \ldots, L_s(\overline{H}(\overline{X})))\end{aligned}$$

Now each $C_t^s(L_u, \overline{H}) \in N$ since $L_u \in M_m$. So $C_l^k(F, \overline{H}) \in N$, and hence $F \in \mathscr{F}$. □

Theorem 13.13. (Lemma 13.7) *For any $H \in$ Gfcn we have*

(i) $u \in H(X, \ldots)$ is Δ_0.

(ii) If φ is Δ_0, then so are $\forall u \in H(X, \ldots)\varphi$ and $\exists u \in H(X, \ldots)\varphi$.

(iii) $Z = H(X, \ldots)$ is Δ_0.

(iv) If φ is Δ_0, then so is $\varphi(H(X, \ldots))$.

(v) $H(X, \ldots) \in u$ is Δ_0.

Proof. With $M_0, \ldots$ as in Proposition 12 we show that each M_m is a subset of the collection of H such that (i)–(iv) hold. We treat M_0 and M_{m+1} simultaneously. Let X, Y be arbitrary sets (in the case of M_0), or arbitrary members of M_m (in the case of M_{m+1}).

First we take (iii),

$$(i):\ z = \{X\} \leftrightarrow \forall a \in z[a = X] \wedge X \in z;$$
$$(ii):\ z = \{X, Y\} \leftrightarrow \forall a \in z[a = X \vee a = Y] \wedge X \in z \wedge Y \in z;$$
$$\begin{aligned}(iii):\ z = (X, Y) \leftrightarrow\ & \exists a \in z \exists b \in z[a = \{X\} \wedge b = \{X, Y\}]\\ & \wedge \forall a \in z[a = \{X\} \vee a = \{X, Y\}]\end{aligned}$$
$$\begin{aligned}(iv):\ z = X \times Y \leftrightarrow\ & \forall a \in z \exists b \in X \exists c \in Y[a = (b, c)]\\ & \wedge \forall a \in X \forall b \in Y \exists c \in z[c = (a, b)];\end{aligned}$$
$$\begin{aligned}(v):\ z = \{(u, v) : u \in X, v \in Y, u \in v\} \leftrightarrow\ & \forall c \in z \exists u \in X \exists v \in Y[u \in v \wedge z = (u, v)]\\ & \wedge \forall u \in X \forall v \in Y[u \in v \rightarrow \exists c \in z[c = (u, v)]];\end{aligned}$$

$$\begin{aligned}
(vi):\ & z = X\backslash Y \leftrightarrow \forall u \in z[u \in X \wedge u \notin Y] \\
& \quad \wedge \forall u \in X[u \notin Y \to u \in z]; \\
(viii):\ & z = X \cap Y \leftrightarrow \forall u \in z[u \in X \wedge u \in Y] \\
& \quad \wedge \forall u \in X[u \in Y \to u \in z]; \\
(viii):\ & z = \bigcup X \leftrightarrow \forall u \in z \exists v \in X[u \in v] \\
& \quad \wedge \forall v \in X \forall u \in v[u \in z]; \\
(ix):\ & z \in \mathrm{dmn}(X) \leftrightarrow \exists u \in X \exists v \in \bigcup u[u = (z,v)]; \\
(x):\ & z = \mathrm{dmn}(X) \leftrightarrow \forall u \in z[u \in \mathrm{dmn}(X)] \wedge \forall v \in \bigcup\bigcup X[v \in \mathrm{dmn}(X) \to v \in z]; \\
(xi):\ & z = \{(u,v) : (v,u) \in X\} \leftrightarrow \forall w \in z \exists a \in X \exists u,v \in \bigcup a[a = (v,u) \wedge w = (u,v)] \\
& \quad \wedge \forall a \in X \forall u,v \in a[a = (v,u) \to \exists c \in z[c = (u,v)]] \\
(xii):\ & z = \{(u,v,w) : (u,w,v) \in X\} \leftrightarrow \forall a \in z \exists b \in X \exists v \in \bigcup b \\
& \quad \exists u,w \in \bigcup\bigcup\bigcup b[b = (u,w,v) \wedge a = (u,v,w)] \wedge \forall b \in X \forall v \in \bigcup b \\
& \quad \forall u,w \in \bigcup\bigcup\bigcup b[(b = (u,w,v) \to \exists a \in z[a = (u,v,w)]] \\
(xiii):\ & \text{similarly}
\end{aligned}$$

Now we treat (i).

$$\begin{aligned}
&(1):\ u \in G_1(X,Y) \leftrightarrow u \in \{X,Y\} \leftrightarrow u = X \vee u = Y; \\
&(2):\ u \in G_2(X,Y) \leftrightarrow u \in X \times Y \leftrightarrow \exists x \in X \exists y \in Y[u = (x,y)]; \\
&(3):\ u \in G_3(X,Y) \leftrightarrow \exists x \in X \exists y \in Y[x \in y \wedge u = (x,y)]; \\
&(4):\ u \in G_4(X,Y) \leftrightarrow u \in X \wedge u \notin Y; \\
&(5):\ u \in G_5(X,Y) \leftrightarrow u \in X \wedge u \in Y; \\
&(6):\ u \in G_6(X) \leftrightarrow \exists x \in X[u \in x]; \\
&(7):\ u \in G_7(X):\ \text{see (iii)(ix)}; \\
&(8):\ u \in G_8(X) \leftrightarrow \exists x \in X \exists u,v \in \bigcup x[x = (v,u) \wedge u = (u,v)]; \\
&(9):\ u \in G_9(X) \leftrightarrow \exists x \in X \exists v \in \bigcup x \exists u', w \in \bigcup\bigcup\bigcup x \\
&\qquad [x = (u',w,v) \wedge u = (u',v,w)]; \\
&(10):\ u \in G_{10}(X) \leftrightarrow \exists x \in X \exists u' \in \bigcup x \exists v,w \in \bigcup\bigcup\bigcup x \\
&\qquad [x = (v,w,u') \wedge u = (u',v,w)].
\end{aligned}$$

Next we do (iv).

(1): Let $\varphi_{\{X,Y\}}$ be obtained from φ by replacing
$v \in u$ by $v = X \vee v = Y$.
$u \in v$ by $\exists w \in v[w = \{X,Y\}]$.
$u = v$ by $\{X,Y\} = v$.

$\exists v \in u\psi$ by $\psi(X) \vee \psi(Y)$.

$\forall v \in u\psi$ by $\psi(X) \wedge \psi(Y)$.

Then $\varphi_{\{X,Y\}}$ is Δ_0, and $\varphi(\{X,Y\}) \leftrightarrow \varphi_{\{X,Y\}}$.

(2) Let $\varphi_{(X,Y)}$ be obtained from φ by replacing

$v \in u$ by $v = \{X,Y\} \vee v = \{Y\}$.

$u \in v$ by $\exists w \in v[w = (X,Y)]$.

$u = v$ by $(X,Y) = v$.

$\exists v \in u\psi$ by $\psi_{\{X,Y\}} \vee \psi_{\{Y\}}$.

$\forall v \in u\psi$ by $\psi_{\{X,Y\}} \wedge \psi_{\{Y\}}$.

Then $\varphi_{(X,Y)}$ is Δ_0, and $\varphi((X,Y)) \leftrightarrow \varphi_{(X,Y)}$.

(3) Let $\varphi_{X\times Y}$ be obtained from φ by replacing

$v \in u$ by $\exists a \in X \exists b \in Y[v = (a,b)]$

$u \in v$ by $\exists w \in v[X \times Y = w]$.

$u = v$ by $v = X \times Y$.

$\exists v \in u\psi$ by $\exists a \in X \exists b \in Y \psi_{(X,Y)}$.

$\forall v \in u\psi$ by $\forall a \in X \forall b \in Y \psi_{(X,Y)}$

Then $\varphi_{X\times Y}$ is Δ_0, and $\varphi(x \times Y) \leftrightarrow \varphi_{X\times Y}$.

(4) Let φ_{G3} be obtained from φ by replacing

$v \in u$ by $\exists a \in X \exists b \in Y[a \in b \wedge v = (a,b)]$

$u \in v$ by $\exists w \in v[X \times Y = w \wedge \exists u \in X \exists v \in Y[u \in v]]$.

$u = v$ by $v = X \times Y \wedge \exists u \in X \exists v \in Y[u \in v]$.

$\exists v \in u\psi$ by $\exists a \in X \exists b \in Y[a \in b \wedge \psi_{(X,Y)}]$.

$\forall v \in u\psi$ by $\forall a \in X \forall b \in Y[a \in b \to \psi_{(X,Y)}]$

Then φ_{G3} is Δ_0, and $\varphi(G_3(X,Y)) \leftrightarrow \varphi_{G3}$.

(5) Let $\varphi_{X\backslash Y}$ be obtained from φ by replacing

$v \in u$ by $v \in X \wedge v \notin Y$

$u \in v$ by $\exists a \in v[a = X\backslash Y]$

$u = v$ by $v = X\backslash Y$

$\exists v \in u\psi$ by $\exists v \in X[v \notin Y \wedge \psi$.

$\forall v \in u\psi$ by $\forall v \in X[v \notin Y \to \psi]$.

Then $\varphi_{X\backslash Y}$ is Δ_0, and $\varphi(G_4(X,Y)) \leftrightarrow \varphi_{X\backslash Y}$.

(6) Let $\varphi_{X\cap Y}$ be obtained from φ by replacing

$v \in u$ by $v \in X \wedge v \in Y$

$u \in v$ by $\exists a \in v[a = X \cap Y]$

$u = v$ by $v = X \cap Y$

$\exists v \in u\psi$ by $\exists v \in X[v \in Y \wedge \psi$.

$\forall v \in u\psi$ by $\forall v \in x[v \in Y \to \psi]$.

Then $\varphi_{X\cap Y}$ is Δ_0, and $\varphi(G_5(X,Y)) \leftrightarrow \varphi_{X\cap Y}$.

(7) Let $\varphi_{\bigcup X}$ be obtained from φ by replacing

$v \in u$ by $\exists a \in X[v \in a]$

$u \in v$ by $\exists a \in v[a = \bigcup X]$

$u = v$ by $v = \bigcup X$

$\exists v \in u\psi$ by $\exists a \in X \exists v \in a\psi$.

$\forall v \in u\psi$ by $\forall a \in X[v \in a \to \psi]$.

Then $\varphi_{\bigcup X}$ is Δ_0, and $\varphi(G_6(X)) \leftrightarrow \varphi_{\bigcup x}$.

(8) Let φ_{dmn} be obtained from φ by replacing
$v \in u$ by $\exists a \in X \exists b \in a \exists d \in b[b = (v, d)]$.
$u \in v$ by $\exists a \in v[a = \text{dmn}X]$
$u = v$ by $v = \text{dmn}X$
$\exists v \in u\psi$ by $\exists a \in X \exists b \in a \exists d \in b[a = (v, d) \wedge \psi]$.
$\forall v \in u\psi$ by $\forall a \in X \forall b \in a \forall d \in b[a = (v, d) \rightarrow \psi]$.

Then φ_{dmn} is Δ_0, and $\varphi(G_7(X)) \leftrightarrow \varphi_{dmn}$.

(9) Let φ_{G8} be obtained from φ by replacing
$v \in u$ by $\exists a \in X \exists b, c \in \bigcup a[a = (c, b) \wedge v = (b, c) \wedge \psi$.
$u \in v$ by $\exists a \in v[a = G_8(X) \wedge \varphi]$.
$u = v$ by $v = G_8(X)$.
$\exists v \in u\psi$ by $\exists a \in X \exists b, c \in \bigcup a[a = (c, b) \wedge v = (b, c) \wedge \psi]$.
$\forall v \in u\psi$ by $\forall a \in X \forall b, c \in \bigcup a[a = (c, b) \wedge v = (b, c) \rightarrow \psi]$.

Then φ_{G8} is Δ_0, and $\varphi(G_8(X)) \leftrightarrow \varphi_{G8}$.

(10) Let φ_{G9} be obtained from φ by replacing
$v \in u$ by $\exists d \in X \exists b \in \bigcup d \exists a, c \in \bigcup\bigcup\bigcup d[d = (a, c, b) \wedge v = (a, b, c)]$
$u \in v$ by $\exists a \in v[a = G_9(X) \wedge \varphi]$.
$u = v$ by $v = G_9(X)$.
$\exists v \in u\psi$ by $\exists d \in X \exists b \in \bigcup d \exists a, c \in \bigcup\bigcup\bigcup d[d = (a, c, b) \wedge v = (a, b, c) \wedge \psi]$
$\forall v \in u\psi$ by $\forall d \in X \forall b \in \bigcup d \forall a, c \in \bigcup\bigcup\bigcup d[d = (a, c, b) \wedge v = (a, b, c) \rightarrow \psi]$

Then φ_{G9} is Δ_0, and $\varphi(G_9(X)) \leftrightarrow \varphi_{G9}$.

(11) G_{10} is treated similarly.

Next, we do (ii), by symmetry only doing $\forall$:

(1): $\forall u \in \{X, Y\}\varphi \leftrightarrow \varphi(X) \wedge \varphi(Y)$.
(2): $\forall u \in G_2(X, Y)\varphi \leftrightarrow \forall x \in X \forall y \in Y \varphi((x, y))]$.
(3): $\forall u \in G_3(X, Y)\varphi \leftrightarrow \forall x \in X \forall y \in Y[x \in y \rightarrow \varphi((x, y))]$.
(4): $\forall u \in G_4(X, Y)\varphi \leftrightarrow \forall u \in X[u \notin y \rightarrow \varphi(u)]$.
(5): $\forall u \in G_5(X, Y)\varphi \leftrightarrow \forall u \in X[u \in Y \rightarrow \varphi(u)]$.
(6): $\forall u \in G_6(X)\varphi \leftrightarrow \forall v \in X \forall u \in v\varphi$.
(7): $\forall u \in G_7(X)\varphi \leftrightarrow \forall x \in X \forall u \in \bigcup x[u \in \text{dmn}(X) \rightarrow \varphi]$.
(8): $\forall u \in G_8(X)\varphi \leftrightarrow \forall x \in X \forall v, w \in \bigcup x[x = (w, v) \wedge u = (v, w) \rightarrow \varphi]$.
(9): $\forall u \in G_9(X)\varphi \leftrightarrow \forall x \in X \forall v \in \bigcup x \forall u', w \in \bigcup\bigcup\bigcup x[x = (u', w, v) \wedge u = (u', v, w) \rightarrow \varphi]$.
(10): G_{10} is similar.

For (v), we have $H(X, \ldots) \in u \leftrightarrow \exists v \in u[u = H(X, \ldots)]$.

We have now shown that each member of M_0 and of M_{m+1} satisfies (i)-(v). □

Proposition 13.14. *For every formula $\varphi(x, \overline{y})$ there is a composition G of Gödel functions such that for every transitive set M and every $\overline{b} \in M$ we have*

$$\{a \in M : M \models \varphi(a, \overline{b})\} = \{a \in M : \varphi^M(a, \overline{b})\} = G(M, b_0, \ldots, b_{m-1}).$$

Proof. By Theorem 11 there is a Gödel function G such that for all $X_0,\ldots,X_m$,

$$G(X_0,\ldots,X_m) = \{(u_0,\ldots,u_m) : u_0 \in X_0,\ldots,u_m \in X_m, \varphi^M(u_0,\ldots,u_m)\}.$$

Hence for all $\overline{b} \in M$,

$$\begin{aligned}&G(M,\{b_0\},\ldots,\{b_{m-1}\})\\ &= \{(u_0,\ldots,u_m) : u_0 \in M, u_1 = b_0,\ldots,u_m = b_{m-1}, \varphi^M(M,u_0,\ldots,u_{m-1})\}\end{aligned}$$

Define $G'(X_0,\ldots,X_m) = G(X_0,G_1(X_1,X_1),\ldots,G_1(X_m,X_m))$. Then G' is a composition of Gödel functions, and for any $\overline{b} \in M$,

$$G'(M,b_0,\ldots,b_{m-1}) = \{(u_0,\ldots,u_m) : u_0 \in M, u_1 = b_0,\ldots,u_m = b_{m-1}, \varphi^M(u_0,\ldots,u_m)\}.$$

Now let $G''(X_0,\ldots,X_m) = \mathrm{dmn}(G'(X_0,\ldots,X_m))$. Then

$$\begin{aligned}G''(X_0,\ldots,X_m) &= \mathrm{dmn}(G'(X_0,\ldots,X_m))\\ &= \{(u_0,\ldots,u_{m-1}) : \exists u_m[(u_0,\ldots,u_m) \in G'(X_0,\ldots,X_m)]\}\end{aligned}$$

and so

$$\begin{aligned}&G''(M,b_0,\ldots,b_{m-1}) = \mathrm{dmn}(G'(M,b_0,\ldots,b_{m-1}))\\ &= \{(u_0,\ldots,u_{m-1}) : \exists u_m[u_0 \in M, u_1 = b_0,\ldots,u_m = b_{m-1}, \varphi^M(u_0,u_1,\ldots,u_m)]\}.\end{aligned}$$

Applying dmn $m-1$ times, we get a Gödel function H such that

$$\begin{aligned}&H(M,b_0,\ldots,b_{m-1})\\ &= \{u_0 : \exists u_1,\ldots,u_m[u_0 \in M, u_1 = b_0,\ldots,u_m = b_{m-1}, \varphi^M(u_0,\ldots,u_m)]\}\\ &= \{u_0 : u_0 \in M, \varphi^M(u_0,b_0,\ldots,b_{m-1})\}\end{aligned}$$ □

For any set X, let $\mathrm{cl}(X)$ be the closure of X under Gödel functions.

Corollary 13.15. *Let M be a transitive set. Then* $\mathrm{def}(M) \subseteq \mathrm{cl}(M \cup \{M\}) \cap \mathscr{P}(M)$.

Proof. Suppose that $X \in \mathrm{def}(M)$. Say $X = \{a \in M : \overline{M} \models \varphi(a,\overline{b})\}$ with $\overline{b} \in M$. By Proposition 14, let G be a composition of Gödel functions such that $\{a \in M : \overline{M} \models \varphi(a,\overline{b})\} = G(M,\overline{b})$. Thus $X \in \mathrm{cl}(M \cup \{M\}) \cap \mathscr{P}(M)$. □

Lemma 13.16. *If H is a composition of Gödel functions, then there is a formula ψ such that if M is a transitive set, $\overline{b} \in M$, and $H(M,\overline{b}) \subseteq M$, then $H(M,\overline{b}) = \{a \in M : \overline{M} \models \psi(a,\overline{b})\}$.*

Proof. Let φ be a Δ_0 formula such that $Y \in H(\overline{X})$ iff $\varphi(Y,\overline{X})$. Then $x \in H(M,\overline{b})$ iff $\varphi(x,M,\overline{b})$. Hence if $X = H(M,\overline{b})$ then $X = \{a \in M : \overline{M} \models \varphi(a,M,\overline{b})\}$. Then let ψ be obtained from φ by replacing

$v_i \in M$ with $v_i = v_i$,
$M \in M$ with $\neg(v_0 = v_0)$,
$M \in v_i$ with $\neg(v_0 = v_0)$.
$\exists v_i \in M$ with $\exists v_i$
$\forall v_i \in M$ with $\forall v_i$.

Then the desired conclusion follows. □

Lemma 13.17. *For any transitive set M, $\mathrm{def}(M) = \{X \subseteq M :$ there is a $\overline{b} \in M$ and a composition H of Gödel functions such that $X = H(M, \overline{b})$.*

Proof. By Proposition 14 and Lemma 16. □

An *inner model* of ZF is a transitive class model of ZF which contains all ordinals.

Lemma 13.18. (Δ_0-comprehension) *If M is a transitive class closed under the Gödel operations, $\varphi(v, \overline{w})$ is a Δ_0-formula, and $a, \overline{b} \in M$, then*

$$Y \stackrel{\text{def}}{=} \{c \in a : \varphi(c, \overline{b})\} \in M.$$

Proof. See the proof of Proposition 14. □

Lemma 13.19. *If $\varphi(u, v, w_0, \ldots, w_{n-1})$ is a formula with free variables among those mentioned, then*

$$ZFC \models \forall X \forall \overline{p} \exists Y \forall u \in X[\exists v \varphi(u, v, p_0, \ldots, p_{n-1}) \to \exists v \in Y \varphi(u, v, p_0, \ldots, p_{n-1})].$$

Proof. Let X be given. Let $X' = \{u \in X : \exists v \varphi(u, v, p_0, \ldots, p_{n-1})\}$. For each $u \in X$ let α_u be minimum such that there is a $v \in V_{\alpha_u}$ such that $\varphi(u, v, p_0, \ldots, p_{n-1})$. Let $\beta = \bigcup_{u \in X'} \alpha_u$. Set $Y = V_\beta$. Take any $u \in X$. If $\neg\exists v \varphi(u, v, p_0, \ldots, p_{n-1})$, then the desired implication holds. If $\exists v \varphi(u, v, p_0, \ldots, p_{n-1})$, then $u \in X'$ and there is a $v \in V_{\alpha_u} \subseteq V_\beta = Y$ such that $\varphi(u, v, p_0, \ldots, p_{n-1})$. □

Lemma 13.20. (Comprehension) *If M is a transitive class closed under the Gödel operations, $\varphi(v, \overline{w})$ is a formula, and $a, \overline{b} \in M$, then*

$$Y \stackrel{\text{def}}{=} \{c \in a : \varphi(c, \overline{b})\} \in M.$$

Proof. First we claim:

(1) If $\varphi(u, v, w_0, \ldots, w_{n-1})$ is a formula with free variables among those mentioned, then

$$\forall X, \overline{p} \in M \exists Y \in M \forall u \in X[(\exists v \varphi(u, v, p_0, \ldots, p_{n-1}))^M \to \exists v \in Y(\varphi(u, v, p_0, \ldots, p_{n-1}))^M].$$

To prove this, we apply Lemma 19 to the formula $v \in M \wedge (\varphi(u, v, p_0, \ldots, p_{n-1}))^M$. This gives for any X and $\overline{p}$ in M a Z such that

$$\begin{aligned}
&\forall u \in X[\exists v[v \in M \wedge (\varphi(u, v, p_0, \ldots, p_{n-1}))^M] \to \\
&\qquad \exists v \in Z[v \in M \wedge \varphi(u, v, p_0, \ldots, p_{n-1}))^M]]; \quad \text{hence} \\
&\forall u \in X[\exists v[v \in M \wedge (\varphi(u, v, p_0, \ldots, p_{n-1}))^M \to \\
&\qquad \exists v \in Z \cap M \wedge \varphi(u, v, p_0, \ldots, p_{n-1}))^M]].
\end{aligned}$$

Choose $Y \in M$ such that $Z \cap M \subseteq Y$. This proves (1).

Now we want to show that if φ is a formula with free variables among $x, z, w_1, \ldots, w_n$, then

$$(\forall z \forall w_1 \ldots \forall w_n \exists y \forall x(x \in y \leftrightarrow x \in z \wedge \varphi))^M.$$

Let $z, w_1, \ldots, w_n \in M$. Suppose that φ has k subformulas of the form $\exists x\psi$ or $\forall x\psi$, and let $\langle \chi_i : i < k \rangle$ list all of them, so that if χ_i is a subformula of χ_j then $i \leq j$. Let $Y_0, \ldots, Y_{k-1}$ be new variables. We define $\overline{\psi}$ for a subformula of φ by recursion. If no χ_i occurs in ψ, then $\overline{\psi} = \psi$. If ψ is χ_i, then

$$\overline{\psi} = \begin{cases} \exists v \in Y_i \overline{\psi'} & \text{if } \chi_i = \exists v \psi', \\ \forall v \in Y_i \overline{\psi'} & \text{if } \chi_i = \forall v \psi'. \end{cases}$$

Further, if some χ_i occurs in ψ, then

$$\overline{\psi} = \begin{cases} \neg\overline{\psi'} & \text{if } \psi = \neg\psi', \\ \overline{\psi'} \wedge \overline{\psi''} & \text{if } \psi = \psi' \wedge \psi''. \end{cases}$$

Now we claim

(2)

$$\exists Y_0, \ldots, Y_{k-1} \in M \forall x \in z[(\varphi(x, z, w_1, \ldots, w_n))^M \leftrightarrow \overline{\varphi}(x, z, w_1, \ldots, w_n, Y_0, \ldots, Y_{k-1})].$$

We prove (2) by induction on k. It is clear for $k = 0$. Now assume it for $k - 1$.

Case 1. χ_0 is $\exists v\psi$. Applying (1) to χ_0 we get

$$\exists Y \in M \forall u \in z[\exists v \in M \chi_i \leftrightarrow \exists v \in Y \chi_i]$$

Together with the inductive hypothesis, this gives (2).

Case 9. χ_0 is $\forall v\psi$. Applying (1) to $\exists v \neg\psi$,

$$\exists Y \in M \forall u \in z[\exists v \in M \neg\psi \leftrightarrow \exists v \in Y \neg\psi];$$

hence

$$\exists Y \in M \forall u \in z[\forall v \in M \psi \leftrightarrow \forall v \in Y \psi];$$

Together with the inductive hypothesis, this gives (2).

Now since $\overline{\varphi}$ is Δ_0, choose $y \in M$ such that

$$\forall x[x \in y \leftrightarrow x \in z \wedge \overline{\varphi}(x, z, w_1, \ldots, w_n, Y_0, \ldots, Y_{k-1})$$

Then by (2),

$$\forall x[x \in y \leftrightarrow x \in z \wedge (\varphi(x, z, w_1, \ldots, w_n)^M. \qquad \square$$

Lemma 13.21. *Suppose that* $\mathbf{M}$ *is a transitive class, and for every formula* φ *with free variables among* $x, y, A, w_1, \ldots, w_n$ *and for any* $A, w_1, \ldots, w_n \in \mathbf{M}$ *the following implication holds:*

$$\begin{aligned}&\forall x \in A \exists! y[y \in \mathbf{M} \wedge \varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n)] \quad \textit{implies that}\\ &\exists Y \in \mathbf{M}[\{y \in \mathbf{M} : \exists x \in A \varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n)\} \subseteq Y]].\end{aligned}$$

Then the replacement axioms hold in $\mathbf{M}$*.*

Proof. Assume the hypothesis of the theorem. We write out the relativized version of an instance of the replacement axiom, remembering to replace the quantifier $\exists!$ by its definition:

$$\begin{aligned}&\forall A \in \mathbf{M} \forall w_1 \in \mathbf{M} \ldots \forall w_n \in \mathbf{M}\\ &[\forall x \in \mathbf{M}[x \in A \to \exists y \in \mathbf{M}[\varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n) \wedge \forall u \in \mathbf{M}\\ &\qquad [\varphi^{\mathbf{M}}(x, u, A, w_1, \ldots, w_n) \to y = u]]] \to\\ &\exists Y \in \mathbf{M} \forall x \in \mathbf{M}[x \in A \to \exists y \in \mathbf{M}[y \in Y \wedge \varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n)]]]].\end{aligned}$$

To prove this, assume that $A, w_1, \ldots, w_n \in \mathbf{M}$ and

$$\begin{aligned}&\forall x \in \mathbf{M}[x \in A \to \exists y \in \mathbf{M}[\varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n) \wedge \forall u \in \mathbf{M}\\ &\qquad [\varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n) \to y = u]]].\end{aligned}$$

Since $\mathbf{M}$ is transitive, we get

$$\forall x \in A \exists y \in \mathbf{M}[\varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n) \wedge \forall u \in \mathbf{M}[\varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n) \to y = u]],$$

so that

$$\forall x \in A \exists! y[y \in \mathbf{M} \wedge \varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n)]. \tag{1}$$

Hence by the hypothesis of the theorem we get $Y \in \mathbf{M}$ such that

$$\{y \in \mathbf{M} : \exists x \in A \varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n)\} \subseteq Y. \tag{2}$$

Suppose that $x \in \mathbf{M}$ and $x \in A$. By (1) we get $y \in \mathbf{M}$ such that $\varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n)$. Hence by (2) we get $y \in Y$, as desired. $\square$

Theorem 13.22. (Theorem 13.9) *A transitive class* M *is an inner model of ZF iff it is closed under the Gödel functions and is* **almost universal***, i.e. for every set* $X \subseteq M \exists Y \in M[X \subseteq Y]$*.*

Proof. $\Rightarrow$: Let M be a transitive class which is an inner model of ZF. Let H be a Gödel function. For any $\overline{X} \in M$ there is a $Z \in M$ such that $Z = H^M(\overline{X})$. By Theorem 13(iii), $Z = H(\overline{X})$. So M is closed under H. Next, if X is a set $\subseteq M$, choose an ordinal α such that $X \subseteq V_\alpha^M$. Note that $V_\alpha^M \in M$.

$\Leftarrow$: Assume that M is closed under the Gödel functions and is almost universal. By Theorem 20, comprehension holds in M. Next we show that every ordinal is in M, by induction. Let $a \in M$. Then $\emptyset = G_4(a,a)$, so $\emptyset \in M$. Suppose that α is an ordinal and $\alpha \in M$. Then $\{\alpha\} = G_1(\alpha,\alpha) \in M$, $\{\alpha,\{\alpha\}\} = G_1(\alpha,\{\alpha\}) \in M$, and $\alpha \cup \{\alpha\} = G_6(\{\alpha,\{\alpha\}\}) \in M$. Finally, suppose that α is limit and $\alpha \subseteq M$. Choose $Y \in M$ such that $\alpha \subseteq Y$. By comprehension in M, there is a $z \in M$ such that for all $a \in M$, $a \in z$ iff $a \in Y$ and a is an ordinal. Then $\bigcup z \in M$. Clearly z is an ordinal and $\alpha \le z$, so $\alpha \in M$.

Now we check the axioms except for comprehension. Extensionality and foundation hold since M is transitive.

Pairing: Suppose that $a, b \in M$. Then $\{a,b\} \subseteq M$, so there is a $z \in M$ such that $\{a,b\} \subseteq z$. So $a, b \in z$.

Union: Suppose that $\mathscr{A} \in M$. Then $\bigcup \mathscr{A} \in M$. If $x \in Y \in \mathscr{A}$ then $x \in \bigcup \mathscr{A}$.

Power set: Suppose that $a \in M$. Then $\mathscr{P}(a) \cap M \subseteq M$, so there is a $b \in M$ such that $\mathscr{P}(a) \cap M \subseteq b$. If $z \in M$ and $z \subseteq a$, then $z \in \mathscr{P}(a) \cap M$, so $z \in b$.

Infinity: Since every ordinal is in M, in particular $\omega \in M$. By absoluteness, the infinity axiom holds.

Replacement: We use Lemma 21. Assume that the hypothesis of Lemma 21 holds. By replacement in the real world, choose Z such that

$$(*) \qquad \forall x[x \in A \to \exists z[z \in Z \text{ and } z \in M \text{ and } \varphi^M(x,z,A,w_1,\ldots,w_n)]]$$

Then let $W = \{y \in Z : y \in M \text{ and } \exists x \in A\varphi^M(x,y,A,w_1,\ldots,w_n)\}$. Then $W \subseteq M$, so there is a $Y \in M$ such that $W \subseteq Y$. Now suppose that $y \in M$, $x \in A$, and $\varphi^M(x,y,A,z_1,\ldots,z_n)$. Choose $z \in Z$ such that $z \in M$ and $\varphi^M(x,z,A,w_1,\ldots,w_n)$, by $(*)$. By the uniqueness condition in the hypothesis of Theorem 14.6, $y = z$. Hence $y \in W$, so $y \in Y$. It follows that $\{y \in \mathbf{M} : \exists x \in A\varphi^{\mathbf{M}}(x,y,A,w_1,\ldots,w_n)\} \subseteq Y$, as desired. $\square$

A formula is Σ_0 and Π_0 iff all its quantifiers are bounded, i.e., it is Δ_0. Then

φ is Σ_{n+1} iff it is equivalent under ZF to a formula $\exists x\psi$ with ψ Π_n

φ is Π_{n+1} iff it is equivalent under ZF to a formula $\forall x\psi$ with ψ Σ_n.

φ is Δ_n iff it is both Σ_n and Π_n.

Lemma 13.23. (Lemma 13.10) *Let $n \ge 1$. Let $\varphi = \varphi(x,\overline{y})$ and $\psi = \psi(x,\overline{y})$.*

(i) If φ and ψ are Σ_n, then so are $\exists x\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\exists u \in x\varphi$, and $\forall u \in x\varphi$.

(ii) If φ is Σ_n, then $\neg\varphi$ is Π_n.

(iii) If φ is Π_n, then $\neg\varphi$ is Σ_n.

(iv) If φ and ψ are Π_n, then so are $\forall x\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\exists u \in x\varphi$, and $\forall u \in x\varphi$.

(v) If φ is Π_n and ψ is Σ_n, then $\varphi \to \psi$ is Σ_n.

(vi) If φ is Σ_n and ψ is Π_n is Π_n, then $\varphi \to \psi$ is Π_n.

(vii) If φ and ψ are Δ_n, then so are $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \to \psi$, $\varphi \leftrightarrow \psi$, $\forall u \in x\varphi$, and $\exists u \in x\varphi$.

Proof. We go by induction on n. First take $n = 1$. For (i), say

$$ZF \models \varphi(x, \overline{y}) \leftrightarrow \exists z \varphi'(z, x, \overline{y});$$
$$ZF \models \psi(x, \overline{y}) \leftrightarrow \exists z \psi'(z, x, \overline{y}),$$

with φ', ψ' Δ_0. Then

$$ZF \models \exists x \varphi(x, \overline{y}) \leftrightarrow \exists x \exists z \varphi'(z, x, \overline{y});$$
$$ZF \models \exists x \varphi(x, \overline{y}) \leftrightarrow \exists v \exists w \in v \exists x \in w \exists z \in w[v = (x, z) \wedge \varphi'(z, x, \overline{y})].$$

In the inductive step we assume that φ' and ψ' are Π_{n-1}, and then by (ii) so is

$$\exists w \in v \exists x \in w \exists z \in w[v = (x, z) \wedge \varphi'(z, x, \overline{y})].$$

Next,

$$\varphi(x, \overline{y}) \wedge \psi(x, \overline{y}) \leftrightarrow \exists z \exists u[\varphi'(z, x, \overline{y}) \wedge \psi'(u, x, \overline{y})]$$
$$\varphi(x, \overline{y}) \vee \psi(x, \overline{y}) \leftrightarrow \exists z \exists u[\varphi'(z, x, \overline{y}) \vee \psi'(u, x, \overline{y})]$$
$$\exists u \in x \varphi(x, \overline{y}) \leftrightarrow \exists z \exists u[u \in x \wedge \varphi'(z, u, \overline{y})]$$

Here the two quantifiers can be collapsed to one as above.

For $\forall u \in x \varphi$ we use the collection principle:

$$\begin{aligned} \forall u \in x \varphi &\leftrightarrow \forall u \in x \exists z \varphi'(z, x, \overline{y}) \\ &\leftrightarrow \exists w \forall u \in x \exists z \in w \varphi'(z, x, \overline{y}). \end{aligned}$$

The inductive step for these formulas is clear.

For (ii) and (iii) we go by induction on n. They are clear for $n = 0$. Now assume them for n. Suppose that $\varphi(x, \overline{y})$ is Σ_{n+1}. Say $ZF \models \varphi \leftrightarrow \exists z \varphi'(z, x, \overline{y})$ with φ' Π_n. Then $ZF \models \neg\varphi \leftrightarrow \forall z \neg\varphi'(z, x, \overline{y})$. By the inductive hypothesis, $\neg\varphi'(z, x, \overline{y})$ is equivalent under ZF to a Σ_n formula. Hence $\neg\varphi$ is equivalent under ZF to a Π_{n+1} formula. This proves (ii) for $n + 1$. (iii) is proved similarly.

(iv): Suppose that φ and ψ are Π_n. Then by (iii), $\neg\varphi$ and $\neg\psi$ are Σ_n. Hence by (1), so are $\exists x \neg\varphi$, $\neg\varphi \wedge \neg\psi$, $\neg\varphi \vee \neg\psi$, $\exists u \in x \neg\varphi$, and $\forall u \in x \neg\varphi$. Hence by (ii) the following are Π_n: $\neg\exists x \neg\varphi$, $\neg(\neg\varphi \wedge \neg\psi)$, $\neg(\neg\varphi \vee \neg\psi)$, $\neg\exists u \in x \neg\varphi$, and $\neg\forall u \in x \neg\varphi$. Simple logical equivalences then give (iv).

(v): $\varphi \rightarrow \psi$ is equivalent under ZF to $\neg\varphi \vee \psi$; now use (ii) and (iv).

(vi): similarly

(vii): By (i)–(vi). □

A function F is Σ_n iff the formula $y = F(\overline{x})$ is Σ_n; F is Π_n iff the formula $y = F(\overline{x})$ is Π_n.

Lemma 13.24. *(i) If F is a Σ_n function, then* dmn(F) *is Σ_n.*

(ii) If F is a Σ_n function and dmn(F) *is Δ_n, then F is Δ_n.*

(iii) If F and G are Σ_n functions of one variable, then so is $F \circ G$.
(iv) If F is a Σ_n function of one variable and $\varphi(x, \overline{y})$ is Σ_n, then $\varphi(F(x), \overline{y})$ is Σ_n.

Proof. (i): $\overline{x} \in \mathrm{dmn}(F) \leftrightarrow \exists y[y = F(\overline{x})]$.
(ii): $y = F(\overline{x}) \leftrightarrow \overline{x} \in \mathrm{dmn}(F) \wedge \forall z[z = F(\overline{x}) \to y = z]$. Now by (i), $\overline{x} \in \mathrm{dmn}(F)$ is Δ_n. $z = F(\overline{x})$ is Δ_n, and $y = z$ is Δ_n. Hence $z = F(\overline{x}) \to y = z$ is Δ_n. Hence by Lemma 23(i),(iv), $\forall z[z = F(\overline{x}) \to y = z]$ is Δ_n.
(iii) $y = F(G(x)) \leftrightarrow \exists z[z = G(x) \wedge y = F(z)]$.
(iv): $ZF \models \varphi(F(x), \overline{y}) \leftrightarrow \exists z[z = F(x) \wedge \varphi(z, \overline{y})]$. □

Lemma 13.25. *"$E \subseteq P \times P$" is Δ_0.*

$$E \subseteq P \times P \leftrightarrow \forall a \in E \exists b, c \in P[a = (b, c)]$$ □

Lemma 13.26. *The following formula $\varphi(E, P, X)$ is Δ_0:*

$$(E \subseteq P \times P) \wedge [X \subseteq P \wedge X \neq \emptyset \to \exists a \in X \forall b \in X[(b, a) \notin E]].$$

Proof. Only the last part of the formula raises a question. We have

$$(b, a) \notin E \leftrightarrow \forall c \in E[c \neq (b, a)].$$ □

Lemma 13.27. *"E is a well-founded relation on P" is Π_1.*

Proof. "E is a well-founded relation on P" iff $E \subseteq P \times P$ and $\forall X \varphi(E, P, X)$. □

Theorem 13.28. *If $E \subseteq P \times P$, then E is well-founded iff there is an $f : P \to$ **ON** such that $\forall a, b \in P[aEb \to f(a) < f(b)]$.*

Proof. $\Rightarrow$: Define $G : A \times V \to V$ as follows. For any $a \in A$ and $f \in V$,

$$G(a, f) = \begin{cases} \bigcup\{f(b) \cup \{f(b)\} : bRa\} & \text{if } f \text{ is a function with domain } \mathrm{pred}_{AR}(a), \\ \emptyset & \text{othewise.} \end{cases}$$

Applying the recursion theorem we obtain $F : A \to V$ such that for all $a \in A$,

$$F(a) = G(a, F \restriction \mathrm{pred}_{AR}(a)) = \bigcup\{F(b) \cup \{F(b)\} : bRa\}.$$

(1) $\forall a \in P[F(a)$ is an ordinal$]$.

Suppose not, and let a be E-minimal such that $F(a)$ is not an ordinal. Then $\forall b[bRa \to F(b)$ is an ordinal$]$, and hence $F(a) = \bigcup\{F(b) \cup \{F(b)\} : bEa\}$ is an ordinal, contradiction.
Clearly aEb implies that $F(a) \in F(b)$.
$\Leftarrow$: Suppose that such an f exists. Let $\emptyset \neq X \subseteq P$. Choose $x \in X$ with $f(x)$ minimum. Clearly there is no y such that yEx. □

Lemma 13.29. *"E is a well-founded relation on P" is Σ_1.*

Proof. "E is a well-founded relation on P" iff $E \subseteq P \times P$ and $\exists f[f$ is a function and $\mathrm{dmn}(f) = P$ and $\forall a \in P[f(a)$ is an ordinal] and for all $a, b \in P[(a,b) \in E \to f(a) \in f(b)]]$. We need to see that the formula within the outer brackets here is Δ_0. For "f is a function" and "$\mathrm{dmn}(f) = P$" see Lemma 19.10. For "$f(a)$ is an ordinal", see the proof off Lemma 19.10. "$(a,b) \in E$" is equivalent to "$\exists c \in E[c = (a,b)]$". Finally, "$f(a) \in f(b)$" is equivalent to

$$\exists u, v \in f \exists s, t \in u \exists s', t' \in v[u = \{s,t\} \wedge v = \{s',t'\} \wedge s = \{a\} \wedge s' = \{b\}$$
$$\wedge \exists p \in t \exists q \in t'[t = \{a,p\} \wedge t' = \{b,q\} \wedge p \in q]]. \qquad \square$$

Theorem 13.30. (Lemma 13.11) *"E is a well-founded relation on P" is* Δ_1. $\square$

Lemma 13.31. *If G is a Δ_0 function, then "$z \in G$" is a Δ_0 formula.*

Proof.

$$ZF \models z \in G \leftrightarrow \exists u, v \in z[\forall a, b \in u[a = b] \wedge$$
$$\exists c, d \in v[\forall e \in v[c = a \vee c = b] \wedge z = (c,d) \wedge d = G(c)]] \qquad \square$$

Theorem 13.32. (Lemma 13.12) *Suppose that $G : V \to V$ is absolute for every model of ZF. We define F by recursion:*

$$F(\alpha) = G(F \restriction \alpha) \quad \textit{for every ordinal } \alpha.$$

Then F is absolute for every model transitive M of ZF.

Proof. Within M define $H(\alpha) = G(H \restriction \alpha)$ for every ordinal α. It suffices to prove that $H = F$. Suppose not, and let α be minimum such that $F(\alpha) \neq H(\alpha)$. Then $F \restriction \alpha = H \restriction \alpha$, so $F(\alpha) = H(\alpha)$, contradiction. $\square$

Theorem 13.33. (Lemma 13.13) *For any ordinal α, "α is a cardinal" is* Π_1.

Proof.

$$\alpha \text{ is a cardinal } \leftrightarrow \forall f[f \text{ is a function } \wedge \mathrm{dmn}(f) \in \alpha \to \mathrm{rng}(f) \neq \alpha$$
$$\leftrightarrow \forall f[f \text{ is a function } \wedge \exists x \in \alpha[x = \mathrm{dmn}(f)] \to \mathrm{rng}(f) \neq \alpha]. \qquad \square$$

Lemma 13.34. *"*$\mathrm{rng}(f) \subseteq a$*" is* Δ_0.

Proof.

$$\mathrm{rng}(f) \subseteq x \leftrightarrow \forall a \in f \exists u, v \in a[\forall b, c \in u[b = c] \wedge \exists b, c \in v[a = (b,c) \wedge c \in x. \qquad \square$$

Theorem 13.35. (Lemma 13.13) *For any ordinal α, "α is a regular cardinal" is* Π_1.

Proof.

$$\begin{aligned}\alpha \text{ is a regular cardinal } \leftrightarrow\ & \alpha \text{ is a cardinal } \wedge \exists x \in \alpha[x = x]\\ & \wedge \forall f[f \text{ is a function } \wedge \exists x \in \alpha[x = \mathrm{dmn}(f)] \wedge \mathrm{rng}(f) \subseteq \alpha\\ & \rightarrow \exists \beta < \alpha[\mathrm{rng}(f) \subseteq \beta]]\end{aligned}$$

Theorem 13.36. (Lemma 13.13) *For any ordinal α, "α is a limit cardinal" is Π_1.*

Proof.

$$\alpha \text{ is a limit cardinal } \leftrightarrow \forall \beta \in \alpha \exists \gamma < \alpha[\beta < \gamma \wedge \gamma \text{ is a cardinal}] \qquad \square$$

Now we define

$$\begin{aligned}\models_0 \varphi[\overline{a}] \quad &\text{iff} \quad \varphi \in \mathrm{Fmla}, \varphi \text{ is } \Delta_0, \text{ and there is an } M \text{ such that } (M, \in) \models \varphi[\overline{a}]\\ \models_{n+1} \exists x\varphi[x, \overline{a}] \quad &\text{iff} \quad \varphi \in \mathrm{Fmla}, \varphi \text{ is } \Pi_n, \text{ and } \exists b(\text{not} \models_n \neg\varphi[b, \overline{a}]).\end{aligned}$$

If $M \subseteq N$, then we define $(M, \in) \prec_{\Sigma_n} (N, \in)$ iff for every Σ_n $\varphi \in$ Fmla and all $\overline{a} \in M$, $\models_n^M \varphi[\overline{a}]$ iff $\models^N \varphi[\overline{a}]$.

Lemma 13.37. *"H is an s-place composition of Gödel functions" is Δ_1.*

Proof. First, it is Σ_1:

$$\begin{aligned}&H \text{ is a } s\text{-place composition of Gödel functions } \leftrightarrow \exists f, f'\Bigg[f \text{ and } f' \text{ are functions}\\ &\wedge \mathrm{dmn}(f) \in \omega \wedge \mathrm{dmn}(f') = \mathrm{dmn}(f) \wedge \Bigg[\bigvee_{i=1}^{5}[f(0) = G_i \wedge f'(0) = 2]\\ &\vee \bigvee_{i=6}^{10}[f(0) = G_i \wedge f'(0) = 1] \vee \exists m \in \omega \exists i < m[f(0) = P_i^m \wedge f'(0) = m]\Bigg]\\ &\wedge \forall k \in \mathrm{dmn}(f)\Bigg[k \neq 0 \rightarrow \Bigg[\bigvee_{i=1}^{5}[f(0) = G_i \wedge f'(0) = 2] \vee \bigvee_{i=6}^{10}[f(0) = G_i \wedge f'(0) = 1]\\ &\vee \exists m \in \omega \exists i < m[f(0) = P_i^m \wedge f'(0) = m]]\\ &\vee \exists m, n \in \omega \exists i < k[f'(i) = m \wedge \exists j[j \text{ is a function } \wedge \mathrm{dmn}(j) = m \wedge \mathrm{rng}(j) \subseteq k\\ &\wedge \forall s < m[f'(j(s)) = n] \wedge f(k) = C_n^m(f(i), f(j_0), \ldots, f(j_{m-1})) \wedge f'(k) = n]\Bigg]\Bigg]\\ &\wedge \exists i \in \mathrm{dmn}(f)[H = f(i) \wedge f'(i) = s].\end{aligned}$$

Second, it is Π_1:

$$H \text{ is an } s\text{-place composition of Gödel functions } \leftrightarrow \forall Y \Bigg[\forall m, i \in \omega[i < m \to P_i^m \in Y] \wedge$$
$$\bigwedge_{i=1}^{5} \forall a, b \in Y[G_i(a,b) \in Y] \wedge \bigwedge_{i=6}^{10} \forall a \in Y[G_i(a) \in Y] \to H \in Y \Bigg] \wedge H \text{ is } s\text{-place} .$$

Lemma 13.38. *Define $G : V \to V$ as follows.*

$$\begin{aligned} G(x) = \{X \subseteq x(\beta) : \exists m \in \omega \exists \overline{b} \in x(\beta) & \textit{ of length } m \textit{ and there is an } (m+1)\textit{-ary} \\ & \textit{composition } H \textit{ of Gödel functions such that } X = H(x(\beta), \overline{b}) \\ & \textit{if } x \textit{ is a function with domain an ordinal } \beta + 1 \\ & = \bigcup x(\gamma) \textit{ if } x \textit{ is a function with domain a limit ordinal } \gamma \\ & = \emptyset \textit{ otherwise.} \end{aligned}$$

Then $y = G(x)$ is absolute for every transitive model of ZF.

Proof. Clearly $y = G(x)$ is Σ_1, so it is absolute upwards. Now suppose that $y = G(x)$, and M is a transitive model of ZF. We claim that $G^M = G$. For, if $x \in M$, then

$$\begin{aligned} X \in G^M(x) \leftrightarrow \exists m \in \omega \exists \overline{b} \in x(\beta) & \text{ of length } m \text{ and there is an } (m+1)\text{-ary} \\ & \text{composition } H \text{ of Gödel functions such that } X = H(x(\beta), \overline{b}) \\ & \text{if } x \text{ is a function with domain an ordinal } \beta + 1 \\ & = \bigcup x(\gamma) \text{ if } x \text{ is a function with domain a limit ordinal } \gamma \\ & = \emptyset \text{ otherwise.} \end{aligned}$$

Since "$(m+1)$-ary composition H of Gödel functions" is Δ_1, this holds in V, as desired. □

Theorem 13.39. (Lemma 13.14) *The function L is absolute for transitive models of ZF.*

Proof. By Lemma 17, Theorem 32, and Lemma 38. □

Theorem 13.40. *$L \models (V = L)$.*

Proof.

$$L \models (V = L) \leftrightarrow (\forall x \exists \alpha [x \in L_\alpha))^L \leftrightarrow \forall x \in L \exists \alpha (x \in L_\alpha) \leftrightarrow T. \qquad \square$$

Theorem 13.41. (Theorem 13.16, minimality) *If M is an inner model of ZF, then $L \subseteq M$.*

Proof. In M we construct L^M.

$$\forall x \in M[x \in L^M \leftrightarrow \exists\alpha[x \in L^M_\alpha] \leftrightarrow \exists\alpha[x \in L_\alpha];$$

so $L_\alpha \subseteq M$, hence $L \subseteq M$. □

Theorem 13.42. (Condensation) *For every limit ordinal α, if $M \preceq L_\alpha$ and N is the transitive collapse of M, then there is a limit ordinal $\beta \le \alpha$ such that $N = L_\beta$.*

Proof. (Following notes of Zilber.) First we claim that M is extensional. For, suppose that $x, y \in M$ with $x \neq y$; say $x \backslash y \neq \emptyset$. Thus $\exists a \in x[a \notin y]$. Since $x, y \in L_\alpha$, by absoluteness $L_\alpha \models \exists a \in x[a \notin y]$, so $M \models \exists a \in x[a \notin y]$. This shows that M is extensional. Hence the Mostowski collapse function $\pi : M \to N$ is an isomorphism.

Now let $\beta = N \cap \mathbf{ON}$. Since N is transitive, β is an ordinal.

(1) For any ordinal γ, if $\Gamma \subseteq \gamma$, then o.t.$(\Gamma) \le \gamma$.

For, let $f : \delta \to \Gamma$ be the strictly increasing enumeration of Γ, with $\delta =$ o.t.(Γ). Then for any $\xi < \delta$ we have $\xi \le f(\xi) < \gamma$, so $\delta \le \gamma$.

(2) $\beta \le \alpha$.

For, suppose that $\alpha < \beta$. Then $\alpha \in N$, and so $\pi^{-1}(\alpha) \in M$. Now α is an ordinal, so $N \models$ (α is an ordinal); hence $M \models$ ($\pi^{-1}(\alpha)$ is an ordinal), and so $L_\alpha \models$ ($\pi^{-1}(\alpha)$ is an ordinal), hence $\pi^{-1}(\alpha)$ is an ordinal. Now $\pi \restriction (\pi^{-1}(\alpha) \cap M)$ is an isomorphism onto α. It follows that o.t.$(\pi^{-1}(\alpha) \cap M) = \alpha$. But by (1), o.t.$(\pi^{-1}(\alpha) \cap M) \le \pi^{-1}(\alpha)$. Since $\pi^{-1}(\alpha) \in M \subseteq L_\alpha$, we have $\pi^{-1}(\alpha) < \alpha$, contradiction.

(3) $0 < \beta$.

For, $L_\alpha \models \exists x \forall y \in x[y \neq y]$, so $M \models \exists x \forall y \in x[y \neq y]$, hence $N \models \exists x \forall y \in x[y \neq y]$. Hence (3) holds.

(4) β is a limit ordinal.

For, suppose that $\beta = \gamma \cup \{\gamma\}$. Then $N \models \exists x[x$ is an ordinal and $\forall y[y \neq x \cup \{x\}]$. Hence $L_\alpha \models \exists x[x$ is an ordinal and $\forall y[y \neq x \cup \{x\}]$, contradiction.

(5) $L_\beta \subseteq N$.

For, $L_\alpha \models \forall \delta \in \mathbf{ON} \exists y[y = L_\delta]$. It follows that $N \models \forall \delta \in \mathbf{ON} \exists y[y = L_\delta]$. By absoluteness, $\forall \delta \in \mathbf{ON} \cap N[L_\delta \in N]$. Hence (5) holds.

(6) $N \subseteq L_\beta$.

For, $L_\alpha \models \forall x \exists y \exists z[y$ is an ordinal and $z = L_y \wedge x \in z]$. Hence $N \models \forall x \exists y \exists z[y$ is an ordinal and $z = L_y \wedge x \in z]$. Now take any $a \in N$. Choose an ordinal $\gamma \in N$ and $z \in N$ such that $z = L_\gamma$ and $x \in z$. (Using the absoluteness of L_γ.) This proves (6). □

Theorem 13.42′. *For every limit ordinal α, if $M \equiv_{ee} L_\alpha$ and M is transitive, then there is a limit ordinal $\beta \le \alpha$ such that $N = L_\beta$.*

Proof. First we claim that M is extensional. For,

$$\begin{aligned} L_\alpha &\models \forall x, y[\forall z[z \in x \leftrightarrow z \in y] \to x = y], \quad \text{so} \\ M &\models \forall x, y[\forall z[z \in x \leftrightarrow z \in y] \to x = y]. \end{aligned}$$

So, suppose that $x.y \in M$ and $\forall z \in M[z \in x \leftrightarrow z \in y]$. Since M is transitive, $\forall z[z \in x \leftrightarrow z \in y]$, hence $x = y$.

Now let $\beta = M \cap \mathbf{ON}$. Since M is transitive, β is an ordinal, and $\beta \subseteq M$.

(1) $0 < \beta$.

For, $L_\alpha \models \exists x \forall y \in x[y \neq y]$, so $M \models \exists x \forall y \in x[y \neq y]$. Choose $x \in M$ so that $\forall y \in M \cap x[y \neq y]$. Thus $M \cap x = \emptyset$. Since M is transitive, $x = \emptyset$. So (1) holds.

(2) β is a limit ordinal.

For,

$$\begin{aligned} L_\alpha &\models \forall \gamma[\gamma \text{ is an ordinal} \to \exists \delta[\delta \text{ is an ordinal} \wedge [\gamma < \delta]]], \quad \text{so} \\ M &\models \forall \gamma[\gamma \text{ is an ordinal} \to \exists \delta[\delta \text{ is an ordinal} \wedge [\gamma < \delta]]]. \end{aligned}$$

Now let $\gamma < \beta$. Then $\gamma \in M$ and by absoluteness $M \models [\gamma \text{ is an ordinal}]$, so $\exists \delta \in M[M \models [\delta \text{ is an ordinal}] \wedge [\gamma < \delta]]$. Thus by absoluteness, $\delta \in M$ and $\gamma < \delta$, so (2) holds

(3) $L_\beta \subseteq N$.

For, $L_\alpha \models \forall \delta \in \mathbf{ON} \exists y[y = L_\delta]$. Hence $M \models \forall \delta \in \mathbf{ON} \exists y[y = L_\delta]$. So for every $\delta < \beta$ there is a $y \in M$ such that $M \models [y = L_\delta]$. By absoluteness, $y - L_\delta$. So (3) holds.

(4) $M \subseteq L_\beta$.

For, $L_\alpha \models \forall x \exists y \exists z[y \text{ is an ordinal and } z = L_y \wedge x \in z]$. Hence $M \models \forall x \exists y \exists z[y \text{ is an ordinal and } z = L_y \wedge x \in z]$. Now take any $a \in M$. Choose an ordinal $\gamma \in M$ and $z \in M$ such that $M \models [z = L_\gamma]$ and $x \in z$. By absoluteness, $z = L_\gamma$. □

Lemma 13.43. $L(\alpha) = V_\alpha$ *for all* $\alpha \leq \omega$.

Proof. $L_n = V_n$ for all $n \in \omega$ by induction, using Proposition 1. $L_\omega = V_\omega$ by taking unions. □

Lemma 13.44. $|\text{def}(A)| = |A|$ *for all infinite* A. □

Theorem 13.45. $|L_\alpha| = |\alpha|$ *for all infinite* α.

Proof. Since $\alpha \subseteq L(\alpha)$ by Proposition 5, we have $|\alpha| \leq |L_\alpha|$. Now we prove $|L_\alpha| = |\alpha|$ for all infinite α by induction on α. It is true for $\alpha = \omega$ by Lemma 43. Now assume that $|L_\alpha| = |\alpha|$. Then $|L_{\alpha+1}| = |\text{def}(L_\alpha)| = |L_\alpha| = |\alpha|$. using Lemma 44. For α limit $> \omega$,

$$|L_\alpha| = \left| \bigcup_{\beta < \alpha} L_\beta \right| \leq \sum_{\beta < \alpha} |L_\beta| = \sum_{\omega \leq \beta < \alpha} |L_\beta| = \sum_{\omega \leq \beta < \alpha} |\beta| = |\alpha|. \qquad \square$$

Theorem 13.46. *If $V = L$, then $\forall\alpha[2^{\aleph_\alpha} = \aleph_{\alpha+1}]$.*

Proof. Assume $V = L$. Let $X \subseteq \omega_\alpha$. We show that there is a $\gamma < \omega_{\alpha+1}$ such that $X \in L_\gamma$. Hence $\mathscr{P}(\omega_\alpha) \subseteq L_{\omega_{\alpha+1}}$, and the theorem follows from Theorem 45.

There is a limit ordinal $\delta > \omega_\alpha$ such that $X \in L_\delta$. Let M be an elementary submodel of L_δ such that $\omega_\alpha \subseteq M$, $X \in M$, and $|M| = \aleph_\alpha$. By Theorem 42, if N is the transitive collapse of M then there is a limit ordinal $\gamma \leq \delta$ such that $N = L_\gamma$. Since $|N| = |M| = \aleph_\alpha$ we have $|L_\gamma| = \aleph_\alpha$. Now $\omega_\alpha \subseteq M$ and the collapsing map is the identity on ω_α. Hence the collapsing map fixes X. So $X \in L_\gamma$, as desired. □

$\diamondsuit$ is the statement that there exists a sequence $\langle A_\alpha : \alpha < \omega_1\rangle$ of sets with the following properties:

(i) $A_\alpha \subseteq \alpha$ for each $\alpha < \omega_1$.

(ii) For every subset A of ω_1, the set $\{\alpha < \omega_1 : A \cap \alpha = A_\alpha\}$ is stationary in ω_1.

Such a sequence is called a *$\diamondsuit$-sequence.*

Theorem 13.47. $V = L \to \diamondsuit$.

Proof. By recursion we define $\langle (S_\alpha, C_\alpha) : \alpha < \omega_1\rangle$ such that $S_\alpha \subseteq \alpha$ and C_α is closed unbounded in α. Let $S_0 = C_0 = \emptyset$ and $S_{\alpha+1} = C_{\alpha+1} = \alpha + 1$ for all $\alpha < \omega_1$. For $\alpha < \omega_1$ limit define

$$(S_\alpha, C_\alpha) = \begin{cases} <_L -\text{least } (S, C) \text{ such that} & \\ C \subseteq \alpha \text{ is club and} & \\ \forall\xi \in C[S \cap \xi \neq S_\xi] & \text{if there is such a } (S, c) \\ (\alpha, \alpha) & \text{otherwise.} \end{cases}$$

We claim that $\langle S_\alpha : \alpha < \omega_1\rangle$ is a $\diamondsuit$-sequence. Suppose not. Then there exist a subset A of ω_1 and a club C in ω_1 such that $C \cap \{\alpha < \omega_1 : A \cap \alpha = S_\alpha\} = \emptyset$. Let (A, C) be the $<_L$-least such pair. Thus (A, C) is $<_L$-least such that

$$(*) \qquad \forall\alpha \in C[A \cap \alpha \neq S_\alpha]$$

(1) $\langle (S_\alpha, C_\alpha) : \alpha < \omega_1\rangle \in L_{\omega_2}$.

In fact, let $g = \langle (S_\alpha, C_\alpha) : \alpha < \omega_1\rangle$, and let $X = \text{trcl}(\{g\})$. Then

$$\begin{aligned} X = \{g\} &\cup \{(\alpha, (S_\alpha, C_\alpha)) : \alpha < \omega_1\} \cup \{\{\alpha\} : \alpha < \omega_1\} \\ &\cup \{\{\alpha, (S_\alpha, C_\alpha)\} : \alpha < \omega_1\} \cup \omega_1 \cup \{(S_\alpha, C_\alpha) : \alpha < \omega_1\} \\ &\cup \{\{S_\alpha\} : \alpha < \omega_1\} \cup \{\{S_\alpha, C_\alpha\} : \alpha < \omega_1\} \\ &\cup \{S_\alpha : \alpha < \omega_1\} \cup \{C_\alpha : \alpha < \omega_1\}. \end{aligned}$$

Thus $|\text{trcl}(\{g\}| = \aleph_1$. We apply the argument in the proof of Theorem 46 and get a $\gamma < \omega_2$ such that $\pi(X) \in L_\gamma$, where π is the collapsing map. π fixes each S_α and C_α, so clearly $\pi(g) = g$. This proves (1).

Let N be a countable elementary submodel of $(L_{\omega_2}, \in)$. Since $\langle (S_\alpha, C_\alpha) : \alpha < \omega_1\rangle$ and (A, C) are definable in $(L_{\omega_2}, \in)$, they are in N. $\omega_1 \cap N$ is an initial segment of ω_1.

Let $\delta = \omega_1 \cap N$. By Theorem 42, the transitive collapse of N is L_γ for some limit ordinal $\gamma < \omega_1$. Let π be the transitive collapse function. Then $\pi(\omega_1) = \delta$, $\pi(A) = A \cap \delta$, $\pi(C) = C \cap \delta$, and $\pi(\langle(S_\alpha, C_\alpha) : \alpha < \omega_1) = \langle(S_\alpha, C_\alpha) : \alpha < \delta\rangle$. Hence

$$(L_\delta, \in) \models (A \cap \delta, C \cap \delta) \text{ is the least pair } (Z, D) \text{ such that}$$
$$Z \subseteq \delta, D \subseteq \delta, D \text{ is club in } \delta, \text{ and } \forall \xi \in D[Z \cap \xi \neq S_\xi]$$

By absoluteness this holds in V. Hence by definition, $A \cap \delta = S_\delta$. Since $C \cap \delta$ is club in δ, it follows that $\delta \in C$. This contradicts the definition of C. □

Now we define

$$\begin{aligned}
\mathrm{def}_A(M) &= \{X \subseteq M : X \text{ is definable over } (M, \in, A \cap M)\};\\
L_0[A] &= \emptyset;\\
L_{\alpha+1}[A] &= \mathrm{def}_A(L_\alpha[A]);\\
L_\gamma[A] &= \bigcup_{\alpha<\gamma} L_\alpha[A] \quad \text{for } \gamma \text{ limit};\\
L[A] &= \bigcup_{\alpha \in \mathbf{ON}} L_\alpha[A].
\end{aligned}$$

Proposition 13.48. $L_\alpha \subseteq L_\alpha[A] \subseteq V_\alpha$. □

Proposition 13.49. *For any ordinal α,*
(i) $L_\alpha[A]$ is transitive.
(ii) $L_\beta[A] \subseteq L_\alpha[A]$ if $\beta < \alpha$.

Proof. See the proof of Proposition 4. □

Proposition 13.50. $\alpha = L_\alpha[A] \cap \mathbf{ON}$.

Proof. See the proof of Proposition 5. □

Proposition 13.51. $L_\alpha[A] \in L_{\alpha+1}[A]$.

Proof. See the proof of Proposition 7. □

Theorem 13.52. *$L[A]$ is a model of ZFC.*

Proof. See the proof of Theorem 8. □

Theorem 13.53. *If $\varphi(v_0, \ldots, v_{n-1})$ is a Δ_0 formula in the expanded language with free variables among $v_0, \ldots, v_{n-1}$, let φ' be obtained from φ by replacing*
$v_i \in \mathbf{P}$ by $v_i \in v_n$;
$\exists v_i \in \mathbf{P}\psi$ by $\exists v_i \in v_n \psi'$;
$\forall v_i \in \mathbf{P}\psi$ by $\forall v_i \in v_n \psi'$.

Then there is an $(n+1)$-ary composition G of Gödel functions such that for all $X_0, \ldots, X_n$,

$$\begin{aligned}&G(X_0, \ldots, X_n)\\&= \{(u_0, \ldots, u_{n-1}) : u_0 \in X_0, \ldots, u_{n-1} \in X_{n-1} \text{ and } \varphi'(u_0, \ldots, u_{n-1}, X_n)\}.\end{aligned}$$

Proof. By Theorem 11 there is an $(n+1)$-ary composition G of Gödel functions such that for all $X_0, \ldots, X_n$,

$$\begin{aligned}&G(X_0, \ldots, X_n)\\&= \{(u_0, \ldots, u_n) : u_0 \in X_0, \ldots, u_n \in X_n \text{ and } \varphi'(u_0, \ldots, u_n)\}.\end{aligned}$$

Then

$$\begin{aligned}&G(X_0, \ldots, G_1(X_n, X_n))\\&= \{(u_0, \ldots, u_n) : u_0 \in X_0, \ldots, u_n \in X_n \text{ and } \varphi'(u_0, \ldots, u_{n-1}, X_n)\}.\end{aligned}$$

Hence

$$\begin{aligned}&\text{dmn}(G(X_0, \ldots, X_n))\\&= \{(u_0, \ldots, u_{n-1}) : u_0 \in X_0, \ldots, u_{n-1} \in X_{n-1} \text{ and } \varphi'(u_0, \ldots, u_{n-1}, X_n)\}.\end{aligned}$$

□

Lemma 13.54. *If $\varphi(v_0, \ldots, v_{n-1})$ is a Δ_0 formula in the expanded language with free variables among $v_0, \ldots, v_{n-1}$, φ' is defined as in Theorem 53, A is any set, M is transitive, and $\overline{a} \in M$, then $(M, \in, M \cap A) \models \varphi(\overline{a})$ iff $(M, \in) \models \varphi'(\overline{a}, M \cap A)$.* □

Proposition 13.55. *For every formula $\varphi(x, \overline{y})$ in the expanded language there is a composition G of Gödel functions such that for every transitive set M, every set A, and every $\overline{b} \in M$ we have*

$$\begin{aligned}&\{a \in M : (M, \in, M \cap A) \models \varphi(a, \overline{b})\}\\&= \{a \in M : \varphi'^M(a, \overline{b}, M \cap A)\} = G(M, b_0, \ldots, b_{n-1}, M \cap A).\end{aligned}$$

Proof. First note that the first equality follows from Lemma 54. Now apply Theorem 11 to φ'^M; we get a composition G of Gödel functions such that

$$G(X_0, \ldots, X_{n+1}) = \{(u_0, \ldots, u_{n+1}) : u_0 \in X_0, \ldots, u_{n+1} \in X_{n+1} \text{ and } \varphi'^M(u_0, \ldots, u_{n+1})\}.$$

Following the proof of Proposition 14 we obtain a composition G' of Gödel functions such that for every $\overline{b} \in M$,

$$\begin{aligned}&G'(M, b_0, \ldots, b_{n-1}, A \cap M) = \{(u_0, \ldots, u_{n+1}) :\\&u_0 \in M, u_1 = b_0, \ldots, u_n = b_{n-1}, u_{n+1} = A \cap M, \varphi'^M(u_0, \ldots, u_{n+1})\}.\end{aligned}$$

Then using dmn we get the desired result. □

Lemma 13.56. *If H is a composition of Gödel functions, then there is a formula ψ such that if M is a transitive set, $\overline{b} \in M$, A is any set, and $H(M, \overline{b}, A \cap M) \subseteq M$, then $H(M, \overline{b}, A \cap M) = \{a \in M : (M, \in, A \cap M) \models \psi(a, \overline{b}, A \cap M)\}$.*

Proof. By Theorem 13, $Y \in H(Z, \overline{X}, W)$ is equivalent to a Δ_0 formula $\varphi(Y, Z, \overline{X}, W)$. So $x \in H(M, \overline{b}, A \cap M)$ iff $\overline{M} \models \varphi(x, M, \overline{b}, A \cap M)$. Let $X = H(M, \overline{b}, A \cap M)$. Then $X = \{a \in M : \overline{M} \models \varphi(a, M, \overline{b}, A \cap M)$. Making replacements as in the proof of Lemma 16, we get a formula ψ such that $X = \{a \in M : \overline{M} \models \psi(a, \overline{b}, A \cap M)$. □

Lemma 13.57. *For any transitive set M, $\mathrm{def}_A(M) = \{X \subseteq M$: there is a $\overline{b} \in M$ and a composition H of Gödel functions such that $X = H(M, \overline{b}, A \cap M)$.*

Proof. By Propositions 55 and 56. □

Lemma 13.58. *If A is any set and M is an inner model of ZF, then $\forall \alpha \in \mathbf{ON}[L_\alpha^M[A \cap M] = L_\alpha[A \cap M]]$.*

Proof. Induction on α. It is clear for $\alpha = 0$ and for α limit. Assume it for α. Then

$$\begin{aligned} L_{\alpha+1}^M[A \cap M] &= \mathrm{def}_A^M(L_\alpha^M[A \cap M]) = \mathrm{def}_A^M(L_\alpha[A \cap M]) \\ &= \{X \subseteq L_\alpha[A \cap M] \cap M : \exists \overline{b} \exists H[X = H(L_\alpha[A \cap M], \in, A \cap L_\alpha[A \cap M]] \\ &= \mathrm{def}_A(L_\alpha[A \cap M]) = L_{\alpha+1}[A \cap M]. \end{aligned}$$ □

Theorem 13.59. *If M is an inner model of ZF and A is any set, then for any x, $x \in L^M[A \cap M]$ iff $x \in L[A \cap M]$.*

Proof.

$$x \in L^M[A \cap M] \leftrightarrow \exists \alpha[x \in L_\alpha^M[A \cap M]] \leftrightarrow \exists \alpha[x \in L_\alpha[A \cap M]] \leftrightarrow x \in L[A \cap M].$$ □

Theorem 13.60. *Let $\overline{A} = A \cap L[A]$. Then $L[\overline{A}] = L[A]$. Moreover, if A is a set, then $\overline{A} \in L[\overline{A}]$.*

Proof. We prove that $L_\alpha[\overline{A}] = L_\alpha[A]$ for all α by induction on α. It is obvious for $\alpha = 0$, and the induction step with α limit is clear. Now assume that $L_\alpha[\overline{A}] = L_\alpha[A]$. Let $U = L[A]$. Then

$$A \cap U = A \cap U \cap L[A] = \overline{A} \cap U.$$

Now $\mathrm{def}_A(U) = \mathrm{def}_{A \cap U}(U)$, so

$$L_{\alpha+1}[A] = \mathrm{def}_A(U) = \mathrm{def}_{A \cap U}(U) = \mathrm{def}_{\overline{A}}(U) = \mathrm{def}_{\overline{A}}(L_\alpha[\overline{A}] = L_{\alpha+1}[A].$$

This completes the induction, and proves that $L[\overline{A}] = L[A]$.

For the "moreover" part, there is an α such that $A \cap L[A] = A \cap L_\alpha[A]$, and

$$A \cap L_\alpha[A] = \{x \in L_\alpha[A] : x \in A \cap L_\alpha[A]\} \in L_{\alpha+1}[A]$$ □

Theorem 13.61. $L[A] \models V = L[A]$.

Proof.

$$\forall x \in L[A][x \in L^{L[A]}[A \cap L[A]] \leftrightarrow x \in L[A \cap L[A]] \text{ (by Theorem 59)}$$
$$\leftrightarrow x \in L[A] \text{ (by Theorem 60) } \leftrightarrow T. \qquad \square$$

Theorem 13.62. (minimality) *If M is an inner model of ZF and $A \cap M \in M$, then $L[A] \subseteq M$.*

Proof. Assume that M is an inner model of ZF and $A \cap M \in M$. We prove by induction on α that $L_\alpha[A] \in M$. The inductive step:

$L_{\alpha+1}[A] = \mathrm{def}_A(L_\alpha[A]) = \{X \subseteq L_\alpha[A] :$ there is a $\overline{b} \in L_\alpha[A]$ and a composition H of Gödel functions such that $X = H(L_\alpha[A], \overline{b}, A \cap L_\alpha[A])$.

Now here we have $H(L_\alpha[A], \overline{b}, A \cap L_\alpha[A]) = H(L_\alpha[A], \overline{b}, A \cap M \cap L_\alpha[A]) \in M$. $\square$

Lemma 13.63. (condensation) *For every limit ordinal α, if $(M, \in, Q) \preceq (L_\alpha[A] \in, A \cap L_\alpha[A])$, then M is extensional, and if N is the transitive collapse of M under the isomorphism π, then*

(i) $Q = A \cap M$.

(ii) There is a limit ordinal $\beta \le \alpha$ such that $N = L_\beta[\pi[A \cap M]]$.

Proof. For (i), $Q = M \cap A \cap L_\alpha[A] = A \cap M$.

For (ii), first we claim that M is extensional. For, suppose that $x, y \in M$ with $x \neq y$; say $x \backslash y \neq \emptyset$. Thus $\exists a \in x[a \notin y]$. Since $x, y \in L_\alpha[A]$, by absoluteness $L_\alpha[A] \models \exists a \in x[a \notin y]$, so $M \models \exists a \in x[a \notin y]$. This shows that M is extensional. Hence the Mostowski collapse function $\pi : M \to N$ is an isomorphism.

Now let $\beta = N \cap \mathbf{ON}$. Since N is transitive, β is an ordinal. We claim that $N = L_\beta[\pi[A \cap M]]$.

(1) For any ordinal γ, if $\Gamma \subseteq \gamma$, then o.t.$(\Gamma) \le \gamma$.

For, let $f : \delta \to \Gamma$ be the strictly increasing enumeration of Γ, with $\delta =$ o.t.(Γ). Then for any $\xi < \delta$ we have $\xi \le f(\xi) < \gamma$, so $\delta \le \gamma$.

(2) $\beta \le \alpha$.

For, suppose that $\alpha < \beta$. Then $\alpha \in N$, and so $\pi^{-1}(\alpha) \in M$. Now α is an ordinal, so $N \models$ (α is an ordinal); hence $M \models$ ($\pi^{-1}(\alpha)$ is an ordinal), and so $L_\alpha[A] \models$ ($\pi^{-1}(\alpha)$ is an ordinal), hence $\pi^{-1}(\alpha)$ is an ordinal. Now $\pi \restriction (\pi^{-1}(\alpha) \cap M)$ is an isomorphism onto α. It follows that o.t.$(\pi^{-1}(\alpha) \cap M) = \alpha$. But by (1), o.t.$(\pi^{-1}(\alpha) \cap M) \le \pi^{-1}(\alpha)$. Since $\pi^{-1}(\alpha) \in M \subseteq L_\alpha[A]$, we have $\pi^{-1}(\alpha) < \alpha$, contradiction.

(3) $0 < \beta$.

For, $L_\alpha[A] \models \exists x \forall y \in x[y \neq y]$, so $M \models \exists x \forall y \in x[y \neq y]$, hence $N \models \exists x \forall y \in x[y \neq y]$. Hence (3) holds.

(4) β is a limit ordinal.

For, suppose that $\beta = \gamma \cup \{\gamma\}$. Then $N \models \exists x[x$ is an ordinal and $\forall y[y \neq x \cup \{x\}]$. Hence $L_\alpha[A] \models \exists x[x$ is an ordinal and $\forall y[y \neq x \cup \{x\}]$, contradiction.

(5) If $Y, \overline{b} \subseteq M$, then $\forall a \in Y[(Y, \in, A \cap M \cap Y) \models \varphi(a, \overline{b}) \leftrightarrow (\pi[Y], \in, \pi[A \cap M \cap Y]) \models \varphi(\pi(a), \pi \circ \overline{b})]$.

(6) For any $Y \subseteq M$, $\pi[\mathrm{def}^M_{A\cap M}(Y)] = \mathrm{def}^N_{\pi[A\cap M]}(\pi[Y])$.

In fact, suppose that $Y \subseteq M$. Suppose also that $Z \in \mathrm{def}^M_{A\cap M}(Y)$. Say $Z = \{a \in Y : (Y, \in, A \cap M \cap Y) \models \varphi(a, \overline{b})\}$, with $\overline{b} \subseteq Y$. Thus by (5),

$$\begin{aligned}\forall a \in Y[a \in Z &\leftrightarrow (Y, \in, A \cap M \cap Y) \models \varphi(a, \overline{b}) \\ &\leftrightarrow (\pi[Y], \in, \pi[A \cap M \cap Y]) \models \varphi(\pi(a), \pi \circ \overline{b})\end{aligned}$$

Thus $\pi[Z] \subseteq \{u \in \pi[Y] : (\pi[Y], \in, \pi[A \cap M \cap Y]) \models \varphi(u, \pi \circ \overline{b})\} \in \mathrm{def}^N_{\pi[A\cap M]}(\pi[Y])$. The other inclusion is symmetric.

(7) If $\gamma \in M$ and γ is an ordinal, then $(\pi(L^M_\gamma[A \cap M], \gamma) = (L^N_{\pi(\gamma)}(\pi(A \cap M), \pi(\gamma))$.

We prove (7) by induction on γ. $\gamma = 0$ and γ limit are clear. The successor step:

$$\begin{aligned}\pi(L^M_{\gamma+1}[A \cap M], \gamma + 1) &= \pi(\mathrm{def}^M_{A\cap M}(L^M_\gamma[A \cap M], \gamma)) \\ &= \mathrm{def}^N_{\pi(A\cap M)}(\pi[L^M_\gamma[A \cap M, \gamma]]) \\ &= \mathrm{def}^N_{\pi(A\cap M)} L^N_\gamma[\pi(A \cap M], \pi(\gamma)) \\ &= L^N_{\gamma+1}[\pi(A \cap M)], \pi(\gamma + 1))\end{aligned}$$

(8) $L_\beta[\pi[A \cap M]] \subseteq N$.

For, $(L_\alpha[A], \in, A \cap L_\alpha[A]) \models \forall \delta \in \mathbf{ON} \exists y[y = L_\delta[A]]$. Hence $(M, \in, Q) \models \forall \delta \in \mathbf{ON} \exists y[y = L^M_\delta[A \cap M]]$. Now suppose that $\gamma < \beta$. Then $M \models [\pi^{-1}(\gamma)$ is an ordinal$]$, so we can choose $y \in M$ such that $y = L^M_{\pi^{-1}(\gamma)}[A \cap M]$. By (7), $\pi(y) = L^N_\gamma[\pi[A \cap M]$. This proves (8).

(9) $N \subseteq L_\beta[\pi[A \cap M]]$.

In fact,

$$\begin{aligned}(L_\alpha[A], \in, A \cap L_\alpha[A]) &\models \forall x \exists y \exists z[y \text{ is an ordinal} \wedge z = L_y[A]) \wedge x \in z], \quad \text{so} \\ (M, \in, Q) &\models \forall x \exists y \exists z[y \text{ is an ordinal} \wedge z = L^M_y[A \cap M] \wedge x \in z].\end{aligned}$$

So, given $x \in M$, choose $y, z \in M$ so that

$$(M, \in, Q) \models [y \text{ is an ordinal} \wedge z = L^M_y[A \cap M] \wedge x \in z].$$

Hence by (7), $(N, \in, \pi[A \cap M]) \models [[\pi(y)$ is an ordinal$] \wedge \pi(z) = L^N_{\pi(y)}(\pi[A \cap M] \wedge \pi(x) \in \pi(z)]$. So $\pi(y) < \beta$ and $\pi(x) \in L^N_{\pi(y)}(\pi[A \cap M]$. This proves (9). $\square$

Theorem 13.64. *If $A \in L[X]$, then $L[A] \subseteq L[X]$.*

Proof. Assume that $A \in L[X]$. Since $L[X]$ is transitive, $A \cap L[X] = A \in L[X]$. Hence $L[A] \subseteq L[X]$ by Theorem 69. $\square$

Theorem 13.65. *For every set X there is a set A of ordinals such that $L[X] = L[A]$.*

Proof. Choose α so that $\overline{X} \in L_\alpha[X]$, with α limit. In $L_\alpha[X]$ let f be a bijection from an ordinal θ onto $trcl(\{\overline{X}\})$. Define $\alpha E \beta$ iff $f(\alpha) \in f(\beta)$. Let Γ be the natural bijection of $\mathbf{ON} \times \mathbf{ON}$ onto $\mathbf{ON}$. Let $A = \Gamma(E)$. Then $A \in L[X]$, so $L[A] \subseteq L[X]$ by Theorem 63.

Now $A \in L_{\sup(A)+1}[A]$. Hence $E = \Gamma^{-1}[A] \in L[A]$. Hence $(\theta, E) \in L[A]$. Let M be the transitive collapse of (θ, E) in $L[A]$. Then $\overline{X} \in M$ and hence $\overline{X} \in L[A]$. So $L[A] = L[X]$. $\square$

Lemma 13.66. *Suppose that $\kappa^+ \leq \alpha$ and $X \in L_\alpha[A]$. Then there is an $M \preceq L_\alpha[A]$ such that $\kappa \subseteq M$, $M \cap \kappa^+ \in \kappa^+$, $X \in M$, and $|M| = \kappa$.*

Proof. Let M_0 be such that $\kappa \subseteq M_0$, $X \in M_0$, $M_0 \preceq L_\alpha[A]$, and $|M_0| = \kappa$. If M_n has been defined so that $M_n \preceq L_\alpha[A]$ and $|M_n| = \kappa$, choose M_{n+1} so that $M_n \cup \bigcup(M_n \cap \kappa^+) \subseteq M_{n+1}$, $M_{n+1} \preceq L_\alpha[A]$, and $|M_{n+1}| = \kappa$. Let $N = \bigcup_{n\in\omega} M_n$. Then $X \in N$, $N \preceq L_\alpha[A]$, and $|N| = \kappa$. $N \cap \kappa^+$ is a collection of ordinals. If $\alpha \in N \cap \kappa^+$, say $\alpha \in M_n$. Then $\alpha \subseteq \bigcup(M_n \cap \kappa^+) \subseteq M_{n+1} \subseteq N$. So $N \cap \kappa^+$ is transitive, and hence it is an ordinal. Since $|N| = \kappa$, also $|N \cap \kappa^+| = \kappa$, and so $N \cap \kappa^+ \in \kappa^+$. $\square$

Theorem 13.67. *If A is a set, then there is an α_0 such that for all $\alpha \geq \alpha_0$, $L[A] \models 2^{\aleph_\alpha} = \aleph_{\alpha+1}$.*

Proof. By Theorem 64 we may assume that A is a set of ordinals. Choose α so that $A \subseteq L_\alpha[A]$. Then $A \subseteq L_\alpha[A] \cap \mathbf{ON} = \alpha$. Let $\kappa > \alpha$. We claim that $2^\kappa = \kappa^+$ in $L[A]$. Take any $X \in L[A]$ such that $X \subseteq \kappa$. Say $X \in L_{\beta_X}[A]$ with $\kappa^+ \leq \beta_X$. By Theorem 42, let $(M_X, \emptyset, Q) \preceq (L_{\beta_X}[A], \emptyset, A)$ be such that $X \in M_X$, $\kappa \subseteq M_X$, $M_X \cap \kappa^+ \in \kappa^+$, and $|M_X| = \kappa$. Thus $Q = A \cap M_X$. Let $\delta_X = M \cap \kappa^+$. Then $A \cap \delta_X = A \cap M_X \cap \kappa^+ = A \cap M_X$. Let $(N_X, \emptyset, \pi[A \cap \delta_X])$ be the transitive collapse of $(M_X, \emptyset, A \cap \delta_X)$ via the function π. Since $\delta_X \subseteq M$, π is the identity on δ_X. Hence $\pi[A \cap \delta_X] = A \cap \delta_X$ and $X \in N_X$. By Theorem 39 there is an ordinal γ_X such that $(N_X, \emptyset, \pi[A \cap \delta_X]) = L_{\gamma_X}[A \cap \delta_X]$. Now $\gamma_X < \kappa^+$ since $|\gamma_X| = |L_{\gamma_X}[A \cap \delta_X]| = |N| = |M| = \kappa$. Also, $\delta_X < \kappa^+$.

Thus $\mathscr{P}(\kappa) \subseteq \bigcup_{\mu,\nu<\kappa^+} L_\mu[A \cap \nu]$, a set of size κ^+. $\square$

Theorem 13.68. *Assume $V = L[A]$ with $A \subseteq \omega_1$. Then GCH holds.*

Proof. First we show that $2^\omega = \omega_1$. Take any $X \subseteq \omega$. Say $X \in L_{\beta_X}[A]$ with $\omega_1 \leq \beta_X$. By Lemma 66, let $(M_X, \emptyset, Q) \preceq (L_{\beta_X}[A], \emptyset, A)$ be such that $X \in M_X$, $\omega \subseteq M_X$, $M_X \cap \omega_1 \in \omega_1$, and $|M_X| = \omega$. Thus $Q = A \cap M_X$. Let $\delta_X = M \cap \omega_1$. Then $A \cap \delta_X = A \cap M_X \cap \omega_1 = A \cap M_X$. Let $(N_X, \emptyset, \pi[A \cap \delta_X])$ be the transitive collapse of $(M_X, \emptyset, A \cap \delta_X)$ via the function π. Since $\delta_X \subseteq M$, π is the identity on δ_X. Hence $\pi[A \cap \delta_X] = A \cap \delta_X$ and $X \in N_X$. By Lemma 63 there is an ordinal γ_X such that $N_X = L_{\gamma_X}[A \cap \delta_X]$. Now $\gamma_X < \omega_1$ since $|\gamma_X| = |L_{\gamma_X}[A \cap \delta_X]| = |N| = |M| = \omega$. Also, $\delta_X < \omega_1$.

Thus $\mathscr{P}(\omega) \subseteq \bigcup_{\mu,\nu<\omega_1} L_\mu[A \cap \nu]$, and this set has size ω_1. So $2^\omega = \omega_1$.

Now suppose that λ is uncountable. We want to show that $2^\lambda = \lambda^+$ in V. Let $Y \subseteq \lambda$. Set $T = \mathrm{trcl}(\{Y\})$. Choose limit θ so that $T \in L_\theta[A]$. Let $(M, \in, N) \preceq (L_\theta[A], \in, A \cap L_\theta[A])$, with $\omega_1 \cup T \subseteq M$ and $|M| = \lambda$. Applying Lemma 63 we get $\gamma < \lambda^+$ such that $Y \in L_\gamma[\pi[A \cap M]]$. Now $A \cap M = A$, so $\pi[A \cap M] = \pi[A] = A$. Thus $Y \in L_\gamma[A] \subseteq L_{\lambda^+}[A]$. This is true for each $Y \subseteq \lambda$, so $\mathscr{P}(\lambda) \subseteq L_{\lambda^+}[A]$. Hence $2^\lambda = \lambda^+$. □

Now we define, for any set A,

$$\begin{aligned} L_0(A) &= \mathrm{trcl}(\{A\}); \\ L_{\alpha+1}(A) &= \mathrm{def}(L_\alpha(A)); \\ L_\gamma(A) &= \bigcup_{\alpha<\gamma} L_\alpha(A) \quad \text{for } \gamma \text{ limit}; \\ L(A) &= \bigcup_{\alpha \in \mathbf{ON}} L_\alpha(A). \end{aligned}$$

Proposition 13.69. $L_\alpha \subseteq L_\alpha(A) \subseteq V_\alpha$. □

Proposition 13.70. *For any ordinal α,*
(i) $L_\alpha(A)$ is transitive.
(ii) $L_\beta(A) \subseteq L_\alpha(A)$ if $\beta < \alpha$.

Proof. See the proof of Proposition 4. □

Proposition 13.71. *(i)* $\mathrm{trcl}(\{A\}) \cap \mathbf{ON}$ *is an ordinal; call it β.*
(ii) $\beta + \alpha = L_\alpha(A) \cap \mathbf{ON}$.

Proof. (i) is clear The conclusion of (ii) is clear for $\alpha = 0$ and inductively for α limit Now assume that $\beta + \alpha = L_\alpha(A) \cap \mathbf{ON}$. If $\alpha + \gamma \in L_{\alpha+1} \cap \mathbf{ON}$, then $\alpha + \gamma \in \mathrm{def}(L_\alpha(A))$, so $\alpha + \gamma \subseteq L_\alpha(A) \cap \mathbf{ON} = \beta + \alpha$; hence $\gamma \leq \alpha$. This shows that $L_{\alpha+1}(A) \cap \mathbf{ON} \subseteq \beta + \alpha + 1$.

If $\gamma < \alpha$, then by the inductive hypothesis, $\beta + \gamma \in L_\alpha \cap \mathbf{ON} \subseteq L_{\alpha+1} \cap \mathbf{ON}$. Thus it remains only to show that $\beta + \alpha \in L_{\alpha+1}$. Now there is a natural Δ_0 formula $\varphi(x)$ which expresses that x is an ordinal:

$$\forall y \in x \forall z \in y (z \in x) \wedge \forall y \in x \forall z \in y \forall w \in z (w \in y);$$

this just says that x is transitive and every member of x is transitive. Now $\varphi(x)$ is absolute, so

$$\beta + \alpha = L_\alpha \cap \mathbf{ON} = \{x \in L_\alpha : (L_\alpha, \in) \models \varphi(x)\} \in \mathrm{def}(L_\alpha(A)) = L_{\alpha+1}(A).$$ □

Proposition 13.72. $L_\alpha(A) \in L_{\alpha+1}(A)$.

Proof. See the proof of Proposition 7. □

Theorem 13.73. *$L(A)$ is an inner model of ZF.*

Proof. See the proof of Theorem 8. $\square$

Theorem 13.74. (minimality) *If M is an inner model of ZF and $A \in M$, then $L(A) \subseteq M$.*

Proof. We prove $L_\alpha(A) \in M$ by induction on α. It is clear for $\alpha = 0$ and inductively for α limit. Now suppose that $L_\alpha(A) \in M$. Then

$$\begin{aligned} L_{\alpha+1}(A) = \operatorname{def}(L_\alpha(A)) = \{x \subseteq L_\alpha(A) : \exists \overline{b} \in L_\alpha(A) \\ \exists H[H \text{ is a composition of Gödel function and } x = H(L_\alpha(A), \overline{b})]\}. \end{aligned}$$

This is in M because H is Δ_0. $\square$

Now we define

$$OD = \bigcup_{\alpha \in \mathbf{ON}} \operatorname{cl}(\{V_\beta : \beta < \alpha\}).$$

Theorem 13.75. *There is a definable well-order of OD and associated with it a definable bijection F of* **ON** *onto OD.*

Proof. We define by recursion a well-order $<_\alpha$ of $\operatorname{cl}(\{V_\beta : \beta < \alpha\})$, Let $<_0 = \emptyset$. If $<_\alpha$ has been defined and it is a well-order of $\operatorname{cl}(\{V_\beta : \beta < \alpha\})$, first define

$$\begin{aligned} Y_0^\alpha &= \operatorname{cl}(\{V_\beta : \beta < \alpha\}) \cup \{V_\alpha\}; \\ Y_{n+1}^\alpha &= Y_n^\alpha \cup \{G_i(X, Z) : X, Z \in Y_n^\alpha, 1 \leq i \leq 10\}. \end{aligned}$$

Note that $\operatorname{cl}(\{V_\beta : \beta < \alpha + 1\}) = \bigcup_{n\in\omega} Y_n^\alpha$. Now we define $<_{\alpha+1}^m$, a well-order of Y_m^α as follows. For $x, y \in Y_0^\alpha$ we define $x <_{\alpha+1}^0$ iff

$x, y \in \operatorname{cl}(\{V_\beta : \beta < \alpha\})$ and $x <_\alpha y$ or
$x \in \operatorname{cl}(\{V_\beta : \beta < \alpha\})$ and $y = V_\alpha$,

Now suppose that $<_{\alpha+1}^n$, a well-order of Y_n^α has been defined. Suppose that $x, y \in Y_{n+1}^\alpha$. Then $x <_{\alpha+1}^{n+1} y$ iff

$x, y \in Y_n^\alpha$ and $x <_{\alpha+1}^n y$ or
$x \in Y_n^\alpha$ and $y \notin Y_n^\alpha$ or
$x, y \in Y_\alpha^{n+1} \backslash Y_\alpha^n$ and $x \neq y$ and $x = G_i(X, Z)$ and $y = G_j(X', Z')$ with $X, Z, X', Z' \in Y_n^\alpha$ and either $i < j$ or $(X, Z) < (X', Z')$ lexicographically.

Then we define $<_{\alpha+1} = \bigcup_{n\in\omega} <_{\alpha+1}^n$.

For γ limit let $<_\gamma = \bigcup_{\alpha<\gamma} <_\alpha$.

Let $<_{OD} = \bigcup_{\alpha\in\mathbf{ON}} <_\alpha$. Clearly this is a well-order of OD, and the bijection F is the natural mapping. $\square$

Theorem 13.76. *If $X \in$OD, then there is a formula $\varphi(y, x_1, \ldots, x_n)$ such that for some ordinal numbers $\alpha_1, \ldots, \alpha_n$, $X = \{u : \varphi(u, \alpha_1, \ldots, \alpha_n)\}$.*

Proof. Let $\varphi(y, x)$ be the formula $y \in F(x)$, where F is given by Theorem 75. Choose α so that $F(\alpha) = X$. Then $X = \{u : u \in F(\alpha)\} = \{u : \varphi(u, \alpha)\}$. $\square$

Theorem 13.77. *If $\varphi(y, x_1, \ldots, x_n)$ is a formula and $\alpha_1, \ldots, \alpha_n$ are ordinal numbers such that $X = \{u : \varphi(u, \alpha_1, \ldots, \alpha_n)\}$, then $X \in$OD.*

Proof. By the reflection theorem, let β be such that

$$X, \alpha_1, \ldots, \alpha_n \in V_\beta \text{ and } \varphi \text{ is absolute for } V_\beta, V.$$

Then $X = \{u \in V_\beta : \varphi^{V_\beta}(u, \alpha_1, \ldots, \alpha_n)\}$. By Theorem 11 there is a composition G of Gödel functions such that for all $X_0, \ldots, X_n$,

$$G(X_0, \ldots, X_n) = \{(u_0, \ldots, u_n) : u_0 \in X_0, \ldots, u_n \in X_n, \varphi(u_0, \ldots, u_n)\}.$$

Hence $X = G(V_\beta, \{\alpha_1\}, \ldots, \{\alpha_n\})$. Now each α_i is definable in V_{α_i} by $\alpha_i = \{u \in V_{\alpha_i} : u$ is an ordinal$\}$. Hence by Proposition 14 there is a composition H_i of Gödel functions such that $\alpha_i = H_i(V_{\alpha_i})$. It follows that $X \in$OD. □

Theorem 13.78. ON $\subseteq$*OD.*

Proof. If α is an ordinal, then $\alpha = \{u : u \in \alpha\}$; use Theorem 77. □

Now we define

$$HOD = \{x : \mathrm{trcl}(\{x\}) \subseteq OD\}.$$

Theorem 13.79. *HOD is an inner model of ZFC.*

Proof. By Theorem 78, every ordinal is in HOD.

Next we show that HOD is closed under the Gödel functions. Clearly OD is closed under the Gödel functions. Assume that $X, Y \in$HOD. Then

$$\mathrm{trcl}(\{X\}), \mathrm{trcl}(\{Y\}) \subseteq OD.$$

In particular $X, Y \in$OD, so $G_i(X, Y) \in$OD.

(1) $G_1(X, Y) = \{X, Y\}$. $\mathrm{trcl}(\{\{X, Y\}\}) = \{X, Y\} \cup \mathrm{trcl}(\{X\}) \cup \mathrm{trcl}(\{Y\}) \subseteq$OD.

(2) $G_2(X, Y) = X \times Y$.

$$\begin{aligned}\mathrm{trcl}(\{X \times Y\}) =&\{X \times Y\} \cup \{(x, y) : x \in X, y \in Y\} \cup \{\{x, y\} : x \in X, y \in Y\} \\ &\cup \bigcup_{x \in X} \mathrm{trcl}(\{x\}) \cup \bigcup_{y \in Y} \mathrm{trcl}(\{y\}).\end{aligned}$$

Now if $x \in X$ and $y \in Y$, then $x, y \in$OD, and so $\{x, y\} \in$OD. Also $(x, y) \in$OD. So $\mathrm{trcl}(\{X \times Y\}) \subseteq$OD.

(3) $G_3(X, Y) = \{(u, v) : u \in X, v \in Y, u \in v\}$. Similar to (2).

(4) $G_4(X, Y) = X \backslash Y$.

$$\mathrm{trcl}(\{X \backslash Y\}) = \{X \backslash Y\} \cup \bigcup_{x \in X \backslash Y} \mathrm{trcl}(\{x\}) \subseteq OD.$$

(5) $G_5(X,Y) = X \cap Y$. $\mathrm{trcl}(\{X \cap Y\}) = \{X \cap Y\} \cup \bigcup_{x \in X \cap Y} \mathrm{trcl}(\{x\}) \subseteq$OD.

(6) $G_6(X) = \bigcup X$. $\mathrm{trcl}(\{\bigcup X\} = \{\bigcup X\} \cup \bigcup_{x \in \bigcup X} \mathrm{trcl}(\{x\}) \subseteq$OD.

(7) $G_7(X) = \mathrm{dmn}(X)$.

$$\mathrm{trcl}(\{\mathrm{dmn}(X)\}) = \{\mathrm{dmn}(X)\} \cup \bigcup_{x \in \mathrm{dmn}(X)} \mathrm{trcl}(\{x\}) \subseteq OD.$$

Here note that if $x \in \mathrm{dmn}(X)$ then $x \in \mathrm{trcl}(X)$.

(8) $G_8(X) = \{(u,v) : (v,u) \in X\}$.

$$\mathrm{trcl}(\{\{(u,v) : (v,u) \in X\}\}) = \{\{(u,v) : (v,u) \in X\}\} \cup \bigcup_{(v,u) \in X} \mathrm{trcl}(\{(u,v)\}) \subseteq OD.$$

(9) $G_9(X) = \{(u,v,w) : (u,w,v) \in X\}$.

$$\begin{aligned} &\mathrm{trcl}(\{\{(u,v,w) : (u,w,v) \in X\}\}) \\ &= \{\{(u,v,w) : (u,w,v) \in X\}\} \cup \bigcup_{(u,w,v) \in X} \mathrm{trcl}(\{(u,v,w)\}) \subseteq OD. \end{aligned}$$

(10) G_{10} is similar.

So HOD is closed under Gödel functions.

Now we verify the condition of Theorem 29. If X is a set $\subseteq$ HOD, then there is an α such that $X \subseteq V_\alpha$. So $X \subseteq V_\alpha \cap$HOD. Hence it suffices to show that $V_\alpha \cap$HOD$\in$HOD. We claim

(11) $V_\alpha \cap HOD = \{u \in V_\alpha : \forall z \in \mathrm{trcl}(\{u\}) \exists \beta [z \in \mathrm{cl}(\{V_\gamma : \gamma < \beta\})]\}$.

In fact, if $u \in V_\alpha \cap$HOD, then by definition $\mathrm{trcl}(\{u\}) \subseteq$OD, and so $\subseteq$ in (11) holds. Conversely, if $u \in V_\alpha$ and $\forall z \in \mathrm{trcl}(\{u\}) \exists \beta [z \in \mathrm{cl}(\{V_\gamma : \gamma < \beta\})]\}$, then $\mathrm{trcl}(\{u\}) \subseteq$OD, and hence $u \in$HOD. So (11) holds.

By (11) and Theorem 77, $V_\alpha \cap$HOD$\in$OD.

Thus by Theorem 22, HOD is an inner model of ZF.

For the axiom of choice, it suffices to find for each α a $g \in$OD which is a one-one mapping of $V_\alpha \cap$HOD into **ON**. By Theorem 75, let F be a definable bijection of **ON** onto OD. Then $F^{-1} \restriction (V_\alpha \cap$HOD) is as desired. □

We define
$$OD[A] = \mathrm{cl}(\{V_\alpha : \alpha \in \mathbf{ON}\} \cup \{A\}).$$

Proposition 13.80. *OD*$\subseteq$*OD*$[A]$. □

Theorem 13.81. *There is a definable well-order of OD$[A]$ and associated with it a definable bijection F of* **ON** *onto OD$[A]$.*

Proof. We define by recursion a well-order $<_\alpha$ of $\mathrm{cl}(\{V_\beta : \beta < \alpha\} \cup \{A\})$. Define

$$\begin{aligned} Y_0^0 &= \{\{A\}\}; \\ Y_{n+1}^0 &= Y_n^0 \cup \{G_i(X,Z) : X, Z \in Y_n^0, 1 \le i \le 10\}. \end{aligned}$$

Define $<_0^0 = \emptyset$. Suppose that $<_0^n$ has been defined and it is a well-order of Y_n^0. For $x, y \in Y_{n+1}^0$ define $x <_0^{n+1} y$ iff

$x, y \in Y_n^0$ and $x <_0^n y$.
$x \in Y_n^0$ and $y \notin Y_n^0$.
$x, y \notin Y_n^0$ and $x \neq y$ and $x = G_i(X,Z)$, $y = G_j(X',Z')$ and either $i < j$ or $i = j$ and $(X,Z) < (X',Z')$ lexicographically.

Now the rest of the construction and proof goes like in the proof of Theorem 75. ☐

Theorem 13.82. *If $X \in OD[A]$, then there is a formula φ such that*

$$X = \{u : \varphi(u, \alpha_1, \ldots, \alpha_n, A)\}.$$

Proof. See the proof of Theorem 76. ☐

Theorem 13.83. *If $X = \{u : \varphi(u, \alpha_1, \ldots, \alpha_n, A)\}$, then $X \in OD[A]$.*

Proof. See the proof of Theorem 77. ☐

We define

$$HOD[A] = \{x : \mathrm{trcl}(\{x\}) \subseteq OD[A]\}.$$

Theorem 13.84. *HOD[A] is a model of ZFC.*

Proof. See the proof of Theorem 79. ☐

Now we define

$$OD(A) = \{X : X \in \mathrm{cl}(\{V_\alpha : \alpha \in \mathbf{ON}\} \cup \{A\} \cup A)\}.$$

Theorem 13.85. *If $X \in OD(A)$, then there is a finite subset E of A such that $X \in \mathrm{cl}(\{V_\alpha : \alpha \in \mathbf{ON}\} \cup \{A\} \cup E)$.*

Proof. Say $X = H(V_{\alpha_1}, \ldots, V_{\alpha_m}, A, \overline{b})$ with H a composition of Gödel functions and $\overline{b} \in A$. Then we can let $E = \mathrm{rng}(\overline{b})$. ☐

Theorem 13.86. *$X \in OD(A)$ iff there exist a formula φ, $x_0, \ldots, x_{m-1} \in A$, and ordinals $\alpha_1, \ldots, \alpha_n$ such that*

$$X = \{u : \varphi(u, \alpha_1, \ldots, \alpha_n, x_0, \ldots, x_{m-1})\}.$$

Proof. We modify the proof of Theorem 81 by starting with $Y_0^\alpha = \{V_\alpha, A\}$. Then we get a bijection F from **ON** onto $\mathrm{OD}(A) \backslash A$, and this gives the desired formula. The other direction is proved like for Theorem 77. ☐

Theorem 13.87. *$\langle L_\alpha(A) : \alpha \in \mathbf{ON}\rangle$ is absolute for transitive models of ZF.*

Proof. See the proof of Theorem 39. $\square$

Theorem 13.88. *If M is an inner model of ZF and $A \in M$, then $L(A) \subseteq M$.*

Proof. See the proof of Theorem 41. $\square$

Define

$$HOD(A) = \{x : \mathrm{trcl}(\{x\}) \subseteq OD\}.$$

Then as for HOD itself we have

Theorem 13.89. *$HOD(A)$ is an inner model of ZF.* $\square$

Theorem 13.90. *If M is a transitive set, then $\mathrm{cl}(M)$ is transitive.*

Proof. Let M be a transitive set, and let $N = \mathrm{cl}(M)$. Let $P = \{y : \mathrm{trcl}(\{y\}) \subseteq N\}$. It suffices to show that $M \subseteq P$ and P is closed under the Gödel operations. Note that $P \subseteq N$ and P is transitive. If $y \in M$, then $\mathrm{trcl}(\{y\}) \subseteq M \subseteq N$; so $y \in P$.

(1) $X, Y \in P \to \{X, Y\} \in P$.

For, $X, Y \in N$, so $\{X, Y\} \in N$. Moreover

$$\mathrm{trcl}(\{X, Y\}) = \{\{X, Y\}\} \cup \mathrm{trcl}(\{X\}) \cup \mathrm{trcl}(\{Y\}) \subseteq N.$$

It follows that

(2) $X, Y \in P \to (X, Y) \in P$.

(3) $X, Y \in P \to X \times Y \in P$.

For, suppose that $X, Y \in P$. Then $X \times Y \in N$. Moreover,

$$\mathrm{trcl}(\{X \times Y\}) = \{\{X \times Y\}\} \cup \bigcup_{\substack{a \in X,\\ b \in Y}} \mathrm{trcl}(\{(a, b)\}) \subseteq N.$$

(4) $X.Y \in P \to \varepsilon(X, Y) \in P$.

For, suppose that $X, Y \in P$. Then $\varepsilon(X, Y) \in N$. Moreover,

$$\mathrm{trcl}(\{\varepsilon(X, Y)\}) = \{\{\varepsilon(X, Y)\}\} \cup \bigcup_{(a,b) \in \varepsilon(X,Y)} \mathrm{trcl}(\{(a, b)\}) \subseteq N.$$

(5) $X, Y \in P \to X \backslash Y \in P$.

For, suppose that $X, Y \in P$. Then $X \backslash Y \in N$, and

$$\mathrm{trcl}(\{X \backslash Y\}) = \{X \backslash Y\} \cup \bigcup_{a \in X \backslash Y} \mathrm{trcl}(\{a\}) \subseteq N.$$

(6) $X, Y \in P \to X \cap Y \in P$.

Treated as in (5).

(7) $X \in P \to \bigcup X \in P$.

For, suppose that $X \in P$. Then $\bigcup X \in N$, and

$$\operatorname{trcl}\left(\left\{\bigcup X\right\}\right) = \left\{\bigcup X\right\} \cup \bigcup_{y \in X} \operatorname{trcl}(y) \subseteq N.$$

(8) $X \in P \to \operatorname{dmn}(X) \in P$.

For, suppose that $X \in P$. Then $X \in N$, so $\operatorname{dmn}(X) \in N$. Now if $(a, b) \in X$ then $a \in \{a\} \in (a, b) \in X \in P$, so $a \in P$. Hence

$$\operatorname{trcl}(\{\operatorname{dmn}(X)\}) = \{\operatorname{dmn}(X)\} \cup \bigcup_{(a,b) \in X} \operatorname{trcl}(\{a\}) \subseteq N.$$

(9) $X \in P \to G_8(X) \in P$.

For, suppose that $X \in P$. Then $X \in N$, so $G_8(X) \in N$. Now as in (8), if $(a, b) \in X$ then $a, b \in P$, and so by (2), $(b, a) \in P$. Hence

$$\operatorname{trcl}(\{G_8(X)\}) = \{G_8(X)\} \cup \bigcup_{(a,b) \in X} \operatorname{trcl}(\{(b, a)\}) \subseteq N.$$

(10) $X \in P \to G_9(X) \in P$.

For, suppose that $X \in P$. Then $X \in N$, so $G_9(X) \in N$. Now if $(u, w, v) \in X$ then as in (8), $u, w, v \in P$, and so by (2), $(u, v, w) \in P$. Hence

$$\operatorname{trcl}(G_9(X)) = \{G_9(X)\} \cup \bigcup_{(u,w,v) \in X} \operatorname{trcl}(\{u, v, w)\}).$$

(11) G_{10} is like G_9.

(12) If $X_0, \ldots, X_{n-1} \in P$, then $P_i^n(X_0, \ldots, X_{n-1}) \in P$.

This is obvious. $\square$

Theorem 13.91. *If M is closed under Gödel functions and is extensional and if $X \in M$ is finite, then $X \subseteq M$.*

Proof.

(1) $\{x\} \in M \to x \in M$.

For, $\bigcup\{x\} = x$.

(2) $\{x, y\} \in M \to x, y \in M$.

In fact, suppose not. By (1), $x \neq y$. If $x \in M$, then $\{x\} \in M$, hence $\{y\} = \{x, y\}\backslash\{x\} \in M$, hence by (1), $y \in M$, contradiction. So $x \notin M$. Similarly, $y \notin M$. Now $\{x, y\} \times \{x, y\} = \{(x, x), (x, y), (y, x), (y, y)\} \in M$. If $\{(x, x), (x, y), (y, x), (y, y)\} \cap M = \emptyset$, then $\{x, y\} \cap M = \emptyset = \{(x, x), (x, y), (y, x), (y, y)\} \cap M$, so $\{x, y\} = \{(x, x), (x, y), (y, x), (y, y)\}$. But $\{x, y\}$ has exactly two elements, and $\{(x, x), (x, y), (y, x), (y, y)\}$ has exactly four elements, contradiction. Hence $\{(x, x), (x, y), (y, x), (y, y)\} \cap M \neq \emptyset$.

Case 1. $(x, x) \in M$. Now $(x, x) = \{\{x\}\}$, so by (1) twice, $x \in M$, contradiction.

Case 9. $(y, y) \in M$. Similar to Case 1.

Case 3. $(x, y) \in M$. Now $(x, y) = \{\{x\}, \{x, y\}\}$ and $\{\{x, y\}\} \in M$. Hence $\{x\} = (x, y)\backslash\{\{x, y\}\} \in M$, so $x \in M$ by (1), contradiction.

Case 4. $(y, x) \in M$. Similar to Case 3.

This proves (2).

Now suppose inductively that $F \in M$, F finite, $|F| > 2$, and $F \not\subseteq M$. If $x \in F \cap M$, then $\{x\} \in M$, and $F\backslash\{x\} \in M$, so by the inductive hypothesis $F \subseteq \{x\} \subseteq M$, so $F \subseteq M$, contradiction. Thus $F \cap M = \emptyset$. Now we claim

(∗) $(F \times F) \cap M \neq \emptyset$.

For, suppose that $(F \times F) \cap M = \emptyset$. Then $F = F \times F$. But $|F| < |F \times F|$, contradiction.

Thus (∗) holds. Say $x, y \in F$ and $(x, y) \in M$. Then $\{x\} \in M$ by (2), so $x \in M$ by (1), contradiction. □

Theorem 13.92. *If M is closed under the Gödel functions and is extensional, and π is the transitive collapse of M, then $\pi(G_i(X, Y)) = G_i(\pi(X), \pi(Y))$ for $i = 1, \ldots, 10$.*

Proof. Recall that

$$\pi(y) = \{\pi(z) : z \in y\}$$

for all $y \in M$.

(1) G_1: $\pi(G_1(X, Y)) = \{\pi(X), \pi(Y)\} = G_1(\pi(X), \pi(Y))$.

(2) $\pi((X, Y)) = \pi(\{\{X\}, \{X, Y\}\}) = \{\pi(\{X\}), \pi(\{X, Y\}) = \{\{\pi(X)\}, \{\pi(X), \pi(Y)\}\} = (\pi(X), \pi(Y))$.

(3) G_2: $\pi(X \times Y) = \{\pi((x, y)) : x \in X, y \in Y\} = \{(\pi(x), \pi(y)) : x \in X, y \in Y\} = \{\pi(x) : x \in X\} \times \{\pi(y) : y \in Y\} = \pi(X) \times \pi(Y)$.

(4) G_3: $\pi(G_3(X, Y)) = \{\pi(x, y) : x \in X, y \in Y, x \in y\} = \{(\pi(x), \pi(y)) : x \in X, y \in Y, x \in y\} = \{(u, v) : u \in \pi(X), v \in \pi(Y), u \in v\} = G_3(\pi(X), \pi(Y))$.

(5) G_4: $\pi(G_4(X, Y)) = \pi(X\backslash Y) = \{\pi(x) : x \in X\backslash Y\} = \{\pi(x) : x \in X\}\backslash\{\pi(y) : y \in Y\} = G_4(\pi(X), \pi(Y))$. Here we used the fact that π is one-one.

(6) G_5: $\pi(G_5(X, Y)) = \pi(X \cap Y) = \{\pi(x) : x \in X \cap Y\} = \{\pi(x) : x \in X\} \cap \{\pi(y) : y \in Y\} = G_5(\pi(X), \pi(Y))$.

(7) G_6: $\pi(G_6(X)) = \pi(\bigcup X) = \{\pi(x) : x \in \bigcup X\} = \{\pi(x) : \exists y \in X[x \in y]\} = \bigcup_{y \in X}\{\pi(x) : x \in y\} = \bigcup \pi(X)$. To see the last equality, first suppose that $y \in X$ and $x \in y$. Then $\pi(x) \in \pi(y) \in \pi(X)$, so $\pi(x) \in \bigcup \pi(X)$. Second, suppose that $u \in \bigcup \pi(X)$.

Say $u \in v \in \pi(X)$. So there is a $y \in X$ such that $v = \pi(y)$. Since $u \in v$, there is an $x \in y$ with $u = \pi(x)$, as desired.

(8) G_7: $\pi(G_7(X)) = \pi(\mathrm{dmn}(X)) = \{\pi(x) : x \in \mathrm{dmn}(X)\} = \{\pi(x) : \exists y[(x,y) \in X]\} = \bigcup_{y\in\mathrm{rng}(X)}\{\pi(x) : (x,y) \in X\}$. Now $\{\pi(x) : (x,y) \in X\} = \{\pi(x) : \pi(x,y) \in \pi(X)\} = \{\pi(x) : (\pi(x),\pi(y)) \in \pi(X)$ and so, continuing the above,

$$\pi(G_7(X)) = \bigcup_{\pi(y)\in\mathrm{rng}(\pi(X))} \{\pi(x) : (\pi(x),\pi(y)) \in \pi(X)\} = G_7(\pi(X)).$$

(9) $G_8 - -G_{10}$: similar to the above. $\square$

Theorem 13.93. *For every transitive class M and every ordinal $\alpha \in M$, $V_\alpha^M = V_\alpha \cap M$.*

Proof. Assume that $M \neq \emptyset$. Then $\emptyset \in M$. In fact, choose $a \in M$, and let $b \in \mathrm{trcl}(\{a\})$ have smallest rank. Then clearly $b = \emptyset$. Now V_α^M is defined as follows, for each $\alpha \in M$:

$$V_\alpha^M = \begin{cases} \emptyset & \text{if } \alpha = 0; \\ \{a \in M : \forall b \in a[b \in V_\beta^M]\} & \text{if } \alpha = \beta+1 \in M; \\ \bigcup_{\beta<\alpha} V_\beta^M & \text{if } \alpha \text{ is a limit ordinal} \in M. \end{cases}$$

Now $V_\alpha^M = V_\alpha \cap M$ for all $\alpha \in M$, by induction.

Theorem 13.94. *"x is finite" is Δ_1.*

Proof. "x is a finite ordinal" is Δ_0. Now x is finite iff there is a finite ordinal m and a bijection from m onto x. This shows that "x" is finite is Σ_1. It is Π_1, since

$$x \text{ is finite iff } \forall Y[x \in Y \wedge \forall b \in Y \forall y \in b[b\backslash\{y\} \in Y] \to \emptyset \in Y]. \qquad \square$$

Theorem 13.95. *$\alpha + \beta$ is Δ_1.*

Proof. $\alpha + \beta$ is Σ_1:

$$\begin{aligned} \alpha+\beta=\gamma \quad \text{iff} \quad & \exists f[f \text{ is a function}, \mathrm{dmn}(f) = \beta+1,\ f(0) = \alpha, \\ & \forall\delta < \beta[f(\delta=1) = f(\delta)+1],\ f(\beta) = \gamma]. \end{aligned}$$

$\alpha + \beta$ is Π_1:

$$\begin{aligned} \alpha+\beta=\gamma \quad \text{iff} \quad & \forall f[f \text{ is a function}, \mathrm{dmn}(f) = \beta+1,\ f(0) = \alpha, \\ & \forall\delta < \beta[f(\delta=1) = f(\delta)+1] \to f(\beta) = \gamma]. \end{aligned} \qquad \square$$

Theorem 13.96. *The canonical well-ordering of* **ON** $\times$ **ON** *is Δ_0.*

Proof.

$$\begin{aligned}(\alpha,\beta)<(\gamma,\delta) \quad \text{iff} \quad & (\max\{\alpha,\beta\}<\max\{\gamma,\delta\})\vee(\max\{\alpha,\beta\}=\max\{\gamma,\delta\}\wedge\alpha<\gamma)\vee\\ & (\max\{\alpha,\beta\}=\max\{\gamma,\delta\}\wedge\alpha=\gamma\wedge\beta<\delta)\\ \text{iff} \quad & (\alpha\le\beta\wedge\gamma\le\delta\wedge\beta<\delta)\vee(\alpha\le\beta\wedge\delta\le\gamma\wedge\beta<\gamma)\vee\\ & (\beta<\alpha\wedge\gamma\le\delta\wedge\alpha<\delta)\vee(\beta<\alpha\wedge\delta<\gamma\wedge\alpha<\gamma)\vee\\ & (\alpha<\gamma\wedge\beta=\gamma\wedge\delta\le\gamma)\vee(\alpha<\gamma\wedge\beta=\delta\wedge\gamma\le\delta)\vee\\ & (\alpha=\gamma\wedge\beta<\delta\wedge\delta\le\gamma).\end{aligned}$$

□

Theorem 13.97. *The function Γ which assigns to each pair (α,β) of ordinals its order type under the canonical ordering, is Δ_1.*

Proof.

$$\begin{aligned}\Gamma(\alpha,\beta)=\varphi \quad \text{iff} \quad & \exists f[f \text{ is a function } \wedge f(0,0)=0\wedge\\ & \forall(\varepsilon,\theta)\le(\alpha,\beta)[\forall(\gamma,\delta)<(\varepsilon,\theta)[(\gamma,\delta)\in\mathrm{dmn}(f)\to\\ & (\varepsilon,\theta)\in\mathrm{dmn}(f)\wedge f(\varepsilon,\theta)=\sup\{f(\gamma,\delta):(\gamma,\delta),(\varepsilon,\theta)\}\\ & \wedge f(\alpha,\beta)=\varphi]]]\end{aligned}$$

Also,

$$\begin{aligned}\Gamma(\alpha,\beta)=\varphi \quad \text{iff} \quad & \forall f[f \text{ is a function } \wedge f(0,0)=0\wedge\\ & \forall(\varepsilon,\theta)\le(\alpha,\beta)[\forall(\gamma,\delta)<(\varepsilon,\theta)[(\gamma,\delta)\in\mathrm{dmn}(f)\to\\ & (\varepsilon,\theta)\in\mathrm{dmn}(f)\wedge f(\varepsilon,\theta)=\sup\{f(\gamma,\delta):(\gamma,\delta),(\varepsilon,\theta)\}\\ & \to f(\alpha,\beta)=\varphi]]]\end{aligned}$$

So Γ is Δ_1. □

Theorem 13.98. *The function assigning to each x its transitive closure is Δ_1.*

Proof.

$$Y=\mathrm{trcl}(X) \quad \text{iff} \quad Y \text{ is transitive } \wedge X\subseteq Y\wedge\forall Z[Z \text{ transitive } \wedge X\subseteq Z\to Y\subseteq Z.$$

Thus $Y=\mathrm{trcl}(X)$ is Π_1.

$$Y=\mathrm{trcl}(X) \quad \text{iff} \quad \exists Z[\mathrm{dmn}(Z)=\omega\wedge Z_0=X\wedge\forall n\in\omega[Z_{n+1}=\bigcup Z_n]\wedge\bigcup_{n\in\omega}Z_n=Y].$$

So $Y=\mathrm{trcl}(X)$ is Σ_1. □

Theorem 13.99. *rank is Δ_1.*

Proof. $y = \text{rank}(x)$ iff $\exists f[f$ is a function and y is an ordinal and $\text{dmn}(f) = y+2$ and $f(0) = \emptyset$ and $\forall z < y+1[f(z+1) = \mathscr{P}(f(z)]$ and for all limit $z < y+2$, $f(z) = \bigcup_{w<z} f(w)$ and $x \in f(y+1)$ and $x \notin f(y)$.

And also $y = \text{rank}(x)$ iff $\forall f[f$ is a function and y is an ordinal and $\text{dmn}(f) = y+2$ and $f(0) = \emptyset$ and $\forall z < y+1[f(z+1) = \mathscr{P}(f(z)]$ and for all limit $z < y+2$, $f(z) = \bigcup_{w<z} f(w)$ and $x \in f(y+1)$ implies that $x \notin f(y)$. □

Let M be a transitive class model of ZF containing all ordinals, and let x be a subset of M. We define $M[x]$ and give its basic properties. A set X is *definable over* $(A, (M \cup \{x\}) \cap A)$ iff there is a formula $\varphi(x, \overline{y})$ and a $\overline{b} \in A$ with length that of $\overline{y}$, such that $X = \{a \in A : (A, (M \cup \{x\}) \cap A) \models \varphi(a, \overline{b})\}$. Now we define

$$\begin{aligned}
\text{defn}(A) &= \{X : X \text{ is definable over } (A, (M \cup \{x\}) \cap A); \\
L_0(M, x) &= \emptyset; \\
L_{\alpha+1}(M, x) &= \text{defn}(L_\alpha(M, x)) \cup ((M \cup \{x\}) \cap V_\alpha); \\
L_\gamma(M, x) &= \bigcup_{\alpha<\gamma} L_\alpha(M, x) \quad \text{for } \gamma \text{ limit}; \\
M[x] &= \bigcup_{\alpha\in\mathbf{ON}} L_\alpha(M, x).
\end{aligned}$$

Proposition 13.100. *For any ordinal α,*
(i) $L_\alpha(M, x)$ is transitive.
(ii) $L_\beta(M, x) \subseteq L_\alpha(M, x)$ if $\beta < \alpha$.

Proof. See the proof of Proposition 4. □

Proposition 13.101. $L_\alpha(M, x) \in L_{\alpha+}(M, x)$.

Proof. See the proof of Proposition 7. □

Proposition 13.102. $\alpha = L_\alpha(M, x) \cap \mathbf{ON}$.

Proof. The proof of Proposition 5 has to be slightly modified:

We prove this by induction on α. It is obvious for $\alpha = 0$, and the inductive step when α is limit is clear. So, suppose the statement holds for β and we want to prove it for $\beta + 1$. If $\gamma \in L_{\beta+1}(M, x) \cap \mathbf{ON}$, then $\gamma \in \text{defn}(L_\beta) \cup ((L_\beta(M, x) \cup \{x\}) \cap V_\beta$, so $\gamma \subseteq L_\beta(M, x) \cap \mathbf{ON} \cup (V_\beta \cap \mathbf{ON}) = \beta$; hence $\gamma \leq \beta$. This shows that $L_{\beta+1} \cap \mathbf{ON} \subseteq \beta + 1$.

If $\gamma < \beta$, then by the inductive hypothesis, $\gamma \in L_\beta \cap \mathbf{ON} \subseteq L_{\beta+1} \cap \mathbf{ON}$. Thus it remains only to show that $\beta \in L_{\beta+1}$. Now there is a natural Δ_0 formula $\varphi(x)$ which expresses that x is an ordinal:

$$\forall y \in x \forall z \in y(z \in x) \wedge \forall y \in x \forall z \in y \forall w \in z(w \in y);$$

this just says that x is transitive and every member of x is transitive. Now $\varphi(x)$ is absolute, so

$$\beta = L_\beta \cap \mathbf{ON} = \{x \in L_\beta : (L_\beta, \in) \models \varphi(x)\} \in \text{def}(L_\beta) = L_{\beta+1}. \qquad \square$$

Theorem 13.103. *$L(M, x)$ is a model of ZF.*

Proof. See the proof of Theorem 8. $\square$

Theorem 13.104. *If $M \models AC$, then $L(M, x) \models AC$*

Proof. See the proof of Theorem 8. $\square$

Lemma 13.105. *$M \cup \{x\} \subseteq L(M, x)$.* $\square$

We abbreviate *countable transitive model of ZFC* by c.t.m.

Lemma 13.106. *If M is a c.t.m., $x \subseteq M$, N is a c.t.m., $M \subseteq N$, and $x \in N$, then $M[x] \subseteq N$.* $\square$

Lemma 13.107. *For every infinite cardinal κ, $L_\kappa \subseteq H_\kappa$.*

Proof. Clearly $L_\omega = H_\omega$. Now suppose that $\kappa > \omega$ and $x \in L_\kappa$. Choose $\alpha < \kappa$ such that $x \in L_\alpha$. Then $\text{trcl}(x) \subseteq L_\alpha$, so $|\text{trcl}(x)| \leq |L_\alpha| = |\alpha| < \kappa$. $\square$

$ZF - P$ is the set of axioms of ZFC minus the axiom of choice and the power set axiom.

Lemma 13.108. *If M is a transitive set and $M \models ZF - P$, then $M \models V = L$ iff $M = L_\gamma$, where γ is the least ordinal such that $\gamma \not\subseteq M$.*

Proof. Clearly γ is a limit ordinal. By absoluteness, $L_\delta \in M$ for all $\delta < \gamma$. It follows that $L_\gamma = \bigcup_{\delta<\gamma} L_\delta \subseteq M$. Now

$$M \models V = L \quad \text{iff} \quad \forall x \in M \exists \delta \in \gamma [x \in L_\delta] \quad \text{iff} \quad M \subseteq L_\gamma \quad \text{iff} \quad M = L_\gamma. \qquad \square$$

Lemma 13.109. *Let κ be an uncountable regular cardinal. Let $\langle A_\xi : \xi \leq \kappa \rangle$ be a system of sets such that*

(i) $\xi < \eta \to A_\xi \subseteq A_\eta$.
(ii) $A_\eta = \bigcup_{\xi<\eta} A_\xi$ for η limit $\leq \kappa$.
(iii) $\forall \xi < \kappa [|A_\xi| < \kappa$.
(iv) $|A_\kappa| = \kappa$.

Then $\forall \xi < \kappa \exists \eta \in (\xi, \kappa)[A_\eta \neq \emptyset \wedge A_\eta \preceq A_\kappa \wedge \eta$ is a limit ordinal $]$.

Proof. Let $\langle \varphi_i : i < \omega \rangle$ list all formulas not using $\forall$. For each $\varphi_i(\overline{x})$ which is of the form $\exists y \varphi_j(\overline{x}, y)$, say with $\overline{x}$ of length r, define $F_i : {}^r A_\kappa$ as follows. If $A_\kappa \models \varphi_i(\overline{a}$, then $F_i(a)$ is the least $\zeta < \kappa$ such that $\exists b \in A_\zeta [A_\kappa \models \varphi_j(\overline{a}, b)$. If $A_\kappa \models \neg\varphi_i(\overline{a}$, then $F_i(\overline{a}) = 0$. Define $G_i : \kappa \to \kappa$ by $G_i(\xi) = \sup\{F_i(\overline{a}) : \overline{a} \in A_\xi\}$ if φ_i is existential, with $G_i(\xi) = 0$ otherwise. Then $G_i(\xi) < \kappa$ since κ is regular. Let $K(\xi)$ be the larger of $\xi + 1$ and $\sup\{G_i(\xi : i, \omega\}$.

Now take any $\xi < \kappa$. Let ζ_0 be the least ordinal greater than ξ such that $A_{\zeta_0} \neq \emptyset$. Let $\zeta_{n+1} = K(\zeta_n)$. Then $\sup_{n\in\omega} \zeta_n$ is as desired in the lemma. $\square$

Lemma 13.110. *If κ is an uncountable regular cardinal, then $L_\kappa \models (ZF - P) + V = L$.*

Proof. Extensionality and foundation hold since L_κ is transitive. Pairing and union and infinity clearly hold. For comprehension, suppose that φ is a formula with free variables among $x, z, w_1, \ldots, w_n$; we want to show that

$$L_\kappa \models \forall z \forall w_1 \ldots \forall w_n \exists y \forall x (x \in y \leftrightarrow x \in z \wedge \varphi).$$

So suppose that $z, w_1, \ldots, w_n \in L_\kappa$. Let $y = \{x \in z : \varphi^{L_\kappa}(x, z, \overline{w})\}$. Say $z, \overline{w} \in L_\xi$ with $\xi < \kappa$. By Lemma 13.109 let $\eta \in (\xi, \kappa)$ be such that $L_\eta \preceq L_\kappa$. Then $y = \{x \in z : \varphi^{L_\eta}(x, z, \overline{w})\} \in L_{\eta+1} \subseteq L_\kappa$, which proves comprehension.

For replacement, assume that $A, \overline{w} \in L_\kappa$ and $\forall x \in L_\kappa[x \in A \rightarrow \exists! y \in L_\kappa \varphi^{L_\kappa}(x, y, \overline{w})$. Say $A, \overline{w} \in L_\alpha$ with $\alpha \in \kappa$. Note that $|A| \leq |L_\alpha| < \kappa$. Let $\mathrm{dmn}(f) = A$ with $f(x) =$ the $y \in L_\kappa$ such that $L_\kappa \models \varphi^{L_\kappa}(x, y, \overline{w})$. Then $\forall x \in A[\rho(f(x)) < \kappa$, so $\beta \stackrel{\text{def}}{=} \sup\{\rho(f(x)) + 1 : x \in A\} < \kappa$. Then $L_\beta \in L_\kappa$. This proves replacement.

Now $L_\kappa \models V = L$ by Lemma 13.108. □

Lemma 13.111. *$V = L$ implies that for every infinite cardinal κ, $L_\kappa = H_\kappa$.*

Proof. First suppose that $\kappa = \lambda^+$. By Lemma 13.107 it suffices to prove that $H_{\lambda^+} \subseteq L_{\lambda^+}$. Suppose that $b \in H_{\lambda^+}$, and let $T = \mathrm{trcl}(b)$. Then $b \in T$ and $|T| \leq \lambda$. Let θ be a regular uncountable cardinal such that $\rho(T) < \theta$. Then $T \subseteq L_\theta$. By Lemma 13.110, $L_\theta \models (ZF - P + V = L$. By the Löwenheim-Skolem theorem let A be such that $A \preceq H_\theta$, $T \subseteq A$, and $|A| \leq \lambda$. Then also $A \models (ZF - P + V = L$. Let $A \cong B$ with B transitive under the collapsing function π. Then $\pi(x) = x$ for all $x \in T$, in particular $\pi(b) = b$. By Lemma 13.108, $B = L_\beta$, where β is the first ordinal not in B. Now $|\beta| = |L_\beta| = |B| = |A| \leq \lambda$, so $\beta < \lambda^+$, and hence $b \in L_\beta \subseteq L_{\lambda^+}$.

Now if κ is a limit ordinal, then

$$L_\kappa = \bigcup_{\lambda < \kappa} L_{\lambda^+} = \bigcup_{\lambda < \kappa} H_{\lambda^+} = H_\kappa$$ □

Theorem 13.112. *If κ is regular limit, then $L_\kappa \models ZFC + V = L$.*

Proof. Clearly (κ is regular limit)^L. Now we work in L. By Lemma 13.110, $L_\kappa \models (ZF - P) + V = L$. By Lemma 13.111, $L_\kappa = H_\kappa$. To check the power set axiom in H_κ, suppose that $X \in H_\kappa$. Then also $\mathscr{P}(\kappa) \in H_\kappa$. □

14. Forcing

Forcing orders and complete BAs

A forcing order is a triple $\mathbb{P} = (P, \leq, 1)$ such that $\leq$ is a reflexive and transitive relation on the nonempty set P, and $\forall p \in P(p \leq 1)$. Note that we do not assume that $\leq$ is antisymmetric. Partial orders are special cases of forcing orders in which this is assumed (but we do not assume the existence of 1 in partial orders). Note that we assume that every forcing order has a largest element. Many set-theorists use "partial order" instead of "forcing order".

Frequently we use just P for a forcing order; $\leq$ and 1 are assumed.

We say that elements $p, q \in P$ are *compatible* iff there is an $r \leq p, q$. We write $p \perp q$ to indicate that p and q are incompatible. A set A of elements of P is an *antichain* iff any two distinct members of A are incompatible. WARNING: sometimes "antichain" is used to mean pairwise incomparable, or in the case of Boolean algebras, pairwise disjoint. A subset Q of P is *dense* iff for every $p \in P$ there is a $q \in Q$ such that $q \leq p$.

Now we are going to describe how to embed a forcing order into a complete BA. We take the regular open algebra of a certain topological space. We assume a very little bit of topology. To avoid assuming any knowledge of topology we now give a minimalist introduction to topology.

A *topology* on a set X is a collection $\mathscr{O}$ of subsets of X satisfying the following conditions:

(1) $X, \emptyset \in \mathscr{O}$.

(2) $\mathscr{O}$ is closed under arbitrary unions.

(3) $\mathscr{O}$ is closed under finite intersections.

The members of $\mathscr{O}$ are said to be *open*. The *interior* of a subset $Y \subseteq X$ is the union of all open sets contained in Y; we denote it by $\mathrm{int}(Y)$.

Proposition 14.1. *(i)* $\mathrm{int}(\emptyset) = \emptyset$.

(ii) $\mathrm{int}(X) = X$.

(iii) $\mathrm{int}(Y) \subseteq Y$.

(iv) $\mathrm{int}(Y \cap Z) = \mathrm{int}(Y) \cap \mathrm{int}(Z)$.

(v) $\mathrm{int}(\mathrm{int}(Y)) = \mathrm{int}(Y)$.

(vi) $\mathrm{int}(Y) = \{x \in X : x \in U \subseteq Y$ *for some open set* $U\}$.

Proof. (i)–(iii), (v), and (vi) are obvious. For (iv), if U is an open set contained in $Y \cap Z$, then it is contained in Y; so $\mathrm{int}(Y \cap Z) \subseteq \mathrm{int}(Y)$. Similarly for Z, so $\subseteq$ holds. For $\supseteq$, note that the right side is an open set contained in $Y \cap Z$. (v) holds since $\mathrm{int}(Y)$ is open. □

A subset C of X is *closed* iff $X \backslash C$ is open.

Proposition 14.2. *(i)* $\emptyset$ *and* X *are closed.*

(ii) The collection of all closed sets is closed under finite unions and intersections of any nonempty subcollection. □

For any $Y \subseteq X$, the *closure* of Y, denoted by $\mathrm{cl}(Y)$, is the intersection of all closed sets containing Y.

Proposition 14.3. *(i)* $\mathrm{cl}(Y) = X\backslash\mathrm{int}(X\backslash Y)$.
(ii) $\mathrm{int}(Y) = X\backslash\mathrm{cl}(X\backslash Y)$.
(iii) $\mathrm{cl}(\emptyset) = \emptyset$.
(iv) $\mathrm{cl}(X) = X$.
(v) $Y \subseteq \mathrm{cl}(Y)$.
(vi) $\mathrm{cl}(Y \cup Z) = \mathrm{cl}(Y) \cup \mathrm{cl}(Z)$.
(vii) $\mathrm{cl}(\mathrm{cl}(Y)) = \mathrm{cl}(Y)$.
(viii) $\mathrm{cl}(Y) = \{x \in X$ *:for every open set* U*, if* $x \in U$ *then* $U \cap Y \neq \emptyset\}$.

Proof. (i): $\mathrm{int}(X\backslash Y)$ is an open set contained in $X\backslash Y$, so Y is a subset of the closed set $X\backslash\mathrm{int}(X\backslash Y)$. Hence $\mathrm{cl}(Y) \subseteq X\backslash\mathrm{int}(X\backslash Y)$. Also. $\mathrm{cl}(Y)$ is a closed set containing Y, so $X\backslash\mathrm{cl}(Y)$ is an open set contained in $X\backslash Y$. Hence $X\backslash\mathrm{cl}(Y) \subseteq \mathrm{int}(X\backslash Y)$. Hence $X\backslash\mathrm{int}(X\backslash Y \subseteq \mathrm{cl}(Y)$. This proves (i).

(ii): Using (i),

$$X\backslash\mathrm{cl}(X\backslash Y) = X\backslash(X\backslash\mathrm{int}(X\backslash(X\backslash Y))) = \mathrm{int}(Y).$$

(iii)–(v): clear.

(vi):

$$\begin{aligned}
\mathrm{cl}(Y \cup Z) &= X\backslash\mathrm{int}(X\backslash(Y \cup Z)) \quad \text{by (i)}\\
&= X\backslash\mathrm{int}((X\backslash Y) \cap (X\backslash Z))\\
&= X\backslash(\mathrm{int}(X\backslash Y) \cap \mathrm{int}(X\backslash Z))\\
&= [X\backslash\mathrm{int}(X\backslash Y)] \cup [X\backslash\mathrm{int}(X\backslash Z)]\\
&= \mathrm{cl}(Y) \cup \mathrm{cl}(Z).
\end{aligned}$$

(vii):

$$\begin{aligned}
\mathrm{cl}(\mathrm{cl}(Y)) &= \mathrm{cl}(X\backslash\mathrm{int}(X\backslash Y))\\
&= X\backslash\mathrm{int}(X\backslash(X\backslash\mathrm{int}(X\backslash Y)))\\
&= X\backslash\mathrm{int}(\mathrm{int}(X\backslash Y))\\
&= X\backslash\mathrm{int}(X\backslash Y)\\
&= \mathrm{cl}(Y).
\end{aligned}$$

(vii): First suppose that $x \in \mathrm{cl}(Y)$, and $x \in U$, U open. By (i) and Proposition 27.15(vi) we have $U \not\subseteq X\backslash Y$, i.e., $U \cap Y \neq \emptyset$, as desired. Second, suppose that $x \notin \mathrm{cl}(Y)$. Then by (i) and 27.15(vi) there is an open U such that $x \in U \subseteq X\backslash Y$; so $U \cap Y = \emptyset$, as desired. □

Now we go beyond this minimum amount of topology and work with the notion of a regular open set, which is not a standard part of topology courses.

We say that Y is *regular open* iff $Y = \text{int}(\text{cl}(Y))$.

Proposition 14.4. *(i) If Y is open, then $Y \subseteq \text{int}(\text{cl}(Y))$.*

(ii) If U and V are regular open, then so is $U \cap V$.

(iii) $\text{int}(\text{cl}(Y))$ is regular open.

(iv) If U is open, then $\text{int}(\text{cl}(U))$ is the smallest regular open set containing U.

(v) If U is open then $U \cap \text{cl}(Y) \subseteq \text{cl}(U \cap Y)$.

(vi) If U is open, then $U \cap \text{int}(\text{cl}(Y)) \subseteq \text{int}(\text{cl}(U \cap Y))$.

(vii) If U and V are open and $U \cap V = \emptyset$, then $\text{int}(\text{cl}(U)) \cap V = \emptyset$.

(viii) If U and V are open and $U \cap V = \emptyset$, then $\text{int}(\text{cl}(U)) \cap \text{int}(\text{cl}(V)) = \emptyset$.

(ix) For any set M of regular open sets, $\text{int}(\text{cl}(\bigcup M)$ is the least regular open set containing each member of M.

Proof. (i): $Y \subseteq \text{cl}(Y)$, and hence $Y = \text{int}(Y) \subseteq \text{int}(\text{cl}(Y))$.

(ii): $U \cap V$ is open, and so $U \cap V \subseteq \text{int}(\text{cl}(U \cap V))$. For the other inclusion, $\text{int}(\text{cl}(U \cap V)) \subseteq \text{int}(\text{cl}(U)) = U$, and similarly for V, so the other inclusion holds.

(iii): $\text{int}(\text{cl}(X)) \subseteq \text{cl}(X)$, so $\text{cl}(\text{int}(\text{cl}(X))) \subseteq \text{cl}(\text{cl}(X)) = \text{cl}(X)$; hence

$$\text{int}(\text{cl}(\text{int}(\text{cl}(X)))) \subseteq \text{int}(\text{cl}(X));$$

the other inclusion is clear.

(iv): By (iii), $\text{int}(\text{cl}(U))$ is a regular open set containing U. If V is any regular open set containing U, then $\text{int}(\text{cl}(U)) \subseteq \text{int}(\text{cl}(V)) = V$.

(v):

$$\begin{aligned}
U \cap (X\backslash(U \cap Y)) &\subseteq X\backslash Y, \quad \text{hence} \\
U \cap \text{int}(X\backslash(U \cap Y)) &= \text{int}(U) \cap \text{int}(X\backslash(U \cap Y)) \\
&= \text{int}(U \cap (X\backslash(U \cap Y))) \\
&\subseteq \text{int}(X\backslash Y), \quad \text{hence} \\
X\backslash\text{int}(X\backslash Y) &\subseteq X\backslash(U \cap \text{int}(X\backslash(U \cap Y))) \\
&= (X\backslash U) \cup (X\backslash\text{int}(X\backslash(U \cap Y))), \quad \text{hence} \\
U \cap (X\backslash\text{int}(X\backslash Y)) &\subseteq (X\backslash\text{int}(X\backslash(U \cap Y))),
\end{aligned}$$

and (v) follows.

(vi):

$$\begin{aligned}
U \cap \text{int}(\text{cl}(Y)) &= \text{int}(U) \cap \text{int}(\text{cl}(Y)) \\
&= \text{int}(U \cap \text{cl}(Y)) \\
&\subseteq \text{int}(\text{cl}(U \cap Y)) \quad \text{by (v)}.
\end{aligned}$$

(vii): $U \subseteq X\backslash V$, hence $\text{cl}(U) \subseteq \text{cl}(X\backslash V) = X\backslash V$, hence $\text{cl}(U) \cap V = \emptyset$, and the conclusion of (vii) follows.

(viii): Apply (vii) twice.

(ix): If $U \in M$, then $U \subseteq \bigcup M \subseteq \text{int}(\text{cl}(\bigcup M)$. Suppose that V is regular open and $U \subseteq V$ for all $U \in M$. Then $\bigcup M \subseteq V$, and so $\text{int}(\text{cl}(\bigcup M)) \subseteq \text{int}(\text{cl}(V) = V$. $\square$

We let $\text{RO}(X)$ be the collection of all regular open sets in X. We define operations on $\text{RO}(X)$ which will make it a Boolean algebra. For any $Y, Z \in \text{RO}(X)$, let

$$\begin{aligned} Y + Z &= \text{int}(\text{cl}(Y \cup Z)); \\ Y \cdot Z &= Y \cap Z; \\ -Y &= \text{int}(X \backslash Y). \end{aligned}$$

Theorem 14.5. *The structure*

$$\langle \text{RO}(X), +, \cdot, -, \emptyset, X \rangle$$

is a complete BA. Moreover, the ordering $\leq$ coincides with $\subseteq$.

Proof. $\text{RO}(X)$ is closed under $+$, and is closed under $\cdot$. Clearly it is closed under $-$, and $\emptyset, X \in \text{RO}(X)$. Now we check the axioms. The following are completely obvious: (A$'$), (C$'$), (C). Now let unexplained variables range over $\text{RO}(X)$. For (A), note that $U \subseteq U + V \subseteq (U + V) + W$; and similarly $V \subseteq (U + V) + W$ and $W \subseteq U + V \subseteq (U + V) + W$. If $U, V, W \subseteq Z$, then $U + V \subseteq Z$ and hence $(U + V) + W \subseteq Z$. Thus $(U + V) + W$ is the least upper bound in $\text{RO}(X)$ of U, V, W. This is true for all U, V, W. So $U + (V + W) = (V + W) + U$ is also the least upper bound of them; so (A) holds. For (L):

$$U + U \cdot V = \text{int}(\text{cl}(U \cup (U \cap V))) = \text{int}(\text{cl}(U)) = U.$$

(L$'$) clearly holds. For (D), first note that

$$\begin{aligned} Y \cdot (Z + W) &= Y \cap \text{int}(\text{cl}(Z \cup W)) \\ &\subseteq \text{int}(\text{cl}(Y \cap (Z \cup W))) \\ &= \text{int}(\text{cl}((Y \cap Z) \cup (Y \cap W))) \\ &= Y \cdot Z + Y \cdot W. \end{aligned}$$

On the other hand, $(Y \cap Z) \cup (Y \cap W) = Y \cap (Z \cup W) \subseteq Y, Z \cup W$, and hence easily

$$\begin{aligned} Y \cdot Z + Y \cdot W &= \text{int}(\text{cl}((Y \cap Z) \cup (Y \cap W))) \\ &\subseteq \text{int}(\text{cl}(Y) = Y \quad \text{and} \\ Y \cdot Z + Y \cdot W &= \text{int}(\text{cl}((Y \cap Z) \cup (Y \cap W))) \\ &\subseteq \text{int}(\text{cl}(Z \cup W) = Z + W; \end{aligned}$$

so the other inclusion follows, and (D) holds.

(K): For any regular open Y we have $-Y = \text{int}(X \backslash Y) = X \backslash \text{cl}(X \backslash (X \backslash Y)) = X \backslash \text{cl}(Y)$. Hence

$$X = \text{cl}(Y) \cup (X \backslash \text{cl}(Y)) \subseteq \text{cl}(Y) \cup \text{cl}((X \backslash \text{cl}(Y)) = \text{cl}(Y \cup (X \backslash \text{cl}(Y))),$$

and hence $X = Y + -Y$.

(K′): Clearly $\emptyset = Y \cap \text{int}(X \backslash Y) = Y \cdot -Y$.

Thus we have now proved that $\langle \text{RO}(X), +, \cdot, -, \emptyset, X \rangle$ is a BA. Since $\cdot$ is the same as $\cap$, $\leq$ is the same as $\subseteq$. Hence $\langle \text{RO}(X), +, \cdot, -, \emptyset, X \rangle$ is a complete BA. $\square$

Now we return to our task of embedding a forcing order into a complete Boolean algebra. Let P be a given forcing order. For each $p \in P$ let $P \downarrow p = \{q : q \leq p\}$. Now we define

$$\mathscr{O}_P = \{X \subseteq P : (P \downarrow p) \subseteq X \text{ for every } p \in X\}.$$

We check that this gives a topology on P. Clearly $P, \emptyset \in \mathscr{O}$. To show that $\mathscr{O}$ is closed under arbitrary unions, suppose that $\mathscr{X} \subseteq \mathscr{O}$. Take any $p \in \bigcup \mathscr{X}$. Choose $X \in \mathscr{X}$ such that $p \in X$. Then $(P \downarrow p) \subseteq X \subseteq \bigcup \mathscr{X}$, as desired. If $X, Y \in \mathscr{O}_P$, suppose that $p \in X \cap Y$. Then $p \in X$, so $(P \downarrow p) \subseteq X$. Similarly $(P \downarrow p) \subseteq Y$, so $(P \downarrow p) \subseteq X \cap Y$. Thus $X \cap Y \in \mathscr{O}_P$, finishing the proof that $\mathscr{O}_P$ is a topology on P.

We denote the complete BA of regular open sets in this topology by $\text{RO}(P)$.

Now for any $p \in P$ we define

$$e(p) = \text{int}(\text{cl}(P \downarrow p)).$$

Thus e maps P into $\text{RO}(P)$.

This is our desired embedding. Actually it is not really an embedding in general, but it has several useful properties, and for many forcing orders it really is an embedding.

The useful properties mentioned are as follows. We say that a subset X of P is *dense below* p iff for every $r \leq p$ there is a $q \leq r$ such that $q \in X$.

Theorem 14.6. *Let P be a forcing order. Suppose that $p, q \in P$, F is a finite subset of P, $a, b \in \text{RO}(P)$, and N is a subset of* $\text{RO}(P)$

(i) $e[P]$ is dense in $\text{RO}(P)$, i.e., for any nonzero $Y \in \text{RO}(P)$ there is a $p \in P$ such that $e(p) \subseteq Y$.

(ii) If $p \leq q$ then $e(p) \subseteq e(q)$.

(iii) $p \perp q$ iff $e(p) \cap e(q) = \emptyset$.

(iv) If $e(p) \leq e(q)$, then p and q are compatible.

(v) The following conditions are equivalent:

(a) $e(p) \leq e(q)$.

(b) $\{r : r \leq p, q\}$ is dense below p.

(vi) The following conditions are equivalent, for F nonempty:

(a) $e(p) \leq \prod_{q \in F} e(q)$.

(b) $\{r : r \leq q$ for all $q \in F\}$ is dense below p.

(vii) The following conditions are equivalent:

(a) $e(p) \leq (\prod_{q \in F} e(q)) \cdot \sum N$.

(b) $\{r : r \leq q$ for all $q \in F$ and $e(r) \leq s$ for some $s \in N\}$ is dense below p.

(viii) $e(p) \leq -a$ iff there is no $q \leq p$ such that $e(q) \leq a$.

(ix) $e(p) \leq -a + b$ iff for all $q \leq p$, if $e(q) \leq a$ then $e(q) \leq b$.

Proof. (i): Assume the hypothesis. By the definition of the topology and since Y is nonempty and open, there is a $p \in P$ such that $P \downarrow p \subseteq Y$. Hence $e(p) = \text{int}(\text{cl}(P \downarrow p)) \subseteq \text{int}(\text{cl}(Y)) = Y$.

(ii): If $p \leq q$, then $P \downarrow p \subseteq P \downarrow q$, and so $e(p) = \text{int}(\text{cl}(P \downarrow p)) \subseteq \text{int}(\text{cl}(P \downarrow q) = e(q))$.

(iii): Assume that $p \perp q$. Then $(P \downarrow p) \cap (P \downarrow q) = \emptyset$, and hence $e(p) \cap e(q) = \emptyset$.

Conversely, suppose that $e(p) \cap e(q) = \emptyset$. Then $(P \downarrow p) \cap (P \downarrow q) \subseteq e(p) \cap e(q) = \emptyset$, and so $p \perp q$.

(iv): If $e(p) \leq e(q)$, then $e(p) \cdot e(q) = e(p) \neq \emptyset$, so p and q are compatible by (iii).

(v): For (a)$\Rightarrow$(b), suppose that $e(p) \leq e(q)$ and $s \leq p$. Then $e(s) \leq e(p) \leq e(q)$, so s and q are compatible by (iv); say $r \leq s, q$. Then $r \leq s \leq p$, hence $r \leq p, q$, as desired.

For (b)$\Rightarrow$(a), suppose that $e(p) \not\leq e(q)$. Thus $e(p) \cdot -e(q) \neq 0$. Hence there is an s such that $e(s) \subseteq e(p) \cdot -e(q)$. Hence $e(s) \cdot e(q) = \emptyset$, so $s \perp q$ by (iii). Now $e(s) \subseteq e(p)$, so s and p are compatible by (iv); say $t \leq s, p$. For any $r \leq t$ we have $r \leq s$, and hence $r \perp q$. So (b) fails.

(vi): We proceed by induction on $|F|$. The case $|F| = 1$ is given by (v). Now assume the result for F, and suppose that $t \in P \backslash F$. First suppose that $e(p) \leq \prod_{q \in F} e(q) \cdot e(t)$. Suppose that $s \leq p$. Now $e(p) \leq \prod_{q \in F} e(q)$, so by the inductive hypothesis there is a $u \leq s$ such that $u \leq q$ for all $q \in F$. Thus $e(u) \leq e(s) \leq e(p) \leq e(t)$, so by (iv), u and t are compatible. Take any $v \leq u, t$. then $v \leq q$ for any $q \in F \cup \{t\}$, as desired.

Second, suppose that (b) holds for $F \cup \{t\}$. In particular, $\{r : r \leq q \text{ for all } q \in F\}$ is dense below p, and so $e(p) \leq \prod_{q \in F} e(q)$ by the inductive hypothesis. But also clearly $\{r : r \leq t\}$ is dense below p, so $e(p) \leq e(t)$ too, as desired.

(vii): First assume that $e(p) \leq (\prod_{q \in F} e(q)) \cdot \sum N$, and suppose that $u \leq p$. By (vi), there is a $v \leq u$ such that $v \leq q$ for each $q \in F$. Now $e(v) \leq e(u) \leq e(p) \leq \sum N$, so $0 \neq e(v) = e(v) \cdot \sum N = \sum_{s \in N} (e(v) \cdot e(s))$. Hence there is an $s \in N$ such that $e(v) \cdot e(s) \neq 0$. Hence by (iii), v and s are compatible; say $r \leq v, s$. Clearly r is in the set described in (b).

Second, suppose that (b) holds. Clearly then $\{r : r \leq q \text{ for all } q \in F\}$ is dense below p, and so $e(p) \leq \prod_{q \in F} e(q)$ by (vi). Now suppose that $e(p) \not\leq \sum N$. Then $e(p) \cdot -\sum N \neq 0$, so there is a q such that $e(q) \leq e(p) \cdot -\sum N$. By (iv), q and p are compatible; say $s \leq p, q$. Then by (b) choose $r \leq s$ and $t \in N$ such that $e(r) \leq t$. Thus $e(r) \leq e(s) \cdot t \leq e(p) \cdot t \leq (-\sum N) \cdot \sum N = 0$, contradiction.

(viii)$\Rightarrow$: Assume that $e(p) \leq -a$. Suppose that $q \leq p$ and $e(q) \leq a$. Then $e(q) \leq -a \cdot a = 0$, contradiction.

(viii)$\Leftarrow$: Assume that $e(p) \not\leq -a$. Then $e(p) \cdot a \neq 0$, so there is a q such that $e(q) \leq e(p) \cdot a$. By (vii) there is an $r \leq p, q$ with $e(r) \leq a$, as desired.

(ix)$\Rightarrow$: Assume that $e(p) \leq -a + b$, $q \leq p$, and $e(q) \leq a$. Then $e(q) \leq a \cdot (-a + b) \leq b$, as desired.

(ix)$\Leftarrow$: Assume the indicated condition, but suppose that $e(p) \not\leq -a + b$. Then $e(p) \cdot a \cdot -b \neq 0$, so there is a q such that $e(q) \leq e(p) \cdot a \cdot -b$. By (vii) with $F = \{p\}$ and $N = \{a \cdot -b\}$ we get q such that $q \leq p$ and $e(q) \leq a \cdot -b$. So $q \leq p$ and $e(q) \leq a$, so by our condition, $e(q) \leq b$. But also $e(q) \leq -b$, contradiction. $\square$

We now expand on the remarks above concerning when e really is an embedding. Note that if P is a simple ordering, then the closure of $P \downarrow p$ is P itself, and hence P has only two regular open subsets, namely the empty set and P itself. If the ordering on P is trivial, meaning that no two elements are comparable, then every subset of P is regular open.

An important condition satisfied by many forcing orders is defined as follows. We say that P is *separative* iff it is a partial order (thus is an antisymmetric forcing order), and for any $p, q \in P$, if $p \not\leq q$ then there is an $r \leq p$ such that $r \perp q$.

Proposition 14.7. *Let P be a forcing order.*

(i) $\mathrm{cl}(P \downarrow p) = \{q : p$ *and* q *are compatible*$\}$.

(ii) $e(p) = \{q :$ *for all* $r \leq q$, r *and* p *are compatible*$\}$.

(iii) The following conditions are equivalent:

(a) P is separative.

(b) e is one-one, and for all $p, q \in P$, $p \leq q$ iff $e(p) \leq e(q)$.

Proof. (i) and (ii) are clear. For (iii), (a)⇒(b), assume that P is separative. Take any $p, q \in P$. If $p \leq q$, then $e(p) \leq e(q)$. Suppose that $p \not\leq q$. Choose $r \leq p$ such that $r \perp q$. Then $r \in e(p)$, while $r \notin e(q)$ by (ii). Thus $e(p) \not\leq e(q)$.

Now suppose that $e(p) = e(q)$. Then $p \leq q \leq p$ by what was just shown, so $p = q$ since P is a partial order.

For (iii), (b)⇒(a), suppose that $p \leq q \leq p$. Then $e(p) \subseteq e(q) \subseteq e(p)$, so $e(p) = e(q)$, and hence $p = q$. So P is a partial order. Suppose that $p \not\leq q$. Then $e(p) \not\subseteq e(q)$. Choose $s \in e(p) \backslash e(q)$. Since $s \notin e(q)$, by (ii) we can choose $t \leq s$ such that $t \perp q$. Since $s \in e(p)$, it follows that t and p are compatible; choose $r \leq t, p$. Clearly $r \perp q$. □

Now we prove a theorem which says that the regular open algebra of a forcing order is unique up to isomorphism.

Theorem 14.8. *Let P be a forcing order, A a complete BA, and j a function mapping P into $A \backslash \{0\}$ with the following properties:*

(i) $j[P]$ is dense in A, i.e., for any nonzero $a \in A$ there is a $p \in P$ such that $j(p) \subseteq a$.

(ii) For all $p, q \in P$, if $p \leq q$ then $j(p) \leq j(q)$.

(iii) For any $p, q \in P$, $p \perp q$ iff $j(p) \cdot j(q) = 0$.

Then there is a unique isomorphism f from $\mathrm{RO}(P)$ *onto A such that $f \circ e = j$. That is, f is a bijection from* $\mathrm{RO}(P)$ *onto A, and for any $x, y \in \mathrm{RO}(P)$, $x \subseteq y$ iff $f(x) \leq f(y)$; and $f \circ e = j$.*

Note that since the Boolean operations are easily expressible in terms of $\leq$ (as least upper bounds, etc.), the condition here implies that f preserves all of the Boolean operations too; this includes the infinite sums and products.

Proof. Before beginning the proof, we introduce some notation in order to make the situation more symmetric. Let $B_0 = \mathrm{RO}(P)$, $B_1 = A$, $k_0 = e$, and $k_1 = j$. Then for each $m < 2$ the following conditions hold:

(1) $k_m[P]$ is dense in B_m.

(2) For all $p, q \in P$, if $p \leq q$ then $k_m(p) \leq k_m(q)$.

(3) For all $p, q \in P$, $p \perp q$ iff $k_m(p) \cdot k_m(q) = 0$.

(4) For all $p, q \in P$, if $k_m(p) \leq k_m(q)$, then p and q are compatible.

In fact, (1)–(3) follow from the assumptions of the theorem. Condition (4) for $m = 0$, so that $k_m = e$, is clear. For $m = 1$, so that $k_m = j$, it follows easily from (iii).

Now we begin the proof. For each $m < 2$ we define, for any $x \in B_m$,

$$g_m(x) = \sum\{k_{1-m}(p) : p \in P,\ k_m(p) \leq x\}.$$

The proof of the theorem now consists in checking the following, for each $m \in 2$:

(5) If $x, y \in B_m$ and $x \leq y$, then $g_m(x) \leq g_m(y)$.

(6) $g_{1-m} \circ g_m$ is the identity on B_m.

(7) $g_0 \circ k_0 = k_1$.

In fact, suppose that (5)–(7) have been proved. If $x, y \in \mathrm{RO}(P)$, then

$x \leq y$ implies that $g_0(x) \leq g_0(y)$ by (5);
$g_0(x) \leq g_0(y)$ implies that $x = g_1(g_0(x)) \leq g_1(g_0(y)) = y$ by (5) and (6).

Also, (6) holding for both $m = 0$ and $m = 1$ implies that g_0 is a bijection from $\mathrm{RO}(P)$ onto A. Moreover, by (7), $g_0 \circ e = g_0 \circ k_0 = k_1 = j$. So g_0 is the desired function f of the theorem.

Now (5) is obvious from the definition. To prove (6), assume that $m \in 2$. We first prove

(8) For any $p \in P$ and any $b \in B_m$, $k_m(p) \leq b$ iff $k_{1-m}(p) \leq g_m(b)$.

To prove (8), first suppose that $k_m(p) \leq b$. Then obviously $k_{1-m}(p) \leq g_m(b)$. Second, suppose that $k_{1-m}(p) \leq g_m(b)$ but $k_m(p) \not\leq b$. Thus $k_m(p) \cdot -b \neq 0$, so by the denseness of $k_m[P]$ in B_m, choose $q \in P$ such that $k_m(q) \leq k_m(p) \cdot -b$. Then p and q are compatible by (4), so let $r \in P$ be such that $r \leq p, q$. Hence

$$k_{1-m}(r) \leq k_{1-m}(p) \leq g_m(b) = \sum\{k_{1-m}(s) : s \in P,\ k_m(s) \leq b\}.$$

Hence $k_{1-m}(r) = \sum\{k_{1-m}(s) \cdot k_{1-m}(r) : s \in P,\ k_m(s) \leq b\}$, so there is an $s \in P$ such that $k_m(s) \leq b$ and $k_{1-m}(s) \cdot k_{1-m}(r) \neq 0$. Hence s and r are compatible; say $t \leq s, r$. Hence $k_m(t) \leq k_m(r) \leq k_m(q) \leq -b$, but also $k_m(t) \leq k_m(s) \leq b$, contradiction. This proves (8).

Now take any $b \in B_m$. Then

$$\begin{aligned} g_{1-m}(g_m(b)) &= \sum\{k_m(p) : p \in P,\ k_{1-m}(p) \leq g_m(b)\} \\ &= \sum\{k_m(p) : p \in P,\ k_m(p) \leq b\} \\ &= b. \end{aligned}$$

Thus (6) holds.

For (7), clearly $k_1(p) \leq g_0(k_0(p))$. Now suppose that $k_0(q) \leq k_0(p)$ but $k_1(q) \not\leq k_1(p)$. Then $k_1(q) \cdot -k_1(p) \neq 0$, so there is an r such that $k_1(r) \leq k_1(q) \cdot -k_1(p)$. Hence q and r are compatible, but $r \perp p$. Say $s \leq q, r$. Then $k_0(s) \leq k_0(q) \leq k_0(p)$, so s and p are compatible. Say $t \leq s, p$. Then $t \leq r, p$, contradiction. This proves (7).

This proves the existence of f. Now suppose that g is also an isomorphism from $\mathrm{RO}(P)$ onto A such that $g \circ e = j$, but suppose that $f \neq g$. Then there is an $X \in \mathrm{RO}(P)$ such that $f(X) \neq g(X)$. By symmetry, say that $f(X) \cdot -g(X) \neq 0$. By (ii), choose $p \in P$ such that $j(p) \leq f(X) \cdot -g(X)$. So $f(e(p)) = j(p) \leq f(X)$, so $e(p) \leq X$, and hence $j(p) = g(e(p)) \leq g(X)$. This contradicts $j(p) \leq -g(X)$. $\square$

From now on we assume that a dense subset of a BA does not contain 0.

Proposition 14.9. *If D is a dense subset of a BA A, then $(D, \leq)$ is separative.*

Proof. Clearly $(D, \leq)$ is a partial order. Now suppose that $p, q \in D$ and $p \not\leq q$. Then $p \cdot -q \neq 0$, so there is an $r \in D$ such that $r \leq p \cdot -q$. Then $r \leq p$. If $s \in D$ and $s \leq r, q$ then $s \leq -q$ also, so $s = 0$, contradiction. Hence $r \perp q$. $\square$

Lemma 14.10. *If $(P, \leq, 1)$ is a forcing poset, then there is a separative forcing poset $(Q, \leq, 1)$ and a mapping h of P onto Q such that:*

(i) $x \leq y$ implies $h(x) \leq h(y)$.

(ii) x and y are compatible in P iff $h(x)$ and $h(y)$ are compatible in Q.

Proof. We define $x \sim y$ iff $x, y \in P$ and

$$\forall z \in P[z \text{ is compatible with } x \leftrightarrow z \text{ is compatible with } y].$$

Clearly $\sim$ is an equivalence relation on P. Let $Q = P/\sim$. Now we define

$$q_1 \preceq q_2 \quad \text{iff} \quad \begin{array}{l} q_1, q_2 \in Q \text{ and there exist } p_1 \in q_1 \text{ and } p_2 \in q_2 \text{ such that} \\ \forall s \leq p_1[s \text{ and } p_2 \text{ are compatible}]. \end{array}$$

(1) $[p_1] \preceq [p_2]$ iff $\forall r \leq p_1[r$ and p_2 are compatible$]$.

In fact, $\Leftarrow$ is clear. Now suppose that $[p_1] \preceq [p_2]$. Choose p_1' and p_2' so that $p_1 \sim p_1'$, $p_2 \sim p_2'$, and $\forall r \leq p_1'[r$ and p_2' are compatible$]$. Suppose that $r \leq p_1$. In particular r and p_1 are compatible, so by $p_1 \sim p_1'$, r and p_1' are compatible; say $s \leq r, p_1'$. Then s and p_2' are compatible. So r and p_2' are compatible, and hence r and p_2 are compatible, as desired in (1).

We define $q_1 \prec q_2$ iff $q_1 \preceq q_2$ and $q_1 \neq q_2$. To show that $\prec$ is transitive, suppose that $[p_1] \prec [p_2] \prec [p_3]$. Take any $r \leq p_1$. Then r and p_2 are compatible by (1); say $s \leq r, p_2$. Then s and p_3 are compatible, so also r and p_3 are compatible. Thus $[p_1] \preceq [p_3]$. Suppose that $[p_1] = [p_3]$. Suppose that r and p_1 are compatible; say $s \leq r, p_1$. Then s and p_2 are compatible, so r and p_2 are compatible. Conversely, suppose that r and p_2 are compatible. Say $s \leq r, p_2$. Then s and p_3 are compatible, so by $[p_1] = [p_3]$, s and p_1 are compatible. So r and p_1 are compatible. We have shown that $\forall r(r$ and p_1 are compatible iff r and p_2 are compatible). So $[p_1] = [p_2]$, contradiction. So $\prec$ is transitive.

Let $h(p) = [p]$ for any $p \in P$. Clearly $p \leq q$ implies that $h(p) \leq h(q)$, using (1). Now suppose that p_1 and p_2 are compatible. Say $r \leq p_1, p_2$. Then $[r] \leq [p_1], [p_2]$. Conversely, suppose that $[p_1]$ and $[p_2]$ are compatible. Say $[r] \leq [p_1], [p_2]$. Then by (1), $\forall s \leq r(s$ and p_1 are compatible), and $\forall s \leq r(s$ and p_2 are compatible). Choose $s \leq r, p_1$; then choose $t \leq s, p_2$. Then $t \leq p_1, p_2$, so p_1 and p_2 are compatible.

Finally, we show that Q is separative. Suppose that $[p_1] \not\leq [p_2]$. Then by (1) there is an $r \leq p_1$ such that r and p_2 are incompatible. So $[r] \leq [p_1]$ and $[r]$ and $[p_2]$ are incompatible, as desired. □

Lemma 14.11. *Let P be a forcing order, and let Q and h be given by Lemma 14.10. Then* $\mathrm{RO}(P) \cong \mathrm{RO}(Q)$*. In fact, there is an isomorphism f from* $\mathrm{RO}(P)$ *onto* $\mathrm{RO}(Q)$ *such that $f(e_P(p)) = e_Q(h(p))$ for all $p \in P$.*

Proof. It suffices to show that $(\mathrm{rng}(e_P), \leq)$ is isomorphic to $(\mathrm{rng}(e_Q), \leq)$, by page 57 of the Handbook of Boolean algebras.

(1) If $e_P(p) = e_P(p')$, then $e_Q(h(p)) = e_Q(h(p'))$.

In fact, assume that $e_P(p) = e_P(p')$. Then by Theorem 14.6(v), $\{r : r \leq p, p'\}$ is dense below both p and p'. Now suppose that $h(q) \leq h(p)$. Then $h(q)$ and $h(p)$ are compatible, so q and p are compatible. Say $r \leq p, q$. Choose $s \leq r$ such that $s \leq p, p'$. Then $h(s) \leq h(p), h(p')$. Since $h(s) \leq h(q)$, this shows that $\{t : t \leq h(p), h(p')\}$ is dense below $h(p)$. Similarly, it is dense below $h(p')$. Hence by Theorem 14.6(v), $e_Q(h(p)) = e_Q(h(p'))$, So (1) holds.

For each $p \in P$ define $f(e_P(p)) = e_Q(h(p))$. Then f is well-defined by (1). To show that f is one-one, suppose that $e_Q(h(p)) = e_Q(h(p'))$. Then $\{r : r \leq h(p), h(p')\}$ is dense below both $h(p)$ and $h(p')$). Now suppose that $q \leq p$. Then $h(q) \leq h(p)$. Then choose r with $h(r) \leq h(q), h(p')$. Then there is an $s \leq r, q$. $h(s) \leq h(p')$, so there is a $t \leq s, p'$. Thus $t \leq q, p, p'$. This shows that $\{r : r \leq p, p'\}$ is dense below p. Similarly it is dense below p', so $e_P(p) = e_P(p')$.

Clearly f maps onto $\mathrm{rng}(e_Q)$.

We have $e_P(p) \subseteq e_P(p')$ iff $e_Q(h(p)) \subseteq e_Q(h(p'))$ by the above argument. □

Boolean-valued models

Now let B be a complete BA. A *Boolean-valued model* is a triple (A, E, F) with A a class and E and F 2-place functions on A satisfying the following conditions, where $E(a, b)$ is abbreviated by $[\![a = b]\!]$ and $F(a, b)$ by $[\![a \in b]\!]$:

(a) $[\![a = a]\!] = 1$.

(b) $[\![a = b]\!] = [\![b = a]\!]$.

(c) $[\![a = b]\!] \cdot [\![b = c]\!] \leq [\![a = c]\!]$.

(d) $[\![a \in b]\!] \cdot [\![c = a]\!] \cdot [\![d = b]\!] \leq [\![c \in d]\!]$.

Then we define by recursion, where $a_1, \ldots, a_n \in A$,

$$\begin{aligned}
\llbracket \neg\varphi(a_1, \ldots, a_n) \rrbracket &= -\llbracket \varphi(a_1, \ldots, a_n) \rrbracket; \\
\llbracket (\varphi \vee \psi)(a_1, \ldots, a_n) \rrbracket &= \llbracket \varphi(a_1, \ldots, a_n) \rrbracket + \llbracket \psi(a_1, \ldots, a_n) \rrbracket; \\
\llbracket (\varphi \wedge \psi)(a_1, \ldots, a_n) \rrbracket &= \llbracket \varphi(a_1, \ldots, a_n) \rrbracket \cdot \llbracket \psi(a_1, \ldots, a_n) \rrbracket; \\
\llbracket \exists x \varphi(x, a_1, \ldots, a_n) \rrbracket &= \sum_{b \in A} \llbracket \varphi(b, a_1, \ldots, a_n) \rrbracket; \\
\llbracket \forall x \varphi(x, a_1, \ldots, a_n) \rrbracket &= \prod_{b \in A} \llbracket \varphi(b, a_1, \ldots, a_n) \rrbracket.
\end{aligned}$$

In any BA we define $a \Rightarrow b = -a + b$.

Now we go into the connection of Boolean-valued models with provability. For the logic notions, we follow my notes on set theory. The following are *logical axioms* in the set theoretic language. φ, ψ, χ are arbitrary formulas.

(A1) $\varphi \to (\psi \to \varphi)$.
(A2) $[[\varphi \to (\psi \to \chi)] \to [(\varphi \to \psi) \to (\varphi \to \chi)]$.
(A3) $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$.
(A4) $\forall v_i(\varphi \to \psi) \to (\forall v_i \varphi \to \forall v_i \chi)$.
(A5) $\varphi \to \forall v_i \varphi$ if v_i does not occur in φ.
(A6) $\exists v_i(v_i = v_j)$ for $i \neq j$.
(A7) $v_i = v_j \to (v_i = v_k \to v_j = v_k)$ for $i, j.k$ distinct.
(A8) $v_i = v_j \to (v_k = v_i \to v_k = v_j)$ for $i, j.k$ distinct.
(A9) $v_i = v_j \to (v_i \in v_k \to v_j \in v_k)$ for $i, j.k$ distinct.
(A10) $v_i = v_j \to (v_k \in v_i \to v_k \in v_j)$ for $i, j.k$ distinct.

Lemma 14.12. *If φ is a logical axiom and $\overline{b} \in A$, then $\llbracket \varphi(\overline{b}) \rrbracket = 1$.*

Proof.

(A1):

$$\begin{aligned}
\llbracket \varphi \to (\psi \to \varphi) \rrbracket &= (\llbracket \varphi \rrbracket \Rightarrow (\llbracket \psi \rrbracket \Rightarrow \llbracket \varphi \rrbracket)) \\
&= -\llbracket \varphi \rrbracket + -\llbracket \psi \rrbracket + \llbracket \varphi \rrbracket \\
&= 1.
\end{aligned}$$

(A2):

$$\begin{aligned}
&\llbracket [\varphi \to (\psi \to \chi)] \to [(\varphi \to \psi) \to (\varphi \to \chi)] \rrbracket = \\
&\qquad - [-\llbracket \varphi \rrbracket + -\llbracket \psi \llbracket + \llbracket \chi \rrbracket] + -(-\llbracket \varphi \rrbracket + \llbracket \psi \rrbracket) + -\llbracket \varphi \rrbracket + \llbracket \chi \rrbracket \\
&\qquad = \llbracket \varphi \rrbracket \cdot \llbracket \psi \llbracket \cdot - \llbracket \chi \rrbracket + \llbracket \varphi \rrbracket \cdot -\llbracket \psi \rrbracket + -\llbracket \varphi \rrbracket + \llbracket \chi \rrbracket \\
&\qquad = \llbracket \varphi \rrbracket \cdot \llbracket \psi \llbracket \cdot - \llbracket \chi \rrbracket + \llbracket \varphi \rrbracket \cdot \llbracket \psi \llbracket \cdot \llbracket \chi \rrbracket + \llbracket \varphi \rrbracket \cdot -\llbracket \psi \rrbracket + -\llbracket \varphi \rrbracket + \llbracket \chi \rrbracket \\
&\qquad = \llbracket \varphi \rrbracket \cdot \llbracket \psi \llbracket + \llbracket \varphi \rrbracket \cdot -\llbracket \psi \rrbracket + -\llbracket \varphi \rrbracket + \llbracket \chi \rrbracket \\
&\qquad = 1.
\end{aligned}$$

(A3):

$$\begin{aligned}[\![(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)]\!] &= -(-- [\![\varphi]\!] + -[\![\psi]\!]) + -[\![\psi]\!] + [\![\varphi]\!] \\ &= -[\![\varphi]\!] \cdot [\![\psi]\!] + -[\![\psi]\!] + [\![\varphi]\!] \\ &= -[\![\varphi]\!] \cdot [\![\psi]\!] + [\![\varphi]\!] \cdot [\![\psi]\!] + -[\![\psi]\!] + [\![\varphi]\!] \\ &= 1.\end{aligned}$$

(A4):

$$\begin{aligned}\prod_{a\in A}([\![\varphi(a) \Rightarrow \psi(a)]\!] \cdot \prod_{a\in A}[\![\varphi(a)]\!] &= \prod_{a\in A}(-[\![\varphi(a)]\!] + [\![\psi(a)]\!]) \cdot \prod_{a\in A}[\![\varphi(a)]\!] \\ &\leq \prod_{a\in A}[\![\psi(a)]\!].\end{aligned}$$

Now (A4) follows.

(A5): For v_i not occurring in φ:

$$\begin{aligned}[\![\varphi(a) \to \forall v_i\varphi(v_i)]\!] &= [\![\varphi(a)]\!] \Rightarrow \prod_{a\in A}[\![\varphi(a)]\!] \\ &= [\![\varphi(a)]\!] \Rightarrow [\![\varphi(a)]\!] = 1.\end{aligned}$$

(A6): For $i \neq j$;

$$[\![\exists v_i(v_i = a)]\!] = \sum_{b\in A}[\![b = a]\!] = 1.$$

(A7): for $i, j.k$ distinct, using (b) and (c),

$$[\![a = b]\!] \cdot [\![a = c]\!] = [\![b = a]\!] \cdot [\![a = c]\!] \leq [\![b = c]\!]$$

(A7) follows.

(A8): for $i, j.k$ distinct, using (c),

$$[\![a = b]\!] \cdot [\![c = a]\!] \leq [\![c = b]\!]$$

(A9): using (d):

$$[\![a = b]\!] \cdot [\![a \in c]\!] \leq [\![b \in c]\!]$$

(A10):

$$[\![a = b]\!] \cdot [\![c \in a]\!] \leq [\![c \in b]\!]$$ □

Now a *logical proof* is a finite sequence $\langle\varphi_0, \ldots, \varphi_{m-1}\rangle$ of formulas such that for each $i < m$ one of the following conditions holds:

(I1) φ_i is a logical axiom
(I2) There are $j, k < i$ such that φ_j is the formula $\varphi_k \to \varphi_i$.

(I3) (generalization) There exist $j < i$ and $k \in \omega$ such that φ_i is the formula $\forall v_k \varphi_j$.

We write $\vdash \varphi$ if there is a logical proof with last entry φ.

Theorem 14.13. *If* $\vdash \varphi$ *then for any* $\overline{b} \in A$, $[\![\varphi(\overline{b})]\!] = 1$.

Proof. Let $\langle \varphi_0, \ldots, \varphi_{m-1} \rangle$ be a logical proof. We prove by induction that $[\![\varphi_i(\overline{b})]\!] = 1$ for all $\overline{b} \in A$ and all $i < m$. Suppose it is true for all $j < i$.

Case 1. φ_i is a logical axiom. Then $[\![\varphi_i]\!] = 1$ by Theorem 14.12.

Case 9. There are $j, k < i$ such that φ_j is $\varphi_k \to \varphi_i$. Then

$$[\![\varphi_i]\!] = [\![\varphi_k]\!] \cdot (-[\![\varphi_k]\!] + [\![\varphi_i]\!]) = 1.$$

Case 3. There exist $j < i$ and $k \in \omega$ such that φ_i is $\forall v_k \varphi_j(v_k, \overline{b})$. Then

$$[\![\varphi_i]\!] = \prod_{a \in A} [\![\varphi_j(ba, \overline{b})]\!] = 1 \qquad \square$$

Corollary 14.14. $[\![x = y]\!] \cdot [\![\varphi(x)]\!] \leq [\![\varphi(y)]\!]$. $\square$

A Boolean valued model (A, E, F) is *full* iff for any formula $\varphi(x, \overline{y})$ we have

$$\forall \overline{b} \in A \exists a \in A [[\![\varphi(a, \overline{b})]\!] =]\!] \exists x \varphi(x, \overline{b})]\!]].$$

If $\mathfrak{A} = (A, E^*, F^*)$ is a Boolean valued model over a complete BA B and F is an ultrafilter on B we define

$$\equiv_F^{\mathfrak{A}} = \{(a, b) : a, b \in A \text{ and } [\![a = b]\!] \in F\}.$$

Lemma 14.15. $\equiv_F^{\mathfrak{A}}$ *is an equivalence relation on* A.

Proof. Clearly $\equiv_F^{\mathfrak{A}}$ is symmetric and reflexive on A. Now suppose that $x \equiv_F^{\mathfrak{A}} y \equiv_F^{\mathfrak{A}} z$. Thus $[\![x = y]\!], [\![y = z]\!] \in F$, so $[\![x = y]\!] \cdot [\![y = z]\!] \in F$. Since $[\![x = y]\!] \cdot [\![y = z]\!] \leq [\![x = z]\!]$, it follows that $x \equiv_F^{\mathfrak{A}} z$. $\square$

Now we define, for $a, b \in A / \equiv_F^{\mathfrak{A}}$, $aE'b$ iff $\exists x \in a \exists y \in b [[\![x \in y]\!] \in F]$.

Lemma 14.16. $\forall x, y \in A[[x]E'[y]$ *iff* $[\![x \in y]\!] \in F]$.

Proof. $\Leftarrow$ is clear. Now suppose that $[x]E'[y]$. Choose $x' \equiv_F^{\mathfrak{A}} x$ amd $y' \equiv_F^{\mathfrak{A}} y$ such that $[\![x' \in y']\!] \in F$. Then

$$[\![x = x']\!], [\![x' \in y']\!], [\![y = y']\!] \in F,$$

so their product is in F, and this product is $\leq [\![x \in y]\!]$, so $[\![x \in y]\!] \in F$. $\square$

We define $\mathfrak{A} / \equiv_F^{\mathfrak{A}} \; = (A / \equiv_F^{\mathfrak{A}}, E')$.

Lemma 14.17. *Let $\mathfrak{A}$ be full. For any formula $\varphi(\overline{x})$ and any $\overline{a} \in A$,*

$$\mathfrak{A}/\equiv_F^{\mathfrak{A}} \models \varphi([a_0], \ldots, [a_{m-1}]) \quad \text{iff} \quad [\![\varphi(a_0, \ldots, a_{m-1})]\!] \in F.$$

Proof. Induction on φ.
Case 1. φ is $v_i = v_j$. Then

$$\mathfrak{A}/\equiv_F^{\mathfrak{A}} \models [a_i] = [a_j] \quad \text{iff} \quad [\![a_i = a_i]\!] \in F.$$

Case 9. φ is $v_i \in v_j$. Then

$$\mathfrak{A}/\equiv_F^{\mathfrak{A}} \models [a_i]E'[a_j] \quad \text{iff} \quad [\![a_i \in a_i]\!] \in F.$$

Case 3. φ is $\neg\psi$. Then

$$\begin{aligned}
\mathfrak{A}/\equiv_F^{\mathfrak{A}} \models \neg\psi \quad &\text{iff} \quad \text{not}(\mathfrak{A}/\equiv_F^{\mathfrak{A}} \models \psi) \\
&\text{iff} \quad \text{not}([\![\psi]\!] \in F) \\
&\text{iff} \quad -[\![\psi]\!] \in F \\
&\text{iff} \quad [\![\neg\psi]\!] \in F.
\end{aligned}$$

Case 4. φ is $\psi \vee \chi$. Then

$$\begin{aligned}
\mathfrak{A}/\equiv_F^{\mathfrak{A}} \models (\psi \vee \chi) \quad &\text{iff} \quad (\mathfrak{A}/\equiv_F^{\mathfrak{A}} \models \psi) \text{ or } (\mathfrak{A}/\equiv_F^{\mathfrak{A}} \models \chi) \\
&\text{iff} \quad ([\![\psi]\!] \in F) \text{ or } ([\![\chi]\!] \in F) \\
&\text{iff} \quad [\![\psi]\!] + ([\![\chi]\!] \in F \\
&\text{iff} \quad [\![\psi \vee \chi]\!] \in F.
\end{aligned}$$

Case 5. $\varphi(\overline{x})$ is $\exists y\psi(y, \overline{x})$. First suppose that $\mathfrak{A}/\equiv_F^{\mathfrak{A}} \models \exists y\psi(y, [a_0], \ldots, [a_{m-1}])$. Say $\mathfrak{A}/\equiv_F^{\mathfrak{A}} \models \psi([b], [a_0], \ldots, [a_{m-1}])$. By the inductive hypothesis, $[\![\psi(b, a_0, \ldots, a_{m-1})]\!] \in F$. Since $[\![\psi(b, a_0, \ldots, a_{m-1})]\!] \leq [\![\exists y\psi(y, a_0, \ldots, a_{m-1})]\!]$, it follows that

$$[\![\exists y\psi(y, a_0, \ldots, a_{m-1})]\!] \in F.$$

Second, suppose that $[\![\exists y\psi(y, a_0, \ldots, a_{m-1})]\!] \in F$. Since $\mathfrak{A}$ is full, choose $b \in A$ such that $[\![\exists y\psi(y, a_0, \ldots, a_{m-1})]\!] = [\![\psi(b, a_0, \ldots, a_{m-1})]\!]$. So $[\![\psi(b, a_0, \ldots, a_{m-1})]\!] \in F$. By the inductive hypothesis, $\mathfrak{A}/\equiv_F^{\mathfrak{A}} \models \psi([b], [a_0], \ldots, [a_{m-1}])$. Hence

$$\mathfrak{A}/\equiv_F^{\mathfrak{A}} \models \exists y\psi(y, [a_0], \ldots, [a_{m-1}]). \qquad \square$$

Let B be a complete BA. We define

$$\begin{aligned}
V_0^B &= \emptyset; \\
V_{\alpha+1}^B &= \{x : x \text{ is a function } \wedge \operatorname{dmn}(x) \subseteq V_\alpha^B \wedge \operatorname{rng}(x) \subseteq B\}; \\
V_\gamma^B &= \bigcup_{\alpha<\gamma} V_\alpha^B \quad \text{for } \gamma \text{ limit}; \\
V^B &= \bigcup_{\alpha \in \mathbf{ON}} V_\alpha^B.
\end{aligned}$$

For $x \in V^B$ we define $\rho(x) = \text{least } \alpha[x \in V^B_{\alpha+1}]$.

Now we define by recursion on $(\rho(x), \rho(y))$ under the canonical order:

$$\begin{aligned}
[\![x \in y]\!] &= \sum_{t \in \mathrm{dmn}(y)} ([\![x = t]\!] \cdot y(t)); \\
[\![x \subseteq y]\!] &= \prod_{t \in \mathrm{dmn}(x)} (x(t) \Rightarrow [\![t \in y]\!]); \\
[\![x = y]\!] &= [\![x \subseteq y]\!] \cdot [\![y \subseteq x]\!],
\end{aligned}$$

Lemma 14.18. $\forall x \in V^B[[\![x = x]\!] = 1]$.

Proof. It suffices to show that $||x \subseteq x|| = 1$, which we do by induction:

$$\begin{aligned}
||x \subseteq x|| = \prod_{t \in \mathrm{dmn}(x)} [x(t) \Rightarrow ||t \in x||] &= \prod_{t \in \mathrm{dmn}(x)} \left[x(t) \Rightarrow \sum_{s \in \mathrm{dmn}(x)} [||t = s|| \cdot x(s)] \right] \\
\geq \prod_{t \in \mathrm{dmn}(x)} [x(t) \Rightarrow (||t = t|| \cdot x(t)] &= \prod_{t \in \mathrm{dmn}(x)} [x(t) \Rightarrow x(t)] = 1. \qquad \square
\end{aligned}$$

Lemma 14.19. *Define $\alpha R \beta$ iff $\alpha, \beta \in {}^3\mathbf{ON}$ and one of the following holds:*

(i) $\max \alpha < \max \beta$.

(ii) $\max \alpha = \max \beta$ *and* $\alpha_0 < \beta_0$.

(iii) $\max \alpha = \max \beta$ *and* $\alpha_0 = \beta_0$ *and* $\alpha_1 < \beta_1$.

(iv) $\max \alpha = \max \beta$ *and* $\alpha_0 = \beta_0$ *and* $\alpha_1 = \beta_1$ *and* $\alpha_2 < \beta_2$.

Then R is a well order of ${}^3\mathbf{ON}$. $\square$

Lemma 14.20. *Let R be as in Lemma A. If $\alpha, \beta \in {}^3\mathbf{ON}$, σ is a permutation of 3, $i < 3$, $\alpha_i < \beta_{\sigma(i)}$, $\alpha_j = \beta_{\sigma(j)}$ for $j \neq i$, then $\alpha R \beta$.*

Proof. If $\max \beta = \beta_{\sigma(i)} > \beta_{\sigma(j)}$ for $j \neq i$, then $\max \alpha < \max \beta$, so $\alpha R \beta$. If $\max \beta = \beta_{\sigma(j)}$ for some $j \neq i$, then $\max \alpha = \max \beta$ and $\alpha R \beta$. $\square$

Lemma 14.21. *For all $x, y, z \in V^B$,*

(i) $[\![x = y]\!] \cdot [\![y = z]\!] \leq [\![x = z]\!]$.

(ii) $[\![x \in y]\!] \cdot [\![x = z]\!] \leq [\![z \in y]\!]$.

(iii) $[\![y \in x]\!] \cdot [\![x = z]\!] \leq [\![y \in z]\!]$.

Proof. It suffices to prove the following:

(i) $||w_0 = w_1|| = ||w_1 = w_0||$,

(ii) $||w_{\sigma(0)} = w_{\sigma(1)}|| \cdot ||w_{\sigma(1)} = w_{\sigma(2)}|| \leq ||w_{\sigma(0)} = w_{\sigma(2)}||$ for any permutation σ of 3.

(iii) $||w_{\sigma(0)} \in w_{\sigma(1)}|| \cdot ||w_{\sigma(0)} = w_{\sigma(2)}|| \leq ||w_{\sigma(2)} \in w_{\sigma(1)}||$ for any permutation σ of 3.

(iv) $||w_{\sigma(0)} \in w_{\sigma(1)}|| \cdot ||w_{\sigma(1)} = w_{\sigma(2)}|| \leq ||w_{\sigma(0)} \in w_{\sigma(2)}||$ for any permutation σ of 3.

(i) is clear. Now we prove (ii)–(iv) by induction on the triples $(\rho(w_0), \rho(w_1), \rho(w_2))$.

To prove (ii), let σ be any permutation of 3 and let $x = w_{\alpha(0)}$, $y = w_{\sigma(1)}$, $z = w_{\sigma(2)}$. We first show that if $t \in \mathrm{dmn}(y)$ then

$$(x(t) \Rightarrow ||t \in y||) \cdot ||y = z|| \leq (x(t) \Rightarrow ||t \in z||) \tag{1}$$

In fact,

$$\begin{aligned}(x(t) \Rightarrow ||t \in y||) \cdot ||y = z|| &= (-x(t) + ||t \in y||) \cdot ||y = z|| \\ &= -x(t) \cdot ||y = z|| + ||t \in y|| \cdot ||y = z|| \\ &\leq -x(t) + ||t \in z|| \quad \text{by (iv)} \\ &= x(t) \Rightarrow ||t \in z||.\end{aligned}$$

This proves (1). Hence

$$\begin{aligned}||x \subseteq y|| \cdot ||y = z|| &= \prod_{t \in \mathrm{dmn}(y)} ((x(t) \Rightarrow ||t \in y||) \cdot ||y = z|| \\ &\leq \prod_{t \in \mathrm{dmn}(y)} (x(t) \Rightarrow ||t \in z||) = ||x \subseteq z||.\end{aligned}$$

Next, if $t \in \mathrm{dmn}(z)$, then

$$(z(t) \Rightarrow ||t \in y||) \cdot ||x = y|| \leq (z(t) \Rightarrow ||t \in x|| \tag{2}$$

In fact,

$$\begin{aligned}(z(t) \Rightarrow ||t \in y||) \cdot ||x = y|| &= (-z(t) + ||t \in y||) \cdot ||x = y|| \\ &= -z(t) \cdot ||x = y|| + ||t \in y|| \cdot ||x = y|| \\ &= -z(t) \cdot ||x = y|| + ||t \in y|| \cdot ||y = x|| \\ &\leq -z(t) + ||t \in x|| \quad \text{by (iv)} \\ &= z(t) \Rightarrow ||t \in x||.\end{aligned}$$

Here we appied (iv) to (z, y, x), thus to $w \circ \sigma \circ (0, 2)$. So (2) holds. It follows that

$$\begin{aligned}||z \subseteq y|| \cdot ||x = y|| &= \prod_{t \in \mathrm{dmn}(z)} (z(t) \Rightarrow ||t \in y|| \cdot ||x = y|| \\ &\leq \prod_{t \in \mathrm{dmn}(z)} ||t \in x|| = ||z \subseteq x||.\end{aligned}$$

Hence

$$||x = y|| \cdot ||y = z|| \leq ||x \subseteq y|| \cdot ||y = z|| \leq ||x \subseteq z||$$

and

$$||x = y|| \cdot ||y = z|| \leq ||z \subseteq y|| \cdot ||x = y|| \leq ||z \subseteq x||.$$

Hence $||x = y|| \cdot ||y = z|| \leq ||x = z||$. Thus (ii) holds.

For (iii), again let σ be any permutation of 3 and let $x = w_{\alpha(0)}$, $y = w_{\sigma(1)}$, $z = w_{\sigma(2)}$. Then

$$\begin{aligned}
||x \in y|| \cdot ||x = z|| &= \sum_{t \in \mathrm{dmn}(y)} ||x = t|| \cdot ||x = z|| \\
&= \sum_{t \in \mathrm{dmn}(y)} ||z = x|| \cdot ||x = t|| \\
&\leq \sum_{t \in \mathrm{dmn}(y)} ||z = t|| \quad \text{by (ii)} \\
&= ||z \in y||.
\end{aligned}$$

For (iv), again let σ be any permutation of 3 and let $x = w_{\alpha(0)}$, $y = w_{\sigma(1)}$, $z = w_{\sigma(2)}$. If $t \in \mathrm{dmn}(y)$, then

$$(3) \quad ||x = t|| \cdot y(t) \cdot (-y(t) + ||t \in z||) \leq ||x = t|| \cdot ||t \in z|| = ||t \in z|| \cdot ||t = x|| \leq ||x \in z||$$

by (iii) applied to (y, z, x), thus to $w \circ \sigma \circ (0, 1, 2)$. Hence

$$\begin{aligned}
||x \in y|| \cdot ||y = z|| &\leq ||x \in y|| \cdot ||y \subseteq z|| \\
&= \left(\sum_{t \in \mathrm{dmn}(y)} (||x = t|| \cdot y(t)) \right) \cdot \prod_{t \in \mathrm{dmn}(y)} (y(t) \Rightarrow ||t \in z||) \\
&= \sum_{t \in \mathrm{dmn}(y)} \left(||x = t|| \cdot y(t)) \cdot \prod_{t \in \mathrm{dmn}(y)} (-y(t) + ||t \in z||) \right) \\
&\leq ||x \in z||. \qquad \square
\end{aligned}$$

Corollary 14.22. *V^B is a Boolean-valued model.*

Proof. In the definition, (c) is given by Lemma 21(i). For (d),

$$\begin{aligned}
||x \in y|| \cdot ||v = x|| \cdot ||w = y|| &= ||x \in y|| \cdot ||y = w|| \cdot ||v = x|| \\
&\leq ||x \in w|| \cdot ||v = x|| \quad \text{by Lemma 21(iii)} \\
&= ||x \in w|| \cdot ||x = v|| \\
&\leq ||v \in w|| \quad \text{by Lemma 21(ii)} \qquad \square
\end{aligned}$$

Lemma 14.23. *V^B is extensional, in the sense that*

$$[\![\forall X, Y[\forall u[u \in X \leftrightarrow u \in Y] \rightarrow X = Y]]\!] = 1.$$

We have

$$\llbracket\forall X, Y[\forall u[u \in X \leftrightarrow u \in Y] \to X = Y]\rrbracket = \prod_{X,Y\in V^B} \llbracket\forall u[u \in X \leftrightarrow u \in Y] \to X = Y]\rrbracket.$$

Hence it suffices to take any $X, Y \in V^B$ and show that

$$\llbracket\forall u[u \in X \leftrightarrow u \in Y]\rrbracket \leq \llbracket X = Y\rrbracket.$$

Here the following argument suffices:

$$\begin{aligned}
||\forall u[u \in X \to u \in Y]|| &= \prod_{a\in V^B} ||\neg a \in X \vee a \in Y|| \\
&= \prod_{a\in V^B} (-||a \in X|| + ||a \in Y||) \\
&= \prod_{a\in V^B} \left(-\sum_{c\in\mathrm{dmn}(X)} (||a = c|| \cdot X(c)) + ||a \in Y||\right) \\
&= \prod_{a\in V^B} \prod_{c\in\mathrm{dmn}(X)} (-||a = c|| + -X(c) + ||a \in Y||) \\
&= \prod_{c\in\mathrm{dmn}(X)} \prod_{a\in V^B} (-||a = c|| + -X(c) + ||a \in Y||) \\
&\leq \prod_{c\in\mathrm{dmn}(X)} (-||c = c|| + -X(c) + ||c \in Y||) \\
&= \prod_{c\in\mathrm{dmn}(X)} (X(c) \Rightarrow ||c \in Y||) \\
&= ||X \subseteq Y||.
\end{aligned}$$

□

Lemma 14.24. *If B is a complete BA, W is a set of pairwise disjoint elements of B, and $\langle a_w : w \in W\rangle$ is a system of elements of V^B, then there is a $b \in V^B$ such that $\forall w \in W[w \leq \llbracket b = a_w\rrbracket]$.*

Proof. Let $D = \bigcup_{w\in W} \mathrm{dmn}(a_w)$, and for each $t \in D$ let $b(t) = \sum\{w \cdot a_w(t) : w \in W, t \in \mathrm{dmn}(a_w)\}$.

(1) $$\forall w \in W \forall t \in \mathrm{dmn}(a_w)[w \cdot b(t) = w \cdot a_w(t)].$$

In fact, if $w \in W$ and $t \in \mathrm{dmn}(a_w)$, then

$$w \cdot b(t) = w \cdot \sum\{v \cdot a_v(t) : v \in W, t \in \mathrm{dmn}(a_v)\} = w \cdot a_w(t)$$

by the disjointness of W.

By (1), $\forall w \in W \forall t \in \mathrm{dmn}(a_w)[w \leq (b(t) \Leftrightarrow a_w(t))]$. Hence for any $w \in W$,

$$\begin{aligned}
[\![b \subseteq a_w]\!] &= \prod_{t \in D} (b(t) \Rightarrow [\![t \in a_w]\!]) \\
&= \prod_{t \in D} \left(b(t) \Rightarrow \sum_{s \in \mathrm{dmn}(a_w)} ([\![t = s]\!] \cdot a_w(s) \right)
\end{aligned}$$

Now if $t \in D$, then $w \cdot b(t) = w \cdot \sum\{v \cdot a_v(t) : v \in W, t \in \mathrm{dmn}(a_v)$, and this is 0 unless $t \in \mathrm{dmn}(a_w)$, in which case it is $w \cdot a_w(t)$; and then $w \cdot a_w(t) \leq [\![t = t]\!] \cdot a_w(t)$. So $w \cdot b(t) \leq \sum_{s \in \mathrm{dmn}(a_w)}([\![t = s]\!] \cdot a_w(s))$. This gives $w \leq [\![b \subseteq a_w]\!]$.

Also, for any $w \in W$,

$$[\![a_w \subseteq b]\!] = \prod_{t \in \mathrm{dmn}(a_w)} \left(a_w(t) \Rightarrow \sum_{s \in D} ([\![t = s]\!] \cdot b(s) \right).$$

For any $t \in \mathrm{dmn}(a_w)$ we have $w \cdot a_w(t) = w \cdot b(t) \leq [\![t = t]\!] \cdot b(t)$, and this gives $w \leq [\![a_w \subseteq b]\!]$. $\square$

Lemma 14.25. *V^B is full.*

Proof. Let $\varphi(x, \overline{w})$ and $\overline{b} \in V^B$ be given. Clearly for any $a \in V^B$, $[\![\varphi(a, \overline{b})]\!] \leq [\![\exists x \varphi(x, \overline{b})]\!]$. Let $c = [\![\exists x \varphi(x, \overline{b})]\!]$. Define

$$D = \{u \in B : \exists a \in V^B[u \leq [\![\varphi(a, \overline{b})]\!]]\}.$$

Then D is dense below c. In fact, $c = \sum_{d \in V^B} [\![\varphi(d, \overline{b})]\!]$, so this is clear. Let W be a maximal disjoint subset of D. Then $c \leq \sum W$. In fact, if $c \cdot -\sum W \neq 0$, choose $d \in D$ with $d \leq c \cdot -\sum W \neq 0$; then $d \notin W$ and $W \cup \{d\}$ is disjoint, contradicting the maximality of W. For each $w \in W$ let $a_w \in V^B$ be such that $w \leq]\!]\varphi(a_w, \overline{b})]\!]$. By Lemma 14.24 let $d \in V^B$ be such that $\forall w \in W[w \leq [\![d = a_w]\!]]$. Then

$$\forall w \in W[w \leq [\![d = a_w]\!] \cdot]\!]\varphi(a_w, \overline{b})]\!] \leq]\!]\varphi(d, \overline{b})]\!]].$$

Hence $c \leq \sum W \leq]\!]\varphi(d, \overline{b})]\!]$. $\square$

Lemma 14.26. $[\![\exists y \in x \varphi(y)]\!] = \sum_{y \in \mathrm{dmn}(x)}(x(y) \cdot [\![\varphi(y)]\!])$.

Proof.

$$\begin{aligned}
[\![\exists y \in x \varphi(y)]\!] &= [\![\exists y[y \in x \wedge \varphi(y)]]\!] \\
&= \sum_{a \in V^B} [\![a \in x \wedge \varphi(a)]\!]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{a\in V^B}(\llbracket a\in x\rrbracket \cdot \llbracket \varphi(a)\rrbracket)\\
&= \sum_{a\in V^B}\left(\left(\sum_{t\in \mathrm{dmn}(x)}(\llbracket a=t\rrbracket \cdot x(t))\right)\cdot \llbracket \varphi(a)\rrbracket\right)\\
&= \sum_{a\in V^B}\sum_{t\in \mathrm{dmn}(x)}(\llbracket a=t\rrbracket \cdot x(t)\cdot \llbracket \varphi(a)\rrbracket)\\
&= \sum_{t\in \mathrm{dmn}(x)}\sum_{a\in V^B}(\llbracket a=t\rrbracket \cdot x(t)\cdot \llbracket \varphi(a)\rrbracket)\\
&\geq \sum_{t\in \mathrm{dmn}(x)}(\llbracket t=t\rrbracket \cdot x(t)\cdot \llbracket \varphi(t)\rrbracket)\\
&= \sum_{t\in \mathrm{dmn}(x)}(x(t)\cdot \llbracket \varphi(t)\rrbracket)
\end{aligned}
$$

Now by Lemma 14.25 choose $a\in V^B$ such that $\llbracket \exists y[y\in x\wedge \varphi(y)]\rrbracket = \llbracket a\in x\wedge\varphi(a)\rrbracket$. Then

$$
\begin{aligned}
\llbracket \exists y[y\in x\wedge \varphi(y)]\rrbracket &= \llbracket a\in x\wedge\varphi(a)\rrbracket\\
&= \llbracket a\in x\rrbracket \cdot \llbracket \varphi(a)\rrbracket\\
&= \left(\sum_{t\in \mathrm{dmn}(x)}(\llbracket a=t\rrbracket \cdot x(t))\right)\cdot \llbracket \varphi(a)\rrbracket\\
&= \sum_{t\in \mathrm{dmn}(x)}(\llbracket a=t\rrbracket \cdot x(t)\cdot \llbracket \varphi(a)\rrbracket)\\
&= \sum_{t\in \mathrm{dmn}(x)}(\llbracket a=t\rrbracket \cdot x(t)\cdot \llbracket \varphi(t)\rrbracket) \quad \text{by Theorem 14.14}\\
&\leq \sum_{t\in \mathrm{dmn}(x)}(x(t)\cdot \llbracket \varphi(t)\rrbracket)
\end{aligned}
$$

Together with the above that gives the desired result. □

Lemma 14.27. $\llbracket \forall y\in x\varphi(y)\rrbracket = \prod_{y\in \mathrm{dmn}(x)}(x(y)\Rightarrow \llbracket \varphi(y)\rrbracket)$.

Proof.

$$
\begin{aligned}
\llbracket \forall y\in x\varphi(y)\rrbracket &= \llbracket \forall y[y\in x\rightarrow \varphi(y)]\rrbracket\\
&= \prod_{y\in V^B}(\llbracket y\in x\rrbracket \Rightarrow \llbracket \varphi(y)\rrbracket)\\
&= -\sum_{y\in V^B}(\llbracket y\in x\rrbracket \wedge -\llbracket \varphi(y)\rrbracket)\\
&= -\llbracket \exists y\in x\neg\varphi(y)\rrbracket
\end{aligned}
$$

$$= -\sum_{y\in \mathrm{dmn}(x)} (x(y)\cdot -[\![\varphi(y)]\!])$$
$$= \prod_{y\in \mathrm{dmn}(x)} (x(y) \Rightarrow [\![\varphi(y)]\!]). \qquad \square$$

Now we define $\check{a}$ for any set a: $\mathrm{dmn}(\check{a}) = \{\check{y} : y \in a\}$, and for any $y \in a$, $\check{a}(\check{y}) = 1$.

Lemma 14.28.
(i) $[\![\check{x} \subseteq \check{y}]\!] = 1$ *or* $[\![\check{x} \subseteq \check{y}]\!] = 0$*;*
(ii) $[\![\check{x} \subseteq \check{y}]\!] = 1$ *iff* $x \subseteq y$*.*
(iii) $[\![\check{x} \in \check{y}]\!] = 1$ *or* $[\![\check{x} \in \check{y}]\!] = 0$*;*
(iv) $[\![\check{x} \in \check{y}]\!] = 1$ *iff* $x \in y$*.*

Proof. We prove these statements by simultaneous induction.

$$\begin{aligned}
[\![\check{x} \subseteq \check{y}[\![= 1 \quad &\text{iff} \quad \prod_{t\in x}(\check{x}(\check{t}) \Rightarrow [\![\check{t} \in \check{y}]\!]) = 1\\
&\text{iff} \quad \forall t \in x[[\![\check{t} \in \check{y}[\![= 1]\\
&\text{iff} \quad \forall t \in x[t \in y]\\
&\text{iff} \quad x \subseteq y;
\end{aligned}$$

In the first equation, by the inductive hypothesis, $[\![\check{t} \in \check{y}]\!]$ is either 0 or 1; so $[\![\check{x} \subseteq \check{y}[\![= 1$ or $[\![\check{x} \subseteq \check{y}]\!] = 0$. Next,

$$\begin{aligned}
[\![\check{x} \in \check{y}[\![= 1 \quad &\text{iff} \quad \sum_{t\in y}([\![\check{x} = \check{t}]\!] \cdot \check{y}(\check{t})) = 1\\
&\text{iff} \quad \exists t \in y[[\![\check{x} = \check{t}[\![= 1 \text{ and } t \in y]\\
&\text{iff} \quad \exists t \in y[x = t \text{ and } t \in y]\\
&\text{iff} \quad x \in y.
\end{aligned}$$

As above, $[\![\check{x} \in \check{y}]\!] = 1$ or $[\![\check{x} \in \check{y}[\![= 0$. $\square$

Lemma 14.29. *For any* Δ_0 *formula* φ*,*
(i) $[\![\varphi(\check{x}, \ldots)]\!] = 0$ *or* $[\![\varphi(\check{x}, \ldots)]\!] = 1$*;*
(ii) $\varphi(x, \ldots)$ *iff* $]\!]\varphi(\check{x}, \ldots)]\!] = 1$.

Proof. Induction on φ.
Case 1. φ is $\check{x} = \check{y}$. See Lemma 14.28.
Case 2. φ is $\check{x} \in \check{y}$. See Lemma 14.28.
Case 3. φ is $\neg\psi$. Clearly then $[\![\varphi(\check{x}, \ldots)]\!] = 0$ or $[\![\varphi(\check{x}, \ldots)]\!] = 1$. Further,

$$\varphi(x, \ldots) \text{ iff not } \psi(x, \ldots) \text{ iff not}([\![\psi(\check{x}, \ldots)]\!] = 1) \text{ iff } [\![\psi(\check{x}, \ldots)]\!] = 0 \text{ iff } [\![\varphi(\check{x}, \ldots)]\!] = 1.$$

Case 4. φ is $\psi \wedge \chi$. Clearly then $[\![\varphi(\check{x},\ldots)]\!] = 0$ or $[\![\varphi(\check{x},\ldots)]\!] = 1$. Further,

$$\begin{aligned}\varphi(x,\ldots) \quad &\text{iff} \quad \psi(x,\ldots) \text{ and } \chi(x,\ldots)\\ &\text{iff} \quad [\![\psi(\check{x},\ldots)]\!] = 1 \text{ and } [\![\chi(\check{x},\ldots)]\!] = 1\\ &\text{iff} \quad [\![\varphi(\check{x},\ldots)]\!] = 1.\end{aligned}$$

Case 5. φ is $\psi \vee \chi$. Similar to Case 4.
Case 6. φ is $\exists y \in x\psi(x,y,\ldots)$. Then

$$\begin{aligned}[\![\exists y \in \check{x}\psi(\check{x},y,\ldots)]\!] = 1 \quad &\text{iff} \quad \sum_{y\in\mathrm{dmn}(x)} (\check{x}(\check{y}) \cdot [\![\psi(\check{x},\check{y},\ldots)]\!] = 1\\ &\text{iff} \quad \text{not } \left(\sum_{y\in\mathrm{dmn}(x)} (\check{x}(\check{y}) \cdot [\![\psi(\check{x},\check{y},\ldots)]\!] = 0\right)\\ &\text{iff} \quad \text{not } ([\![\exists y \in \check{x}\psi(\check{x},y,\ldots)]\!] = 0).\end{aligned}$$

Moreover,

$$\begin{aligned}\exists y \in x\psi(x,y,\ldots) \quad &\text{iff} \quad \exists y \in x([\![\psi(\check{x},y,\ldots)]\!] = 1)\\ &\text{iff} \quad \sum_{y\in\mathrm{dmn}(x)} (\check{x}(\check{y}) \cdot [\![\psi(\check{x},\check{y},\ldots)]\!]) = 1\\ &\text{iff} \quad [\![\exists y \in \check{x}\psi(\check{x},y,\ldots)]\!] = 1.\end{aligned}$$

Other cases can be derived from the above. □

Corollary 14.30. *If φ is Σ_1 and φ holds, then $[\![\varphi]\!] = 1$.*

Proof. Say $\varphi \leftrightarrow \exists x\psi(x,\overline{b})$ with ψ Δ_0. Choose a so that $\psi(a,\overline{b})$. Then by Lemma 14.29, $[\![\psi(\check{a},\check{b}_0,\ldots,\check{b}_{m-1})]\!] = 1$. so $[\![\varphi]\!] = 1$. □

Lemma 14.31. *Replacement is equivalent on the basis of the other axioms to the following statement:*

$$\forall w_1,\ldots,w_n[\forall X\exists Y\forall u \in X[\exists v\varphi(u,v,w_1,\ldots,w_n) \to \exists v \in Y\varphi(u,v,w_1,\ldots,w_n)],$$

where φ is a formula with free variables among $u,v,w_1,\ldots,w_n$.

Proof. First assume replacement. Let $w_1,\ldots,w_n,X$ be given. Define

$$\begin{aligned}Y = \{v : \exists u \in X[&\varphi(u,v,w_1,\ldots,w_n) \text{ and}\\ &\forall z[\varphi(u,z,w_1,\ldots,w_n) \to \mathrm{rank}(v) \leq \mathrm{rank}(z)]]\}.\end{aligned}$$

By the replacement axiom, Y is a set. Suppose that $u \in X$ and $\exists v\varphi(u,v,w_1,\ldots,w_n)$. Taking such a v of smallest rank, we get $v \in Y$, as desired.

Now assume the given statement, and suppose that $A, w_1, \ldots, w_n$ are given, and $\forall x \in A\exists! y\varphi(x, y, A, w_1, \ldots, w_n)$. By the statement, choose Y so that

$$\forall x \in A[\exists y\varphi(x, y, A, w_1, \ldots, w_n) \to \exists y \in Y\varphi(x, y, A, w_1, \ldots, w_n).$$

Clearly Y is as desired. □

Theorem 14.32. *If φ is an axiom of ZFC, then $[\![\varphi]\!] = 1$.*

Proof.

Case 1. Extensionality. See Lemma 14.23

Case 9. Comprehension. Let φ be a formula with free variables among $x, z, w_1, \ldots, w_n$. We want to show that

$$||\forall z, w_1, \ldots, w_n\exists y\forall x[x \in y \leftrightarrow x \in z \wedge \varphi]|| = 1.$$

We have

$$\begin{aligned}&||\forall z, w_1, \ldots, w_n\exists y\forall x[x \in y \leftrightarrow x \in z \wedge \varphi]||\\ &= \prod\{||\exists y\forall x[x \in y \leftrightarrow x \in z \wedge \varphi]|| : z, w_1, \ldots, w_n \in V^B\}\end{aligned}$$

So, let $z, w_1, \ldots, w_n \in V^B$. we want to show that

$$||\exists y\forall x[x \in y \leftrightarrow x \in z \wedge \varphi]|| = 1.$$

Let $\mathrm{dmn}(y) = \mathrm{dmn}(z)$ and for all $t \in \mathrm{dmn}(z)$ let $y(t) = z(t) \cdot ||\varphi||$. Then

$$||\forall x[x \in y \leftrightarrow x \in z \wedge \varphi]|| = \prod_{x \in V^B}(||x \in y|| \Leftrightarrow ||x \in z|| \cdot ||\varphi||)$$

So it suffices to show that for any $x \in V^B$,

$$||x \in y|| \Leftrightarrow ||x \in z|| \cdot ||\varphi|| = 1.$$

We have

$$\begin{aligned}||x \in y|| &= \sum_{t \in \mathrm{dmn}(y)}(y(t) \cdot ||x = t||) = \sum_{t \in \mathrm{dmn}(y)}(z(t) \cdot ||\varphi|| \cdot ||x = t||)\\ &= \sum_{t \in \mathrm{dmn}(z)}(z(t) \cdot ||\varphi|| \cdot ||x = t||) = ||x \in z|| \cdot ||\varphi||,\end{aligned}$$

as desired.

Case 3. Pairing. We want to prove

$$||\forall x, y\exists z[x \in z \wedge y \in z]|| = 1;$$

so, given $x, y \in V^B$, we want to prove that $||\exists z[x \in z \wedge y \in z]|| = 1$. Let $\mathrm{dmn}(z) = \{x, y\}$ with $z(x) = z(y) = 1$. Then

$$
\begin{aligned}
||x \in z|| \cdot ||y \in z|| &= \left(\sum_{t \in \mathrm{dmn}(z)} (z(t) \cdot ||x = t||)\right) \cdot \sum_{t \in \mathrm{dmn}(z)} (z(t) \cdot ||y = t||) \\
&= ((z(x) \cdot ||x = x||) + (z(y) \cdot ||x = y||)) \\
&\cdot ((z(x) \cdot ||y = x||) + (z(y) \cdot ||y = y||)) \\
&= 1.
\end{aligned}
$$

Case 4. Union. We want to prove

$$||\forall \mathscr{A} \exists A \forall Y \in \mathscr{A} \forall x \in Y (x \in A)|| = 1.$$

Given $\mathscr{A} \in V^B$, let $\mathrm{dmn}(A) = \bigcup_{u \in \mathrm{dmn}(\mathscr{A})} \mathrm{dmn}(u)$, and for each $u \in \mathrm{dmn}\mathscr{A}$ and $v \in \mathrm{dmn}(u)$ let $A(v) = 1$. Then

$$
\begin{aligned}
&||\forall Y \in \mathscr{A} \forall x \in Y (x \in A)|| \\
&= \prod_{Y \in \mathrm{dmn}(\mathscr{A})} \left(\mathscr{A}(Y) \Rightarrow \prod_{x \in \mathrm{dmn}(Y)} (Y(x) \Rightarrow ||x \in A||)\right) \\
&= \prod_{Y \in \mathrm{dmn}(\mathscr{A})} \left(\mathscr{A}(Y) \Rightarrow \prod_{x \in \mathrm{dmn}(Y)} (Y(x) \Rightarrow \sum_{u \in \mathrm{dmn}(A)} (A(u) \cdot ||x = u||))\right)
\end{aligned}
$$

Now if $Y \in \mathrm{dmn}(\mathscr{A}$ and $x \in \mathrm{dmn}(Y)$, then $x \in \mathrm{dmn}(A)$ and $A(x) = 1$. It follows that the big product here is equal to 1.

Case 5. Power set. We want to prove

$$||\forall x \exists y \forall z[z \subseteq x \rightarrow z \in y]|| = 1.$$

So, let $x \in V^B$. Let

$$
\begin{aligned}
&\mathrm{dmn}(y) = \{u \in V^B : \mathrm{dmn}(u) = \mathrm{dmn}(x) \text{ and } \forall t \in \mathrm{dmn}(x)[u(t) \leq x(t)]\}; \\
&\qquad \forall u \in \mathrm{dmn}(y)[y(u) = 1].
\end{aligned}
$$

Now suppose that $z \in V^B$. Define $\mathrm{dmn}(z') = \mathrm{dmn}(x)$ and for any $t \in \mathrm{dmn}(x)$, $z'(t) = x(t) \cdot ||t \in z||$. Then

(1) $||z' \subseteq z|| = 1$.

In fact,

$$
\begin{aligned}
||z' \subseteq z|| &= \prod_{t \in \mathrm{dmn}(z')} (z'(t) \Rightarrow ||t \in z||) \\
&= \prod_{t \in \mathrm{dmn}(x)} (x(t) \cdot ||t \in z|| \Rightarrow ||t \in z||) = 1
\end{aligned}
$$

So (1) holds. Note that $z' \in \mathrm{dmn}(y)$

(2) If $t \in \mathrm{dmn}(z)$, then $z(t) \Rightarrow ||t \in z|| = 1$.

In fact, if $t \in \mathrm{dmn}(z)$, then

$$z(t) \Rightarrow ||t \in z|| = z(t) \Rightarrow \sum_{s \in \mathrm{dmn}(z)} (z(s) \cdot ||t = s||) \geq z(t) \Rightarrow (z(t) \cdot ||t = t||) = 1.$$

(3) $||z \subseteq x|| = ||z \subseteq z'||$.

In fact,

$$\begin{aligned}
||z \subseteq x|| &= \prod_{t \in \mathrm{dmn}(z)} (z(t) \Rightarrow ||t \in x||) \\
&= \prod_{t \in \mathrm{dmn}(z)} (z(t) \Rightarrow ||t \in x||) \cdot (z(t) \Rightarrow ||t \in z||) \\
&= \prod_{t \in \mathrm{dmn}(z)} (z(t) \Rightarrow (||t \in x|| \cdot ||t \in z||) \\
&= \prod_{t \in \mathrm{dmn}(z)} \left(z(t) \Rightarrow \sum_{s \in \mathrm{dmn}(z')} (x(s) \cdot ||t = s|| \cdot ||t \in z|| \right) \\
&= \prod_{t \in \mathrm{dmn}(z)} \left(z(t) \Rightarrow \sum_{s \in \mathrm{dmn}(z')} (x(s) \cdot ||t = s|| \cdot ||s \in z|| \right) \\
&= \prod_{t \in \mathrm{dmn}(z)} \left(z(t) \Rightarrow \sum_{s \in \mathrm{dmn}(z')} (z'(s) \cdot ||t = s|| \right) \\
&= \prod_{t \in \mathrm{dmn}(z)} (z(t) \Rightarrow ||t \in z'||) \\
&= ||z \subseteq z'||.
\end{aligned}$$

Now $||z \in y|| = \sum_{u \in \mathrm{dmn}(y)} (y(u) \cdot ||z = u|| \geq ||z = z'|| = ||z \subseteq x||$, as desired.

Case 6. Infinity. $\check{\omega}$ works here.

Case 7. Replacement.

Now given $X, w_1, \ldots, w_n \in V^B$, for each $u \in \mathrm{dmn}(X)$ let S_u be a set such that

$$\sum_{v \in S_u} ||\varphi(u, v, w_1, \ldots, w_n)|| = \sum_{v \in V^B} ||\varphi(u, v, w_1, \ldots, w_n)||.$$

Let $\mathrm{dmn}(Y) = \bigcup_{u \in \mathrm{dmn}(X)} S_u$ and let $Y(v) = 1$ for each $v \in \mathrm{dmn}(Y)$. Then

$$||\forall u \in X[\exists v \varphi(u, v, w_1, \ldots, w_n) \to \exists v \in Y \varphi(u, v, w_1, \ldots, w_n)]||$$

$$= \prod_{u\in \mathrm{dmn}(X)} \Bigg(X(u) \Rightarrow \Bigg(\sum_{v\in V^B} ||\varphi(u,v,w_1,\ldots,w_n)|| \Rightarrow$$

$$\sum_{v\in \mathrm{dmn}(Y)} (Y(v)\cdot ||\varphi(u,v,w_1,\ldots,w_n)||)$$

$$= \prod_{u\in \mathrm{dmn}(X)} \Bigg(X(u) \Rightarrow \Bigg(\sum_{v\in V^b} ||\varphi(u,v,w_1,\ldots,w_n)|| \Rightarrow$$

$$\sum_{v\in V^B} ||\varphi(u,v,w_1,\ldots,w_n)||)\Bigg)\Bigg)$$

$$= 1.$$

Case 8. Foundation. Suppose that

$$||\forall x[x\neq\emptyset \to \exists y\in x[x\cap y=\emptyset]]||\neq 1.$$

Then

$$\begin{aligned}
0 \neq& -||\forall x[x\neq\emptyset \to \exists y\in x[x\cap y=\emptyset]]||\\
&= -||\forall x[\exists y[y\in x] \to \exists y[y\in x\wedge \forall z\in y[z\notin x]]]||\\
&= -\prod_{x\in V^B}\Bigg(\Bigg(\sum_{y\in V^B}||y\in x||\Bigg)\Rightarrow\\
&\sum_{y\in V^B}\Bigg(||y\in x||\cdot \prod_{z\in \mathrm{dmn}(y)}(y(z)\Rightarrow -||z\in x||)\Bigg)\Bigg)\\
&= \sum_{x\in V^B}\Bigg(\Bigg(\sum_{y\in V^B}||y\in x||\Bigg)\cdot\\
&\prod_{y\in V^B}\Bigg(||y\in x||\Rightarrow \sum_{z\in \mathrm{dmn}(y)}(y(z)\cdot ||z\in x||)\Bigg)\Bigg).
\end{aligned}$$

Choose $x\in V^B$ so that

$$\Bigg(\sum_{y\in V^B}||y\in x||\Bigg)\cdot \prod_{y\in V^B}\Bigg(||y\in x||\Rightarrow \sum_{z\in \mathrm{dmn}(y)}(y(z)\cdot ||z\in x||)\Bigg)\neq 0$$

Let $y\in V^B$ have smallest rank such that

$$||y\in x||\cdot \prod_{y\in V^B}\Bigg(||y\in x||\Rightarrow \sum_{z\in \mathrm{dmn}(y)}(y(z)\cdot ||z\in x||)\Bigg)\neq 0$$

Then

$$0 \neq ||y\in x||\cdot \prod_{y'\in V^B}\Bigg(||y'\in x||\Rightarrow \sum_{z\in \mathrm{dmn}(y')}(y'(z)\cdot ||z\in x||)\Bigg)$$

$$\cdot\left(||y\in x||\Rightarrow\sum_{z\in\mathrm{dmn}(y)}(y(z)\cdot||z\in x||\right)$$

$$=||y\in x||\cdot\prod_{y'\in V^B}\left(||y'\in x||\Rightarrow\sum_{z\in\mathrm{dmn}(y')}(y'(z)\cdot||z\in x||)\right)$$

$$\cdot\sum_{z\in\mathrm{dmn}(y)}(y(z)\cdot||z\in x||)$$

Hence there is a $z\in\mathrm{dmn}(y)$ such that

$$0\neq||y\in x||\cdot\prod_{y'\in V^B}\left(||y'\in x||\Rightarrow\sum_{z\in\mathrm{dmn}(y')}(y'(z)\cdot||z\in x||)\right)\cdot y(z)\cdot||z\in x||)$$

So

$$||z\in x||\cdot\prod_{y\in V^B}\left(||y\in x||\Rightarrow\sum_{z\in\mathrm{dmn}(y)}(y(z)\cdot||z\in x||)\right)\neq 0$$

contradicting the minimality of the rank of y.

Case 9. Choice. Let $\varphi(\alpha,g,S)$ say that α is an ordinal, and g is a function mapping α onto S. Thus S is well-ordered by xRy iff $x,y\in S$ and $\min\{\xi<\alpha:g(\xi)=x\}<\min\{\xi<\alpha:g(\xi)=y\}$.

Given $X\in V^B$, let $S=\mathrm{dmn}(X)$ and choose α,g so that $\varphi(\alpha,g,S)$. Then by Lemma 14.30, $||\varphi(\check{\alpha},\check{g},\check{S})||=1$. Define $f\in V^B$ by

$$\mathrm{dmn}(f)=\{(\check{x},x)^B:x\in S\};\ \forall t\in\mathrm{dmn}(f)[f(t)=1].$$

Then

$$||f\text{ is a function }\wedge\check{S}\subseteq\mathrm{dmn}(f)\wedge X\subseteq\mathrm{rng}(f)\wedge\varphi(\check{\alpha},\check{g},\check{S})||=1.$$

Some details on this:

(1) For any $x\in V^B$, $\{x\}^B$ is the function with domain $\{x\}$ and value 1.

(2) For any $x\in V^B$, $||\{x\}^B=\{x\}||=1$. In fact,

$$||\{x\}^B=\{x\}||=||\forall y\in\{x\}^B[y=x]\wedge x\in\{x\}^B||=||\forall y\in\{x\}^B[y=x]||\cdot||x\in\{x\}^B||$$
$$=\prod_{y\in\{x\}}||y=x||\cdot\sum_{y\in\{x\}}||x=y||=||x=x||\cdot||x=x||=1.$$

(3) For any $x,y\in V^B$, $\{x,y\}^B$ is the function with domain $\{x,y\}$ and value 1 for each argument.

(4) For any $x, y \in V^B$, $||\{x,y\}^B = \{x,y\}|| = 1$. In fact,

$$\begin{aligned}
||\{x,y\}^B = \{x,y\}|| &= ||\forall z \in \{x,y\}^B(z = x \vee z = y) \wedge x \in \{x,y\}^B \wedge y \in \{x,y\}^B|| \\
&= ||\forall z \in \{x\}^B[z = x \vee z = y]|| \cdot ||x \in \{x,y\}^B|| \cdot ||y \in \{x,y\}^B|| \\
&= \prod_{z \in \{x,y\}} (||z = x|| + ||z = y||) \cdot \sum_{z \in \{x,y\}} ||x = z|| \cdot \sum_{z \in \{x,y\}} ||z = y|| \\
&= (||x = x|| + ||x = y||) \cdot (||y = x|| + ||y = y||) \\
&\cdot (||x = x|| + ||x = y||) \cdot (||x = y|| + ||y = y|| \\
&= 1.
\end{aligned}$$

(5) $||f$ is a relation$|| = 1$.

In fact,

$$\begin{aligned}
||f \text{ is a relation } || &= ||\forall w \in f \exists u, v[w = (u,v)]|| \\
&= \prod_{w \in \mathrm{dmn}(f)} [f(w) \Rightarrow \sum_{u,v \in V^B} ||w = (u,v)]||.
\end{aligned}$$

(6)

$$\begin{aligned}
||\check{S} \subseteq \mathrm{dmn}(f)|| &= ||\forall a \in \check{S} \exists b[(a,b) \in f]|| \\
&= \prod_{a \in S} \sum_{b \in V^B} \sum_{t \in \mathrm{dmn}(f)} (f(t) \cdot ||(b,a) = t||
\end{aligned}$$

Now, given $a \in S$ let $b = \check{a}$. Then $(b,a)^B \in \mathrm{dmn}(f)$ and $||(b,a) = (b,a)^B|| = ||(\check{a},a) = (\check{a},a)^B|| = 1$ by (4).

(7)

$$\begin{aligned}
||X \subseteq \mathrm{rng}(f)|| &= ||\forall x \in X \exists a[(a,x) \in f]|| \\
&= \prod_{x \in \mathrm{dmn}(X)} \sum_{a \in V^B} \sum_{y \in V^B} ||(a,y) \in f|| \\
&= \prod_{x \in S} \sum_{a \in V^B} \sum_{y \in V^B} \sum_{t \in \mathrm{dmn}(f)} ||(a,y) = t||
\end{aligned}$$

Now let $x \in S$. Let $a = \check{x}$, $y = x$, and $t = (\check{x},x)^B$. Then $||(a,y) = t|| = ||(\check{x},x) = (\check{x},x)^B|| = 1$ by (4).

This completes the proof of Theorem 36. ☐

Corollary 14.33. *If $ZFC \vdash \varphi$, then $[\![\varphi]\!] = 1$.*

Proof. See the proofs of Theorems 14.13 and 14.32. ☐

Lemma 14.34. *For every $x \in V^B$,*

$$[\![x \text{ is an ordinal}]\!] = \sum_{\alpha \in \mathbf{ON}} [\![x = \check{\alpha}]\!].$$

Proof. If α is an ordinal, then by Lemma 14.30, $[\![\check{\alpha} \text{ is an ordinal}]\!] = 1$. Hence by Theorem 14.14, for any $x \in V^B$,

$$[\![x = \check{\alpha}]\!] \leq [\![x = \check{\alpha}]\!] \cdot [\![\check{\alpha} \text{ is an ordinal}]\!] = [\![x = \check{\alpha}]\!] \cdot [\![x \text{ is an ordinal}]\!] \leq [\![x \text{ is an ordinal}]\!].$$

Hence $\sum_{\alpha \in \mathbf{ON}} [\![x = \check{\alpha}]\!] \leq [\![x \text{ is an ordinal}]\!]$.

(1) $\{\alpha : [\![\check{\alpha} \in x]\!] \neq 0\}$ is a set.

For, let A be this class. For each $\alpha \in A$ we have $0 \neq \sum_{t \in \mathrm{dmn}(x)} ([\![\check{\alpha} = t]\!] \cdot x(t))$, so there is a $t_\alpha \in \mathrm{dmn}(x)$ such that $[\![\check{\alpha} = t_\alpha]\!] \neq 0$. Define $\alpha \equiv \beta$ iff $t_\alpha = t_\beta$. Then there are at most $|\mathrm{dmn}(x)|$ equivalence classes. If $\alpha \equiv \beta$ and $\alpha \neq \beta$, then $[\![\check{\alpha} = t_\alpha]\!] \cdot [\![\check{\beta} = t_\alpha]\!] \leq [\![\check{\alpha} = \check{\beta}]\!] = 0$. Thus each equivalence class has size at most the supremum of the cardinalities of disjoint subsets of B. This proves (1).

By (1), let γ be such that $\forall \alpha \geq \gamma [[\![\check{\alpha} \in x]\!] = 0]$. In particular, $[\![\check{\gamma} \in x]\!] = 0$. Now by Corollary 35, $[\![\forall u, v[u, v \text{ ordinals} \rightarrow (u \in v \vee u = v \vee v \in u)]\!] = 1$. So

$$\prod_{u,v \in V^B} [[\![u \text{ is an ordinal}]\!] \cdot [\![v \text{ is an ordinal}]\!] \Rightarrow ([\![u \in v]\!] + [\![u = v]\!] + [\![v \in u]\!])] = 1.$$

Now $[\![\check{\gamma} \text{ is an ordinal}]\!] = 1$. So

$$\begin{aligned} 1 &= [\![x \text{ is an ordinal}]\!] \Rightarrow ([\![x \in \check{\gamma}]\!] + [\![x = \check{\gamma}]\!] + [\![\gamma \in x]\!]) \\ &= [\![x \text{ is an ordinal}]\!] \Rightarrow (\sum_{\alpha < \gamma} [\![x = \check{\alpha}]\!] + [\![x = \check{\gamma}]\!]; \end{aligned}$$

hence $[\![x \text{ is an ordinal}]\!] \leq \sum_{\alpha \leq \gamma} [\![x = \check{\alpha}]\!] \leq \sum_{\alpha \in \mathbf{ON}} [\![x = \check{\alpha}]\!]$. ☐

Lemma 14.35. $[\![\exists x[x$ *is an ordinal and* $\varphi(x)]]\!] = \sum_{\alpha \in \mathbf{ON}} [\![\varphi(\check{\alpha})]\!]$.

Proof.

$$\begin{aligned} &[\![\exists x[x \text{ is an ordinal and } \varphi(x)]\!] = \sum_{x \in M^P} ([\![x \text{ is an ordinal}]\!] \cdot [\![\varphi(x)]\!]) \\ &= \sum_{x \in M^P} \left(\left(\sum_{\alpha \in \mathbf{ON}} [\![x = \check{\alpha}]\!] \right) \cdot [\![\varphi(x)]\!] \right) \\ &= \sum_{x \in M^P} \sum_{\alpha \in \mathbf{ON}} ([\![x = \check{\alpha}]\!] \cdot [\![\varphi(x)]\!]) \\ &= \sum_{x \in M^P} \sum_{\alpha \in \mathbf{ON}} ([\![x = \check{\alpha}]\!] \cdot [\![\varphi(\check{\alpha})]\!]) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\alpha\in\mathbf{ON}} \sum_{x\in M^P} ([\![x=\check{\alpha}]\!]\cdot[\![\varphi(\check{\alpha})]\!])\\
&= \sum_{\alpha\in\mathbf{ON}} \left(\left(\sum_{x\in M^P} ([\![x=\check{\alpha}]\!]\right)\cdot[\![\varphi(\check{\alpha})]\!]\right)\\
&= \sum_{\alpha\in\mathbf{ON}} [\![\varphi(\check{\alpha})]\!]. \qquad \square
\end{aligned}
$$

Generic sets and forcing

Generic sets and forcing can be defined for forcing orders or complete Boolean algebras. We give both versions, and conections between them. Both versions are considered together with a countable transitive model of ZFC,

If P is a forcing poset, a subset G of P is a *filter* iff $1\in G$, $\forall p,q\in P[p\in G$ and $p\le q$ imply that $q\in G]$, and $\forall p,q\in G\exists r\in G[r\le p,q]$.

Let M be a c.t.m. off ZFC and let $P\in M$ be a forcing poset. We say that G is a *P-generic filter over M* provided that the following conditions hold:

(1) G is a filter on $\mathbb{P}$.

(2) For every dense $D\subseteq P$ such that $D\in M$ we have $G\cap D\ne\emptyset$.

Theorem 14.36. *Let M be a c.t.m. of ZFC, $P\in M$ a forcing order, and $p\in P$. Then there is a P-generic filter over M such that $p\in P$.*

Proof. Let $\langle D_m : m\in\omega\rangle$ enumerate all the dense subsets of P which are in M. Let $p_0=p$. If p_m has been defined, let $p_{m+1}\le p_m$ be in D_m. Then let

$$G=\{q\in P:\exists m\in\omega[p_m\le q]\}$$

Clearly G is as desired. $\square$

Given a c.t.m. M of ZFC, a Boolean algebra $A\in M$, and an ultrafilter U on A (U not necessarily in M), we say that U is *M-generic* iff $\forall X\in M[X\subseteq U\Rightarrow\prod X\in U]$.

Proposition 14.37. *U is M-generic iff $\forall X\in M[\sum X\in U\Rightarrow\exists x\in X\cap U]$.*

Proof. $\Rightarrow$: Assume that U is M-generic, $X\in M$, $\sum X\in U$, but $\forall x\in X[x\notin U]$. Let $Y=\{-x:x\in X\}$. Then $Y\in M\cap U$, so $\prod Y\in U$. But $\prod Y=-\sum X$, contradiction.

$\Leftarrow$: Assume the indicated condition, $X\in M$, $X\subseteq U$, but $\prod X\notin U$. So with $Y=\{-x:x\in X\}$ we have $\sum Y\in U$. Choose $x\in Y\cap U$. But $-x\in X\subseteq U$, contradiction. $\square$

Proposition 14.38. *U is a generic ultrafilter on A iff U is a generic filter in A^+.*

Proof. $\Rightarrow$: Assume that U is a generic ultrafilter on A. Clearly U is a filter on A^+. Now suppose that $D\subseteq A^+$ is dense in A^+, $D\in M$, but $\forall x\in D[x\notin U]$. Then

$\{x : -x \in D\}$ is a subset of U, so $a \stackrel{\text{def}}{=} \prod_{-x\in D} x \in D$. Hence $a \neq 0$. Choose $x \in D$ with $x \leq a$. Then $x \leq -x$, contradiction.

$\Leftarrow$: Suppose that U is a generic filter in A^+. Thus U is a filter on A. Suppose that $a \neq 0, 1$. Now $\{x : x \leq a \text{ or } x \leq -a\}$ is dense, and it follows that $a \in U$ or $-a \in U$. Thus U is an ultrafilter on A. Now suppose that $X \in M$ and $X \subseteq U$. Let $D = \{a \in A^+ : \exists x \in X[a \cdot x = 0 \text{ or } \forall x \in X[a \leq x]]\}$. Then D is dense in A^+, since if $a \in A^+$ and $\neg\forall x \in X[a \leq x]$, then there is an $x \in X$ such that $a \not\leq x$ and so $a \cdot -x \neq 0$ and $a \cdot -x \leq a$ with $a \cdot -x \in D$. It follows that there is an $a \in U$ such that $\exists x \in X[a \cdot x = 0]$ or $\forall x \in X[a \leq x]$. If $x \in X$ and $a \cdot x = 0$. Since $X \subseteq U$ and $a \in U$, this is a contradiction. Hence $\forall x \in X[a \leq x]$, so $a \leq \prod X$ and so $\prod X \in U$. □

Corollary 14.39. *Let $A \in M$ be a complete BA in the sense of M, and $a \in A^+$. Then there is a generic ultrafilter U on A such that $a \in U$.* □

Proposition 14.40. *If G is P-generic over M, let $G' = \{a \in \mathrm{RO}(P) : \exists p \in G[e(p) \leq a]\}$. Then G' is a generic ultrafilter on $\mathrm{RO}(P)$.*

Proof. We apply Proposition 14.38. Thus we want to show that G' is a generic filter on $\mathrm{RO}(P)^+$. Clearly G' is closed upwards. Suppose that $a, b \in G'$. Say $p, q \in G$ with $e(p) \leq a$ and $e(q) \leq b$. Choose $r \in G$ with $r \leq p, q$. Then $e(r) \leq a \cdot b$, so $a \cdot b \in G'$. So G' is a filter on $\mathrm{RO}(P)^+$. Now suppose that D is dense in $\mathrm{RO}(P)^+$, with $D \in M$. Let $D' = \{p \in P : \exists x \in D[e(p) \leq x]\}$. Then D' is dense in P. For, let $q \in P$. Choose $a \in D$ such that $a \leq e(q)$. Then choose $r \in P$ such that $e(r) \leq a$. Let p be such that $p \leq r, q$. Then $e(p) \leq e(r) \leq a$, so $p \in D'$. Also, $p \leq q$. So D' is dense in P. Choose $p \in D' \cap G$. Choose $x \in D$ such that $e(p) \leq x$. Then $x \in G' \cap D$, as desired. □

For M a c.t.m. we consider $M^{\mathrm{RO}(P)}$, the interpretation of $V^{\mathrm{RO}(P)}$ in M. For $p \in P$ and $\dot{a}_0, \ldots \dot{a}_{m-1} \in M^{\mathrm{RO}(P)}$ we define

$$p \Vdash \varphi(\dot{a}_0, \ldots \dot{a}_{m-1}) \quad \text{iff} \quad e(p) \leq [\![\varphi(\dot{a}_0, \ldots \dot{a}_{m-1})]\!].$$

Lemma 14.41. *Let P be a forcing poset, and let Q, h be as in Lemma 14.10.*

(i) If G is P-generic over M, let G' be the filter on Q generated by $h(G)$. Then G' is Q-generic over M.

(ii) If G is Q-generic over M and G' is the filter on P generated by $h^{-1}[G]$, then G' is P-generic over M.

Proof. For (i), suppose that $D \in M$ is dense in Q. Let $D' = \{p \in P : \exists q \in D[h(p) \leq q]\}$. Then D' is dense in P. For, suppose that $p \in P$. Choose $q \in D$ such that $q \leq h(p)$. Say $q = h(r)$. Then $h(r) \leq h(p)$, so r and p are compatible. Say $s \leq r, p$. Then $h(s) \leq h(r) = q$, so $s \in D'$. So D' is dense in P. Choose $p \in D' \cap G$. Say $q \in D$ and $h(p) \leq q$. Then $q \in D \cap G'$, as desired.

For (ii), suppose that D is dense in P. Let $D' = \{q \in Q : \exists p \in D[q \leq h(p)]\}$. Then D' is dense in Q. For, let $r \in Q$. Say $r = h(p)$. Choose $s \in D$ such that $s \leq p$. Then

$h(s) \leq h(p) = r$ and $h(s) \in D'$, as desired. Now take $q \in D' \cap G$. Say $p \in D$ and $q \leq h(p)$. Then $h(p) \in G$, so $p \in G'$, as desired. $\square$

It is important to realize that usually generic filters are not in the ground model M; this is expressed in the following lemma.

Lemma 14.42. *Suppose that M is a c.t.m. of ZFC and $\mathbb{P} = (P, \leq, 1) \in M$ is a forcing order. Assume the following:*

(1) For every $p \in P$ there are $q, r \in P$ such that $q \leq p$, $r \leq p$, and $q \perp r$.

Also suppose that G is $\mathbb{P}$-generic over M.

Then $G \notin M$.

Proof. Suppose to the contrary that $G \in M$. Then also $P \backslash G \in M$, since M is a model of ZFC and by absoluteness. We claim that $P \backslash G$ is dense. In fact, given $p \in P$, choose q, r as in (1). Then q, r cannot both be in G, by the definition of filter. So one at least is in $P \backslash G$, as desired. Since $P \backslash G$ is dense and in M, we contradict G being generic. $\square$

We now give several equivalent definitions of generic.

A subset E if P is *open dense* iff it is dense, and $\forall p \in E \forall q \leq p[q \in E]$.

E is *predense in P* iff $\forall p \in P \exists q \in E[p$ and q are compatible$]$.

Lemma 14.43. *Suppose that M is a c.t.m. of ZFC and $\mathbb{P}$ is a forcing order in M. Suppose that $G \subseteq P$ satisfies the following condition:*

(i) $\forall p \in G \forall q \geq p[q \in G]$.

Then the following conditions are equivalent:

(ii) $G \cap D \neq \emptyset$ whenever $D \in M$ and D is dense in $\mathbb{P}$.

(iii) $G \cap A \neq \emptyset$ whenever $A \in M$ and A is a maximal antichain of $\mathbb{P}$.

(iv) $G \cap E \neq \emptyset$ whenever $E \in M$ and E is predense in $\mathbb{P}$.

(v) $G \cap D \neq \emptyset$ whenever $D \in M$ and D is open dense in $\mathbb{P}$.

Moreover, suppose that G satisfies (i) and one, hence all, of the conditions (ii)–(v). Then G is $\mathbb{P}$-generic over M iff the following condition holds:

(vi) For all $p, q \in G$, p and q are compatible.

Proof. (ii)⇒(iii): Assume (ii), and suppose that $A \in M$ is a maximal antichain of $\mathbb{P}$. Let $D = \{p \in P : p \leq q$ for some $q \in A\}$. We claim that D is dense. Suppose that r is arbitrary. Choose $q \in A$ such that r and q are compatible. Say $p \leq r, q$. Thus $p \in D$. So, indeed, D is dense. Clearly $D \in M$, since $A \in M$. By (ii), choose $p \in D \cap G$. Say $p \leq q \in A$. Then $q \in G \cap A$, as desired.

(iii)⇒(iv): Assume (iii), and suppose that $E \in M$ is predense in $\mathbb{P}$. By Zorn's lemma, let A be a maximal member of

(1) $\{B \subseteq P : B$ is an antichain, and for every $p \in B$ there is a $q \in E$ such that $p \leq q\}$.

We claim that A is a maximal antichain. For, suppose that $p \perp q$ for all $q \in A$. Choose $s \in E$ such that p and s are compatible. Say $r \leq p, s$. Hence $r \perp q$ for all $q \in A$, so $r \notin A$. Thus $A \cup \{r\}$ is a member of (1), contradiction.

Clearly $A \in M$, since $E \in M$. So, since A is a maximal antichain, choose $p \in A \cap G$. Then choose $q \in E$ such that $p \leq q$. So $q \in E \cap G$, as desired.

(iv)⇒(v): Clearly open dense → predense. So this implication is clear.

(v)⇒(ii): Assume (v), and suppose that D is dense in $\mathbb{P}$. Let $E = \{p : \exists q \in D[p \leq q]$. Clearly E is open dense, so there is a $p \in G \cap E$. Say $q \in D$ with $p \leq q$. Then $q \in G \cap D$, as desired.

Now we assume (i)–(v).

If G is $\mathbb{P}$-generic over M, clearly (vi) holds.

Now asume that (i)–(vi) hold, and suppose that $p, q \in G$; we want to find $r \in G$ such that $r \leq p, q$. Let

$$D = \{r : r \perp p \text{ or } r \perp q \text{ or } r \leq p, q\}.$$

We claim that D is dense in $\mathbb{P}$. For, let $s \in P$ be arbitrary. If $s \perp p$, then $s \leq s$ and $s \in D$, as desired. So suppose that s and p are compatible; say $t \leq s, p$. If $t \perp q$, then $t \leq s$ and $t \in D$, as desired. So suppose that t and q are compatible. Say $r \leq t, q$. Then $r \leq t \leq p$ and $r \leq t \leq s$, so $r \leq s$ and $r \leq p, q$, hence $r \in D$, as desired. This proves that D is dense.

Now by (ii) choose $r \in D \cap G$. By (vi), r is compatible with p and r is compatible with q. So $r \leq p, q$, as desired. □

Proposition 14.44. *Let P be a forcing poset in M, and suppose that G is a generic ultrafilter on* RO(P)*. Let $G' = \{p \in P : e(p) \in G\}$. Then G' is a P-generic filter over M.*

Proof. Suppose that $p \in G'$ and $p \leq q$. Then $e(p) \in G$ and $e(p) \leq e(q)$, so $e(q) \in G$ and hence $q \in G'$.

Suppose that D is dense in P. Then $\{e(p) : p \in D\}$ is dense in RO(P). Then $\sum_{p \in D} e(p) = 1 \in G$, so there is a $p \in D$ such that $e(p) \in G$. Thus $p \in G' \cap D$.

Hence by Lemma 14.43 it suffices to show that any elements $p, q \in G'$ are compatible. Thus $e(p), e(q) \in G$, so they are compatible. By Theorem 14.6(iii), p and q are compatible. □

Theorem 14.45. *Let P be a forcing poset.*

(i) If $p \Vdash \varphi$ and $q \leq p$, then $q \Vdash \varphi$.

(ii) There is no p such that $p \Vdash \varphi$ and $p \Vdash \neg\varphi$.

(iii) $\forall \varphi \forall p \exists q \leq p[q \Vdash \varphi$ or $q \Vdash \neg\varphi]$.

(iv) $p \Vdash \neg\varphi$ iff there is no $q \leq p$ such that $q \Vdash \varphi$.

(v) $p \Vdash \varphi \wedge \psi$ iff $p \Vdash \varphi$ and $p \Vdash \psi$.

(vi) $p \Vdash \forall x \varphi(x)$ iff for all $\dot{a} \in M^{\mathrm{RO}(P)}[p \Vdash \varphi(\dot{a})]$.

(vii) $p \Vdash \varphi \vee \psi$ iff $\forall q \leq p \exists r \leq q[r \Vdash \varphi$ or $r \Vdash \psi]$.

(viii) $p \Vdash \exists x \varphi(x)$ iff $\forall q \leq p \exists r \leq q \exists \dot{a} \in M^{\mathrm{RO}(P)}[r \Vdash \varphi(\dot{a})]$.

(ix) If $p \Vdash \exists x \varphi$ then there is an $\dot{a} \in M^{\mathrm{RO}(P)}$ such that $p \Vdash \varphi(\dot{a})$.

Proof. (i), (ii), (v), and (vi) are clear.

(iii): Suppose that φ and p are given. Now $e(p) \leq 1 = [\![\varphi]\!] + [\![\neg\varphi]\!]$, so $e(p) \cdot [\![\varphi]\!] \neq 0$ or $e(p) \cdot [\![\neg\varphi]\!] \neq 0$. Wlog $e(p) \cdot [\![\varphi]\!] \neq 0$. Choose q with $e(q) \leq e(p) \cdot [\![\varphi]\!] \neq 0$. Then p and q are compatible; say $r \leq p, q$. Then $e(r) \leq [\![\varphi]\!]$. so $r \Vdash [\![\varphi]\!]$.

(iv): $\Rightarrow$: assume that $p \Vdash \neg\varphi$, $q \leq p$, and $q \Vdash \varphi$. By (i), $q \Vdash \neg\varphi$, contradiction.

$\Leftarrow$: Assume that there is no $q \leq p$ such that $q \Vdash \varphi$, while $p \nVdash \neg\varphi$. Thus $e(p) \nleq [\![\varphi]\!]$, so $e(p) \cdot [\![\neg\varphi]\!] \neq 0$. Then there is a $q \leq p$ such that $e(q) \leq [\![\varphi]\!]$, contradiction.

(vii): $\Rightarrow$: Assume that $p \Vdash \varphi \vee \psi$. Then $e(p) \leq [\![\varphi]\!] + [\![\psi]\!]$, so $e(p) \cdot [\![\varphi]\!] \neq 0$ or $e(p) \cdot [\![\psi]\!] \neq 0$. Hence clearly $\forall q \leq p \exists r \leq q[r \Vdash \varphi \text{ or } r \Vdash \psi]$.

$\Leftarrow$: Assume that $\forall q \leq p \exists r \leq q[r \Vdash \varphi \text{ or } r \Vdash \psi]$, but $p \nVdash \varphi \vee \psi$. Then $e(p) \cdot [\![\neg\varphi]\!] \cdot]\!]\neg\psi]\!] \neq 0$. Take q' with $e(q') \leq e(p) \cdot [\![\neg\varphi]\!] \cdot]\!]\neg\psi]\!] \neq 0$, and then take $q \leq p, q'$. Choose $r \leq q$ such that $r \Vdash \varphi$ or $r \Vdash \psi$. But clearly $r \Vdash \neg\varphi$ and $r \Vdash \neg\psi$, so this contradicts (ii).

(viii): $\Rightarrow$: Assume that $p \Vdash \exists x\varphi(x)$. Thus for any $q \leq p$, $e(q) \leq [\![\exists x\varphi(x)]\!] = \sum_{\dot{a} \in M^{\mathrm{RO}(P)}} [\![\varphi(\dot{a})]\!]$. Hence there is an $\dot{a} \in M^{\mathrm{RO}(P)}$ such that $e(q) \cdot [\![\varphi(\dot{a})]\!] \neq 0$, and this easily gives the right side of the equivalence.

$\Leftarrow$: Suppose that $p \nVdash \exists x\varphi(x)$. Then $e(p) \cdot -[\![\exists x\varphi(x)]\!] \neq 0$. So we easily get $q \leq p$ such that $e(q) \leq -[\![\exists x\varphi(x)]\!]$. For any $r \leq q$ we also have $e(r) \leq -[\![\exists x\varphi(x)]\!]$. Hence $e(r) \cdot \sum_{\dot{a} \in M^{\mathrm{RO}(P)}} [\![\varphi(\dot{a}) \uparrow = 0$. So easily there is no $\dot{a} \in M^{\mathrm{RO}(P)}$ such that $r \Vdash \varphi(\dot{a})$.

(ix): Assume that $p \Vdash \exists x\varphi(x)$. So $e(p) \leq [\![\exists x\varphi(x)]\!]$. By Lemma 27 there is a $\dot{a} \in M^{\mathrm{RO}(P)}$ such that $[\![\exists x\varphi(x)]\!] = [\![\varphi(\dot{a})]\!]$, as desired. □

Now if B is a complete BA in M and G is a generic ultrafilter on B, for each $x \in M^B$ we define x^G by recursion:x^G

$$x^G = \{y^G : y \in \mathrm{dmn}(x) \wedge x(y) \in G\}.$$

If P is a forcing poset in M and G is a P-generic filter over M, for each $x \in M^{\mathrm{RO}(P)}$ we define x_G by recursion:

$$x_G = \{y_G : y \in \mathrm{dmn}(x) \wedge \exists p \in G[e(p) \leq x(y)]\}.$$

If B is a complete BA, then the Γ is the Boolean valued function defined by

$$\mathrm{dmn}(\Gamma) = \{\check{u} : u \in B\}; \quad \forall u \in B[\Gamma(\check{u}) = u].$$

For any forcing poset P, let $B = \mathrm{RO}(P)$. Then the Γ' is the B-valued function defined by

$$\mathrm{dmn}(\Gamma') = \{\check{p} : p \in P\}; \quad \forall p \in P[\Gamma'(\check{p}) = e(p)].$$

Now we define $\mathrm{dmn}(\check{M}) = \{\check{a} : a \in M\}$, and for any $a \in M$, $\check{M}(\check{a}) = 1$. Then

Lemma 14.46. *$p \Vdash \dot{a} \in \check{M}$ iff $\forall q \leq p \exists b \in M \exists r \leq q[r \Vdash \dot{a} = \check{b}]$.*

Proof.

$$\begin{aligned} p \Vdash \dot{a} \in \check{M} \quad & \text{iff} \quad e(p) \leq ||\dot{a} \in \check{M}|| \\ & \text{iff} \quad e(p) \leq \sum_{b \in M} ||\dot{a} = \check{b}|| \\ & \text{iff} \quad \forall q \leq p \exists b \in M[e(q) \cdot ||\dot{a} = \check{b}|| \neq 0] \\ & \text{iff} \quad \forall q \leq p \exists b \in M \exists r \leq q[e(r) \leq ||\dot{a} = \check{b}|| \\ & \text{iff} \quad \forall q \leq p \exists b \in M \exists r \leq q[r \Vdash \dot{a} = \check{b}] \end{aligned}$$ □

Lemma 14.47. *If G is a generic ultrafilter on B, then $\Gamma^G = G$.*

Proof. $\Gamma^G = \{\check{a}^G : a \in G\} = \{a : a \in G\} = G$. $\square$

Lemma 14.48. *If G is P-generic over M, then $\Gamma'_G = G$.*

Proof. $\Gamma'_G = \{(\check{p}_G : \exists q \in G[e(q) \leq e(p)]\} = \{p : \exists q \in G[e(q) \leq e(p)]\} = G$.

Lemma 14.49. *If G is a generic ultrafilter on B in M, then $\check{M}^G = M$.*

Proof. $\check{M}^G = \{\check{a}^G : a \in M\} = M$. $\square$

Lemma 14.50. *If G is a P-generic filter over M, then $\check{M}_G = M$.*

Proof. $\check{M}_G = \{\check{a}_G : a \in M\} = M$. $\square$

Lemma 14.51. $p \Vdash e(q)\check{} \in \Gamma'$ *iff* $e(p) \leq e(q)$ *iff* $\forall r \leq p \exists s \leq r[s \leq q]$.

Proof. Note that

$$[\![e(q)\check{} \in \Gamma']\!] = \sum_{s \in P} ([\![e(q)\check{} = e(s)\check{}]\!] \cdot e(s) = e(q).$$

Hence $p \Vdash e(q)\check{} \in \Gamma'$ iff $e(p) \leq e(q)$.

If $e(p) \leq e(q)$ and $r \leq p$, then $e(r) \leq e(q)$ and so there is an $s \leq r$ such that $s \leq q$.

If $e(p) \not\leq e(q)$, then there is a t such that $e(t) \leq e(p) \cdot -e(q)$. Then there is an $r \leq t, p$, and $e(r) \cdot e(q) = 0$, hence there is no $s \leq r$ such that $s \leq q$. $\square$

Lemma 14.52. *Let P be a forcing poset over M, and G P-generic over M. Let G' be the generic ultrafilter on $\mathrm{RO}(P)$ given by Proposition 14.40: $G' = \{a \in \mathrm{RO}(P) : \exists p \in G[e(p) \leq a\}$. Then for any $a \in M^{\mathrm{RO}(P)}$, $a_G = a^{G'}$.*

Proof. Induction:

$$a_G = \{b_G : b \in \mathrm{dmn}(a) \wedge \exists p \in G[e(p) \leq a(b)]\} = \{b^{G'} : b \in \mathrm{dmn}(a) \wedge a(b) \in G'\} = a^{G'}. \square$$

Lemma 14.53. *If G is a generic ultrafilter on B, then $\forall x[\check{x}^G = x]$.*

Proof. $\check{x}^G = \{\check{y}^G : y \in x\} = \{y : y \in x\} = x$. $\square$

Lemma 14.54. *If G is a P-generic filter over M, then $\forall x[\check{x}_G = x]$.*

Proof. $\check{x}_G = \{\check{y}_G : y \in x$ and $\exists p \in G[e(p) \leq 1]\} = \{y : y \in x\} = x$. $\square$

For G a generic ultrafilter on B we define

$$M[G] = \{x^G : x \in M^B\}.$$

Also for G a generic filter over P we define

$$M[G] = \{x_G : x \in M^{\mathrm{RO}(P)}\}$$

Lemma 14.55. *If G is P-generic over M and G' is the generic ultrafilter on $\mathrm{RO}(P)$ given by Lemma 14.40. then $M[G] = M[G']$.*

Proof. $M[G] = \{x_G : x \in M^{\mathrm{RO}(P)}\} = \{x^{G'} : x \in M^{\mathrm{RO}(P)}\} = M[G']$. $\square$

Lemma 14.56. *Let M be a c.t.m., let $B \in M$ be a complete BA in the sense of M, and let G be a generic ultrafilter on B. Recall from Proposition 14.38 that G is a B^+-generic filter over M. Let e be the embedding of B^+ into $\mathrm{RO}(B^+)$.*

(i) $\forall b \in B^+[e(b) = B^+ \downarrow b]$.

(ii) For any $b \in B^+$ let $f(B^+ \downarrow b) = b$. Then f is an isomorphism of $\mathrm{RO}(B^+)$ onto B such that $f \circ e$ is the identity on B^+.

(iii) Let $G' = f^{-1}[G]$. Then G' is a generic ultrafilter on $\mathrm{RO}(B^+)$.

(iv) For any $x \in V^{\mathrm{RO}(B^+)}$ let $x' \in V^B$ be defined by:

$$\mathrm{dmn}(x') = \{y' : y \in \mathrm{dmn}(x)\} \quad \textit{and} \quad x'(y') = f(x(y)).$$

Then for any $x \in V^{\mathrm{RO}(B^+)}$ we have $x^{G'} = x'^G$.

Proof. For (i), take any $b \in B^+$. By Proposition 14.7(i), $\mathrm{cl}(B^+ \downarrow b) = \{c \in B^+ : b \cdot c \neq 0\}$. Hence

$$\begin{aligned} e(b) &= \mathrm{int}(\mathrm{cl}(B^+ \downarrow b)) = \mathrm{int}(\{c \in B^+ : b \cdot c \neq 0\}) \\ &= \{d \in B^+ : B^+ \downarrow d \subseteq \{c \in B^+ : b \cdot c \neq 0\}\} = B^+ \downarrow b \end{aligned}$$

For (ii), see the proof of Theorem 14.8.

(iii) is immediate from (ii).

For (iv), we have

$$\begin{aligned} x^{G'} &= \{y^{G'} : y \in \mathrm{dmn}(x) \wedge x(y) \in G'\} = \{y'^G : y' \in \mathrm{dmn}(x') \wedge x'(y') \in f(G')\} \\ &= \{y'^G : y' \in \mathrm{dmn}(x') \wedge x'(y') \in G\} = x'^G. \end{aligned}$$

$\square$

Lemma 14.57. *If G is a generic ultrafilter on B and $x, y \in M^B$, then:*

(i) $x^G \in y^G$ iff $[\![x \in y]\!] \in G$;

(ii) $x^G = y^G$ iff $[\![x = y]\!] \in G$.

Proof. By simultaneous induction:

$$\begin{aligned} [\![x \in y]\!] \in G \quad &\text{iff} \quad \sum_{t \in \mathrm{dmn}(y)} ([\![x = t]\!] \cdot y(t)) \in G \\ &\text{iff} \quad \exists t \in \mathrm{dmn}(y)[([\![x = t]\!] \cdot y(t)) \in G] \\ &\text{iff} \quad \exists t \in \mathrm{dmn}(y)[[\![x = t]\!] \in G \text{ and } y(t)) \in G] \\ &\text{iff} \quad \exists t \in \mathrm{dmn}(y)[x^G = t^G \text{ and } y(t) \in G] \\ &\text{iff} \quad x^G \in \{t^G : t \in \mathrm{dmn}(y) \text{ and } y(t) \in G\} \\ &\text{iff} \quad x^G \in y^G; \end{aligned}$$

$$
\begin{aligned}
[\![x \subseteq y]\!] \in G \quad \text{iff} \quad & \prod_{t \in \mathrm{dmn}(x)} (x(t) \Rightarrow [\![t \in y]\!]) \in G \\
\text{iff} \quad & \forall t \in \mathrm{dmn}(x)[x(t) \in G \to [\![t \in y]\!]) \in G] \\
\text{iff} \quad & \forall t \in \mathrm{dmn}(x)[x(t) \in G \to t^G \in y^G] \\
\text{iff} \quad & \{t^G : t \in \mathrm{dmn}(x), x(t) \in G\} \subseteq y^G \\
\text{iff} \quad & x^G \subseteq y^G. \qquad \square
\end{aligned}
$$

Theorem 14.58. *If G is a generic ultrafilter on B in M, if $\varphi(x_0, \ldots, x_{m-1})$ is a formula, and $a_0, \ldots, a_{m-1} \in V^B$, then*

$$
M[G] \models \varphi(a_0^G, \ldots, a_{m-1}^G) \quad \text{iff} \quad [\![\varphi(a_0, \ldots, a_{m-1})]\!] \in G.
$$

Proof. Induction on φ. φ atomic is given by Lemma 14.57.
φ is $\neg\psi$:

$$
[\![\neg\psi]\!] \in G \quad \text{iff} \quad [\![\psi]\!] \notin G \quad \text{iff} \quad \text{not}(M[G] \models \psi) \quad \text{iff} \quad M[G] \models \neg\psi.
$$

φ is $\psi \vee \chi$:

$$
\begin{aligned}
[\![\psi \vee \chi]\!] \in G \quad \text{iff} \quad & [\![\psi]\!] + [\![\chi]\!] \in G \quad \text{iff} \quad [\![\psi]\!] \in G \text{ or } [\![\chi]\!] \in G \\
\text{iff} \quad & M[G] \models \psi \text{ or } M[G] \models \chi \quad \text{iff} \quad M[G] \models \psi \vee \chi.
\end{aligned}
$$

φ is $\exists x\psi(x)$:

$$
\begin{aligned}
[\![\exists x\psi(x)]\!] \in G \quad \text{iff} \quad & \sum_{x \in M^B} [\![\psi(x)]\!] \in G \\
\text{iff} \quad & \exists x \in M^B[[\![\psi(x)]\!] \in G] \\
\text{iff} \quad & \exists x \in M^B[M[G] \models \psi(x^G)] \\
\text{iff} \quad & M[G] \models \exists x\psi(x). \qquad \square
\end{aligned}
$$

Theorem 14.59. *If G is a generic filter on M, $\varphi(x_0, \ldots, x_{m-1})$ is a formula, and $a_0, \ldots, a_{m-1} \in V^{\mathrm{RO}(P)}$. then*

$$
M[G] \models \varphi(a_0, \ldots, a_{m-1}) \quad \textit{iff} \quad \exists p \in G[p \Vdash \varphi(a_0, \ldots, a_{m-1}).
$$

Proof.

$$
\begin{aligned}
M[G] \models \varphi(a_0, \ldots, a_{m-1}) \quad \text{iff} \quad & [\![\varphi(a_0, \ldots, a_{m-1})[\![\in \mathrm{RO}(P) \\
\text{iff} \quad & \exists p \in G[p \Vdash \varphi(a_0, \ldots, a_{m-1}).
\end{aligned}
$$

Here we use Theorem 14.58 and the fact that $[\![\varphi(a_0, \ldots, a_{m-1})$ is the sum of all $e(p)$ below it. $\square$

Theorem 14.60. *For G a generic ultrafilter on $B \in M$, $M[G]$ is a model of ZFC.*

Proof. If φ is an axiom of ZFC, then by Theorem 14.32, $[\![\varphi]\!] = 1 \in G$. By Theorem 14.59, $M[G] \models \varphi$. $\square$

Theorem 14.61. *For G a generic ultrafilter on $B \in M$, $M[G]$ is transitive, $M \subseteq M[G]$, and $G \in M[G]$.*

Proof. If $x \in y \in M[G]$, say $y = a^G$. Now $a^G = \{b^G : b \in \text{dmn}(a) \text{ and } a(b) \in G\}$, so $x = b^G \in M[G]$ for some b. So $M[G]$ is transitive. Now by Lemma 14.53, for any $a \in M$ we have $\check{a}^G = a$; so $M \subseteq M[G]$. By Lemma 14.47, $G \in M[G]$. $\square$

Theorem 14.62. *M and $M[G]$ have the same ordinals.*

Proof. By absoluteness, ordinals in the sense of M or in the sense of $M[G]$ are really ordinals. Since $M \subseteq M[G]$ by Theorem 14.54, every ordinal in M is in $M[G]$. Now suppose that x is an ordinal in $M[G]$. Say $x = a^G$. Thus $M[G] \models$ "a^G is an ordinal", so by Theorem 14.58, $[\![a \text{ is an ordinal}]\!] \in G$. By Lemma 14.34, $\sum_{\alpha \in \mathbf{ON}}]\!]a = \check{\alpha}]\!] \in G$; the sum is taken in M, so the α's are in M. Hence there is an ordinal $\alpha \in M$ such that $]\!]a = \check{\alpha}]\!] \in G$. Then by Theorem 14.58, $a^G = \alpha$. $\square$

Theorem 14.63. () *If N is a transitive model of ZFC and $M \cup \{G\} \subseteq N$, then $M[G] \subseteq N$.*

Proof. The mapping $a \mapsto a^G$ is absolute. $\square$

Lemma 14.64. *Let G be P-generic over M. Let $\varphi(\overline{w})$ be a formula, and $a_0, \ldots, a_{m-1} \in M^{\mathrm{RO}(P)}$. Then $p \Vdash \varphi(a_0, \ldots, a_{m-1})$ iff for every G which is P-generic over M, if $p \in G$ then $M[G] \models \varphi(a_{0G}, \ldots, a_{(m-1)G})$.*

Proof. $\Rightarrow$: Assume that $p \Vdash \varphi(a_0, \ldots, a_{m-1})$, and suppose that $p \in G$. Then by Theorem 14.59, $M[G] \models \varphi(a_{0G}, \ldots, a_{(m-1)G})$.

$\Leftarrow$: assume the indicated condition, but $p \not\Vdash \varphi(a_0, \ldots, a_{m-1})$. Thus

$$e(p) \cdot -[\![\varphi(a_0, \ldots, a_{m-1})]\!] \neq 0,$$

so we easily get $q \leq p$ such that $q \Vdash \neg\varphi(a_0, \ldots, a_{m-1})$. Let G be generic with $q \in G$. Then by the $\Rightarrow$ already proved, $M[G] \models \neg\varphi(a_{0G}, \ldots, a_{(m-1)G})$. But $p \in G$, so this contradicts our condition. $\square$

Lemma 14.65. *Suppose that α is a limit ordinal, κ and λ are regular cardinals, $f : \kappa \to \alpha$ is strictly increasing with $\text{rng}(f)$ cofinal in α, and $g : \lambda \to \alpha$ is strictly increasing with $\text{rng}(g)$ cofinal in α. Then $\kappa = \lambda$.*

Proof. Suppose not; say by symmetry $\kappa < \lambda$. For each $\xi < \kappa$ choose $\eta_\xi < \lambda$ such that $f(\xi) < g(\eta_\xi)$. Let $\rho = \sup_{\xi<\kappa} \eta_\xi$. Thus $\rho < \lambda$ by the regularity of λ. But then $f(\xi) < g(\rho) < \alpha$ for all $\xi < \kappa$, contradiction. $\square$

Lemma 14.66. *Let M be a c.t.m. of ZFC, $\mathbb{P} \in M$ be a forcing order, and κ be a cardinal of M.*

(i) If $\mathbb{P}$ preserves regular cardinals $\geq \kappa$, then it preserves cofinalities $\geq \kappa$.

(ii) If $\mathbb{P}$ preserves cofinalities $\geq \kappa$ and κ is regular, then $\mathbb{P}$ preserves cardinals $\geq \kappa$.

(iii) If $\mathbb{P}$ preserves cofinalities, then $\mathbb{P}$ preserves cardinals.

Proof. (i): Let α be a limit ordinal of M with $(\mathrm{cf}(\alpha))^M \geq \kappa$. Then $(\mathrm{cf}(\alpha))^M$ is a regular cardinal of M which is $\geq \kappa$ and hence is also a regular cardinal of $M[G]$. Now we can apply Lemma 14.65 within $M[G]$ to $\kappa = (\mathrm{cf}(\alpha))^M$ and $\lambda = (\mathrm{cf}(\alpha))^{M[G]}$ to infer that $(\mathrm{cf}(\alpha))^M = (\mathrm{cf}(\alpha))^{M[G]}$.

(ii): Suppose that cardinals $\geq \kappa$ are not preserved, and let λ be the least cardinal of M which is $\geq \kappa$ but which is not a cardinal of $M[G]$. If λ is regular in M, then

$$\lambda = (\mathrm{cf}(\lambda))^M = (\mathrm{cf}(\lambda))^{M[G]},$$

and so λ is a regular cardinal in $M[G]$, contradiction. If λ is singular in M, then $\lambda > \kappa$ since κ is regular and $\lambda \geq \kappa$. So λ is the supremum of a set S of cardinals of M which are regular and $\geq \kappa$, so each member of S is a cardinal of $M[G]$ by the minimality of λ, so λ is a cardinal of $M[G]$.

(iii): follows from (ii), with $\kappa = \omega$. □

Now if $\sigma, \tau \in M^P$ we define

$$\begin{aligned}
&\mathrm{dmn}(\mathrm{up}(\sigma,\tau)) = \{\sigma,\tau\}; \quad (\mathrm{up}(\sigma,\tau))(\sigma) = (\mathrm{up}(\sigma,\tau))(\tau) = 1;\\
&\mathrm{op}(\sigma,\tau) = \mathrm{up}(\mathrm{up}(\sigma,\sigma), \mathrm{up}(\sigma,\tau)),
\end{aligned}$$

Lemma 14.67. *(i)* $(\mathrm{up}(\sigma,\tau))_G = \{\sigma_G, \tau_G\}$.

(ii) $(\mathrm{op}(\sigma,\tau))_G = (\sigma_G, \tau_G)$. □

A forcing order $\mathbb{P}$ satisfies the *κ-chain condition*, abbreviated κ-c.c., iff every antichain in $\mathbb{P}$ has size less than κ.

The following theorem is very useful in forcing arguments.

Theorem 14.68. *Let M be a c.t.m. of ZFC, $\mathbb{P} \in M$ be a forcing order, κ be a cardinal of M, G be $\mathbb{P}$-generic over M, and suppose that $\mathbb{P}$ satisfies the κ-c.c. Suppose that $f \in M[G]$, $A, B \in M$, and $f : A \to B$. Then there is an $F : A \to \mathscr{P}(B)$ with $F \in M$ such that:*

(i) $f(a) \in F(a)$ for all $a \in A$.

(ii) $(|F(a)| < \kappa)^M$ for all $a \in A$.

Proof. Let $\tau \in M^P$ be such that $\tau_G = f$. Thus the statement "$\tau_G : A \to B$" holds in $M[G]$. Hence by Theorem 14.64 there is a $p \in G$ such that

$$p \Vdash \tau : \check{A} \to \check{B}.$$

Now for each $a \in A$ let

$$F(a) = \{b \in B : \text{there is a } q \leq p \text{ such that } q \Vdash \mathrm{op}(\check{a}, \check{b}) \in \tau\}.$$

To prove (i), suppose that $a \in A$. Let $b = f(a)$. Thus $(a, b) \in f$, i.e. $\mathrm{op}(\check{a}, \check{b})_G \in \tau_G$, so by Theorem 53 there is an $r \in G$ such that $r \Vdash \mathrm{op}(\check{a}, \check{b}) \in \tau$. Let $q \in G$ with $q \leq p, r$. Then q shows that $b \in F(a)$.

To prove (ii), again suppose that $a \in A$. By the axiom of choice in M, there is a function $Q : F(a) \to P$ such that for any $b \in F(a)$, $Q(b) \leq p$ and $Q(b) \Vdash \mathrm{op}(\check{a}, \check{b}) \in \tau$.

(1) If $b, b' \in F(a)$ and $b \neq b'$, then $Q(b) \perp Q(b')$.

In fact, suppose that $r \leq Q(b), Q(b')$. Then

$$r \Vdash \mathrm{op}(\check{a}, \check{b}) \in \tau \wedge \mathrm{op}(\check{a}, \check{b'}) \in \tau; \tag{2}$$

but also $r \leq Q(b) \leq p$, so $r \Vdash \tau : \check{A} \to \check{B}$, hence

$$r \Vdash \forall x, y, z[\mathrm{op}(x, y) \wedge \mathrm{op}(x, z) \to y = z]$$

and hence

$$r \Vdash \mathrm{op}(\check{a}, \check{b}) \in \tau \wedge \mathrm{op}(\check{a}, \check{b'}) \in \tau \to \check{b} = \check{b'}. \tag{3}$$

Now let H be $\mathbb{P}$-generic over M with $r \in H$. By Lemma 14.67 we have $(a, b) = (\mathrm{op}(\check{a}, \check{b}))_G \in \tau_G$ and $(a, b') = (\mathrm{op}(\check{a}, \check{b'}) \in \tau_G$. By (3) and Lemma 14.67 it follows that $b = b'$. Thus (1) holds.

By (1), $\langle Q(b) : b \in F(a) \rangle$ is a one-one function onto an antichain of P. Hence $(|F(a)| < \kappa)^M$ by the κ-cc. $\square$

Proposition 14.69. *If M is a c.t.m. of ZFC, κ is a cardinal of M, and $\mathbb{P} \in M$ satisfies κ-cc in M, then $\mathbb{P}$ preserves regular cardinals $\geq \kappa$, and also preserves cofinalities $\geq \kappa$. If also κ is regular in M, then $\mathbb{P}$ preserves cardinals $\geq \kappa$.*

Proof. First we want to show that if $\lambda \geq \kappa$ is regular in M then also λ is regular in $M[G]$ (and hence is a cardinal of $M[G]$). Suppose that this is not the case. Hence in $M[G]$ there is an $\alpha < \lambda$ and a function $f : \alpha \to \lambda$ such that the range of f is cofinal in λ. Now $\alpha \in M$. By Theorem 14.68, let $F : \alpha \to \mathscr{P}(\lambda)$ be such that $f(\xi) \in F(\xi)$ and $(|F(\xi)| < \lambda)^M$ for all $\xi < \alpha$. Let $S = \bigcup_{\xi<\alpha} F(\xi)$. Then S is a subset of λ which is cofinal in λ and has size less than λ, contradiction.

The rest of the proposition follows from Lemma 14.66. $\square$

Theorem 14.70. *There is a forcing poset and a G which is P-generic over M such that $M[G] \models 2^{\aleph_0} > \aleph_1$ and M and $M[G]$ have the same cardinals.*

Proof. Let P consist of all functions p such that $\mathrm{dmn}(p)$ is a finite subset of $\omega_2 \times \omega$ and $\mathrm{rng}(p) \subseteq \{0, 1\}$. We order P by $\supseteq$. Let G be any P-generic filter over M. Any two members of G are compatible, so $f \overset{\text{def}}{=} \bigcup G$ is a function.

(1) $\mathrm{dmn}(f) = \omega_2 \times \omega$.

In fact, for $(\alpha, n) \in \omega_2 \times \omega$ let $D_{\alpha n} = \{p \in P : (\alpha, n) \in \mathrm{dmn}(p)\}$. Clearly $D_{\alpha n}$ is dense. Hence (1) follows.

Now for each $\alpha < \omega_2$ let $g_\alpha \in {}^\omega 2$ be defined by $g_\alpha(n) = f(\alpha, n)$.

(2) If $\alpha, \beta \in \omega_2$ and $\alpha \neq \beta$, then $g_\alpha \neq g_\beta$.

In fact, let $\alpha, \beta \in \omega_2$ with $\alpha \neq \beta$. Let $E = \{p \in P : \exists n[(\alpha, n), (\beta, n) \in \mathrm{dmn}(p)$ and $p(\alpha, n) \neq p(\beta, n)]\}$. Clearly E is dense. Hence (2) follows.

(3) P satisfies the ω_1-chain condition in M.

In fact, suppose that $X \in [P]^{\omega_1}$ is an antichain. Define $p \equiv q$ iff $f, g \in X$ and $\mathrm{dmn}(f) = \mathrm{dmn}(g)$. This is an equivalence relation, and each equvalence class is finite. Let Y consist of one element from each equivalence class. So $|Y| = \omega_1$. Let $Z = \{\mathrm{dmn}(p) : p \in Y\}$. So Z is a collection of finite sets, and $|Z| = \omega_1$. By Theorem 9.20 let $W \in [Z]^{\omega_1}$ be a Δ-system. Say $x \cap y = z$ for distinct $x, y \in W$. Now take two distinct members $\mathrm{dmn}(p), \mathrm{dmn}(q)$ of W. We define

$$\mathrm{dmn}(r) = \mathrm{dmn}(p) \cup \mathrm{dmn}(q) \quad \text{and} \quad r(\alpha, n) = \begin{cases} p(\alpha, n) & \text{if } \alpha, n) \in \mathrm{dmn}(p), \\ q(\alpha, n) & \text{otherwise.} \end{cases}$$

Then $p, q \subseteq r$, so p and q are compatible, contradiction. So (3) holds.

Now by (3) and Proposition 14.69, every cardinal of M is a cardinal of $M[G]$. $\square$

Proposition 14.71. *If $\{q : q \Vdash \varphi\}$ is dense below p, then $p \Vdash \varphi$.*

Proof. Suppose that $\{q : q \Vdash \varphi\}$ is dense below p but $p \nVdash \varphi$. Then $e(p) \cdot -[\![\varphi]\!] \neq 0$. Choose s with $e(s) \leq e(p) \cdot -[\![\varphi]\!]$. Say $t \leq s, p$. Choose $r \leq t$ with $r \Vdash \varphi$. Then $e(r) \leq e(t) \leq e(s) \leq -[\![\varphi]\!]$, contradiction.

Proposition 14.72. *Suppose that G is an ultrafilter on $B \in M$. Then G is a generic ultrafilter on B iff for every partition W of B with $W \in M$ there is a unique $a \in G \cap W$.*

Proof. $\Rightarrow$: Suppose that G is a generic ultrafilter on B and W is a partition of B with $W \in M$. Then there is a $w \in G \cap W$. Clearly w is unique.

$\Leftarrow$: Suppose that for every partition W of B with $W \in M$ there is a unique $a \in G \cap W$. Suppose that $\sum X \in G$. Write $X = \{x_\alpha : \alpha < \kappa\}$. Define $y_\alpha = x_\alpha \cdot \prod_{\beta < \alpha} -x_\beta$ for each $\alpha < \kappa$, and let $y_\kappa = -\sum X$. Then $\{y_\alpha : \alpha \leq \kappa\}$ is a partition with $\sum_{\alpha \leq \kappa} y_\alpha = 1 \in G$. Choose $\alpha \leq \kappa$ such that $y_\alpha \in G$. Since $\sum X \in G$, we have $\alpha < \kappa$. Hence $x_\alpha \in G$. $\square$

Proposition 14.73. *Let G be a generic ultrafilter on B over M, and $\mathfrak{A} = (M^B, E^*, F^*)$, where $E^*(a, b) = [\![a = b]\!]$ and $F^*(a, b) = [\![a \in b]\!]$, for $a, b \in M^B$. Let $\mathfrak{A}/\equiv_G^{\mathfrak{A}}$ be defined as before Lemma 14.24. Then $\mathfrak{A}/\equiv_G^{\mathfrak{A}} \cong M[G]$.*

Proof.

(1) If $a, b \in M^B$ and $a \equiv_G^{\mathfrak{A}}$, then $a^G = b^G$.

In fact, assume that $a, b \in M^B$ and $a \equiv_G^{\mathfrak{A}}$. Then $[\![a = b]\!] \in G$. By Lemma 52, $a^G = b^G$.

We define $f([a]) = a^G$. By Lemma 52, f is one-one. It is clearly onto. $[a] \in [b]$ iff $[\![a \in b]\!] \in G$ iff $a_G \in b_G$, by Lemma 14.59. $\square$

We now give a generalization of the .

Theorem 14.74. *Suppose that κ and λ are cardinals, $\omega \le \kappa < \lambda$, λ is regular, and for all $\alpha < \lambda$, $|[\alpha]^{<\kappa}| < \lambda$. Suppose that $\mathscr{A}$ is a collection of sets, with each $A \in \mathscr{A}$ of size less than κ, and with $|\mathscr{A}| \ge \lambda$. Then there is a $\mathscr{B} \in [\mathscr{A}]^\lambda$ which is a Δ-system.*

Proof.

(1) There is a regular cardinal μ such that $\kappa \le \mu < \lambda$.

In fact, if κ is regular, we may take $\mu = \kappa$. If κ is singular, then $\kappa^+ \le |[\kappa]^{<\kappa}| < \lambda$, so we may take $\mu = \kappa^+$.

We take μ as in (1). Let $S = \{\alpha < \lambda : \alpha \text{ is a limit ordinal and } \mathrm{cf}(\alpha) = \mu\}$. Then S is a stationary subset of λ.

Let $\mathscr{A}_0$ be a subset of $\mathscr{A}$ of size λ. Now $\left|\bigcup_{A\in\mathscr{A}_0} A\right| \le \lambda$ since $\kappa < \lambda$. Let a be an injection of $\bigcup_{A\in\mathscr{A}_0} A$ into λ, and let A be a bijection of λ onto $\mathscr{A}_0$. Set $b_\alpha = a[A_\alpha]$ for each $\alpha < \lambda$. Now if $\alpha \in S$, then $|b_\alpha \cap \alpha| \le |b_\alpha| = |A_\alpha| < \kappa \le \mu = \mathrm{cf}(\alpha)$, so there is an ordinal $g(\alpha)$ such that $\sup(b_\alpha \cap \alpha) < g(\alpha) < \alpha$. Thus g is a regressive function on S. By Fodor's theorem, there exist a stationary $S' \subseteq S$ and a $\beta < \lambda$ such that $g[S'] = \{\beta\}$. For each $\alpha \in S'$ let $F(\alpha) = b_\alpha \cap \alpha$. Thus $F(\alpha) \in [\beta]^{<\kappa}$, and $|[\beta]^{<\kappa}| < \lambda$, so there exist an $S'' \in [S']^\lambda$ and a $B \in [\beta]^{<\kappa}$ such that $b_\alpha \cap \alpha = B$ for all $\alpha \in S''$.

Now we define $\langle \alpha_\xi : \xi < \lambda \rangle$ by recursion. For any $\xi < \lambda$, α_ξ is a member of S'' such that

(2) $\alpha_\eta < \alpha_\xi$ for all $\eta < \xi$, and
(3) $\delta < \alpha_\xi$ for all $\delta \in \bigcup_{\eta<\xi} b_{\alpha_\eta}$.

Since $\left|\bigcup_{\eta<\xi} b_{\alpha_\eta}\right| < \lambda$, this is possible by the regularity of λ.

Now let $\mathscr{A}_1 = A[\{\alpha_\xi : \xi < \lambda\}]$ and $r = a^{-1}[B]$. We claim that $C \cap D = r$ for distinct $C, D \in \mathscr{A}_1$. For, write $C = A_{\alpha_\xi}$ and $D = A_{\alpha_\eta}$. Without loss of generality, $\eta < \xi$. Suppose that $x \in r$. Thus $a(x) \in B \subseteq b_{\alpha_\xi}$, so by the definition of b_{α_ξ} we have $x \in A_{\alpha_\xi} = C$. Similarly $x \in D$. Conversely, suppose that $x \in C \cap D$. Thus $x \in A_{\alpha_\xi} \cap A_{\alpha_\eta}$, and hence $a(x) \in b_{\alpha_\xi} \cap b_{\alpha_\eta}$. By the definition of α_ξ, since $a(x) \in b_{\alpha_\eta}$ we have $a(x) < \alpha_\xi$. So $a(x) \in b_{\alpha_\xi} \cap \alpha_\xi = B$, and hence $x \in r$.

Clearly $|\mathscr{A}_1| = \lambda$. □

Another form of this theorem is as follows. An *indexed Δ-system* is a system $\langle A_i : i \in I \rangle$ of sets such that there is a set r (the *root*) such that $A_i \cap A_j = r$ for all distinct $i, j \in I$. Some, or even all, the A_i's can be equal.

Theorem 14.75. *Suppose that κ and λ are cardinals, $\omega \le \kappa < \lambda$, λ is regular, and for all $\alpha < \lambda$, $|[\alpha]^{<\kappa}| < \lambda$. Suppose that $\langle A_i : i \in I \rangle$ is a system of sets, with each A_i of size less than κ, and with $|I| \ge \lambda$. Then there is a $J \in [I]^\lambda$ such that $\langle A_i : i \in J \rangle$ is an indexed Δ-system.*

Proof. Define $i \equiv j$ iff $i, j \in I$ and $A_i = A_j$. If some equivalence class has λ or more elements, a subset J of that class of size λ is as desired. If every equivalence class has fewer than λ elements, then there are at least λ equivalence classes. Let $\mathscr{A}$ have exactly one element in common with λ equivalence classes. We apply Theorem 9.20 to get a subset $\mathscr{B}$ of $\mathscr{A}$ of size λ which is a Δ-system, say with kernel r. Say $\mathscr{B} = \{A_i : i \in J\}$ with $J \in [I]^\lambda$ and $A_i \neq A_j$ for $i \neq j$. Then $\langle A_i : i \in J\rangle$ is an indexed Δ-system with root r. $\square$

15. Applications of forcing

Theorem 15.1. *Let M be a c.t.m. In M let κ be an infinite cardinal, and let P consist of all finite functions p with $\mathrm{dmn}(p) \subseteq \kappa \times \omega$ and $\mathrm{rng}(p) \subseteq \{0.1\}$, with ordering $\supseteq$. Let G be P-generic over M. Let $f = \bigcup G$. Then $f : \kappa \times \omega \to 2$. For each $\alpha < \kappa$ define $g_\alpha(n) = f(\alpha, n)$. Then $g_\alpha \neq g_\beta$ for $\alpha \neq \beta$. M and $M[G]$ have the same cardinals. For each $\alpha < \kappa$ let $a_\alpha = \{n \in \omega : g_\alpha(n) = 1\}$. Then $a_\alpha \neq a_\beta$ for $\alpha \neq \beta$.*

(i) $\forall \alpha < \kappa[a_\alpha \notin M]$.

(ii) $(2^{\aleph_0})^{M[G]} \geq (\kappa^{\aleph_0})^M$.

Proof. Suppose that $\alpha, \beta < \kappa$ and $\alpha \neq \beta$. Let

$$E = \{p \in P : \exists \in \omega[(\alpha, n), (\beta.n) \in \mathrm{dmn}(p)[p(\alpha, n) \neq p(\beta, n)]$$

Clearly E is dense in P. Hence $g_\alpha \neq g_\beta$ and $a_\alpha \neq a_\beta$ for $\alpha \neq \beta$. By Proposition 14.69, M and $M[G]$ have the same cardinals. For (i), suppose that $\alpha < \kappa$ and $g_\alpha \in M$. Let $D = \{p \in P : \exists n \in \omega[(\alpha, n) \in \mathrm{dmn}(p) \wedge p(\alpha, n) \neq g_\alpha(n)]\}$. Then D is dense in P, and it follows that $f(\alpha, n) \neq g_\alpha(n)$, contradiction.

For (ii),

$$(2^{\aleph_0})^{M[G]} = ((2^{\aleph_0})^{\aleph_0})^{M[G]} \geq (\kappa^{\aleph_0})^{M[G]} \geq (\kappa^{\aleph_0})^M \qquad \square$$

Lemma 15.2. *If λ is a cardinal in M and G is a generic ultrafilter on B, then*

$$(2^\lambda)^{M[G]} \leq (|B|^\lambda)^M.$$

Proof. In M, let X be the set of all functions $f : \lambda \to B$ such that for some $\dot{A} \in M^B$, $\forall \alpha < \lambda[f(\alpha) = [\![\check{\alpha} \in \dot{A}]\!]]$. In $M[G]$, for each $f \in X$ choose such a $\dot{A}$, and let $g(f) = \dot{A}_G$. This definition does not depend on the $\dot{A}$ chosen. In fact, if $\dot{A}$ and $\dot{C}$ both satisfy the definition, then

$$\forall \alpha \in \lambda[[\![\check{\alpha} \in \dot{A}]\!] = f(\alpha) = [\![\check{\alpha} \in \dot{C}]\!]],$$

and hence

$$\forall \alpha \in \lambda[\alpha \in \dot{A}_G \quad \text{iff} \quad [\![\check{\alpha} \in \dot{A}[\![\in G \quad \text{iff} \quad [\![\check{\alpha} \in \dot{C}[\![\in G \quad \text{iff} \quad \alpha \in \dot{C}_G],$$

so $\dot{A}_G = \dot{C}_G$.

Now in $M[G]$, $\mathscr{P}(\lambda) \subseteq \mathrm{rng}(g)$. In fact, if $A \subseteq \lambda$, choose $\dot{A}$ so that $\dot{A}_G = A$. Define $f(\alpha) = [\![\check{\alpha} \in \dot{A}]\!]$ for all $\alpha < \lambda$. Then $g(f) = A$.

Hence $(2^\lambda)^{M[G]} \leq |X|^M \leq (|B|^\lambda)^M$. $\square$

Theorem 15.3. *Let κ be an infinite cardinal, and let P consist of all finite functions with domain contained in $\kappa \times \omega$ and range contained in $\{0, 1\}$, ordered by $\supseteq$. Let G be P-generic over M. Then $(2^{\aleph_0})^{M[G]} = (\kappa^{\aleph_0})^M$.*

Proof. Let $B = \mathrm{RO}(P)$. Since P has ccc, we have $|B| = \kappa^{\aleph_0}$. By Theorem 15.1, $(2^{\aleph_0})^{M[G]} \geq (\kappa^{\aleph_0})^M$. The other inequality follows from Lemma 15.2. $\square$

Lemma 15.4. *If I, J are sets and λ is an infinite cardinal, let P be the set of all p such that p is a function with domain a subset of I of size $< \lambda$, and range contained in J, ordered by $\supseteq$. Then P has the $(|J|^{<\lambda})^+$-cc.*

Proof. Let $\theta = (|J|^{<\lambda})^+$, and suppose that $\{p_\xi : \xi < \theta\}$ is a collection of elements of P; we want to show that there are distinct $\xi, \eta < \theta$ such that p_ξ and p_η are compatible. We want to apply Theorem 14.75 with κ, λ, $\langle A_i : i \in I\rangle$ replaced by λ, θ, $\langle \mathrm{dmn}(p_\xi) : \xi < \theta\rangle$ respectively. Obviously θ is regular. If $\alpha < \theta$, then $|[\alpha]^{<\lambda}| \le |\alpha|^{<\lambda} \le (|J|^{<\lambda})^{<\lambda} = |J|^{<\lambda} < \theta$. Thus we can apply Theorem 14.75, and we get $J \in [\theta]^\theta$ such that $\langle \mathrm{dmn}(p_\xi) : \xi \in J\rangle$ is an indexed Δ-system, say with root r. Now $|{}^rJ| \le |J|^{<\lambda} < \theta$, so there exist a $K \in [J]^\theta$ and an $f \in {}^rJ$ such that $p_\xi \restriction r = f$ for all $\xi \in K$. Clearly p_ξ and p_η are compatible for any two $\xi, \eta \in K$. □

P is *κ-distributive* iff the intersection of κ open dense sets is open dense. Recall that open means closed downwards.

Lemma 15.5. *For P separative, P is κ-distributive iff* $\mathrm{RO}(P)$ *is κ-distributive.*

Proof. We will apply 14.9(c), page 217, of the Handbook of Boolean Algebras.

First suppose that P is κ-distributive; this direction does not need P separative. Suppose that $\mathscr{Q}$ is a collection of $\le \kappa$ partitions of B. For each $Q \in \mathscr{Q}$, let $X_Q = \{p \in P : e(p) \le a \text{ for some } a \in Q\}$. Clearly X_Q is open. To show that it is dense, suppose that $q \in P$. Choose $u \in Q$ such that $e(q) \cap u \ne \emptyset$, and then choose $r \in P$ with $e(r) \le e(q) \cap u$. Clearly $e(r)$ and $e(q)$ are compatible, so by Theorem 14.6(iii) also r and q are compatible. Choose $p \le r, q$. Then $e(p) \le e(r) \le u$, so $p \in X_Q$. So X_Q is dense open. So $Y \stackrel{\text{def}}{=} \bigcap_{Q \in \mathscr{Q}} X_Q$ is also open dense. Let $Z \subseteq Y$ be maximal pairwise disjoint. By denseness of Y, Z is a partition of B. Clearly Z refines each $Q \in \mathscr{Q}$.

Second suppose that $\mathrm{RO}(P)$ is κ-distributive. Suppose that $\mathscr{Q}$ is a collection of $\le \kappa$ open dense subsets of P. For each $Q \in \mathscr{Q}$ let $S_Q = \{e(p) : p \in Q\}$ and let R_Q be a maximal disjoint subset of S_Q. Then R_Q is a partition of $\mathrm{RO}(P)$. For, suppose that $0 \ne u \in \mathrm{RO}(P)$. Choose $q \in P$ so that $e(q) \le u$. Since Q is dense, choose $p \in Q$ with $p \le q$. Then $e(p) \in S_Q$, so there is a $v \in R_Q$ such that $e(p) \cdot v \ne 0$. Say $v = e(r)$ with $r \in Q$. Then $e(p) \cdot e(r) \ne 0$, so p and r are compatible. Say $s \le p, r$. Then $e(s) \le e(p) \le e(q) \le u$ and $e(s) \le e(r) = v$. So $u \cdot v \ne 0$. This verifies that R_Q is a partition of $\mathrm{RO}(P)$. If follows that there is some partition Y which refines all R_Q for $Q \in \mathscr{Q}$.

Now clearly $\bigcap \mathscr{Q}$ is open. To show that it is dense, take any $p \in P$. Choose $u \in Y$ such that $e(p) \cap u \ne 0$. Then choose $q \in P$ such that $e(q) \le e(p) \cap u$. By Proposition 14.7(iii)(b) we have $q \le p$. We claim that $q \in \bigcap \mathscr{Q}$. For, suppose that $Q \in \mathscr{Q}$. Then there is a $v \in R_Q$ such that $u \le v$. Say $v = e(s)$ with $s \in Q$. Then $e(q) \le u \le v = e(r)$, so by Proposition 14.7(iii)(b) $q \le r$. Hence $q \in Q$, as desired. □

Lemma 15.6. *Suppose that $f : A \to M$ with $f \in M[G]$. Then there is a $B \in M$ such that $f : A \to B$.*

Proof. Let $f = \tau_G$ and define $B = \{b : \exists p \in P[p \Vdash [\check{b} \in \mathrm{rng}(\tau)]]\}$. The definition of B takes place in M; so $B \in M$. Suppose that b is in the range of f. Thus $\check{b}_G = b \in \mathrm{rng}(\tau_G)$, so we can choose $p \in B$ such that $p \Vdash \check{b} \in \mathrm{rng}(\tau)$. So $b \in B$, as desired. □

Theorem 15.7. *Let κ be an infinite cardinal, assume that P is κ-distributive, G is P-generic over M, and $f : \kappa \to M$ with $f \in M[G]$. Then $f \in M$.*

Proof. Note the following:

(1) If $A \in M[G]$ and $A \subseteq M$, then $A \subseteq B$ for some $B \in M$.

This is obtained from Lemma 15.6 by taking f to be the identity on A.

(2) We say that $D \subseteq P$ is *dense open below* $p \in P$ iff it is open and $\forall q \leq p \exists r \in D[r \leq q]$. If P is κ-distributive, then for each $p \in P$, the intersection of κ sets which are dense open below p is also dense open below p.

In fact, let $\mathscr{D}$ be a collection of κ sets which are dense open below p. For each $D \in \mathscr{D}$ let $D' = D \cup \{q : p \perp q\}$. Then D' is dense open. In fact, it is clearly open. Given $r \in P$, if $r \perp p$ then $r \in D'$. If r and p are compatible, say $s \leq r, p$. Then choose $t \in D$ such that $t \leq s$. Then $t \in D'$ and $t \leq r$. So D' is dense open. It follows that $E \stackrel{\text{def}}{=} \bigcap_{D \in \mathscr{D}} D'$ is dense open. We claim that $\bigcap \mathscr{D} = E \cap \{q : q \leq p\}$. For, suppose that $q \in \bigcap \mathscr{D}$. Then $\forall D \in \mathscr{D}[q \in D']$, so $q \in E$; and clearly $q \leq p$. Conversely, suppose that $q \in E$ and $q \leq p$. Then for each $D \in \mathscr{D}$ we have $q \in D'$ and $q \leq p$, so $q \in D$. Thus $q \in \bigcap \mathscr{D}$.

(3) A general fact: If $r \Vdash \exists b \in \check{B}\varphi(b)$, then there exist a $q \leq r$ and a $b \in B$ such that $q \Vdash \varphi(\check{b})$.

In fact, $e(r) \leq ||\exists b \in \check{B}\varphi(b)||$, so if $e(r) \in H$ with H generic over $\mathrm{RO}(P)$, then $||\exists b \in \check{B}\varphi(b)|| \in H$; thus $\sum_{b \in B} ||\varphi(\check{b})|| \in H$. So there is a $b \in B$ such that $||\varphi(\check{b}|| \in H$. Choose q with $e(q) \leq e(r), ||\varphi(\check{b})|| \in H$. Say $s \leq q, r$. Then $s \leq r$ and $s \Vdash \varphi(\check{b})$.

Now we turn to the actual proof. Assume that κ is an infinite cardinal, P is κ-distributive, G is P-generic over M, and $f : \kappa \to M$ with $f \in M[G]$. By Lemma 15.6 there is a $B \in M$ such that $f : \kappa \to B$. Then there exist a $\dot{f} \in M^{\mathrm{RO}(P)}$ and a $p \in G$ such that $\dot{f}_G = f$ and

$$p \Vdash \dot{f} \text{ is a function mapping } \check{\kappa} \text{ into } \check{B}.$$

For each $\alpha < \kappa$ let

$$D_\alpha = \{q \leq p : \exists b \in B[q \Vdash \dot{f}(\check{\alpha}) = \check{b}]\}.$$

We claim that D_α is open dense below p. Clearly it is open. Now suppose that $r \leq p$. Then $r \Vdash \exists x \in \check{B}[\dot{f}(\check{\alpha}) = \check{x}]$. So by (3), there exist $x \in B$ and $q \leq r$ such that $q \Vdash \dot{f}(\check{\alpha}) = \check{x}$. Thus $q \in D_\alpha$. So D_α is open dense below p.

By (2), $\bigcap_{\alpha \in \kappa} D_\alpha$ is open dense below p. Choose $q \in G \cap \bigcap_{\alpha \in \kappa} D_\alpha$. Then for each $\alpha < \kappa$ there is a $b_\alpha \in B$ such that $q \Vdash \dot{f}(\check{\alpha}) = \check{b_\alpha}$. Let $g(\alpha) = b_\alpha$ for all $\alpha < \kappa$. But clearly $f(\alpha) = b_\alpha$ for all $\alpha < \kappa$, so $f = g \in M$. $\square$

P is *κ-closed* iff every decreasing sequence $p_0 \geq p_1 \geq \cdots \geq p_\alpha \geq \cdots$ with $\alpha < \lambda \leq \kappa$, has a lower bound.

Lemma 15.8. *If P is κ-closed, then it is κ-distributive.*

Proof. Assume that P is κ-closed. Let $\langle D_\alpha : \alpha < \kappa\rangle$ be a system of dense open subsets of P. We claim that $\bigcap_{\alpha<\kappa} D_\alpha$ is dense open. It is clearly open. To show that it is closed, suppose that $p \in P$. We construct q_α for $\alpha < \kappa$ by induction. Let $q_0 = p$. If q_ξ has been constructed for all $\xi < \alpha$ with $q_\xi \in D_\alpha$ and $q_\xi \leq q_\eta$ if $\eta < \xi < \alpha$, let r be a lower bound for all q_ξ, and choose $q_\alpha \in D_\alpha$ with $q_\alpha \leq r$. Finally, let $s \leq q_\alpha$ for all $\alpha < \kappa$. Clearly $s \in \bigcap_{\alpha<\kappa} D_\alpha$. □

Lemma 15.9. *In M, let κ be a regular cardinal such that $2^{<\kappa} = \kappa$. Let λ be a cardinal greater than κ such that $\lambda^\kappa = \lambda$. Let P be the set of all functions p such that* $\mathrm{dmn}(p) \subseteq \lambda \times \kappa$, $|\mathrm{dmn}(p)| < \kappa$, *and* $\mathrm{rng}(p) \subseteq \{0, 1\}$. *The order on P is $\supseteq$. Let G be P-generic over M. Define $f = \bigcup G$. Then*

(i) f is a function mapping $\lambda \times \kappa$ into $\{0, 1\}$.

For each $\alpha < \lambda$ let

$$a_\alpha = \{\xi < \kappa : f(\alpha, \xi) = 1\}.$$

(ii) $\forall \alpha < \lambda[a_\alpha \notin M]$.

(iii) If $\alpha < \beta < \lambda$ then $a_\alpha \neq a_\beta$.

(iv) All cardinals are preserved in $M[G]$.

(v) In $M[G]$, $2^\kappa = \lambda$.

Proof. (i) is clear. For (ii), suppose that $\alpha < \lambda$ and $a_\alpha \in M$. Let $D = \{p \in P : \exists \xi < \kappa : \xi \in a_\alpha, (\alpha, \xi) \in \mathrm{dmn}(p)$, and $p(\alpha, x) = 0\}$. Clearly D is dense in P. Hence there is a $\xi < \kappa$ such that $\xi \in a_\alpha$ and $f(a, \xi) = 0$, contradiction.

(iii) is clear.

Now by Lemma 15.4, P has the $(2^{<\kappa})^+$-cc, i.e. by an assumption of the Lemma, the κ^+-cc. Hence by Theorem 14.64, P preserves cardinals $\geq \kappa^+$. Now if $\lambda \leq \kappa$ and λ is a cardinal in M but not in $M[G]$, then there exist a $\mu < \lambda$ and a function $f \in M[G]$ mapping μ onto λ. By Theorem 15.7, $f \in M$, contradiction. So (iv) holds.

By (i) and (iii), $\lambda \leq (2^\kappa)^{M[G]}$. Now $P = \bigcup\{{}^X2 : X \in [\lambda]^{<\kappa}\}$ and for each such X, $|{}^X2| \leq \kappa$. Moreover, $|\lambda]^{<\kappa}| = \lambda$. So $|P| = \lambda$. By Corollary 10.5 of the Handbook of Boolean Algebras, $|\mathrm{RO}(P)| = \lambda$. Hence by Lemma 15.2, $(2^\kappa)^{M[G]} \leq \lambda$. Thus (v) holds. □

If P and Q are forcing posets we define

$$(p_1, q_1) \leq (p_2, q_2) \quad \text{iff} \quad p_1 \leq p_2 \text{ and } q_1 \leq q_9.$$

Lemma 15.10. *Let P and Q be forcing posets in M, and suppose that $G \subseteq P \times Q$. Then G is $(P \times Q)$-generic over M iff there exist $G_1 \subseteq P$ and $G_2 \subseteq Q$ such that*

(i) $G = G_1 \times G_2$.

(ii) G_1 is P-generic over M.

(iii) G_2 is Q-generic over $M[G_1]$.

Proof. $\Rightarrow$: Suppose that G is $(P \times Q)$-generic over M. Define

$$G_1 = \{p : \exists q[(p, q) \in G]\} \quad \text{and} \quad G_2 = \{q : \exists p[(p, q) \in G]\}.$$

G_1 is closed upwards: Suppose that $p \in G_1$ and $p \leq p'$. Say $(p, q) \in G$. Then $(p, q) \leq (p', q)$, so $(p', q) \in G$ and hence $p' \in G_1$. Similarly G_2 is closed upwards.

Suppose that $p_1, p_2 \in G_1$. Say $(p_1, q_1) \in G$ and $(p_2, q_2) \in G$. Choose $(p_3, q_3) \in G$ such that $(p_3, q_3) \leq (p_1, q_1), (p_2, q_2)$. Then $p_3 \in G_1$ and $p_3 \leq p_1, p_2$. So G_1 is a filter. Similarly for G_2.

Clearly $G \subseteq G_1 \times G_2$.

Now suppose that $(p_1, p_2) \in G_1 \times G_2$. Since $p_1 \in G_1$, choose p_2' such that $(p_1, p_2') \in G$. Similarly we get p_1' such that $(p_1', p_2) \in G$. Choose $(q_1, q_2) \in G$ such that $(q_1.q_2) \leq (p_1, p_2'), (p_1', p_2)$. Then $(q_1, q_2) \leq (p_1, p_2)$, so $(p_1, p_2) \in G$. Thus $G = G_1 \times G_2$.

Next, G_1 is P-generic over M. For, let $D \in M$ be dense in P. Then $D \times Q$ is dense in $P \times Q$, so we can choose $(p, q) \in G \cap (D \times Q)$. Then $p \in D \cap G_1$, as desired.

Next, G_2 is Q-generic over $M[G_1]$. For, let $D_2 \in M[G_1]$ be dense in Q. Say $\dot{D}_2 \in M^{\mathrm{RO}(P)}$ and $\dot{D}_{2G_1} = D_2$. Say $p_1 \in G_1$ and $p_1 \Vdash \dot{D}_2$ is dense in Q. Take any $p_2 \in G_2$ and let

$$D = \{(r_1, r_2) : r_1 \leq p_1 \text{ and } r_1 \Vdash \check{r}_2 \in \dot{D}_2\}.$$

Then D is dense below (p_1, p_2). For, suppose that $(s, t) \leq (p_1, p_2)$. Thus $s \Vdash \dot{D}_2$ is dense in Q. So $s \Vdash \forall t \in \dot{Q} \exists u \in \dot{D}_2[u \leq t]$. In particular, $s \Vdash \exists u \in \dot{D}_2[u \leq \check{p}_2]$. Thus by Lemma 14.26,

$$e(s) \leq \sum_{y \in \mathrm{dmn}(\dot{D}_2)} (\dot{D}_2(y) \cdot [\![y \leq \check{p}_2]\!])$$

Hence we easily get $s' \leq s$ and $y \in \mathrm{dmn}(\dot{D}_2)$ such that $e(s') \leq \dot{D}_2(y) \cdot [\![y \leq \check{p}_2]\!]$. Now $p_1 \Vdash \dot{D}_2 \subseteq \check{Q}$, and $s' \leq s \leq p$, so $e(s') \leq (\dot{D}_2(y) \Rightarrow [\![y \in \check{Q}]\!]$; hence $e(s') \leq (\dot{D}_2(y) \Rightarrow \sum_{u \in Q} [\![y = \check{u}]\!]$. So $e(s') \leq \dot{D}_2(y) \cdot \sum_{u \in Q} [\![y = \check{u}]\!] \cdot [\![y \leq \check{p}_2]\!]$. Hence there exist a $s'' \leq s'$ and a $u \in Q$ such that $e(s'') \leq \dot{D}_2(y) \cdot [\![y = \check{u}]\!] \cdot [\![y \leq \check{p}_2]\!] \leq \dot{D}_2(y) \cdot [\![\check{u} \leq \check{p}_2]\!]$. Hence by an earlier lemma, $u \leq p_2$. Also, $e(s'') \leq [\![y = \check{u}]\!] \cdot [\![y \in \dot{D}_2]\!] \leq [\![\check{u} \in \dot{D}_2]\!]$. Now $(s'', u) \leq (p_1, p_2)$ and $(s'', u) \in D$. So D is dense below (p_1, p_2).

Now $(p_1, p_2) \in G_1 \times G_2 = G$, so there is a $(r_1, r_2) \in G \cap D$. Now $r_1 \in G_1$ and $r_1 \Vdash \check{r}_2 \in \dot{D}_2$, so $r_2 \in D_2 \cap G_2$. So G_2 is Q-generic over $M[G_1]$.

Conversely, suppose that $G_1 \subseteq P$, $G_2 \subseteq Q$, $G = G_1 \times G_2$, G_1 is P-generic over M, and G_2 is Q-generic over $M[G_1]$.

G is closed upwards: suppose that $(p, q) \in G$ and $(p, q) \leq (p', q')$. Then $p \leq p'$ and $q \leq q'$, so $p' \in G_1$ and $q' \in G_2$, hence $(p', q') \in G$.

Suppose that $(p, q), (p', q') \in G$. Choose $p'' \in G_1$ with $p'' \leq p, p'$ and choose $q'' \in G_2$ such that $q'' \leq q, q'$. Then $(p'', q'') \in G$ and $(p'', q'') \leq (p, q), p', q')$.

Now suppose that $D \in M$ is dense in $P \times Q$. Let

$$D_2 = \{p_2 : \exists p_1 \in G_1[(p_1, p_2) \in D\}.$$

Thus $D_2 \in M[G_1]$.

(1) D_2 is dense in Q

In fact, let $q_2 \in Q$. Then

(2) $D_1 \stackrel{\text{def}}{=} \{p_1 : \exists p_2 \leq q_2[(p_1, p_2) \in D\}$ is dense in P.

In fact, suppose that $p \in P$. Choose $(s, t) \in D$ such that $(s, t) \leq (p, q_2)$. Then $s \in D_1$. So (2) holds.

Choose $p_1 \in G_1 \cap D_1$. Then there is a $p_2 \leq q_2$ such that $(p_1, p_2) \in D$. This proves (1).

Choose $p_2 \in G_2 \cap D_2$. and choose $p_1 \in G_1$ such that $(p_1, p_2) \in D$. Thus $(p_1, p_2) \in G$, and so G is $(P \times Q)$-generic over M. $\square$

Lemma 15.11. *Let P and Q be forcing posets in M, and suppose that $G \subseteq P \times Q$. Suppose that G is $(P \times Q)$-generic over M, and define G_1 and G_2 as in the proof of Lemma 15.10. Then $M[G] = M[G_1][G_2] = M[G_2][G_1]$.*

Proof. Since $G \in M[G_1][G_2]$, by an earlier theorem we have $M[G] \subseteq M[G_1][G_2]$. Also, $G_1 \in M[G]$, so $M[G_1] \subseteq M[G]$. And $G_2 \in M[G]$, so $M[G_1][G_2] \subseteq M[G]$. So $M[G] = M[G_1][G_2]$. Similarly, $M[G] = M[G_2][G_1]$. $\square$

If $\langle P_i : i \in I \rangle$ is a system of forcing posets and κ is any cardinal, then we define

$$\prod_{i \in I}^{w} P_i = \{p \in \prod_{i \in I} P_i : |\{i \in I : p_i \neq 1\}| < \omega\};$$

$$\prod_{i \in I}^{<k} P_i = \{p \in \prod_{i \in I} P_i : |\{i \in I : p_i \neq 1\}| < \kappa\}.$$

For any $p \in \prod_{i \in I} P_i$ let *supp*$(p) = \{i \in I : p(i) \neq 1\}$.

Lemma 15.12. *If P and Q are λ-closed, then so is $P \times Q$.* $\square$

Lemma 15.13. *If $\langle P_i : i \in I \rangle$ is a system of forcing posets, $\lambda < \mathrm{cf}(\kappa)$, and each P_i is λ-closed, then $\prod_{i \in I}^{<\kappa} P_i$ is λ-closed.*

Proof. Suppose that $\langle p_\alpha : \alpha < \lambda \rangle$ is a descending sequence in $\prod_{i \in I}^{<\kappa} P_i$. For each $\alpha < \lambda$ let $M_\alpha = \{i \in I : p_\alpha(i) \neq 1\}$, and let $N = \bigcup_{\alpha < \lambda} M_\alpha$. Then $|N| < \kappa$. For each $i \in I$ let $q_i \in P_i$ be such that $\forall \alpha < \lambda[q_i \leq p_\alpha(i)]$. We may assume that $q_i = 1$ for all $i \in I \backslash N$. Hence $|\{i \in I : q_i \neq 1\}| < \kappa$. Then $\forall \alpha < \lambda[q \leq p_\alpha]$, and $q \in \prod_{i \in I}^{<\kappa} P_i$. $\square$

A forcing poset P has *property* (K) iff every uncountable subset of P has an uncountable subset consisting of pairwise compatible elements.

Lemma 15.14. *If P and Q both have property (K), then so does $P \times Q$.*

Proof. Let $X \subseteq P \times Q$ be uncountable.

Case 1. $\exists p \in P[\{q \in Q : (p, q) \in X\}$ is uncountable]. Then the desired conclusion is clear.

Case 9. $\exists q \in Q[\{p \in P : (p, q) \in X\}$ is uncountable]. This is symmetric to Case 1.

Case 3. (a) $\forall p \in P[\{q \in Q : (p, q) \in X\}$ is countable] and (b) $\forall q \in Q[\{p \in P : (p, q) \in X\}$ is countable]. Let $Y = \{p \in P : \exists q[(p, q) \in X]\}$. By (a), Y is uncountable. We now define $(p_\alpha, q_\alpha) \in X$ for $\alpha < \omega_1$, with $p_\alpha \in Y$, by recursion. Suppose defined for all $\beta < \alpha$. Now $\{p \in Y : \exists \beta < \alpha[(p, q_\beta) \in X]\}$ is countable by (b). Hence $Z \stackrel{\text{def}}{=} \{p \in Y : \forall \beta <$

$\alpha[(p, q_\beta) \notin X]\}$ is uncountable. Take $p_\alpha \in Z \backslash \{p_\beta < \beta < \alpha\}$. Since $p_\alpha \in Y$, choose q_α such that $(p_\alpha, q_\alpha) \in X$.

Then $\{(p_\alpha, q_\alpha) : \alpha < \omega_1\}$ is a one-one function f. Let $U \subseteq \mathrm{dmn}(f)$ be uncountable and pairwise compatible. Let $V \subseteq f[U]$ be uncountable and pairwise compatible. Then $\{(p_\alpha, q_\alpha) : q_\alpha \in V\}$ is a pairwise compatible subset of X. □

Lemma 15.15. *If $\forall i \in I[P_i$ has property (K)], then $\prod^w_{i \in I} P_i$ has property (K).*

Proof. Suppose that $\forall i \in I[P_i$ has property (K)], and suppose that $X \subseteq \prod^w_{i \in I} P_i$ is uncountable. Let $W = \{supp(p) : p \in X\}$.

Case 1. W is countable. Then exist is a finite $J \subseteq I$ and an uncountable $W' \subseteq W$ such that $supp(p) = J$ for all $p \in W'$. By Lemma 15.14, $\prod_{i \in J} P_i$ has property (K). Let $W'' = \{p \restriction J : p \in W'\}$. Then $|W''| = |W'|$, so let W''' be an uncountable pairwise uncountable subset of W''. Now there is a subset W^{iv} of W' such that $W''' = \{p \restriction J : p \in W^{iv}\}$. Clearly W^{iv} is an uncountable pairwise compatible subset of W.

Case 2. W is uncountable. By the Δ-system lemma, there exist an uncountable $W' \subseteq W$ and a set J such that $Z \cap Y = J$ for all distinct $Z, Y \in W'$. Wlog $Z \neq J$ for all $X \in W'$. For each $Z \in W'$ let $p_Z \in X$ be such that $supp(p_Z) = Z$. Let $W'' = \{p_Z \restriction J : Z \in W'\}$.

Subcase 2.1. W'' is countable. Then there exist a $q \in \prod_{i \in J} P_i$ and an uncountable $W''' \subseteq W''$ such that $p_Z \restriction J = q$ for all $Z \in W'''$. Then $\{p_Z : Z \in W'''\}$ is an uncountable pairwise compatible subset of W.

Subcase 2.2. W'' is uncountable. By Lemma 15.14, there is an uncountable $W''' \subseteq W''$ such that W''' consists of pairwise compatible elements. Say $W''' = \{p_Z \restriction J : Z \in W^{iv}\}$. Then W^{iv} is an uncountable pairwise compatible subset of W.

Corollary 15.16. *If each P_i is countable, then $\prod^w_{i \in I} P_i$ has property (K).* □

Theorem 15.17. *If κ is regular, $\lambda \geq \kappa$, $\lambda^{<\kappa} = \lambda$, and $\forall i \in I[|P_i| \leq \lambda]$, then $\prod^{<\kappa}_{i \in I} P_i$ has the λ^+ chain condition.*

Proof. Let $Q = \prod^{<\kappa}_{i \in I} P_i$ and let W be an antichain in Q. For each $p \in W$ let $p' = p \restriction supp(p)$. Then for distinct $p, q \in W$ there is an $i \in supp(p) \cap supp(q)$ such that p_i and q_i are incompatible. Let $W' = \{p' : p \in W\}$. Then W' is a set of functions such that if $p, q \in W$ are distinct, then there is an $i \in \mathrm{dmn}(p') \cap \mathrm{dmn}(q')$ such that p_i and q_i are incompatible.

We assume that $|W| = \lambda^+$. Note that $|W'| = |W|$. Let $\mathscr{A} = \{supp(p) : p \in W\}$.

Case 1. $|\mathscr{A}| \geq \lambda^+$. In Theorem 14.69, replace λ by λ^+. $\forall \alpha < \lambda^+[|\alpha|^{<\kappa} \leq \lambda^{<\kappa} = \lambda < \lambda^+$. Each member of $\mathscr{A}$ has size less than κ. By Theorem 14.69 there is a $\mathscr{B} \in [\mathscr{A}]^{\lambda^+}$ which is a Δ-system. Say $J \cap K = L$ for all distinct $J, K \in \mathscr{B}$. Then $\prod_{i \in L} P_i$ has size at most $\lambda^{<\kappa} = \lambda$ but for distinct $p, q \in W$ with $supp(p), supp(q) \in \mathscr{B}$ we have $p \restriction l \neq q \restriction L$, contradiction.

Case 9. $|\mathscr{A}| \leq \lambda$. Then there exist an $L \in [I]^{<\kappa}$ and a $W'' \subseteq W'$ such that $\mathrm{dmn}(p) = L$ for all $p \in W''$ and $|W''| = \lambda^+$. But then $W'' \subseteq \prod_{i \in L} P_i$ and $|\prod_{i \in L} P_i| \leq \lambda$, contradiction. □

Corollary 15.18. *If each P_i has size at most λ, with λ infinite, then $\prod_{i\in I}^{w} P_I$ has the λ^+ chain condition.*

Proof. Apply Theorem 15.17 with $\kappa = \omega$. $\square$

Theorem 15.19. *If λ is inaccessible, $\kappa < \lambda$ is regular, and $\forall i \in I[|P_i| < \lambda]$, then $\prod_{i\in I}^{<\kappa} P_i$ has the λ chain condition.*

Proof. Let $Q = \prod_{i\in I}^{<\kappa} P_i$ and let W be an antichain in Q. For each $p \in W$ let $p' = p \restriction supp(p)$. Then for distinct $p, q \in W$ there is an $i \in supp(p) \cap supp(q)$ such that p_i and q_i are incompatible. Let $W' = \{p' : p \in W\}$. Then W' is a set of functions such that if $p, q \in W$ are distinct, then there is an $i \in \mathrm{dmn}(p') \cap \mathrm{dmn}(q')$ such that p_i and q_i are incompatible.

We assume that $|W| = \lambda$. Note that $|W'| = |W|$. Let $\mathscr{A} = \{supp(p) : p \in W\}$.

Case 1. $|\mathscr{A}| = \lambda$. By Theorem 14.69 there is a $\mathscr{B} \in [\mathscr{A}]^\lambda$ which is a Δ-system. Say $J \cap K = L$ for all distinct $J, K \in \mathscr{B}$. Then $\prod_{i\in L} P_i$ has size less than λ, but for distinct $p, q \in W$ with $supp(p), supp(q) \in \mathscr{B}$ we have $p \restriction l \neq q \restriction L$, contradiction.

Case 9. $|\mathscr{A}| < \lambda$. Then there exist an $L \in [I]^{<\kappa}$ and a $W'' \subseteq W'$ such that $\mathrm{dmn}(p) = L$ for all $p \in W''$ and $|W''| = \lambda$. But then $W'' \subseteq \prod_{i\in L} P_i$ and $|\prod_{i\in L} P_i| < \lambda$, contradiction. $\square$

Lemma 15.20. *For any cardinals κ, λ, $|[\kappa]^{<\lambda}| \leq \kappa^{<\lambda}$.*

Proof. For each cardinal $\mu < \lambda$ define $f : {}^\mu\kappa \to [\kappa]^{\leq\mu}\backslash\{\emptyset\}$ by setting $f(x) = \mathrm{rng}(x)$ for any $x \in {}^\mu\kappa$. Clearly f is an onto map. It follows that $|[\kappa]^{\leq\mu}| \leq |{}^\mu\kappa| \leq \kappa^{<\lambda}$. Hence

$$\begin{aligned}
|[\kappa]^{<\lambda}| &= \left| \bigcup_{\substack{\mu<\lambda,\\ \mu \text{ a cardinal}}} [\kappa]^{\leq\mu} \right| \\
&\leq \sum_{\substack{\mu<\lambda,\\ \mu \text{ a cardinal}}} |[\kappa]^{\leq\mu}| \\
&\leq \sum_{\substack{\mu<\lambda,\\ \mu \text{ a cardinal}}} \kappa^{<\lambda} \\
&\leq \lambda \cdot \kappa^{<\lambda} \\
&= \kappa^{<\lambda}.
\end{aligned}$$

$\square$

Lemma 15.21. *If λ is regular, then $\lambda^{<\lambda} = 2^{<\lambda}$.*

Proof. Note that if $\alpha < \lambda$, then by the regularity of λ,

$$|{}^\alpha\lambda| = \left|\bigcup_{\beta<\lambda} {}^\alpha\beta\right| \leq \sum_{\beta<\lambda} |\beta|^{|\alpha|} \leq \sum_{\beta<\lambda} |\max(\alpha,\beta)|^{|\max(\alpha,\beta)|} \leq \sum_{\beta<\lambda} 2^{|\max(\alpha,\beta)|} \leq 2^{<\lambda} \leq \lambda^{<\lambda};$$

hence the lemma follows. $\square$

An *index function* is a function E such that $\mathrm{dmn}(E)$ is a set of regular cardinals. An *Easton index function* is an index function E such that:

(1) $\forall\kappa \in \mathrm{dmn}(E)[E(\kappa)$ is an infinite cardinal such that $\mathrm{cf}(E(\kappa)) > \kappa]$.

(2) $\forall\kappa, \lambda \in \mathrm{dmn}(E)[\kappa < \lambda \to E(\kappa) \le E(\lambda)]$.

If I and J are sets and κ is a cardinal, then $\mathrm{Fn}(I, J, \kappa) = \{f \in [I \times J]^{<\kappa} : f$ is a function$\}$. If E is an Easton index function with domain I and $\mathbb{E} = \prod_{\kappa\in I} \mathrm{Fn}(E(\kappa), 2, \kappa)$, then the *Easton poset* $\mathbb{P}(E)$ is defined by

$$p \in \mathbb{P}(E) \quad \text{iff} \quad p \in \mathbb{E} \text{ and } \forall\lambda[\lambda \text{ regular } \to |\{\kappa \in \lambda \cap I : p(\kappa) \ne \mathbb{1}\}| < \lambda].$$

Note that $\mathbb{1} = \emptyset$.

Proposition 15.22. *Let E be an Easton index function such that there is no regular limit cardinal λ such that there is a $p \in \mathbb{R}$ such that $|\{\kappa \in \lambda \cap \mathrm{dmn}(E) : p(\kappa) \ne \mathbb{1}\}| = \lambda$. Then $\mathbb{P}(E) = \mathbb{E}$, with $\mathbb{E}$ as above.*

Proof. Assume the hypothesis, but suppose that λ is regular and there is a $p \in \mathbb{R}$ such that $|\{\kappa \in \lambda \cap \mathrm{dmn}(E) : p(\kappa) \ne \mathbb{1}\}| = \lambda$. Then λ is a successor cardinal $\aleph_{\alpha+1}$. But then $|\{\kappa \in \lambda \cap \mathrm{dmn}(E) : p(\kappa) \ne \mathbb{1}\}| \le \max(\omega, |\alpha|) < \lambda$, contradiction. □

Lemma 15.23. (*Suppose that E is an Easton index function such that* $\mathrm{dmn}(E) \subseteq \lambda^+$, *where λ is a regular cardinal such that $2^{<\lambda} = \lambda$. Then $\mathbb{P}(E)$ has the λ^+-cc.*

Proof. Say $\mathrm{dmn}(E) = I$. Let $W = \{p_\alpha : \alpha < \lambda^+\} \subseteq \mathbb{P}(E)$; we want to show that W is not an antichain. Thus each p_α is a function with domain I, with $p_\alpha(\kappa) \in \mathrm{Fn}(E(\kappa), 2, \kappa)$ for each $\kappa \in I$. For each $\alpha < \lambda^+$ let $D_\alpha = \{(\kappa, x) : \kappa \in I, x \in \mathrm{dmn}(p_\alpha(\kappa))\}$.

(1) $|D_\alpha| < \lambda$ for each $\alpha < \lambda^+$.

In fact, let $X = \{\kappa \in \lambda \cap I : p_\alpha(\kappa) \ne \mathbb{1}\}$. Then $|X| < \lambda$. If $\lambda \notin I$, then

$$|D_\alpha| = \sum_{\kappa\in I} |\mathrm{dmn}(p_\alpha(\kappa))| = \sum_{\kappa\in X} |\mathrm{dmn}(p_\alpha(\kappa))| < \lambda,$$

since each $|\mathrm{dmn}(p_\alpha(\kappa))| < \kappa < \lambda$. If $\lambda \in I$, then

$$|D_\alpha| = \sum_{\kappa\in I} |\mathrm{dmn}(p_\alpha(\kappa))| = \sum_{\kappa\in X} |\mathrm{dmn}(p_\alpha(\kappa))| + |\mathrm{dmn}(p_\alpha(\lambda))| < \lambda.$$

Note by Lemmas 20 and 21 that for $\alpha < \lambda^+$ we have $|[\alpha]^{<\lambda} \le \lambda^{<\lambda} = 2^{<\lambda} = \lambda$. Hence we can apply Theorem 14.69 with κ, λ replaced by λ, λ^+ to obtain $B \in [\lambda^+]^{\lambda^+}$ and R such that $D_\alpha \cap D_\beta = R$ for all distinct $\alpha, \beta \in B$. Now $2^{|R|} \le 2^{<\lambda} = \lambda$ and

$$B = \bigcup_{h\in Q} \{\alpha \in B : \forall(\kappa, s) \in R[(p_\alpha(\kappa))(s) = h(\kappa, s)]\},$$

where $Q = {}^R2$, so there exist distinct $\alpha, \beta \in B$ such that $\forall(\kappa, s) \in R[(p_\alpha(\kappa))(s) = (p_\beta(\kappa))(s)]$. Thus p_α and p_β are compatible. $\square$

If E is an Easton index function and λ is an ordinal, then $E^+_\lambda = E \restriction \{\kappa : \kappa > \lambda\}$ and $E^-_\lambda = E \restriction \{\kappa : \kappa \leq \lambda\}$.

Lemma 15.24. $\mathbb{P}(E) \cong \mathbb{P}(E^-_\lambda) \times \mathbb{P}(E^+_\lambda)$.

Proof. Clearly E^-_λ and E^+_λ are Easton index functions. For any $x \in \mathbb{P}(E)$ let $f(x) = (x \restriction \{\kappa : \kappa \leq \lambda\}, x \restriction \{\kappa : \kappa > \lambda\})$. Clearly $f(x) \in \mathbb{P}(E^-_\lambda) \times \mathbb{P}(E^+_\lambda)$, and f is one-one and onto. Clearly also f preserves $\leq$. $\square$

Lemma 15.25. *Assuming GCH, if E is any Easton index function, then $\mathbb{P}(E)$ preserves cofinalities and cardinals.*

Proof. By Lemma 14.61 it suffices to show that every uncountable regular cardinal in M remains regular in $M[K]$ whenever K is $\mathbb{P}(E)$-generic over M. Suppose not; say θ is uncountable and regular in M while $\lambda \stackrel{\text{def}}{=} (\text{cf}(\theta))^{M[K]} < \theta$. Thus λ is regular in $M[K]$. Let $f \in M[K]$, $f : \lambda \to \theta$ with $\sup(\text{rng}(f)) = \theta$.

By earlier lemmas we can write $M[K] = M[H][G]$ with H $(\mathbb{P}(E^+_\lambda))^M$-generic over M and G $(\mathbb{P}(E^-_\lambda))^M$-generic over $M[H]$.

Now $(\mathbb{P}(E^+_\lambda))^M$ is λ-closed in M. For, if $\alpha < \lambda$ and $\langle p_\xi : \xi < \alpha\rangle$ is decreasing in $(\mathbb{P}(E^+_\lambda))^M$, recall that $(\mathbb{P}(E^+_\lambda))^M \subseteq \prod_{\kappa\in I, \lambda<\kappa} \text{Fn}(E(\kappa), 2, \kappa)$; hence we can define $q(\kappa) = \bigcup_{\xi<\alpha} p_\xi(\kappa)$ for all $\kappa \in I$ with $\kappa > \lambda$ and we get an extension of $\langle p_\xi : \xi < \alpha\rangle$. It follows earlier results that $(\mathbb{P}(E^+_\lambda))^M$ does not add λ-sequences. Hence $2^{<\lambda} = \lambda$ in $M[H]$ and $(\mathbb{P}(E^-_\lambda))^{M[H]} = (\mathbb{P}(E^-_\lambda))^M$. Now by an earlier lemma $(\mathbb{P}(E^-_\lambda))^M$ is λ^+-cc in $M[H]$. Now by an earlier theorem there is an $F : \lambda \to \mathscr{P}(\theta)$ such that $\forall\xi < \lambda[f(\xi) \in F(\xi)$ and $(|F(\xi)| \leq \lambda)^{M[H]}$. Now again $(\mathbb{P}(E^+_\lambda))^M$ is λ-closed in M, so by earlier theorems we get $F \in M$ and $\forall\xi < \lambda[(|F(\xi)| \leq \lambda)^M]$. Now in M, $\bigcup_{\xi<\lambda} F(\xi)$ is of size $\leq \lambda$ and is cofinal in θ, contradiction. $\square$

Proposition 15.26. *Assume GCH, and let E be an Easton index function with domain I. For any infinite cardinal θ, $|\mathbb{P}(E^-_\theta)| \leq \prod_{\kappa\in I,\kappa\leq\theta} E(\kappa)$.*

Proof. In fact, $\mathbb{P}(E^-_\theta) \subseteq \prod_{\kappa\in I,\kappa\leq\theta} \text{Fn}(E(\kappa), 2, \kappa)$. Now if $\kappa \in I$ and $\kappa \leq \theta$, then $|[E(\kappa)]^{<\kappa}| = E(\kappa)$ by (1). Hence

$$\begin{aligned}
|\text{Fn}(E(\kappa), 2, \kappa)| &= |\{f : f \text{ is a function, } \text{dmn}(f) \in [E(\kappa)]^{<\kappa}, \text{rng}(f) \subseteq 2\}| \\
&= |\{f : \exists X \in [E(\kappa)]^{<\kappa}[f \in {}^X2]\}| \\
&= \left|\bigcup_{X\in[E(\kappa)]^{<\kappa}} {}^X2\right| \\
&\leq \sum_{X\in[E(\kappa)]^{<\kappa}} 2^{|X|} \\
&\leq |E(\kappa)|.
\end{aligned}$$

It follows that $|\mathbb{P}(E_\theta^-)| \le \prod_{\kappa\in I,\kappa\le\theta} E(\kappa)$. $\square$

Theorem 15.27. *Let $M \models$ GCH. In M let E be an Easton index function and let $\mathbb{P} = \mathbb{P}(E)$. Let K be $\mathbb{P}$-generic over M. Then $\mathbb{P}$ preserves cofinalities and cardinals, and $M[K] \models \forall\kappa \in \operatorname{dmn}(E)[2^\kappa = E(\kappa)]$.*

Proof. Preservation of cofinalities and cardinals is given by Lemma 15.25.

Now let $\kappa \in \operatorname{dmn}(E)$. We define $F_\kappa : E(\kappa) \to 2$ by saying for $\delta < E(\kappa)$ that $F_\kappa(\delta) = i$ iff there is a $p \in K$ such that $\delta \in \operatorname{dmn}(p(\kappa))$ and $(p(\kappa))(\delta) = i$. Then we define for $\alpha < E(\kappa)$ $h_\alpha \in {}^\kappa 2$ by defining $h_\alpha(\xi) = F_\kappa(\kappa\cdot\alpha+\xi)$. Clearly for any $\delta < \kappa$ the set $D_\delta \stackrel{\text{def}}{=} \{p \in \mathbb{P} : \delta \in \operatorname{dmn}(p(\kappa))\}$ is dense, so $F_\kappa(\delta)$ is defined. If $\alpha, \beta \in \kappa$ and $\alpha \ne \beta$, then the set

$$N_{\alpha\beta} \stackrel{\text{def}}{=} \{p \in \mathbb{P} : \exists\xi < \kappa[\kappa\cdot\alpha+\xi, \kappa\cdot\beta+\xi \in \operatorname{dmn}(p(\kappa)) \text{ and } (p(\kappa))(\kappa\cdot\alpha+\xi) \ne (p(\kappa))(\kappa\cdot\beta+\xi)]\}$$

is dense. It follows that $h_\alpha \ne h_\beta$ for $\alpha \ne \beta$. Hence $E(\kappa) \le (2^\kappa)^{M[K]}$.

(1) $|\mathbb{P}(E_\kappa^-)| \le E(\kappa)$.

In fact, by Proposition 15.26, $|\mathbb{P}(E_\kappa^-)| \le \prod_{\mu\in I,\mu\le\kappa} |E(\mu)| \le (E(\kappa))^{<\kappa} = E(\kappa)$. Now by Lemma 15.23, $\mathbb{P}(E_\kappa^-)$ has the κ^+-cc. It follows that $|\mathrm{RO}(\mathbb{P}(E_\kappa^-)| \le E(\kappa)$. Hence by Lemma 15.2, $(2^\kappa)^{M[K]} \le E(\kappa)$. $\square$

Example 15.28. *In a c.t.m. M let λ be an uncountable cardinal, and let P be the set of all finite sequences of members of λ, ordered by $\supseteq$. Let G be P-generic over M. Then $\lambda^{M[G]} = \omega$.*

Proof. Assume the hypotheses, and let $f = \bigcup G$. For $m \in \omega$ let $D_m = \{p \in P : m \le \operatorname{dmn}(p)\}$. Clearly D is dense, so f is a function with domain ω. For each $\alpha < \lambda$ let $E_\alpha = \{p \in P : \alpha \in \operatorname{rng}(p)\}$. Clearly E is dense, so f maps onto λ. $\square$

Lemma 15.29. *Let M be a c.t.m., and in M let $\kappa < \lambda$ be cardinals, with κ regular. Then there is a forcing poset P such that if G is P-generic over M then:*

(i) In $M[G]$, $|\lambda| = \kappa$.

(ii) For every cardinal $\mu \le \kappa$ in M, μ is a cardinal in $M[G]$.

(iii) If $\lambda^{<\kappa} = \lambda$ in M, then every cardinal $\mu > \lambda$ in M is a cardinal in $M[G]$.

Proof. Let P be the set of all functions p such that $\operatorname{dmn}(p) \in [\kappa]^{<\kappa}$ and $\operatorname{rng}(p) \subseteq \lambda$. The order on P is $\supseteq$. Let G be P-generic over M, and let $f = \bigcup G$. Clearly f maps κ onto λ, so (i) holds.

P is $(<\kappa)$-closed, and so by Theorem 15.7 and Lemma 14.17, (ii) holds.

If $\lambda^{<\kappa} = \lambda$ in M, then $|P| = \lambda$. Hence P has the λ^+-cc, and so (iii) holds by Proposition 14.64. $\square$

Theorem 15.30. *Let M be a c.t.m., and in M let $\kappa < \lambda$ be regular cardinals, with λ inaccessible. Then there is a forcing poset P such that if G is P-generic over M then:*

(i) If $\kappa \le \alpha < \lambda$, then $|\alpha|^{M[G]} = \kappa$.

(ii) Every cardinal $\leq \kappa$ in M remains a cardinal in $M[G]$.
(iii) Every cardinal $\geq \lambda$ in M remains a cardinal in $M[G]$.
(iv) $M[G] \models \kappa^+ = \lambda$.

Proof. Let P consist of all functions with $\mathrm{dmn}(p) \subseteq \lambda \times \kappa$ such that $|\mathrm{dmn}(p)| < \kappa$ and $\forall(\alpha, \xi) \in \mathrm{dmn}(p)[p(\alpha, \xi) < \alpha]$, with order $\supseteq$. Let G be P-generic over M. For each $\alpha < \lambda$ let $P_\alpha = \{q \subseteq \kappa \times \alpha : q$ is a function and $|\mathrm{dmn}(q)| < \kappa\}$, with order $\supseteq$. For $p \in G$ and $\alpha < \lambda$, let p'^{α} be the function with domain $\{\xi < \kappa : (\alpha, \xi) \in \mathrm{dmn}(p)\}$, with $p'^{\alpha}(\xi) = p(\alpha, \xi)$. Let $G_\alpha = \{p'^{\alpha} : p \in G\}$.

(1) $\forall \alpha < \lambda[G_\alpha$ is a P_α-generic filter over $M]$.

For, first suppose that $q \in G_\alpha$ and $q \leq r \in P_\alpha$. Say $q = p'^{\alpha}$ with $p \in G$. Define $s \in P$ by

$$\begin{aligned}
&\mathrm{dmn}(s) = \{(\alpha, \xi) : \xi \in \mathrm{dmn}(r)\} \cup \{(\beta, \xi) \in \mathrm{dmn}(p) : \beta \neq \alpha\} \quad \text{and} \\
&s(\alpha, \xi) = r(\xi) \quad \text{for } \xi \in \mathrm{dmn}(r); \\
&s(\beta, \xi) = p(\beta, \xi) \quad \text{for } \beta \neq \alpha \text{ and } (\beta, \xi) \in \mathrm{dmn}(p).
\end{aligned}$$

Then $p \subseteq s$ since if $(\alpha, \xi) \in \mathrm{dmn}(p)$ then $\xi \in \mathrm{dmn}(p'^{\alpha})$, and $p(\alpha, \xi) = p'^{\alpha}(\xi) = q(\xi) = r(\xi) = s(\alpha, \xi)$, while if $(\beta, \xi) \in \mathrm{dmn}(p)$ with $\beta \neq \alpha$ then $p(\beta, \xi) = s(\beta, \xi)$. It follows that $s \in G$. Now $s'^{\alpha} = r$, for $\mathrm{dmn}(s'^{\alpha}) = \{\xi < \kappa : (\alpha, \xi) \in \mathrm{dmn}(s)\} = \mathrm{dmn}(r)$, and $s'^{\alpha}(\xi) = s(\alpha, \xi) = r(\xi)$. Thus $r \in G_\alpha$. So G_α is closed upwards.

Now suppose that $D \subseteq P_\alpha$ is dense. Let $D' = \{p \in P : p'^{\alpha} \in D\}$. Then D' is dense in P. For, suppose that $q \in P$. Choose $p \in D$ such that $p \leq q'^{\alpha}$. Clearly $p \in D'$ and $q \subseteq p$. So, choose $p \in D' \cap G$. Then $p'^{\alpha} \in D \cap G_\alpha$.

Next, suppose that $p, q \in G_\alpha$. Say $p = r'^{\alpha}$ with $r \in G$ and $q = s'^{\alpha}$ with $s \in G$. Choose $t \in G$ such that $t \leq r, s$. Then $t'^{\alpha} \in G_\alpha$ and $t'^{\alpha} \leq p, q$.

Thus G_α is P_α-generic over M; (1) holds.

Let $f_\alpha = \bigcup G_\alpha$. Clearly f_α is a mapping of κ onto α. Hence $|\alpha|^{M[G]}|\alpha|^{M[G_\alpha]} = \kappa$, so that (i) holds.

(2) P is $(< \kappa)$-closed.

In fact, suppose that $\langle p_\alpha : \alpha, \mu \rangle$ with $\mu < \kappa$ is decreasing. Clearly $\bigcup_{\alpha<\mu} p_\alpha$ is below each p_α. Hence all cardinals $\mu \leq \kappa$ are preserved, by Theorem 15.7 and Lemma 15.8. So (ii) holds.

(3) $P \cong \prod^{<\kappa}_{\alpha<\lambda} P_\alpha$.

In fact, for each $p \in P$ define $h(p)$ setting $((h(p))(\alpha))(\xi) = p(\alpha, \xi)$ for all $\alpha < \lambda$ and $\xi < \kappa$ such that $(\alpha, \xi) \in \mathrm{dmn}(p)$. Clearly $h(p) \in \prod_{\alpha<\lambda} P_\alpha$. For any $\alpha < \lambda$

$$|\{\xi < \kappa : ((h(p))(\alpha))(\xi) \neq 1\}| = |\{(\alpha, \xi) \in \lambda \times \kappa : p(\alpha, \xi) \neq 1\}| < \kappa.$$

Thus $h(p) \in \prod^{<\kappa}_{\alpha<\lambda} P_\alpha$. Clearly h is one-one, onto, and preseves $\leq$. Clearly $\forall \alpha < \lambda[|P_\alpha| < \lambda]$. Hence by Theorem 15.18, P has the λ chain condition. Hence by Proposition 14.64, (iii) holds.

(iv) follows from (i) and (iii). $\square$

A *special normal tree* is a set T such that for some $\alpha < \omega_1$ the following conditions hold:

(i) Each $t \in T$ is is a function $t : \beta \to \omega$ for some $\beta < \alpha$.
(ii) $\forall t \in T \forall s[s$ is an initial segment of $t \to s \in T]$.
(iii) $\forall \beta < \alpha[t : \beta \to \omega$ and $t \in T$ and $\beta + 1 < \alpha \to \forall n \in \omega[t^\frown\langle n\rangle \in T]]$.
(iv) $\forall \beta < \alpha \forall \gamma \in [\beta, \alpha) \forall t[t : \beta \to \omega$ and $t \in T \to \exists s \in T[s : \gamma \to \omega$ and $t \subseteq s]]$.
(v) $\forall \beta < \alpha[|\{T \cap {}^\beta\omega\}| \leq \aleph_0]$.

The ordinal α is the *height* of T. A special normal tree T_1 *extends* a special normal tree T_2 iff $\exists \alpha \leq \mathit{height}(T_1)[T_2 = \{t \restriction \alpha : t \in T_1\}]$; we denote this by $T_1 \leq T_2$.

A *normal Suslin tree* is a Suslin tree T satisfying the following conditions:

(i) T has a unique root.
(ii) If x is not maximal in T, then there are infinitely many elements at the next level.
(iii) For each $x \in T$ there is a $y > x$ at every level above that of x.
(iv) If $\alpha < \omega_1$ is a limit ordinal, $x, y \in T$ are at level α, and $\{z : z < x\} = \{z : z < y\}$, then $x = y$.

Lemma 15.31. *If A is a maximal antichain in a special normal tree T, and there is an $\alpha < \mathit{height}(T)$ such that every member of A has domain less than α, then A is maximal in every extension of T.*

Proof. Assume the hypotheses. Let T' be an extension of T. Take any $t \in T' \backslash T$. Then $t \restriction \alpha \in T$, and so there is an $s \in A$ such that $s \subset t \restriction \alpha \subseteq t$. □

Lemma 15.32. *Suppose that α is a countable limit ordinal, T is a special normal tree of height α, and A is a maximal antichain in T. Then there is an extension T' of T of height $\alpha + 1$ such that A is a maximal antichain in T'.*

Proof.

(1) For each $t \in T$ there exist an $a \in A$ and a maximal chain b_t of length α such that $a, t \in b_t$.

In fact, let $t \in T$. Then there is an $a \in A$ such that $a \subseteq t$ or $t \subseteq a$. Hence there is an initial chain c in T with $a, t \in c$. Now by (iv) we can construct the desired maximal chain b_t extending c, by taking a sequence $\langle \beta_n : n \in \omega\rangle$ with supremum α and β_0 the height of c.

Now we let $T' = T \cup \{\bigcup b_t : t \in T\}$. □

Lemma 15.33. *If T is a special normal tree of height ω_1 and T has no uncountable antichain, then T is a Suslin tree.*

Proof. Suppose to the contrary that C is a chain of length ω_1. We may assume that C is maximal, so that it has elements of each level less than ω_1. For each $t \in T$ choose $f(t) \in T$ such that $t < f(t) \notin C$; this is possible by (iii). Now we define $\langle s_\alpha : \alpha < \omega_1\rangle$ by recursion, choosing

$$s_\alpha \in \left\{t \in C : \sup_{\beta < \alpha} \mathrm{ht}(f(s_\beta), T) < \mathrm{ht}(t, T)\right\}.$$

Now $\langle f(s_\alpha) : \alpha < \omega_1\rangle$ is an antichain. In fact, if $\beta < \alpha$ and $f(s_\beta)$ and $f(s_\alpha)$ are comparable, then by construction $\mathrm{ht}(f(s_\beta), T) < \mathrm{ht}(s_\alpha, T) < \mathrm{ht}(f(s_\alpha), T)$, and so $f(s_\beta) < f(s_\alpha)$. But then the tree property yields that $f(s_\beta) < s_\alpha$ and so $f(s_\beta) \in C$, contradiction.

Thus we have an antichain of size ω_1, contradiction. $\square$

Lemma 15.34. *Suppose that G is a generic ultrafilter on B over M and $\tau \in M^B$. Let $\mathrm{dmn}(\bigcup' \tau) = \bigcup_{y\in\mathrm{dmn}(\tau)} \mathrm{dmn}(y)$, and for any $x \in \mathrm{dmn}(\bigcup' \tau)$ let $(\bigcup' \tau)(x) = \sum\{y(x)\cdot\tau(y) : y \in \mathrm{dmn}(\tau) \wedge x \in \mathrm{dmn}(y)\}$. Then $(\bigcup' \tau)^G = \bigcup \tau^G$.*

Proof. Suppose that $u \in (\bigcup' \tau)^G$. Then there is an $x \in \mathrm{dmn}(\bigcup' \tau)$ such that $u = x^G$ and $(\bigcup' \tau)(x) \in G$. Thus $\sum\{y(x) \cdot \tau(y) : y \in \mathrm{dmn}(\tau) \wedge x \in \mathrm{dmn}(y)\} \in G$, so there is a $y \in \mathrm{dmn}(\tau)$ with $x \in \mathrm{dmn}(y)$ such that $y(x) \cdot \tau(y) \in G$, hence $\tau(y) \in G$ and $y(x) \in G$. It follows that $y^G \in \tau^G$ and $x^G \in y^G$, so $u = x^G \in \bigcup \tau^G$.

Conversely, suppose that $u = x^G \in \bigcup \tau^G$. Then there is a $v \in \tau^G$ such that $u \in v$. Say $v = y^G$ with $y \in \mathrm{dmn}(\tau)$ and $\tau(y) \in G$. Also $x \in \mathrm{dmn}(y)$ and $y(x) \in G$. So $\tau(y) \cdot y(x) \in G$. Hence $\sum\{y(x) \cdot \tau(y) : y \in \mathrm{dmn}(\tau) \wedge x \in \mathrm{dmn}(y)\} = (\bigcup' \tau)(x) \in G$ and so $u = x^G \in (\bigcup' \tau)^G$. $\square$

Lemma 15.35. *Suppose that G is P-generic over M and $\tau \in M^{\mathrm{RO}(P)}$. Then $(\bigcup' \tau)_G = \bigcup \tau_G$.*

Proof. Let $G' = \{a \in \mathrm{RO}(P) : \exists p \in G[e(p) \le a]\}$. Then $(\bigcup' \tau)_G = (\bigcup' \tau)^{G'} = \bigcup \tau^{G'} = \bigcup \tau_G$. $\square$

Lemma 15.36. *Suppose that G is a generic ultrafilter on B over M. Then $(\bigcup' \Gamma)^G = \bigcup G$.*

Proof. $\Gamma^G = G$ by Proposition 14.46, so $(\bigcup' \Gamma)^G = \bigcup G$ by Lemma 1. $\square$

Lemma 15.37. *Suppose that G is P-generic over M. Then $(\bigcup' \Gamma')_G = \bigcup G$.*

Proof. $\Gamma'_G = G$ by Lemma 14.47. so $(\bigcup' \Gamma')_G = \bigcup G$ by Lemma 9. $\square$

Theorem 15.38. *In M let P be the set of all special normal trees, with the indicated order. Let G be P-generic over M. Then $M[G] \models \bigcup G$ is a normal Suslin tree.*

Proof.

(1) If $T_1, T_2 \in P$, then either one is an extension of the other, or they are incompatible.

In fact, suppose that they are compatible. Say $T_3 \le T_1, T_2$. Choose α and β so that $\alpha, \beta \le \mathrm{height}(T_3)$, $T_1 = \{t \restriction \alpha : t \in T_3\}$ with α minimum, and $T_2 = \{t \restriction \beta : t \in T_3\}$ with β minimum. By (iv), T_1 has members with domain any ordinal less than α. Similarly for T_2 and β. Say $\alpha \le \beta$. We claim that

(2) $T_1 = \{t \restriction \alpha : t \in T_2\}$.

(Hence $T_2 \le T_1$.) For, let $s \in T_1$. Choose $t \in T_3$ such that $s = t \restriction \alpha$. Then $t \restriction \beta \in T_2$, and $(t \restriction \beta) \restriction \alpha = t \restriction \alpha = s$. So $\subseteq$ holds in (2). Conversely, suppose that $t \in T_2$. Say $t = s \restriction \beta$ with $s \in T_3$. Then $t \restriction \alpha = (s \restriction \beta) \restriction \alpha = s \restriction \alpha \in T_1$. This proves $\supseteq$ in (2).

Thus (1) holds.

Now

(3) If $T_{n+1} \leq T_n$ for all $n \in \omega$, then $\bigcup_{n\in\omega} T_n$ is a special normal tree.

This is clear. So P is $\aleph_0$-closed. Hence by Theorem 15.7 and Lemma 15.8, P preserves $\aleph_1$.

(4) $\bigcup G$ is a special normal tree of height ω_1.

In fact, by (1) $\bigcup G$ is a special normal tree. By (iii) and (3), it has height ω_1.

Now by Lemma 15.33 it remains only to show that $\bigcup G$ has no uncountable antichain. Suppose A is one; wlog it is maximal. Then there exist a name $\dot{A}$ and a $T \in G$ such that $T \Vdash \dot{A}$ is a maximal antichain in P. Let

$$D = \{T' \leq T : \text{ there is a bounded maximal antichain } A' \text{ in } T' \text{ such that } T' \Vdash A' \subseteq A\}$$

We claim that D is dense below T. For, let $T_0 \leq T$. Now

(5) $T_0 \Vdash [\dot{A}$ is a maximal antichain in P and $\bigcup' \Gamma' \leq T_0]$.

In fact, let $T_0 \in H$ generic. Then $T_0 \subseteq \bigcup H = (\bigcup' \Gamma')_H$ by Lemma 37; so (5) holds.

Now $T_0 \Vdash \forall s \in \check{T}_0 \exists t \in \dot{A}[t$ and s are comparable], and also $T_0 \Vdash \forall t \in \dot{A}[t \in \bigcup' \Gamma']$. So for each $s \in T_0$,

$$\begin{aligned} e(T_0) &\leq \sum_{y \in \mathrm{dmn}(\dot{A})} (\dot{A}(y) \cdot [\![\check{s} \text{ and } y \text{ are comparable}) \text{ and} \\ e(T_0) &\leq \prod_{t \in \mathrm{dmn}(\dot{A})} [\dot{A}(y) \Rightarrow [\![y \in \bigcup{}' \Gamma']\!]. \end{aligned}$$

hence

$$\begin{aligned} e(T_0) \leq & \sum_{y \in \mathrm{dmn}(\dot{A})} (\dot{A}(y) \cdot [\![\check{s} \text{ and } y \text{ are comparable}) \cdot ([\dot{A}(y) \Rightarrow [\![y \in \bigcup{}' \Gamma']\!] \\ = & \sum_{y \in \mathrm{dmn}(\dot{A})} (\dot{A}(y) \cdot [\![\check{s} \text{ and } y \text{ are comparable}) \cdot [\![y \in \bigcup{}' \Gamma']\!]). \end{aligned}$$

Hence there is a $T_0' \leq T_0$ and a $y \in \mathrm{dmn}(\dot{A})$ such that

$$e(T_0') \leq (\dot{A}(y) \cdot [\![\check{s} \text{ and } y \text{ are comparable}) \cdot [\![y \in \bigcup{}' \Gamma']\!])$$

So

$$T_0' \Vdash y \in \dot{A} \wedge (\check{s} \text{ and } y \text{ are comparable}) \wedge y \in \bigcup{}' \Gamma'.$$

Let $T_0' \in H$ generic. Then $y_H \in \dot{A}_H$, y_H and s are comparable, and $y_H \in \bigcup H$. Say $T_0'' \in H$ and $y_H \in T_0''$. Let $t = y_H$. Then $T_0'' \leq T_0'$ and $t \in T_0''$. Moreover, $T_0'' \Vdash \check{t} \in \dot{A}$. Thus

(6) $\forall s \in T_0 \exists T_0' \leq T_0 \exists t \in T_0'[s$ and t are comparable and $T_0' \Vdash \check{t} \in \dot{A}]$.

Repeating the argument for (6) we get for each $n \in \omega$,

(7) $\forall s \in T_n \exists T_{n+1} \leq T_n \exists t_s \in T_{n+1}[s$ and t_s are comparable and $T_{n+1} \Vdash \check{t}_s \in \dot{A}]$.

Let $T_\infty = \bigcup_{n\in\omega} T_n$ and $A' = \{t_s : s \in T_\infty\}$. A' is a maximal antichain in T_∞. Then $T_\infty \Vdash \check{A}' \subseteq \dot{A}$. By Lemma 15,32 there is an extension T' of T_∞ such that A' is a bounded maximal antichain in T' Clearly $T' \Vdash \check{A}' \subseteq \dot{A}$. So $T' \in D$. This shows that D is dense.

Choose $T' \in D \cap G$. Let A' be a bounded maximal chain in T' such that $A' \subseteq A$. Now $\bigcup G$ is an extension of T'. By Lemma 15.31, $A = A'$. Thus A is countable. □

Lemma 15.39. *Suppose that κ is an uncountable regular cardinal, $X \in [\kappa]^{<\kappa}$, and $\mathscr{F}$ is a collection of finitary partial operations on κ, with $|\mathscr{F}| < \kappa$. Then $\{\alpha < \kappa : X \subseteq \alpha$ and α is closed under each $f \in \mathscr{F}\}$ is club in κ.*

Proof. Denote the indicated set by C. To show that it is closed, suppose that α is a limit ordinal less than κ, and $C \cap \alpha$ is unbounded in α. To show that α is closed under any partial operation $f \in \mathscr{F}$, suppose that $\mathrm{dmn}(f) \subseteq {}^m\kappa$ and $a \in ({}^m\alpha) \cap \mathrm{dmn}(f)$. For each $i < m$ choose $\beta_i < \alpha$ such that $a_i \in \beta_i$. Since α is a limit ordinal, the ordinal $\gamma \stackrel{\text{def}}{=} \bigcup_{i<m} \beta_i$ is still less than α. Since $C \cap \alpha$ is unbounded in α, choose $\delta \in C \cap \alpha$ such that $\gamma < \delta$. Then $a \in {}^m\delta$ so, since $\delta \in C$, we have $f(a) \in \delta \subseteq \alpha$. Thus α is closed under f. Hence $\alpha \in C$; so C is closed in κ.

To show that C is unbounded in κ, take any $\alpha < \kappa$. We now define a sequence $\langle \beta_n : n \in \omega\rangle$ by recursion. Let $\beta_0 = \alpha$. Having defined $\beta_i < \kappa$, consider the set

$$\{f(a) : f \in \mathscr{F}, a \in \mathrm{dmn}(f), \text{ and each } a_j \text{ is in } \beta_i\}.$$

This set clearly has fewer than κ members. Hence we can take β_{i+1} to be some ordinal less than κ and greater than each member of this set. This finishes the construction.

Let $\gamma = \bigcup_{i\in\omega} \beta_i$. We claim that $\gamma \in C$, as desired. For, suppose that $f \in \mathscr{F}$, f has domain $\subseteq {}^n\kappa$, and $a \in ({}^n\gamma) \cap \mathrm{dmn}(f)$. Then for each $i < n$ choose $m_i \in \omega$ such that $a_i \in \beta_{m_i}$. Let p be the maximum of all the β_i's. Then $a \in ({}^n\beta_p) \cap \mathrm{dmn}(f)$, so by construction $f(a) \in \beta_{p+1} \subseteq \gamma$. □

An ω_1-*tree* is a tree of height ω_1 with every level countable. A tree is *eventually branching* iff $\forall t[\{s : t \leq s\}$ is not a chain].

Lemma 15.40. *Suppose that $T = (\omega_1, \prec)$ is an ω_1-tree and A is a maximal antichain in T. Then*

$$\{\alpha < \omega_1 : T_\alpha = \alpha \text{ and } A \cap \alpha \text{ is a maximal antichain in } T_\alpha\}$$

is club in ω_1.

Proof. Let C be the indicated set. Suppose that $A \subseteq \omega_1$ is a maximal antichain in T. To see that C is closed in ω_1, let $\alpha < \omega_1$ be a limit ordinal, and suppose that $C \cap \alpha$ is unbounded in α. If $\beta \in T_\alpha$, then there is a $\gamma < \alpha$ such that $\beta \in T_\gamma$. Choose $\delta \in (C \cap \alpha)$ such that $\gamma < \delta$. Then $\beta \in T_\delta = \delta$, so also $\beta \in \alpha$. This shows that $T_\alpha \subseteq \alpha$. Conversely, suppose that $\beta \in \alpha$. Choose $\gamma \in C \cap \alpha$ such that $\beta < \gamma$. Then $\beta \in \gamma = T_\gamma \subseteq T_\alpha$. Thus $T_\alpha = \alpha$.

To show that $A \cap \alpha$ is a maximal antichain in T_α, note first that at least it is an antichain. Now take any $\beta \in T_\alpha$; we show that β is comparable under $\prec$ to some member of $A \cap \alpha$, which will show that $A \cap \alpha$ is a maximal antichain in T_α. Choose $\gamma < \alpha$ such that $\beta \in T_\gamma$, and then choose $\delta \in (C \cap \alpha)$ such that $\gamma < \delta$. Thus $\beta \in T_\delta$. Now $A \cap \delta$ is a maximal antichain in T_δ since $\delta \in C$, so β is comparable with some $\varepsilon \in (A \cap \delta) \subseteq (A \cap \alpha)$, as desired.

To show that C is unbounded in κ we will apply Lemma 15.39 to the following three functions $f, g, h : \kappa \to \kappa$:

$$\begin{aligned} f(\beta) &= \mathrm{ht}(\beta, T); \\ g(\beta) &= \sup(\mathrm{Lev}_\beta(T)); \\ h(\beta) &= \text{ some member of } A \text{ comparable with } \beta \text{ under } \prec. \end{aligned}$$

By Lemma 15.39, the set D of all $\alpha < \kappa$ which are closed under each of f, g, h is club in κ. We now show that $D \subseteq C$, which will prove that C is unbounded in κ. So, suppose that $\alpha \in D$. If $\beta \in T_\alpha$, let $\gamma = \mathrm{ht}(\beta, T)$. Then $\gamma < \alpha$ and $\beta \in \mathrm{Lev}_\gamma(T)$, and so $\beta \leq g(\gamma) < \alpha$. Thus $T_\alpha \subseteq \alpha$. Conversely, suppose that $\beta < \alpha$. Then $f(\beta) < \alpha$, i.e., $\mathrm{ht}(\beta, T) < \alpha$, so $\beta \in T_\alpha$. Therefore $T_\alpha = \alpha$. Now suppose that $\beta \in T_\alpha$; we want to show that β is comparable with some member of $A \cap \alpha$, as this will prove that $A \cap \alpha$ is a maximal antichain in T_α. Since $\beta \in \alpha$ by what has already been shown, we have $h(\beta) < \alpha$, and so the element $h(\beta)$ is as desired. □

Lemma 15.41. *Let $T = (\omega_1, \prec)$ be an eventually branching ω_1-tree and let $\langle A_\alpha : \alpha < \omega_1 \rangle$ be a $\diamondsuit$-sequence. Assume that for every limit $\alpha < \omega_1$, if $T_\alpha = \alpha$ and A_α is a maximal antichain in T_α, then for every $x \in \mathrm{Lev}_\alpha(T)$ there is a $y \in A_\alpha$ such that $y \prec x$.*

Then T is a Suslin tree.

Proof. By Lemma 15.33 it suffices to show that every maximal antichain A of T is countable. By Lemma 15.40, the set

$$C \stackrel{\text{def}}{=} \{\alpha < \omega_1 : T_\alpha = \alpha \text{ and } A \cap \alpha \text{ is a maximal antichain in } T_\alpha\}$$

is club in ω_1. Now by the definition of the $\diamondsuit$-sequence, the set $\{\alpha < \omega_1 : A \cap \alpha = A_\alpha\}$ is stationary, so we can choose $\alpha \in C$ such that $A \cap \alpha = A_\alpha$. Now if $\beta \in T$ and $\mathrm{ht}(\beta, T) \geq \alpha$, then there is a $\gamma \in \mathrm{Lev}(\alpha, T)$ such that $\gamma \preceq \beta$, and the hypothesis of the lemma further yields a $\delta \in A_\alpha$ such that $\delta \prec \gamma$. Since $\delta \prec \beta$, it follows that $\beta \notin A$. So we have shown that for all $\beta \in T$, if $\mathrm{ht}(\beta, T) \geq \alpha$ then $\beta \notin A$. Hence for any $\beta \in T$, if $\beta \in A$ then $\beta \in T_\alpha = \alpha$. So $A \subseteq \alpha$ and hence $A = A_\alpha$, so that A is countable. □

Theorem 15.42. $\diamondsuit$ *implies that there is a Suslin tree.*

Proof. Assume $\diamondsuit$, and let $\langle A_\alpha : \alpha < \omega_1 \rangle$ be a $\diamondsuit$-sequence. We are going to construct a Suslin tree of the form $(\omega_1, \prec)$ in which for each $\alpha < \omega_1$ the α-th level is the set

$\{\omega\cdot\alpha+m : m\in\omega\}$. We will do the construction by completely defining the tree up to heights $\alpha<\omega_1$ by recursion. Thus we define by recursion trees $(\omega\cdot\alpha,\prec_\alpha)$, so that really we are just defining the partial orders $\prec_\alpha$ by recursion.

We let $\prec_0=\prec_1=\emptyset$. Now suppose that $\beta>1$ and $\prec_\alpha$ has been defined for all $\alpha<\beta$ so that the following conditions hold whenever $0<\alpha<\beta$:

(1) $(\omega\cdot\alpha,\prec_\alpha)$ is a tree, denoted by T_α for brevity.

(2) If $\gamma<\alpha$ and $\xi,\eta\in T_\gamma$, then $\xi\prec_\gamma\eta$ iff $\xi\prec_\alpha\eta$.

(3) For each $\gamma<\alpha$, $\mathrm{Lev}_\gamma(T_\alpha)=\{\omega\cdot\gamma+m : m\in\omega\}$.

(4) If $\gamma<\delta<\alpha$ and $m\in\omega$, then there is an $n\in\omega$ such that $\omega\cdot\gamma+m\prec_\alpha\omega\cdot\delta+n$.

(5) If $\delta<\alpha$, δ is a limit ordinal, $\omega\cdot\delta=\delta$, and A_δ is a maximal antichain in T_δ, then for every $x\in\mathrm{Lev}_\delta(T_\alpha)$ there is a $y\in A_\delta$ such that $y\prec_\alpha x$.

Note that conditions (1)–(3) just say that the trees constructed have the special form indicated at the beginning, and are an increasing chain of trees. Condition (4) is to assure that the final tree is well-pruned. Conditions (1)–(5) imply that if $x\in T_\alpha$, then it has the form $\omega\cdot\beta+m$ for some $\beta<\alpha$, and then $x\in\mathrm{Lev}_\beta(T_\alpha)$ and for each $\gamma<\beta$ there is a unique element $\omega\cdot\gamma+n$ in T_α such that $\omega\cdot\gamma+n\prec_\alpha x$.

If β is a limit ordinal, let $\prec_\beta=\bigcup_{\alpha<\beta}\prec_\alpha$. Conditions (1)–(5) are then clear for any $\alpha\le\beta$.

Next suppose that $\beta=\gamma+2$ for some ordinal γ. Then we define

$$\begin{aligned}\prec_\beta=\prec_{\gamma+1}&\cup\{(\xi,\omega\cdot(\gamma+1)+2m) : \xi\preceq_{\gamma+1}\omega\cdot\gamma+m,\ m\in\omega\}\\&\cup\{(\xi,\omega\cdot(\gamma+1)+2m+1) : \xi\preceq_{\gamma+1}\omega\cdot\gamma+m,\ m\in\omega\}.\end{aligned}$$

Clearly (1)–(5) hold for all $\alpha<\beta$.

The most important case is $\beta=\gamma+1$ for some limit ordinal γ. To treat this case, we first associate with each $x\in T_\gamma$ a chain $B(x)$ in T_γ, and to do this we define by recursion a sequence $\langle y^x_n : n\in\omega\rangle$ of elements of T_γ. To define y^x_0 we consider two cases.

Case 1. $\omega\cdot\gamma=\gamma$ and A_γ is a maximal antichain in T_γ. Then x is comparable with some member z of A_γ, and we let y^x_0 be some element of T_γ such that $x,z\prec_\gamma y^x_0$.

Case 9. Otherwise, we just let $y^x_0=x$.

Now let $\langle\xi_m : m\in\omega\rangle$ be a strictly increasing sequence of ordinals less than γ such that $\xi_0=\mathrm{ht}(y^x_0,T_\gamma)$ and $\sup_{m\in\omega}\xi_m=\gamma$. Now if y^x_i has been defined of height ξ_i, by (4) let y^x_{i+1} be an element of height ξ_{i+1} such that $y^x_i\prec_\gamma y^x_{i+1}$. Then we define

$$B(x)=\{z\in\omega\cdot\gamma : z\prec_\gamma y^x_i \text{ for some } i\in\omega\}.$$

Finally, let $\langle x(n) : n\in\omega\rangle$ be a one-one enumeration of $\omega\cdot\gamma$, and set

$$\prec_\beta=\prec_\gamma\cup\{(z,\omega\cdot\gamma+n) : n\in\omega,\ z\in B(x_n)\}.$$

Clearly (1)–(3) hold with γ in place of α. For (4), suppose that $\delta<\gamma$ and $m\in\omega$. Let $z=\omega\cdot\delta+m$. Thus $z\in\omega\cdot\gamma$, and hence there is an $n\in\omega$ such that $z=x(n)$. Hence $z\in B(x(n))$ and $z\prec_\beta\omega\cdot\gamma+n$, as desired.

For (5), suppose that $\omega\cdot\gamma=\gamma$, and A_γ is a maximal antichain in T_γ. Suppose that $w\in \mathrm{Lev}_\gamma(T_\beta)$. Choose n so that $w=\omega\cdot\gamma+n$. Then there is an $s\in A_\gamma$ such that $s<y_0^{x(n)}$. So $s\in B(x(n))$ and $s\prec_\beta \omega\cdot\gamma+n=w$, as desired.

Thus the construction is finished. Now we let $\prec=\bigcup_{\alpha<\omega_1}\prec_\alpha$. Clearly $T\stackrel{\text{def}}{=}(\omega_1,\prec)$ is an ω_1-tree. It is eventually branching by (4) and the $\beta=\gamma+2$ step in the construction. The hypothesis of Lemma 15.41 holds by the step $\beta=\gamma+1$, γ limit, in the construction. Therefore T is a Suslin tree by Lemma 15.41. □

Proposition 15.43. *If $(T,<)$ is a Suslin tree, let $P=(T,>)$. Then elements are compatible in P iff they are comparable in T. Hence P satisfies ccc.* □

Lemma 15.44. *If $(T,<)$ is a normal Suslin tree, then $P=(T,>)$ is $\aleph_0$-distributive.*

Proof. First note:

(1) $D\subseteq T$ is open in P iff it is closed upwards in the sense of T.

(2) $D\subseteq T$ is dense in P iff $\forall t\in T\exists s\in D[t\le s]$.

(3) If $D\subseteq T$ is dense open, then $\exists\alpha<\omega_1\forall t\in T[height(t)>\alpha\to t\in D]$.

In fact, let A be a maximal antichain in D. So A is countable. Choose $\alpha<\omega_1$ so that each member of A has height less than α. If $t\in D$ and $height(t)>\alpha$, then there is an $s\in D$ such that $s<t$, and hence $t\in D$. So (3) holds.

Now let D_n be dense open for all $n\in\omega$. Clearly $\bigcap_{n\in\omega}D_n$ is open. To show that it is dense, take any $s\in T$. By (3), for each n choose $\alpha_n<\omega_1$ such that $\{t\in T: height(t)>\alpha_n\to t\in D_n\}$. Let $\beta=\sup_{n\in\omega}\alpha_n$. Then $\{t\in T: height(t)>\beta\}\subseteq\bigcap_{n\in\omega}D_n$. Since T is normal, choose $t>s$ with $height(t)>\beta$. So $t\in\bigcap_{n\in\omega}D_n$, proving that $\bigcap_{n\in\omega}D_n$ is dense. □

Lemma 15.45. *If $(T,<)$ is a normal Suslin tree, then $\mathrm{RO}(P,<)$ is an $\aleph_0$-distributive, atomless, ccc, complete BA.*

Proof. Let $B=\mathrm{RO}(P,<)$. By Lemma 15.44 B is $\aleph_0$-distributive. To see that it is atomless, let $0\ne b\in B$. Choose $t\in T$ such that $e(t)\le b$. Choose s,u distinct at the next level above t. Then $e(s),e(u)\le e(t)\le b$ and $e(s)\cap e(u)=\emptyset$. So B is atomless. Now suppose that $X\subseteq B$ is pairwise disjoint. For each $b\in X$ choose t_b such that $e(t_b)\le b$. Then t_b and t_c are incomparable for $b\ne c$. So X is countable. Clearly B is complete. □

Let P_{rand} be the set of all Borel sets of reals with positive Lebesgue measure, ordered by $\subseteq$.

Lemma 15.46. *If A is a σ-complete BA, I is a σ-complete ideal in A, and A/I has ccc, then A/I is complete.*

Proof. Let $B=A/I$. First note that B is σ-complete. For, let $\{[x]:x\in X\}$ be given, with X countable. We claim that $\sum_{x\in X}[x]=[\sum X]$. In fact, clearly $[\sum X]$ is an upper bound for $\{[x]:x\in X\}$. Suppose that $[y]$ is any upper bound. Then $\forall x\in X[x\cdot -y\in I]$, so $(\sum X)\cdot -y\in I$. Hence $[\sum X]\le[y]$.

Now suppose that $X \subseteq B$. Let $X' = \{a \in B : a \leq x \text{ for some } x \in X\}$. Let $Y \subseteq X'$ be maximal pairwise disjoint. Then $\sum Y$ exists. If $x \in X$ and $x \not\leq \sum Y$, then $x \cdot -\sum Y \neq 0$ and $x \cdot -\sum Y \in X'$, so $Y \cup \{x \cdot -\sum Y\} \subseteq X'$ is pairwise disjoint, contradicting the maximality of Y. Hence $\sum Y$ is an upper bound for X. Clearly $\sum Y \leq z$ for any upper bound z for X, so $\sum Y = \sum X$. □

Lemma 15.47. *Let B be the σ-algebra of Borel sets of reals, and let I_μ be the σ-ideal of B of Lebesgue measure 0 sets. Then B/I_μ has ccc, and so B/I_μ is complete.*

Proof. Suppose that $\langle [a_\alpha] : \alpha < \omega_1\rangle$ is a pairwise disjoint system of nonzero elements of B/I_μ. Define $b_\alpha = a_\alpha \backslash \bigcup_{\beta<\alpha} a_\beta$. Then

$$\mu(a_\alpha) = \mu(b_\alpha) + \mu(a_\alpha \cap \bigcup_{\beta<\alpha} a_\alpha) = \mu(b_\alpha).$$

Choose a positive integer m and an uncountable subset M of ω_1 such that $\mu(b_\alpha) \geq \frac{1}{m}$ for all $\alpha \in M$. Then $\mu(\bigcup_{\alpha\in M} b_\alpha) = \infty$, contradiction. □

Lemma 15.48. $\mathrm{RO}(P_{\mathrm{rand}}) \cong B/I_\mu$*, where B is the BA of Borel sets of reals and I_μ is the set of $b \in B$ of Lebesgue measure 0.*

Proof.

(1) $e(p) = \{s : s\backslash p \text{ has measure } 0\}$.

In fact,

$$\begin{aligned} e(p) &= \mathrm{int}(\mathrm{cl}(P \downarrow p)) \\ &= \mathrm{int}(\{q : (P \downarrow q) \cap (P \downarrow p) \neq \emptyset\}) \\ &= \mathrm{int}(\{q : p \text{ and } q \text{ are compatible}\}) \\ &= \{r : \forall q \leq r[p \text{ and } q \text{ are compatible}\}) \end{aligned}$$

Now to prove (1), first suppose that $r \in e(p)$, but suppose that $r\backslash p$ has positive measure. Then $r\backslash p \leq r$ but p and $r\backslash p$ are not compatible, contradiction. Second, suppose that $s\backslash p$ has measure 0. Hence for all $q \leq s$, $\mu(q\backslash p) = 0$ and so $\mu(q \cap p) > 0$, and so $q \cap p \leq p, q$, as desired.

(2) $-e(p) = \{r : \mu(r \cap p) = 0\}$.

In fact, $-e(p) = \mathrm{int}(P\backslash e(p)) = \{r : \forall s \leq r[\mu(s\backslash p) > 0\} = \{r : \mu(r \cap p) = 0\}$.

Now we turn to the proof of the Lemma. For each $p \in P$ define $f(e(p)) = [p]$. First we use Sikorski's extension criterion to show that f is well-defined and extends to an isomorphism of $\langle\{e(p) : p \in P\}\rangle^{\mathrm{RO}(P)}$ into B/I_μ. So, we want to show that

$$e(p_0) \cap \ldots \cap e(p_{m-1}) \cap -e(q_0) \cap \ldots \cap -e(q_{n-1}) = \emptyset \tag{4}$$

is equivalent to

$$p_0 \cap \ldots \cap p_{m-1} \cap -q_0 \cap \ldots \cap -q_{n-1} \in I_\mu. \tag{5}$$

Taking any r, we have

$$\begin{aligned}
& r \in e(p_0) \cap \ldots \cap e(p_{m-1}) \cap -e(q_0) \cap \ldots \cap -e(q_{n-1}) \\
\text{iff} \quad & \forall i < m[\mu(r \backslash p_i) = 0] \text{ and } \forall i < n[\mu(r \cap q_i) = 0] \\
\text{iff} \quad & \mu\left(\bigcup_{i<m}(r \backslash p_i) \cup \bigcup_{i<n}(r \cap q_i)\right) = 0 \\
\text{iff} \quad & \mu\left(r \cap \left(\left(P \backslash \bigcap_{i<m} p_i\right) \cup \bigcup_{i<n} q_i\right)\right) = 0 \\
\text{iff} \quad & \mu(r) = \mu\left(r \cap \bigcap_{i<m} p_i \cap \bigcap_{i<n}(P \backslash q_i)\right)
\end{aligned}$$

Now the equivalence of (4) and (5) follows.

So f is well-defined and is an isomorphism of $\langle\{e(p) : p \in P\}\rangle^{\mathrm{RO}(P)}$ into B/I_μ. Restricted to $\{e(p) : p \in P\}$ onto B/I_μ. Since $\{e(p) : p \in P\}$ is dense in RO(P), the Lemma follows. $\square$

Lemma 15.49. *For G generic over P_{rand}, there is a unique $a \in \mathbb{R}$ in $M[G]$ such that $a \in [r,s]^M$ for all rationals $r < s$ such that $[r,s]^M \in G$.*

Proof. Note that $\mathbb{R} = \bigcup\{[r,s]^M : r,s \in \mathbb{Q}, r < s\} \in G$, so there are rationals $r < s$ such that $[r,s]^M \in G$. We define $\langle[r_i,s_i]^M : i \in \omega\rangle$ by recursion. Let $[r_0,s_0]^M = [m,m+1]^M$ such that $m \in \mathbb{Z}$ and $[m,m+1]^M \in G$. If $[r_i,s_i]^M \in G$ has been defined, then

$$[r_i,s_i]^M = \left[r_i, \frac{r_i+s_i}{2}\right]^M \cup \left[\frac{r_i+s_i}{2}, s_i\right]^M,$$

and

$$D \stackrel{\text{def}}{=} \left\{p : p \subseteq \left[r_i, \frac{r_i+s_i}{2}\right]^M \text{ or } p \subseteq \left[\frac{r_i+s_i}{2}, s_i\right]^M\right\}$$

is dense, so we can define

$$[r_{i+1}, s_{i+1}]^M = \begin{cases} \left[r_i, \frac{r_i+s_i}{2}\right]^M & \text{if } \left[r_i, \frac{r_i+s_i}{2}\right]^M \in G, \\ \left[\frac{r_i+s_i}{2}, s_i\right]^M & \text{if } \left[\frac{r_i+s_i}{2}, s_i\right]^M \in G. \end{cases}$$

Note that each $r_i, s_i \in M$, but the sequence $\langle r_i : i \in \omega\rangle$ is not. Now $\langle r_i : i \in \omega\rangle$ is Cauchy, so it has a limit a. Suppose that $u < v$ are rationals with $[u,v]^M \in G$ and $a \notin [u,v]^M$. Say $a < u$. Then there is an i such that $s_i < u$, so $\emptyset = [r_i,s_i]^M \cap [u,v]^M \in G$, contradiction. $\square$

Lemma 15.50. *For G generic over P_{rand}, if $f \in {}^\omega\omega$ in $M[G]$ then there is a $g \in {}^\omega\omega$ in M such that $\forall n \in \omega[f(n) < g(n)]$.*

Proof. Let $\dot{f} \in M^{\text{RO}|(P_{\text{rand}})}$ and $p \in P_{\text{rand}}$ be such that $\dot{f}_G = f$ and $p \Vdash \dot{f} : \omega \to \omega$. Let

$$D = \{q \in P_{\text{rand}} : \exists h : \omega \to \omega[q \Vdash \forall n \in \omega[\dot{f}(n) < \check{h}(n)]]\}$$

We claim that D is dense below p; clearly this will prove the Lemma.

So suppose that $p' \le p$. We have $e(p') \le \sum_{m\in\omega} ||\dot{f}(\check{n}) = \check{m}||$, so $e(p') = \sum_{m\in\omega}(e(p') \cdot ||\dot{f}(\check{n}) = \check{m}||$. Hence $\mu(e(p')) = \sum_{m\in\omega} \mu(e(p') \cdot ||\dot{f}(\check{n}) = \check{m}||)$. Choose k so that

$$\mu(e(p')) - \sum_{m\le k} \mu(e(p') \cdot ||\dot{f}(\check{n}) = \check{m}||) < \frac{1}{2^n} \cdot \frac{1}{4} \cdot \mu(e(p)).$$

Let $g(n) = k + 1$. Then

$$\sum_{m\le k} \mu(e(p') \cdot ||\dot{f}(\check{n}) = \check{m}||) = \mu\left(\sum_{m\le k}(e(p') \cdot ||\dot{f}(\check{n}) = \check{m}||)\right) = \mu(e(p') \cdot ||\dot{f}(\check{n}) < g(n)\check{}\,||).$$

Hence $\mu(e(p')) - \mu(e(p') \cdot ||\dot{f}(\check{n}) < g(n)\check{}\,||) < \frac{1}{2^n} \cdot \frac{1}{4} \cdot \mu(e(p'))$. Now

$$\mu(e(p')) = \mu(e(p') \cdot ||\dot{f}(\check{n}) < g(n)\check{}\,||) + \mu(e(p') \cdot -||\dot{f}(\check{n}) < g(n)\check{}\,||),$$

so $\mu(e(p') \cdot -||\dot{f}(\check{n}) < g(n)\check{}\,||) < \frac{1}{2^n} \cdot \frac{1}{4} \cdot \mu(e(p'))$. Thus

$$\begin{aligned}\mu\left(e(p') \cap -\bigcap_{n\in\omega} ||\dot{f}(\check{n}) < g(n)\check{}\,||\right) &= \mu\left(\bigcup_{n\in\omega}(e(p') \cap -||\dot{f}(\check{n}) < g(n)\check{}\,||\right) \\ &\le \sum_{n\in\omega} \mu(e(p') \cap -||\dot{f}(\check{n}) < g(n)\check{}\,||) \le \frac{1}{2}\mu((p)).\end{aligned}$$

It follows that $\mu\left(e(p') \cap \bigcap_{n\in\omega} ||\dot{f}(\check{n}) < g(n)\check{}\,||\right) \ge \frac{1}{2}e(p) > 0$. Clearly $e(p') \cap \bigcap_{n\in\omega} ||\dot{f}(\check{n}) < g(n)\check{}\,|| \Vdash \forall n[\dot{f}(\check{n}) < g(n)\check{}\,]$. $\square$

Lemma 15.51. *Suppose that M is a c.t.m. of ZFC, $P = \{p \subseteq \omega \times 2 : p$ is a finite function$\}$ and G is P-generic over M. Let $g = \bigcup G$ (so that g is a Cohen real). Then for any $f \in {}^\omega 2$ which is in M, the set $\{m \in \omega : f(m) < g(m)\}$ is infinite.*

Proof. For each $n \in \omega$ let in M

$$D_n = \{h \in P : \text{there is an } m > n \text{ such that } m \in \text{dmn}(h) \text{ and } f(m) < h(m)\}.$$

Clearly D_n is dense. Hence the desired result follows. $\square$

For the next results we need to develop more measure theory. Let κ be an infinite cardinal, and $P = \{f \subseteq \kappa \times 2 : f \text{ a finite function}\}$. For $f \in P$ let $U_f = \{g \in {}^{\kappa}2 : f \subseteq g\}$. Hence $U_\emptyset = {}^{\kappa}2$. Note that the function taking f to U_f is one-one. For each $f \in P$ let $\theta_0(U_f) = 1/2^{|\mathrm{dmn}(f)|}$. Thus $\theta_0(U_\emptyset) = 1$. Let $\mathcal{C} = \{U_f : f \in P\}$. Note that ${}^{\kappa}2 \in \mathcal{C}$. For any $A \subseteq {}^{\kappa}2$ let

$$\theta(A) = \inf\left\{\sum_{n\in\omega} \theta_0(C_n) : C \in {}^{\omega}\mathcal{C} \text{ and } A \subseteq \bigcup_{n\in\omega} C_n\right\}.$$

Proposition 15.52. *θ is an outer measure on ${}^{\kappa}2$.*

Proof. For the definition of outer measure see page 88. For (1), for any $m \in \omega$ let $f \in P$ have domain of size m. Then $\emptyset \subseteq U_f$ and $\theta_0(U_f) = \frac{1}{2^m}$. Hence $\theta(\emptyset) = 0$.

For (2), if $A \subseteq B \subseteq {}^{\kappa}2$, then

$$\left\{C \in {}^{\omega}\mathcal{C} : B \subseteq \bigcup_{n\in\omega} C_n\right\} \subseteq \left\{C \in {}^{\omega}\mathcal{C} : A \subseteq \bigcup_{n\in\omega} C_n\right\},$$

and hence $\mu(A) \leq \mu(B)$.

For (3), assume that $A \in {}^{\omega}\mathscr{P}({}^{\kappa}2)$. We may assume that $\sum_{n\in\omega} \theta(A_n) < \infty$. Let $\varepsilon > 0$; we show that $\theta(\bigcup_{n\in\omega} A_n) \leq \sum_{n\in\omega} \theta(A_n) + \varepsilon$, and the arbitrariness of ε then gives the desired result. For each $n \in \omega$ choose $C^n \in {}^{\omega}\mathcal{C}$ such that $A_n \subseteq \bigcup_{m\in\omega} C^n_m$ and $\sum_{m\in\omega} \theta_0(C^n_m) \leq \theta(A_n) + \frac{\varepsilon}{2^n}$. Then $\bigcup_{n\in\omega} A_n \subseteq \bigcup_{n\in\omega} \bigcup_{m\in\omega} C^n_m$ and

$$\theta\left(\bigcup_{n\in\omega} A_n\right) \leq \sum_{n\in\omega}\sum_{n\in\omega} \theta_0(C^n_m) \leq \sum_{n\in\omega} \theta(A_n) + \varepsilon,$$

as desired. ☐

Let Σ_0 be the set of all θ-measurable subsets of ${}^{\omega}2$.

Proposition 15.53. *If $\varepsilon \in 2$ and $\alpha < \kappa$, then $\{f \in {}^{\kappa}2 : f(\alpha) = \varepsilon\} \in \Sigma_0$.*

Proof. Let $E = \{f \in {}^{\kappa}2 : f(\alpha) = \varepsilon\}$, and let $X \subseteq {}^{\kappa}2$; we want to show that $\theta(X) = \theta(X \cap E) + \theta(X \backslash E)$. $\leq$ holds by the definition of outer measure. Now suppose that $\delta > 0$. Choose $C \in {}^{\omega}\mathcal{C}$ such that $X \subseteq \bigcup_{n\in\omega} C_n$ and $\sum_{n\in\omega} \theta_0(C_n) < \theta(X) + \delta$. For each $n \in \omega$ let $C_n = U_{f_n}$ with $f_n \in P$. For each $n \in \omega$, if $\alpha \notin \mathrm{dmn}(f_n)$, replace C_n by U_g and U_h, where $g = f_n \cup \{(\alpha, 0)\}$ and $h = f_n \cup \{(\alpha, 1)\}$; let the new sequence be $C' \in {}^{\omega}\mathcal{C}$. Note that

$$\theta_0(C_n) = \theta_0(U_{f_n}) = \frac{1}{2^{|\mathrm{dmn}(f_n)|}} = \theta_0(U_g) + \theta_0(U_h).$$

Then $\sum_{n\in\omega} \theta(C_n) = \sum_{n\in\omega} \theta(C'_n)$ and $X \subseteq \bigcup_{n\in\omega} C'_n$. Say $C'_n = U_{g_n}$ for each $n \in \omega$. Note that $\alpha \in \mathrm{dmn}(g_n)$ for each $n \in \omega$. Let $M = \{n \in \omega : g_n(\alpha) = \varepsilon\}$ and $N = \{n \in \omega : g_n(\alpha) = 1 - \varepsilon\}$. Then M, N is a partition of ω such that $X \cap E \subseteq \bigcup_{n\in M} C'_n$ and $X \backslash E \subseteq \bigcup_{n\in N} C'_n$. Hence

$$\theta(X \cap E) + \theta(X \backslash E) \leq \sum_{n\in M} \theta(C'_n) + \sum_{n\in N} \theta(C'_n) = \sum_{n\in\omega} \theta(C'_n) < \theta(X) + \delta.$$

Since δ is arbitrary, it follows that $\theta(X) = \theta(X \cap E) + \theta(X \backslash E)$. $\square$

For $f : 2 \to \mathbb{R}$ we define $\int f = \frac{1}{2} f(0) + \frac{1}{2} f(1)$.

Proposition 15.54. *If $f_n : 2 \to [0, \infty)$ for each $n \in \omega$ and $\forall t < 2[\sum_{n\in\omega} f_n(t) < \infty]$, then $\sum_{n\in\omega} \int f_n < \infty$, and $\sum_{n\in\omega} \int f_n = \int \sum_{n\in\omega} f_n$.*

Proof.

$$\int \sum_{n\in\omega} f_n = \frac{1}{2}\sum_{n\in\omega} f_n(0) + \frac{1}{2}\sum_{n\in\omega} f_n(1) = \sum_{n\in\omega}\left(\frac{1}{2}f_n(0) + \frac{1}{2}f_n(1)\right) = \sum_{n\in\omega}\int f_n.$$

Proposition 15.55. $\theta({}^\kappa 2) = 1$.

Proof. It is obvious that ${}^\kappa 2 \in \Sigma_0$, and that $\theta({}^\kappa 2) \leq \theta_0({}^\kappa 2) = 1$. Suppose that $\theta({}^\kappa 2) < 1$. Choose $C \in {}^\omega\mathcal{C}$ such that $2^\kappa = \bigcup_{n\in\omega} C_n$ and $\sum_{n\in\omega} \theta_0(C_n) < 1$, with C one-one. For each $n \in \omega$ let $C_n = U_{f_n}$, where $f_n \in P$.

(1) $\forall g \in \mathrm{Fn}(\kappa, 2, \omega)\exists n \in \omega[f_n \subseteq g$ or $g \subseteq f_n]$.

In fact, let $g \in \mathrm{Fn}(\kappa, 2, \omega)$. Let $h \in {}^\kappa 2$ with $g \subseteq h$. Choose n such that $h \in C_n$. Then $f_n \subseteq h$. So $f_n \subseteq g$ or $g \subseteq f_n$.

(2) Let $M = \{n \in \omega : \forall m \neq n[f_m \not\subseteq f_n]\}$. Then ${}^\kappa 2 \subseteq \bigcup_{n\in M} U_{f_n}$.

For, given $g \in {}^\kappa 2$ choose $m \in \omega$ such that $g \in C_m$. Thus $f_m \subseteq g$. Let $n \in \omega$ with $f_n \subseteq f_m$ and $|\mathrm{dmn}(f_n)|$ minimum. Then $f_n \subseteq g$ and $n \in M$, as desired.

(3) $|M| \geq 2$.

In fact, obviously $M \neq \emptyset$. Suppose that $M = \{n\}$. Since $\sum_{n\in M} \theta_0(C_n) < 1$, we have $f_n \neq \emptyset$. Then ${}^\kappa 2 \subseteq U_{f_n}$, contradiction.

(4) M is infinite.

In fact, suppose that M is finite, and let $m = \sup\{|\mathrm{dmn}(f_n)| : n \in M\}$. Let $g \in \mathrm{Fn}(\kappa, 2, \omega)$ be such that $|\mathrm{dmn}(g)| = m+1$. Then by (1), $f_n \subseteq g$ for all $n \in M$. By (3), this contradicts the definition of M.

Let $J = \bigcup_{n\in M} \mathrm{dmn}(f_n)$.

(5) J is infinite.

For, suppose that J is finite. Now $M = \bigcup_{G\subseteq J}\{n \in M : \mathrm{dmn}(f_n) = G\}$, so there is a $G \subseteq J$ such that $\{n \in M : \mathrm{dmn}(f_n) = G\}$ is infinite. But clearly $|\{n \in M : \mathrm{dmn}(f_n) = G\}| \leq 2^{|G|}$, contradiction.

Let $i : \omega \to J$ be a bijection. For $n, k \in \omega$ let f'_{nk} be the restriction of f_n to the domain $\{\alpha \in \mathrm{dmn}(f_n) : \forall j < k[\alpha \neq i_j]\}$, and let

$$\alpha_{nk} = \frac{1}{2^{|\mathrm{dmn}(f'_{nk})|}}.$$

Now for $n, k \in \omega$ and $t < 2$ we define

$$\varepsilon_{nk}(t) = \begin{cases} \alpha_{n,k+1} & \text{if } i_k \notin \mathrm{dmn}(f_n), \\ \alpha_{n,k+1} & \text{if } i_k \in \mathrm{dmn}(f_n) \text{ and } f_n(i_k) = t, \\ 0 & \text{otherwise.} \end{cases}$$

(6) $\int \varepsilon_{nk} = \alpha_{nk}$ for all $n, k \in \omega$.

In fact,

$$\begin{aligned} \int \varepsilon_{nk} &= \frac{1}{2}\varepsilon_{nk}(0) + \frac{1}{2}\varepsilon_{nk}(1) \\ &= \begin{cases} \alpha_{n,k+1} & \text{if } i_k \notin \mathrm{dmn}(f_n), \\ \frac{1}{2}\alpha_{n,k+1} & \text{if } i_k \in \mathrm{dmn}(f_n) \end{cases} \\ &= \alpha_{nk}. \end{aligned}$$

Now we define by induction elements $t_k \in 2$ and subsets M_k of M. Let $M_0 = M$. Note that

$$\alpha_{n0} = \frac{1}{2^{|\mathrm{dmn}(f_n)|}}; \quad \sum_{n \in M} \alpha_{n0} = \sum_{n \in M} \frac{1}{2^{|\mathrm{dmn}(f_n)|}} = \sum_{n \in M} \theta_0(C_n) < 1.$$

Now suppose that M_k and t_i have been defined for all $i < k$, so that $\sum_{n \in M_k} \alpha_{nk} < 1$. Note that this holds for $k = 0$. Now

$$\begin{aligned} 1 > \sum_{n \in M_k} \alpha_{nk} &= \sum_{n \in M_k} \int \varepsilon_{nk} \quad \text{by (6)} \\ &= \int \sum_{n \in M_k} \varepsilon_{nk} \quad \text{by Proposition 54.} \end{aligned}$$

It follows that there is a $t_k < 2$ such that $\left(\sum_{n \in M_k} \varepsilon_{nk}\right)(t_k) < 1$. Let

$$M_{k+1} = \{n \in M : \forall j < k+1[i_j \notin \mathrm{dmn}(f_n), \text{ or } i_j \in \mathrm{dmn}(f_n) \text{ and } f_n(i_j) = t_j]\}.$$

If $n \in M_{k+1}$, then $\varepsilon_{nk}(t_k) = \alpha_{n,k+1}$. Hence

$$\sum_{n \in M_{k+1}} \alpha_{n,k+1} = \sum_{n \in M_{k+1}} \varepsilon_{nk}(t_k) \leq \left(\sum_{n \in M_k} \varepsilon_{nk}\right)(t_k) < 1.$$

Also, $M_{k+1} \neq \emptyset$. For, let $g \in {}^{\kappa}2$ such that $g(i_j) = t_j$ for all $j \leq k$. Say $g \in C_n$ with $n \in M$. Then $f_n \subseteq g$. Hence $i_j \notin \mathrm{dmn}(f_n)$, or $i_j \in \mathrm{dmn}(f_n)$ and $f_n(i_j) = t_j$. Thus $n \in M_{k+1}$.

This finishes the construction. Now let $g \in {}^{\kappa}2$ be such that $g(i_j) = t_j$ for all $j \in \omega$. Say $g \in C_n$ with $n \in M$. Then $f_n \subseteq g$. The domain of f_n is a finite subset of J. Choose $k \in \omega$ so that $\mathrm{dmn}(f_n) \subseteq \{i_j : j < k\}$. Then $n \in M_k$. Hence $f'_{nk} = \emptyset$ and so $\alpha_{nk} = 1$. This contradicts $\sum_{m \in M_k} \alpha_{mk} < 1$. $\square$

Let ν be the tiny function with domain 2 which interchanges 0 and 1. For any $f \in {}^{\kappa}2$ let $F(f) = \nu \circ f$.

Proposition 15.56.
(i) F is a permutation of ${}^{\kappa}2$.
(ii) For any $f \in \mathrm{Fn}(\kappa, 2, \omega)$ we have $F[U_f] = U_{\nu \circ f}$.
(iii) For any $X \subseteq {}^{\kappa}2$ we have $\theta(X) = \theta(F[X])$.
(iv) $\forall E \in \Sigma_0[F[E] \in \Sigma_0]$.

Proof. (i): Clearly F is one-one, and $F(F(f)) = f$ for any $f \in {}^{\kappa}2$. So (i) holds.
(ii): For any $g \in {}^{\kappa}2$,

$$\begin{aligned} g \in F[U_f] \quad &\text{iff} \quad \exists h \in U_f[g = F(h)] \\ &\text{iff} \quad \exists h \in {}^{\kappa}2[f \subseteq h \text{ and } g = \nu \circ h] \\ &\text{iff} \quad \exists h \in {}^{\kappa}2[\nu \circ f \subseteq \nu \circ h \text{ and } g = \nu \circ h] \\ &\text{iff} \quad \nu \circ f \subseteq g \\ &\text{iff} \quad g \in U_{\nu \circ f} \end{aligned}$$

(iii): Clearly $\theta_0(U_f) = \theta_0(F[U_f])$ for any $f \in \mathrm{Fn}(\kappa, 2, \omega)$. Also, $A \subseteq \bigcup_{n\in\omega} C_n$ iff $F[A] \subseteq \bigcup_{n\in\omega} F[C_n]$. So (iii) holds.
(iv): Suppose that $E \in \Sigma_0$. Let $X \subseteq {}^{\kappa}2$. Then

$$\begin{aligned} \theta(X \cap F[E]) + \theta(X \backslash F[E]) &= \theta(F[F[X]] \cap F[E]) + \theta(F[F[X]] \backslash F[E]) \\ &= \theta(F[F[X] \cap E]) + \theta(F[F[X] \backslash E]) \\ &= \theta(F[X] \cap E) + \theta(F[X] \backslash E) \\ &= \theta(E) = \theta(F[E]). \end{aligned}$$

□

Proposition 15.57. *If $\alpha < \kappa$ and $\varepsilon < 2$, then $\theta(U_{\{(\alpha,\varepsilon)\}}) = \frac{1}{2}$.*

Proof. By Proposition 15.55 we have $\theta(U_{\{(\alpha,\varepsilon)\}}) = \theta(U_{\{(\alpha,1-\varepsilon)\}})$, so the result follows from Proposition 15.54. □

Proposition 15.58. *For each $f \in P$ we have $U_f \in \Sigma_0$ and $\theta(U_f) = \frac{1}{2^{|\mathrm{dmn}(f)|}}$.*

Proof. We have $U_f = \bigcap_{\alpha\in\mathrm{dmn}(f)} U_{\{(\alpha,f(\alpha))\}}$. Note that if $\alpha \in \mathrm{dmn}(f)$, then $U_{\{(\alpha,f(\alpha))\}} = \{g \in {}^{\kappa}2 : g(\alpha) = f(\alpha)\}$; hence $U_{\{(\alpha,f(\alpha))\}} \in \Sigma_0$ by Proposition 15.52, and so $U_f \in \Sigma_0$. We prove that $\theta(U_f) = \frac{1}{2^{|\mathrm{dmn}(f)|}}$ by induction on $|\mathrm{dmn}(f)|$. For $|\mathrm{dmn}(f)| = 1$, this holds by Proposition 15.57. Now assume that it holds for $|\mathrm{dmn}(f)| = m$. For any f with $|\mathrm{dmn}(f)| = m$ and $\alpha \notin \mathrm{dmn}(f)$ we have $2^{-|\mathrm{dmn}(f)|} = \theta(U_f) = \theta(U_{f\cup\{(\alpha,0)\}}) + \theta(U_{f\cup\{(\alpha,1)\}})$. Since $\theta(U_{f\cup\{(\alpha,\varepsilon)\}}) \le \theta_0(U_{f\cup\{(\alpha,\varepsilon)\}}) = 2^{-|\mathrm{dmn}(f)|-1}$ for each $\varepsilon \in 2$, it follows that $\theta(U_{f\cup\{(\alpha,\varepsilon)\}}) = 2^{-|\mathrm{dmn}(f)|-1}$ for each $\varepsilon \in 2$. □

Proposition 15.59. *If F is a finite subset of ${}^{\kappa}2$, then $F \in \Sigma_0$ and $\theta(F) = 0$.*

Proof. This is obvious if $F = \emptyset$. For $F = \{f\}$ we have $F \subseteq U_{f\restriction n}$ for each $n \in \omega$, and so $\theta(F) = 0$. Then it is clear that $F \in \Sigma_0$. Now the general case follows easily. $\square$

Proposition 15.60. *If $X \subseteq {}^\kappa 2$ is measurable, then $\theta(X) = \inf\{\varphi(U) : X \subseteq U$ and U is open$\}$.*

Proof. By Proposition 15.58, $\theta(U_f) = \theta_0(U_f)$ for each $f \in \mathrm{Fn}(\kappa, 2, \omega)$. Hence by the definition preceding Proposition 15.52,

$$\begin{aligned}
\theta(X) &\leq \inf\left\{\theta\left(\bigcup_{n\in\omega} U_{f_n}\right) : f \in {}^\omega\mathrm{Fn}(\kappa, 2, \omega),\ X \subseteq \bigcup_{n\in\omega} U_{f_n}\right\}\\
&\leq \inf\left\{\sum\{\theta(U_{f_n}) : f \in {}^\omega\mathrm{Fn}(\kappa, 2, \omega),\ X \subseteq \bigcup_{n\in\omega} U_{f_n}\right\}\\
&= \inf\left\{\sum\{\theta_0(U_{f_n}) : f \in {}^\omega\mathrm{Fn}(\kappa, 2, \omega),\ X \subseteq \bigcup_{n\in\omega} U_{f_n}\right\}\\
&= \theta(X).
\end{aligned}$$

$\square$

Proposition 15.61. *If $X \subseteq {}^\kappa 2$ is measurable, then there is a system $\langle f^n_m : n, m \in \omega\rangle$ with each $f^n_m \in P$ such that $X \subseteq \bigcap_{n\in\omega}\bigcup_{m\in\omega} U_{f^n_m}$ and $\theta((\bigcap_{n\in\omega}\bigcup_{m\in\omega} U_{f^n_m})\backslash X) = 0$.*

Proof. By the proof of Proposition 15.60, for each $n \in \omega$ let $\langle f^n_m : m \in \omega\rangle$ be such that each $f^n_m \in \mathrm{Fn}(\kappa, 2, \omega)$, $X \subseteq \bigcup_{m\in\omega} U_{f^n_m}$, and $\theta(\bigcup_{m\in\omega} U_{f^n_m}) - \theta(X) \leq \frac{1}{n+1}$. Then

$$\forall n \in \omega\left[X \subseteq \bigcap_{p\in\omega}\bigcup_{m\in\omega} U_{f^p_m} \subseteq \bigcup_{m\in\omega} U_{f^n_m}\right];$$

$$\forall n \in \omega\left[\theta\left(\bigcap_{n\in\omega}\bigcup_{m\in\omega} U_{f^n_m}\right) - \theta(X) \leq \theta\left(\bigcup_{m\in\omega} U_{f^n_m}\right) - \theta(X) \leq \frac{1}{n+1}\right];$$

$$\theta\left(\bigcap_{n\in\omega}\bigcup_{m\in\omega} U_{f^n_m}\right) - \theta(X) = 0;$$

$$\theta\left(\bigcap_{n\in\omega}\bigcup_{m\in\omega} U_{f^n_m}\right) = \theta\left(\left(\bigcap_{n\in\omega}\bigcup_{m\in\omega} U_{f^n_m}\right)\backslash X\right) + \theta(X);$$

$$\theta\left(\left(\bigcap_{n\in\omega}\bigcup_{m\in\omega} U_{f^n_m}\right)\backslash X\right) = 0.$$

$\square$

By Proposition 15.58, each set U_f for $f \in P$ is measurable. Let $\mathscr{S}$ be the σ-algebra of subsets of ${}^\kappa 2$ generated by $\{U_f : f \in P\}$, and let I_θ be the ideal of elements of $\mathscr{S}$ of measure 0.

Lemma 15.62. *$\mathscr{S}/I_\theta$ has ccc.*

Proof. See the proof of Lemma 15.46. $\square$

Theorem 15.63. *Suppose that M is a ctm for ZFC, κ is an infinite cardinal in M. Let G be a $(\mathscr{S}/I_\theta)$-generic ultrafilter over M.*

Then M and $M[G]$ have the same cardinals and cofinalities,and $(2^{\aleph_0})^{M[G]} \geq \kappa$.

Proof. M and $M[G]$ have the same cardinals and cofinalities by Lemma 15.69. Let $k : \kappa \times \omega \to \kappa$ be a bijection (in M). Now for $\alpha < \kappa$, $n \in \omega$ and $i \in 2$ let $p_i^{\alpha n} = U_{\{(k(\alpha,n),i)\}}$. Then $p_i^{\alpha n}$ is measurable and $\mu(p_i^{\alpha n}) = \frac{1}{2}$, by Proposition 15.57 $\{p_0^{\alpha n}, p_1^{\alpha n}\}$ is a maximal antichain, so exactly one of them is in G. Define $F(\alpha, n)$ to be the $i \in 2$ such that $p_i^{\alpha n} \in G$. Let $h_\alpha(n) = F(\alpha, n)$ for all $\alpha < \kappa$ and $n \in \omega$. Now suppose that $\alpha, \beta \in \kappa$, $\alpha \neq \beta$, and $f \in {}^m 2$. Define

$$\begin{aligned}&\mathrm{dmn}(g_{\alpha\beta f}) = \{k(\alpha, n) : n < m\} \cup \{k(\beta, n) : n < m\} \quad \text{and}\\ &\forall n < m[g_{\alpha\beta f}(k(\alpha, n)) = g_{\alpha\beta f}(k(\beta, n)) = f(n)].\end{aligned}$$

Then $|\mathrm{dmn}(g_{\alpha\beta f})| = 2m$ and so $|U_{g_{\alpha\beta f}}| = \frac{1}{2^{2m}}$. If $f, f' \in {}^m 2$ and $f \neq f'$ then $U_{\alpha\beta f} \cap U_{\alpha\beta f'} = \emptyset$. Hence

$$\theta\left(\bigcup_{f \in {}^m 2} U_{g_{\alpha\beta f}}\right) = 2^m \cdot \frac{1}{2^{2m}} = \frac{1}{2^m}.$$

Now

$$\bigcup_{f \in {}^m 2} U_{g_{\alpha\beta f}} = \{s \in {}^\kappa 2 : \forall n < m[s(k(\alpha, n)) = s(k(\beta, n))]\}.$$

Hence

$$\bigcap_{m \in \omega} \bigcup_{f \in {}^m 2} U_{g_{\alpha\beta f}}$$

has measure 0, and hence

$$\bigcup_{m \in \omega} \bigcap_{f \in {}^m 2} -U_{g_{\alpha\beta f}} \in G.$$

so we can choose $m \in \omega$ such that $\bigcap_{f \in {}^m 2} -U_{g_{\alpha\beta f}} \in G$. Now

$$\begin{aligned}\bigcap_{f \in {}^m 2} -U_{g_{\alpha\beta f}} &= \{s \in {}^\kappa 2 : \exists n < m[s(k(\alpha, n)) \neq s(k(\beta, n))]\}\\ &= \bigcup_{n<m} \{s \in {}^\kappa 2 : s(k(\alpha, n)) \neq s(k(\beta, n))\}\end{aligned}$$

so we can choose $n < m$ so that $\{s \in {}^\kappa 2 : s(k(\alpha, n)) \neq s(k(\beta, n))\} \in G$. Now

$$\begin{aligned}\{s \in {}^\kappa 2 : s(k(\alpha, n)) \neq s(k(\beta, n))\} &= \{s \in {}^\kappa 2 : s(k(\alpha, n)) = 0 \text{ and } s(k(\beta, n)) = 1\}\\ &\cup \{s \in {}^\kappa 2 : s(k(\alpha, n)) = 1 \text{ and } s(k(\beta, n)) = 0\}\end{aligned}$$

By symmetry say $\{s \in {}^\kappa 2 : s(k(\alpha, n)) = 0 \text{ and } s(k(\beta, n)) = 1\} \in G$. Hence $\{s \in {}^\kappa 2 : s(k(\alpha, n)) = 0\} \in G$. Now $h_\alpha(n) = F(\alpha, n)$ is the unique i such that $p_i^{\alpha n} \in G$, and

$p_i^{\alpha n} = U_{\{(k(\alpha,n),i)\}} = \{s \in {}^\kappa 2 : s(k(\alpha,n)) = i\}$. Hence $h_\alpha(n) = 0$. Similarly, $h_\beta(n) = 1$. So $h_\alpha \neq h_\beta$.

It follows that $(2^\omega)^{M[G]} \geq \kappa$. $\square$

A *special tree* is a subset T of ${}^{<\omega}2$ such that if $t \in T$ and $m \in \mathrm{dmn}(t)$ then $t \restriction m \in T$. A nonempty special tree T is *perfect* iff $\forall t \in T \exists s \supseteq t[s^\frown\langle 0\rangle, s^\frown\langle 1\rangle \in T]$. A *path* in T is a sequence $a \in {}^\omega 2$ such that $\forall m \in \omega[a \restriction m \in T]$.

Lemma 15.64. *The collection of paths in a perfect tree T is a perfect subset of ${}^\omega 2$.*

Proof. Let P be the collection of paths in the perfect tree T. P is closed: suppose that $a \in {}^\omega 2 \backslash P$. Then there is an $m \in \omega$ such that $a \restriction m \notin T$. Then $a \in U_{a \restriction m} \subseteq {}^\omega 2 \backslash P$.

It is dense in itself: suppose that $p \in P$ and $p \in U_{p \restriction m}$. Now $p \restriction m \in T$. Choose $s \in T$ with $p \restriction m \subseteq s$ and $s0, s1 \in T$. Say $s\varepsilon \not\subseteq p$. Extend $s\varepsilon$ to a path q. Then $q \in U_{p \restriction m} \backslash \{p\}$. $\square$

P_{perf} is the set of all perfect trees with the order $\subseteq$.

Note that an intersection of perfect trees does not have to be perfect. For example (with $\varepsilon_1, \varepsilon_2, \ldots$ any members of 2):

$$\begin{aligned} p &= \{\emptyset, \langle 0\rangle, \langle 0\varepsilon_1\rangle, \langle 0\varepsilon_1\varepsilon_2\rangle, \ldots\}; \\ q &= \{\emptyset, \langle 1\rangle, \langle 1\varepsilon_1\rangle, \langle 1\varepsilon_1\varepsilon_2\rangle, \ldots\}. \end{aligned}$$

Also, one can have p, q perfect, $p \cap q$ not perfect, but $r \subseteq p \cap q$ for some perfect r:

$$\begin{aligned} p = \{&\emptyset, \langle 1\rangle, \langle 1\varepsilon_1\rangle, \langle 1\varepsilon_1\varepsilon_2\rangle, \ldots \\ &\langle 0\rangle, \langle 01\rangle, \langle 01\varepsilon_2\rangle, \langle 01\varepsilon_2\varepsilon_3\rangle \ldots\}; \\ q = \{&\emptyset, \langle 1\rangle, \langle 1\varepsilon_1\rangle, \langle 1, \varepsilon_1\varepsilon_2\rangle, \ldots \\ &\langle 0\rangle, \langle 00\rangle, \langle 00\varepsilon_2\rangle, \langle 00\varepsilon_2\varepsilon_3\rangle \ldots\}; \\ r = \{&\emptyset, \langle 1\rangle, \langle 1\varepsilon_1\rangle, \langle 1, \varepsilon_1\varepsilon_2\rangle, \ldots\}. \end{aligned}$$

Theorem 15.65. *Suppose that M is a c.t.m. of ZFC. Consider P_{perf} within M, and let G be P_{perf}-generic over M. Then the set*

$$\{s \in {}^{<\omega}2 : s \in p \textit{ for all } p \in G\}$$

is a function from ω into 9.

Proof. For each $n \in \omega$ let

$$\begin{aligned} D_n = \{p \in P_{perf} : &\text{ there is an } s \in {}^{<\omega}2 \text{ such that } \mathrm{dmn}(s) = n \\ &\text{and } s \subseteq t \text{ or } t \subseteq s \text{ for all } t \in p\}. \end{aligned}$$

Then D_n is dense: if $q \in P_{perf}$, choose any $s \in q$ such that $\mathrm{dmn}(s) = n$, and let $p = \{t \in q : s \subseteq t \text{ or } t \subseteq s\}$. Clearly $p \in D_n$ and $p \subseteq q$.

Now for each $n \in \omega$ let $p^{(n)}$ be a member of $G \cap D_n$, and choose $s^{(n)}$ accordingly.

(1) If $m < n$, then $s^{(m)} \subseteq s^{(n)}$.

In fact, choose $r \in G$ such that $r \subseteq p^{(m)} \cap p^{(n)}$. Then $s^{(m)} \subseteq t$ and $s^{(n)} \subseteq t$ for all $t \in r$ with $\mathrm{dmn}(t) \geq n$, so $s^{(m)} \subseteq s^{(n)}$.

(2) $s^{(m)} \in q$ for all $q \in G$.

In fact, let $q \in G$, and choose $r \in G$ such that $r \subseteq q$ and $r \subseteq p^{(m)}$. Take $t \in r$ with $\mathrm{dmn}(t) = m$. then $t = s^{(m)}$ since $r \subseteq p^{(m)}$. Thus $s^{(m)} \in q$ since $r \subseteq q$.

(3) If $t \in q$ for all $q \in G$, then $t = s^{(m)}$ for some m.

For, let $\mathrm{dmn}(t) = m$. Since $t \in p^{(m)}$, we have $t = s^{(m)}$.

From (1)–(3) the conclusion of the theorem follows. □

The function described in Theorem 15.65 is called a *Sacks real.*

If $p \in P_{perf}$, a member f of p is a *branching point* iff $f^\frown\langle 0\rangle, f^\frown\langle 1\rangle \in p$. Sacks forcing does not satisfy ccc:

Proposition 15.66. *There is a family of 2^ω pairwise incompatible members of P_{perf}.*

Proof. Let $\mathscr{A}$ be a family of 2^ω infinite pairwise almost disjoint subsets of ω. With each $A \in \mathscr{A}$ we define a sequence $\langle P_{A,n} : n \in \omega\rangle$ of subsets of ${}^{<\omega}2$, by recursion:

$$P_{A,0} = \{\emptyset\};$$
$$P_{A,n+1} = \begin{cases} \{f^\frown\langle 0\rangle : f \in P_{A,n}\} & \text{if } n \notin A, \\ \{f^\frown\langle 0\rangle : f \in P_{A,n}\} \cup \{f^\frown\langle 1\rangle : f \in P_{A,n}\} & \text{if } n \in A. \end{cases}$$

Note that all members of $P_{A,n}$ have domain n. We set $p_A = \bigcup_{n\in\omega} P_{A,n}$. We claim that p_A is a perfect tree. The first condition is clear. For the second condition, suppose that $f \in p_A$; say $f \in P_{A,n}$. Let m be the least member of A greater than n. If g extends f by adjoining 0's from n to $m-1$, then $g^\frown\langle 0\rangle, g^\frown\langle 1\rangle \in p_A$, as desired in the second condition.

We claim that if $A, B \in \mathscr{A}$ and $A \neq B$, then p_A and p_B are incompatible. For, suppose that q is a perfect tree and $q \subseteq p_A, p_B$. Now $A \cap B$ is finite. Let m be an integer greater than each member of $A \cap B$. Let f be a branching point of q with $\mathrm{dmn}(f) \geq m$; it exists by the definition of perfect tree. Let $\mathrm{dmn}(f) = n$. Then $f \in P_{A,n}$ and $f^\frown\langle 0\rangle, f^\frown\langle 1\rangle \in P_{A,n+1}$, so $n \in A$ by construction. Similarly, $n \in B$, contradiction. □

Proposition 15.67. *P_{perf} is not ω_1-closed.*

Proof. For each $n \in \omega$ let

$$p_n = \{f \in {}^{<\omega}2 : f(i) = 0 \text{ for all } i < n\}.$$

Clearly p_n is perfect, $p_n \subseteq p_m$ if $n > m$, and $\bigcap_{n\in\omega} P_n$ is $\{f\}$ with $f(i) = 0$ for all i, so that the descending sequence $\langle p_n : n \in \omega\rangle$ does not have any member of P_{perf} below it. □

By Propositions 15.66 and 15.67, the previous methods cannot be used to show that forcing with P_{perf} preserves cardinals, even if we assume CH in the ground model. Nevertheless, we will show that it does preserve cardinals. To do this we will prove a modified version of ω_1-closure.

If p is a perfect tree, an *n-th branching point* of p is a branching point f of p such that there are exactly n branching points g such that $g \subseteq f$. Thus $n > 0$. For perfect trees p, q and n a positive integer, we write $p \leq_n q$ iff $p \subseteq q$ and every n-th branching point of q is a branching point of p. Also we write $p \leq_0 q$ iff $p \subseteq q$.

Lemma 15.68. *Suppose that $p \subseteq q$ are perfect trees, and $n \in \omega$. Then:*

(i) If $p \leq_n q$, then $p \leq_i q$ for every $i < n$.

(ii) If $p \leq_n q$ and f is an n-th branching point of q, then f is an n-th branching point of p.

(iii) For each positive integer n there is an $f \in p$ such that f is an n-th branching point of q.

(iv) The following conditions are equivalent:

(a) $p \leq_n q$.

(b) For every $f \in {}^{<\omega}2$, if f is an n-th branching point of q, then $f^\frown\langle 0\rangle, f^\frown\langle 1\rangle \in p$.

(v) For each positive integer n there are exactly 2^{n-1} n-th branching points of a perfect tree p.

(vi) If p and q are perfect trees, then so is $p \cup q$.

(vii) If p and q are perfect trees, then $\{r : r$ is a perfect tree and $r \subseteq p$ or $r \subseteq q\}$ is dense below $p \cup q$.

Proof. (i): Assume that $p \leq_n q$, $i < n$, and f is an i-th branching point of q. Then since q is perfect there are n-th branching points g, h of q such that $f^\frown\langle 0\rangle \subseteq g$ and $f^\frown\langle 1\rangle \subseteq h$. So $g, h \in p$, hence $f \in p$. This shows that $p \leq_i q$.

(ii): Suppose that $p \leq_n q$ and f is an n-th branching point of q. Let $r_0, \ldots, r_{n-1}$ be all of the branching points g of q such that $g \subseteq f$. Then by (i), $r_0, \ldots, r_{n-1}$ are all branching points of p. Hence f is an n-th branching point of p.

(iii): Let f be an n-th branching point of p. Then it is an m-th branching point of q for some $m \geq n$. Let r be an n-th branching point of q below f. Then $r \in p$, as desired. [But r might not be a branching point of p.]

(iv), (v), (vi): Immediate from the definitions.

(vii): Suppose that p, q, t are perfect trees and $t \subseteq p \cup q$; we want to find a perfect tree $r \subseteq t$ such that $r \subseteq p$ or $r \subseteq q$. If $t \subseteq p \cap q$, then $r = t$ works. Otherwise, there is some member f of t which is not in both p and q; say $f \in p \backslash q$. Then $r \stackrel{\text{def}}{=} \{g \in t : g \subseteq f$ or $f \subseteq g\}$ is a perfect tree with $r \subseteq t$ and $r \subseteq p$. $\square$

Lemma 15.69. (Fusion lemma) *If $\langle p_n : n \in \omega\rangle$ is a sequence of perfect trees and $\cdots \leq_n p_n \leq_{n-1} \cdots \leq_2 p_2 \leq_1 p_1 \leq_0 p_0$, then $q \stackrel{\text{def}}{=} \bigcap_{n\in\omega} p_n$ is a perfect tree, and $q \leq_n p_n$ for all $n \in \omega$.*

Proof. Let n be a positive integer, and let s be an n-th branching point of p_n. If $n \leq m$, then $p_m \leq_n p_n$, so s is an n-th branching point of p_m; hence $s, s^\frown\langle 0\rangle, s^\frown\langle 1\rangle \in p_m$. It follows that $s, s^\frown\langle 0\rangle, s^\frown\langle 1\rangle \in q$, and s is a branching point of q. Thus we just need to show that q is a perfect tree.

Clearly if $t \in q$ and $n < \mathrm{dmn}(t)$, then $t \restriction n \in q$. Now suppose that $s \in q$; we want to find a $t \in q$ with $s \leq t$ and t is a branching point of q. Let $n = \mathrm{dmn}(s)$. Now $s \in p_n$, and p_n has fewer than n elements less than s, so p_n has an n-th branching point $t \geq s$. By the first paragraph, $t \in q$. $\square$

Let p be a perfect tree and $s \in p$. We define

$$p \restriction s = \{t \in p : t \subseteq s \text{ or } s \subseteq t\}.$$

Clearly $p \restriction s$ is still a perfect tree. Now for any positive integer n, let $t_0, \ldots, t_{2^n-1}$ be the collection of all immediate successors of n-th branching points of p. Suppose that for each $i < 2^n$ we have a perfect tree $q_i \leq p \restriction t_i$. Then we define the *amalgamation of* $\{q_i : i < 2^n\}$ *into* p to be the set $\bigcup_{i<2^n} q_i$.

Lemma 15.70. *Under the above assumptions, the amalgamation r of $\{q_i : i < 2^n\}$ into p has the following properties:*

(i) r is a perfect tree.

(ii) $r \leq_n p$.

Proof. (i): Suppose that $f \in r$, $g \in {}^{<\omega}2$, and $g \subseteq f$. Say $f \in q_i$ with $i < 2^n$. Then $g \in q_i$, so $g \in r$. Now suppose that $f \in r$; we want to find a branching point of r above f. Say $f \in q_i$. Let g be a branching point of q_i with $f \subseteq g$. Clearly g is a branching point of r.

(ii): Suppose that f is an n-th branching point of p. Then there exist $i, j < 2^n$ such that $f^\frown\langle 0\rangle = t_i$ and $f^\frown\langle 1\rangle = t_j$. So $f^\frown\langle 0\rangle \in q_i \subseteq r$ and $f^\frown\langle 1\rangle = t_j \in q_j \subseteq r$, and so f is a branching point of r. $\square$

Lemma 15.71. *Suppose that M is a c.t.m. of ZFC and we consider the Sacks partial order P_{perf} within M. Suppose that $B \in M$, $\tau \in M^{P_{perf}}$, $p \in P_{perf}$, and $p \Vdash \tau : \check\omega \to \check B$. Then there is a $q \leq p$ and a function $F : \omega \to [B]^{<\omega}$ in M such that $q \Vdash \tau(\check n) \in \check F_n$ for every $n \in \omega$.*

Proof. We work entirely within M, except as indicated. We construct two sequences $\langle q_n : n \in \omega\rangle$ and $\langle F_n : n \in \omega\rangle$ by recursion. Let $q_0 = p$. Suppose that q_n has been defined; we define F_n and q_{n+1}. Assume that $q_n \leq p$. Then $q_n \Vdash \tau : \check\alpha \to \check B$, so $q_n \Vdash \exists x \in \check B \tau(\check n) = x)$. Let $t_0, \ldots, t_{2^n-1}$ list all of the functions $f^\frown\langle 0\rangle$ and $f^\frown\langle 1\rangle$ such that f is an n-th branching point of q_n. Then for each $i < 2^n$ we have $q_n \restriction t_i \subseteq q_n$, and so $q_n \restriction t_i \Vdash \exists x \in \check B \tau(\check n) = x)$. Hence there exist an $r_i \subseteq q_n \restriction t_i$ and a $b_i \in B$ such that $r_i \Vdash \tau(\check n) = \check b_i$. Let q_{n+1} be the amalgamation of $\{r_i : i < 2^n\}$ into q_n, and let $F_n = \{b_i : i < 2^n\}$. Thus $q_{n+1} \leq_n q_n$ by Lemma 15.70. Moreover:

$$q_{n+1} \Vdash \tau(\check n) \in \check F_n. \tag{1}$$

In fact, let G be P_{perf}-generic over M with $q_{n+1} \in G$. Then there is an i such that $r_i \in G$. Since $r_i \Vdash \tau(\check n) = \check b_i$, it follows that $\tau_G(n) \in F_n$, as desired in (1).

Now with (1) the construction is complete.

By the fusion Lemma 15.69 we get $s \leq_n q_n$ for each n. Hence the conclusion of the lemma follows. $\square$

Theorem 15.72. *If M is a c.t.m. of $ZFC + CH$ and $P_{perf} \in M$ is the Sacks forcing partial order, and if G is P_{perf}-generic over M, then cofinalities and cardinals are preserved in $M[G]$.*

Proof. Since $|P_{perf}| \leq 2^\omega = \omega_1$ by CH, the poset P_{perf} satisfies the ω_2-chain condition, and so preserves cofinalities and cardinals $\geq \omega_2$. Hence it suffices to show that ω_1^M remains regular in $M[G]$. Suppose not: then there is a function $f : \omega \to \omega_1^M$ in $M[G]$ such that $\mathrm{rng}(f)$ is cofinal in ω_1^M. Hence there is a name τ such that $f = \tau_G$, and hence there is a $p \in G$ such that $p \Vdash \tau : \check{\omega} \to \check{\omega}_1^M$. By Lemma 15.71 choose $q \leq p$ and $F : \omega \to [\omega_1^M]^{<\omega}$ in M such that $q \Vdash \tau(\check{n}) \in \check{F}_n$ for every $n \in \omega$. Take $\beta < \omega_1^M$ such that $\bigcup_{n\in\omega} F_n < \beta$. Now $q \Vdash \exists n \in \omega(\check{\beta} < \tau(\check{n})$, so there exist an $r \leq q$ and an $n \in \omega$ such that $r \Vdash \check{\beta} < \tau(\check{n})$. So we have:

(2) $r \Vdash \tau(\check{n}) \in \check{F}_n$;

(3) $\bigcup_{n\in\omega} F_n < \beta$;

(4) $r \Vdash \check{\beta} < \tau(\check{n})$.

These three conditions give the contradiction $r \Vdash \tau(\check{n}) < \tau(\check{n})$. $\square$

Suppose that G is generic over M. We say that G is *minimal over M* iff for every inner model N of ZFC such that $M \subseteq N \subseteq M[G]$, either $M = N$ or $N = M[G]$.

Theorem 15.73. *If G is P_{perf}-generic over M, then G is minimal over M.*

Proof. The conclusion is equivalent to saying that if $X \in M[G]\backslash M$ and $X \in N$ with N an inner model with $M \subseteq N \subseteq M[G]$, then $G \in N$, and hence $N = M[G]$.

We may assume that X is a set of ordinals, by AC in N.

So, suppose that X is a set of ordinals in $M[G]$ and $X \notin M$ and $X \in N$, an inner model of ZFC; we want to show that $G \in N$. Let $\dot{X}$ be a name such that $\dot{X}^G = X$. Now $1 \Vdash \neg[\dot{X} \in \check{M}]$. By Lemma 14.40(iv) this means that for all s $s \not\Vdash \dot{X} \in \check{M}$. Hence by Lemma 14.42,

$$\forall s \neg \forall q \leq s \exists b \in M \exists r \leq q[r \Vdash \dot{X} = \check{b}],$$

so

$$\forall s \exists q \leq s \forall b \in M \forall r \leq q[r \not\Vdash \dot{X} = \check{b}].$$

If s is a node in a tree $p \in P$, we define $p \restriction s = \{t \in p : t \leq s \text{ or } s \leq t\}$.

Now we define $\langle p_n : n \in \omega\rangle$ by recursion. Let $p_0 = p$. Suppose that $n > 0$ and p_{n-1} has been defined. Let S_n be the set of all n-th spliting nodes of p_{n-1}.

(1) For each $s \in S_n$ there is an ordinal γ_s such that $p_{n-1} \restriction s \not\Vdash \check{\gamma}_s \in \dot{X}$ and $p_{n-1} \restriction s \not\Vdash \check{\gamma}_s \notin \dot{X}$.

In fact, suppose not. Say $s \in S_n$ and for every ordinal γ, $p_{n-1} \upharpoonright s \Vdash \check{\gamma} \in \dot{X}$ or $p_{n-1} \upharpoonright s \Vdash \check{\gamma} \notin \dot{X}$. Let $x = \{\gamma : p_{n-1} \upharpoonright s \Vdash \check{\gamma} \in \dot{X}\}$. We claim that $p_{n-1} \upharpoonright s \Vdash \dot{X} = \check{x}$, contradiction. For,

$$||\dot{X} = \check{x}|| = ||\dot{X} \subseteq \check{x}|| \cdot ||\check{x} \subseteq \dot{X}||.$$

Now

$$||\check{x} \subseteq \dot{X}|| = \prod_{\gamma \in x} ||\check{\gamma} \in \dot{X}||,$$

and for each $\gamma \in x$, $e(p_{n-1} \upharpoonright s) \leq ||\check{\gamma} \in \dot{X}||$. Hence $e(p_{n-1} \upharpoonright s) \leq ||\check{x} \subseteq \dot{X}||$.

A general fact about complete BAs:

(2) $((\sum_{i \in I} x_i) \Rightarrow y) = \prod_{i \in I}(x_i \Rightarrow y)$.

In fact,

$$\left(\left(\sum_{i \in I} x_i\right) \Rightarrow y\right) = \left(\prod_{i \in I} -x_i\right) + y = \prod_{i \in I}(-x_i + y) = \prod_{i \in I}(x_i \Rightarrow y).$$

Next we claim

(3) $e(p) \leq \prod_{y \in \mathrm{dmn}(\dot{X})} \left(\dot{X}(y) \Rightarrow \sum_{\alpha \in \mathbf{ON}} ||y = \check{\alpha}||\right)$.

In fact, $p \Vdash \forall y \in \dot{X}[y \text{ is an ordinal}]$, so

$$\begin{aligned}
e(p) &\leq \prod_{y \in V^B} (||y \in \dot{X}|| \Rightarrow ||y \text{ is an ordinal}||) \\
&= \prod_{y \in V^B} \left(\left(\sum_{z \in \mathrm{dmn}(\dot{X})} (\dot{X}(z) \cdot ||y = z||)\right) \Rightarrow \sum_{\alpha \in \mathbf{ON}} ||y = \check{\alpha}||\right) \\
&= \prod_{y \in V^B} \prod_{z \in \mathrm{dmn}(\dot{X})} \left(\dot{X}(z) \cdot ||y = z|| \Rightarrow \sum_{\alpha \in \mathbf{ON}} ||y = \check{\alpha}||\right) \\
&= \prod_{z \in \mathrm{dmn}(\dot{X})} \prod_{y \in V^B} \left(\dot{X}(z) \cdot ||y = z|| \Rightarrow \sum_{\alpha \in \mathbf{ON}} (||y = z|| \cdot ||y = \check{\alpha}||)\right) \\
&= \prod_{z \in \mathrm{dmn}(\dot{X})} \prod_{y \in V^B} \left(\dot{X}(z) \cdot ||y = z|| \Rightarrow \sum_{\alpha \in \mathbf{ON}} (||y = z|| \cdot ||z = \check{\alpha}||)\right) \\
&\leq \prod_{z \in \mathrm{dmn}(\dot{X})} \left(\dot{X}(z) \Rightarrow \sum_{\alpha \in \mathbf{ON}} ||z = \check{\alpha}||\right).
\end{aligned}$$

Thus (3) holds.

Now

$$
\begin{aligned}
e(p_{n-1} \restriction s) &\leq \prod_{\substack{\gamma \in \mathrm{ON} \\ \gamma \notin x}} -||\check{\gamma} \in \dot{X}|| \\
&= \prod_{\substack{\gamma \in \mathrm{ON} \\ \gamma \notin x}} - \sum_{y \in \mathrm{dmn}(\dot{X})} (\dot{X}(y) \cdot ||\check{\gamma} = y||) \\
&= \prod_{\substack{\gamma \in \mathrm{ON} \\ \gamma \notin x}} \prod_{y \in \mathrm{dmn}(\dot{X})} (-\dot{X}(y) + -||\check{\gamma} = y||) \\
&= \prod_{y \in \mathrm{dmn}(\dot{X})} \prod_{\substack{\gamma \in \mathrm{ON} \\ \gamma \notin x}} (-\dot{X}(y) + -||\check{\gamma} = y||)
\end{aligned}
$$

Hence by (3),

$$
\begin{aligned}
e(p_{n-1} \restriction s) \leq &\prod_{y \in \mathrm{dmn}(\dot{X})} \prod_{\substack{\gamma \in \mathrm{ON} \\ \gamma \notin x}} (-\dot{X}(y) + -||\check{\gamma} = y||) \\
&\cdot \prod_{y \in \mathrm{dmn}(\dot{X})} \left(\dot{X}(y) \Rightarrow \sum_{\alpha \in \mathbf{ON}} ||y = \check{\alpha}|| \right) \\
= &\prod_{y \in \mathrm{dmn}(\dot{X})} \left(\prod_{\substack{\gamma \in \mathrm{ON} \\ \gamma \notin x}} (-\dot{X}(y) + -||\check{\gamma} = y||) \right. \\
&\left. \cdot \left(\dot{X}(y) \Rightarrow \sum_{\alpha \in \mathbf{ON}} ||y = \check{\alpha}|| \right) \right)
\end{aligned}
$$

Now if $y \in \mathrm{dmn}(\dot{X})$, then

$$
\begin{aligned}
&\left(\prod_{\substack{\gamma \in \mathrm{ON} \\ \gamma \notin x}} (-\dot{X}(y) + -||\check{\gamma} = y||) \right) \cdot \left(\dot{X}(y) \Rightarrow \sum_{\alpha \in \mathbf{ON}} ||y = \check{\alpha}|| \right) \\
&= \left(\prod_{\substack{\gamma \in \mathrm{ON} \\ \gamma \notin x}} (-\dot{X}(y) + -||\check{\gamma} = y||) \right) \cdot \left(-\dot{X}(y) + \sum_{\alpha \in \mathbf{ON}} ||y = \check{\alpha}|| \right) \\
&= \prod_{\substack{\gamma \in \mathrm{ON} \\ \gamma \notin x}} \left((-\dot{X}(y) + -||\check{\gamma} = y||) \cdot \left(-\dot{X}(y) + \sum_{\alpha \in \mathbf{ON}} ||y = \check{\alpha}|| \right) \right) \\
&= \prod_{\substack{\gamma \in \mathrm{ON} \\ \gamma \notin x}} \left(-\dot{X}(y) + -||\check{\gamma} = y|| \cdot \sum_{\alpha \in \mathbf{ON}} ||y = \check{\alpha}|| \right)
\end{aligned}
$$

$$= -\dot{X}(y) + \prod_{\substack{\gamma\in\mathbf{ON}\\ \gamma\notin x}} \left(-||\check{\gamma} = y|| \cdot \sum_{\alpha\in\mathbf{ON}} ||y = \check{\alpha}||\right)$$
$$= -\dot{X}(y) + -\sum_{\substack{\alpha\in\mathbf{ON}\\ \alpha\in x}} ||y = \check{\alpha}||.$$

Thus
$$e(p_{n-1} \restriction s) \le \prod_{y\in\mathrm{dmn}(\dot{X})} (-\dot{X}(y) + \sum_{\substack{\alpha\in\mathbf{ON}\\ \alpha\in x}} ||y = \check{\alpha}|| = ||\dot{X} \subseteq \check{x}||.$$

It follows that $p_{n-1} \restriction s \Vdash \dot{X} = \check{x}$, contradiction. Hence (1) holds.

Next we claim

(4) $\exists q_{s^\frown\langle 0\rangle} \le p \restriction (s^\frown\langle 0\rangle)[q_{s^\frown\langle 0\rangle} \Vdash \check{\gamma}_s \in \dot{X}]$ and $\exists q_{s^\frown\langle 1\rangle} \le p \restriction (s^\frown\langle 1\rangle)[q_{s^\frown\langle 1\rangle} \Vdash \check{\gamma}_s \notin \dot{X}]$

or

(5) $\exists q_{s^\frown\langle 0\rangle} \le p \restriction (s^\frown\langle 0\rangle)[q_{s^\frown\langle 0\rangle} \Vdash \check{\gamma}_s \notin \dot{X}]$ and $\exists q_{s^\frown\langle 1\rangle} \le p \restriction (s^\frown\langle 1\rangle)[q_{s^\frown\langle 1\rangle} \Vdash \check{\gamma}_s \in \dot{X}]$.

For, by (1) we have
$$e(p \restriction s) \cdot ||\check{\gamma}_s \in \dot{P}|| \ne 0 \ne e(p \restriction s) \cdot -||\check{\gamma}_s \in \dot{P}||.$$

Hence there are $q, r \le p \restriction s$ such that $e(q) \le ||\check{\gamma}_s \in \dot{P}||$ and $e(r) \le -||\check{\gamma}_s \in \dot{P}||$.

Case 1. $s^\frown\langle 0\rangle, s^\frown\langle 1\rangle \in q \cap r$. Let $q' = q \restriction (s^\frown\langle 0\rangle)$ and $r' = r \restriction (s^\frown\langle 1\rangle)$. Then (4) holds.

Case 9. $s^\frown\langle 0\rangle \in q\backslash r$. Then $s^\frown\langle 1\rangle \in r$. Let $q' = q \restriction (s^\frown\langle 0\rangle)$. Then q' and r satisfy (4).

Case 3. $s^\frown\langle 0\rangle \in r\backslash q$. Then $s^\frown\langle 1\rangle \in q$. Let $r' = r \restriction s^\frown\langle 0\rangle$. Then r' and q satisfy (5).

This proves our claim.

Now we let p_n be the amalgamation of $\langle q_{s^\frown\langle\varepsilon\rangle} : s \in A\rangle$ into p_{n-1}:
$$\{t \in P : \exists s \in S_n \exists \varepsilon \in 2[t \in q_{s^\frown\langle\varepsilon\rangle}]\}.$$

(6) p_n is a perfect tree.

In fact, suppose that $t \in p_n$, $g \in \mathrm{Seq}$, and $g \subseteq t$. Say $f \in q_{s^\frown\langle\varepsilon\rangle}$ with $\varepsilon < 2$. Then $g \in q_{s^\frown\langle\varepsilon\rangle}$, so $g \in p_n$. Now suppose that $f \in p_n$; we want to find a branching point of r above f. Say $f \in q_{s^\frown\langle\varepsilon\rangle}$. Let g be a branching point of $q_{s^\frown\langle\varepsilon\rangle}$ with $f \subseteq g$. Clearly g is a branching point of p_n.

(7) $p_n \le_n p_{n-1}$

In fact, this is clear.

Now let $q = \bigcap_{n\in\omega} p_n$. By Theorem 15.70, q is a perfect tree and $q \le_n p_n$ for all $n \in \omega$.

Now suppose that $q \in G$.

(8) If s is a 0th splitting node of q, then $s \subseteq f$.

For, suppose not. Then there is a $t < s$ such that ($t^\frown\langle 0\rangle \in q$ and $t^\frown\langle 1\rangle \subseteq f$) or ($t^\frown\langle 1\rangle \in q$ and $t^\frown\langle 0\rangle \subseteq f$); say $t^\frown\langle 0\rangle \in q$ and $t^\frown\langle 1\rangle \subseteq f$. Say $\mathrm{dmn}(t) = n$. Choose $u \in G$ with $t^\frown\langle 1\rangle \subseteq u$ such that u does not split through domain $n+1$. Let $v \in G$ with $v \leq q, u$. Then $v(n)$ should be 0 since $v \leq q$, while it should be 1 since $v \leq u$, contradiction. Thus (8) holds.

Now let s be an mth splitting node of q. Say $\mathrm{dmn}(s) = n$. Say $f(n) = \varepsilon$ with $\varepsilon \in \{0, 1\}$.

Case 1. $\gamma_s \in X$. Choose $u \in G$ such that $u \Vdash \check{\gamma}_s \in \dot{X}$ and u does not split through domain $n+1$. Choose $v \in G$ such that $v \leq u, q$. Then $v(n) = u(n) = \varepsilon$.

(9) $q \restriction s^\frown\langle\varepsilon\rangle \Vdash \check{\gamma}_s \in \dot{X}$.

For, suppose that $q \restriction s^\frown\langle\varepsilon\rangle \Vdash \check{\gamma}_s \notin \dot{X}$. Since $v(n) = \varepsilon$ and $v \leq q$ and v does not split through domain $n+1$, we have $v \leq q \restriction s^\frown\langle\varepsilon\rangle$, so $v \Vdash \check{\gamma}_s \notin \dot{X}$, contradicting $v \leq u$.

Case 9. $\gamma_s \notin X$. Choose $u \in G$ such that $u \Vdash \check{\gamma}_s \notin \dot{X}$ and u does not split through domain $n+1$. Choose $v \in G$ such that $v \leq u, q$. Then $v(n) = u(n) = \varepsilon$.

(10) $q \restriction s^\frown\langle\varepsilon\rangle \Vdash \check{\gamma}_s \notin \dot{X}$.

In fact, if $q \restriction s^\frown\langle\varepsilon\rangle \Vdash \check{\gamma}_s \in \dot{X}$, then $v \leq q \restriction s^\frown\langle\varepsilon\rangle$, so $v \Vdash \check{\gamma}_s \in \dot{X}$, contradicting $v \leq u$.

It follows now that $f(n) = \varepsilon$ iff $\gamma_s \in X$ and $q \restriction s^\frown\langle\varepsilon\rangle \Vdash \check{\gamma}_s \in \dot{X}$, or $\gamma_s \notin X$ and $q \restriction s^\frown\langle\varepsilon\rangle \Vdash \check{\gamma}_s \notin \dot{X}$.

(11) If u is an mth splitting node and $u^\frown\langle\varepsilon\rangle \subseteq f$ and $u^\frown\langle\varepsilon\rangle \subseteq s$ with s an $(m+1)$th splitting node, then $s \subseteq f$.

For, suppose not. Then there is a $t < s$ with $u \leq t$ such that ($t^\frown\langle 0\rangle \in q$ and $t^\frown\langle 1\rangle \subseteq f$) or ($t^\frown\langle 1\rangle \in q$ and $t^\frown\langle 0\rangle \subseteq f$); say $t^\frown\langle 0\rangle \in q$ and $t^\frown\langle 1\rangle \subseteq f$. Say $\mathrm{dmn}(t) = n$. Choose $u \in G$ with $t^\frown\langle 1\rangle \subseteq u$ such that u does not split through domain $n+1$. Let $v \in G$ with $v \leq q, u$. Then $v(n)$ should be 0 since $v \leq q$, while it should be 1 since $v \leq u$, contradiction. Thus (11) holds.

We have shown that f can be defined from $q \in M$ and X. Now back to the beginning of this proof, we assume that $p \in G$. The above construction can be applied to any $r \leq p$, producing q_r from which, using also X, f can be defined. Now $\{q_r : r \leq p\}$ is dense below p, so there is an $r \leq p$ with $q_r \in G$. It follows that $f \in N$, and hence $G \in N$, completing the proof. $\square$

Theorem 15.74. *B is (κ, λ)-distributive iff for every generic ultrafilter on B over M, if $f : \kappa \to \lambda$ in $M[G]$ then $f \in M$.*

Proof. $\Rightarrow$: Assume that B is (κ, λ)-distributive, G is B-generic over M, and $f \in M[G]$ with $f : \kappa \to \lambda$. Let τ be a name such that $\tau_G = f$. Choose $p \in G$ such that $p \Vdash \tau : \check{\kappa} \to \check{\lambda}$.

Then

$$e(p) \le \prod_{\alpha<\kappa} \sum_{\beta<\lambda} [\![\tau(\check{\alpha}) = \check{\beta}]\!]$$
$$= \sum_{g:\kappa\to\lambda} \prod_{\alpha<\kappa}]\!]\tau(\check{\alpha}) = g(\alpha)\check{}]\!];$$

it follows that $\{q$:there is a $g : \kappa \to \lambda$ such that $q \Vdash \forall\alpha < \kappa(\tau(\alpha) = g(\alpha)\check{}\}$ is dense below p, and hence we can choose $g : \kappa \to \lambda$ and $q \le p$ such that $q \in G$ and $q \Vdash \forall\alpha < \kappa(\tau(\alpha) = g(\alpha)\check{}\}$. Hence $f(\alpha) = g(\alpha)$ for all $\alpha < \kappa$. So $f = g \in M$.

$\Leftarrow$: We use 14.9(a) of the BA handbook. Thus suppose that $\langle a_{\alpha\beta} : \alpha < \kappa, \beta < \lambda\rangle$ is a system of elements of B such that $\langle a_{\alpha\beta} : \beta < \lambda\rangle$ is a partition of unity for each $\alpha < \kappa$. Note that the right side of the equation (1) of 14.9(a) is obviously $\le$ the left side. Suppose that $p \le l \cdot -r$, where l and r are the left and right sides respectively. Let G be a B-generic filter over M with $p \in G$. For each $\alpha < \kappa$ choose $f(\alpha) < \lambda$ such that $a_{\alpha f(\alpha)} \in G$. Then $f \in M$ by assumption. Now clearly the set

$$C \stackrel{\text{def}}{=} \{r \in B : \forall\alpha < \kappa(r \le a_{\alpha f(\alpha)}) \text{ or } \exists\alpha < \kappa(r \cdot a_{\alpha f(\alpha)} = 0)\}$$

is dense below p. So we can choose $r \in G \cap C$. Hence clearly $\prod_{\alpha<\kappa} a_{\alpha f(\alpha)} \in G$, contradiction, since $p \le -\prod_{\alpha<\kappa} a_{\alpha f(\alpha)}$. $\square$

A complete BA B is *weakly* (κ, λ)*-distributive* iff

$$\prod_{\alpha<\kappa} \sum_{\beta<\lambda} u_{\alpha\beta} = \sum_{g:\kappa\to\lambda} \prod_{\alpha<\kappa} \sum_{\beta<g(\alpha)} u_{\alpha\beta},$$

Lemma 15.75. *B is weakly (κ, λ)-distributive iff every $f : \kappa \to \lambda$ in $M[G]$ is dominated by some $g : \kappa \to \lambda$ which is in M.*

Proof. $\Rightarrow$: Assume that B is weakly (κ, λ)-distributive, G is B-generic over M, and $f \in M[G]$ with $f : \kappa \to \lambda$. Let τ be a name such that $\tau_G = f$. Choose $p \in G$ such that $p \Vdash \tau : \check{\kappa} \to \check{\lambda}$. Then

$$e(p) \le \prod_{\alpha<\kappa} \sum_{\beta<\lambda} [\![\tau(\check{\alpha}) = \check{\beta}]\!]$$
$$= \sum_{g:\kappa\to\lambda} \prod_{\alpha<\kappa} \sum_{\beta<g(\alpha)}]\!]\tau(\check{\alpha}) = \check{\beta}]\!]$$
$$= \sum_{g:\kappa\to\lambda} \prod_{\alpha<\kappa} [\![\tau(\check{\alpha}) < g(\alpha)\check{}]\!];$$

it follows that $\{q$:there is a $g : \kappa \to \lambda$ such that $q \Vdash \forall\alpha < \kappa(\tau(\alpha) < g(\alpha)\check{}\}$ is dense below p, and hence we can choose $g : \kappa \to \lambda$ and $q \le p$ such that $q \in G$ and $q \Vdash \forall\alpha < \kappa(\tau(\alpha) < g(\alpha)\check{}\}$. Hence $f(\alpha) < g(\alpha)$ for all $\alpha < \kappa$, as desired.

$\Leftarrow$: Assume the indicated condition, and suppose that $e(p) \leq l \cdot -r$, where l and r are the left and right sides of the weak distributivity equation. (Clearly $r \leq l$.) Let G be generic with $p \in G$. Then

$$e(p) \leq \prod_{\alpha<\kappa} \sum_{\beta<\lambda} u_{\alpha\beta}.$$

Hence for all $\alpha < \kappa$ there is an $f(\alpha) < \lambda$ and a $q_\alpha \in G$ such that $e(q_\alpha) \leq u_{\alpha f(\alpha)}$. So $f : \kappa \to \lambda$ and $f \in M[G]$. By our assumed condition, let $g : \kappa \to \lambda$ be a member of M such that $f(\alpha) < g(\alpha)$ for all $\alpha < \kappa$. Since $e(p) \leq -r$, we have

$$e(p) \leq \sum_{\alpha<\kappa} \prod_{\beta<g(\alpha)} -u_{\alpha\beta}.$$

Hence there is an $r \leq p$ with $r \in G$ such that $e(r) \leq \prod_{\beta<g(\alpha)} -u_{\alpha\beta}$. But $q_\alpha \in G$, $e(q_\alpha) \leq u_{\alpha f(\alpha)}$, and $f(\alpha) < g(\alpha)$, contradiction. $\square$

Lemma 15.76. *Let X be a subset of a complete BA B in M which completely generates B, and let G be a generic ultrafilter on B. Then $M[G]$ is the smallest model N of ZFC such that $M \subseteq N$ and $X \cap G \in N$.*

Proof. Assume that X be a subset of the complete BA B in M which completely generates B, let G be a generic ultrafilter on B, and let N be a model of ZFC such that $M \subseteq N$ and $X \cap G \in N$. We want to show that $M[G] \subseteq N$. By the minimality of $M[G]$, it suffices to show that $G \in N$. Define in M

$$\begin{aligned}
Y_0 &= X;\\
Y_{2\alpha+1} &= Y_{2\alpha} \cup \{x : -x \in Y_{2\alpha}\};\\
Y_{2\alpha+2} &= Y_{2\alpha+1} \cup \left\{\sum Z : Z \subseteq Y_{2\alpha+1}\right\};\\
Y_\lambda &= \bigcup_{\alpha<\lambda} Y_\alpha \quad \text{for } \lambda \text{ limit.}
\end{aligned}$$

Then there is a $\theta \leq |B|^+$ such that $B = Y_\theta$. Now in N define

$$\begin{aligned}
H_0 &= X \cap G;\\
H_{2\alpha+1} &= \{x : -x \in Y_{2\alpha} \backslash H_{2\alpha}\};\\
H_{2\alpha+2} &= \left\{\sum Z : Z \subseteq Y_{2\alpha+1} \wedge Z \cap H_{2\alpha+1} \neq \emptyset\right\};\\
H_\lambda &= \bigcup_{\alpha<\lambda} H_\alpha \quad \text{for } \lambda \text{ limit.}
\end{aligned}$$

Now we claim that $\forall\alpha[Y_\alpha \cap G = H_\alpha]$. (Hence $G = B \cap G = Y_\theta \cap G = H_\theta \in N$, as desired.) We prove this by induction on α. It is clear for $\alpha = 0$, and the limit step is clear.

$$Y_{2\alpha+1} \cap G = (Y_{2\alpha} \cap G) \cup \{x : x \in G \wedge -x \in Y_{2\alpha}\}$$

$$
\begin{aligned}
&= H_{2\alpha} \cup \{x : x \in G \wedge -x \in Y_{2\alpha}\backslash G\} \\
&= H_{2\alpha} \cup \{x : x \in G \wedge -x \in Y_{2\alpha}\backslash(G \cap Y_{2\alpha})\} \\
&= H_{2\alpha+1}; \\
&= H_{2\alpha} \cup \{x : -x \in Y_{2\alpha}\backslash H_{2\alpha}\} \\
Y_{2\alpha+2} \cap G &= (Y_{2\alpha+1} \cap G) \cup \left\{\sum Z : \sum Z \in G,\ Z \subseteq Y_{2\alpha+1}\right\} \\
&= H_{2\alpha+1} \cup \left\{\sum Z : Z \cap (G \cap Y_{2\alpha+1}) \neq \emptyset,\ Z \subseteq Y_{2\alpha+1}\right\} \\
&= H_{2\alpha+1} \cup \left\{\sum Z : Z \cap H_{2\alpha+1} \neq \emptyset,\ Z \subseteq Y_{2\alpha+1}\right\} \\
&= H_{2\alpha+2}.
\end{aligned}
$$
□

Corollary 15.77. *If B is completely generated by a set X of size $\leq \kappa$, then $M[G] = M[A]$ for some $A \subseteq \kappa$.*

Proof. In M let $A \subseteq \kappa$ with f a bijection from A onto X. By Lemma 76 we have $M[G] = M[X \cap G]$. Hence $M[G] = M[f^{-1}[X \cap G]]$. □

Lemma 15.78. *Let κ be a cardinal in M, B a complete BA in M, G B-generic over M, and $A \in M[G]$ a subset of κ. Then there is a κ-generated complete subalgebra D of B such that $M[D \cap G] = M[A]$.*

Proof. Let τ be a name such that $\tau_G = A$. Define $u_\alpha = [\![\check{\alpha} \in \tau]\!]$, and $X = \{u_\alpha : \alpha < \kappa\}$. If $\alpha \in A$, then there is a $p \in G$ such that $p \Vdash \check{\alpha} \in \tau$. So $u_\alpha = [\![\check{\alpha} \in \tau]\!] \in G$. So $A \subseteq \{\alpha : u_\alpha \in X \cap G\}$. Now suppose that $u_\alpha \in G$. Then $\alpha \in \tau_G = A$. Hence $A = \{\alpha : u_\alpha \in X \cap G\}$.

Hence if $u_\alpha \in G$, then $\alpha \in A$. So $X \cap G \subseteq \{u_\alpha : \alpha \in A\}$. Conversely, if $\alpha \in A$, then $u_\alpha \in X \cap G$.

Lemma 15.79. *Suppose that M is a c.t.m. of* ZFC*, B is a complete BA in M, G is B-generic over M, N is a c.t.m. of* ZFC*, and $M \subseteq N \subseteq M[G]$. Then there is a complete subalgebra D of B such that $N = M[D \cap G]$.*

Proof. First we prove the following independently interesting fact:

Fact. *Suppose that M is a c.t.m. of* ZFC*, B is a complete BA in M, and G is a generic ultrafilter over M. Then for every $X \in M[G]$ there exist an ordinal α, a subset A of α, and a complete subalgebra D of B, such that $M[X] = M[A] = M[D \cap G]$.*

Proof. In $M[G]$, let f be a bijection from a cardinal α onto $\mathrm{trcl}(\{X\})$. Define $E = \{(\beta, \gamma) \in \alpha \times \alpha : f(\beta) \in f(\gamma)\}$. Let Γ be the standard bijection from On $\times$ On onto On, and let $A = \Gamma[E]$. Then by Lemma 15.78 we get a complete subalgebra D of B such that $M[A] = M[D \cap G]$. Clearly $A \in M[X]$, so $M[A] \subseteq M[X]$. So it suffices to show that $X \in M[A]$. Note that E is well-founded on α, and $E \in M[A]$. In $M[A]$, let G be the Mostowski collapse function for α, E. Thus for any $\beta \in \alpha$, $G(\beta) = \{G(\gamma) : \gamma E \beta\}$. We claim that $G[\alpha] = \text{trcl}(\{X\})$, as desired. In fact, we claim that $G = f$. Suppose that $\beta \in \alpha$ and $G(\gamma) = f(\gamma)$ for all $\gamma E \beta$. Then

$$\begin{aligned} G(\beta) &= \{G(\gamma) : \gamma E \beta\} \\ &= \{f(\gamma) : f(\gamma) \in f(\beta)\} \\ &= f(\beta). \end{aligned}$$

Now we turn to the proof of the Lemma. We apply the fact to $\mathscr{P}(B) \cap N = \mathscr{P}^N(B) \in N \subseteq M[G]$; so we obtain A, D such that $M[\mathscr{P}(B) \cap N] = M[A] = M[D \cap G]$. We claim that this is equal to N. Clearly it is a subset of N. Now suppose that $X \in N$. Then by the fact again we get C, E such that $M[X] = M[C] = M[E \cap G]$. Now $E \cap G \in M[X] \subseteq N$, so $E \cap G \in \mathscr{P}(B) \cap N$. Hence $M[X] = M[E \cap G] \subseteq M[\mathscr{P}(B) \cap N]$, and hence $X \in M[\mathscr{P}(B) \cap N]$, as desired. $\square$

Proposition 15.80. *If B is a complete subalgebra of C, then $M^B \subseteq M^C$.*

Proof. $M^B_\alpha \subseteq M^C_\alpha$ by induction. It is clear for $\alpha = 0$ and the case of limit α is clear. Now suppose that $M^B_\alpha \subseteq M^C_\alpha$. Then

$$\begin{aligned} M^B_{\alpha+1} &= \{x : x \text{ is a function } \wedge \text{dmn}(x) \subseteq M^B_\alpha\} \\ &\subseteq \{x : x \text{ is a function } \wedge \text{dmn}(x) \subseteq M^C_\alpha\} \\ &= M^C_{\alpha+1} \end{aligned}$$

$\square$

Proposition 15.81. *If B_1 is a complete subalgebra of B_2 and G is a generic ultrafilter over B_2 and M, then $G \cap B_1$ is a generic ultrafilter over B_1 and M.*

Proof. Clearly $G \cap B_1$ is an ultrafilter on B_1. Now suppose that $X \in M$ and $X \subseteq G \cap B_1$. Then $\prod^{B_2} X \in G$, so $\prod^{B_2}(X) = \prod^{B_1}(X) \in G \cap B_1$. $\square$

Proposition 15.82. *If B_1 is a complete subalgebra of B_2 and G is a generic ultrafilter over B_2 and M, then $\forall x \in M^{B_1}[x^{G \cap B_1} = x^G]$.*

Proof. By induction: if $x \in M^{B_1}$ then $x^{G \cap B_1} = \{y^{G \cap B_1} : y \in \text{dmn}(x), x(y) \in G \cap B_1\} = \{y^G : y \in \text{dmn}(x), x(y) \in G\} = x^G$. $\square$

Proposition 15.83. *If B is a complete subalgebra of C and G is generic over C and M, and if $\{a \in C : B \upharpoonright a = C \upharpoonright a\}$ is dense in C, then $M[G \cap B] = M[G]$.*

Proof. With each $x \in M^C$ we associate $x' \in M^B$ by induction. Suppose that $x \in M^C_{\alpha+1}$. Let $\text{dmn}(x') = \{y' : y \in \text{dmn}(x)\}$, and for each $y \in \text{dmn}(x)$, if $x(y) \in G$ choose $a_{xy} \in G$ such that $B \upharpoonright a_{xy} = C \upharpoonright a_{xy}$ and let $X_{xy} = \{b \in B : b \cdot a_{xy} \in G\}$, and let $x'(y') = \prod^B X_{xy}$. Note that $\prod X_{xy} \cdot a_{xy} \in G$, and so $\prod X_{xy} \in G$. If $x(y) \notin G$ let $X_{xy} = B$ and $x'(y') = 0$. Then by induction, for all $x \in M^C$, $x^G = x'^{(G \cap B)}$:

$$\begin{aligned} x^G &= \{y^G : y \in \text{dmn}(x) \wedge x(y) \in G\} \\ &= \{y'^{(G \cap B)} : y' \in \text{dmn}(x') \wedge \prod X_{xy} \in G \cap B\} = x'^{(G \cap B)}. \end{aligned}$$

$\square$

We write $M^P \subseteq' M^Q$ iff for every generic filter G on Q over M there is a generic filter H on P over M such that $H \in M[G]$.

Lemma 15.84. *Let $i : P \to Q$ be such that*
(i) $\forall p_1, p_2 \in P[p_1 \le p_2 \to i(p_1) \le i(p_2)]$.
(ii) $\forall p_1, p_2 \in P[p_1$ and p_2 are incompatible $\to i(p_1)$ and $i(p_2)$ are incompatible$]$.
(iii) $\forall q \in Q \exists p \in P \forall p' \le p[i(p')$ is compatible with $q]$.

Then $M^P \subseteq' M^Q$.

Proof. Suppose that G is a generic filter on Q over M.

$i^{-1}[G]$ is closed upwards: suppose that $p \in i^{-1}[G]$ and $p \le q$. Then $i(p) \in G$ and $i(p) \le i(q)$, so $i(q) \in G$ and hence $q \in i^{-1}[G]$. Suppose that $p_1, p_2 \in i^{-1}[G]$. Then $i(p_1), i(p_2) \in G$, so $i(p_1), i(p_2)$ are compatible. Hence p_1, p_2 are compatible. Now suppose that D is dense in P. Then we claim that $i[D]$ is predense in Q. For, suppose that $q \in Q$; we want to find $p \in D$ such that q and $i(p)$ are compatible. Choose $p \in P$ so that $\forall p' \le p[i(p'), q$ are compatible.]. Choose $p' \in D$ so that $p' \le p$. then q and $i(p')$ are compatible. Now choose $q \in i[D] \cap G$. Say $q = i(p)$ with $p \in D$. Then $p \in i^{-1}[G] \cap D$. Thus $i^{-1}[G]$ is generic on P. So $V^p \subseteq' V^Q$. ☐

Lemma 15.85. *Let $h : Q \to P$ be such that*
(i) $\forall q_1, q_2 \in Q[q_1 \le q_2 \to h(q_1) \le h(q_2)]$.
(ii) $\forall q \in Q \forall p \le h(q) \exists q' \in Q[q$ and q' are compatible and $h(q') \le p]$.

Then $M^P \subseteq' M^Q$.

Proof.

(1) If $D \subseteq P$ is open dense, then $h^{-1}[D]$ is predense in Q.

In fact, suppose that $D \subseteq P$ is open dense, and suppose that $q \in Q$; we want to find $q' \in Q$ such that q, q' are compatible and $h(q') \in D$. Choose $p \in D$ such that $p \le h(q)$. Then choose q' compatible with q such that $h(q') \le p$. Say $r \le q, q'$. Then $h(r) \le h(q')$, so $h(r) \in D$, as desired. So (1) holds.

Now suppose that G is generic on Q. Let $H = \{p \in P : \exists q \in G[h(q) \le p]\}$. Clearly H is closed upwards. Suppose that $p_1, p_2 \in H$. Choose $q_1, q_2 \in G$ such that $h(q_1) \le p_1$ and $h(q_2) \le p_2$. Choose $q_3 \in G$ such that $q_3 \le q_1, q_2$. Then $h(q_3) \in H$ and $h(q_3) \le h(q_1) \le p_1$ and similarly $h(q_3) \le p_2$. So H is a filter. Now suppose that D is open dense in P. Then by (1), $h^{-1}[D]$ is predense in Q. Choose $q \in G \cap h^{-1}[D]$. Then $h(q) \in D$ and also $h(q) \in H$, as desired. ☐

Lemma 15.86. *If P satisfies the κ-chain condition, then $|\mathrm{RO}(P)| \le |P|^{<\kappa}$.*

Proof.

(1) For every $x \in \mathrm{RO}(P)$ there is an antichain C in P such that $x = \sum_{p \in C} e(p)$.

In fact, let $C \subseteq P$ be maximal pairwise incompatible such that $\forall p \in C[e(p) \le x]$. Clearly $x = \sum_{p \in C} e(p)$.

Now $|P|^{<\kappa}$ is the number of antichains in P. Hence by (1), $|\mathrm{RO}(P)| \le |P|^{<\kappa}$. $\square$

Lemma 15.87. *Let P consist of all functions p such that $\mathrm{dmn}(p) \in [\kappa]^{<\kappa}$ and $\mathrm{rng}(p) \subseteq 2$, and let $Q = \{p \in P : \mathrm{dmn}(p)$ is an initial segment of κ. Then*

(i) Q is dense in P.

(ii) There is an isomorphism f of $\mathrm{RO}(Q)$ onto $\mathrm{RO}(P)$ such that $f(e_Q(q)) = e_P(q)$ for all $q \in Q$.

Proof. Clearly Q is dense in P. Next we show that the mapping $q \mapsto e_P(q)$ extends to a homomorphism of $\mathrm{RO}(Q)$ to $\mathrm{RO}(P)$. To apply Sikorski's criterion, suppose that

$$\prod_{q \in F} e_Q(q) \cdot \prod_{q \in G} -e_Q(q) \neq 0, \tag{1}$$

with F and G disjoint finite subsets of Q. Choose $r \in Q$ such that

$$e_Q(r) \le \prod_{q \in F} e_Q(q) \cdot \prod_{q \in G} -e_Q(q).$$

Then

$$e_Q(r) \cdot \left(\sum_{q \in F} -e_Q(q) + \sum_{q \in G} e_Q(q) \right) = 0.$$

Hence $\forall q \in F \forall s \in G[e_Q(r) \cdot (-e_Q(q) + e_Q(s)) = 0$. Suppose that $q \in F$. By Theorem 14.6(v), $\{t \in Q : t \le r, q\}$ is dense below r, in the sense of Q, and hence in the sense of P. So $e_P(r) \cdot -e_P(q) = 0$.

Suppose that $s \in G$. Then r and s are incompatible in Q, hence also in P. So $e_P(r) \cdot e_P(s) = 0$.

It follows that

$$e_P(r) \le \prod_{q \in F} e_P(q) \cdot \prod_{q \in G} -e_P(q).$$

Hence

$$\prod_{q \in F} e_P(q) \cdot \prod_{q \in G} -e_P(q) \neq 0. \tag{2}$$

So (1) implies (2). Similarly, (2) implies (1). It follows that there is an isomorphism from $\langle\{e_Q(q) : q \in Q\}\rangle$ onto $\langle\{e_P(q) : q \in Q\}\rangle$. Now the desired conclusion follows by the remark at the bottom of page 57 of the Handbook. $\square$

Lemma 15.88. *Let κ be a singular cardinal and let P consist of all functions p such that $\mathrm{dmn}(p) \in [\kappa]^{<\kappa}$ and $\mathrm{rng}(p) \subseteq 2$, with the order $\supseteq$. With G P-generic over M, in $M[G]$ there is a one-one function from κ into $\mathrm{cf}(\kappa)$.*

Proof. Let $\langle \lambda_\xi : \xi < \mathrm{cf}(\kappa) \rangle$ be strictly increasing and continuous with supremum κ. Let G be M-generic over P. Any two members of G are compatible, so $g \stackrel{\text{def}}{=} \bigcup G$ is a function. For any $\alpha < \kappa$ the set $D_\alpha \stackrel{\text{def}}{=} \{p \in P : \alpha \in \text{dmn}(p)\}$ is clearly dense. Hence $g \in {}^\kappa 2$. Let $A = g^{-1}[\{1\}]$.

For any $p \in P$ and $\xi < \text{cf}(\kappa)$ let

$$Q(p, \xi) = \{\beta \in \text{dmn}(p) : p(\beta) = 1\} \cap (\lambda_{\xi+1} \backslash \lambda_\xi).$$

For each $\alpha < \kappa$ let $E_\alpha = \{p \in P : \exists \xi < \text{cf}(\kappa)[(\lambda_{\xi+1} \backslash \lambda_\xi) \subseteq \text{dmn}(p) \text{ and } \text{ot}(Q(p, \xi)) = \lambda_\xi + \alpha]\}$.

(1) $\forall \alpha < \kappa[E_\alpha$ is dense in $P]$.

In fact, let $\alpha < \kappa$ and let $q \in P$ be given.

(2) $\exists \xi < \text{cf}(\kappa)[\text{ot}(Q(q, \xi)) \leq \lambda_\xi + \alpha]$.

For, otherwise we have $\forall \xi < \text{cf}(\kappa)[|\text{dmn}(q) \cap (\lambda_{\xi+1} \backslash \lambda_\xi)| \geq \lambda_\xi]$, and hence $|\text{dmn}(q)| = \kappa$, contradiction.

(1) follows from (2).

For each $\alpha < \kappa$ there is a $p \in G$ such that $p \in E_\alpha$. Hence we can define $h(\alpha) = \min\{\xi < \text{cf}(\kappa) : \text{ot}(A \cap (\lambda_{\xi+1} \backslash \lambda_\xi)) = \lambda_\xi + \alpha\}$. Then h is one-one and maps κ onto $\text{cf}(\kappa)$. □

Lemma 15.89. *Assume that for every generic G and every function $f \in M[G]$ with domain κ and range contained in M, $f \in M$. Then B is κ-distributive.*

Proof. Assume the hypotheses, and suppose that $\langle W_\alpha : \alpha < \kappa \rangle$ is a system of partitions of B. Note that if $f, g \in \prod_{\alpha<k} W_\alpha$ and $f \neq g$ then $\prod_{\alpha<\kappa} f(\alpha) \cdot \prod_{\alpha<\kappa} g(\alpha) = 0$. Moreover, if $f \in \prod_{\alpha<\kappa} W_\alpha$ then $\forall \beta < \kappa \exists w \in W_\beta[\prod_{\alpha<\kappa} f(\alpha) \subseteq w]$. Hence it suffices to show that $\sum\{\prod_{\alpha<\kappa} f(\alpha) : f \in \prod_{\alpha<\kappa} W_\alpha\} = 1$; and for this it suffices to take any $a \neq 0$ and find $f \in \prod_{\alpha<\kappa} W_\alpha$ such that $a \cdot \prod_{\alpha<\kappa} f(\alpha) \neq 0$. Let G be a generic ultrafilter such that $a \in G$. For each $\alpha < \kappa$ let $W'_\alpha = \{-a\} \cup (\{a \cdot u : u \in W_\alpha\} \backslash \{0\})$. W'_α is a partition, so we can choose $w_\alpha \in W'_\alpha \cap G$. Clearly there is a $u_\alpha \in W_\alpha$ such that $w_\alpha = a \cdot u_\alpha$. For each $\alpha < \kappa$ let $f(\alpha) = u_\alpha$. By hypothesis, $f \in M$. Then $a \cdot \prod_{\alpha<\kappa} u_\alpha \in G$, and so $a \cdot \prod_{\alpha<\kappa} f(\alpha) \neq 0$. □

Lemma 15.90. *If $\text{RO}(P_1) \cong \text{RO}(P_2)$ and $\text{RO}(Q_1) \cong \text{RO}(Q_2)$ then $\text{RO}(P_1 \times Q_1) \cong \text{RO}(P_2 \times Q_2)$.*

Proof. Let $f : \text{RO}(P_1) \to \text{RO}(P_2)$ be an isomorphism, and let $g : \text{RO}(Q_1) \to \text{RO}(Q_2)$ be an isomorphism. It suffices to show that the following two conditions are equivalent:

$$e(p_0, q_0) \cdot \ldots \cdot e(p_{m-1}, q_{m-1}) \cdot -e(p_m, q_m) \cdot \ldots \cdot -e(p_n, q_n) = 0; \tag{1}$$

$$e(f(p_0), g(q_0)) \cdot \ldots \cdot e(f(p_{m-1}), g(q_{m-1})) \cdot -e(f(p_m), g(q_m)) \cdot \ldots \cdot -e(f(p_n), g(q_n)) = 0; \tag{2}$$

In fact, by symmetry it suffices to show that (2) implies (1). So assume that

$$e(p_0, q_0) \cdot \ldots \cdot e(p_{m-1}, q_{m-1}) \cdot -e(p_m, q_m) \cdot \ldots \cdot -e(p_n, q_n) \neq 0.$$

Choose (r,s) with

$$e(r,s) \leq e(p_0,q_0)\cdot\ldots\cdot e(p_{m-1},q_{m-1})\cdot -e(p_m,q_m)\cdot\ldots\cdot -e(p_n,q_n).$$

Then

$$e(r,s)\cdot(-e(p_0,q_0)+\cdots+-e(p_{m-1},q_{m-1})+e(p_m,q_m)+\cdots+e(p_n,q_n))=0.$$

Take any $i<m$. Then $e(r,s)\leq e(p_i,q_i)$. We claim:

(3) $\{(a,b):(a,b)\leq(f(r),g(s)),(f(p_i),g(q_i))\}$ is dense below $(f(r),f(s))$.

For, suppose that $(c,d)\leq(f(r),f(s))$. Say $(c,d)=(f(u),g(v))$. Then $(u,v)\leq(r,s)$. By Theorem 14.6(v), there is a $(x,y)\leq(u,v)$ with $(x,y)\leq(p_i,q_i)$. Hence $(f(x),g(y))\leq(f(r),g(s)),(f(p_i),g(q_i))$, as desired in (3).

By (3) and Theorem 14.6(v), $e(f(r),g(s))\leq e(f(p_i),g(q_i))$.

Now take i with $m\leq i,n$. Then $e(r,s)\cdot e(p_i,q_i)=0$, so (r,s) and (p_i,q_i) are incompatible. Hence $(f(r),g(s))$ and $(f(p_i),g(q_i))$ are incompatible. So $e(f(r),g(s))\cdot e(f(p_i),g(q_i))=0$.

We have now shown that (2) fails, completing the proof. □

Lemma 15.91. $\mathrm{RO}(P\times Q)\cong\mathrm{RO}(P)\oplus\mathrm{RO}(Q)$.

Proof. We want to show that there is an isomorphism from $\mathrm{RO}(P)\oplus\mathrm{RO}(Q)$ onto a dense subalgebra of $\mathrm{RO}(P\times Q)$. We claim that the following two conditions are equivalent:

$$\begin{aligned}(1)\qquad &(e_P(p_0)\cdot e_Q(q_0))\cdot\ldots\cdot(e_P(p_{m-1})\cdot e_Q(q_{m-1})\cdot\\ &(-e_P(p_m)+-e_Q(q_m))\cdot\ldots\cdot(-e_P(p_{n-1})+-e_Q(q_{n-1}))=0\\ (2)\qquad &e_{P\times Q}(p_0,q_0)\cdot\ldots\cdot e_{P\times Q}(p_{m-1},q_{m-1})\cdot\\ &-e_{P\times Q}(p_m,q_m)\cdot\ldots\cdot -e_{P\times Q}(p_{n-1},q_{n-1})=0.\end{aligned}$$

First suppose that (1) is nonzero. Take $r\in P$ and $s\in Q$ so that

$$\begin{aligned}e_P(r)\cdot e_Q(s)\leq&(e_P(p_0)\cdot e_Q(q_0))\cdot\ldots\cdot(e_P(p_{m-1})\cdot e_Q(q_{m-1})\cdot\\ &(-e_P(p_m)+-e_Q(q_m))\cdot\ldots\cdot(-e_P(p_{n-1})+-e_Q(q_{n-1})).\end{aligned}$$

Hence

$$\begin{aligned}&\forall i<m[e_P(r)\cdot e_Q(s)\leq e_P(p_i)\cdot e_Q(q_i)]\quad\text{and}\\ &\forall i\in[m,n)[e_P(r)\cdot e_Q(s)\cdot e_P(p_i)\cdot e_Q(q_i)]=0.\end{aligned}$$

Take any $i<m$. Then $e_P(r)\leq e_P(p_i)$ and $e_Q(s)\leq e_Q(q_i)$. Hence by Theorem 14.6(v), $\{t:t\leq r,p_i\}$ is dense below r, and $\{t:t\leq s,q_i\}$ is dense below s. Hence $\{(t,u):(t,u)\leq(r,s),(p_i,q_i)\}$ is dense below (r,s). It follows that $e_{P\times Q}(r,s)\leq e_{P\times Q}(p_i,q_i)$,

Take any $i\in[m,n)$. Then r and p_i are incompatible, or s and q_i are incompatible. Hence (r,s) and (p_i,q_i) are incompatible. Hence $e_{P\times Q}(r,s)\cdot e_{P\times Q}(p_i,q_i)=0$.

It follows that $e(r,s)$ is leq (2). Thus (2) implies (1).

The above arguments reverse to show that (1) implies (2).

Now the Lemma follows by the remark at the bottom of page 57 of the Handbook.

$\square$

Lemma 15.92. *If* $\mathrm{RO}(P_i) \cong \mathrm{RO}(Q_i)$ *for all* $i \in I$*, then* $\mathrm{RO}(\prod_{i\in I}^{\mathrm{fin}} P_i) \cong \mathrm{RO}(\prod_{i\in I}^{\mathrm{fin}} Q_i)$.

Proof. Let $f_i : \mathrm{RO}(P_i) \to \mathrm{RO}(Q_i)$ be an isomorphism, for each $i \in I$. For $x \in \prod_{i\in I}^{\mathrm{fin}} P_i$ and $i \in I$, let $M(x,i) = f(x_i)$, and let $N(x) = \langle M(x,i) : i \in I\rangle$. We claim that the following two conditions are equivalent, for $x^0, \ldots, x^{n-1} \in \prod_{i\in I}^{\mathrm{fin}} P_i$:

$$e(x^0) \cdot \ldots \cdot e(x^{m-1}) \cdot -e(x^m) \cdot \ldots \cdot -e(x^{n-1}) = 0; \tag{1}$$

$$e(N(x^0)) \cdot \ldots \cdot e(N(x^{m-1})) \cdot -e(N(x^m)) \cdot \ldots \cdot -e(N(x^{n-1})) = 0; \tag{2}$$

Suppose that (1) is false; choose $y \in \prod_{i\in I}^{\mathrm{fin}} P_i$ such that

$$e(y) \le e(x^0) \cdot \ldots \cdot e(x^{m-1}) \cdot -e(x^m) \cdot \ldots \cdot -e(x^{n-1}).$$

Thus

$$\forall i < m[e(y) \le e(x^i)] \quad \text{and} \quad \forall i \in [m,n)[e(y) \cdot e(x_i) = 0].$$

Take any $i < m$. Then by Theorem 14.6(v), $\{z : z \le y, x^i\}$ is dense below y. Hence for all $j \in I$, $\{z : z \le y_j, x^i_j\}$ is dense below y_j. so for all $j \in I$, $\{z : z \le f(y_j), f(x^i_j\}$ is dense below $f(y_i)$. Hence $e(N(y)) \le e(N(x^i))$.

Next, take any $i \in [m,n)$. Then y and x^i are incompatible. So there is a $j \in I$ such that y_j and x^i_j are incompatible. So $f_j(y_j)$ and $f_j(x^i_j)$ are incompatible; hence $N(y)$ and $N(x^i)$ are incompatible.

So we have shown that (2) fails.

Similarly, (2) fails implies that (1) fails. So (1) and (2) are equivalent. Hence there is an isomorphism of $\langle\{e(x) : x \in \prod_{i\in I}^{\mathrm{fin}} P_i\rangle$ onto $\langle\{e(x) : x \in \prod_{i\in I}^{\mathrm{fin}} Q_i\rangle$. By Sikorkski's extension theorem this extends to an isomorphism f of $\mathrm{RO}(\prod_{i\in I}^{\mathrm{fin}} P_i)$ into $(\prod_{i\in I}^{\mathrm{fin}} Q_i)$.

Now the Lemma follows by the remark at the bottom of page 57 of the Handbook.

$\square$

Lemma 15.93. *Let P be the set $\bigcup_{n\in\omega} {}^n2$ with order $\supseteq$. Let P' be the set of all finite functions $\subseteq \omega \times 2$, with order $\supseteq$. Then P is dense in P', and so* $\mathrm{RO}(P) \cong \mathrm{RO}(P')$*; see the proof of Lemma 15.88.*

Let Q be the set of all functions p whose domain is a finite subset of $\kappa \times \omega$ and range a subset of 2; with order $\supseteq$. Then $Q \cong \prod_{\alpha<\kappa}^{\mathrm{fin}}(P')$.

Proof. Suppose that $q \in Q$ and $\alpha < \kappa$. Let p^α have domain $\{m \in \omega : (\alpha, m) \in \mathrm{dmn}(q)\}$, with $p^\alpha(m) = q(\alpha, m)$. Then we set $f(q) = p$. Clearly f is a bijection from Q onto $\prod_{\alpha<\kappa}^{\mathrm{fin}}(P')$. Also it is clear that $q_0 \le q_1$ iff $f(q_0) \le f(q_1)$. $\square$

Lemma 15.94. *If P satisfies ccc and Q has property (K), then $P \times Q$ satisfies ccc.*

Proof. Let $X \subseteq P \times Q$ be uncountable and pairwise incompatible.

Case 1. $\exists q \in Q[Y_q \stackrel{\text{def}}{=} \{p \in P : (p,q) \in X\}$ is uncountable$\}$. For such a q we have $(p_1,q) \perp (p_2,q)$ for all distinct $p_1,p_2 \in Y_q$, so $p_1 \perp p_2$ for all distinct $p_1,p_2 \in Y_q$, contradicting P being ccc.

Case 9. $\forall q \in Q[Y_q \stackrel{\text{def}}{=} \{p \in P : (p,q) \in X\}]$ is countable.

(1) $Z \stackrel{\text{def}}{=} \{q \in Q : Y_q \neq \emptyset\}$ is uncountable.

In fact, otherwise $X = \{(p,q) : p \in Y_q\}$ is countable, contradiction.

Let W be an uncountable pairwise compatible subset of Z. For each $q \in W$ choose $p_q \in P$ such that $(p_q,q) \in X$. For distinct $q,q' \in W$ we have $(p_q,q) \perp (p_{q'},q')$, hence $p_q \perp p_{q'}$, contradicting ccc of P. $\square$

Lemma 15.95. *Let P be the set of all functions p with* $\mathrm{dmn}(p) \in [\omega_1]^{<\omega_1}$ *and range* $\subseteq \aleph_\omega$. *Then in $M[G]$ there is a one-one function $g : \aleph_\omega^{\aleph_0} \to \aleph_1$.*

Proof. By usual arguments, if $g = \bigcup G$ then g is a function from ω_1 onto $\aleph_\omega$. Now note that every countable subset of $\aleph_\omega$ in $M[G]$ is a member of M. For each countable subset X of $\aleph_\omega$ let

$$D_X = \{p \in P : \exists \alpha < \omega_1[(\alpha+\omega)\backslash\alpha \subseteq \mathrm{dmn}(p) \wedge p[(\alpha+\omega)\backslash\alpha] = X]\}$$

We claim that D_X is dense. For, suppose that $p \in P$. Take any $\alpha \in \omega_1\backslash\mathrm{dmn}(p)$, and let q extend p in such a way that $q[(\alpha+\omega)\backslash\alpha] = X$. So $q \in D$, as desired.

Now for each countable subset X of $\aleph_\omega$ let $f(X)$ be the least α such that $g[(\alpha+\omega)\backslash\alpha] = X$. So f is a one-one function from $[\aleph_\omega]^\omega$ into ω_1. $\square$

Lemma 15.96. *Let P consist of all functions p such that* $\mathrm{dmn}(p) \in [\aleph_\omega]^{<\aleph_\omega}$ *and* $\mathrm{rng}(p) \subseteq \lambda$. *Then in $M[G]$ there is a one-one function from λ into ω.*

Proof. $g = \bigcup G$ is a function mapping $\aleph_\omega$ onto λ. For any $\alpha < \lambda$ let

$$D_\alpha = \{p \in P : \exists n \exists \beta \in [\omega_n, \omega_{n+1})\forall \gamma \in [\omega_n, \omega_{n+1})[\beta \leq \gamma \to \gamma \in \mathrm{dmn}(p) \wedge p(\gamma) = \alpha]]$$

Then D_α is dense. For, suppose that $p \in P$. Then there is an $n \in \omega$ such that $|\mathrm{dmn}(p) \cap [\omega_n, \omega_{n+1})| < \omega_{n+1}$. Extend p to q in which n exhibits that $q \in D_\alpha$.

It follows that for every $\alpha < \lambda$ there exist $n \in \omega$ and $\beta \in [\omega_n, \omega_{n+1})$ such that $\forall \gamma \in [\beta, \omega_{n+1})[g(\gamma) = \alpha]$. This induces the desired one-one function from λ into ω.

Lemma 15.97. *Let P consist of all p such that* $\mathrm{dmn}(p) \in [\omega_1]^{<\omega_1}$ *and range* $\subseteq 2$. *Let Q consist of all p such that* $\mathrm{dmn}(p) \in [\omega_1]^{<\omega_1}$ *and range* $\subseteq {}^\omega 2$. *Then* $\mathrm{RO}(P) \cong \mathrm{RO}(Q)$.

Proof. Let $Q' = \{p \in Q : \mathrm{dmn}(p)$ is an initial segment of $\omega_1\}$. Clearly Q' is dense in Q. Also, let $P' = \{p \in P : \exists \alpha < \omega_1[\mathrm{dmn}(p) = \omega \cdot \alpha]\}$. Clearly P' is dense in P. Take any $p \in P'$; we define $q \stackrel{\text{def}}{=} f(p) \in Q'$. Say $\mathrm{dmn}(p) = \omega \cdot \alpha$ with $\alpha < \omega_1$. Let $\mathrm{dmn}(q) = \alpha$, and for any $\xi < \alpha$ and $n \in \omega$ let $(q(\xi))(n) = p(\omega \cdot \xi + n)$. f is one-one: suppose that $p_1, p_2 \in P'$ and $p_1 \neq p_2$. Say p_1 has domain $\omega \cdot \alpha_1$ and p_2 has domain $\omega \cdot \alpha_2$. If $\alpha_1 \neq \alpha_2$, then $f(p_1) \neq f(p_2)$. Suppose that $\alpha_1 = \alpha_2$. Suppose that $p_1 \neq p_2$. Say $\xi < \alpha$, $n \in \omega$, and $p_1(\omega \cdot \xi + n) \neq p_2(\omega \cdot \xi + n)$. Then clearly $f(p_1) \neq f(p_2)$. f is onto: leq $q \in Q'$ be given. Say $\mathrm{dmn}(q) = \alpha < \omega_1$. Let p have domain $\omega \cdot \alpha$, with $p(\omega \cdot \xi + n) = (q(\xi))(n)$ for all $\xi < \alpha$ and $n \in \omega$. Clearly $f(p) = q$.

Now suppose that $p_1 \supseteq p_2$. Let $f(p_1) = q_1$ and $f(p_2) = q_2$. Say p_1 has domain $\omega \cdot \alpha_1$ and p_2 has domain $\omega \cdot \alpha_2$. Then $\alpha_2 \leq \alpha_1$. Take any $\xi < \alpha_2$ and any $n \in \omega$. Then $(q_2(\xi))(n) = p_2(\omega \cdot \xi + n) = p_1(\omega \cdot \xi + n) = (q_1(\xi))(n)$. Thus $q_1 \supseteq q_2$. The other direction is similar. □

Lemma 15.98. *Let T be a special normal α-tree, with $\alpha < \omega_1$, and suppose that π is a non-trivial automorphism of T. Then T has an extension T' of height $\alpha + 1$ such that π cannot be extended to an automorphism of T'*

Proof. Let $t_0 \in T$ with $\pi(t_0) \neq t_0$. Fix a branch b_{t_0} through t_0. Let T' extend T by adding a vertex above each branch of T except $\pi[b_{t_0}]$. If T' is normal it is clearly as desired. For normality, only (v) is questionable. Given $t \in T$, if $t \neq \pi(t_0)$ then obviously there is a node at level α above t. For $t = \pi(t_0)$, choose an immediate successor s of $\pi(t_0)$ with $s \notin \pi[b_{t_0}]$. Then there is a node of level α above s, hence above $\pi(t_0)$. □

Lemma 15.99. *Let P be the set of all special normal trees with the order defined before Lemma 15.31. Let G be P-generic over M. Then $\bigcup G$ is rigid.*

Proof. Let $\mathscr{T} = \bigcup G$. Recall from Theorem 15.38 that $\mathscr{T}$ is a normal Suslin tree. Suppose that π (in $M[G]$) is a nontrivial automorphism of $\mathscr{T}$; say $\pi(t_0) \neq t_0$.

Now by Lemma 15.37 we have $(\bigcup' \Gamma')_G = \bigcup G = \mathscr{T}$. Then the following statement holds in $M[G]$:

$$\begin{gathered}\dot{\pi}_G \text{ is a bijection from } (\textstyle\bigcup' \Gamma')_G \text{ to } (\textstyle\bigcup' \Gamma')_G \text{ and } t_0 \in (\textstyle\bigcup' \Gamma')_G \wedge \\ \forall t_1, t_2 \in (\textstyle\bigcup' \Gamma')_G [t_1 \leq t_2 \leftrightarrow \pi_G(t_1) \leq \pi_G(t_2)] \wedge \dot{\pi}_G(t_0) \neq t_0.\end{gathered}$$

Hence there is a $T \in G$ such that

$$\begin{gathered}T \Vdash [\dot{\pi} \text{ is a bijection from } (\textstyle\bigcup' \Gamma') \text{ to } (\textstyle\bigcup' \Gamma') \text{ and } \wedge \check{t}_0 \in (\textstyle\bigcup' \Gamma') \text{ and} \\ \forall t_1, t_2 \in (\textstyle\bigcup' \Gamma')[t_1 \leq t_2 \leftrightarrow \pi(t_1) \leq \pi(t_2)]] \wedge [\dot{\pi}(\check{t}_0) \neq \check{t}_0]]\end{gathered}$$

Now $t_0 \in \mathscr{T} = \bigcup G$, so there is a $T' \in G$ such that $t_0 \in T'$. Choose $T'' \in G$ such that $T'' \leq T, T'$, Now let

$$\begin{aligned}D = \{T''' \in P : T''' \leq T'' \wedge \exists \dot{\sigma} \exists T^{iv}, T^v [T''' \leq T^v \leq T^{iv} \wedge T''' \Vdash \dot{\sigma} \text{ is an automorphism} \\ \text{of } \check{T}^{iv} \text{ which cannot be extended to an automorphism of } \check{T}^v \wedge \dot{\sigma} \subseteq \dot{\pi}]\}\end{aligned}$$

We claim that D is dense below T''. For, suppose that $T''' \leq T''$. Let T^{iv} extend T''' so that T^{iv} is an α-tree with α limit. Let T^v extend T^{iv} by adding nodes above all branches except one, call it b, above t_0. Let K be generic such that $T^v \in K$. Let $\dot\sigma$ be a name such that $\dot\sigma^K = \dot\pi^K \restriction T^{iv}$. Then

$$\begin{aligned}&\dot\sigma^K \text{ is an automorphism of } T^{iv} \text{ which cannot be extended}\\&\text{to an automorphism of } T^v, \text{ and } \dot\sigma^K \subseteq \dot\pi^K.\end{aligned}$$

Hence there is a $T^{vi} \in K$ such that

$$\begin{aligned}&T^{vi} \Vdash \dot\sigma \text{ is an automorphism of } \check T^{iv} \text{ which cannot be extended}\\&\text{to an automorphism of } \check T^v, \text{ and } \dot\sigma \subseteq \dot\pi.\end{aligned}$$

Let $T^{vii} \in K$ be such that $T^{vii} \le T^v, T^{vi}$. Then $T^{vii} \le T^v \le T^{iv} \le T'''$, and

$$\begin{aligned}&T^{vii} \Vdash \dot\sigma \text{ is an automorphism of } \check T^{iv} \text{ which cannot be extended}\\&\text{to an automorphism of } \check T^v, \text{ and } \dot\sigma \subseteq \dot\pi.\end{aligned}$$

Then $T^{vii} \le T^v \le T^{iv} \le T'''$ and $T^{viii} \in D$. This shows that D is dense.

Choose $T''' \in D \cap G$. Then choose $\dot\sigma$, T^{iv}, T^v so that $T''' \le T^v \le T^{iv}$ and

$$\begin{aligned}&T''' \Vdash \dot\sigma \text{ is an automorphism of } \check T^{iv}\\&\text{which cannot be extended to an automorphism of } \check T^v \wedge \dot\sigma \subseteq \dot\pi]\}\end{aligned}$$

It follows that $\dot\sigma_G$ is an automorphism of T^{iv} which cannot be extended to an automorphism of T^v, and $\dot\sigma_G \subseteq \pi$, contradiction. □

Lemma 15.100. *Let P consist of finite trees $(T, <_T)$ such that $T \subseteq \omega_1$ and $\forall \alpha, \beta < \omega_1[\alpha <_T \beta \to \alpha < b]$. We define $(T_1, <_{T_1}) \le (T_2, <_{T_2})$ iff $T_1 \supseteq T_2$ and $<_{T_2} = <_{T_1} \cap (T_2 \times T_2)$. Then*

(i) If G is P-generic over M, then $\bigcup G$ is a tree of height ω_1.

(ii) If G is P-generic over M, then for every $\alpha \in \bigcup G$ there are β, γ with $\alpha <_{\bigcup G} \beta, \gamma$ and β and γ are incomparable.

(iii) If G is P-generic over M, then $\bigcup G$ has no uncountable antichain.

(iv) If G is P-generic over M, then $\bigcup G$ is a Suslin tree.

Proof. (i): Clearly $\bigcup G$ is a tree. Now suppose that $\alpha < \omega_1$. Let $D = \{T \in P : \exists \beta \ge \alpha[\beta \in T]\}$. Clearly D is dense, and it follows that $\bigcup G$ has height ω_1.

(ii): Suppose that $\alpha \in \bigcup G$. Say $\alpha \in T \in G$. Let $D = \{T' \in P : T' \le T$ and $\exists \beta, \gamma \in T'[\alpha < T'\beta, \beta$ and β and γ are incomparable in $T']\}$. Clearly D is dense below T, and (iii) follows.

(iii): Suppose that $\langle \alpha_\xi : \xi < \omega_1 \rangle$ is an uncountable antichain. For each $\xi < \omega_1$ let $T_\xi \in G$ be such that $\alpha_\xi \in T_\xi$. Let A and $W \in [\omega_1]^{\omega_1}$ be such that A is finite and $T_\xi \cap T_\eta = A$ for all distinct $\xi, \eta \in W$. For each $\xi \in W$ let $C_\xi = \{p \in T_\xi : p \le_{T_\xi} \alpha_\xi\}$. Now

$$W = \bigcup_{C \subseteq A} \{\xi \in W : T_\xi \cap A = C\}.$$

So there is a $W' \in [W]^{\omega_1}$ and a $C \subseteq A$ such that $\forall \xi \in W'[T_\xi \cap A = C]$. Define $\xi \equiv \eta$ iff $\xi, \eta \in W'$ and $C_\xi \cap A = C_\eta \cap A$. This is an equivalence relation with finitely many classes. Hence there is a $W'' \in [W']^{\omega_1}$ such that $\forall \xi, \eta \in W''[C_\xi \cap A = C_\eta \cap A]$. Now for each $\eta \leq \max(A)$ there is at most one $\xi \in W''$ such that $\min(T_\xi \backslash A) = \eta$; this is because $T_\xi \cap T_\mu = A$ for $\xi \neq \mu$. Let $W''' = W'' \backslash \{\xi \in W'' : \min(T_\xi \backslash A) \leq \max(A)\}$. So $|W'''| = \omega_1$.

Now fix $\xi \in W'''$. By the argument producing W''', there is an $\eta \in W'''$ such that $\min(T_\eta \backslash A) > \max(T_\xi)$. Now let $T' = T_\xi \cup T_\eta$ with the ordering

$$\gamma \leq_{T'} \beta \quad \text{iff} \quad \begin{cases} \gamma \leq_{T_\xi} \beta & \text{if } \gamma, \beta \in T_\xi, \\ \gamma \leq_{T_\eta} \beta & \text{if } \gamma, \beta \in T_\eta, \\ \gamma \leq_{T_\xi} \alpha_\xi \text{ and } \beta \in T_\eta \backslash A & \text{if } \gamma \in T_\xi \text{ and } \beta \in T_\eta. \end{cases}$$

Then $T' \leq T_\xi, T_\eta$ and $\alpha_\xi \leq_{T'} \alpha_\eta$. Now $T_\xi \cup T_\eta \Vdash \alpha_\sigma$ and α_η are incomparable, so this is a contradiction.

(iv): This follows from (ii) and (iii) by the argument in the proof of Lemma 15.35.

$\square$

Lemma 15.101. *Let Q consist of all countable sequences $\langle S_\xi : \xi < \alpha \rangle$ with $\alpha < \omega_1$ and $\forall \xi < \alpha[S_\xi \subseteq \xi]$. The order is $\supseteq$. Suppose that G is Q-generic over M. Then $M[G] \models \diamondsuit$.*

Proof. Let $g = \bigcup G$. Say $g = \langle S_\xi : \xi < \omega_1 \rangle$. We claim that g is a $\diamondsuit$-sequence in $M[G]$. Let $A \subseteq \omega_1$ in $M[G]$, and let C be club in $M[G]$. Say $A = \dot{A}^G$ and $C = \dot{C}^G$. Choose $p \in G$ so that $p \Vdash \dot{A} \subseteq \check{\omega}_1$ and $\dot{C}$ is club in $\check{\omega}_1$. Define

$$D = \{q \leq p : \exists \alpha < \omega_1[\alpha \in \mathrm{dmn}(q) \wedge q \Vdash \check{\alpha} \in \dot{C} \wedge \dot{A} \cap \check{\alpha} = \dot{H}_\alpha]\}.$$

We claim that D is dense below p. For, suppose that $r \leq p$. Take $s \leq r$ so that for some $\alpha \in C$, $\mathrm{dmn}(s) = \alpha + 1$ and $s_\alpha = A \cap \alpha$. Suppose that $s \in K$ generic. Then $\alpha \in C$ and $s_\alpha = \dot{A}^K \cap \alpha$. Let $t \leq s$ so that $t \Vdash \check{\alpha} \in \dot{C}$ and $\dot{A} \cap \check{\alpha} = \dot{H}_\alpha$. Then $t \in D$; so D is dense below p.

Choose $q \in G$ with $q \in D$. Choose $\alpha < \omega_1$ so that $\alpha \in \mathrm{dmn}(q)$ and $q \Vdash \check{\alpha} \in \dot{C} \wedge \dot{A} \cap \check{\alpha} = \dot{H}_\alpha$. Then $\alpha \in C$ and $A \cap \alpha = g_\alpha$, as desired. $\square$

Lemma 15.102. *Assume $\diamondsuit$. Then there exist sets $A_\alpha \subseteq \alpha \times \alpha$ such that for every $A \subseteq \omega_1 \times \omega_1$ the set $\{\alpha < \omega_1 : A \cap (\alpha \times \alpha) = A_\alpha\}$ is stationary.*

Proof. Let $f : \omega_1 \to \omega_1 \times \omega_1$ be a bijection. Let $C = \{\alpha < \omega_1 : f[\alpha] = (\alpha \times \alpha)\}$. Then C is club in ω_1: to prove closure, suppose that $\gamma < \omega_1$ is a limit ordinal and $C \cap \gamma$ is unbounded in γ. Take any $\beta < \gamma$. Choose $\alpha \in C$ with $\beta < \alpha < \gamma$. Then $f(\beta) \in f[\alpha] = (\alpha \times \alpha) \subseteq (\gamma \times \gamma)$. This shows that $f[\gamma] \subseteq (\gamma \times \gamma)$. Now take any $(\varepsilon, \delta) \in (\gamma \times \gamma)$. Choose $\alpha \in C$ so that $\varepsilon, \delta < \alpha < \gamma$. Then $f[\alpha] = (\alpha \times \alpha)$, so there is a $\psi < \alpha$ such that $f(\psi) = (\varepsilon, \delta)$. This shows that $(\gamma \times \gamma) \subseteq f[\gamma]$. So C is closed.

To prove that C is unbounded, take any $\alpha < \omega_1$. Define $\beta_0 = \alpha$. Choose β_{2n+1} so that $\beta_{2n} < \beta_{2n+1}$ and $f[\beta_{2n}] \subseteq (\beta_{2n+1} \times \beta_{2n+1})$. Then choose $\beta_{2n+2} > \beta_{2n+1}$ so that $(\beta_{2n+1} \times \beta_{2n+1}) \subseteq f[\beta_{2n+2}]$. Let $\gamma = \bigcup_{n \in \omega} \beta_n$. Then $\alpha < \gamma \in C$.

Let $\langle A_\alpha : \alpha < \omega_1\rangle$ be a $\diamondsuit$-sequence. For each $\alpha < \omega_1$ let $A'_\alpha = f[A_\alpha] \cap (\alpha \times \alpha)$. Take any $A \subseteq \omega_1 \times \omega_1$. To show that $D \stackrel{\text{def}}{=} \{\alpha < \omega_1 : A \cap (\alpha \times \alpha) = A'_\alpha\}$ is stationary it suffices to show that $D \cap C$ is stationary. Let $A' = f^{-1}[A]$. Then $E \stackrel{\text{def}}{=} \{\alpha < \omega_1 : A' \cap \alpha = A_\alpha\}$ is stationary So also $C \cap E$ is stationary. Now note that if $\alpha \in C$, then

$$\begin{aligned} A' \cap \alpha = A_\alpha \quad &\text{iff} \quad f^{-1}[A] \cap f^{-1}[\alpha \times \alpha] = f^{-1}[f[A_\alpha]] \cap f^{-1}[\alpha \times \alpha] \\ &\text{iff} \quad f^{-1}[A \cap (\alpha \times \alpha)] = f^{-1}[f[A_\alpha] \cap (\alpha \times \alpha)] \\ &\text{iff} \quad A \cap (\alpha \times \alpha) = f[A_\alpha] \cap (\alpha \times \alpha) \\ &\text{iff} \quad A \cap (\alpha \times \alpha) = A'_\alpha. \end{aligned}$$

Hence

$$\begin{aligned} \alpha \in C \cap E \quad &\text{iff} \quad \alpha \in C \text{ and } A' \cap \alpha = A_\alpha \\ &\text{iff} \quad \alpha \in C \text{ and } A \cap (\alpha \times \alpha) = A'_\alpha \\ &\text{iff} \quad \alpha \in C \text{ and } \alpha \in D \\ &\text{iff} \quad \alpha \in C \cap D. \end{aligned}$$

So $C \cap D$ is stationary. □

Lemma 15.103. *Let $\diamondsuit'$ be the statement that there is a sequence $\langle h_\alpha : \alpha < \omega_1\rangle$ of functions with $\forall\alpha < \omega_1[h_\alpha : \alpha \to \omega_1]$ such that for every $f : \omega_1 \to \omega_1$ the set $\{\alpha < \omega_1 : f \restriction \alpha = h_\alpha\}$ is stationary.*

Then $\diamondsuit$ is equivalent to $\diamondsuit'$.

Proof. $\Rightarrow$. Assume $\diamondsuit$. By Lemma 15.102 choose $A_\alpha \subseteq \alpha \times \alpha$ for $\alpha < \omega_1$ so that for every $A \subseteq \omega_1 \times \omega_1$ the set $\{\alpha < \omega_1 : A \cap (\alpha \times \alpha) = A_\alpha\}$ is stationary. If $A_\alpha : \alpha \to \alpha$ let $g_\alpha = A_\alpha$; otherwise let $g_\alpha = \emptyset$. Suppose that $g : \omega_1 \to \omega_1$. Let $C = \{\alpha < \omega_1 : g \restriction \alpha = g \cap (\alpha \times \alpha)\}$. We claim that C is club. Closed: suppose that γ is a limit ordinal and $C \cap \gamma$ is unbounded in γ. If $\beta < \gamma$, choose $\alpha \in C$ with $\beta < \alpha < \gamma$. Then $g \restriction \beta \subseteq g \restriction \alpha = g \cap (\alpha \times \alpha) \subseteq g \cap (\gamma \times \gamma)$. This shows that $g \restriction \gamma \subseteq g \cap (\gamma \times \gamma)$. Now suppose that $(\alpha, \beta) \in g \cap (\gamma \times \gamma)$. Choose $\delta \in C$ so that $\alpha, \beta < \delta < \gamma$. Then $(\alpha, \beta) \in g \cap (\delta \times \delta) = g \restriction \delta \subseteq g \restriction \gamma$. This shows that $g \cap (\gamma \times \gamma) \subseteq g \restriction \gamma$. So C is closed.

Unbounded: Let $\alpha < \omega_1$. Define $\beta_0 = \alpha$. Let $\beta_{2n+1} > \beta_{2n}$ be such that $g \restriction \beta_{2n} \subseteq (\beta_{2n+1} \times \beta_{2n+1})$. Let $\beta_{2n+2} > \beta_{2n+1}$ be such that $g \cap (\beta_{2n+1} \times \beta_{2n+1}) \subseteq g \restriction \beta_{2n+2}$. Let $\gamma = \bigcup_{n\in\omega} \beta_n$. Then $\alpha < \gamma \in C$.

Now $D \stackrel{\text{def}}{=} \{\alpha < \omega_1 : g \cap (\alpha \times \alpha) = A_\alpha\}$ is stationary. Hence so is $D \cap C$. For any $\alpha \in D \cap C$ we have $g \restriction \alpha = g \cap (\alpha \times \alpha) = A_\alpha = g_\alpha$, as desired.

$\Leftarrow$. Assume $\diamondsuit'$. For each $\alpha < \omega_1$ let $A_\alpha = \{\beta < \alpha : h_\alpha(\beta) = 1\}$. Suppose that $A \subseteq \omega_1$. Define

$$f(\alpha) = \begin{cases} 1 & \text{if } \alpha \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Then we claim that $\{\alpha < \omega_1 : f \restriction \alpha = h_\alpha\} \subseteq \{\alpha < \omega_1 : A \cap \alpha = A_\alpha\}$. (As desired) For, suppose that $f \restriction \alpha = h_\alpha$. Then for all $\beta < \alpha$,

$$\beta \in A_\alpha \quad \text{iff} \quad h_\alpha(\beta) = 1 \quad \text{iff} \quad f(\beta) = 1 \quad \text{iff} \quad \beta \in A. \qquad \square$$

Lemma 15.104. *If $(T,<)$ is a normal Suslin tree, then $(T,>)\times(T,>)$ is not ccc.*

Proof. Assume that $(T,<)$ is a normal Suslin tree. For each $x\in T$ let p_x and q_x be two distinct immediate successors of x. We claim that $\{(p_x,q_x):x\in T\}$ is an antichain in $(T,>)\times(T,>)$. For, suppose that $(u,v)\ge(p_x,q_x),(p_y,q_y)$ with $x\ne y$; say $x<y$. Then $x<p_x\le u$, $x\le q_x\le v$, $y\le p_y\le u$, and $y\le q_y\le v$. So x,p_x,y,p_y are comparable. Hence $p_x\le y$. Also, x,q_x,y,q_y are comparable. So $q_x\le y$, contradiction. □

Lemma 15.105. *Let G be P_{perf}-generic over M, and $f:\omega\to\omega$ in $M[G]$. Then there is an $h:\omega\to\omega$ in M such that $\forall n\in\omega[f(n)<h(n)]$.*

Proof. Say $\dot f^G=f$. Let $p\in G$ be such that $p\Vdash\dot f:\check\omega\to\check\omega$.

(1) For all $r\le p$ there is a $q\le r$ and a $g:\omega\to\omega$ in M such that for all $n\in\omega$, $q\Vdash\dot f(\check n)<\check g(\check n)$.

In fact, let $F\in M$ be such that $F:\omega\to[\omega]^{<\omega}$ with $q\le r$ such that for all $n\in\omega$, $q\Vdash\dot f(\check n)\in\check F(\check n)$. Define $h(n)=\max(F(n))+1$ for all $n\in\omega$. ($h(n)=1$ if $F(n)=\emptyset$). Then $q\Vdash\dot f(\check n)<\check h(n)$ for all $n\in\omega$. So (1) holds.

In particular,

$$\{q:\exists g:\omega\to\omega[g\in M\wedge\forall n\in\omega[q\Vdash\dot f(\check n)<\check g(\check n)]\}$$

is dense below p. Hence we can choose $q\in G$ and $g\in M$ such that for all $n\in\omega$, $q\Vdash\dot f(\check n)<\check g(\check n)$. Hence for all $n\in\omega$, $f(n)<g(n)$. □

Lemma 15.106. *A complete BA is $(\kappa,2)$-distributive iff it is $(\kappa,2^\kappa)$-distributive.*

Proof. See Handbook Theorem 14.10. □

Lemma 15.107. *If κ is singular and B is a complete BA which is $(<\kappa)$-distributive, then B is κ-distributive.*

Proof. . We will apply Theorem 15.75 Suppose that $f\in M[G]$ with $\mathrm{dmn}(f)=\kappa$. Let $\langle\mu_\alpha:\alpha<\mathrm{cf}(\kappa)\rangle$ be strictly increasing with supremum κ. By Theorem 15.75, $f\restriction\mu_\alpha$ is in M for all $\alpha<\mathrm{cf}(\kappa)$, so $f\in M$. Thus B is κ-distributive by Theorem 15.75. □

Lemma 15.108. *Let P consist of finite functions contained in $\omega\times2$. Then* $\mathrm{RO}(P)$ *is not weakly (ω,ω)-distributive.* □

Lemma 15.109. *Let B be a complete BA and G B-generic over M. Then B is weakly $(\omega.\omega_1)$-distributive iff ω_1 is a cardinal in $M[G]$.*

Proof. By Lemma 15.75, B is weakly (ω,ω_1)-distributive iff every $f\in M[G]$ with $f:\omega\to\omega_1$ is dominated by some $g\in M$ with $g:\omega\to\omega_1$.

First suppose that ω_1^M is a cardinal in $M[G]$. Given $f:\omega\to\omega_1$ in $M[G]$, let $\alpha<\omega_1$ be such that $f:\omega\to\alpha$. Define $g(n)=\alpha$ for all $n\in\omega$.

Second suppose that ω_1^M is not a cardinal in $M[G]$. Let $f:\omega\to\omega_1^M$ be a bijection in $M[G]$. Clearly f is not dominated by a function in M. □

Lemma 15.110. *Suppose that B is a complete BA which is completely κ-generated and λ-saturated. Then $|B| \le \kappa^{<\lambda}$.*

Proof. Let $\mu = \mathrm{sat}(B)$. So $\mu \le \lambda$ and μ is regular. Say X generates B, with $|X| \le \kappa$. Define for $\alpha < \mu$

$$Y_\alpha = \begin{cases} X & \text{if } \alpha = 0, \\ \{\sum Z : Z \subseteq Y_\beta, |Z| < \mu\} \cup \{-Z : Z \in Y_\beta\} & \text{if } \alpha = \beta + 1, \\ \bigcup_{\gamma<\beta} Y_\gamma & \text{if } \alpha \text{ is limit.} \end{cases}$$

By induction. $|Y_\alpha| \le \kappa^{<\mu}$ for all α. Clearly $B = \bigcup_{\alpha<\mu} Y_\alpha$ and $|B| \le \kappa^{<\mu}$. □

Lemma 15.111. *If B is an infinite complete BA which is ccc and countably completely generated, then $|B| = 2^{\aleph_0}$.*

Proof. By Lemma 15.110, $|B| \le 2^{\aleph_0}$. Obviously $2^{\aleph_0} \le |B|$. □

16. Iterated forcing and Martin's axiom

Let P be a forcing poset, and let $\dot{Q} \in M^{\mathrm{RO}(P)}$ be such that $\Vdash_P \dot{Q}$ is a forcing order. We define

$$P * \dot{Q} = \{(p, \dot{q}) : p \in P \text{ and } p \Vdash \dot{q} \in \dot{Q}\};$$
$$(p_1, \dot{q}_1) \leq (p_2, \dot{q}_2) \quad \text{iff} \quad p_1 \leq p_2 \text{ and } p_1 \Vdash \dot{q}_1 \leq \dot{q}_9.$$

Theorem 16.1. *Let P be a forcing poset, and let $\dot{Q} \in M^{\mathrm{RO}(P)}$ be such that $\Vdash_P \dot{Q}$ is a forcing order. Suppose that G is P-generic over M. Let $Q = \dot{Q}_G$, and let H be $H[G]$-generic over M. Then*

$$G * H \stackrel{\text{def}}{=} \{(p, \dot{q}) \in P * \dot{Q} : p \in G \text{ and } \dot{q}_G \in H\}$$

is $(P * \dot{Q})$-generic over M, and $M[G * H] = M[G][H]$.

Proof. $G * H$ is upwards closed: Suppose that $(p, \dot{q}) \in G * H$ and $(p, \dot{q}) \leq (p', \dot{q}')$. Thus $p \leq p'$ and $p \Vdash \dot{q} \leq \dot{q}'$. Also, $p \in G$ and $\dot{q}_G \in H$. So $p' \in G$. Since $p \Vdash \dot{q} \leq \dot{q}'$, we have $\dot{q}_G \leq \dot{q}'_G$. Hence $\dot{q}'_G \in H$.

Next, suppose that $(p_1, \dot{q}_1), (p_2, \dot{q}_2) \in G * H$. Thus $p_1, p_2 \in G$ and $\dot{q}_{1G}, \dot{q}_{2G} \in H$. So there exist $p_3 \in G$ with $p_3 \leq p_1, p_2$ and $\dot{q}_{3G} \in H$ with $\dot{q}_{3G} \leq \dot{q}_{1G}, \dot{q}_{2G}$. Now there is an $r \in G$ such that $r \Vdash \dot{q}_3 \leq \dot{q}_1$, and there is an $s \in G$ such that $s \Vdash \dot{q}_3 \leq \dot{q}_2$. Choose $p_4 \in G$ so that $p_4 \leq p_3, r, s$. Then $(p_4, \dot{q}_3) \in G * H$ and $(p_4, \dot{q}_3) \leq (p_1, \dot{q}_1), (p_2, \dot{q}_2)$. Thus $G * H$ is a filter.

Now suppose that D is a dense subset of $P*Q$; we want to find a member of $D \cap (G*H)$. Let

$$E_1 = \{\dot{q}_G : \exists p \in G[(p, \dot{q}) \in D\}.$$

(1) E_1 is dense in Q.

To prove (1), we first show that

(2) $\quad \forall s \in P \forall \dot{q}_0 [s \Vdash \dot{q}_0 \in \dot{Q} \to \{p \in P : \exists \dot{q}_1 [p \Vdash \dot{q}_1 \leq \dot{q}_0 \wedge (p, \dot{q}_1) \in D]\}$ is dense below s.]

To prove this, suppose $s \in P$, $s \Vdash \dot{q}_0 \in \dot{Q}$, and $r \leq s$. Then $r \Vdash \dot{q}_0 \in \dot{Q}$, so $(r, \dot{q}_0) \in P * \dot{Q}$. Choose $(t, \dot{q}_1) \in D$ such that $(t, \dot{q}_1) \leq (r, \dot{q}_0)$. Thus $t \leq r$ and $t \Vdash \dot{q}_1 \leq \dot{q}_0$. This proves (2).

Now to prove (1), let $\dot{q}_{0G} \in Q = \dot{Q}_G$ be given. Then there is a $p \in G$ such that $p \Vdash \dot{q}_0 \in \dot{Q}$. By (2), choose $r \leq p$ with $r \in G$ so that for some $\dot{q}_1$ we have $r \Vdash \dot{q}_1 \leq \dot{q}_0$ and $(r, \dot{q}_1) \in D$. Then $\dot{q}_{1G} \leq \dot{q}_{0G}$ and $\dot{q}_{1G} \in E_1$, proving (1).

By (1), choose $\dot{q}_G \in E_1 \cap H$. So there is a $p \in G$ such that $(p, \dot{q}) \in D$. Hence $(p, \dot{q}) \in D \cap (G * H)$. This proves that $G * H$ is a generic filter on $P * \dot{Q}$.

Clearly $G * H \in M[G][H]$, so by the minimality of $M[G * H]$ we have $M[G * H] \subseteq M[G][H]$. Since $M[G] \subseteq M[G * H]$ and $H \in M[G * H]$, we have $M[G][H] \subseteq M[G * H]$ by the minimality of $M[G][H]$. $\square$

Lemma 16.2. *If $\dot{A}$ is a B-name and $x \in \mathrm{dmn}(\dot{A})$, then $\dot{A}(x) \leq ||x \in \dot{A}||$.*

Proof. We have $||x \in \dot{A}|| = \sum_{y\in \mathrm{dmn}(\dot{A})}(\dot{A}(y)\cdot||x = y|| \geq \dot{A}(x)\cdot||x = x|| = \dot{A}(x)$. $\square$

Lemma 16.3. *If $r \Vdash \exists x \in \dot{A}\varphi(x)$, then there exist $q \leq r$ and $x \in \mathrm{dmn}(\dot{A})$ such that $q \Vdash x \in \dot{A} \wedge \varphi(x)$.*

Proof. By Lemma 14.31 we have $e(r) = \sum_{x\in\mathrm{dmn}(\dot{A})}(e(r)\cdot \dot{A}(x)\cdot||\varphi(x)||$. Hence there exist p and $x \in \mathrm{dmn}(\dot{A})$ such that $e(p) \leq e(r)\cdot\dot{A}(x)\cdot||\varphi(x)||$. Take $q \leq p, r$. Then by Lemma 16.2, $e(q) \leq \dot{A}(x)\cdot||\varphi(x)|| \leq ||x \in \dot{A}||\cdot||\varphi(x)||$, and the lemma follows. $\square$

Theorem 16.4. *Let P be a forcing poset, and let $\dot{Q} \in M^{\mathrm{RO}(P)}$ be such that $\Vdash_P \dot{Q}$ is a forcing order. Suppose that K is $(P * \dot{Q})$-generic over M. Let*

$$G = \{p \in P : \exists\dot{q}[(p, \dot{q}) \in K]\} \quad \textit{and} \quad H = \{\dot{q}_G : \exists p[(p, \dot{q}) \in K]\}.$$

Then G is P-generic over M, H is $\dot{Q}_G$-generic over $M[G]$, and $K = G * H$.

Proof. First we show that G is a filter. Upwards closed: suppose that $p \in G$ and $p \leq p'$. Choose $\dot{q}$ so that $(p, \dot{q}) \in K$. Then $(p, \dot{q}) \leq (p', \dot{q})$, so $(p', \dot{q}) \in K$ and hence $p' \in G$. Next, suppose that $p_1, p_2 \in G$. Choose $\dot{q}_1, \dot{q}_2$ so that $(p_1, \dot{q}_1), (p_2, \dot{q}_2) \in K$. Then choose $(p_3, \dot{q}_3) \in K$ so that $(p_3, \dot{q}_3) \leq (p_1, \dot{q}_1), (p_2, \dot{q}_2)$. Then $p_3 \in G$ and $p_3 \leq p_1, p_2$. So G is a filter.

Next, suppose that D is dense in P. Let $D_1 = \{(p, \dot{q}) : p \Vdash \dot{q} \in \dot{Q} \text{ and } p \in D\}$.

(1) D_1 is dense in $P * \dot{Q}$.

In fact, let $(p, \dot{q}) \in P * \dot{Q}$. Choose $q \in D$ with $q \leq p$. Then $(q, \dot{q}) \in D_1$ and $(q, \dot{q}) \leq (p, \dot{q})$. So (1) holds.

Now choose $(p, \dot{q}) \in D_1 \cap K$. Then $p \in D \cap G$. So G is generic over P.

Next we show that H is a filter. Upwards closed: suppose that $\dot{q}_G \in H$ and $\dot{q}_G \leq \dot{q}'^G$. Choose p so that $(p, \dot{q}) \in K$. Choose $p' \in G$ so that $p' \Vdash \dot{q} \leq \dot{q}'$. Say $(p', \dot{q}'') \in K$. Choose $(p'', \dot{q}''') \in K$ so that $(p'', \dot{q}''') \leq (p, \dot{q}), (p', \dot{q}'')$. Then $p'' \Vdash \dot{q}''' \leq \dot{q}$. Also, $p'' \leq p'$, so $p'' \Vdash \dot{q} \leq \dot{q}'$. Hence $p'' \Vdash \dot{q}''' \leq \dot{q}'$. Hence $(p'', \dot{q}''') \leq (p, \dot{q}')$, so $(p, \dot{q}') \in K$ and so $\dot{q}'^G \in H$.

Next, suppose that $\dot{q}_{1G}, \dot{q}_{2G} \in H$. Say $(p_1, \dot{q}_1), (p_2, \dot{q}_2) \in K$. Choose $(p_3, \dot{q}_3) \in K$ with $(p_3, \dot{q}_3) \leq (p_1, \dot{q}_1), (p_2, \dot{q}_2)$. Then $\dot{q}_{3G} \in H$. Also, $p_1, p_2, p_3 \in G$ and $p_3 \Vdash \dot{q}_3 \leq \dot{q}_1$, so $\dot{q}_{3G} \leq \dot{q}_{1G}$. Similarly $\dot{q}_{3G} \leq \dot{q}_{2G}$.

Now let $D \in M[G]$ be dense in $\dot{Q}_G$. Let $\dot{D}$ be a name and $p_0 \in G$ such that $p_0 \Vdash \dot{D}$ is dense in $\dot{Q}$. Say $(p_0, \dot{q}) \in K$. Let $E = \{(p_1, \dot{q}') : p_1 \leq p_0 \text{ and } p_1 \Vdash \dot{q}' \in \dot{D} \wedge \dot{q}' \in \dot{Q}\}$.

(2) E is dense below $(p_0, \dot{q})$.

In fact, suppose that $(p_1, \dot{q}') \leq (p_0, \dot{q})$. Then $p_1 \Vdash \dot{D}$ is dense in $\dot{Q}$, so $p_1 \Vdash \exists x \in \dot{D}[x \in \dot{Q} \wedge x \leq \dot{q}']$. Hence by Lemma 16.3 there exist $p_2 \leq p_1$ and $\dot{q}'' \in \mathrm{dmn}(\dot{D})$ such that

$$p_2 \Vdash \dot{q}'' \in \dot{D} \wedge \dot{q}'' \leq \dot{q}' \wedge \dot{q}'' \in \dot{Q}.$$

Hence $(p_2, \dot{q}'') \in E$ and $(p_2, \dot{q}'') \leq (p_1, \dot{q}')$.

So (2) holds and hence we can choose $(p_1, \dot{q}') \in E \cap K$. Hence $\dot{q}'^G \in D \cap H$.

Now if $(p, \dot{q}) \in K$, then clearly $p \in G$ and $\dot{q}_G \in H$, so $(p, \dot{q}) \in G * H$. Now suppose that $(p, \dot{q}) \in G * H$. So $p \in G$ and $\dot{q}_G \in H$. Say $(p, \dot{r}) \in K$. Say $(p', \dot{s}) \in K$, with $\dot{q}_G = \dot{s}_G$. Choose $p'' \in G$ such that $p'' \Vdash \dot{q} = \dot{s}$. Say $(p'', \dot{t}) \in K$. Choose $(p''', \dot{u}) \in K$ such that $(p''', \dot{u}) \leq (p, \dot{r}), (p', \dot{s}), (p'', \dot{t})$. Then $p''' \Vdash \dot{u} \leq \dot{s} \wedge \dot{q} = \dot{s}$, so $p''' \Vdash \dot{u} \leq \dot{q}$. Hence $(p''', \dot{u}) \leq (p, \dot{q})$ and so $(p, \dot{q}) \in K$. $\square$

Now suppose that B is a complete BA, $[\![\dot{C}$ is a complete BA$]\!]_B = 1$, and there are operations $+, \cdot, -, 0, 1$ on $A \stackrel{\text{def}}{=} \{\dot{c} : [\![\dot{c} \in \dot{C}]\!]_B = 1\}$ such that for all $\dot{c}_0, \dot{c}_1, \dot{c}_2 \in A$,

$$[\![\dot{c}_0 + (\dot{c}_1 + \dot{c}_2) = (\dot{c}_0 + \dot{c}_1) + \dot{c}_2]\!]_B = 1,$$

and similarly for the other axioms for BAs.

Define $\dot{c}_0 \equiv \dot{c}_1$ iff $\dot{c}_0, \dot{c}_1 \in A$ and $[\![\dot{c}_0 = \dot{c}_1]\!]_B = 1$.

(1) $\equiv$ is an equivalence relation on $A \stackrel{\text{def}}{=} \{\dot{c} : [\![\dot{c} \in \dot{C}]\!]_B = 1\}$.

For, if $\dot{c} \in A$, then $[\![\dot{c} = \dot{c}]\!]_B = 1$. Clearly $\equiv$ is symmetric and transitive.

Let D be the set $\{[\dot{c}] : \dot{c} \in A\}$.

(2) $[\![\dot{c}_1 = \dot{c}_2]\!]_B \leq [\![\dot{c}_3 + \dot{c}_1 = \dot{c}_3 + \dot{c}_2]\!]_B$.

In fact, suppose that $[\![\dot{c}_1 = \dot{c}_2]\!]_B \not\leq [\![\dot{c}_3 + \dot{c}_1 = \dot{c}_3 + \dot{c}_2]\!]_B$, and let G be a generic ultrafilter such that $[\![\dot{c}_1 = \dot{c}_2]\!]_B \cdot -[\![\dot{c}_3 + \dot{c}_1 = \dot{c}_3 + \dot{c}_2]\!]_B \in G$. Then $\dot{c}_1^G = \dot{c}_2^G$, so

$$(\dot{c}_3 + \dot{c}_1)^G = \dot{c}_3^G + \dot{c}_1^G = \dot{c}_3^G + \dot{c}_2^G = (\dot{c}_3 + \dot{c}_2)^G.$$

Hence there is a $p \in G$ such that $p \leq [\![\dot{c}_3 + \dot{c}_1 = \dot{c}_3 + \dot{c}_2]\!]_B$. Since $-[\![\dot{c}_3 + \dot{c}_1 = \dot{c}_3 + \dot{c}_2]\!]_B \in G$, this is a contradiction. So (2) holds.

Similarly,

(3) $[\![\dot{c}_1 = \dot{c}_2]\!]_B \leq [\![\dot{c}_3 \cdot \dot{c}_1 = \dot{c}_3 \cdot \dot{c}_2]\!]_B$.

(4) $[\![\dot{c}_1 = \dot{c}_2]\!]_B \leq [\![-\dot{c}_1 = -\dot{c}_2]\!]_B$.

For $\dot{c}_0, \dot{c}_1 \in A$ we define

$$\begin{aligned} &[\dot{c}_0] + [\dot{c}_1] = [\dot{c}_0 + \dot{c}_1]; \quad [\dot{c}_0] \cdot [\dot{c}_1] = [\dot{c}_0 \cdot \dot{c}_1]; \quad -[\dot{c}_0] = [-\dot{c}_0]; \\ &0_D = [0]; \quad 1_D = [1]. \end{aligned}$$

(5) These operations on D are well-defined.

For example, suppose that $[\dot{c}_0] = [\dot{c}_0']$ and $[\dot{c}_1] = [\dot{c}_1']$. Thus $[\![\dot{c}_0 = \dot{c}_0']\!]_B = 1$ and $[\![\dot{c}_1 = \dot{c}_1']\!]_B = 1$. Hence by (2), $[\![\dot{c}_0 + \dot{c}_1 = \dot{c}_0' + \dot{c}_1]\!]_B = 1 = [\![\dot{c}_0' + \dot{c}_1 = \dot{c}_0' + \dot{c}_1']\!]_B$. Thus $[\![\dot{c}_0 + \dot{c}_1 = \dot{c}_0' + \dot{c}_1']\!]_B = 1$. This proves that $+$ is well-defined.

(6) $(D, +, \cdot, -, 0, 1)$ is a BA.

It is routine to check this. For example, the associative law for $+$ is checked like this:

$$[\dot{c}_1] + ([\dot{c}_2] + [\dot{c}_3]) = [\dot{c}_1 + (\dot{c}_2 + \dot{c}_3)] = [(\dot{c}_1 + \dot{c}_2) + \dot{c}_3] = ([\dot{c}_1] + [\dot{c}_2]) + [\dot{c}_3].$$

(7) D is complete.

For, suppose that $X \subseteq D$. Say $X = \{[\dot{c}] : \dot{c} \in X'\}$. Let $\dot{X}'$ be a name such that $\mathrm{dmn}(\dot{X}') = X'$ and $\dot{X}'(\dot{c}) = 1$ for all $\dot{c} \in X'$. Now

$$[\![\forall X[X \subseteq \dot{C} \to \exists y \in \dot{C}[\forall x \in X[x \leq y] \wedge \forall z \in \dot{C}[\forall x \in X[x \leq z] \to y \leq z]]]]\!]_B = 1.$$

Hence

$$[\![\dot{X}' \subseteq \dot{C} \to \exists y \in \dot{C}[\forall x \in \dot{X}'[x \leq y] \wedge \forall z \in \dot{C}[\forall x \in \dot{X}'[x \leq z] \to y \leq z]]]]\!]_B = 1.$$

Now $[\![\dot{X}' \subset \dot{C}]\!]_B = \prod_{\dot{c} \in X'} [\![\dot{c} \in \dot{C}]\!]_B = 1$. Hence

$$[\![\exists y \in \dot{C}[\forall x \in \dot{X}'[x \leq y] \wedge \forall z \in \dot{C}[\forall x \in \dot{X}'[x \leq z] \to y \leq z]]]]\!]_B = 1.$$

By Lemma 14.30 there is a name $\dot{d}$ such that

$$[\![\dot{d} \in \dot{C} \wedge \forall x \in \dot{X}'[x \leq \dot{d}] \wedge \forall z \in \dot{C}[\forall x \in \dot{X}'[x \leq z] \to \dot{d} \leq z]]]]\!]_B = 1.$$

Thus $[\![\dot{d} \in \dot{C}]\!]_B = 1$, so $\dot{d} \in A$ and $[\dot{d}] \in D$. Now $[\![\forall x \in \dot{X}'[x \leq \dot{d}]]\!]_B = 1$, so for all $x \in X'[[\![x \leq \dot{d}]\!]_B = 1]$. Hence $\forall x \in X[x \leq [\dot{d}]]$. Now suppose that $\forall x \in X[x \leq [\dot{e}]]$. Thus $\forall x \in \dot{X}'[[\![x \leq \dot{e}]\!]_B = 1]$. Hence $[\![\dot{d} \leq \dot{e}]\!]_B = 1$, so $[\dot{d}] \leq [\dot{e}]$. All of this verifies that $[\dot{d}] = \sum X$, proving (7).

(8) $\forall b \in B \exists \dot{c}[[[\![\dot{c} = 1]\!]_B = b$ and $[\![\dot{c} = 0]\!]_B = -b]$ and $\forall \dot{d}[[\![\dot{d} = 1]\!]_B = b$ and $[\![\dot{d} = 0]\!]_B = -b \to [\![\dot{c} = \dot{d}]\!]_B = 1$.

In fact, by Lemma 14.29 choose $\dot{c}$ such that $b \leq [\![\dot{c} = 1]\!]_B$ and $-b \leq [\![\dot{c} = 0]\!]_B$. Now $[\![\dot{c} = 1]\!]_B \cdot [\![\dot{c} = 0]\!]_B = 0$, so $[\![\dot{c} = 1]\!]_B = b$ and $[\![\dot{c} = 0]\!]_B = -b]$. Suppose that also $[\![\dot{d} = 1]\!]_B = b$ and $[\![\dot{d} = 0]\!]_B = -b]$. Hence

$$1 = b + -b = [\![\dot{c} = 1]\!]_B \cdot [\![\dot{d} = 1]\!]_B + [\![\dot{c} = 0]\!]_B \cdot [\![\dot{d} = 0]\!]_B \leq [\![\dot{c} = \dot{d}]\!]_B,$$

proving (8).

Now for any $b \in B$, let $\pi(b) = [\dot{c}]$ with $\dot{c}$ as in (8); this is justified by (8).

(9) π preserves $+$.

For, suppose that $b_1, b_1 \in B$. Say $\pi(b_i) = [\dot{c}_i]$ for $i < 2$. Then

$$\begin{aligned} 1 = [\![\dot{c}_0 = 1]\!]_B \cdot [\![\dot{c}_1 = 1]\!]_B \leq [\![\dot{c}_0 + \dot{c}_1 = 1]\!]_B \quad &\text{and} \\ 1 = [\![\dot{c}_0 = 0]\!]_B \cdot [\![\dot{c}_1 = 0]\!]_B \leq [\![\dot{c}_0 + \dot{c}_1 = 0]\!]_B. & \end{aligned}$$

Hence $\pi(b_0 + b_1) = \pi(b_0) + \pi(b_1)$, and (9) holds.

(10) π preserves $-$.

In fact, suppose that $b \in B$. Say $\pi(b) = [\dot{c}]$. Then

$$[\![-\dot{c} = 1]\!]_B = [\![\dot{c} = 0]\!]_B = -b \quad \text{and} \quad [\![-\dot{c} = 0]\!]_B = [\![\cdot c = 1]\!]_B = b;$$

Hence (10) holds.

(11) π is one-one.

For, suppose that $b_0, b_1 \in B$ and $\pi(b_0) = \pi(b_1)$. Say $\pi(b_i) = [\dot{c}_i]$ for $i < 2$. Thus $[\dot{c}_0] = [\dot{c}_1]$, so $[\![\dot{c}_0 = \dot{c}_1]\!]_B = 1$. Hence

$$b_1 = [\![\dot{c}_1 = 1]\!]_B \cdot [\![\dot{c}_0 = \dot{c}_1]\!]_B \leq [\![\dot{c}_0 = 1]\!]_B = b_0,$$

and similarly $b_0 \leq b_1$. So (11) holds.

(12) π is a complete embedding.

For, suppose that $X \subseteq B$ and $b = \sum X$. Say $\pi(b) = [\dot{c}]$ and $\forall x \in X[\pi(x) = [\dot{d}_x]]$, with

$$\begin{aligned} &[\![\dot{c} = 1]\!]_B = b \quad \text{and} \quad [\![\dot{c} = 0]\!]_B = -b; \\ &[\![\dot{d}_x = 1]\!]_B = x \quad \text{and} \quad [\![\dot{d}_x = 0]\!]_B = -x. \end{aligned}$$

Since π is an isomorphic embedding, $\pi(b)$ is an upper bound of $\pi[X]$. Now suppose that $[\dot{e}]$ is another upper bound; we want to show that $\pi(b) \leq [\dot{e}]$, i.e., $\dot{c} \leq \dot{e}$. Now if $x \in X$, then $x = [\![\dot{d}_x \cdot \dot{e} = \dot{d}_x]\!]_B \cdot [\![\dot{d}_x = 1]\!]_B \leq [\![\dot{e} = 1]\!]_B$. Hence $[\![\dot{c} = 1]\!]_B = b \leq [\![\dot{e} = 1]\!]_B$. Hence

$$\begin{aligned} [\![\dot{c} \cdot \dot{e} = \dot{c}]\!]_B &= [\![\dot{c} \cdot \dot{e} = \dot{c}]\!]_B \cdot (b + -b) \\ &= [\![\dot{c} \cdot \dot{e} = \dot{c}]\!]_B \cdot ([\![\dot{c} = 1]\!]_B + [\![\dot{c} = 0]\!]_B) \\ &= [\![\dot{e} = 1]\!]_B + [\![0 = 0]\!]_B = 1. \end{aligned}$$

So $\dot{c} \leq \dot{e}$.

The algebra D constructed here is denoted by $B * \dot{C}$.

Lemma 16.5. $P \times Q \cong P * \check{Q}$.

Proof. Define $f(p,q) = (p, \check{q})$. Then $p \Vdash \check{q} \in \check{Q}$, so $(p, \check{q}) \in P * \check{Q}$. Given $(p, \check{q}) \in P * \check{Q}$, we have $f(p,q) = (p, \check{q})$. Finally,

$$\begin{aligned} (p_1, q_1) \leq (p_2, q_2) \quad &\text{iff} \quad p_1 \leq p_2 \text{ and } q_1 \leq q_2 \\ &\text{iff} \quad p_1 \leq p_2 \text{ and } p_1 \Vdash \check{q}_1 \leq \check{q}_2 \\ &\text{iff} \quad (p_1, \check{q}_1) \leq (p_2, \check{q}_2). \end{aligned}$$

□

Theorem 16.6. *Let κ be regular and uncountable. Assume that $\mathbb{1}_P \Vdash \dot{Q}$ has the κ-cc. Then $P * \dot{Q}$ has the κ-cc.*

Proof. Suppose that $\langle (p_\alpha, \dot{q}_\alpha) : \alpha < \kappa \rangle$ is a system of pairwise incompatible elements of $P * \dot{Q}$. Define $\operatorname{dmn}(\dot{Z}) = \{\check{\alpha} : \alpha < \kappa\}$ and for each $\alpha < \kappa$, $\dot{Z}(\check{\alpha}) = e(p_\alpha)$. Then for G generic over P, $\dot{Z}_G = \{\alpha < \kappa : \exists r \in G[e(r) \leq e(p_\alpha)\}$. Hence for any $\alpha < \kappa$,

$$||\check{\alpha} \in \dot{Z}|| = \sum_{\beta < \kappa} (\dot{Z}(\check{\beta}) \cdot ||\check{\alpha} = \check{\beta}) = \dot{Z}(\check{\alpha}) = e(p_\alpha).$$

(1) $\forall\alpha,\beta<\kappa[\alpha\neq\beta\to[p_\alpha\perp p_\beta \text{ or } \forall r\leq p_\alpha,p_\beta[r\Vdash\dot q_\alpha\perp\dot q_\beta]]]$.

In fact, suppose that $\alpha,\beta<\kappa$, $\alpha\neq\beta$, p_α and p_β are compatible, $r\leq p_\alpha,p_\beta$, and $r\not\Vdash\dot q_\alpha\perp\dot q_\beta$. Thus

$$r\not\Vdash\forall x\in\dot Q[x\not\leq\dot q_\alpha \text{ or } x\not\leq\dot q_\beta].$$

Hence

$$e(r)\not\leq||\forall x\in\dot Q[x\not\leq\dot q_\alpha \text{ or } x\not\leq\dot q_\beta]||,$$

so we can choose s so that

$$e(s)\subseteq e(r)\cdot-||\forall x\in\dot Q[x\not\leq\dot q_\alpha \text{ or } x\not\leq\dot q_\beta]||,$$

and then choose $t\leq s,r$; then

$$\begin{aligned}e(t)&\leq-||\forall x\in\dot Q[x\not\leq\dot q_\alpha \text{ or } x\not\leq\dot q_\beta]||\\&=||\exists x\in\dot Q[x\leq\dot q_\alpha \text{ and } x\leq\dot q_\beta];\end{aligned}$$

so $t\Vdash\exists x\in\dot Q[x\leq\dot q_\alpha \text{ and } x\leq\dot q_\beta]$. It follows by Lemma 14.30 that there exist $\dot q'\in\mathrm{dmn}(\dot Q)$ and $u\leq t$ such that

$$u\Vdash\dot q'\in\dot Q\wedge\dot q'\leq\dot q_\alpha\wedge\dot q'\leq\dot q_\beta.$$

Hence $(u,\dot q')\leq(p_\alpha,\dot q_\alpha),(p_\beta,\dot q_\beta)$, contradiction. So (1) holds.

(2) If G is generic and $\alpha,\beta\in\dot Z_G$, with $\alpha\neq\beta$, then $\dot q_\alpha\perp\dot q_\beta$.

In fact, choose $r,s\in G$ such that $e(r)\leq e(p_\alpha)$ and $e(s)\leq e(p_\beta)$. Then p_α and p_β are compatible, so if $r\leq p_\alpha,p_\beta$, then $\dot q_G\perp\dot q_\beta$ by (1).

From (2) it follows that $|\dot Z^G|<\kappa$ for any generic G, since $\dot Q$ has the κ-cc. Thus

(3) $\mathbb{1}\Vdash\exists\gamma<\kappa[\dot Z\subseteq\gamma]$.

(4) $\forall p\exists q\leq p\exists\gamma<\kappa[q\Vdash\dot Z\subseteq\gamma]$.

To see this, use (3) and Lemma 14.30.

Now let $W\subseteq P$ be maximal such that its elements are pairwise incompatible, and for each $p\in W$ there is a $\gamma_p<\kappa$ such that $p\Vdash\dot Z\subseteq\gamma_p$. $|W|<\kappa$ by the κ-cc for P, so $\delta\stackrel{\text{def}}{=}\sup_{p\in W}\gamma_p<\kappa$. Now

(5) $\mathbb{1}\Vdash\dot Z\subseteq\delta$.

For, let G be generic. Since W is a maximal antichain, let $p\in G\cap W$. Then $\dot Z^G\subseteq\gamma_p\subseteq\delta$. So (5) holds. But $p_\delta\Vdash\check\delta\in\dot Z$, so $p_\delta\Vdash\check\delta<\check\delta$, contradiction. □

A function $f:P\to Q$ is a *complete embedding* iff the following conditions hold:

(i) $\forall p_1,p_2\in P[p_1\leq p_2\to f(p_1)\leq f(p_2)]$.

(ii) $\forall p_1,p_2\in P[p_1\perp p_2\leftrightarrow f(p_1)\perp f(p_2)]$.

(iii) $\forall q\in Q\exists p\in P\forall p'\in P[p'\leq p\to f(p')$ and q are compatible$]$.

Lemma 16.7. *Let M be a transitive model of ZFC, with forcing posets $P, Q \in M$, and suppose that $i : P \to Q$ is a complete embedding, with $i \in M$. Let G be Q-generic over M. Then $i^{-1}[G]$ is P-generic over M.*

Proof. First $i^{-1}[G]$ is upwards closed. For, suppose that $p \in i^{-1}[G]$ and $p \leq q$. Then $i(p) \in G$ and $i(p) \leq i(q)$ by (ii) in the definition of complete embedding, so $i(q) \in G$ and hence $q \in i^{-1}[G]$.

Now suppose that D is dense in P. We claim that $i[D]$ is predense in Q. For, suppose that $q \in Q$. Choose $p \in P$ so that $\forall p' \leq p[i(p')$ and q are compatible. Choose $p' \in D$ with $p' \leq p$. Then $i(p') \in i[D]$ and $i(p')$ and q are compatible. Since $i[D]$ is predense, choose $p \in D$ so that $i(p) \in G$. Then $p \in D \cap i^{-1}[G]$.

Finally, any two elements of $i^{-1}[G]$ are compatible. For, suppose that $p, q \in i^{-1}[G]$. Thus $i(p), i(q) \in G$, so they are compatible. By (ii) in the definition of complete embedding, also p and q are compatible. □

Theorem 16.8. *If $f : P \to Q$ is a complete embedding, then there is a complete embedding $g : \mathrm{RO}(P) \to \mathrm{RO}(Q)$ such that $\forall p \in P[g(e(p)) = e(f(p))]$.*

Proof. For all $p \in P$ let $g(e(p)) = e(f(p))$. First we show that g is a well-defined isomorphism of $e[P]$ onto $e[f[P]]$. By Sikorski's criterion, it suffices to show that the following two conditions are equivalent:

(1) $e(p_0) \cap \ldots \cap e(p_{m-1}) \cap -e(p_m) \cap \ldots \cap -e(p_n) = 0$;

(2) $e(f(p_0)) \cap \ldots \cap e(f(p_{m-1})) \cap -e(f(p_m)) \cap \ldots \cap -e(f(p_n)) = 0$.

First suppose that $e(p_0) \cap \ldots \cap e(p_{m-1}) \cap -e(p_m) \cap \ldots \cap -e(p_n) \neq 0$; say

$$e(r) \leq e(p_0) \cap \ldots \cap e(p_{m-1}) \cap -e(p_m) \cap \ldots \cap -e(p_n);$$

thus

$$e(r) \cap (-e(p_0) + \cdots + -e(p_{m-1}) + e(p_m) + \cdots + e(p_n)) = 0.$$

Suppose that $i < m$. Then $e(r) \leq e(p_i)$, so by Theorem 14.6(v), $\{s : s \leq r, p_i\}$ is dense below r. We claim that $\{q : q \leq f(r), f(p_i)\}$ is dense below $f(r)$. For, suppose that $q \leq f(r)$. By (iii) choose $s \in P$ such that $\forall s' \in P[s' \leq s \to f(s')$ and q are compatible]. Then $\forall s' \leq s \exists u \leq f(s'), f(r)$, so $\forall s' \leq s[s'$ and r are compatible], hence $\forall s' \leq s \exists u \leq s', r$, hence $\exists v \leq u, p_i$. So there is a $v \leq s, p_i$. Hence $f(v)$ and q are compatible, so $f(p_i)$ and q are compatible. This proves the claim. By Lemma 14.6(v), $e(f(r)) \leq e(f(p_i))$.

Suppose that $m \leq i \leq n$. Then $e(r) \cap e(p_i) = 0$, so $r \perp p_i$, hence $f(r) \perp f(p_i)$, hence $e(f(r)) \cap e(f(p_i)) = 0$.

We have shown that

$$(*) \qquad e(f(p_0)) \cap \ldots \cap e(f(p_{m-1})) \cap -e(f(p_m)) \cap \ldots \cap -e(f(p_n)) \neq 0$$

Conversely, suppose that $(*)$ holds, and let

$$e(r) \leq e(f(p_0)) \cap \ldots \cap e(f(p_{m-1})) \cap -e(f(p_m)) \cap \ldots \cap -e(f(p_n));$$

Hence

$$e(r) \cap (-e(f(p_0)) + \cdots + -e(f(p_{m-1})) + e(f(p_m)) + \cdots + e(f(p_n))) = 0.$$

Suppose that $i < m$. Then $e(r) \leq e(f(p_i))$, so by Theorem 14.6(v), $\{s : s \leq r, f(p_i)\}$ is dense below r. Choose $s \in P$ so that $\forall s' \leq s[f(s')$ and r are compatible$]$. We claim that $\{t : t \leq s, p_i\}$ is dense below s. For, let $s' \leq s$. Then $f(s')$ and r are compatible; say $u \leq f(s'), r$. Then there is a $v \leq u$ such that $v \leq f(p_i)$. So $v \leq f(s'), f(p_i)$, so s' and p_i) are compatible. This proves the claim. It follows from Theorem 14.6(v) that $e(s) \leq e(p_i)$.

Suppose that $m \leq i \leq n$. Then $e(r) \cap e(f(p_i)) = 0$, so $r \perp f(p_i)$. If s and p_i are compatible, choose $s' \leq s, p_i$. Then $f(s')$ and r are compatible, so $f(p_i))$ and r are compatible, contradiction. Hence $s \perp p_i$, and so $e(s) \cap e(p_i) = 0$.

We have shown that $e(p_0) \cap \ldots \cap e(p_{m-1}) \cap -e(p_m) \cap \ldots \cap -e(p_n) \neq 0$. Thus (1) and (2) are equivalent. Now the Theorem follows from the remark at the bottom of page 57 of the Handbook. □

Lemma 16.9. *Let f be a complete embedding of A into B. We define $f^* : V^A \to V^B$, defining $f^* \restriction v^A_\alpha$ and proving it is one-one by recursion. The case $\alpha = 0$ is trivial. The induction step for α limit is clear. Now suppose that $x \in V^A_{\alpha+1} \backslash V^A_\alpha$. So x is a function with $\mathrm{dmn}(x) \subseteq V^A_\alpha$ and range $\subseteq A$. Let $\mathrm{dmn}(f^*(x)) = \{z : \exists y \in \mathrm{dmn}(x)[z = f^*(y)]\}$ and set $(f^*(x))(z) = f(x(y))$ with $z = f^*(y)$. Suppose that $x, x' \in V^A_{\alpha+1}$, $f^*(x) = f^*(x')$, and $x \neq x'$. Say $y \in x \backslash x'$. Then $f^*(y) \in f^*(x) = f^*(x')$, so there is a $v \in x'$ such that $f^*(y) = f^*(v)$. Now $y, v \in V^A_\alpha$, so it follows that $y = v \in x'$, contradiction.*

Then

(i) For any $p_0, \ldots p_{m-1} \in V^A$ and any formula $\varphi(v_0, \ldots, v_{m-1})$ we have

$$f([\![\varphi(p_0, \ldots, p_{m-1})]\!]_A = \langle\varphi(f^*(p_0), \ldots, f^*(p_{m-1}))]\!]_B.$$

(ii) $f^*(\check{a}) = \check{a}$.

(iii) If $p : \check{a} \to A$, then $f^*(p) = f \circ p$.

Proof. First we show:

Claim. *If $x, y \in V^A$, then*

(i) $f([\![x = y]\!]_A) = [\![f^*(x) = f^*(y)]\!]_B$.

(ii) $f([\![x \in y]\!]_A) = [\![f^*(x) \in f^*(y)]\!]_B$.

(iii) $f([\![x \subseteq y]\!]_A) = [\![f^*(x) \subseteq f^*(y)]\!]_B$.

Proof. Induction:

$$\begin{aligned} f([\![x \subseteq y]\!]_A) &= f\left(\prod^A_{t \in \mathrm{dmn}(x)} (x(t) \Rightarrow [\![t \in y]\!]_A)\right) \\ &= f\left(\prod^A_{t \in \mathrm{dmn}(x)} \left(x(t) \Rightarrow \sum^A_{s \in \mathrm{dmn}(y)} (y(s) \cdot [\![t = s]\!]_A)\right)\right) \end{aligned}$$

$$= \prod_{t\in\mathrm{dmn}(x)}^{B}\left(f(x(t)) \Rightarrow \sum_{s\in\mathrm{dmn}(y)}^{B}(f(y(s))\cdot[\![f^*(t)=f^*(s)]\!]_B)\right)$$
$$= \prod_{v\in\mathrm{dmn}(f^*(x))}^{B}\left((f^*(x))(v) \Rightarrow \sum_{u\in\mathrm{dmn}(f^*(y))|}^{B}((f^*(y))(u))\cdot[\![v=u]\!]_B)\right)$$
$$= [\![f^*(x)\subseteq f^*(y)]\!].$$

The other parts of the claim are proved similarly, and the first part of Lemma itself follows by an easy induction on formulas.

For (ii), we have $\mathrm{dmn}(f^*(\check{a})) = \{z : \exists b \in a[z = f^*(\check{b})]\} = \{z : \exists b \in a[z = \check{b}]\} = \check{a}$.

For (iii), we have $\mathrm{dmn}(f^*(p)) = \{\{z : \exists b \in \mathrm{dmn}(p)[z = f^*(b)]\} = \{z\exists b \in a[z = \check{b}]\} = \check{a}$. For any $b \in a$, $(f^*(p))(\check{b}) = f(p(\check{b}))$. ☐

Lemma 16.10. *For any $p \in P$ let $f(p) = (p, 1)$. Then f is a complete embedding of P into $P * \dot{Q}$.*

Proof. Obviously $p_1 \le p_2 \to (p_1, 1) \le (p_2, 1)$. Next,

$$\begin{array}{rll} p_1 \not\perp p_2 & \text{iff} & \exists p_3 \le p_1.p_2 \\ & \text{iff} & \exists (p_3, \dot{q}) \le (p_1, 1), (p_2, 1) \\ & \text{iff} & (p_1, 1) \not\perp (p_2, 1). \end{array}$$

Now, given $(p, \dot{q}) \in P * \dot{Q}$, suppose that $p' \le p$. Then $(p', 1)$ and $(p, \dot{q})$ are compatible, since $(p', \dot{q}) \le (p', 1), (p, \dot{q})$. ☐

Corollary 16.11. $\mathrm{RO}(P)$ *can be completely embedded in* $\mathrm{RO}(P * \dot{Q})$. ☐

Lemma 16.12. *If $P * \dot{Q}$ satisfies the κ-cc, then P satisfies the κ-cc.*

Proof. Assume that $P * \dot{Q}$ satisfies the κ-cc. By Lemma 16.10, the function f given by $f(p) = (p, 1)$ is a complete embedding if P into $P * \dot{Q}$. It $\langle p_\alpha : \alpha < \kappa\rangle$ is a system of pairwise incompatible elements, then by (ii) in the definition, $\langle (p_\alpha, 1) : \alpha < \omega_1\rangle$ is a system of pairwise incompatible elements, contradiction. ☐

Lemma 16.13. *If $P * \dot{Q}$ satisfies the κ-cc, then $\mathbb{1} \Vdash \dot{Q}$ satisfies the κ-cc.*

Proof. It suffices to prove the following claim:

Claim Suppose that $\dot{W}$ is a name and $p \in P$ is such that $p \Vdash \dot{W} \subseteq \dot{Q}$ and $|\dot{W}| = \kappa$. Then there is a $q \le p$ such that $q \Vdash \dot{W}$ is not an antichain.

Assuming the claim holds, suppose that $\mathbb{1} \not\Vdash \dot{C}$ has the κ-cc. Thus

$$[\![\forall W \subseteq \dot{Q}[|W| = \check{\kappa} \to W \text{ is not an antichain}]\!] \neq 1],$$

so there is a p such that

$$p \Vdash \exists W \subseteq \dot{Q}[|W| = \check{\kappa} \wedge W \text{ is an antichain}]$$

Now by Lemma 14.30 there is a name $\dot{U}$ such that $p \Vdash \dot{U} \subseteq \dot{Q} \wedge |\dot{U}|| = \check{\kappa} \wedge \check{U}$ is an antichain. This contradicts the claim.

Proof of claim. Assume that $\dot{W}$ is a name and $p \in P$ is such that $p \Vdash \dot{W} \subseteq \dot{Q}$ and $|\dot{W}| = \kappa$. Let $\dot{f}$ be a name such that $p \Vdash \dot{f}$ is a one-one function mapping $\check{\kappa}$ onto $\dot{W}$. For every $\alpha < \kappa$, $p \Vdash \exists x \in \dot{W}[\dot{f}(\check{\alpha}) = x]$. By Lemma 14.30 there is a $\dot{c}_\alpha \in M^{\mathrm{RO}(P*\dot{Q})}$ such that $p \Vdash \dot{c}_\alpha \in \dot{W}$ and $\dot{f}(\check{\alpha}) = \dot{c}_\alpha$. Then $p \Vdash \dot{c}_\alpha \in \dot{Q}$, so $(p, \dot{c}_\alpha) \in P * \dot{Q}$. Now $\forall \alpha, \beta < \kappa[\alpha \neq \beta \to p \Vdash \dot{c}_\alpha \neq \dot{c}_\beta]$. Take any $\alpha, \beta < \kappa$ with $\alpha \neq \beta$. If $(p, \dot{c}_\alpha) = (p, \dot{c}_\beta)$, then $p \Vdash \dot{c}_\alpha = \dot{c}_\beta$, contradiction. So $(p, \dot{c}_\alpha) \neq (p, \dot{c}_\beta)$. Since $P * \dot{Q}$ satisfies the κ-cc, choose $\alpha, \beta < \kappa$ with $\alpha \neq \beta$ such that $(p, \dot{c}_\alpha)$ and $(p, \dot{c}_\beta)$ are compatible. Say $(p', \dot{d}) \leq (p, \dot{c}_\alpha), (p, \dot{c}_\beta)$. Then $p' \Vdash \dot{d} \leq \dot{c}_\alpha$ and $p' \Vdash \dot{d} \leq \dot{c}_\beta$. Hence $p' \leq p$ and $p' \Vdash \dot{W}$ is not an antichain. $\square$

Corollary 16.14. *If P and Q satisfy the κ-cc, then $P \times Q$ satisfies the κ-cc iff $\mathbb{1} \Vdash \check{Q}$ satisfies the κ-cc.*

Proof. Assume that P and Q satisfy the κ-cc.

Suppose that $P \times Q$ satisfies the κ-cc. By Lemmas 16.5 and 16.12, $\mathbb{1} \Vdash \check{Q}$ satisfies the κ-cc.

Suppose that $\mathbb{1} \Vdash \check{Q}$ satisfies the κ-cc. By Theorem 16.6 and Lemma 16.5, $P \times Q$ satisfies the κ-cc. $\square$

Lemma 16.15. *If P is κ-closed and $\mathbb{1} \Vdash \dot{Q}$ is $\check{\kappa}$-closed, then $P * \dot{Q}$ is κ-closed.*

Proof. Suppose that $\lambda \leq \kappa$ and $\langle (p_\alpha, \dot{q}_\alpha) : \alpha < \lambda \rangle$ is a decreasing sequence. Then $p_0 \geq p_1 \geq \cdots \geq p_\alpha \geq \ldots$ for $\alpha < \lambda$, so choose $p' \leq p_\alpha$ for all $\alpha < \lambda$. Then $p' \Vdash \dot{q}_\varphi \leq \dot{q}_\beta$ for $\beta < \lambda$, so $p' \Vdash \exists \dot{r}[\dot{r} \in \dot{Q}$ and $\forall \alpha < \lambda[\dot{r} \leq \dot{q}_\alpha]]$. By Lemma 14.30 there is a $\dot{s}$ such that $p' \Vdash \dot{s} \in \dot{Q}$ and $\forall \alpha < \lambda[\dot{s} \leq \dot{q}_\alpha]]$. Hence $(p', \dot{s}) \leq (p_\alpha, \dot{q}_\alpha)$ for all $\alpha < \lambda$. $\square$

For each $\alpha \geq 1$ a *finite support iteration of length* α is a pair $(\langle P_\xi : \xi \leq \alpha \rangle, \langle Q_\xi : \xi < \alpha \rangle)$ with the following properties:

(i) Each P_ξ is a forcing poset.

(ii) Each Q_ξ is a P_ξ-name, and $\mathbb{1}_{P_\xi} \Vdash Q_\xi$ is a forcing poset.

(iii) Each $p \in P_\xi$ is a sequence of length ξ such that $\forall \eta < \xi[p_\eta \in \mathrm{dmn}(Q_\eta)]$

(iv) If $\eta < \xi$ and $p \in P_\xi$, then $p \restriction \eta \in P_\eta$.

(v) If $\eta < \xi$, $p \in P_\eta$, and p' is the function with domain ξ such that $p' \restriction \eta = p$ and $\forall \mu \in [\eta, \xi)[p'_\mu = 1]$, then $p' \in P_\xi$.

(vi) $1_\xi = \langle 1_\eta : \eta < \xi \rangle$.

(vii) If $p, p' \in P_\xi$, then $p \leq_\xi p'$ iff $\forall \mu < \xi[p \restriction \mu \Vdash_{P_\mu} p_\mu \leq p'_\mu]$.

(viii) If $\xi + 1 \leq \alpha$, then $P_{\xi+1} = \{p^\frown\langle q \rangle : p \in P_\xi$ and $q \in \mathrm{dmn}(Q_\xi)$ and $p \Vdash_{P_\xi} q \in Q_\xi$.

(ix) $\forall \xi \leq \alpha[\xi$ limit $\to P_\xi = \{p \in \prod_{\eta<\xi} P_\eta : \{\eta < \xi : p_\eta \neq 1\}$ is finite$\}]$.

Note that $P_0 = \{\emptyset\}$ with the natural order. Then $\emptyset\downarrow = \{\emptyset\}$ and $\mathscr{O}_{P_0} = \{\emptyset, \{\emptyset\}\}$. This is a topology on $\{\emptyset\}$. We have $\mathrm{cl}(\emptyset\downarrow) = \{\emptyset\}$ and $\mathrm{int}(\mathrm{cl}(\emptyset\downarrow)) = \{\emptyset\}$. Thus $\mathrm{RO}(P_0) = \{\emptyset, \{\emptyset\}\}$. Let $B = \mathrm{RO}(P_0)$.

$$\begin{aligned}
&V_0^B = \emptyset;\\
&V_1^B = \{\emptyset\};\\
&V_2^B = \{\emptyset, \{\{(\emptyset, \emptyset)\}\}\}
\end{aligned}$$

V_3^B has 9 elements: $\emptyset$, four functions with one element, and four with two elements. Altogether there is a proper class of P_0-names.

Given an iteration as above, with $p \in P_\xi$ let $\mathrm{supp}(p) = \{\eta < \xi : p_\eta \neq 1\}$.

Lemma 16.16. $\mathrm{supp}(p)$ *is finite for all* $\xi \leq \alpha$ *and all* $p \in P_\xi$.

Proof. An easy induction on ξ. □

Lemma 16.17. $P_0 = \{\emptyset\}$. □

Lemma 16.18. $P_{\xi+1} \cong P_\xi * Q_\xi$.

Proof. For any $p \in \mathbb{P}_{\xi+1}$ let $f(p) = (p \restriction \xi, p_\xi)$. By Definition (iv), $p \restriction \xi \in \mathbb{P}_\xi$. By Definition (iii), $p_\xi \in \mathrm{dmn}(\dot{\mathbb{Q}}_\xi)$. By Definition (viii), $(p \restriction \xi) \Vdash [p_\xi \in \dot{\mathbb{Q}}_\xi]$. Hence $f(p) \in \mathbb{P}_\xi * \dot{\mathbb{Q}}_\xi$. Clearly f is a bijection. Suppose that $p^1, p^2 \in \mathbb{P}_{\xi+1}$. Then

$$\begin{aligned}
p^1 \leq p^2 \quad &\text{iff} \quad \forall \mu \leq \xi [p \restriction \mu \Vdash [p^1_\mu \leq p^2_\mu]] \quad \text{by Definition (vii)}\\
&\text{iff} \quad [p^1 \restriction \xi \leq p^2 \restriction \xi] \text{ and } p \restriction \xi \Vdash [p^1_\xi \leq p^2_\xi]\\
&\text{iff} \quad (p^1 \restriction \xi, p^1_\xi) \leq (p^2 \restriction \xi, p^2_\xi)\\
&\text{iff} \quad f(p^1) \leq f(p^2).
\end{aligned}$$

□

Lemma 16.19. *Suppose that* $\eta < \xi$. *Define* $i_{\eta\xi}(p) = p'$ *as in (v). Then* $i_{\eta\xi}$ *is a complete embedding of* P_η *into* P_ξ.

Proof. See the definition of complete embedding before Theorem 16.7. Clearly (i) holds. For $\Leftarrow$ in (ii), suppose that $p_1 \not\perp p_2$; say $p_3 \leq p_1, p_2$. Then $i_{\eta\xi}(p_3) \leq i_{\eta\xi}(p_1), i_{\eta\xi}(p_2$ by (i), and so $i_{\eta\xi}(p_1) \not\perp i_{\eta\xi}(p_2)$. Now suppose that $p_1, p_2 \in \mathbb{P}_\xi$ and $i^\eta_\xi(p_1)$ and $i^\eta_\xi(p_2)$ are compatible. Say $r \leq i^\eta_\xi(p_1), i^\eta_\xi(p_2)$. By Definition (iv), $r \restriction \xi \in \mathbb{P}_\xi$, and by Definition (vii), $r \restriction \xi \leq p_1, p_2$.

Finally, suppose that $p \in \mathbb{P}_\xi$. We claim that $p \restriction \eta$ works to verify (iii). For, suppose that $p' \in P_\eta$ and $p' \leq p \restriction \eta$. Then clearly $i_{\eta\xi}(p')$ and p are compatible. □

Lemma 16.20. *If* $\xi < \eta < \rho$, *then* $i_{\xi\rho} = i_{\eta\rho} \circ i_{\xi\eta}$. □

Lemma 16.21. *If* $\xi \leq \eta$, $p, p' \in P_\xi$, *and* $p \leq p'$, *then* $i_{\xi\eta}(p) \leq i_{\xi\eta}(p')$.

Proof. Assume the hypothesis. Then by the definition (vii),

$$\begin{aligned} i_{\xi\eta}(p) \le i_{\xi\eta}(p') \quad &\text{iff} \quad \forall\mu < \eta[(i_{\xi\eta}(p) \restriction \mu \Vdash (i_{\xi\eta}(p))_\mu \le (i_{\xi\eta}(p'))_\mu \\ &\text{iff} \quad \forall\mu < \xi[p \restriction \mu \Vdash p_\mu \le p'_\mu] \\ &\text{iff} \quad p \le p' \end{aligned}$$ □

Lemma 16.22. *If $\xi \le \eta$, $p, p' \in P_\eta$, and $p \le p'$, then $p \restriction \xi \le p' \restriction \xi$.*

Proof. Assume the hypothesis. Then by (vii), $\forall\mu < \eta[p \restriction \mu \Vdash_{P_\mu} p_\mu \le p'_\mu]$, hence $\forall\mu < \xi[p \restriction \mu \Vdash_{P_\mu} p_\mu \le p'_\mu]$, hence $p \restriction \xi \le p' \restriction \xi$. □

Lemma 16.23. *If $\xi \le \eta$, $p, q \in P_\eta$, and $p \restriction \xi \perp q \restriction \xi$, then $p \perp q$.*

Proof. Immediate from Lemma 16.29. □

Lemma 16.24. *If $\xi < \eta$, $p, q \in P_\eta$, and supp$(p) \cap$ supp$(q) \subseteq \xi$, then $p \restriction \xi \perp q \restriction \xi$ iff $p \perp q$.*

Proof. Assume the hypothesis. $\Rightarrow$ holds by Lemma 16.23. For $\Leftarrow$, suppose that $r \in P_\xi$ and $r \le p \restriction \xi, q \restriction \xi$; we want to show that p and q are compatible.

Define s with domain η by setting, for each $\rho < \eta$,

$$s(\rho) = \begin{cases} r(\rho) & \text{if } \rho < \xi, \\ p(\rho) & \text{if } \xi \le \rho \in \text{supp}(p), \\ q(\rho) & \text{if } \xi \le \rho \in \text{supp}(q)\backslash\text{supp}(p), \\ 1 & \text{otherwise.} \end{cases}$$

Now it suffices to prove the following statement:

(*) For all $\gamma \le \eta$ we have $s \restriction \gamma \in P_\gamma$, $s \restriction \gamma \le p \restriction \gamma$, and $s \restriction \gamma \le q \restriction \gamma$.

We prove (*) by induction on γ. If $\gamma \le \xi$, then $s \restriction \gamma = r \restriction \gamma \in P_\gamma$ by (iv), and $s \restriction \gamma \le p \restriction \gamma, q \restriction \gamma$ since $r \le p \restriction \xi, q \restriction \xi$, by Lemma 16.29.

Now assume inductively that $\xi < \gamma \le \eta$. First suppose that γ is a successor ordinal $\gamma' + 1$. Then $s \restriction \gamma' \in P_{\gamma'}$ by the inductive hypothesis. Now we consider several cases.

Case 1. $\gamma' \in$ supp(p). Then $s(\gamma') = p(\gamma') \in$ dmn$(Q_{\gamma'})$ by (iii). Moreover, by the inductive hypothesis $s \restriction \gamma' \le p \restriction \gamma'$, and $p \restriction \gamma' \Vdash p(\gamma') \in Q_{\gamma'}$ by (viii). It follows that $s \restriction \gamma' \Vdash s(\gamma') \in Q_{\gamma'}$. Thus $s \restriction \gamma \in P_\gamma$ by (viii).

Case 9. $\gamma' \in$ supp$(q)\backslash$supp(p). This is treated similarly to Case 1.

Case 3. $\gamma' \notin$ supp$(p) \cup$ supp(q). Then $s(\gamma') = 1$, and hence clearly $s \restriction \gamma \in P_\gamma$ by (viii).

So, we have shown that $s \restriction \gamma \in P_\gamma$ in any case.

To show that $s \restriction \gamma \le p \restriction \gamma$, first note that $s \restriction \gamma' \le p \restriction \gamma'$ by the inductive hypothesis. If $\gamma' \in$ supp(p), then $s(\gamma') = p(\gamma')$ and so obviously $p \restriction \gamma' \Vdash s(\gamma') \le p(\gamma')$ and hence $s \restriction \gamma \le p \restriction \gamma$ by (vii). If $\gamma' \notin$ supp(p), then $p(\gamma') = 1$ and again the desired conclusion holds. Thus $s \restriction \gamma \le p \restriction \gamma$.

For $s \restriction \gamma \le q \restriction \gamma$, first note that $s \restriction \gamma' \le q \restriction \gamma'$ by the inductive hypothesis. If $\gamma' \in$ supp(q), then $\gamma' \notin$ supp(p) by the hypothesis of the lemma, since $\xi \le \gamma'$. Hence

$s(\gamma') = q(\gamma')$ and so obviously $q \upharpoonright \gamma' \Vdash s(\gamma') \leq q(\gamma')$ and hence $s \upharpoonright \gamma \leq q \upharpoonright \gamma$ by (vii). If $\gamma' \notin \text{supp}(q)$, then $q(\gamma') = 1$ and again the desired conclusion holds. Thus $s \upharpoonright \gamma \leq q \upharpoonright \gamma$.

This finishes the successor case $\gamma = \gamma' + 1$. Now suppose that γ is a limit ordinal. By the inductive hypothesis, $s \upharpoonright \rho \in P_\rho$ for each $\rho < \gamma$. Since clearly $\text{supp}(s) \subseteq \text{supp}(r) \cup \text{supp}(p) \cup \text{supp}(q)$, $\text{supp}(s)$ is finite. Hence $s \in P_\gamma$ by (ix). Finally, by (vii),

$$\begin{aligned} s \upharpoonright \gamma \leq p \upharpoonright \gamma \quad &\text{iff} \quad \forall \mu < \gamma[s \upharpoonright \mu \Vdash s_\mu \leq p_\mu] \\ &\text{iff} \quad \forall \mu < \gamma[s \upharpoonright \mu \leq p \upharpoonright \mu], \end{aligned}$$

and the last statement is true by the inductive hypothesis. Similarly, $s \upharpoonright \gamma \leq q \upharpoonright \gamma$. $\square$

Lemma 16.25. *Suppose that a finite support iterated forcing construction of length α is given, with notation as above. Also suppose that κ is an uncountable regular cardinal. Suppose that for each $\xi < \alpha$, $1 \Vdash_{\mathbb{P}_\xi} (Q_\xi$ is $\check{\kappa} - cc)$. Then for each $\xi \leq \alpha$ the forcing order $\mathbb{P}_\xi$ is κ-cc in M.*

Proof. We proceed by induction on ξ. It is trivially true for $\xi = 0$. The inductive step from ξ to $\xi + 1$ follows from Corollary 16.13 and Lemma 16.18. Now suppose that ξ is limit and the assertion is true for all $\eta < \xi$. Suppose that $\langle p^\beta : \beta < \kappa \rangle$ is an antichain in $\mathbb{P}_\xi$. Let $M \in [\kappa]^\kappa$ be such that $\langle \text{supp}(p_\xi) : \xi \in M \rangle$ is a Δ-system, say with root r. Choose $\eta < \xi$ such that $r \subseteq \eta$. Then by Lemma 16.23, $\langle p_\nu \upharpoonright \eta : \nu \in M \rangle$ is a system of incompatible elements of $\mathbb{P}_\eta$, contradiction. $\square$

Lemma 16.26. *Suppose that a finite support iterated forcing construction of length α is given, with notation as above.*

(i) Suppose that G is $\mathbb{P}_\alpha$-generic over M. For each $\xi \leq \alpha$ let $G_\xi = i_{\xi\alpha}^{-1}[G]$. Then

(a) For each $\xi \leq \alpha$, the set G_ξ is $\mathbb{P}_\xi$-generic over M.

(b) If $\xi \leq \eta \leq \alpha$, then $M[G_\xi] \subseteq M[G_\eta] \subseteq M[G]$.

(ii) Let $\xi < \alpha$. Define

$$\begin{aligned} &\mathbb{Q}_\xi = (\pi_\xi)_{G_\xi}; \\ &H_\xi = \{\rho_{G_\xi} : \rho \in \text{dmn}(Q_\xi) \text{ and } \exists p(p^\frown\langle\rho\rangle \in G_{\xi+1})\}. \end{aligned}$$

Then $H_\xi \in M[G_{\xi+1}]$ and H_ξ is $\mathbb{Q}_\xi$-generic over $M[G_\xi]$.

Proof. (i)(a) holds by Lemmas 16.7 and 16.18; and (i)(b) follows from these theorems too.

To prove (ii) we are going to apply Theorem 16.4 with P and $\dot{Q}$ replaced by P_ξ and Q_ξ; by (ii), Q_ξ is a P_ξ-name for a forcing order. Let j be the complete embedding of P_ξ into $P_\xi * Q_\xi$ given by $j(p) = (p, 1)$. Now $G_{\xi+1}$ is $\mathbb{P}_{\xi+1}$-generic over M by (i). Let f be the isomorphism of $P_\xi * Q_\xi$ with $P_{\xi+1}$ given in the proof of Lemma 16.17. Clearly then $f^{-1}[G_{\xi+1}]$ is $P_\xi * Q_\xi$-generic over M, and we apply Theorem 16.4 with it in place of K. Note that $f \circ j = i_{\xi,\xi+1}$, and hence $j^{-1}[f^{-1}[G_{\xi+1}]] = G_\xi$,

$$\begin{aligned} H_\xi &= \{\rho_{G_\xi} : \rho \in \text{dmn}(Q_\xi) \text{ and } \exists p(p^\frown\langle\rho\rangle \in G_{\xi+1})\} \\ &= \{\rho_{G_\xi} : \rho \in \text{dmn}(Q_\xi) \text{ and } \exists p((p, \rho) \in f^{-1}[G_{\xi+1}])\}, \end{aligned}$$

so that Theorem 16.4 applies to yield that G_ξ is P_ξ-generic over M (we already know this by (i)) and H_ξ is Q_ξ-generic over $M[G_\xi]$. Clearly $H_\xi \in M[G_{\xi+1}]$. $\square$

Lemma 16.27. *Suppose that a finite support iterated forcing construction of length α is given, with notation as above. Suppose that G is P_α-generic over M. Let $\xi < \eta \leq \alpha$ and let $p \in i_{\eta\kappa}^{-1}[G]$. Then $p \restriction \xi \in i_{\xi\kappa}^{-1}[G]$.*

Proof. Assume the hypotheses. Then $p \subseteq i_{\xi\eta}(p \restriction \xi)$, so $i_{\xi\eta}(p \restriction \xi) \in i_{\eta\kappa}^{-1}[G]$ by Lemma 16.26. Thus $i_{\xi\kappa}(p \restriction \xi) = i_{\eta\kappa}(i_{\xi\eta}(p \restriction \xi)) \in G$, and it follows that $p \restriction \xi \in i_{\xi\kappa}^{-1}[G]$. $\square$

Lemma 16.28. *Suppose that a finite support iterated forcing construction of length α is given, with notation as above. Suppose that G is P_α-generic over M. Let $\beta < \alpha$. By Lemma 16.18 let l be an isomorphism of $P_{\beta+1}$ onto $P_\beta * Q_\beta$. Let $G_\xi = i_{\xi\alpha}^{-1}[G]$ for all $\xi \leq \alpha$. Now clearly $l[G_{\beta+1}]$ is $(P_\beta * Q_\beta)$-generic over M. Let*

$$G' = \{p \in P_\beta : \exists \dot{q}[(p, \dot{q}) \in l[G_{\beta+1}]]\} \quad \text{and} \quad H = \{\dot{q}_{G'} : \exists p[(p, \dot{q}) \in l[G_{\beta+1}]\}.$$

(See Theorem 16.4.) Then $G' = G_\beta$.

Proof. Assume the hypotheses. First suppose that $p \in G'$. Say $(p, \dot{q}) \in l[G_{\beta+1}]$. Choose $p' \in G_{\beta+1}$ such that $(p, \dot{q}) = l(p')$. Thus $p = p' \restriction \beta$, so by Lemma 16.27, $p \in G_\beta$.

Second suppose that $p \in G_\beta$. Thus $i_{\beta\kappa}(p) \in G$, so $i_{\beta+1,\kappa}(i_{\beta,\beta+1}(p)) = i_{\beta\kappa}(p) \in G$, hence $i_{\beta,\beta+1}(p) \in G_{\beta+1}$. The first coordinate of $l(i_{\beta,\beta+1}(p))$ is $i_{\beta,\beta+1}(p) \restriction \beta = p$, so $p \in G'$. $\square$

Lemma 16.29. *Suppose that a finite support iterated forcing construction is given, with notation as above. Suppose that G is $\mathbb{P}_\alpha$-generic over M, $S \in M$, $X \subseteq S$, $X \in M[G]$, and $(|S| < \mathrm{cf}(\alpha))^{M[G]}$.*

Then there is an $\eta < \alpha$ such that $X \in M[i_{\eta\alpha}^{-1}[G]]$.

Proof. Let σ be a $\mathbb{P}_\alpha$-name such that $X = \sigma_G$. Thus for any $s \in S$, $s \in X$ iff there is a $p \in G$ such that $p \Vdash_{\mathbb{P}_\alpha} \check{s} \in \sigma$. Now clearly $P_\alpha = \bigcup_{\xi<\alpha} i_{\xi\alpha}[P_\xi]$, and $G = \bigcup_{\xi<\alpha} i_{\xi\alpha}[i_{\xi\alpha}^{-1}[G]]$.

Hence for each $s \in X$ we can find $\xi(s) < \alpha$ such that there is a $p \in i_{\xi(s)\alpha}^{-1}[G]$ such that $i_{\xi(s)\alpha}(p) \Vdash_{\mathbb{P}_\alpha} \check{s} \in \sigma$. Let $\eta = \sup_{s\in X} \xi(s)$; so $\eta < \alpha$ by assumption.

Thus $X = \{s \in S : \exists p \in G_\eta(i_{\eta\alpha}(p) \Vdash_{\mathbb{P}_\alpha} \check{s} \in \sigma\}$. Hence $X \in M[G_\eta]$. $\square$

Theorem 16.30. *Let $B_0 \subseteq B_1 \subseteq \cdots \subseteq B_\xi \subseteq \cdots$ be sequence of complete BAs for $\xi < \alpha$ such that*

(i) $\forall \xi, \eta < \alpha[\xi < \eta \rightarrow B_\xi$ is a complete subalgebra of $B_\eta]$.

(ii) $\forall \xi < \alpha[\xi$ limit $\rightarrow \bigcup_{\eta<\xi} B_\eta$ is dense in $B\xi]$.

Let κ be an uncountable regular cardinal. Then: if each B_ξ has the κ-cc, then so does $\bigcup_{\xi<\alpha} B_\xi$.

Proof. This is trivial if α is a successor ordinal; so assume that α is a limit ordinal. If $\kappa < \mathrm{cf}(\alpha)$ the conclusion is also clear. Suppose that $\mathrm{cf}(\alpha) < \kappa$ and X is a disjoint subset of $\bigcup_{\xi<\alpha} B_\xi$ of size κ. Let $\langle \beta_\xi : \xi < \mathrm{cf}(\alpha) \rangle$ be strictly increasing with supremum α.

Then $\forall\xi < \mathrm{cf}(\alpha)[|X \cap B_\xi| < \kappa]$, and since $\mathrm{cf}(\alpha) < \kappa$, also $|\bigcup_{\xi<\alpha} B_\xi| = |\bigcup_{\xi<\mathrm{cf}(\alpha)} B_\xi| < \kappa$, contradiction. Hence we may assume that $\mathrm{cf}(\alpha) = \kappa$, and even that $\alpha = \kappa$.

Let $A = \bigcup_{\xi<\kappa} B_\xi$. For each $\xi < \kappa$ we define $c_\xi : A \to B_\xi$ by

$$c_\xi(x) = \prod_{x \le a \in B_\xi}^{B_\xi} a.$$

Suppose that X is a disjoint subset of A of size $\ge \kappa$. We may assume that X is maximal disjoint. Then $\Sigma^A X = 1$. Hence $\Sigma^A_{x\in X} c_\xi(x) = 1$. Hence clearly $\Sigma^{B_\xi}_{x\in X} c_\xi(x) = 1$. Since B_ξ satisfies the κ-cc, choose $Y_\xi \in [X]^{<\kappa}$ so that $\Sigma^{B_\xi}_{x\in Y_\xi} c_\xi(x) = 1$. (See Lemma 3.12 of the Handbook.) Choose $\beta_\xi < \kappa$ such that $Y_\xi \subseteq B_{\beta_\xi}$.

Now let $\delta_0 = 0$. If $\delta_n < \kappa$ has been defined, let $\delta_{n+1} > \delta_n$ be such that $\delta_{n+1} < \kappa$ and $\forall\xi < \delta_n[\beta_\xi < \delta_{n+1}]$. Let $\gamma = \bigcup_{n\in\omega} \delta_n$. Then

$(*)$ γ is a limit ordinal less than κ, and $\forall\xi < \gamma[\beta_\xi < \gamma]$.

We claim that $X \subseteq B_\gamma$. (Contradiction.) For, let $x \in X$. Since $\bigcup_{\eta<\gamma} B_\eta$ is dense in B_γ, choose a non-zero $b \in \bigcup_{\eta<\gamma} B_\eta$ such that $b \le c_\gamma(x)$. Say $b \in B_\eta$ with $\eta < \gamma$. Since $\sum^{B_\eta}_{x\in Y_\eta} c_\eta(x) = 1$, choose $a \in Y_\eta$ so that $c_\eta(a) \cdot b \ne 0$. We claim

$(**)$ $a \cdot b \ne 0$.

For, assume that $a \cdot b = 0$. Then $a \le -b \in B_\eta$, so $c_\eta(a) \le -b$, hence $c_\eta(a) \cdot b = 0$, contradiction.

So $(**)$ holds. Since $b \le c_\gamma(x)$, we have $c_\gamma(x) \cdot a \ne 0$. By the above argument, $x \cdot a \ne 0$. Since both x and a are in X, it follows that $x = a \in Y_\eta \subseteq B_{\beta_\eta} \subseteq B_\gamma$. □

Let κ be an infinite cardinal. MA_κ is the following statement:

For every poset P that satisfies the ccc, and for every family $\mathscr{D}$ of size at most κ consisting of dense subsets of P, there is a $\mathscr{D}$-generic filter on P, i.e. there is a $G \subseteq P$ satisfying the following conditions:

(i) $\forall x \in G \forall y \ge x[y \in G]$.

(ii) $\forall x, y \in G \exists z \in G[z \le x, y]$.

(iii) $\forall D \in \mathscr{D}[D \cap G \ne \emptyset]$.

MA is the statement that MA_κ holds for all $\alpha < 2^{\aleph_0}$.

Lemma 16.31. MA_ω.

Proof. Let P be a ccc poset, and let $\langle D_n : n \in \omega\rangle$ be a system of dense subsets of P. Define p_n for $n \in \omega$ by recursion, as follows. Let $p_0 \in D_0$. If p_n has been defined, let $p_{n+1} \le p_n$ with $p_{n+1} \in D_{n+1}$. Now let $G = \{q \in P : \exists n \in \omega[p_n \le q]\}$. Clearly G works. □

Lemma 16.32. MA_{2^ω} *fails.*

Proof. Let P consist of all finite functions contained in $\omega \times 2$, with order $\supseteq$. For each $g \in {}^{\omega}2$ let $D_g = \{p \in P : p \not\subseteq g\}$. Clearly each D_g is dense. Suppose that G is $\{D_g : g \in {}^{\omega}2\}$-generic. Clearly $\bigcup G \in {}^{\omega}2$. But $D_{\bigcup G} \cap G = \emptyset$. $\square$

Lemma 16.33. MA_κ *is equivalent to* MA_κ *restricted to ccc forcing orders of cardinality* $\leq \kappa$.

Proof. We assume the indicated special form of MA_κ, and assume given a ccc forcing order P and a family $\mathscr{D}$ of at most κ dense sets in P; we want to find a filter on P intersecting each member of $\mathscr{D}$. We introduce some operations on P. For each $D \in \mathscr{D}$ define $f_D : P \to P$ by setting, for each $p \in P$, $f_D(p)$ to be some element of D which is $\leq p$. Also we define $g : P \times P \to P$ by setting, for all $p, q \in P$,

$$g(p,q) = \begin{cases} p & \text{if } p \text{ and } q \text{ are incompatible,} \\ r & \text{with } r \leq p, q \text{ if there is such an } r. \end{cases}$$

Here, as in the definition of f_D, we are implicitly using the axiom of choice; for g, we choose any r of the indicated form.

We may assume that $\mathscr{D} \neq \emptyset$. Choose $D \in \mathscr{D}$, and choose $s \in D$. Now let Q be the intersection of all subsets of P which have s as a member and are closed under all of the operations f_D and g. We take the order on Q to be the order induced from P.

(1) $|Q| \leq \kappa$.

To prove this, we give an alternative definition of Q. Define

$$\begin{aligned} R_0 &= \{s\}; \\ R_{n+1} &= R_n \cup \{g(a,b) : a, b \in R_n\} \cup \{f_D(a) : D \in \mathscr{D} \text{ and } a \in R_n\}. \end{aligned}$$

Clearly $\bigcup_{n\in\omega} R_n = Q$. By induction, $|R_n| \leq \kappa$ for all $n \in \omega$, and hence $|Q| \leq \kappa$, as desired in (1).

We also need to check that Q is ccc. Suppose that X is a collection of pairwise incompatible elements of Q. Then these elements are also incompatible in P, since $x, y \in X$ with x, y compatible in P implies that $g(x,y) \leq x, y$ and $g(x,y) \in Q$, so that x, y are compatible in Q. It follows that X is countable. So Q is ccc.

Next we claim that if $D \in \mathscr{D}$ then $D \cap Q$ is dense in Q. For, suppose $p \in Q$. Then $f_D(q) \in D \cap Q$. as desired.

Now we can apply our special case of MA_κ to Q and $\{D \cap Q : D \in \mathscr{D}\}$; we obtain a filter G on Q such that $G \cap D \cap Q \neq \emptyset$ for all $D \in \mathscr{D}$. Let

$$G' = \{p \in P : q \leq p \text{ for some } q \in G\}.$$

We claim that G' is the desired filter on P intersecting each $D \in \mathscr{D}$.

Clearly if $p \in G'$ and $p \leq r$, then $r \in G'$.

Suppose that $p_1, p_2 \in G'$. Choose $q_1, q_2 \in G$ such that $q_i \leq p_1$ for each $i = 1, 2$. Then there is an $r \in G$ such that $r \leq q_1, q_2$. Then $r \in G'$ and $r \leq p_1, p_2$. So G' is a filter on P.

Now for any $D \in \mathscr{D}$. Take $q \in G \cap D \cap Q$. Then $q \in G' \cap D$, as desired. $\square$

Theorem 16.34. *Suppose that M is a c.t.m. of GCH, and in M we have an uncountable regular cardinal κ.*

Then there is a forcing order $\mathbb{P}$ in M such that $\mathbb{P}$ satisfies ccc, and for any $\mathbb{P}$-generic G over M, the extension $M[G]$ satisfies MA and $2^\omega = \kappa$.

Proof. The overall idea of the proof runs like this. We do a finite support iterated forcing which has the effect of producing a chain

$$M = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_\alpha \subseteq M_{\alpha+1} \subseteq \cdots \subseteq M_\kappa$$

of length $\kappa + 1$ of c.t.m.s of ZFC. We carry along in the construction a list of names of forcing orders. This list is of length κ. At the step from M_α to $M_{\alpha+1}$ we take care of one entry in this list, say $\mathbb{Q}$, by taking a $\mathbb{Q}$-generic filter G and setting $M_{\alpha+1} = M_\alpha[G]$, and we add to our list all names of forcing orders in M_α. By proper coding, we can do this so that at end we have taken care of all forcing orders in any model M_α. Then we show that any ccc forcing order in M_κ appeared already in an earlier stage and so a generic filter for it was added.

We begin by defining the coding which will be used.

Claim. There is a function f in M with the following properties:
(1) $f : \kappa \to \kappa \times \kappa$.
(2) For all $\xi, \beta, \gamma < \kappa$ there is an $\eta > \xi$ such that $f(\eta) = (\beta, \gamma)$.
(3) $1^{st}(f(\xi)) \leq \xi$ for all $\xi < \kappa$.

Proof of Claim. Let $g : \kappa \to \kappa \times \kappa \times \kappa$ be a bijection. For each $\xi < \kappa$ let $g(\xi) = (\alpha, \beta, \gamma)$, and set

$$f(\xi) = \begin{cases} (\beta, \gamma) & \text{if } \beta \leq \xi, \\ (0, 0) & \text{otherwise.} \end{cases}$$

So (1) and (3) obviously hold. For (2), suppose that $\xi, \beta, \gamma < \kappa$. Now $g^{-1}[\{(\alpha, \beta, \gamma) : \alpha < \kappa\}]$ has size κ, so there is an $\eta \in g^{-1}[\{(\alpha, \beta, \gamma) : \alpha < \kappa\}]$ such that $\xi, \beta < \eta$. Say $g(\eta) = (\alpha, \beta, \gamma)$. Then $\beta < \eta$, so $f(\eta) = (\beta, \gamma)$ and $\xi < \eta$, as desired.

For brevity, we let $\mathrm{pord}(\lambda, W)$ abbreviate the statement that W is the order relation of a ccc forcing order on the set λ, with largest element 0.

Now we are going to define by recursion functions $\mathbb{P}$, π, λ, and σ with domain κ.

Let $\mathbb{P}_0$ be the trivial partial order $(\{0\}, 0, 0)$.

Now suppose that $\mathbb{P}_\alpha$ has been defined, so that it is a ccc forcing order in M, with $|\mathbb{P}_\alpha| \leq \kappa$. We now define π_α, λ^α, σ^α, and $\mathbb{P}_{\alpha+1}$. Clearly $|\mathrm{RO}(\mathbb{P}_\alpha)| \leq \kappa$. Let $N = \{(\beta, g) : \beta < \kappa, g : (\beta \times \beta)\check{} \to \mathrm{RO}(\mathbb{P}_\alpha)\}$. Then $|N| \leq \kappa$. We let $\{(\lambda^\alpha_\xi, \sigma^\alpha_\xi) : \xi < \kappa\}$ enumerate N. This defines λ^α and σ^α. Now let $f(\alpha) = (\beta, \gamma)$. So $\beta \leq \alpha$, and hence λ^β_γ and σ^β_γ are defined. We consider the complete embedding $i_{\beta\alpha}$ given in Lemma 16.18. By Theorem 16.8 there is a complete embedding $j_{\beta\alpha} : \mathrm{RO}(P_\beta) \to \mathrm{RO}(P_\alpha)$ such that $\forall p \in P_\beta[j_{\beta\alpha}(e(p)) = e(i_{\beta\alpha}(p))]$. By Lemma 16.9 we have

$$j^*_{\beta\alpha}(\sigma^\beta_\gamma) = j_{\beta\alpha} \circ \sigma^\beta_\gamma : (\beta \times \beta)\check{} \to \mathrm{RO}(P_\alpha).$$

We let $\pi'_\alpha = j_{\beta\alpha} \circ \sigma^\beta_\gamma$.

(4) There is a $\pi_\alpha : \lambda^\beta_\gamma \times \lambda^\beta_\gamma \to \mathrm{RO}(P_\alpha)$ such that

$$\mathbb{1}_{P_\alpha} \Vdash \mathrm{pord}((\lambda^\beta_\gamma)\check{}, \pi_\alpha) \wedge [\mathrm{pord}((\lambda^\beta_\gamma)\check{}, \pi'_\alpha) \to \pi_\alpha = \pi'_\alpha].$$

In fact, clearly

$$\mathbb{1}_{P_\alpha} \Vdash \exists W[\mathrm{pord}((\lambda^\beta_\gamma)\check{}, W) \wedge [\mathrm{pord}((\lambda^\beta_\gamma)\check{}, \pi'_\alpha) \to \pi'_\alpha = W]].$$

Hence (4) follows from the maximal principle.

Finally, $\mathbb{P}_{\alpha+1}$ is determined by (viii).

For limit $\alpha \le \kappa$ we define $\mathbb{P}_\alpha$ by (ix).

This finishes the construction.

By Lemma 16.18, each forcing order $\mathbb{P}_\alpha$ for $\alpha \le \kappa$ satisfies ccc.

Now take any $\mathbb{P}_\kappa$-generic G over M; we want to show that $\mathrm{MA}(\mu)$ holds in $M[G]$ for every $\mu < \kappa$. Note that, by ccc, $\mathbb{P}_\kappa$ preserves cofinalities and cardinalities. Let $G_\xi = i^{-1}_{\xi\kappa}[G]$ for each $\xi < \kappa$.

Suppose that $\mathbb{Q}$ is a ccc forcing order in $M[G]$, $|Q| \le \mu$, and $\mathscr{D}$ is a family of at most μ subsets of Q dense in $\mathbb{Q}$, with $\mathscr{D} \in M[G]$. By taking an isomorphic image, we may assume that $\mathbb{Q} = (\varphi, \le_\mathbb{Q})$ with $\varphi < \kappa$. By Lemma 14.26 there is a $\beta < \kappa$ such that $\mathbb{Q} \in M[G_\beta]$ and $\mathscr{D} \in M[G_\beta]$. Then there is a $g : (\varphi \times \varphi)\check{} \to \mathrm{RO}(P_\beta)$ such that $g_{G_\beta} = \le_Q$. Say $(\varphi, g) = (\lambda^\beta_\gamma, \sigma^\beta_\gamma)$. Let $\alpha = f^{-1}(\beta, \gamma)$. Then we have $\pi'_\alpha = j^*_{\beta\alpha}(\sigma^\beta_\gamma)$. By Lemma 16.25, $G_{\alpha+1}$ is $P_{\alpha+1}$-generic over M. For each $p \in P_\alpha$ let $k(p) = (p, 1)$. By Lemma 16.9, k is a complete embedding of P_α into $P_\alpha * \pi_\alpha$. For $p \in P_{\alpha+1}$ let $l(p) = (p \restriction \alpha, p(\alpha))$. Then by the proof Lemma 16.17, l is an isomorphism from $P_{\alpha+1}$ onto $P_\alpha * \pi_\alpha$. Then $l[G_{\alpha+1}]$ is $(P_\alpha * \pi_\alpha)$-generic over M. Let $G' = \{p \in P_\alpha : \exists \dot{q}[(p, \dot{q}) \in l[G_{\alpha+1}]]\}$. Then $G' = G_\alpha$ by Lemma 16.28. Also, let $K = \{\dot{q}_{G'} : \exists p \in P_\alpha[(p, \dot{q}) \in l[G_{\alpha+1}]]\}$. Then by Theorem 16.4, K is $\pi_{\alpha G'}$-generic over $M[G']$.

(5) If $q \Vdash_{P_\beta} \varphi$, then $j_{\beta\alpha}(e(q)) \le j_{\beta\alpha}([\![\varphi]\!]_{P_\beta})$.

In fact, assume that $q \Vdash_{P_\beta} \varphi$. Then $e(q) \le [\![\varphi]\!]_{P_\beta}$, and so (5) follows.

Now $g_{G_\beta} = \le_Q$, so there is a $p \in G_\beta$ such that

$$p \Vdash \mathrm{pord}((\lambda^\beta_\gamma)\check{}, \sigma^\beta_\gamma);$$

so

$$e(p) \le [\![\mathrm{pord}((\lambda^\beta_\gamma)\check{}, \sigma^\beta_\gamma)]\!]_{\mathrm{RO}(P_\beta)}.$$

By Lemma 16.9,

$$j_{\beta\alpha}(e(p)) \le [\![\mathrm{pord}(j^*_{\beta\alpha}((\lambda^\beta_\gamma))\check{}), j^*_{\beta\alpha}(\sigma^\beta_\gamma))]\!]_{\mathrm{RO}(P_\alpha)}.$$

Now by Lemma 16.9, $j^*_{\beta\alpha}((\lambda^\beta_\gamma))\check{}) = (\lambda^\beta_\gamma)\check{}$ and $j^*_{\beta\alpha}(\sigma^\beta_\gamma)) = \pi'_\alpha$, so

$$j_{\beta\alpha}(e(p)) \le [\![\mathrm{pord}((\lambda^\beta_\gamma))\check{}), \pi'_\alpha))]\!].$$

By (4) it follows that $j_{\beta\alpha}(e(p)) \leq [\![\pi_\alpha = \pi'_\alpha]\!]$. Hence $e(i_{\beta\alpha}(p)) \leq [\![\pi_\alpha = \pi'_\alpha]\!]$. Thus $i_{\beta\alpha}(p) \Vdash \pi_\alpha = \pi'_\alpha$. Now $p \in G_\beta = i^{-1}_{\beta\kappa}[G]$, so $i_{\beta\kappa}(p) \in G$, hence $i_{\alpha\kappa}(i_{\beta\alpha}(p)) = i_{\beta\kappa}(p) \in G$, so $i_{\beta\alpha}(p) \in i^{-1}_{\alpha\kappa}[G] = G_\alpha$. It follows that $\pi_{\alpha G_\alpha} = \pi'_{\alpha G_\alpha}$. Hence K is $\pi'_{\alpha G_\alpha}$-generic over $M[G_\alpha]$, i.e. it is $(j^*_{\beta\alpha}(\sigma^\beta_\gamma))_{G_\alpha}$-generic over $M[G_\alpha]$. We claim that $j^{*-1}_{\beta\alpha}[K]$ is $\sigma^\beta_{\gamma G_\beta}$-generic over $M[G_\alpha]$. Clearly it is a filter. Now suppose that $D \in \mathscr{D}$. So D is dense in $\sigma^\beta_{\gamma G_\beta}$. Then $j^*_{\beta\alpha}[D]$ is dense in $(j^*_{\beta\alpha}(\sigma^\beta_\gamma))_{G_\alpha}$. For, suppose that $\xi \in \lambda^\beta_\gamma$. Choose $\eta \in D$ such that $(\eta, \xi) \in (\sigma^\beta_\gamma))_{G_\alpha}$. Hence there is a $p \in G_\beta$ such that $e(p) \leq \sigma^\beta_\gamma(\eta, \xi)$. So $j_{\beta\alpha}(e(p)) \leq j_{\beta\alpha}(\sigma^\beta_\gamma(\eta, \xi))$. Now

(6) $\forall x \in G_\beta[j_{\beta\alpha}(x) \in G_\alpha]$.

In fact, if $x \in G_\beta$ then $x \in j^{-1}_{\beta\kappa}[G]$, so $j_{\alpha\kappa}(j_{\beta\alpha}(x)) = j_{\beta\kappa}(x) \in G$, and hence $j_{\beta\alpha}(x) \in j^{-1}_{\alpha\kappa}[G] = G_\alpha$.

It follows that $j^*_{\beta\alpha}(\sigma^\beta_\gamma(\eta, \xi)) \in G_\alpha$ Hence $(\eta, \xi) \in (j^*_{\beta\alpha}(\sigma^\beta_\gamma)_{G_\alpha}$. Thus $j^*_{\beta\alpha}[D]$ is dense in $(j^*_{\beta\alpha}(\sigma^\beta_\gamma))_{G_\alpha}$. Choose $\xi \in K \cap j^*_{\beta\alpha}[D]$. Say $\xi = j^*_{\beta\alpha}(\eta)$ with $\eta \in D$. Then $\xi = \eta \in D \cap K$. This finishes the proof that MA_μ holds in $M[G]$ for all $\mu < \kappa$.

Since MA(μ) holds for every $\mu < \kappa$, it follows from Lemma 26.32 that in $M[G]$, $\kappa \leq 2^\omega$.

Now in M we have $\kappa^\omega = \kappa$ by GCH. Hence by Lemma 15.2 it follows that $2^\omega \leq \kappa$ in $M[G]$. Thus $2^\omega = \kappa$ in $M[G]$. □

Theorem 16.35. *If $MA_{\aleph_1}$ holds then there is no Suslin tree.*

Proof. Suppose that T is a normal Suslin tree, and let P be T with order reversed. Then elements of P are compatible iff they are comparable in T, so P is ccc. For each $\alpha < \omega_1$ let D_α be the union of all levels greater than α. Clearly D_α is dense in P. Let G be a filter intersecting each D_α. Then $\bigcup G$ is a chain of size $\aleph_1$, contradiction. □

Lemma 16.36. *If T is an Aronszajn tree and W is an uncountable collection of finite pairwise disjoint subsets of T, then there exist distinct $S, S' \in W$ such that $\forall x \in S \forall y \in S'[x$ and y are incomparable$]$.*

Proof. We may assume that there is an $m \in \omega$ such that $\forall S \in W[|S| = m]$. For each $S \in W$ write $S = \{x^S_0, \ldots, x^S_{m-1}\}$.

(1) There is an ultrafilter D on W such that each member of D is uncountable.

In fact, $\{W \backslash T : T \in [W]^{\leq\omega}\}$ has fip.

Suppose that the lemma is false. For each $x \in T$ and each $i < m$ let $Y_{xi} = \{S \in W : x$ is comparable with $x^S_i\}$. Then

(2) $\forall S \in W \left[\bigcup_{x \in S} \bigcup_{i<m} Y_{xi} = W\right]$.

In fact, let $S' \in W$. Then there are $x \in S$ and $x' \in S'$ such that x and x' are comparable. Then $S' \in Y_{xi}$.

By (2), for each $S \in W$ we can choose $x_S \in S$ and $i_S < m$ such that $Y_{x_S i_S} \in D$. Then there is an uncountable $Z \subseteq W$ and a $k < m$ such that $i_S = k$ for all $S \in Z$. We claim not that $\{x_S : S \in Z\}$ is linearly ordered, which is a contradiction.

Suppose $S_1, S_2 \in Z$ with $S_1 \neq S_2$. Then $V \stackrel{\text{def}}{=} Y_{x_{S_1}k} \cap Y_{x_{S_2}k} \in D$. Hence V is uncountable. If $S_3 \in V$, then x_{S_1} is comparable with $x_k^{S_3}$, and x_{S_2} is comparable with $x_k^{S_3}$. Choose $S_3 \in V$ such that $x_k^{S_3}$ has level above the levels of x_{S_1} and x_{S_2}. Hence $x_{S_1} < x_k^{S_3}$ and $x_{S_2} < x_k^{S_3}$. It follows that x_{S_1} and x_{S_2} are comparable. $\square$

Lemma 16.37. *Let T be an Aronszajn tree. Define $(P, <)$ as follows P consists of all functions p such that*

(i) $\mathrm{dmn}(p)$ *is a finite subset of T.*

(ii) $\mathrm{rng}(p) \subseteq \omega$.

(iii) $\forall x, y \in \mathrm{dmn}(p)[x$ *and* y *are comparable* $\rightarrow p(x) \neq p(y)]$.

Then for $p, q \in P$ we say that $p < q$ iff $q \subseteq p$.

Conclusion: P is ccc.

Proof. Let W be an uncountable subset of P. Then there is an uncountable $W' \subseteq W$ and an $m \in \omega$ such that $|\mathrm{dmn}(p)| = m$ for all $p \in W'$. By the indexed Δ-system theorem (Theorem 14.70) there is an uncountable $W'' \subseteq W'$ and a finite $S \subseteq T$ such that $\mathrm{dmn}(p) \cap \mathrm{dmn}(q) = S$ for all distinct $p, q \in W''$. Then

$$W'' = \bigcup_{t \in {}^S\omega} \{p \in W'' : p \restriction S = t\},$$

so there is a $t \in {}^S\omega$ and an uncountable $W''' \subseteq W''$ such that $p \restriction S = t$ for all $p \in W'''$. Let $U = \{\mathrm{dmn}(p) \backslash S : p \in W''\}$. Then U is a collection of pairwise disjoint nonempty finite subsets of T. By Lemma 16.36 there are distinct $S_1, S_2 \in U$ such that $\forall x \in S_1 \forall y \in S_2[x$ and y are incomparable]. Say $S_1 = \mathrm{dmn}(p) \backslash S$ and $S_2 = \mathrm{dmn}(q) \backslash S$. Then $p \cup q \in P$. In fact, clearly (i) and (ii) hold. Now suppose that $x, y \in \mathrm{dmn}(p \cup q)$ and x and y are comparable. If $x, y \in S$ then $(p \cup q)(x) = p(x) \neq p(y) = (p \cup q)(y)$. If $x \notin S$ and $y \in S$, then $x, y \in \mathrm{dmn}(p)$ and so $(p \cup q)(x) = p(x) \neq p(y) = (p \cup q)(y$. Similarly if $x \in S$ and $y \notin S$. Finally, suppose that $x, y \notin S$. Then x and y are incomparable, so (iii) holds. $\square$

An Aronszajn tree T is *special* iff there is a function $f : T \rightarrow \omega$ such that $\forall n \in \omega[f^{-1}[\{n\}]$ is an antichain].

Theorem 16.38. *Under $MA_{\aleph_1}$ every Aronszajn tree is special.*

Proof. We take the poset given in the statement of Lemma 16.37. For each $t \in T$ let $D_t = \{p \in P : t \in \mathrm{dmn}(p)\}$. Clearly each D_t is dense in P. By $MA_{\aleph_1}$, let G be a $\{D_t : t \in T\}$-generic filter on P. Clearly $\bigcup G : T \rightarrow \omega$. If $n \in \omega$, then $(\bigcup G)^{-1}[\{n\}] = \{t \in T : (\bigcup G)(t) = n\}$. If $s, t \in T$ and $(\bigcup G)(t) = (\bigcup G)(s)$, choose $p \in G$ such that $s, t \in \mathrm{dmn}(p)$. Then $p(s) = n = p(t)$, so s and t are incomparable. $\square$

Let $\mathscr{A} \subseteq \mathscr{P}(\omega)$. The *almost disjoint partial order for $\mathscr{A}$* is defined as follows:

$$\begin{gathered}
P_{\mathscr{A}} = \{(s, F) : s \in [\omega]^{<\omega} \text{ and } F \in [\mathscr{A}]^{<\omega}\}; \\
(s', F') \leq (s, F) \text{ iff } s \subseteq s',\ F \subseteq F',\ \text{and } x \cap s' \subseteq s \text{ for all } x \in F; \\
\mathbb{P}_{\mathscr{A}} = (P_{\mathscr{A}}, \leq).
\end{gathered}$$

We give some useful properties of this construction.

Lemma 16.39. *Let $\mathscr{A} \subseteq \mathscr{P}(\omega)$.*

(i) $\mathbb{P}_{\mathscr{A}}$ is a forcing poset.

(ii) Let $(s,F),(s',F') \in P_{\mathscr{A}}$. Then the following conditions are equivalent:

(a) (s,F) and (s',F') are compatible.

(b) $\forall x \in F(x \cap s' \subseteq s)$ and $\forall x \in F'(x \cap s \subseteq s')$.

(c) $(s \cup s', F \cup F') \leq (s,F),(s',F')$.

(iii) Suppose that $x \in \mathscr{A}$, and let $D_x = \{(s,F) \in P_{\mathscr{A}} : x \in F\}$. Then D_x is dense in $\mathbb{P}_{\mathscr{A}}$.

(iv) $\mathbb{P}_{\mathscr{A}}$ has ccc.

Proof. (i): Clearly $\leq$ is reflexive on $P_{\mathscr{A}}$ and it is antisymmetric, i.e. $(s,F) \leq (s',F') \leq (s,F)$ implies that $(s,F) = (s',F')$. Now suppose that $(s'',F'') \leq (s',F') \leq (s,F)$. Thus $s \subseteq s' \subseteq s''$, so $s \subseteq s''$. Similarly, $F \subseteq F''$. Now take any $x \in F$. Then $x \in F'$, so $x \cap s'' \subseteq s'$ because $(s'',F'') \leq (s',F')$. Hence $x \cap s'' \subseteq x \cap s'$. And $x \cap s' \subseteq s$ because $(s',F') \leq (s,F)$. So $x \cap s'' \subseteq s$, as desired.

(ii): For (a)$\Rightarrow$(b), assume (a). Choose $(s'',F'') \leq (s,F),(s',F')$. Now take any $x \in F$. Then $x \cap s' \subseteq x \cap s''$ since $s' \subseteq s''$, and $x \cap s'' \subseteq s$ since $(s'',F'') \leq (s,F)$; so $x \cap s' \subseteq s''$. The other part of (b) follows by symmetry.

(b)$\Rightarrow$(c): By symmetry it suffices to show that $(s \cup s', F \cup F') \leq (s,F)$, and for this we only need to check the last condition in the definition of $\leq$. So, suppose that $x \in F$. Then $x \cap (s \cup s') = (x \cap s) \cup (x \cap s') \subseteq s$ by (b).

(c)$\Rightarrow$(a): Obvious.

(iii): For any $(s,F) \in P_{\mathscr{A}}$, clearly $(s, F \cup \{x\}) \leq (s,F)$.

(iv) Suppose that $\langle (s_\xi, F_\xi) : \xi < \omega_1 \rangle$ is a pairwise incompatible system of elements of $P_{\mathscr{A}}$. Clearly then $s_\xi \neq s_\eta$ for distinct $\xi, \eta < \omega_1$, contradiction. $\square$

Theorem 16.40. *Let κ be an infinite cardinal, and assume $\mathrm{MA}(\kappa)$. Suppose that $\mathscr{A}, \mathscr{B} \subseteq \mathscr{P}(\omega)$, and $|\mathscr{A}|, |\mathscr{B}| \leq \kappa$. Also assume that*

(i) For all $y \in \mathscr{B}$ and all $F \in [\mathscr{A}]^{<\omega}$ we have $|y \backslash \bigcup F| = \omega$.

Then there is a $d \subseteq \omega$ such that $|d \cap x| < \omega$ for all $x \in \mathscr{A}$ and $|d \cap y| = \omega$ for all $y \in \mathscr{B}$.

Proof. For each $y \in \mathscr{B}$ and each $n \in \omega$ let

$$E_n^y = \{(s,F) \in P_{\mathscr{A}} : s \cap y \not\subseteq n\}.$$

We claim that each such set is dense. For, suppose that $(s,F) \in \mathbb{P}_{\mathscr{A}}$. Then by assumption, $|y \backslash \bigcup F| = \omega$, so we can pick $m \in y \backslash \bigcup F$ such that $m > n$. Then $(s \cup \{m\}, F) \leq (s,F)$, since for each $z \in F$ we have $z \cap (s \cup \{m\}) \subseteq s$ because $m \notin z$. Also, $m \in y \backslash n$, so $(s \cup \{m\}, F) \in E_n^y$. This proves our claim.

There are clearly at most κ sets E_n^y; and also there are at most κ sets D_x with $x \in \mathscr{A}$, with D_x as in Lemma 16.39(iii). Hence by $\mathrm{MA}(\kappa)$ we can let G be a filter on $\mathbb{P}_{\mathscr{A}}$ intersecting all of these dense sets. Let $d = \bigcup_{(s,F) \in G} s$.

(1) For all $x \in \mathscr{A}$, the set $d \cap x$ is finite.

For, by the denseness of D_x, choose $(s, F) \in G \cap D_x$. Thus $x \in F$. We claim that $d \cap x \subseteq s$. To prove this, suppose that $n \in d \cap x$. Choose $(s', F') \in G$ such that $n \in s'$. Now (s, F) and (s', F') are compatible. By Lemma 16.39(ii), $\forall y \in F(y \cap s' \subseteq s)$; in particular, $x \cap s' \subseteq s$. Since $n \in x \cap s'$, we get $n \in s$. This proves our claim, and so (1) holds.

The proof will be finished by proving

(2) For all $y \in \mathscr{B}$, the set $d \cap y$ is infinite.

To prove (2), given $n \in \omega$ choose $(s, F) \in E^y_n \cap G$. Thus $s \cap y \not\subseteq n$, so we can choose $m \in s \cap y \backslash n$. Hence $m \in d \cap y \backslash n$, proving (2). $\square$

Corollary 16.41. *Let κ be an infinite cardinal and assume* MA(κ). *Suppose that $\mathscr{A} \subseteq \mathscr{P}(\omega)$ is an almost disjoint set of infinite subsets of ω of size κ. Then $\mathscr{A}$ is not maximal.*

Proof. If F is a finite subset of $\mathscr{A}$, then we can choose $a \in \mathscr{A} \backslash F$; then $a \cap \bigcup F = \bigcap_{b \in F}(a \cap b)$ is finite. Thus $\omega \backslash \bigcup F$ is infinite. Hence we can apply Theorem 16.40 to $\mathscr{A}$ and $\mathscr{B} \stackrel{\text{def}}{=} \{\omega\}$ to obtain the desired result. $\square$

Corollary 16.42. *Assuming MA, every maximal almost disjoint set of infinite sets of natural numbers has size 2^ω.* $\square$

Lemma 16.43. *Suppose that $\mathscr{B} \subseteq \mathscr{P}(\omega)$ is an almost disjoint family of infinite sets, and $|\mathscr{B}| = \kappa$, where $\omega \le \kappa < 2^\omega$. Also suppose that $\mathscr{A} \subseteq \mathscr{B}$. Assume* MA($\kappa$).

Then there is a $d \subseteq \omega$ such that $|d \cap x| < \omega$ for all $x \in \mathscr{A}$ and $|d \cap x| = \omega$ for all $x \in \mathscr{B} \backslash \mathscr{A}$.

Proof. We apply 16.40 with $\mathscr{B} \backslash \mathscr{A}$ in place of $\mathscr{B}$. If $y \in \mathscr{B} \backslash \mathscr{A}$ and $F \in [\mathscr{A}]^{<\omega}$, then $y \cup F \subseteq \mathscr{B}$, and hence $y \cap z$ is finite for all $y \in F$. Hence also $y \cap \bigcup F$ is finite. Since y itself is infinite, it follows that $y \backslash \bigcup F$ is infinite.

Thus the hypotheses of 16.40 hold, and it then gives the desired result. $\square$

We now come to two of the most striking consequences of Martin's axiom.

Theorem 16.44. *If κ is an infinite cardinal and* MA(κ) *holds, then $2^\kappa = 2^\omega$.*

Proof. By Lemma 9.21 let $\mathscr{B}$ be an almost disjoint family of infinite subsets of ω such that $|\mathscr{B}| = \kappa$. For each $d \subseteq \omega$ let $F(d) = \{b \in \mathscr{B} : |b \cap d| < \omega\}$. We claim that F maps $\mathscr{P}(\omega)$ onto $\mathscr{P}(\mathscr{B})$; from this it follows that $2^\kappa \le 2^\omega$, hence $2^\kappa = 2^\omega$. To prove the claim, suppose that $\mathscr{A} \subseteq \mathscr{B}$. A suitable d with $F(d) = \mathscr{A}$ is then given by Theorem 16.40. $\square$

Corollary 16.45. MA *implies that 2^ω is regular.*

Proof. Assume MA, and suppose that $\omega \le \kappa < 2^\omega$. Then $2^\kappa = 2^\omega$ by Theorem 16.44, and so $\mathrm{cf}(2^\omega) = \mathrm{cf}(2^\kappa) > \kappa$. $\square$

A forcing poset P is said to have ω_1 as a *pre-caliber* iff for every system $\langle p_\alpha : \alpha < \omega_1\rangle$ of elements of P there is an $X \in [\omega_1]^{\omega_1}$ such that for every finite subset F of X there is a $q \in P$ such that $q \leq p_\alpha$ for all $\alpha \in F$. Clearly this implies property (K).

Theorem 16.46. *MA(ω_1) implies that every ccc forcing poset P has ω_1 as a pre-caliber.*

Proof. Let $\langle p_\alpha : \alpha < \omega_1\rangle$ be a system of elements of P; we want to come up with a set X as indicated. For each $\alpha < \omega_1$ let

$$W_\alpha = \{q \in P : \exists\beta > \alpha(q \text{ and } p_\alpha \text{ are compatible})\}.$$

Clearly if $\alpha < \beta < \omega_1$ then $W_\beta \subseteq W_\alpha$. Now we claim

$$\exists\alpha\forall\beta \in (\alpha, \omega_1)[W_\alpha = W_\beta]. \tag{1}$$

In fact, otherwise we get a strictly increasing sequence $\langle\alpha_\xi : \xi < \omega_1\rangle$ of ordinals such that $W_{\alpha_{\xi+1}} \subset W_{\alpha_\xi}$ for all $\xi < \omega_1$. Choose $q_\xi \in W_{\alpha_\xi}\backslash W_{\alpha_{\xi+1}}$ for all $\xi < \omega_1$. Then there is an ordinal β_ξ such that $\alpha_\xi < \beta_\xi \leq \alpha_{\xi+1}$ and q_ξ and p_{β_ξ} are compatible; say $r_\xi \leq q_\xi, p_{\beta_\xi}$. We claim that r_ξ and r_η are incompatible for $\xi < \eta < \omega_1$ (contradicting ccc for P). For, if $s \leq r_\xi, r_\eta$, then q_ξ and p_{β_η} are compatible, and hence $q_\xi \in W_{\alpha_{\xi+1}}$, contradiction. Thus (1) holds.

We are going to apply MA(ω_1) to W_α. The dense sets are as follows. For each $\beta \in (\alpha, \omega_1)$, let

$$D_\beta = \{q \in W_\alpha : \exists\gamma \in (\beta, \omega_1)[q \leq p_\gamma]\}.$$

To prove density, suppose that $r \in W_\alpha$. Then, since $W_\alpha = W_\beta$ it follows that r and p_γ are compatible for some $\gamma > \beta$, as desired.

So, let G be a filter on W_α intersecting each set D_β. It follows that there exist a strictly increasing sequence $\langle\beta_\xi : \xi < \omega_1\rangle$ and a sequence $\langle q_\xi : \xi < \omega_1\rangle$ such that $q_\xi \leq p_{\beta_\xi}$ with $q_\xi \in G$ for all $\xi < \omega_1$. Clearly then $\{p_{\beta_\xi} : \xi < \omega_1\}$ has the desired property. ☐

Corollary 16.47. *$MA_{\aleph_1}$ implies that if each P_i for $i \in I$ satisfies ccc, then so does $\prod^w_{i\in I} P_i$.*

Proof. By Lemma 15.17 and Theorem 16.46. ☐

Theorem 16.48. *Martin's axiom implies that the intersection of fewer than 2^ω dense open subsets of $\mathbb{R}$ is dense.*

Proof. Let $\kappa < 2^\omega$ and let $\langle U_\alpha : \alpha < \kappa\rangle$ be a system of dense open subsets of $\mathbb{R}$. Let (a, b) be given; we will show that $\bigcap_{\alpha<\kappa} U_\alpha \cap (a, b) \neq \emptyset$.

Let P consist of all nonempty open sets p such that $p \subseteq (a, b)$, with order $\subseteq$. Note that $p, q \in P$ are compatible iff $p \cap q \neq \emptyset$. So P satisfies ccc. For each $\alpha < \kappa$ let $D_\alpha = \{p \in P : \overline{p} \subseteq U_\alpha\}$. Then D_α is dense in P. For, suppose that $p \in P$. Since U_α is dense, we have $p \cap U_\alpha \neq \emptyset$. Choose $c < d$ so that $(c, d) \subseteq p \cap U_\alpha$. Then choose $e < f$ with $c < e < f < d$. Then $\overline{(e, f)} = [e, f] \subseteq U_\alpha$. Thus D_α is dense. Let G be a $\{D_\alpha : \alpha < \kappa\}$-generic filter on P. Then $\bigcap_{p\in G} \overline{p} \neq \emptyset$. Clearly $\forall\alpha < \kappa[\bigcap_{p\in G} \overline{p} \subseteq U_\alpha]$. ☐

A set $\mathscr{A} \subseteq [\omega]^\omega$ has the *strong finite intersection property* (SFIP) iff $\forall X \in [\mathscr{A}]^{<\omega}[\bigcap X$ is infinite].

Proposition 16.49. *There is a $\mathscr{A} \subseteq [\omega]^\omega$ which has the SFIP and is such that there is no infinite $A \subseteq \omega$ such that $\forall B \in \mathscr{A}[A \backslash B$ is finite].*

Proof. The set $\{\omega\backslash F : F$ finite$\}$ has the fip. Suppose that $A \subseteq \omega$ and $\forall F \in [\omega]^{<\omega}[A\backslash(\omega\backslash F)$ is finite]. Thus $A \cap F = \emptyset$ for all finite $F \subseteq \omega$. Hence $A = \emptyset$. $\square$

Let $\mathfrak{p} = \min\{|X| : X \subseteq \omega^\omega$ has SFIP and $\neg\exists A \in [\omega]^\omega \forall B \in X[A\backslash B$ is finite $]\}$.

The following theorem generalizes Theorem 16.24 in Jech.

Theorem 16.50. *MA_κ implies that $\kappa < \mathfrak{p}$.*

Proof. Let $\mathscr{E} \subseteq [\omega]^\omega$ have SFIP, with $|\mathscr{E}| = \kappa$. We want to find a pseudo-intersection of $\mathscr{E}$. Let

$$\mathbb{P} = \{(s_p, W_p) : s_p \in [\omega]^{<\omega} \text{ and } W_p \in [\mathscr{E}]^{<\omega}\}.$$

We define $q \leq p$ iff the following hold:

(1) $s_p \subseteq s_q$.
(2) $W_p \subseteq W_q$.
(3) $\forall Z \in W_p[(s_q\backslash s_p) \subseteq Z]$.

This is a forcing order. For transitivity, suppose that $r \leq q \leq p$. Clearly (1) and (2) for r and p hold. Now suppose that $Z \in W_p$. Then $Z \in W_q$, and $(s_r\backslash s_p) = (s_r\backslash s_q) \cup (s_q\backslash s_p) \subseteq Z$.

If $s_p = s_q$, then p and q are compatible, since $(s_p, W_p \cup W_q)$ extends both of them. Since $[\omega]^{<\omega}$ is countable, it follows that $\mathbb{P}$ has ccc.

Now for each $n \in \omega$ let $D_n = \{p \in \mathbb{P} : |s_p| \geq n\}$. Then D_n is dense, for if $p \in \mathbb{P}$, then $\bigcap W_p$ is infinite, so we can choose $t \subseteq \bigcap W_p$ with $|t| = n$, and then $(s_p \cup t, W_p) \in D_n$ and $(s_p \cup t, W_p) \leq p$.

For any $Z \in \mathscr{E}$ let $E_Z = \{p \in \mathbb{P} : Z \in W_p\}$. Then E_Z is dense, since if $p \in \mathbb{P}$, then $(s_p, W_p \cup \{Z\}) \in E_Z$ and $(s_p, W_p \cup \{Z\}) \leq p$.

Let G be a filter intersecting all of these dense sets. Let $K_G = \bigcup_{p \in G} s_p$. Then G intersecting all sets D_n for $n \in \omega$ implies that K_G is infinite.

Given $Z \in \mathscr{E}$, choose $p \in G \cap E_Z$. Suppose that $m \in K_G\backslash Z$. Say $m \in s_q$ with $q \in G$. Choose $r \in G$ such that $r \leq p, q$. Then $m \in s_r$ since $r \leq q$. If $m \notin s_p$, then $m \in Z$ since $r \leq p$. Thus $K_G \subseteq Z \subseteq s_p$ and hence $K_G\backslash Z$ is finite. $\square$

Corollary 16.51. *MA implies that $\mathfrak{p} = 2^\omega$.* $\square$

For $f, g \in {}^\omega\omega$ we write $f \leq^* g$ iff $\exists m \forall n \geq m[f(n) \leq g(n)]$. A family $\mathscr{D} \subseteq {}^\omega\omega$ is *dominating* iff $\forall f \in {}^\omega\omega \exists g \in \mathscr{D}[f \leq^* g]$. Obviously ${}^\omega\omega$ is dominating. $\mathfrak{d}$ is the smallest size of a dominating family.

A subset $\mathscr{B}$ of ${}^\omega\omega$ is *unbounded* iff there is no $g \in {}^\omega\omega$ such that $\forall f \in \mathscr{B}[f \leq^* g]$. ${}^\omega\omega$ is unbounded; for suppose that $g \in {}^\omega\omega$ is a bound for ${}^\omega\omega$. Let $f(n) = g(n) + 1$ for all $n \in \omega$. Clearly this gives a contradiction. idx$\mathfrak{b}$ is the smallest size of an unbounded family.

Proposition 16.52. $\mathfrak{b} \leq \mathfrak{d}$.

Proof. Let $\mathscr{D}$ be a dominating family of size $\mathfrak{d}$. Then $\mathscr{D}$ is unbounded. For, suppose that g is a bound. Let $f(n) = g(n) + 1$ for all n. Choose $h \in \mathscr{D}$ such that $f \leq^* h$. Thus $h \leq^* g$. Choose m so that $\forall n \geq m[h(n) \leq g(n)]$ and $f(n) \leq h(n)$. Then $g(n)+1 = f(n) \leq h(n) \leq g(n)$, contradiction. $\square$

Theorem 16.53. $\mathfrak{p} \leq \mathfrak{b}$.

Proof. We show that $\forall \kappa[\kappa < \mathfrak{p} \to \kappa < \mathfrak{b}]$. Take any $\langle f_\xi : \xi \leq \kappa \rangle \in {}^{\kappa+1}({}^\omega\omega)$. We define $\langle g_\xi : \xi \leq \kappa \rangle$ by recursion. Suppose that g_ξ has been defined for all $\xi < \eta$, with $\eta \leq \kappa$ so that

(1) $\forall \xi < \eta[g_\xi \in {}^\omega\omega$ and it is strictly increasing$]$.

(2) $\forall \xi, \xi' < \eta[\xi < \xi' \to \text{rng}(g_{\xi'}) \backslash \text{rng}(g_\xi)$ is finite$]$.

Then

(3) $\{\text{rng}(g_\xi) : \xi < \eta\}$ has SFIP.

In fact, suppose that $F \in [\eta]^{<\omega}$. We may assume that F is nonempty. Let ξ be the greatest member of F. Then by (2), $\bigcup\{\text{rng}(g_\xi)\backslash\text{rng}(g_{\xi'}) : \xi' \in F, \xi' < \xi\}$ is finite. Hence

$$\begin{aligned}
\text{rng}(g_\xi) &= \bigcup\{\text{rng}(g_\xi)\backslash\text{rng}(g_{\xi'}) : \xi' \in F, \xi' < \xi\} \cup \\
&\quad \Big(\text{rng}(g_\xi)\backslash \bigcup\{\text{rng}(g_\xi)\backslash\text{rng}(g_{\xi'}) : \xi' \in F, \xi' < \xi\}\Big) \\
&= \bigcup\{\text{rng}(g_\xi)\backslash\text{rng}(g_{\xi'}) : \xi' \in F, \xi' < \xi\} \cup \bigcap\{\text{rng}(g'_\xi) : \xi' \in F\},
\end{aligned}$$

and so $\bigcap\{\text{rng}(g'_\xi) : \xi' \in F\}$ is infinite, proving (3).

Now let $A \in [\omega]^\omega$ be such that $\forall \xi < \eta[A \subseteq \text{rng}(g_\xi)$ is finite$]$. For each $n \in \omega$ let

$$g_\eta(n) = \min\{a \in A : \forall k \leq 2n[f_\eta(k) \leq a] \text{ and } \forall m < n[g_\eta(m) < a]\}.$$

Clearly g_η is strictly increasing. If $\xi < \eta$, then $\text{rng}(g_\eta) \subseteq A$ and $A\backslash\text{rng}(g_\xi)$ is finite, so $\text{rng}(g_\eta)\backslash\text{rng}(g_\xi)$ is finite. So (2) holds for η.

We claim that g_κ bounds $\langle f_\xi : \xi \leq \kappa\rangle$. For, take any $\xi \leq \kappa$.

(4) There is an $m \in \omega$ such that $\forall n[g_\kappa(m+n) \in \text{rng}(g_\xi)]$.

In fact, choose $m \in \omega$ such that $\forall n \geq m[n \in \text{rng}(g_\kappa) \to n \in \text{rng}(g_\xi)]$. Hence for all n, $m \leq m + n \leq g_\kappa(m+n) \in \text{rng}(g_\kappa)$ and so $g_\kappa(m+n) \in \text{rng}(g_\xi)$.

Take m as in (4). Choose p so that $g_\kappa(m) = g_\xi(p)$.

(5) For all n, $g_\kappa(m+n) \geq g_\xi(p+n)$.

We prove this by induction on n. It is clear for $n = 0$. Assume it for n. Then $g_\xi(p+n) \leq g_\kappa(m+n) < g_\kappa(m+n+1$ and $g_\kappa(m+n) \in \text{rng}(g_\xi)$, so $g_\xi(p+n+1) \leq g_\kappa(m+n+1)$.

Now if $n \geq m$, then $n \geq m - p$, hence $2p + 2n - 2m \geq n$ and

$$g_\kappa(n) = g_\kappa(m+n-m) \geq g_\xi(p+n-m) \geq f_\xi(2p+2n-2m) \geq f_\xi(n). \qquad \square$$

Corollary 16.54. *MA implies that* $\mathfrak{p} = \mathfrak{b} = \mathfrak{d} = 2^\omega$. $\square$

For $f, g \in {}^\omega\omega$ we define $f < g$ iff $\forall n[f(n) < g(n)]$.

Lemma 16.55. $\mathfrak{d} = \min\{|X| : X \subseteq {}^\omega\omega$ *and* $\forall f \in {}^\omega\omega \exists g \in X[f < g]\}$.

Proof. Let Y be a dominating family with $|Y| = \mathfrak{d}$. Let $X = \{f \in {}^\omega\omega : \exists g \in Y[\{n \in \omega : f(n) \neq g(n)\}$ is finite$]\}$. Clearly $|X| = \mathfrak{d}$ and $\forall f \in {}^\omega\omega \exists g \in X[f < g]$. $\square$

Theorem 16.56. *If* $\mathfrak{b} = \mathfrak{d} = \kappa$*, then there is a* κ*-scale.*

Proof. Let $\mathscr{D} = \{f_\xi : \xi < \mathfrak{b}\}$ be a dominating family with the additional property given in Lemma 16.55. We define $\langle g_\xi : \xi < \kappa \rangle$ by recursion. Suppose defined for all $\eta < \xi$, with $\xi < \kappa$, so that $\forall \eta, \eta' < \xi[\eta <\leq \eta' \to g_\eta \leq^* g_{\eta'}]$. Then $\{g_\eta : \eta < \xi\}$ is bounded, so choose $g_\xi \in \mathscr{D}$ such that $f_\eta \leq^* g_\xi$ for all $\eta < \xi$ and $g_\eta \leq^* g'_\xi$ for all $\eta < \xi$. Now for every $h \in {}^\omega\omega$ there is a $\xi < \mathfrak{b}$ such that $h \leq^* f_\xi$, and $f_\xi \leq^* g_{\xi+1}$. $\square$

Corollary 16.57. *MA implies that there is a* 2^ω*-scale.* $\square$

Corollary 16.58. *MA implies that* 2^ω *is not real-valued measurable.*

Proof. By Lemma 10.16 and Corollary 16.57. $\square$

Lemma 16.59. *Assume MA, and let* $\mathscr{G}$ *be a family of infinite subsets of* ω *with the fip such that* $|\mathscr{G}| < 2^\omega$*. Let* $A_0 \supseteq A_1 \supseteq \cdots$ *be a decreasing sequence of elements of* $\mathscr{G}$*. Then there is a* $Z \subseteq \omega$ *such that*

(i) $\mathscr{G} \cup \{Z\}$ *has fip.*

(ii) $\forall n \in \omega[Z \backslash A_n$ *is finite*$]$.

Proof. We may assume that $\forall A, B \in \mathscr{G}[A \cap B \in \mathscr{G}]$. Thus $\forall X \in \mathscr{G} \forall n \in \omega[X \cap A_n \in \mathscr{G}]$ and hence $\forall X \in \mathscr{G} \forall n \in \omega[X \cap A_n \neq \emptyset]$. For each $X \in \mathscr{G}$ and each $n \in \omega$ let $h_X(n) \in X \cap A_n$. By Corollary 16.54, $\mathfrak{b} = 2^\omega$, and so $\{h_X : X \in \mathscr{G}\}$ is bounded; say $h_X \leq^* f$ for all $X \in \mathscr{G}$. Hence for all $X \in \mathscr{G}$ there is an n_X such that $\forall m \geq n_X[h_X(m) \leq f(m)]$. Let $Z = \bigcup_{n \in \omega}\{k \in A_n : k \leq f(n)\}$.

(1) $\forall X \in \mathscr{G}[Z \cap X \neq \emptyset]$.

In fact, let $X \in \mathscr{G}$. Then $h_X(n_X) \leq f(n_X)$ and $h_X(n_X) \in X \cap A_{n_x}$. Hence $h_X(n_X) \in Z \cap X$.

(2) $\forall n \in \omega[Z \backslash A_n$ is finite$]$.

For, let $n \in \omega$. Then

$$Z \backslash A_n = \bigcup_{m \in \omega}\{k \in A_m \backslash A_n : k \leq f(m)\} = \bigcup_{m < n}\{k \in A_m \backslash A_n : k \leq f(m)\},$$

and this set is finite.

(1) and (2) give the lemma. $\square$

Theorem 16.60. *MA implies that there is a P-point*

Proof. Assume MA. We will apply exercise 7.7. Let $\mathscr{A} = \{A \in {}^\omega([\omega]^\omega) : A$ is strictly decreasing$\}$. Clearly $|\mathscr{A}| \leq 2^\omega$. Let $\langle A^\alpha : \alpha < 2^\omega\rangle$ enumerate $\mathscr{A}$. Now we construct $\langle \mathscr{G}_\alpha : \alpha < 2^\omega\rangle$ by recursion so that

(i) Each $\mathscr{G}_\alpha$ is a collection of nonempty subsets of ω.
(ii) Each $\mathscr{G}_\alpha$ is closed under $\cap$.
(iii) $\mathscr{G}_\alpha \subseteq \mathscr{G}_\beta$ for $\alpha < \beta$.
(iv) $\forall \alpha < 2^\omega[|\mathscr{G}_\alpha| < 2^\omega]$.

Let $\mathscr{G}_0 = \{\omega\backslash F : F \in [\omega]^{<\omega}\}$. For $\alpha < 2^\omega$ limit let $\mathscr{G}_\alpha = \bigcup_{\beta<\alpha}\mathscr{G}_\beta$. Note that (iv) then holds for α since 2^ω is regular by Corollary 16.45. Now if $\mathscr{G}_\alpha$ has been constructed, we construct $\mathscr{G}_{\alpha+1}$. If $\exists n \in \omega\exists X \in \mathscr{G}_\alpha[A^\alpha_n \cap X = \emptyset]$, let $\mathscr{G}_{\alpha+1} = \mathscr{G}_\alpha$. Suppose that $\forall n \in \omega\forall X \in \mathscr{G}_\alpha[A^\alpha_n \cap X \neq \emptyset]$. Then $\mathscr{G}_\alpha \cup \{A^\alpha_n : n \in \omega\}$ has fip. Hence by Lemma 16.59 there is a $Z \subseteq \omega$ such that $\mathscr{G}_\alpha \cup \{A^\alpha_n : n \in \omega\} \cup \{Z\}$ has fip and $\forall n \in \omega[Z\backslash A^\alpha_n$ is finite$]$. Let $\mathscr{G}_{\alpha+1}$ consist of all finite intersections of members of $\mathscr{G}_\alpha \cup \{A^\alpha_n : n \in \omega\} \cup \{Z\}$.

Finally, let $\mathscr{G} = \bigcup_{\alpha<2^\omega}\mathscr{G}_\alpha$, and let D be an ultrafilter such that $\mathscr{G} \subseteq D$. We claim that D is a p-point. To apply exercise 7.7, suppose that $B_0 \supseteq B_1 \supseteq \cdots$ are members of D. Say $B = A^\alpha$. Choose $Z \in \mathscr{D}_{\alpha+1}$ such that $\forall n \in \omega[Z \subseteq A^\alpha_n$ is finite$]$. So by exercise 7.7 D is a p-point. □

For each $\alpha \geq 1$ an *iteration of length* α is a pair $(\langle P_\xi : \xi \leq \alpha\rangle, \langle Q_\xi : \xi < \alpha\rangle)$ with the following properties:

(i) Each P_ξ is a forcing poset.
(ii) Each Q_ξ is a P_ξ-name for a forcing poset.
(iii) Each $p \in P_\xi$ is a sequence of length ξ such that $\forall \eta < \xi[p_\eta \in \mathrm{dmn}(Q_\eta)]$
(iv) If $\eta < \xi$ and $p \in P_\xi$, then $p \restriction \eta \in P_\eta$.
(v) If $\eta < \xi$, $p \in P_\eta$, and p' is the function with domain ξ such that $p' \restriction \eta = p$ and $\forall \mu \in [\eta, \xi)[p'_\mu = 1]$, then $p' \in P_\xi$.
(vi) $1_\xi = \langle 1_\eta : \eta < \xi\rangle$.
(vii) If $p, p' \in P_\xi$, then $p \leq_\xi p'$ iff $\forall \mu < \xi[p \restriction \mu \Vdash_{P_\mu} p_\mu \leq p'_\mu]$.
(viii) If $\xi + 1 \leq \alpha$, then $P_{\xi+1} = \{p^\frown\langle q\rangle : p \in P_\xi$ and $q \in \mathrm{dmn}(Q_\xi)$ and $p \Vdash_{P_\xi} q \in Q_\xi$.

We have merely omitted clause (ix) in the definition of finite support iteration; so what happens at limit ordinals is not determined. For α a limit ordinal, we say that P_α is a *direct limit* iff

$$\forall p[p \in P_\alpha \quad \text{iff} \quad \exists \beta < \alpha[p \restriction \beta \in P_\beta \text{ and } \forall \xi \in [\beta, \alpha)[p(\xi) = 1].$$

Theorem 16.61. *Let κ be a regular uncountable cardinal and let α be a limit ordinal. Let an iteration of length α be given, with notation as above. Suppose that P_α is a direct limit and $\mathrm{cf}(\alpha) \neq \kappa$. For all $\beta < \alpha$ let $P_\alpha \restriction \beta = \{p \restriction \beta : p \in P_\alpha\}$. Suppose that $\forall \beta < \alpha[P_\alpha \restriction \beta$ satisfies the κ-cc$]$.*

Then P_α satisfies κ-cc.

Proof. Suppose that $\langle p^\beta : \beta < \kappa\rangle$ is an antichain in P_α. For each $\beta < \kappa$ let $\gamma_\beta < \alpha$ be such that $\forall \xi \in [\gamma_\beta, \alpha)[p^\beta_\xi = 1]$.

Case 1. $\kappa < \mathrm{cf}(\alpha)$. Let $\delta = \sup_{\beta<\alpha}\gamma_\beta$. So $\delta < \alpha$ and $\forall\beta < \kappa\forall\xi \in [\delta,\alpha)[p^\beta_\xi = 1]$. Hence $\langle p^\beta \restriction \delta : \beta < \kappa\rangle$ is an antichain in P_δ. contradiction.

Case 9. $\mathrm{cf}(\alpha) < \kappa$. Let $\langle\delta_\xi : \xi < \mathrm{cf}(\alpha)\rangle$ be strictly increasing with supremum α. Then $\kappa = \bigcup_{\xi<\mathrm{cf}(\alpha)}\{\beta < \kappa : \gamma_\beta < \delta_\xi\}$. Hence there is a $\xi < \mathrm{cf}(\alpha)$ such that $|\{\beta < \kappa : \gamma_\beta < \delta_\xi\}| = \kappa$. Hence $\langle p^\beta \restriction \delta_\xi : \beta < \kappa, \gamma_\beta < \delta_\xi\rangle$ is an antichain in P_{δ_ξ}, contradiction. $\square$

Lemma 16.62. *In M suppose that κ is uncountable and regular, and $\mathbb{P}$ is ccc. Suppose that $\dot S$ is a name, and $\mathbb{1} \Vdash [\dot S \subseteq \kappa \wedge |\dot S| < \kappa]$. Then there is a $\beta < \kappa$ such that $\mathbb{1} \Vdash [\dot S \subseteq \check\beta]$.*

Proof. Let $E = \{\alpha < \kappa : \exists p[p \Vdash [\check\alpha = \sup(\dot S)]]\}$. For each $\alpha \in E$ let p_α be such that $p_\alpha \Vdash [\check\alpha = \sup(\dot S)]$. Clearly $\{p_\alpha : \alpha \in E\}$ is an antichain, so E is countable. Hence there is a $\beta < \kappa$ such that $E \subseteq \beta$. We claim that $\mathbb{1} \Vdash [\dot S \subseteq \check\beta]$.

Suppose not. Then there is a p such that $p \Vdash \neg[\dot S \subseteq \check\beta]$. So $p \Vdash [\sup(\dot S) \geq \beta]$, hence $p \Vdash \exists x \in \kappa[x \geq \beta \wedge \sup(\dot S) = x]$. Hence there exist a $q \leq p$ and an $\alpha < \kappa$ such that $q \Vdash [\check\alpha \geq \check\beta \wedge \sup(\dot S) = \check\alpha]$. Hence $\alpha \geq \beta$ and $q \Vdash \sup(\dot S) = \alpha$, so $\alpha \in E$, contradicting $E \subseteq \beta$. $\square$

Lemma 16.63. *If P has property (K) and $\mathbb{1}_P \Vdash \dot Q$ has property (K), then $P * \dot Q$ has property (K).*

Proof. Assume the hypotheses. Let $\langle(p_\xi, \dot q_\xi) : \xi < \omega_1\rangle$ be a system of elements of $\mathbb{P} * \dot{\mathbb{Q}}$. Let $\dot S$ be the $\mathbb{P}$-name with $\mathrm{dmn}(\dot S) = \{\check\xi : \xi < \omega_1\}$ and $\dot S(\check\xi) = e(p_\xi)$. Then for any generic G, $\dot S_G = \{\xi : \exists q \in G[e(q) \leq e(p_\xi)]\}$. Note that

(1) $\exists q \in G[e(q) \leq e(p_\xi)]$ iff $p_\xi \in G$.

In fact, $\Leftarrow$ is clear. Now suppose that $q \in G$ and $e(q) \leq e(p_\xi)$. Then $\{r : r \leq p_\xi\}$ is dense below q, since if $s \leq q$ the $e(s) \leq e(p_\xi)$ hence there is an $r \leq s, p_\xi$, as desired. It follows that $p_\xi \in G$. So (1) holds.

Hence $\dot S_G = \{\xi : p_\xi \in G\}$.

Let $\mathrm{dmn}(\dot F) = \{(\mathrm{op}(\check\xi, \dot q_\xi) : \xi < \omega_1$ and $\dot F(\mathrm{op}(\check\xi, \dot q_\xi) = e(p_\xi)$. Thus for any generic G, $\dot F_G = \{(\xi, \dot q_{\xi G}) : p_\xi \in G\}$. So $\dot F_G$ is the function with domain $\dot S_G$ such that $\dot F_G(\xi) = \dot q_{\xi G}$ for any ξ with $p_\xi \in G$.

(2) There is no $\beta < \omega_1$ such that $\mathbb{1} \Vdash [\dot S \subseteq \beta]$.

In fact, otherwise let G be generic with $p_\beta \in G$. Then $\beta \in \dot S_G \subseteq \beta$, contradiction.

Now clearly $\mathbb{1} \Vdash [\dot S \subseteq \omega_1]$. Hence by Lemma 16.62 we have $\mathbb{1} \nVdash [|\dot S| < \omega_1]$. So there is a p such that $p \Vdash [|\dot S| = \omega_1]$. Let G be generic with $p \in G$. Then

(3) $M[G] \models [|\dot S_G| = \omega_1]$.

Now $\dot F_G$ maps $\dot S_G$ into $\dot{\mathbb{Q}}_G$ and $\dot{\mathbb{Q}}_G$ has property K, so in $M[G]$ there is a set $B \in [\dot S_G]^{\omega_1}$ such that in $\{\dot F_G(\xi) : \xi \in B\}$ any two elements are compatible. Say $B = \dot B_G$. Take $p^* \in G$ with $p^* \leq p$ and

$$p^* \Vdash [\dot B \subseteq \dot S \text{ and any two elements of } \{\dot F(\xi) : \xi \in \dot B\} \text{ are compatible}].$$

In M let

$$A = \{\xi < \omega_1 : \exists q[(q \le p^*) \wedge (q \le p_\xi) \wedge q \Vdash [\xi \in \dot{B}]\}.$$

(4) $B \subseteq A$.

In fact, suppose that $\xi \in B$. Since $B \subseteq \dot{S}_G$, we have $\xi \in \dot{S}_G$ and hence $p_\xi \in G$. Also, there is an $r \in G$ such that $r \Vdash [\xi \in \dot{B}]$. Choose $q \in G$ so that $q \le r, p^*, p_\xi$. Thus $\xi \in A$.

It follows that A is uncountable, as otherwise $A \subseteq \beta$ for some $\beta < \omega_1$, hence by (4) $B \subseteq \beta$, contradicting $|B| = \omega_1$. Now in M, for any $\xi \in A$ choose p'_ξ such that $p'_\xi \le p^*, p_\xi$ and $p'_\xi \Vdash \xi \in \dot{B}$. Since A is uncountable and $\mathbb{P}$ has property K, choose $L \in [A]^{\omega_1}$ such that any two elements of $\{p'_\xi : \xi \in L\}$ are compatible.

Finally, to show that any two elements of $\{(p_\xi, \dot{q}_\xi) : \xi \in L\}$ are compatible, take any $\xi, \eta \in L$. Take any $p'' \le p'_\xi, p'_\eta$. Then $p'' \Vdash [\xi \in \dot{B} \wedge \eta \in \dot{B}]$. Then $p'' \Vdash [\dot{q}_\xi \not\perp \dot{q}_\eta]$, so $p'' \Vdash \exists q'[(q' \le \dot{q}_\xi) \wedge (q' \le \dot{q}_\eta)]$. Hence by Lemma 14.31, there is a $p''' \le p''$ and a $q'' \in \operatorname{dmn}(\dot{\mathbb{Q}})$ such that $p''' \Vdash [(q'' \le \dot{q}_\xi) \wedge (q'' \le \dot{q}_\eta)]$. By the definition of the order on $\mathbb{P} * \dot{\mathbb{Q}}$, this shows that $(p''', q'') \le (p_\xi, \dot{q}_\xi), (p_\eta, \dot{q}_\eta)$. □

Theorem 16.64. *If P is κ-distributive and $\mathbb{1} \Vdash \dot{Q}$ is κ-distributive, then $P * \dot{Q}$ is κ-distributive.*

Proof. We will apply Lemma 15.90. Suppose that K is $(P * \dot{Q})$-generic over M and $f \in M[K]$ with $\operatorname{dmn}(f) = \kappa$ and $\operatorname{rng}(f) \subseteq M$. Let G and H be as in Theorem 16.4. Since G is P-generic over M, it follows that $\dot{Q}_G$ is κ-distributive. Now $f \in M[G][H]$, so by Lemma 15.90, $f \in M[G]$. Then by Lemma 15.90 again, $f \in M$. □

Lemma 16.65. *Given an iteration of length α and $\xi < \eta < \alpha$, let $P_\eta \restriction \xi \stackrel{\text{def}}{=} \{p \restriction \xi : p \in P_\eta\}$. Then $P_\xi = P_\eta \restriction \xi$.*

Proof. By (iv) and (v). □

Lemma 16.66. *Given a finite support iteration of length α and a limit ordinal $\beta \le \alpha$, for $\xi < \eta \le \beta$ let $f_{\xi\eta}$ be the complete embedding of* $\mathrm{RO}(P_\xi)$ *into* $\mathrm{RO}(P_\eta)$ *given by 16.8 and 16.19. Then there is a sequence $B_0 \subseteq B_1 \subseteq \cdots \subseteq B_\xi \subseteq \cdots$ for $\xi < \beta$ and isomorphisms $g_\xi : \mathrm{RO}(P_\xi) \to B_\xi$ such that B_ξ is a complete subalgebra of B_η for $\xi < \eta$, B_β is the completion of $\bigcup_{\xi<\beta} B_\xi$, and the following diagram commutes:* xxx **Proof.** We check that B_β is the completion of $\bigcup_{\xi<\beta} B_\xi$. Suppose that $a \in B_\beta$. Then $g_\beta^{-1}(a) \in \mathrm{RO}(P_\beta)$, and so there is a $p \in P_\beta$ such that $e(p) \le g_\beta^{-1}(a)$. By (x) in the definition there exist $\xi < \beta$ and $q \in P_\xi$ such that $i_{\xi\beta}(q) = p$. Then $e(q) \in \mathrm{RO}(P_\xi)$ and $g_\xi(e(q)) \le a$, as desired. □

17. Large cardinals

Let U be an ultrafilter on a set S. Then is the class of all functions with domain S. We define

$$\begin{aligned} f =^* g \quad &\text{iff} \quad f, g \in \mathrm{Fcn}(S) \text{ and } \{x \in S : f(x) = g(x)\} \in U; \\ f \in^* g \quad &\text{iff} \quad f, g \in \mathrm{Fcn}(S) \text{ and } \{x \in S : f(x) \in g(x)\} \in U. \end{aligned}$$

If the exact S and U are not clear, we write $=^*_{SU}$ and $\in^*_{SU}$. Clearly $=^*$ is an equivalence relation on $\mathrm{Fcn}(S)$. We denote by $[f]$ the Scott equivalence class of f:

$$[f] = \{g : f =^* g \text{ and } \forall h(h =^* f \to \mathrm{rank}(g) \leq \mathrm{rank}(h))\}.$$

Then we define $\mathrm{Ult} = \mathrm{Ult}_{SU}$ to be the collection of all equivalence classes, with $\in_{\mathrm{Ult}} = \{([f], [g]) : f \in^* g\}$. We can write this as $\in_{\mathrm{Ult}} = \{(x, y) : \exists f, g[x = [f],\ y = [g]$, and $f \in^* g\}$.

Proposition 17.1. *If $f, g \in \mathrm{Fcn}(S)$ and $f \in^* g$, then there is an $f' \in \prod_{s \in S}(g(s) \cup \{\emptyset\})$ such that $f =^* f'$.*

Proof. For each $s \in S$, let

$$f'(s) = \begin{cases} f(s) & \text{if } f(s) \in g(s), \\ \emptyset & \text{otherwise.} \end{cases}$$

Clearly $f =^* f'$. □

Proposition 17.2. *$\in_{\mathrm{Ult}}$ is set-like.*

Proof. Let $x \in \mathrm{Ult}_{SU}$. Choose $g \in \mathrm{Fcn}(S)$ such that $x = [g]$. We claim

$$(*) \qquad \{y \in \mathrm{Ult}_{SU} : y \in_{\mathrm{Ult}} x\} = \left\{[f] : f \in \prod_{s \in S}(g(s) \cup \{\emptyset\}) \text{ and } f \in^* g\right\}.$$

(Clearly this will prove the proposition.) To prove $(*)$, first suppose that $y \in \mathrm{Ult}_{SU}$ and $y \in_{\mathrm{Ult}} x$. Choose $f, g' \in \mathrm{Fcn}(S)$ such that $x = [g']$, $y = [f]$, and $f \in^* g'$. Then $[g] = [g']$ and $f \in^* g$. By Proposition 17.1, choose $f' \in \prod_{s \in S}(g(s) \cup \{\emptyset\})$ such that $f =^* f'$. Then $f' \in^* g$ and $y = [f] = [f']$. So y is in the right side of $(*)$.

Second suppose that $f \in \prod_{s \in S}(g(s) \cup \{\emptyset\})$ and $f \in^* g$. Then $[f] \in \mathrm{Ult}_{SU}$ and $[f] \in_{\mathrm{Ult}} x$, as desired. □

Theorem 17.3. *For any formula $\varphi(x_1, \ldots, x_n)$ of set theory and any $f_1, \ldots, f_n \in Fcn(S)$ we have*

$$\mathrm{Ult} \models \varphi([f_1], \ldots, [f_n]) \quad \text{iff} \quad \{s \in S : \varphi(f_1(s), \ldots, f_n(s))\} \in U$$

Note that this is a theorem schema in ZFC.

Proof. Induction on φ:

$$\begin{aligned}
[f]=[g] \quad &\text{iff} \quad f =^* g\\
&\text{iff} \quad \{s\in S : f(s)=g(s)\}\in U;\\
[f]\in[g] \quad &\text{iff} \quad f \in^* g\\
&\text{iff} \quad \{s\in S : f(s)\in g(s)\}\in U;\\
\text{Ult}\models\neg\varphi([f_1],\ldots,[f_n]) \quad &\text{iff} \quad \text{not}(\text{Ult}\models\varphi([f_1],\ldots,[f_n]))\\
&\text{iff} \quad \text{not}(\{s\in S:\varphi(f_1(s),\ldots,f_n(s))\}\in U\\
&\text{iff} \quad \{s\in S:\varphi(f_1(s),\ldots,f_n(s))\}\notin U\\
&\text{iff} \quad \{s\in S:\neg\varphi(f_1(s),\ldots,f_n(s))\}\in U.
\end{aligned}$$

$\vee$ is treated similarly. Now suppose that $\text{Ult}\models\exists y\varphi([f_1],\ldots,[f_n],y)$. Choose $g\in \text{Fcn}(S)$ such that $\text{Ult}\models\varphi([f_1],\ldots,[f_n],[g])$. Then by the inductive hypothesis, $\{s\in S:\varphi(f_1(s),\ldots,f_n(s),g(s))\}\in U$. Now

$$\{s\in S:\varphi(f_1(s),\ldots,f_n(s),g(s))\}\subseteq\{s\in S:\exists y\varphi(f_1(s),\ldots,f_n(s),y)\},$$

so the latter set is in U.

Conversely, suppose that $\{s\in S:\exists y\varphi(f_1(s),\ldots,f_n(s),y)\}\in U$. Then by the axiom of choice, choose $g\in\text{Fcn}(S)$ so that $\{s\in S:\varphi(f_1(s),\ldots,f_n(s),g(s))\}\in U$. By the inductive hypothesis, $\text{Ult}\models\varphi([f_1],\ldots,[f_n],[g])$ and hence $\text{Ult}\models\exists y\varphi([f_1],\ldots,[f_n],y)$. □

Let S,a be any sets. We define c^S_a, a function with domain S, by $c^S_a(s)=a$ for all $s\in S$.

($*$) If U is a σ-complete ultrafilter on S, then $\in_{\text{Ult}}$ is well-founded.

For, suppose that $\ldots f_{n+1}\in^* f_n\ldots\in^* f_0$. Then $\forall n\in\omega\{x\in S: f_{n+1}(x)\in f_n(x)\}\in U$, so $\bigcap_{n\in\omega}\{x\in S: f_{n+1}(x)\in f_n(x)\}\in U$. This set is hence nonempty, and for any x in it, $\cdots f_{n+1}(x)\in f_n(x)\ldots\in f_0(x)$, contradiction.

($**$) $\in_{\text{Ult}}$ is extensional on Ult.

For, suppose that $[f],[g]\in\text{Ult}$ and $[f]\neq[g]$. Then $A\stackrel{\text{def}}{=}\{x\in S: f(x)\neq g(x)\}\in U$. For each $x\in A$ choose $a_x\in f(x)\triangle g(x)$. Then $A=\{x\in S: a_x\in f(x)\backslash g(x)\}\cup\{x\in S: a_x\in g(x)\backslash f(x)$. By symmetry say $\{x\in S: a_x\in f(x)\backslash g(x)\}\in U$. Then $[a]\in_{\text{Ult}}[f]$ and $[a]\notin_{\text{Ult}}[g]$. This proves ($**$).

Suppose that U is a σ-complete ultrafilter on S. Then $\in_{\text{Ult}}$ is well-founded and set-like and extensional on Ult. Thus by Theorem 6.15, the Mostowski collapse is an isomorphism. It is defined by

$$\pi([f])=\{\pi([g]):[g]\in^*[f]\}$$

for any $f\in\text{Fcn}(S)$. Thus $\pi([g])\in\pi([f])$ iff $[g]\in^*[f]$ iff $g\in^* f$ iff $\{s\in S: g(s)\in f(s)\}\in U$. We denote this Mostowski collapse by M^S_U. Recall that M^S_U is transitive.

$j^{S\prime}_U$ is the natural elementary embedding of V into Ult, given by $j^{S\prime}_U(a)=[c^S_a]$ for any set a. Also $j^S_U(a)=\pi([c^S_a])$. Here c^S_a is the function with domain S and constant value a.

We now assume that U is non-principal and σ-complete.

Proposition 17.4. *For any formula $\varphi(x_1,\ldots,x_n)$ of set theory and any $a_1,\ldots,a_n$,*

$$\varphi(a_1,\ldots,a_n) \quad \textit{iff} \quad \mathrm{Ult}\models\varphi(j_U^{S\prime}(a_1),\ldots,j_U^{S\prime}(a_n)).$$

Proof.

$$\begin{aligned}
\mathrm{Ult}\models\varphi(j_U^{S\prime}(a_1),\ldots,j_U^{S\prime}(a_n)) \quad &\text{iff} \quad \mathrm{Ult}\models\varphi([c^S_{a_1}],\ldots,[c^S_{a_n}])\\
&\text{iff} \quad \{s\in S:\varphi((c^S_{a_1})(s),\ldots,(c^S_{a_n})(s))\}\in U\\
&\text{iff} \quad \{s\in S:\varphi(a_1,\ldots,a_n)\}\in U\\
&\text{iff} \quad \varphi(a_1,\ldots,a_n).
\end{aligned}$$

$\square$

Proposition 17.5. *For any formula $\varphi(x_1,\ldots,x_n)$ of set theory and any $a_1,\ldots,a_n\in S$,*

$$\varphi(a_1,\ldots,a_n) \quad \textit{iff} \quad M_U^S\models\varphi(j_U^S(a_1),\ldots,j_U^S(a_n)).$$

$\square$

We write j_U if S is understood. Note that j_U is defined only if U is σ-complete.

Proposition 17.6. *$a\in b$ iff $j_U(a)\in j_U(b)$.*

Proof.

$$\begin{aligned}
a\in b \quad &\text{iff} \quad \{s\in S:a\in b\}\in U\\
&\text{iff} \quad \{s\in S:c^S_a(s)\in c^S_b(s)\}\in U\\
&\text{iff} \quad c^S_a\in^* c^S_b\\
&\text{iff} \quad [c^S_a]\in_{\mathrm{Ult}}[c^S_b]\\
&\text{iff} \quad j_U^{S\prime}(a)\in_{\mathrm{Ult}} j_U^{S\prime}(b)\\
&\text{iff} \quad \pi(j_U^S(a))\in\pi(j_U^S(b))\\
&\text{iff} \quad j_U(a)\in j_U(b).
\end{aligned}$$

$\square$

(1) If α is an ordinal, then so is $j_U(\alpha)$.

In fact, assume that α is an ordinal. Then $(j_U(\alpha)\text{ is an ordinal})^{M_U^S}$, so by the transitivity of M_U^S and absoluteness, $j_U(\alpha)$ is an ordinal.

Since j_U is strictly increasing on On, it follows that $\alpha\le j_U(\alpha)$ for every ordinal α.

(2) $j_U(x\cup\{x\})=j_U(x)\cup\{j_U(x)\}$.

In fact, by Proposition 17.5, $b=x\cup\{x\}$ iff $j_U(b)=j_U(x)\cup\{j_U(x)\}$, so (2) holds.

(3) $j_U(n)=n$ for all $n\in\omega$.

This holds by induction, using (2).

(4) If U is λ-complete, then $j_U(\alpha) = \alpha$ for every $\alpha < \lambda$.

In fact, suppose that this is true for all $\alpha < \beta$, with $\beta < \lambda$, and suppose that $\beta < j_U(\beta)$. Say $\beta = j_U(a)$. Thus $[c^S_a] \in^* [c^S_\beta]$, so $\{s \in S : a \in \beta\} \in U$. Now

$$\{s \in S : a \in \beta\} = \bigcup_{\alpha<\beta} \{s \in S : a = \alpha\},$$

so by λ-completeness there is an $\alpha < \beta$ such that $\{s \in S : a = \alpha\} \in U$. Hence $[c^S_a] = [c^S_\alpha]$, so $\beta = j_U(a) = j_U(\alpha) = \alpha$, contradiction.

Now suppose that $S = \kappa$, a measurable cardinal. Let d be the diagonal function: $d(\alpha) = \alpha$ for every $\alpha < \kappa$.

(5) $\kappa \le \pi([d])$.

In fact, if $\gamma < \kappa$, then $\{\alpha < \kappa : \gamma < \alpha\} = \kappa \backslash (\gamma + 1) \in U$, and so $[c^S_\gamma] \in^* [d]$, and hence by (4), $\gamma = j_U(\gamma) = \pi([c^S_\gamma]) < \pi([d])$, as desired in (5).

(6) $\pi([d]) < j_U(\kappa)$ and hence $\kappa < j_U(\kappa)$.

For, $\{\alpha < \kappa : d(\alpha) < c^S_\kappa\} = \kappa \in U$, so $[d] \in^* [c^S_\kappa]$ and (6) follows, using (5).

An *inner model of ZFC* is a transitive proper class model of ZFC containing all ordinals. Thus L is an inner model, in fact it is the least such.

Theorem 17.7. *If U is a σ-complete ultrafilter, then M^S_U is an inner model of ZFC.*

Proof. By Proposition 17.5, M^S_U is a a transitive model of ZFC. Since $\alpha \le j_U(\alpha)$ for all α, M^S_U contains all ordinals. $\square$

Theorem 17.8. (Dana Scott) *If there is a measurable cardinal, then $V \neq L$.*

Proof. Suppose that κ is the least measurable cardinal, and $V = L$. Now $L \subseteq M^S_U$ by Theorem 13.41. Hence $V = M^S_U = L$. Since j_U is an elementary embedding, it follows that $j_U(\kappa)$ is the least measurable cardinal. Since $\kappa < j_U(\kappa)$, this is a contradiction. $\square$

A *nontrivial elementary embedding of the universe* is a class function f which is an elementary embedding of V into some transitive class M such that f is not the identity.

Theorem 17.9. *The following conditions are equivalent:*

(i) There is a measurable cardinal.

(ii) There is a nontrivial elementary embedding of the universe into some transitive model of ZFC.

Proof. (i)$\Rightarrow$(ii): See (6) above.

Now assume (ii); let f be a nontrivial elementary embedding of the universe V into M.

(1) There is an ordinal α such that $f(\alpha) \neq \alpha$.

In fact, suppose not. Then we claim

(2) $\text{rank}(f(x)) = \text{rank}(x)$ for every set x.

For, let $\varphi(x, y)$ be the formula which defines rank; so

$$V \models \forall x, y[\varphi(x, y) \leftrightarrow y \text{ is an ordinal and } \text{rank}(x) = y].$$

Suppose that $x \in V$. Let $\text{rank}(x) = \alpha$. Then $V \models \varphi(x, \alpha)$, so $M \models \varphi(f(x), f(\alpha))$, hence by the "suppose not" above, $M \models \varphi(f(x), \alpha)$. Since $f(x), \alpha \in V$, by elementarity we have $V \models \varphi(f(x), \alpha)$, so $\text{rank}(f(x)) = \alpha$, as desired in (2).

Now since f is nontrivial, let x be such that $f(x) \neq x$, and choose such an x of minimal rank. If $y \in x$, then $f(y) = y$ by the minimality of $\text{rank}(x)$, and $f(y) \in f(x)$, so $y \in f(x)$. Thus $x \subseteq f(x)$. Since $f(x) \neq x$, we can thus choose $y \in f(x) \backslash x$. But by (2) we have $\text{rank}(f(x)) = \text{rank}(x)$, so $\text{rank}(y) < \text{rank}(x)$, and hence $f(y) = y$ by the minimality of $\text{rank}(x)$. So $f(y) \in f(x)$, hence $y \in x$, contradiction.

This proves (1). Let κ be the least such α.

The arguments for j_U apply for f to show that $f(n) = n$ for each $n \in \omega$, $f(\alpha + 1) = f(\alpha) + 1$ for all α, and $f(\omega) = \omega$. It follows that $\kappa > \omega$. Define

$$D = \{X \subseteq \kappa : \kappa \in f(X)\}.$$

Now $\kappa < f(\kappa)$, so $\kappa \in D$. Since $f(\emptyset) = \emptyset$, we have $\emptyset \notin D$. Now $\forall y(y \in X \cap Y \leftrightarrow y \in X$ and $y \in Y)$, so by elementarity, $\forall y(y \in f(X \cup Y) \leftrightarrow y \in f(X)$ and $y \in f(Y))$. So $f(X \cap Y) = f(X) \cap f(Y)$. Also, if $X \subseteq Y$, then $\forall x(x \in X \rightarrow x \in Y)$, so by elementarity, $\forall x(x \in f(X) \rightarrow x \in f(Y))$. So $X \subseteq Y$ implies that $f(X) \subseteq f(Y)$. From these facts it follows that D is a filter. Also, for any $X \subseteq \kappa$ we have $\forall y(y \in \kappa \leftrightarrow y \in X$ or $y \in (\kappa \backslash X))$, so by elementarity $\forall y(y \in f(\kappa) \leftrightarrow y \in f(X)$ or $y \in f(\kappa \backslash X))$. Hence $f(\kappa) = f(X) \cup f(\kappa \backslash X)$. Since $\kappa \in f(\kappa)$, it follows that $\kappa \in f(X)$ or $\kappa \in f(\kappa \backslash X)$. So D is an ultrafilter.

To show that D is nonprincipal, suppose that $\alpha < \kappa$. Now $\forall x(x \in \{\alpha\} \leftrightarrow x = \alpha)$, so $\forall x(x \in f(\{\alpha\}) \leftrightarrow x = f(\alpha))$. Since $f(\alpha) = \alpha$, it follows that $f(\{\alpha\}) = \{\alpha\}$. So $\kappa \notin f(\{\alpha\})$, and consequently $\{\alpha\} \notin D$.

Next assume that $\gamma < \kappa$ and $X = \langle X_\alpha : \alpha < \gamma \rangle$ is a sequence of members of D. Thus X is a function with domain γ. Let $Y = \bigcap_{\alpha<\gamma} X_\alpha$. Now $f(X)$ is a function with domain $f(\gamma)$, which is γ.

(3) If $\alpha < \gamma$, then $(f(X))_\alpha = f(X_\alpha)$.

For, $(\alpha, X_\alpha) \in X$, so $(f(\alpha), f(X_\alpha)) \in f(X)$. Since $f(\alpha) = \alpha$, (3) follows.

(4) $f(Y) = \bigcap_{\alpha<\gamma} f(X_\alpha)$.

For, $\forall y[y \in Y \leftrightarrow \forall \alpha < \gamma(y \in X_\alpha)]$, so $\forall y[y \in f(Y) \rightarrow \forall \alpha < f(\gamma)(y \in (f(X))_\alpha)]$. By (3) and the fact that $f(\gamma) = \gamma$ it follows that $\forall y[y \in f(Y) \leftrightarrow \forall \alpha < \gamma(y \in f(X_\alpha))]$. Thus (4) holds.

Hence $\kappa \in f(Y)$ and so $Y \in D$.

Finally, we show that κ is a cardinal. For suppose it isn't. Then there is a function g mapping some ordinal $\alpha < \kappa$ onto κ. Thus $\kappa \backslash \{g(\xi)\} \in D$ for every $\xi < \alpha$, so also $\bigcap_{\xi<\alpha}(\kappa \backslash \{g(\xi)\}) \in D$. But $\bigcap_{\xi<\alpha}(\kappa \backslash \{g(\xi)\}) = \emptyset$, contradiction. $\square$

Theorem 17.10. *Suppose that $j : V \to M$ is a nontrivial elementary embedding, and let κ be the first ordinal moved. Define D as above:*

$$D = \{X \subseteq \kappa : \kappa \in j(X)\}.$$

Then there is an elementary embedding k of M^κ_D into M such that $k \circ j^\kappa_D = j$: xxx

Proof. We would like to define k as follows. Let u be any member of Ult. Choose f, a function with domain κ, such that $u = [f]$. Then define

$$k(\pi(u)) = (j(f))(\kappa).$$

To show that this is possible, suppose that $f, g \in u$. Then $f =^* g$, and so $X \stackrel{\text{def}}{=} \{\alpha \in \kappa : f(\alpha) = g(\alpha)\} \in D$. Now

$$\forall x[x \in X \leftrightarrow x \in \kappa \text{ and } f(x) = g(x)],$$

so

$$\forall x[x \in j(X) \leftrightarrow x \in j(\kappa) \text{ and } (j(f))(x) = (j(g))(x)].$$

Now $X \in D$, so $\kappa \in j(X)$, so it follows that $(j(f))(\kappa) = (j(g))(\kappa)$. This shows that k is well-defined.

Next, let $\varphi(v_0, \ldots, v_{n-1})$ be given. Suppose that $M^\kappa_D \models \varphi(\pi([f_0]), \ldots, \pi([f_{n-1}]))$. Then Ult $\models \varphi([f_0], \ldots, [f_{n-1}])$, so by Theorem 3, $X \stackrel{\text{def}}{=} \{\alpha \in \kappa : \varphi(f_0(\alpha), \ldots, f_{n-1}(\alpha))\} \in D$. So $\kappa \in j(X)$. Now

$$\forall\alpha[\alpha \in X \leftrightarrow \alpha \in \kappa \wedge \varphi(f_0(\alpha), \ldots, f_{n-1}(\alpha))],$$

and hence

$$\forall\alpha[\alpha \in j(X) \leftrightarrow \alpha \in j(\kappa) \wedge M \models \varphi((j(f_0))(\alpha), \ldots, (j(f_{n-1}))(\alpha))].$$

Hence $M \models \varphi((j(f_0))(\kappa), \ldots, (j(f_{n-1}))(\kappa))$. So $M \models \varphi(k(\pi([f_0])), \ldots, k(\pi([f_{n-1}])))$, as desired.

Finally, $k(j_D(a)) = k(\pi(j^\kappa_D(a))) = k(\pi([c^\kappa_a])) = (j(c^\kappa_a))(\kappa)$. Now $\forall\alpha < \kappa(c^\kappa_a(\alpha) = a)$, so $\forall\alpha < j(\kappa)((j(c^\kappa_a))(\alpha) = j(a))$. Since $\kappa < j(\kappa)$, we have $(j(c^\kappa_a))(\kappa) = j(a)$, as desired. □

Recall from exercise 8.8, page 104, an equivalent definition of normality.

Theorem 17.11. *Let j be a nontrivial elementary embedding with κ the first ordinal moved, and define*

$$D = \{X \subseteq \kappa : \kappa \in j(X)\}.$$

Then D is normal.

Proof. Let f be a regressive function on some $X \in D$. Thus $\forall\alpha \in X[f(\alpha) < \alpha]$, so $\forall\alpha \in j(X)](j(f))(\alpha) < \alpha]$. Now $\kappa \in j(X)$ since $X \in D$, so $(j(f))(\kappa) < \kappa$. Let

$\gamma = (j(f))(\kappa)$, and let $Y = \{\alpha < \kappa : f(\alpha) = \gamma\}$. Thus $\forall\alpha < \kappa[\alpha \in Y \leftrightarrow f(\alpha) = \gamma]$, so $\forall\alpha < j(\kappa)[\alpha \in j(Y) \leftrightarrow (j(f))(\alpha) = \gamma]$. Since $\kappa < j(\kappa)$ and $(j(f))(\kappa) = \gamma$, it follows that $\kappa \in j(Y)$. Hence $Y \in D$. □

Theorem 17.12. *Let D be a nonprincipal κ-complete ultrafilter on an uncountable cardinal κ. Then the following are equivalent:*

(i) D is normal.

(ii) $\kappa = \pi([d])$, where d is the identity on κ.

(iii) For every $X \subseteq \kappa$, $X \in D$ iff $\kappa \in j_D(X))$.

Proof. (i)⇒(ii): Assume (i). Let a be any member of M_D^κ. Write $a = \pi([f])$. Then

$$\begin{aligned}
a \in \pi([d]) \quad &\text{iff} \quad [f] \in^* [d] \\
&\text{iff} \quad f \in^* d \\
&\text{iff} \quad \{\alpha < \kappa : f(\alpha) < \alpha\} \in D \\
&\text{iff} \quad \exists\gamma < \kappa[\{\alpha < \kappa : f(\alpha) = \gamma\} \in D] \\
&\qquad \text{by exercise 8.8, page 104} \\
&\text{iff} \quad \exists\gamma < \kappa[f =^* c_D^\kappa(\gamma)] \\
&\text{iff} \quad \exists\gamma < \kappa[[f] = [c_\gamma^\kappa] \\
&\text{iff} \quad \exists\gamma < \kappa[\pi([f]) = j_D^\kappa(\gamma)] \\
&\text{iff} \quad \exists\gamma < \kappa[\pi([f]) = \gamma] \\
&\text{iff} \quad a \in \kappa.
\end{aligned}$$

(ii)⇒(iii): Assume (ii), and suppose that $X \subseteq \kappa$. Then

$$\begin{aligned}
X \in D \quad &\text{iff} \quad \{\alpha < \kappa : \alpha \in X\} \in D \\
&\text{iff} \quad \{\alpha < \kappa : d(\alpha) \in X\} \in D \\
&\text{iff} \quad [d] \in_D j_D^\kappa(X) \\
&\text{iff} \quad \kappa \in \pi(j_D^\kappa(X)).
\end{aligned}$$

(iii)⇒(i): This holds by Theorem 11.

Lemma 17.13. *Assume that κ is an infinite cardinal, and $2^\lambda < \kappa$ for every cardinal $\lambda < \kappa$. Then $2^\kappa = \kappa^{\mathrm{cf}(\kappa)}$.*

Proof. If κ is a successor cardinal, then $\mathrm{cf}(\kappa) = \kappa$ and the desired conclusion is clear. Suppose that κ is a limit cardinal. Let $\langle \mu_\xi : \xi < \mathrm{cf}(\kappa)\rangle$ be a strictly increasing sequence of cardinals with supremum κ. Then

$$2^\kappa = 2^{\sum_{\xi<\mathrm{cf}(\kappa)} \mu_\xi} = \prod_{\xi<\mathrm{cf}(\kappa)} 2^{\mu_\xi} \le \prod_{\xi<\mathrm{cf}(\kappa)} \kappa = \kappa^{\mathrm{cf}(\kappa)} \le \kappa^\kappa = 2^\kappa. \qquad \square$$

Lemma 17.14. *Let j be a nontrivial elementary embedding of the universe, and let κ be the least ordinal moved. Suppose that C is a club of κ. Then $\kappa \in j(C)$.*

Proof. First we claim

(1) $j(C) \cap \kappa = C$.

For, if $\alpha \in C$, then $\alpha = j(\alpha) \in j(C)$, so $\alpha \in j(C) \cap \kappa$; this proves $\supseteq$. Now suppose that $\alpha \in j(C) \cap \kappa$. Then $j(\alpha) = \alpha \in j(C)$, so $\alpha \in C$. This proves $\subseteq$.

Now since C is closed, we have

$$\forall \text{limit } \gamma \in \kappa[\forall \delta \in \gamma \exists \varepsilon(\varepsilon \in C \text{ and } \delta < \varepsilon < \gamma) \to \gamma \in C];$$

hence by elementarity,

$$(*) \qquad \forall \text{limit } \gamma \in j(\kappa)[\forall \delta \in \gamma \exists \varepsilon[\varepsilon \in j(C) \text{ and } \delta < \varepsilon < \gamma) \to \gamma \in j(C)].$$

Now κ is a limit ordinal less than $j(\kappa)$. If $\delta \in \kappa$, then there is a $\varepsilon \in C$ such that $\delta < \varepsilon < \kappa$. By (1), $\varepsilon \in j(C)$. It follows by $(*)$ that $\kappa \in j(C)$. □

Lemma 17.15. *Let λ be an infinite cardinal such that $2^\lambda = \lambda^{\aleph_0}$. Then there is a function $F : {}^\omega\lambda \to \lambda$ such that for all $A \in [\lambda]^\lambda$ and all $\gamma < \lambda$ there is an $s \in {}^\omega A$ such that $F(s) = \gamma$.*

Proof. Let $\langle (A_\alpha, \gamma_\alpha) : \alpha < 2^\lambda \rangle$ enumerate all pairs (A, γ) with $A \in [\lambda]^\lambda$ and $\gamma < \lambda$. We define $\langle s^\alpha : \alpha < 2^\lambda \rangle$ by recursion, each $s^\alpha \in {}^\omega\lambda$. If s^β has been defined for all $\beta < \alpha$, where $\alpha < 2^\lambda$, then $|\alpha| < 2^\lambda = \lambda^\omega$, so we may choose $s^\alpha \in {}^\omega A_\alpha$ with $s^\alpha \neq s^\beta$ for all $\beta < \alpha$.

Now for each $\alpha < 2^\lambda$ define $F(s_\alpha) = \gamma_\alpha$, and for $t \in {}^\omega\lambda$ and $\forall \alpha < 2^\lambda[t \neq s^\alpha)$ let $F(t) = 0$. Now if $A \in [\lambda]^\lambda$ and $\gamma < \lambda$, choose α such that $(A_\alpha, \gamma_\alpha) = (A, \gamma)$; then $F(s^\alpha) = \gamma$. □

Theorem 17.16. (Kunen inconsstency) *If $j : V \to M$ is a nontrivial elementary embedding, then $M \neq V$.*

Proof. Assume that j is an elementary embedding of V into V. Let $\kappa = \kappa_0$ be the first ordinal moved. So κ_0 is measurable. Define $\kappa_{n+1} = j(\kappa_n)$ for each $n \in \omega$. By induction, $\kappa_n < \kappa_{n+1}$ for all $n \in \omega$. Let $\lambda = \sup_{n\in\omega} \kappa_n$. Thus

$$\begin{aligned}
&\forall x \in \lambda[x < \kappa_0 \text{ or } \exists n \in \omega[\kappa_n \leq x < \kappa_{n+1}]\\
&\text{and } \forall x[x < \kappa_0 \to x \in \lambda]\\
&\text{and } \forall x \forall n \in \omega[\kappa_n \leq x < \kappa_{n+1} \to x \in \lambda].
\end{aligned}$$

Hence

$$\begin{aligned}
&\forall x \in j(\lambda)[x < j(\kappa_0) \text{ or } \exists n \in j(\omega)[j(\kappa_n) \leq x < j(\kappa_{n+1})]\\
&\text{and } \forall x[x < j(\kappa_0) \to x \in j(\lambda)]\\
&\text{and } \forall x \forall n \in j(\omega)[j(\kappa_n) \leq x < j(\kappa_{n+1}) \to x \in j(\lambda)],
\end{aligned}$$

so

$$\begin{aligned}
&\forall x \in j(\lambda)[x < \kappa_1 \text{ or } \exists n \in \omega[\kappa_{n+1} \leq x < \kappa_{n+2})]\\
&\text{and } \forall x[x < \kappa_1 \to x \in j(\lambda)]\\
&\text{and } \forall x \forall n \in \omega)[\kappa_{n+1}) \leq x < \kappa_{n+2}) \to x \in j(\lambda)],
\end{aligned}$$

so $j(\lambda) = \{\kappa_{n+1} : n \in \omega\} = \lambda$. Let $G = \{j(\alpha) : \alpha < \lambda\}$. Now each κ_n is measurable and hence by Lemma 10.4 is strongly inaccessible. It follows that λ is strong limit, and hence by Lemma 17.13, $2^\lambda = \lambda^\omega$. Hence we can apply Lemma 17.15 and get a function $F : {}^\omega\lambda \to \lambda$ such that $\forall A \in [\lambda]^\lambda \forall\gamma < \lambda\exists s \in {}^\omega A[F(s) = \gamma]$. Thus $F[{}^\omega A] = \lambda$ for each $A \in [\lambda]^\lambda$. Now

$$\begin{aligned}&F \text{ is a function } \wedge \forall s[s \in \mathrm{dmn}(F) \leftrightarrow s : \omega \to \lambda]\\&\wedge \forall s\forall\gamma[(s, \gamma) \in F \to \gamma < \lambda]\wedge\\&\forall A \subseteq \lambda[|A| = \lambda \to \forall\gamma < \lambda\exists s[s : \omega \to A \wedge (s, \gamma) \in F]]\end{aligned}$$

Hence

$$\begin{aligned}&j(F) \text{ is a function } \wedge \forall s[s \in \mathrm{dmn}(j(F)) \leftrightarrow s : \omega \to \lambda]\\&\wedge \forall s\forall\gamma[(s, \gamma) \in j(F) \to \gamma < \lambda]\wedge\\&\forall A \subseteq \lambda[|A| = \lambda \to \forall\gamma < \lambda\exists s[s : \omega \to A \wedge (s, \gamma) \in j(F)]\end{aligned}$$

Hence there is an $s \in {}^\omega G$ such that $(j(F))(s) = \kappa$. Let $t : \omega \to \lambda$ be such that $s(n) = j(t(n))$ for all $n \in \omega$. Thus $\forall n \in \omega[(n, t(n)) \in t]$, so $\forall n \in \omega[(n, j(t(n))) \in j(t)]$, i.e., $\forall n \in \omega[(n, s(n)) \in j(t)]$. So $j(t) = \{(n, s(n)) : n \in \omega\} = s$. Then $\kappa = ((j(F))(j(t)) = j(F(t))$. But $F(t) < \lambda$, and κ is not in the range of $j \restriction \lambda$, contradiction. □

Lemma 17.17. *If $j : V \to M$ and κ is the least ordinal moved, then*
(i) $\forall x \in V_\kappa[j(x) = x]$.
(ii) $\forall X \subseteq V_\kappa[j[X] \cap V_\kappa = X]$.
(iii) $V^M_\kappa = V_\kappa$.
(iv) $V^M_{\kappa+1} = V_{\kappa+1}$.
(v) $\mathscr{P}^M(\kappa) = \mathscr{P}(\kappa)$.

Proof. (i): For any $x \in V_\kappa$, say $\mathrm{rank}(x) = \alpha < \kappa$. Then $V \models \mathrm{rank}(x) = \alpha$, so $M \models \mathrm{rank}(j(x)) = \alpha$. By absoluteness, $\mathrm{rank}(j(x)) = \alpha$. So $\mathrm{rank}(j(x)) = \mathrm{rank}(x)$ for all $x \in V_\kappa$.

Now suppose that $j(y) = y$ for all y of rank less than the rank of x, where $\mathrm{rank}(x) < \kappa$. Then $y \in x$ implies that $y = j(y) \in j(x)$; so $x \subseteq j(x)$. Suppose that $z \in j(x)$. Since $\mathrm{rank}(j(x)) = \mathrm{rank}(x)$, we have $j(z) = z$. So $j(z) \in j(x)$, hence $z \in x$. This shows that $j(x) \subseteq x$. So $j(x) = x$. Hence by induction on rank we have $j(x) = x$ for all $x \in V_\kappa$, as desired.

(ii): If $X \subseteq V_\kappa$, then $j(X) \cap V_\kappa = X$: suppose that $x \in X$. Then $x \in V_\kappa$, so by the preceding paragraph, $x = j(x) \in j(X)$. Thus $X \subseteq j(X) \cap V_\kappa$. Conversely, suppose that $x \in j(X) \cap V_\kappa$. Then $j(x) = x \in j(X)$, so $x \in X$. So $j(X) \cap V_\kappa = X$.

(iii): For any x, $x \in V^M_\kappa$ iff $x \in M$ and $(\mathrm{rank}(x) < \kappa)^M$ iff $x \in M$ and $\mathrm{rank}(x) < \kappa$ iff $\mathrm{rank}(x) < \kappa$ iff $x \in V_\kappa$.

(iv): Now if $X \in V^M_{\kappa+1}$, then $X \in M$ and $X \subseteq V^M_\kappa = V_\kappa$. So $X \in V_{\kappa+1}$. Conversely, suppose that $X \in V_{\kappa+1}$. Then $X \subseteq V_\kappa$, so by the above, $j(X) \cap V_\kappa = X$. Since $j(X), V_\kappa \in M$, it follows that $X \in M$. So $X \in V^M_{\kappa+1}$. This shows that $V^M_{\kappa+1} = V_{\kappa+1}$.

(v) $\subseteq$ is clear. Now suppose that $X \subseteq \kappa$. Now κ has rank κ, so $\kappa \in V_{\kappa+1}$, hence $\kappa \subseteq V_\kappa$, hence $X \subseteq V_\kappa$, hence $X \in V_{\kappa+1} \subseteq M$ by the above, as desired. $\square$

Lemma 17.18. *Let U be a non-principal κ-complete ultrafilter on κ and let M^κ_U be as above, with $S = \kappa$. Also let π, j^κ_U and j_U be as above. Then*
(i) ${}^\kappa(M^\kappa_U) = M^\kappa_U$.
(ii) $U \notin M^\kappa_U$.
(iii) $2^\kappa \le (2^\kappa)^{M^\kappa_U} < j_U(\kappa) < (2^\kappa)^+$.
(iv) If λ is a limit ordinal and $\mathrm{cf}(\lambda) = \kappa$, then $j_U(\lambda) > \bigcup_{\alpha<\lambda} j_U(\kappa)$.
(v) If λ is a limit ordinal and $\mathrm{cf}(\lambda) \ne \kappa$, then $j_U(\lambda) = \bigcup_{\alpha<\lambda} j_U(\kappa)$.
(vi) If $\lambda > \kappa$ is a strong limit cardinal and $\mathrm{cf}(\lambda) \ne \kappa$, then $j_U(\lambda) = \lambda$.

Proof.

(1) $\mathbf{ON} \subseteq M^\kappa_U$.

For, by (1) in the proof of Proposition 17.6, $j_U(\alpha)$ is an ordinal for every ordinal α, and by Proposition 17.6 itself; $\alpha \in M^\kappa_U$ since M^κ_U is transitive.

For (i), let $a \in {}^\kappa(M^\kappa_U)$. Then for each $\xi < \kappa$ there is a function $g_\xi \in {}^\kappa V$ such that $\pi([g_\xi]) = a_\xi$. Since $\kappa \in M^\kappa_U$, there is an $h \in {}^\kappa V$ such that $\pi([h]) = \kappa$. Thus $\pi([h])$ is an ordinal, hence Ult $\models [h]$ is an ordinal, hence $\{\alpha < \kappa : h(\alpha) \text{ is an ordinal}\} \in U$. Also,

(2) $\{\alpha < \kappa : h(\alpha) \text{ is an ordinal and } h(\alpha) < \kappa\} \in U$.

In fact, otherwise we have $\{\alpha < \kappa : h(\alpha) \text{ is an ordinal and } \kappa \le h(\alpha)\} \in U$, hence $\{\alpha < \kappa : h(\alpha) \text{ is an ordinal and } c^\kappa_\kappa(\alpha) \le h(\alpha)\} \in U$, hence $[c^\kappa_\kappa] < [h]$, hence $\kappa < j(\kappa) \le \pi([h]) = \kappa$, contradiction. So (2) holds.

Let $A = \{\alpha < \kappa : h(\alpha) \text{ is an ordinal and } h(\alpha) < \kappa\}$. Now we define $F : \kappa \to V$ by

$$F(\alpha) = \begin{cases} \langle g_\xi : \xi < h(\alpha)\rangle & \text{if } \alpha \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Now

$$A \subseteq \{\alpha < \kappa : \forall x(x \in \mathrm{dmn}(F(\alpha)) \leftrightarrow x \in h(\alpha))\},$$

and hence Ult $\models \forall x(x \in \mathrm{dmn}([F]) \leftrightarrow x \in [h])$, i.e., Ult $\models \mathrm{dmn}([F]) = [h]$, hence, using absoluteness, $\mathrm{dmn}(\pi([F]) = \kappa$. Now take any $\xi < \kappa$. If $\alpha \in A$, then $\xi \in \mathrm{dmn}(F(\alpha))$, and $(F(\alpha))(\xi) = g_\xi$. Thus

$$\begin{aligned} A &\subseteq \{\alpha < \kappa : \xi \in \mathrm{dmn}(F(\alpha)) \text{ and } (F(\alpha))(\xi) = g_\xi\} \\ &= \{\alpha < \kappa : c_\xi(\alpha) \in \mathrm{dmn}(F(\alpha)) \text{ and } (F(\alpha))(c_\xi(\alpha)) = g_\xi\} \end{aligned}$$

so it follows that Ult $\models [c_\xi] \in \mathrm{dmn}([F]) \wedge [F]([c_\xi]) = [g_\xi]$, and so, using absoluteness, $\xi \in \mathrm{dmn}(\pi([F])$ and $(\pi([F]))(\xi) = a_\xi$. Thus $\pi([F]) = \langle a_\xi : \xi < \kappa\rangle$, as desired.

(ii): For any $f \in {}^\kappa\kappa$ let $F(f) = \pi([f])$. Now $\{\alpha \in \kappa : f(\alpha) < \kappa\} = \kappa \in U$, so $F(f) \in \pi([c_\kappa]) = j(\kappa)$. If $\pi([g]) \in j(\kappa)$, then $\{\alpha \in \kappa : g(\alpha) \in \kappa\} \in U$. Let $g' \in {}^\kappa\kappa$ be such that $[g'] = [g]$. Then $F(g') = \pi([g])$. Thus F maps ${}^\kappa\kappa$ onto $j(\kappa)$. Now ${}^\kappa\kappa \in M$ by (i). Now for any $f \in {}^\kappa\kappa$,

$$\begin{aligned}[f] = \{g : g \text{ is a function with domain } \kappa \wedge \{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in U \\ \wedge \forall h[h \text{ is a function with domain } \kappa \wedge \{\alpha < \kappa : f(\alpha) = h(\alpha)\} \in U \\ \to \text{rank}(g) \leq \text{rank}(h)].\}\end{aligned}$$

Thus $[f] \in M^\kappa_U$ if $U \in M^\kappa_U$. Hence also $F \in M^\kappa_U$. So $M^\kappa_U \models j_U(\kappa) \leq 2^\kappa$. But $\kappa < j(\kappa)$ and $j(\kappa)$ is inaccessible in M^κ_U, contradiction.

(iii): We have $\mathscr{P}^{M^\kappa_U}(\kappa) = \mathscr{P}(\kappa)$ by Lemma 17.17(v).

Thus there is a bijection in M^κ_U from $\mathscr{P}(\kappa)$ onto $(2^\kappa)^{M^\kappa_U}$, and by absoluteness this is really a bijection. So $(2^\kappa)^{M^\kappa_U}$ is at least $\geq 2^\kappa$. Thus $2^\kappa \leq (2^\kappa)^{M^\kappa_U}$.

Since $\kappa < j_U(\kappa)$ and $j_U(\kappa)$ is strongly inaccessible, $j_U(\kappa)$ is strongly inaccessibke in N^k_U and we have $(2^\kappa)^{M^\kappa_U} < j_U(\kappa)$.

The mapping $f \mapsto \pi([f])$ for $f \in {}^\kappa\kappa$ maps ${}^\kappa\kappa$ onto $j_U(\kappa)$, so $|j_U(\kappa)| \leq 2^\kappa$ and hence $j_U(\kappa) < (2^\kappa)^+$.

(iv): Write $\lambda = \bigcup_{\alpha<\kappa} \lambda_\alpha$ with each $\lambda_\alpha < \lambda$. Let $f(\alpha) = \lambda_\alpha$ for all $\alpha < \kappa$. Then

$$\{\beta < \kappa : c^\kappa_{\lambda_\alpha}(\beta) < f(\beta)\} = \{\beta < \kappa : \lambda_\alpha < \lambda_\beta\} \in U,$$

and so $c^\kappa_{\lambda_\alpha} \in^* f$, so $[c^\kappa_{\lambda_\alpha}] <^* [f]$ and hence $j_U(\lambda_\alpha) < \pi([f]$. Thus $\bigcup_{\alpha<\lambda} j_U(\alpha) = \bigcup_{\alpha<\kappa} j_U(\lambda_\alpha) \leq \pi([f])$. Clearly $\pi([f]) < j_U(\lambda)$, so $\bigcup_{\alpha<\lambda} j_U(\alpha) < j_U(\lambda)$.

(v) Suppose that $\text{cf}(\lambda) > \kappa$. Take any $\beta < j_U(\lambda)$. Then there is an f with domain κ such that $\pi([f]) = \beta < \pi([c^\kappa_\lambda])$. Hence $[f] <^* [c^\kappa_\lambda]$, and so $f <^* c^\kappa_\lambda$. So $\{\alpha < \kappa : f(\alpha) < \lambda\} \in U$. Define for $\alpha < \kappa$

$$f'(\alpha) = \begin{cases} f(\alpha) & \text{if } f(\alpha) < \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Choose $\gamma < \lambda$ so that $f'(\alpha) < \gamma$ for all $\alpha < \kappa$. Then $[f] <^* [c^\kappa_\gamma]$, so $\beta = \pi([f])) < j_U(\gamma)$. This shows that $\bigcup_{\gamma<\lambda} j_U(\gamma) = j_U(\gamma)$.

Now suppose that $\text{cf}(\lambda) < \kappa$. Say $\lambda = \bigcup_{\alpha<\text{cf}(\lambda)} \lambda_\alpha$. Now $j(\lambda)$ is an upper bound for $\{j(\lambda_\alpha) : \alpha < \kappa\}$. Suppose that $\pi([f])$ is another upper bound. Then for all $\alpha < \text{cf}(\lambda)$, $X_\alpha \stackrel{\text{def}}{=} \{\beta < \kappa : \lambda_\alpha < f(\beta)\} \in U$. Hence $Y \stackrel{\text{def}}{=} \bigcap_{\alpha<\text{cf}(\lambda)} X_\alpha \in U$. For $\beta \in Y$ we have $\forall \alpha < \text{cf}(\lambda)[\lambda_\alpha < f(\beta)]$. Hence $j_U(\lambda) \leq \pi([f])$, as desired.

(vi): We prove: if $\text{cf}(\lambda) \neq \kappa$ and $\forall \mu < \lambda[\mu^\kappa < \lambda]$, then $j_U(\lambda) = \lambda$. Take any $\mu < \lambda$. For each $f \in {}^\kappa\mu$ define

$$F(f) = \begin{cases} j_U(\beta) & \text{if } [f] = [c^\kappa_\beta] \text{ and } \beta < \mu, \\ 0 & \text{otherwise.} \end{cases}$$

Then F maps ${}^\kappa\mu$ onto $j_U(\mu)$. So $|j_U(\mu)| \leq \mu^\kappa < \lambda$. By (iv), $j_U(\lambda) = \bigcup_{\alpha<\lambda} j_U(\alpha) = \lambda$. $\square$

Theorem 17.19. *If D is a normal ultrafilter on κ, then $\{\alpha < \kappa : \alpha$ is weakly compact$\} \in D$.*

Proof. Let F be the collection of all functions $f : [\kappa]^2 \to 2$. Then $F^{M^\kappa_D} = F$. In fact, $\subseteq$ is clear by absoluteness. For the other direction, let $g : \kappa \to [\kappa]^2$ be a bijection. Then $\{f \circ g : f \in F\} \subseteq {}^\kappa M^\kappa_D$, so by Lemma 17.18(i), $\{f \circ g : f \in F\} \subseteq M^\kappa_D$, and so $F \subseteq M^\kappa_D$. Hence $F^{M^\kappa_D} = F$.

Now κ is weakly compact, so

$$\begin{aligned}&\forall f[f \in F \Rightarrow \exists A \subseteq \kappa[|A| = \kappa \\ &\quad \text{and } \exists \varepsilon \in 2 \forall \alpha, \beta < \kappa[\alpha, \beta \in A \text{ and } \alpha \neq \beta \Rightarrow f(\{\alpha, \beta\}) = \varepsilon]]].\end{aligned}$$

Now $A \subseteq \kappa$ implies that $A \in M^\kappa_D$ by Lemma 17.18(i). Hence κ is weakly compact in M^κ_D. Thus $\pi([d]) = \kappa$ is weakly compact, so by Łoś's theorem, $\{\alpha < \kappa : \alpha \text{ is weakly compact}\}$ is in D. $\square$

Lemma 17.20. *Let κ be a measurable cardinal, with D a normal κ-complete ultrafilter on κ. Assume that $2^\kappa > \kappa^+$. Then $\{\alpha < \kappa : 2^\alpha > \alpha^+\} \in D$.*

Proof. We prove the contrapositive. Assume that $\{\mu < \kappa : 2^\mu = \mu^+\} \in D$. Let $\varphi(x)$ be the formula "x is a cardinal and $2^x = x^+$". Thus $\{\mu < \kappa : \varphi(\mu)\} \in D$, that is, $\{\mu < \kappa : \varphi(d(\mu))\} \in D$. By Proposition 17.5, $M^\kappa_D \models \varphi(\pi([d]))$. Now $\pi([d]) = \kappa$ by Theorem 17.12 above, so $(\kappa \text{ is a cardinal and } 2^\kappa = \kappa^+)^{M^\kappa_D}$. By Lemma 17.18, $2^\kappa = \kappa^+$. $\square$

Lemma 17.21. *Let κ be a measurable cardinal, and D a normal κ-complete ultrafilter on κ. Let $\lambda > \kappa$ be strong limit with* $\mathrm{cf}(\lambda) = \kappa$. *Then $2^\lambda < j_D(\lambda)$.*

Proof. Since $\alpha < \beta \to j(\alpha) < j(\beta)$, we have always $\alpha \le j(\alpha)$. By Lemma 17.18(iv), $\lambda \le \lim_{\alpha \to \lambda} j(\alpha) < j(\lambda)$. Let $\langle \mu_\xi : \xi < \kappa \rangle$ be strictly increasing with supremum λ. Next,

$$2^\lambda = 2^{\sum_{\xi<\kappa} \mu_\xi} = \prod_{\xi<\kappa} 2^{\mu_\xi} \le \prod_{\xi<\kappa} \lambda = \lambda^\kappa \le \lambda^\lambda = 2^\lambda.$$

Thus $2^\lambda = \lambda^\kappa$. Now $\lambda^\kappa \le (\lambda^\kappa)^M$ by Lemma 17.18(i). Since $\kappa < j(\kappa)$ we have $(\lambda^\kappa)^M \le (\lambda^{j(\kappa)})^M$. Now by Łoś's theorem, $M \models j(\lambda)$ is strong limit. Since $\lambda < j(\lambda)$ and $j(\kappa) < j(\lambda)$, it follows that $(\lambda^{j(\kappa)})^M < j(\lambda)$. $\square$

• A cardinal κ is *weakly compact* iff $\kappa > \omega$ and $\kappa \to (\kappa, \kappa)^2$. There are several equivalent definitions of weak compactness. The one which justifies the name "compact" involves infinitary logic, and it will be discussed later. Right now we consider equivalent conditions involving trees and linear orderings.

• A cardinal κ has the *tree property* iff every κ-tree has a chain of size κ.

Equivalently, κ has the tree property iff there is no κ-Aronszajn tree.

• A cardinal κ has the *linear order property* iff every linear order $(L, <)$ of size κ has a subset with order type κ or κ^* under $<$.

Lemma 17.22. *For any regular cardinal κ, the linear order property implies the tree property.*

Proof. Assume the linear order property, and let $(T, <)$ be a κ-tree. For each $x \in T$ and each $\alpha \leq \mathrm{ht}(x, T)$ let x^α be the element of height α below x. Thus x^0 is the root which is below x, and $x^{\mathrm{ht}(x)} = x$. For each $x \in T$, let $T \restriction x = \{y \in T : y < x\}$. If x, y are incomparable elements of T, then let $\chi(x, y)$ be the smallest ordinal $\alpha \leq \min(\mathrm{ht}(x), \mathrm{ht}(y))$ such that $x^\alpha \neq y^\alpha$. Let $<'$ be a well-order of T. Then we define, for any distinct $x, y \in T$,

$$x <'' y \quad \text{iff} \quad x < y, \text{ or } x \text{ and } y \text{ are incomparable and } x^{\chi(x,y)} <' y^{\chi(x,y)}.$$

We claim that this gives a linear order of T. To prove transitivity, suppose that $x <'' y <'' z$. Then there are several possibilities. These are illustrated in diagrams below.

Case 1. $x < y < z$. Then $x < z$, so $x <'' z$.

Case 9. $x < y$, while y and z are incomparable, with $y^{\chi(y,z)} <' z^{\chi(y,z)}$.

Subcase 9.1. $\mathrm{ht}(x) < \chi(y, z)$. Then $x = x^{\mathrm{ht}(x)} = y^{\mathrm{ht}(x)} = z^{\mathrm{ht}(x)}$ so that $x < z$, hence $x <'' z$.

Subcase 9.9. $\chi(y, z) \leq \mathrm{ht}(x)$. Then x and z are incomparable. In fact, if $z < x$ then $z < y$, contradicting the assumption that y and z are incomparable; if $x \leq z$, then $y^{\mathrm{ht}(x)} = x = x^{\mathrm{ht}(x)} = z^{\mathrm{ht}(x)}$, contradiction. Now if $\alpha < \chi(x, z)$ then $y^\alpha = x^\alpha = z^\alpha$; it follows that $\chi(x, z) \leq \chi(y, z)$. If $\alpha < \chi(y, z)$ then $\alpha \leq \mathrm{ht}(x)$, and hence $x^\alpha = y^\alpha = z^\alpha$; this shows that $\chi(y, z) \leq \chi(x, z)$. So $\chi(y, z) = \chi(x, z)$. Hence $x^{\chi(x,z)} = y^{\chi(x,z)} = y^{\chi(y,z)} <' z^{\chi(y,z)} = z^{\chi(x,z)}$, and hence $x <'' z$.

Case 3. x and y are incomparable, and $y < z$. Then x and z are incomparable. Now if $\alpha < \chi(x, y)$, then $x^\alpha = y^\alpha = z^\alpha$; this shows that $\chi(x, y) \leq \chi(x, z)$. Also, $x^{\chi(x,y)} <' y^{\chi(x,y)} = z^{\chi(x,y)}$, and this implies that $\chi(x, z) \leq \chi(x, y)$. So $\chi(x, y) = \chi(x, z)$. It follows that $x^{\chi(x,z)} = x^{\chi(x,y)} <' y^{\chi(x,y)} = z^{\chi(x,z)}$, and hence $x <'' z$.

Case 4. x and y are incomparable, and also y and z are incomparable. We consider subcases.

Subcase 4.1. $\chi(y, z) < \chi(x, y)$. Now if $\alpha < \chi(y, z)$, then $x^\alpha = y^\alpha = z^\alpha$; so $\chi(y, z) \leq \chi(x, z)$. Also, $x^{\chi(y,z)} = y^{\chi(y,z)} <' z^{\chi(y,z)}$, so that $\chi(x, z) \leq \chi(y, z)$. Hence $\chi(x, z) = \chi(y, z)$, and $x^{\chi(x,z)} = y^{\chi(y,z)} <' z^{\chi(y,z)}$, and hence $x <'' z$.

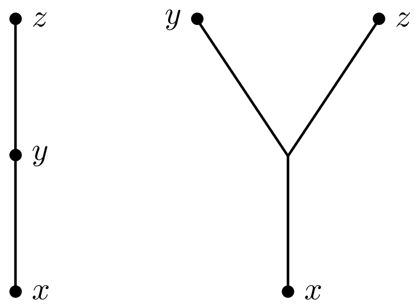
Subcase 4.9. $\chi(y, z) = \chi(x, y)$. Now $x^{\chi(x,y)} <' y^{\chi(x,y)} = y^{\chi(y,z)} <' z^{\chi(y,z)} = z^{\chi(x,y)}$. It follows that $\chi(x, z) \leq \chi(x, y)$. For any $\alpha < \chi(x, y)$ we have $x^\alpha = y^\alpha = z^\alpha$ since $\chi(y, z) = \chi(x, y)$. So $\chi(x, y) = \chi(x, z)$. Hence $x^{\chi(x,z)} = x^{\chi(x,y)} <' y^{\chi(x,y)} = y^{\chi(y,z)} <' z^{\chi(y,z)} = z^{\chi(x,z)}$, so $x <'' z$.

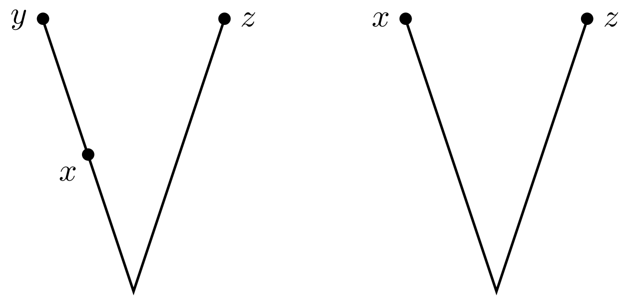
Subcase 4.3. $\chi(x, y) < \chi(y, z)$. Then $x^{\chi(x,y)} <' y^{\chi(x,y)} = z^{\chi(x,y)}$, and if $\alpha < \chi(x, y)$ then $x^\alpha = y^\alpha = z^\alpha$. It follows that $x <'' z$

Clearly any two elements of T are comparable under $<''$, so we have a linear order. The following property is also needed.

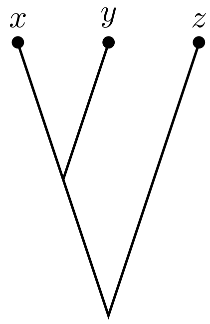
(*) If $t < x, y$ and $x <'' a <'' y$, then $t < a$.

In fact, suppose not. If $a \leq t$, then $a < x$, hence $a <'' x$, contradiction. So a and t are incomparable. Then $\chi(a, t) \leq \mathrm{ht}(t)$, and hence $x <'' y <'' a$ or $a <'' x <'' y$, contradiction.

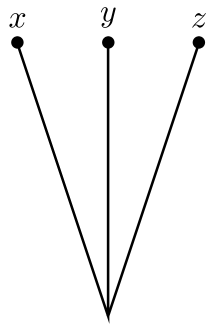




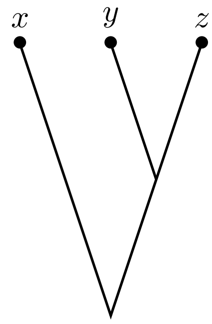
Case 1	Subcase 2.1	Subcase 2.2	Case 3



Subcase 4.1



Subcase 4.2



Subcase 4.3

Now by the linear order property, $(T, <'')$ has a subset L of order type κ or κ^*. First suppose that L is of order type κ. Define

$$B = \{t \in T : \exists x \in L \forall a \in L[x \leq'' a \to t \leq a]\}.$$

We claim that B is a chain in T of size κ. Suppose that $t_0, t_1 \in B$ with $t_0 \neq t_1$, and choose $x_0, x_1 \in L$ correspondingly. Say wlog $x_0 <'' x_1$. Now $t_0 \in B$ and $x_0 \leq'' x_1$, so $t_0 \leq x_1$. And $t_1 \in B$ and $x_1 \leq x_1$, so $t_1 \leq x_1$. So t_0 and t_1 are comparable.

Now let $\alpha < \kappa$; we show that B has an element of height α. For each t of height α let $V_t = \{x \in L : t \leq x\}$. Then

$$\{x \in L : \mathrm{ht}(x) \geq \alpha\} = \bigcup_{\mathrm{ht}(t)=\alpha} V_t;$$

since there are fewer than κ elements of height less than κ, this set has size κ, and so there is a t such that $\mathrm{ht}(t) = \alpha$ and $|V_t| = \kappa$. We claim that $t \in B$. To prove this, take any $x \in V_t$ such that $t < x$. Suppose that $a \in L$ and $x \leq'' a$. Choose $y \in V_t$ with $a <'' y$ and $t < y$. Then $t < x$, $t < y$, and $x \leq'' a <'' y$. If $x = a$, then $t \leq a$, as desired. If $x <'' a$, then $t < a$ by (*).

This finishes the case in which L has a subset of order type κ. The case of order type κ^* is similar, but we give it. So, suppose that L has order type κ^*. Define

$$B = \{t \in T : \exists x \in L \forall a \in L[a \leq'' x \to t \leq a]\}.$$

We claim that B is a chain in T of size κ. Suppose that $t_0, t_1 \in B$ with $t_0 \neq t_1$, and choose $x_0, x_1 \in L$ correspondingly. Say wlog $x_0 <'' x_1$. Now $t_0 \in B$ and $x_0 \leq x_0$, so $t_0 \leq x_0$. and $t_1 \in B$ and $x_0 \leq'' x_1$, so $t_1 \leq x_0$. So t_0 and t_1 are comparable.

Now let $\alpha < \kappa$; we show that B has an element of height α. For each t of height α let $V_t = \{x \in L : t \leq x\}$. Then

$$\{x \in L : \mathrm{ht}(x) \geq \alpha\} = \bigcup_{\mathrm{ht}(t)=\alpha} V_t;$$

since there are fewer than κ elements of height less than κ, this set has size κ, and so there is a t such that $\mathrm{ht}(t) = \alpha$ and $|V_t| = \kappa$. We claim that $t \in B$. To prove this, take any $x \in V_t$ such that $t < x$. Suppose that $a \in L$ and $a \leq'' x$. Choose $y \in V_t$ with $y <'' a$ and $t < y$. Then $t < x$, $t < y$, and $y <'' a \leq'' x$. If $a = x$, then $t < a$, as desired. If $a <'' x$, then $t < a$ by (*). $\square$

Theorem 17.23. *For any uncountable cardinal κ the following conditions are equivalent:*
(i) κ is weakly compact.
(ii) κ is inaccessible, and it has the linear order property.
(iii) κ is inaccessible, and it has the tree property.
(iv) For any cardinal λ such that $1 < \lambda < \kappa$ we have $\kappa \to (\kappa)^2_\lambda$.

Proof. (i)⇒(ii): Assume that κ is weakly compact. First we need to show that κ is inaccessible.

To show that κ is regular, suppose to the contrary that $\kappa = \sum_{\alpha<\lambda} \mu_\alpha$, where $\lambda < \kappa$ and $\mu_\alpha < \kappa$ for each $\alpha < \lambda$. By the definition of infinite sum of cardinals, it follows that we can write $\kappa = \bigcup_{\alpha<\lambda} M_\alpha$, where $|M_\alpha| = \mu_\alpha$ for each $\alpha < \lambda$ and the M_α's are pairwise disjoint. Define $f : [\kappa]^2 \to 2$ by setting, for any distinct $\alpha, \beta < \kappa$,

$$f(\{\alpha, \beta\}) = \begin{cases} 0 & \text{if } \alpha, \beta \in M_\xi \text{ for some } \xi < \lambda, \\ 1 & \text{otherwise.} \end{cases}$$

Let H be homogeneous for f of size κ. First suppose that $f[[H]^2] = \{0\}$. Fix $\alpha_0 \in H$, and say $\alpha_0 \in M_\xi$. For any $\beta \in H$ we then have $\beta \in M_\xi$ also, by the homogeneity of H. So $H \subseteq M_\xi$, which is impossible since $|M_\xi| < \kappa$. Second, suppose that $f[[H]^2] = \{1\}$. Then any two distinct members of H lie in distinct M_ξ's. Hence if we define $g(\alpha)$ to be the $\xi < \lambda$ such that $\alpha \in M_\xi$ for each $\alpha \in H$, we get a one-one function from H into λ, which is impossible since $\lambda < \kappa$.

To show that κ is strong limit, suppose that $\lambda < \kappa$ but $\kappa \leq 2^\lambda$. Now by Lemma 9.4 we have $2^\lambda \not\to (\lambda^+, \lambda^+)^2$. So choose $f : [2^\lambda]^2 \to 2$ such that there does not exist an $X \in [2^\lambda]^{\lambda^+}$ with $f \restriction [X]^2$ constant. Define $g : [\kappa]^2 \to 2$ by setting $g(A) = f(A)$ for any $A \in [\kappa]^2$. Choose $Y \in [\kappa]^\kappa$ such that $g \restriction [Y]^2$ is constant. Take any $Z \in [Y]^{\lambda^+}$. Then $f \restriction [Z]^2$ is constant, contradiction.

So, κ is inaccessible. Now let $(L, <)$ be a linear order of size κ. Let $\prec$ be a well order of L. Now we define $f : [L]^2 \to 2$; suppose that $a, b \in L$ with $a \prec b$. Then

$$f(\{a, b\}) = \begin{cases} 0 & \text{if } a < b, \\ 1 & \text{if } b > a. \end{cases}$$

Let H be homogeneous for f and of size κ. If $f[[H]^2] = \{0\}$, then H is well-ordered by $<$. If $f[[H]^2] = \{1\}$, then H is well-ordered by $>$.

(ii)⇒(iii): By Lemma 17.29.

(iii)⇒(iv): Assume (iii). Suppose that $F : [\kappa]^2 \to \lambda$, where $1 < \lambda < \kappa$; we want to find a homogeneous set for F of size κ. We construct by recursion a sequence $\langle t_\alpha : \alpha < \kappa\rangle$ of members of ${}^{<\kappa}\kappa$; these will be the members of a tree T. Let $t_0 = \emptyset$. Now suppose that $0 < \alpha < \kappa$ and $t_\beta \in {}^{<\kappa}\kappa$ has been constructed for all $\beta < \alpha$. We now define t_α by recursion; its domain will also be determined by the recursive definition, and for this purpose it is convenient to actually define an auxiliary function $s : \kappa \to \kappa + 1$ by recursion. If $s(\eta)$ has been defined for all $\eta < \xi$, we define

$$s(\xi) = \begin{cases} F(\{\beta, \alpha\}) & \text{where } \beta < \alpha \text{ is minimum such that } s \restriction \xi = t_\beta, \text{ if there is such a } \beta, \\ \\ \kappa & \text{if there is no such } \beta. \end{cases}$$

Now eventually the second condition here must hold, as otherwise $\langle s \restriction \xi : \xi < \kappa\rangle$ would be a one-one function from κ into $\{t_\beta : \beta < \alpha\}$, which is impossible. Take the least ξ such that $s(\xi) = \kappa$, and let $t_\alpha = s \restriction \xi$. This finishes the construction of the t_α's. Let $T = \{t_\alpha : \alpha < \kappa\}$, with the partial order $\subseteq$. Clearly this gives a tree.

By construction, if $\alpha < \kappa$ and $\xi < \mathrm{dmn}(t_\alpha)$, then $t_\alpha \restriction \xi \in T$. Thus the height of an element t_α is $\mathrm{dmn}(t_\alpha)$.

(2) The sequence $\langle t_\alpha : \alpha < \kappa\rangle$ is one-one.

In fact, suppose that $\beta < \alpha$ and $t_\alpha = t_\beta$. Say that $\mathrm{dmn}(t_\alpha) = \xi$. Then $t_\alpha = t_\alpha \restriction \xi = t_\beta$, and the construction of t_α gives something with domain greater than ξ, contradiction. Thus (2) holds, and hence $|T| = \kappa$.

(3) The set of all elements of T of level $\xi < \kappa$ has size less than κ.

In fact, let U be this set. Then

$$|U| \leq \prod_{\eta<\xi} \lambda = \lambda^\xi < \kappa$$

since κ is inaccessible. So (3) holds, and hence, since $|T| = \kappa$, T has height κ and is a κ-tree.

(4) If $t_\beta \subset t_\alpha$, then $\beta < \alpha$ and $F(\{\beta, \alpha\}) = t_\alpha(\mathrm{dmn}(t_\beta))$.

This is clear from the definition.

Now by the tree property, there is a branch B of size κ. For each $\xi < \lambda$ let

$$H_\xi = \{\alpha < \kappa : t_\alpha \in B \text{ and } t_\alpha^\frown\langle\xi\rangle \in B\}.$$

We claim that each H_ξ is homogeneous for F. In fact, take any distinct $\alpha, \beta \in H_\xi$. Then $t_\alpha, t_\beta \in B$. Say $t_\beta \subset t_\alpha$. Then $\beta < \alpha$, and by construction $t_\alpha(\mathrm{dmn}(t_\beta)) = F(\{\alpha, \beta\})$. So $F(\{\alpha, \beta\}) = \xi$ by the definition of H_ξ, as desired. Now

$$\{\alpha < \kappa : t_\alpha \in B\} = \bigcup_{\xi<\lambda} \{\alpha < \kappa : t_\alpha \in H_\xi\},$$

so since $|B| = \kappa$ it follows that $|H_\xi| = \kappa$ for some $\xi < \lambda$, as desired.

(iv)$\Rightarrow$(i): obvious. $\square$

Now we go into the connection of weakly compact cardinals with logic, thereby justifying the name "weakly compact".

Let κ and λ be infinite cardinals. The language $L_{\kappa\lambda}$ is an extension of ordinary first order logic as follows. The notion of a model is unchanged. In the logic, we have a sequence of λ distinct individual variables, and we allow quantification over any one-one sequence of fewer than λ variables. We also allow conjunctions and disjunctions of fewer than κ formulas. It should be clear what it means for an assignment of values to the variables to satisfy a formula in this extended language. We say that an infinite cardinal κ is *logically weakly compact* iff the following condition holds:

(*) For any language $L_{\kappa\kappa}$ with at most κ basic symbols, if Γ is a set of sentences of the language and if every subset of Γ of size less than κ has a model, then also Γ has a model.

Notice here the somewhat unnatural restriction that there are at most κ basic symbols. If we drop this restriction, we obtain the notion of a strongly compact cardinal. These cardinals are much larger than even the measurable cardinals discussed later. See below for more about strongly compact cardinals.

Theorem 17.24. *An infinite cardinal is logically weakly compact iff it is weakly compact.*

Proof. Suppose that κ is logically weakly compact.

(1) κ is regular.

Suppose not; say $X \subseteq \kappa$ is unbounded but $|X| < \kappa$. Take the language with individual constants c_α for $\alpha < \kappa$ and also one more individual constant d. Consider the following set Γ of sentences in this language:

$$\{d \neq c_\alpha : \alpha < \kappa\} \cup \left\{ \bigvee_{\beta\in X} \bigvee_{\alpha<\beta} (d = c_\alpha) \right\}.$$

If $\Delta \in [\Gamma]^{<\kappa}$, let A be the set of all $\alpha < \kappa$ such that $d \neq c_\alpha$ is in Δ. So $|A| < \kappa$. Take any $\alpha \in \kappa \backslash A$, and consider the structure $M = (\kappa, \gamma, \alpha)_{\gamma<\kappa}$. There is a $\beta \in X$ with $\alpha < \beta$, and this shows that M is a model of Δ.

Thus every subset of Γ of size less than κ has a model, so Γ has a model; but this is clearly impossible.

(2) κ is strong limit.

In fact, suppose not; let $\lambda < \kappa$ with $\kappa \leq 2^\lambda$. We consider the language with distinct individual constants c_α, d^i_α for all $\alpha < \kappa$ and $i < 2$. Let Γ be the following set of sentences in this language:

$$\left\{ \bigwedge_{\alpha<\lambda} [(c_\alpha = d^0_\alpha \vee c_\alpha = d^1_\alpha) \wedge d^0_\alpha \neq d^1_\alpha] \right\} \cup \left\{ \bigvee_{\alpha<\lambda} (c_\alpha \neq d^{f(\alpha)}_\alpha) : f \in {}^\lambda 2 \right\}.$$

Suppose that $\Delta \in [\Gamma]^{<\kappa}$. We may assume that Δ has the form

$$\left\{\bigwedge_{\alpha<\lambda}[(c_\alpha = d^0_\alpha \vee c_\alpha = d^1_\alpha) \wedge d^0_\alpha \neq d^1_\alpha]\right\} \cup \left\{\bigvee_{\alpha<\lambda}(c_\alpha \neq d^{f(\alpha)}_\alpha) : f \in M\right\},$$

where $M \in [{}^\lambda 2]^{<\kappa}$. Fix $g \in {}^\lambda 2 \backslash M$. Let $d^0_\alpha = \alpha$, $d^1_\alpha = \alpha + 1$, and $c_\alpha = d^{g(\alpha)}_\alpha$, for all $\alpha < \lambda$. Clearly $(\kappa, c_\alpha, d^i_\alpha)_{\alpha<\lambda, i<2}$ is a model of Δ.

Thus every subset of Γ of size less than κ has a model, so Γ has a model, say $(M, u_\alpha, v^i_\alpha)_{\alpha<\lambda, i<2}$. By the first part of Γ there is a function $f \in {}^\lambda 2$ such that $u_\alpha = d^{f(\alpha)}_\alpha$ for every $\alpha < \lambda$. this contradicts the second part of Γ.

Hence we have shown that κ is inaccessible.

Finally, we prove that the tree property holds. Suppose that $(T, \leq)$ is a κ-tree. Let L be the language with a binary relation symbol $\prec$, unary relation symbols P_α for each $\alpha < \kappa$, individual constants c_t for each $t \in T$, and one more individual constant d. Let Γ be the following set of sentences:

all $L_{\kappa\kappa}$-sentences holding in the structure $M \overset{\text{def}}{=} (T, <, \mathrm{Lev}_\alpha(T), t)_{\alpha<\kappa, t\in T}$;
$\exists x[P_\alpha x \wedge x \prec d]$ for each $\alpha < \kappa$.

Clearly every subset of Γ of size less than κ has a model. Hence Γ has a model $N \overset{\text{def}}{=} (A, <', S'_\alpha, a_t, b)_{\alpha<\kappa, t\in T}$. For each $\alpha < \kappa$ choose $e_\alpha \in S'_\alpha$ with $e_\alpha <' b$. Now the following sentence holds in M and hence in N:

$$\forall x \left[P_\alpha x \leftrightarrow \bigvee_{s\in \mathrm{Lev}_\alpha(T)} (x = c_s)\right].$$

Hence for each $\alpha < \kappa$ we can choose $t(\alpha) \in T$ such that $e_a = a_{t(\alpha)}$. Now the sentence

$$\forall x, y, z[x < z \wedge y < z \to x \text{ and } y \text{ are comparable}]$$

holds in M, and hence in N. Now fix $\alpha < \beta < \kappa$. Now $e_\alpha, e_\beta <' b$, so it follows that e_α and e_β are comparable under $\leq'$. Hence $a_{t(\alpha)}$ and $a_{t(\beta)}$ are comparable under $\leq'$. It follows that $t(\alpha)$ and $t(\beta)$ are comparable under $\leq$. So $t(\alpha) < t(\beta)$. Thus we have a branch of size κ.

Now suppose that κ is weakly compact. Let L be an $L_{\kappa\kappa}$-language with at most κ symbols, and suppose that Γ is a set of sentences in L such that every subset Δ of Γ of size less than κ has a model M_Δ. We will construct a model of Γ by modifying Henkin's proof of the completeness theorem for first-order logic.

First we note that there are at most κ formulas of L. This is easily seen by the following recursive construction of all formulas:

$$\begin{aligned}
F_0 &= \text{ all atomic formulas;}\\
F_{\alpha+1} &= F_\alpha \cup \{\neg\varphi : \varphi \in F_\alpha\} \cup \left\{\bigvee \Phi : \Phi \in [F_\alpha]^{<\kappa}\right\} \cup \{\exists\overline{x}\varphi : \varphi \in F_\alpha, \overline{x} \text{ of length } < \kappa\};\\
F_\alpha &= \bigcup_{\beta<\alpha} F_\beta \text{ for } \alpha \text{ limit.}
\end{aligned}$$

By induction, $|F_\alpha| \leq \kappa$ for all $\alpha \leq \kappa$, and F_κ is the set of all formulas. (One uses that κ is inaccessible.)

Expand L to L' by adjoining a set C of new individual constants, with $|C| = \kappa$. Let Θ be the set of all subformulas of the sentences in Γ. Let $\langle \varphi_\alpha : \alpha < \kappa \rangle$ list all sentences of L' which are of the form $\exists \overline{x} \psi_\alpha(\overline{x})$ and are obtained from a member of Θ by replacing variables by members of C. Here $\overline{x}$ is a one-one sequence of variables of length less than κ; say that $\overline{x}$ has length β_α. Now we define a sequence $\langle d_\alpha : \alpha < \kappa \rangle$; each d_α will be a sequence of members of C of length less than κ. If d_β has been defined for all $\beta < \alpha$, then

$$\bigcup_{\beta<\alpha} \operatorname{rng}(d_\beta) \cup \{c \in C : c \text{ occurs in } \varphi_\beta \text{ for some } \beta < \alpha\}$$

has size less than κ. We then let d_α be a one-one sequence of members of C not in this set; d_α should have length β_α. Now for each $\alpha \leq \kappa$ let

$$\Omega_\alpha = \{\exists \overline{x} \psi_\beta(\overline{x}) \rightarrow \psi_\beta(\overline{d_\beta}) : \beta < \alpha\}.$$

Note that $\Omega_\alpha \subseteq \Omega_\gamma$ if $\alpha < \gamma \leq \kappa$. Now we define for each $\Delta \in [\Gamma]^{<\kappa}$ and each $\alpha \leq \kappa$ a model N_α^Δ of $\Delta \cup \Omega_\alpha$. Since $\Omega_0 = \emptyset$, we can let $N_0^\Delta = M_\Delta$. Having defined N_α^Δ, since the range of d_α consists of new constants, we can choose denotations of those constants, expanding N_α^Δ to $N_{\alpha+1}^\Delta$, so that the sentence

$$\exists \overline{x} \psi_\alpha(\overline{x}) \rightarrow \psi_\alpha(\overline{d_\alpha})$$

holds in $N_{\alpha+1}^\Delta$. For $\alpha \leq \kappa$ limit we let $N_\alpha^\Delta = \bigcup_{\beta<\alpha} N_\beta^\Delta$.

It follows that N_κ^Δ is a model of $\Delta \cup \Omega_\kappa$. So each subset of $\Gamma \cup \Omega_\kappa$ of size less than κ has a model.

It suffices now to find a model of $\Gamma \cup \Omega_\kappa$ in the language L'. Let $\langle \psi_\alpha : \alpha < \kappa \rangle$ be an enumeration of all sentences obtained from members of Θ by replacing variables by members of C, each such sentence appearing κ times. Let T consist of all f satisfying the following conditions:

(3) f is a function with domain $\alpha < \kappa$.

(4) $\forall \beta < \alpha[(\psi_\beta \in \Gamma \cup \Omega_\kappa \rightarrow f(\beta) = \psi_\beta)$ and $\psi_\beta \notin \Gamma \cup \Omega_\kappa \rightarrow f(\beta) = \neg\psi_\beta)]$.

(5) $\operatorname{rng}(f)$ has a model.

Thus T forms a tree $\subseteq$.

(6) T has an element of height α, for each $\alpha < \kappa$.

In fact, $\Delta \stackrel{\text{def}}{=} \{\psi_\beta : \beta < \alpha, \psi_\beta \in \Gamma \cup \Omega_\kappa\} \cup \{\neg\psi_\beta : \beta < \alpha, \neg\psi_\beta \in \Gamma \cup \Omega_\kappa\}$ is a subset of $\Gamma \cup \Omega_\kappa$ of size less than κ, so it has a model P. For each $\beta < \alpha$ let

$$f(\beta) = \begin{cases} \psi_\beta & \text{if } P \models \psi_\beta, \\ \neg\psi_\gamma & \text{if } P \models \neg\psi_\beta. \end{cases}$$

Clearly f is an element of T with height α. So (6) holds.

Thus T is clearly a κ-tree, so by the tree property we can let B be a branch in T of size κ. Let $\Xi = \{f(\alpha) : \alpha < \kappa,\ f \in B,\ f \text{ has height } \alpha + 1\}$. Clearly $\Gamma \cup \Omega_\kappa \subseteq \Xi$ and for every $\alpha < \kappa$, $\psi_\alpha \in \Xi$ or $\neg\psi_\alpha \in \Xi$.

(7) If $\varphi, \varphi \to \chi \in \Xi$, then $\chi \in \Xi$.

In fact, say $\varphi = f(\alpha)$ and $\varphi \to \chi = f(\beta)$. Choose $\gamma > \alpha, \beta$ so that ψ_γ is χ. We may assume that $\mathrm{dmn}(f) \geq \gamma + 1$. Since $\mathrm{rng}(f)$ has a model, it follows that $f(\gamma) = \chi$. So (7) holds.

Let S be the set of all terms with no variables in them. We define $\sigma \equiv \tau$ iff $\sigma, \tau \in S$ and $(\sigma = \tau) \in \Xi$. Then $\equiv$ is an equivalence relation on S. In fact, let $\sigma \in S$. Say that $\sigma = \sigma$ is ψ_α. Since ψ_α holds in every model, it holds in any model of $\{f(\beta) : \beta \leq \alpha\}$, and hence $f(\alpha) = (\sigma = \sigma)$. So $(\sigma = \sigma) \in \Xi$ and so $\sigma \equiv \sigma$. Symmetry and transitivity follow by (7).

Let M be the collection of all equivalence classes. Using (7) it is easy to see that the function and relation symbols can be defined on M so that the following conditions hold:

(8) If F is an m-ary function symbol, then

$$F^M(\sigma_0/\equiv, \ldots, \sigma_{m-1}/\equiv) = F(\sigma_0, \ldots, \sigma_{m-1})/\equiv .$$

(9) If R is an m-ary relation symbol, then

$$\langle \sigma_0/\equiv, \ldots, \sigma_{m-1}/\equiv \rangle \in R^M \quad \text{iff} \quad R(\sigma_0, \ldots, \sigma_{m-1}) \in \Xi.$$

Now the final claim is as follows:

(10) If φ is a sentence of L', then $M \models \varphi$ iff $\varphi \in \Xi$.

Clearly this will finish the proof. We prove (10) by induction on φ. It is clear for atomic sentences by (8) and (9). If it holds for φ, it clearly holds for $\neg\varphi$. Now suppose that Q is a set of sentences of size less than κ, and (10) holds for each member of Q. Suppose that $M \models \bigwedge Q$. Then $M \models \varphi$ for each $\varphi \in Q$, and so $Q \subseteq \Xi$. Hence there is a $\Delta \in [\kappa]^{<\kappa}$ such that $Q = f[\Delta]$, with $f \in B$. Choose α greater than each member of Δ such that ψ_α is the formula $\bigwedge Q$. We may assume that $\alpha \in \mathrm{dmn}(f)$. Since $\mathrm{rng}(f)$ has a model, it follows that $f(\alpha) = \bigwedge Q$. Hence $\bigwedge Q \in \Xi$.

Conversely, suppose that $\bigwedge Q \in \Xi$. From (7) it easily follows that $\varphi \in \Xi$ for every $\varphi \in Q$, so by the inductive hypothesis $M \models \varphi$ for each $\varphi \in Q$, so $M \models \bigwedge Q$.

Finally, suppose that φ is $\exists \overline{x}\psi$, where (10) holds for shorter formulas. Suppose that $M \models \exists \overline{x}\psi$. Then there are members of S such that when they are substituted in ψ for $\overline{x}$, obtaining a sentence ψ', we have $M \models \psi'$. Hence by the inductive hypothesis, $\psi' \in \Xi$. (7) then yields $\exists \overline{x}\psi \in \Xi$.

Conversely, suppose that $\exists \overline{x}\psi \in \Xi$. Now there is a sequence $\overline{d}$ of members of C such that $\exists \overline{x}\psi \in \Xi \to \psi(\overline{d})$ is also in Ξ, and so by (7) we get $\psi(\overline{d}) \in \Xi$. By the inductive hypothesis, $M \models \psi(\overline{d})$, so $M \models \exists \overline{x}\psi \in \Xi$. $\square$

Next we want to show that every weakly compact cardinal is a Mahlo cardinal. To do this we need two lemmas.

Lemma 17.25. *Let A be a set of infinite cardinals such that for every regular cardinal κ, the set $A \cap \kappa$ is non-stationary in κ. Then there is a one-one regressive function with domain A.*

Proof. We proceed by induction on $\gamma \stackrel{\text{def}}{=} \bigcup A$. Note that γ is a cardinal; it is 0 if $A = \emptyset$. The cases $\gamma = 0$ and $\gamma = \omega$ are trivial, since then $A = \emptyset$ or $A = \{\omega\}$ respectively.

Next, suppose that γ is a successor cardinal κ^+. Then $A = A' \cup \{\kappa^+\}$ for some set A' of infinite cardinals less than κ^+. Then $\bigcup A' < \kappa^+$, so by the inductive hypothesis there is a one-one regressive function f on A'. We can extend f to A by setting $f(\kappa^+) = \kappa$, and so we get a one-one regressive function defined on A.

Suppose that γ is singular. Let $\langle \mu_\xi : \xi < \mathrm{cf}(\gamma) \rangle$ be a strictly increasing continuous sequence of infinite cardinals with supremum γ, with $\mathrm{cf}(\gamma) < \mu_0$. Note then that for every cardinal $\lambda < \gamma$, either $\lambda < \mu_0$ or else there is a unique $\xi < \mathrm{cf}(\gamma)$ such that $\mu_\xi \leq \lambda < \mu_{\xi+1}$. For every $\xi < \mathrm{cf}(\gamma)$ we can apply the inductive hypothesis to $A \cap \mu_\xi$ to get a one-one regressive function g_ξ with domain $A \cap \mu_\xi$. We now define f with domain A. In case $\mathrm{cf}(\gamma) = \omega$ we define, for each $\lambda \in A$,

$$f(\lambda) = \begin{cases} g_0(\lambda) + 2 & \text{if } \lambda < \mu_0, \\ \mu_\xi + g_{\xi+1}(\lambda) + 1 & \text{if } \mu_\xi < \lambda < \mu_{\xi+1}, \\ \mu_\xi & \text{if } \lambda = \mu_{\xi+1}, \\ 1 & \text{if } \lambda = \mu_0, \\ 0 & \text{if } \lambda = \gamma \in A. \end{cases}$$

Here the addition is ordinal addition. Clearly f is as desired in this case. If $\mathrm{cf}(\gamma) > \omega$, let $\langle \nu_\xi : \xi < \mathrm{cf}(\gamma) \rangle$ be a strictly increasing sequence of limit ordinals with supremum $\mathrm{cf}(\gamma)$. Then we define, for each $\lambda \in A$,

$$f(\lambda) = \begin{cases} g_0(\lambda) + 1 & \text{if } \lambda < \mu_0, \\ \mu_\xi + g_{\xi+1}(\lambda) + 1 & \text{if } \mu_\xi < \lambda < \mu_{\xi+1}, \\ \nu_\xi & \text{if } \lambda = \mu_\xi, \\ 0 & \text{if } \lambda = \gamma \in A. \end{cases}$$

Clearly f works in this case too.

Finally, suppose that γ is a regular limit cardinal. By assumption, there is a club C in γ such that $C \cap \gamma \cap A = \emptyset$. We may assume that $C \cap \omega = \emptyset$. Let $\langle \mu_\xi : \xi < \gamma \rangle$ be the strictly increasing enumeration of C. Then we define, for each $\lambda \in A$,

$$f(\lambda) = \begin{cases} g_0(\lambda) + 1 & \text{if } \lambda < \mu_0, \\ \mu_\xi + g_{\xi+1}(\lambda) + 1 & \text{if } \mu_\xi < \lambda < \mu_{\xi+1}, \\ 0 & \text{if } \lambda = \gamma \in A. \end{cases}$$

Clearly f works in this case too. □

Lemma 17.26. *Suppose that κ is weakly compact, and S is a stationary subset of κ. Then there is a regular $\lambda < \kappa$ such that $S \cap \lambda$ is stationary in λ.*

Proof. Suppose not. Thus for all regular $\lambda < \kappa$, the set $S \cap \lambda$ is non-stationary in λ. Let C be the collection of all infinite cardinals less than κ. Clearly C is club in κ, so

$S \cap C$ is stationary in κ. Clearly still $S \cap C \cap \lambda$ is non-stationary in λ for every regular $\lambda < \kappa$. So we may assume from the beginning that S is a set of infinite cardinals.

Let $\langle \lambda_\xi : \xi < \kappa \rangle$ be the strictly increasing enumeration of S. Let

$$T = \left\{ s : \exists \xi < \kappa \left[s \in \prod_{\eta < \xi} \lambda_\eta \text{ and } s \text{ is one-one} \right] \right\}.$$

For every $\xi < \kappa$ the set $S \cap \lambda_\xi$ is non-stationary in every regular cardinal, and hence by Lemma 26.9 there is a one-one regressive function s with domain $S \cap \lambda_\xi$. Now $S \cap \lambda_\xi = \{\lambda_\eta : \eta < \xi\}$. Hence $s \in T$.

Clearly T forms a tree of height κ under $\subseteq$. Now for any $\alpha < \kappa$,

$$\prod_{\beta < \alpha} \lambda_\beta \leq \left(\sup_{\beta < \alpha} \lambda_\beta \right)^{|\alpha|} < \kappa.$$

Hence by the tree property there is a branch B in T of size κ. Thus $\bigcup B$ is a one-one regressive function with domain S, contradicting Fodor's theorem. $\square$

Theorem 17.27. *Every weakly compact cardinal is Mahlo, hyper-Mahlo, hyper-hyper-Mahlo, etc.*

Proof. Let κ be weakly compact. Let $S = \{\lambda < \kappa : \lambda \text{ is regular}\}$. Suppose that C is club in κ. Then C is stationary in κ, so by Lemma 17.26 there is a regular $\lambda < \kappa$ such that $C \cap \lambda$ is stationary in λ; in particular, $C \cap \lambda$ is unbounded in λ, so $\lambda \in C$ since C is closed in κ. Thus we have shown that $S \cap C \neq \emptyset$. So κ is Mahlo.

Let $S' = \{\lambda < \kappa : \lambda \text{ is a Mahlo cardinal}\}$. Suppose that C is club in κ. Let $S'' = \{\lambda < \kappa : \lambda \text{ is regular}\}$. Since κ is Mahlo, S'' is stationary in κ. Then $C \cap S''$ is stationary in κ, so by Lemma 17.26 there is a regular $\lambda < \kappa$ such that $C \cap S'' \cap \lambda$ is stationary in λ. Hence λ is Mahlo, and also $C \cap \lambda$ is unbounded in λ, so $\lambda \in C$ since C is closed in κ. Thus we have shown that $S' \cap C \neq \emptyset$. So κ is hyper-Mahlo.

Let $S''' = \{\lambda < \kappa : \lambda \text{ is a hyper-Mahlo cardinal}\}$. Suppose that C is club in κ. Let $S^{iv} = \{\lambda < \kappa : \lambda \text{ is Mahlo}\}$. Since κ is hyper-Mahlo, S^{iv} is stationary in κ. Then $C \cap S^{iv}$ is stationary in κ, so by Lemma 17.26 there is a regular $\lambda < \kappa$ such that $C \cap S^{iv} \cap \lambda$ is stationary in λ. Hence λ is hyper-Mahlo, and also $C \cap \lambda$ is unbounded in λ, so $\lambda \in C$ since C is closed in κ. Thus we have shown that $S''' \cap C \neq \emptyset$. So κ is hyper-hyper-Mahlo.

Etc. $\square$

We now give another equivalent definition of weak compactness. For it we need several lemmas.

Lemma 17.28. *Suppose that* $\mathbf{R}$ *is a well-founded class relation on a class* $\mathbf{A}$*, and it is set-like and extensional. Also suppose that* $\mathbf{B} \subseteq \mathbf{A}$*,* $\mathbf{B}$ *is transitive,* $\forall a, b \in \mathbf{A}[a\mathbf{R}b \in \mathbf{B} \rightarrow a \in \mathbf{B}]$*, and* $\forall a, b \in \mathbf{B}[a\mathbf{R}b \leftrightarrow a \in b]$*. Let* $\mathbf{G}, \mathbf{M}$ *be the Mostowski collapse of* $(\mathbf{A}, \mathbf{R})$*. Then* $\mathbf{G} \restriction \mathbf{B}$ *is the identity.*

Proof. Suppose not, and let $\mathbf{X} = \{b \in \mathbf{B} : \mathbf{G}(b) \neq b\}$. Since we are assuming that $\mathbf{X}$ is a nonempty subclass of $\mathbf{A}$, choose $b \in \mathbf{X}$ such that $y \in \mathbf{A}$ and $y\mathbf{R}b$ imply that $y \notin \mathbf{X}$. Then

$$\begin{aligned}
\mathbf{G}(b) &= \{\mathbf{G}(y) : y \in \mathbf{A} \text{ and } y\mathbf{R}b\} \\
&= \{\mathbf{G}(y) : y \in \mathbf{B} \text{ and } y\mathbf{R}b\} \\
&= \{y : y \in \mathbf{B} \text{ and } y\mathbf{R}b\} \\
&= \{y : y \in \mathbf{B} \text{ and } y \in b\} \\
&= \{y : y \in b\} \\
&= b,
\end{aligned}$$

contradiction. □

Lemma 17.29. *Let κ be weakly compact. Then for every $U \subseteq V_\kappa$, the structure $(V_\kappa, \in, U)$ has a transitive elementary extension $(M, \in, U')$ such that $\kappa \in M$.*

(This means that $V_\kappa \subseteq M$ and a sentence holds in the structure $(V_\kappa, \in, U, x)_{x \in V_\kappa}$ iff it holds in $(M, \in, U', x)_{x \in V_\kappa}$.)

Proof. Let Γ be the set of all $L_{\kappa\kappa}$-sentences true in the structure $(V_\kappa, \in, U, x)_{x \in V_\kappa}$, together with the sentences

$$\begin{aligned}
&c \text{ is an ordinal}, \\
&\alpha < c \text{ (for all } \alpha < \kappa),
\end{aligned}$$

where c is a new individual constant. The language here clearly has κ many symbols. Every subset of Γ of size less than κ has a model; namely we can take $(V_\kappa, \in, U, x, \beta)_{x \in V_\kappa}$, choosing β greater than each α appearing in the sentences of Γ. Hence by weak compactness, Γ has a model $(M, E, W, k_x, y)_{x \in V_\kappa}$. This model is well-founded, since the sentence

$$\neg \exists v_0 v_1 \ldots \left[\bigwedge_{n \in \omega} (v_{n+1} \in v_n) \right]$$

holds in $(V_\kappa, \in, U, x)_{x \in V_\kappa}$, and hence in $(M, E, W, k_x, y)_{x \in V_\kappa}$.

Note that k is an injection of V_κ into M. Let F be a bijection from $M \backslash \text{rng}(k)$ onto $\{(V_\kappa, u) : u \in M \backslash \text{rng}(k)\}$. Then $G \stackrel{\text{def}}{=} k^{-1} \cup F^{-1}$ is one-one, mapping M onto some set N such that $V_\kappa \subseteq N$. We define, for $x, z \in N$, $xE'z$ iff $G^{-1}(x) E G^{-1}(z)$. Then G is an isomorphism from $(M, E, W, k_x, y)_{x \in V_\kappa}$ onto $\overline{N} \stackrel{\text{def}}{=} (N, E', G[W], x, G(y))_{x \in V_\kappa}$. Of course $\overline{N}$ is still well-founded. It is also extensional, since the extensionality axiom holds in $(V_\kappa, \in)$ and hence in (M, E) and (N, E'). Let H, P be the Mostowski collapse of (N, E'). Thus P is a transitive set, and

(1) H is an isomorphism from (N, E') onto $(P, \in)$.

(2) $\forall a, b \in N[aE'b \in V_\kappa \to a \in b]$.

In fact, suppose that $a, b \in N$ and $aE'b \in V_\kappa$. Let the individual constants used in the expansion of $(V_\kappa, \in, U)$ to $(V_\kappa, \in, U, x)_{a \in V_\kappa}$ be $\langle c_x : x \in V_\kappa \rangle$. Then

$$(V_\kappa, \in, U, x)_{a \in V_\kappa} \models \forall z \left[z \in k_b \to \bigvee_{w \in b} (z = k_w) \right],$$

and hence this sentence holds in $(N, E', G[W], x, G(y))_{x \in V_\kappa}$ as well, and so there is a $w \in b$ such that $a = w$, i.e., $a \in b$. So (2) holds.

(3) $\forall a, b \in V_\kappa [a \in b \to aE'b]$

In fact, suppose that $a, b \in V_\kappa$ and $a \in b$. Then the sentence $k_a \in k_b$ holds in $(V_\kappa, \in, U, x)_{x \in V_\kappa}$, so it also holds in $(N, E', G[W], x, G(y))_{x \in V_\kappa}$, so that $aE'b$.

We have now verified the hypotheses of Lemma 17.28. It follows that $H \restriction V_\kappa$ is the identity. In particular, $V_\kappa \subseteq P$. Now take any sentence σ in the language of $(V_\kappa, \in, U, x)_{x \in V_\kappa}$. Then

$$\begin{aligned} (V_\kappa, \in, U, x)_{x \in V_\kappa} \models \sigma \quad &\text{iff} \quad (M, E, W, k_x)_{x \in V_\kappa} \models \sigma \\ &\text{iff} \quad (N, E', G[W], x)_{x \in V_\kappa} \models \sigma \\ &\text{iff} \quad (P, \in, H[G[W]], x)_{x \in V_\kappa} \models \sigma. \end{aligned}$$

Thus $(P, \in, H[G[W]])$ is an elementary extension of $(V_\kappa, \in, U)$.

Now for $\alpha < \kappa$ we have

$$\begin{aligned} &(M, E, W, k_x, y)_{x \in V_\kappa} \models [y \text{ is an ordinal and } k_\alpha E y], \quad \text{hence} \\ &(N, E', G[W], x, G(y))_{x \in V_\kappa} \models [G(y) \text{ is an ordinal and } \alpha E' G(y)], \quad \text{hence} \\ &(P, \in, H[G[W]], x, H(G(y)))_{x \in V_\kappa} \models [H(G(y)) \text{ is an ordinal and } \alpha \in H(G(y))]. \end{aligned}$$

Thus $H(G(y))$ is an ordinal in P greater than each $\alpha < \kappa$, so since P is transitive, $\kappa \in P$. ☐

An infinite cardinal κ is *first-order describable* iff there is a $U \subseteq V_\kappa$ and a sentence σ in the language for $(V_\kappa, \in, U)$ such that $(V_\kappa, \in, U) \models \sigma$, while there is no $\alpha < \kappa$ such that $(V_\alpha, \in, U \cap V_\alpha) \models \sigma$.

Theorem 17.30. *If κ is infinite but not inaccessible, then it is first-order describable.*

Proof. ω is describable by the sentence that says that κ is the first limit ordinal; absoluteness is used. The subset U is not needed for this. Now suppose that κ is singular.

Let $\lambda = \text{cf}(\kappa)$, and let f be a function whose domain is some ordinal $\gamma < \kappa$ with $\text{rng}(f)$ cofinal in κ. Let $U = \{(\lambda, \beta, f(\beta)) : \beta < \lambda\}$. Let σ be the sentence expressing the following:

For every ordinal γ there is an ordinal δ with $\gamma < \delta$, U is nonempty, and there is an ordinal μ and a function g with domain μ such that U consists of all triples $(\mu, \beta, g(\beta))$ with $\beta < \mu$.

Clearly $(V_\kappa, \in, U) \models \sigma$. Suppose that $\alpha < \kappa$ and $(V_\alpha, \in, V_\alpha \cap U) \models \sigma$. Then α is a limit ordinal, and there is an ordinal $\gamma < \alpha$ and a function g with domain γ such that $V_\alpha \cap U$ consists of all triples $(\gamma, \beta, g(\beta))$ with $\beta < \gamma$. (Some absoluteness is used.) Now $V_\alpha \cap U$ is nonempty; choose $(\gamma, \beta, g(\beta))$ in it. Then $\gamma = \lambda$ since it is in U. It follows that $g = f$. Choose $\beta < \lambda$ such that $\alpha < f(\beta)$. Then $(\lambda, \beta, f(\beta)) \in U \cap V_\alpha$. Since $\alpha < f(\beta)$, it follows that α has rank less than α, contradiction.

Now suppose that $\lambda < \kappa \le 2^\lambda$. A contradiction is reached similarly, as follows. Let f be a function whose domain is $\mathscr{P}(\lambda)$ with range κ. Let $U = \{(\lambda, B, f(B)) : B \subseteq \lambda\}$. Let σ be the sentence expressing the following:

For every ordinal γ there is an ordinal δ with $\gamma < \delta$, U is nonempty, and there is an ordinal μ and a function g with domain $\mathscr{P}(\mu)$ such that U consists of all triples $(\mu, B, g(B))$ with $B \subseteq \mu$.

Clearly $(V_\kappa, \in, U) \models \sigma$. Suppose that $\alpha < \kappa$ and $(V_\alpha, \in, V_\alpha \cap U) \models \sigma$. Then α is a limit ordinal, and there is an ordinal $\gamma < \alpha$ and a function g with domain $\mathscr{P}(\gamma)$ such that $V_\alpha \cap U$ consists of all triples $(\gamma, B, g(B))$ with $B \subseteq \gamma$. (Some absoluteness is used.) Clearly $\gamma = \lambda$; otherwise $U \cap V_\alpha$ would be empty. Note that $g = f$. Choose $B \subseteq \lambda$ such that $\alpha = f(B)$. Then $(\lambda, B, f(B)) \in U \cap V_\alpha$. Again this implies that α has rank less than α, contradiction. □

The new equivalent of weak compactness involves second-order logic. We augment first order logic by adding a new variable S ranging over subsets rather than elements. There is one new kind of atomic formula: Sv with v a first-order variable. This is interpreted as saying that v is a member of S.

Now an infinite cardinal κ is Π^1_1-*indescribable* iff for every $U \subseteq V_\kappa$ and every second-order sentence σ of the form $\forall S\varphi$, with no quantifiers on S within φ, if $(V_\kappa, \in, U) \models \sigma$, then there is an $\alpha < \kappa$ such that $(V_\alpha, \in, U \cap V_\alpha) \models \sigma$. Note that if κ is Π^1_1-indescribable then it is not first-order describable.

Theorem 17.31. *An infinite cardinal κ is weakly compact iff it is Π^1_1-indescribable.*

Proof. First suppose that κ is Π^1_1-indescribable. By Theorem 17.30 it is inaccessible. So it suffices to show that it has the tree property. By the proof of Theorem 17.23(iii)⇒(iv) it suffices to check the tree property for a tree $T \subseteq {}^{<\kappa}\kappa$. Note that ${}^{<\kappa}\kappa \subseteq V_\kappa$. Let σ be the following sentence in the second-order language of $(V_\kappa, \in, T)$:

$$\exists S[T \text{ is a tree under } \subset, \text{ and}$$
$$S \subseteq T \text{ and } S \text{ is a branch of } T \text{ of unbounded length}].$$

Thus for each $\alpha < \kappa$ the sentence σ holds in $(V_\alpha, \in, T \cap V_\alpha)$. Hence it holds in $(V_\kappa, \in, T)$, as desired.

Now suppose that κ is weakly compact. Let $U \subseteq V_\kappa$, and let σ be a Π^1_1-sentence holding in $(V_\kappa, \in, U)$. By Lemma 17.29, let $(M, \in, U')$ be a transitive elementary extension of $(V_\kappa, \in, U)$ such that $\kappa \in M$. Say that σ is $\forall S\varphi$, with φ having no quantifiers on S. Now

$$\forall X \subseteq V_\kappa[(V_\kappa, \in, U) \models \varphi(X)]. \tag{1}$$

Now since $\kappa \in M$ and $(M, \in)$ is a model of ZFC, V_κ^M exists, and by absoluteness it is equal to V_κ. Hence by (1) we get

$$(M, \in, U') \models \forall X \subseteq V_\kappa \varphi^{V_\kappa}(U' \cap V_\kappa).$$

Hence

$$(M, \in, U') \models \exists\alpha\forall X \subseteq V_\alpha \varphi^{V_\alpha}(U' \cap V_\alpha),$$

so by the elementary extension property we get

$$(V_\kappa, \in, U) \models \exists\alpha\forall X \subseteq V_\alpha \varphi^{V_\alpha}(U' \cap V_\alpha).$$

We choose such an α. Since $V_\kappa \cap \mathbf{On} = \kappa$, it follows that $\alpha < \kappa$. Hence $(V_\alpha, \in, U' \cap V_\alpha) \models \sigma$, as desired. □

Lemma 17.32. *Every measurable cardinal is Π^2_1-indescribable.*

Proof. We claim:

Suppose that κ is a measurable cardinal, $U \subseteq V_\kappa$, and σ is a Π^2_1 sentence such that $(V_\kappa, \in, U) \models \sigma$. Then there is an $\alpha < \kappa$ such that $(V_\alpha, \in, U \cap V_\alpha) \models \sigma$.

Proof. Say σ is $\forall X\varphi(X)$, where X is a third order variable and $\varphi(X)$ contains only second and first order quantifiers. Then

$$\forall X \in \mathscr{P}(\mathscr{P}(V_\kappa))[(V_\kappa. \in, U) \models \varphi(X)],$$

i.e.

$$\forall X \subseteq V_{\kappa+1}[(V_\kappa. \in, U) \models \varphi(X)].$$

Let $\overline{\varphi}$ be obtained from φ by replacing first order quantifiers $\exists x$ and $\forall x$ by $\exists x \in V_\kappa$ and $\forall x \in V_\kappa$, and treating second order variables as first order. Then

$$\forall X \subseteq V_{\kappa+1}[(V_{\kappa+1}, \in, V_\kappa, U) \models \overline{\varphi}].$$

Now let D be a normal ultrafilter on κ. Then $V_\kappa^{M_D^\kappa} = V_\kappa$ and $V_{\kappa+1}^{M_D^\kappa} = V_\kappa$ by Lemma 17.17. Hence

(1) $$(\forall X \subseteq V_{\kappa+1}[(V_{\kappa+1}, \in, V_\kappa, U) \models \overline{\varphi}])^{M_D^\kappa}.$$

(2) $\forall a \in V_\kappa[j_U(a) = a]$.

Now let $f(\alpha) = V_\alpha$ for all $\alpha < \kappa$.

(3) $\pi([f]) = V_\kappa$.

In fact, suppose that $\pi([k]) \in \pi([f])$. Then $X \stackrel{\text{def}}{=} \{\alpha < \kappa : k(\alpha) \in V_\alpha\} \in D$. Now $X \subseteq \{\alpha < \kappa : \text{rank}(k(\alpha)) < \alpha\}$, so by exercise 8.8 on page 104, there is a $\gamma < \kappa$ such that $Y \stackrel{\text{def}}{=} \{\alpha < \kappa : \text{rank}(k(\alpha)) = \gamma\} \in D$. Now $Y \subseteq \{\alpha < \kappa : k(\alpha) \in V_{\gamma+1}\}$ and $|V_{\gamma+1}| < \kappa$, so there is a $y \in V_{\gamma+1}$ such that $\{\alpha < \gamma : k(\alpha) = y\} \in D$. Thus $[k] = [c^\kappa_y]$, so $\pi([k]) \in V_\kappa$. This shows that $\pi([f]) \subseteq V_\kappa$. The other inclusion is clear, so (3) holds.

Now let $g(\alpha) = V_{\alpha+1}$ for all $\alpha < \kappa$.

(4) $\pi([g]) = V_{\kappa+1}$.

In fact, let $a \in M^\kappa_D$. Say $a = \pi([k])$. Suppose that $\pi([k]) \in \pi([g])$. Thus $P \stackrel{\text{def}}{=} \{\alpha < \kappa : k(\alpha) \in V_{\alpha+1}\} \in D$. Now we claim that $\pi([k]) \subseteq V_\kappa$. For, suppose that $\pi([s]) \in \pi([k])$. Then $N \stackrel{\text{def}}{=} \{\alpha < \kappa : s(\alpha) \in k(\alpha)\} \in D$. If $\alpha \in P \cap N$, then $s(\alpha) \in k(\alpha) \in V_{\alpha+1} = \mathscr{P}(V_\alpha)$. So $k(\alpha) \subseteq V_\alpha$, and hence $s(\alpha) \in V_\alpha$. Thus $P \cap N \subseteq \{\alpha : s(\alpha) \in V_\alpha\}$. Hence by (3), $\pi([s]) \in V_\kappa$. This proves the claim. Hence $\pi([k]) \in V_{\kappa+1}$. So we have shown $\subseteq$ in (4).

Conversely, if $\pi([k]) \in V_{\kappa+1}$, then $\pi([k]) \subseteq V_\kappa$. So by (3), $\pi([k]) \subseteq \pi([f])$. We claim that $\{\alpha < \kappa : k(\alpha) \subseteq V_\alpha\} \in D$. For, suppose not. Then $P \stackrel{\text{def}}{=} \{\alpha < \kappa : k(\alpha) \not\subseteq V_\alpha\} \in D$. For each $\alpha \in P$ choose $s(\alpha) \in k(\alpha)\backslash V_\alpha$. Then $P \subseteq \{\alpha < \kappa : s(\alpha) \in k(\alpha)\}$, so $\pi([s]) \in \pi([k])$ and hence $\pi([s]) \in \pi([f])$. So $\{\alpha < \kappa : s(\alpha) \in V_\alpha\} \in D$. But $P \subseteq \{\alpha :< \kappa : s(\alpha) \notin V_\alpha\}$, contradiction. This proves the claim. Hence $\{\alpha < \kappa : k(\alpha) \in V_{\alpha+1}\} \in D$, so $\pi([k]) \in \pi([g])$, as desired. This proves (4).

Next, let $h(\alpha) = U \cap V_\alpha$ for all $\alpha < \kappa$.

(5) $\pi([h]) = U$.

For, first suppose that $u \in U$. Since $U \subseteq V_\kappa$, there is a $\beta < \kappa$ such that $u \in V_\beta$. Hence $\{\alpha < \kappa : u \in V_\alpha\} \supseteq (\kappa\backslash\beta)$, and hence $\{\alpha < \kappa : u \in V_\alpha\} \in D$. Thus $\{\alpha < \kappa : c^\kappa_u(\alpha) \in U \cap V_\alpha\} \in D$, so $u = j_D(u) \in \pi([h])$. This proves $\supseteq$ in (5).

Conversely, suppose that $\pi([s]) \in \pi([h])$. Thus $\{\alpha < \kappa : s(\alpha) \in U \cap V_\alpha\} \in D$, so $\{\alpha < \kappa : s(\alpha) \in f(\alpha)\} \in D$, and so $\pi([s]) \in \pi([f]) = V_\kappa$. Hence there is an $\alpha < \kappa$ such that $\pi([s]) \in V_\alpha$. Let $u = \pi([s])$. Then $\pi([s]) = \pi([c^\kappa_u])$, so $[s] = [c^\kappa_u]$. Hence $\{\alpha < \kappa : s(\alpha) \in U$ and $s(\alpha) = u\} \in D$, so $u \in U$. Since $\pi([s]) = j_U(u) = u$, this finishes the proof of (5).

Now by (1)-(5) and Łoś's theorem,

$$\{\alpha < \kappa : \forall X \subseteq V_{\alpha+1}[(V_{\alpha+1}, \in, V_\alpha, U \cap V_\alpha) \models \overline{\varphi}\} \in D.$$

Converting this to the third order language, we get

$$\{\alpha < \kappa : (V_\alpha, \in, U \cap V_\alpha) \models \varphi\} \in D$$ □

Lemma 17.33. *If κ is weakly compact, $A \subseteq \kappa$, and $\forall \alpha < \kappa[A \cap \alpha \in L]$, then $A \in L$.*

Proof. We have $\forall\alpha\exists x[x$ is constructible and $x = A \cap \alpha]$. By absoluteness, $(V_\kappa, \in, A) \models \forall\alpha\exists x[x$ is constructible and $x = A \cap \alpha]$. By Lemma 17.29 let $(M, \in, A')$ be a transitive elementary extension of $(V_\kappa, \in, A)$ with $\kappa \in M$. Hence $(M, \in, A') \models \forall\alpha\exists x[x$ is constructible and $x = A \cap \alpha]$. Hence $(M, \in, A') \models \exists x[x$ is constructible and $x = A \cap \kappa]$. By absoluteness, A is constructible. □

Theorem 17.34. *If κ is weakly compact, then (κ is weakly compact)^L.*

Proof. In L, let $T = (\kappa, <)$ be a tree of height κ with each level of size less than κ. Then there is a branch A through T with $|A| = \kappa$. Now $\forall\alpha < \kappa[A \cap \alpha$ is constructible. Hence by Lemma 17.33, $A \in L$. $\square$

Let $\mathscr{L}$ be a first-order language, let κ be an infinite cardinal, let $\mathfrak{A}$ be an $\mathscr{L}$-structure, and assume that $\kappa \subseteq A$. A set $I \subseteq \kappa$ is a *set of indiscernibles* for $\mathfrak{A}$ iff for every $n \in \omega$, every formula $\varphi(v_0, \ldots, v_{n-1})$, and all $\alpha, \beta \in {}^n\kappa$ such that $\alpha_0 < \cdots < \alpha_{n-1}$ and $\beta_0 < \cdots \beta_{n-1}$,

$$\mathfrak{A} \models \varphi(\alpha_0, \ldots, \alpha_{n-1}) \quad \text{iff} \quad \mathfrak{A} \models \varphi(\beta_0, \ldots, \beta_{n-1}).$$

Lemma 17.35. *Suppose that κ and λ are infinite cardinals, and α is a limit ordinal. Assume that*

$$\kappa \to (\alpha)^{<\omega}_{2^\lambda},$$

(See page 121.) Let $\mathscr{L}$ be a language of size $\leq \lambda$ and let $\mathscr{A}$ be an $\mathscr{L}$-structure such that $\kappa \subseteq A$.

Then $\mathfrak{A}$ has a set of indiscernibles with order type α.

Proof. Let Φ be the set of all formulas of $\mathscr{L}$. Define $F : [\kappa]^{<\omega} \to \mathscr{P}(\Phi)$ as follows. For $n \in \omega$ and $x \in [\kappa]^n$, write $x = \{\alpha_0, \ldots, \alpha_{n-1}\}$ with $\alpha_0 < \cdots < \alpha_{n-1}$ and define

$$F(x) = \{\varphi(v_0, \ldots, v_{n-1}) \in \Phi : \mathfrak{A} \models \varphi(\alpha_0, \ldots, \alpha_{n-1})\}.$$

Let I be a homogeneous set with order type α. Then I is a set of indiscernibles for $\mathfrak{A}$. For suppose that $n \in \omega$, $\varphi(v_0, \ldots, v_{n-1})$ is a formula of $\mathscr{L}$, and $\langle\alpha_0 < \ldots < \alpha_{n-1}\rangle$ and $\langle\beta_0 < \ldots < \beta_{n-1}\rangle$ are ordinals from I. Suppose that $\mathfrak{A} \models \varphi(\alpha_0, \ldots, \alpha_{n-1})$. Then

$$\begin{aligned}\varphi(v_0, \ldots, v_{n-1}) &\in F(\{\alpha_0, \ldots, \alpha_{n-1}\}) \\ &= F(\{\beta_0, \ldots, \beta_{n-1}\}),\end{aligned}$$

and so $\mathfrak{A} \models \varphi(\beta_0, \ldots, \beta_{n-1})$. The converse is similar. $\square$

Lemma 17.36. *If $\kappa \to (\kappa)^{<\omega}$ and if $\lambda < \kappa$ is a cardinal, then $\kappa \to (\kappa)^{<\omega}_\lambda$.*

Proof. Let $F : [\kappa]^{<\omega} \to \lambda$ be given. We define $G : [\kappa]^{<\omega} \to 2$ as follows. Given $x \in [\kappa]^{<\omega}$, let

$$G(x) = \begin{cases} 1 & \text{if } \exists k \in \omega\backslash\{0\}[x = \{\alpha_0 < \cdots < \alpha_{k-1} < \alpha_k < \cdots < \alpha_{2k-1}\}] \\ & \text{and } F(\{\alpha_0, \ldots, \alpha_{k-1}) = F(\{\alpha_k, \ldots, \alpha_{2k-1}\}) \\ 0 & \text{otherwise.} \end{cases}$$

Let H be homogeneous for G with $|H| = \kappa$.

(1) $\forall k \in \omega\backslash\{0\}\forall\alpha \in {}^{2k}H[G(\{x_0, \ldots, x_{2k-1}\}) = 1]$.

For, let $\alpha \in {}^\kappa({}^kH)$ be such that $\forall\xi < \kappa[\alpha_{\xi 0} < \cdots \alpha_{\xi(k-1)}]$ and $\forall\xi, \eta < \kappa[\xi < \eta \to \alpha_{\xi(k-1)} < \alpha_{\eta 0}]$. Then $\langle F(\{\alpha_{\xi 0}, \ldots, \alpha_{\xi(k-1)}\}) : \xi < \kappa\rangle$ maps κ into λ, so there exist distinct $\xi < \eta$ such that $F(\{\alpha_{\xi 0}, \ldots, \alpha_{\xi(k-1)}\}) = F(\{\alpha_{\eta 0}, \ldots, \alpha_{\eta(k-1)}\})$. Hence

$$G(\{\alpha_{\xi 0}, \ldots, \alpha_{\xi(k-1)}, \alpha_{\eta 0}, \ldots, \alpha_{\eta(k-1)}\}) = 1,$$

and (1) follows.

Now we show that H is homogeneous for F. Suppose that $\alpha_0 < \ldots < \alpha_{m-1}$ and $\beta_0 < \ldots < \beta_{m-1}$, all of these ordinals in H. Choose $\gamma_0 < \cdots < \gamma_{m-1}$ in H with $\alpha_{m-1}, \beta_{m-1} < \gamma_0$. Then

$$G(\{\alpha_0, \ldots, \alpha_{m-1}, \gamma_0, \ldots, \gamma_{m-1}\}) = 1 = G(\{\beta_0, \ldots, \beta_{m-1}, \gamma_0, \ldots, \gamma_{m-1}\}),$$

and so

$$F(\{\alpha_0, \ldots, \alpha_{m-1}\}) = F(\{\gamma_0, \ldots, \gamma_{m-1}\}) = F(\{\beta_0, \ldots, \beta_{m-1}\}). \qquad \square$$

Corollary 17.37. *If κ is a Ramsey cardinal and $\mathfrak{A} \supseteq \kappa$ is an $\mathscr{L}$-structure with $|\mathscr{L}| < \kappa$, then $\mathfrak{A}$ has a set of indiscernibles of size κ.* $\square$

Let $\mathscr{L}$ be a first-order language, and κ an infinite cardinal. A *κ-special* $\mathscr{L}$-structure is an $\mathscr{L}$-structure $\mathscr{A}$ such that $\kappa \subseteq A$, there is an indiscernible set $I \subseteq \kappa$ for $\mathscr{A}$, and for every formula $\varphi(y, x_0, \ldots, x_{m-1})$ with $m \geq 0$ there is an n-ary function h_φ on A such that

(i) There is a formula $\psi(y, x_0, \ldots, x_{m-1})$ such that

$$\forall y, x_0, \ldots, x_{m-1} \in A[y = h_\varphi(x_0, \ldots, x_{m-1}) \quad \text{iff} \quad \mathfrak{A} \models \psi(y, x_0, \ldots, x_{m-1}).$$

(ii) For all $x_0, \ldots, x_{m-1} \in A$ there is a $y \in A$ such that if $\mathfrak{A} \models \varphi(y, x_0, \ldots, x_{m-1})$, then $\mathfrak{A} \models \varphi(h_\varphi(x_0, \ldots, x_{m-1}), x_0, \ldots, x_{m-1})]$.

Lemma 17.38. *Let $\mathfrak{A}$ be κ-special with associated set I and functions h_φ. Let B be the closure of I under the functions of $\mathscr{L}$ and the functions h_φ. Then $\mathfrak{B}$ is an elementary substructure of $\mathfrak{A}$.* $\square$

Let $\mathfrak{A}$ be κ-special with associated set I and functions h_φ. Augment $\mathscr{L}$, forming $\mathscr{L}'$, by adding function symbols for all the functions h_φ. Terms in $\mathscr{L}'$ are called *Skolem terms*. Let $\mathfrak{A}'$ be the expansion of $\mathfrak{A}$ by letting the denotation of the function symbol corresponding to h_φ be h_φ. Clearly $\mathfrak{B}$ can be expanded to $\mathfrak{B}'$ so that $\mathfrak{B}'$ is a substructure of $\mathfrak{A}'$.

Lemma 17.39. *Then $\mathfrak{B}'$ is an elementary substructure of $\mathfrak{A}'$.* $\square$

Lemma 17.40. *Let $\mathfrak{A}, \mathfrak{A}', I, \mathfrak{B}, \mathfrak{B}'$ be as above. Let $\psi(x_0, \ldots, x_{m-1})$ be a formula in the expanded language. Suppose that $\alpha_0 < \cdots < \alpha_{m-1}$ and $\beta_0 < \cdots < \beta_{m-1}$.*

(i) $\mathfrak{A}' \models \psi(\alpha_0, \ldots, \alpha_{m-1})$ iff $\mathfrak{A}' \models \psi(\beta_0, \ldots, \beta_{m-1})$.

(ii) $\mathfrak{A}' \models \psi(\alpha_0, \ldots, \alpha_{m-1})$ iff $\mathfrak{B}' \models \psi(\beta_0, \ldots, \beta_{m-1})$.

(iii) $\mathfrak{B}' \models \psi(\alpha_0, \ldots, \alpha_{m-1})$ iff $\mathfrak{A}' \models \psi(\beta_0, \ldots, \beta_{m-1})$.

(iv) $\mathfrak{B}' \models \psi(\alpha_0, \ldots, \alpha_{m-1})$ iff $\mathfrak{B}' \models \psi(\beta_0, \ldots, \beta_{m-1})$.

Proof. There is a formula $\chi(v_0, \ldots, v_{m-1})$ in the original language such that for all $a_0, \ldots, a_{m-1} \in A$, $\mathfrak{A}' \models \psi(a_0, \ldots, a_{m-1})$ iff $\mathfrak{A} \models \chi(a_0, \ldots, a_{m-1})$. Similarly for $\mathfrak{B}$ and $\mathfrak{B}'$.

(i):

$$\begin{aligned} \mathfrak{A}' \models \psi(\alpha_0, \ldots, \alpha_{m-1}) \quad &\text{iff} \quad \mathfrak{A} \models \chi(\alpha_0, \ldots, \alpha_{m-1}) \\ &\text{iff} \quad \mathfrak{A} \models \chi(\beta_0, \ldots, \beta_{m-1}) \\ &\text{iff} \quad \mathfrak{A}' \models \psi(\beta_0, \ldots, \beta_{m-1}). \end{aligned}$$

(ii)-(iv) are proved similarly. $\square$

Theorem 17.41. *If κ is a Ramsey cardinal and λ is an infinite cardinal less than κ, then $|\mathscr{P}^L(\lambda)| = \lambda$.*

Proof.

(1) $\mathscr{P}^L(\lambda) \subseteq L_\kappa$.

In fact, suppose that $A \in \mathscr{P}^L(\lambda)$. So $A \subseteq \lambda$ and A is constructible. Say $\lambda = \aleph_\alpha$. Then there is a limit ordinal $\delta > \alpha$ such that $A \in L_\delta$. Let M be an elementary submodel of $(L_\delta, \in)$ such that $\omega_\alpha \subseteq M$, $A \in M$, and $|M| = \aleph_\alpha$. By Theorem 13.42, the transitive collapse of M has the form L_β with $\beta \le \delta$. Since $\omega_\alpha \subseteq M$, it follows that the transitive collapse function fixes ω_α pointwise. Since $A \subseteq \lambda = \omega_\alpha$, it follows that $A \in L_\beta$. Say $\kappa = \aleph_\gamma$. Since κ is inaccessible and $\lambda < \kappa$, we have $\alpha < \gamma$. Now $|\beta| = |L_\beta|$ by Theorem 13.45. Also, $\aleph_\alpha = |M| = |L_\beta| = |\beta|$. So $\beta < \aleph_{\alpha+1} < \kappa$, so $L_\beta < L_\kappa$. So (1) holds.

Let $\mathscr{L}$ be the language with non-logical constants $\in$, Q (a unary relation symbol) and constants c_α for $\alpha \le \lambda$. Let $\mathfrak{A} = (L_\kappa, \in, \mathscr{P}^L(\lambda), \alpha)_{\alpha \le \lambda}$. By Corollary 17.37, $\mathfrak{A}$ has a set I of indiscernibles of size κ. By Theorem 13.112, $\mathfrak{A}$ is a model of $ZFC + V = L$. Hence it has a definable well-ordering, and hence has definable Skolem functions. Let $\mathfrak{B}$ be the elementary submodel of $\mathfrak{A}$ generated from I by those Skolem functions.

(2) $S \stackrel{\text{def}}{=} \mathscr{P}^L(\lambda) \cap B$ has at most λ elements.

To prove this, first note that $\mathscr{P}^L(\lambda) \cap B$ is the interpretation in $\mathfrak{B}$ of Q.

(3) For any Skolem term $t(x_0, \ldots, x_{n-1})$, $\{t(\alpha_0, \ldots, \alpha_{n-1}) : \alpha_0 < \cdots < \alpha_{n-1} \text{ in } I\}$ has either just one element, or it has κ elements.

In fact,

$$(*) \qquad t(\alpha_0, \cdots, \alpha_{n-1}) = t(\beta_0, \cdots, \beta_{n-1})$$

is either true for all increasing sequences $\alpha_0 < \cdots < \alpha_{n-1}) < \beta_0 < \cdots \beta_{n-1}$ of elements of I, or false for all such. If $(*)$ is true for all increasing sequences $\alpha_0 < \cdots < \alpha_{n-1}) < \beta_0 < \cdots \beta_{n-1}$ of elements of I, then $(*)$ holds whenever $\alpha_0 < \cdots < \alpha_{n-1})$ in I and $\beta_0 < \cdots \beta_{n-1}$ in I. For, suppose that $\alpha_0 < \cdots < \alpha_{n-1})$ in I and $\beta_0 < \cdots \beta_{n-1}$ in I. Choose $\gamma_0 < \cdots < \gamma_{n-1}$ in I with $\alpha_{n-1} < \gamma_0$ and $\beta_{n-1} < \gamma_0$. Then

$$t(\alpha_0 < \cdots < \alpha_{n-1}) = t(\gamma_0 < \cdots \gamma_{n-1}) = t(\beta_0 < \cdots \beta_{n-1}).$$

Thus $\{t(\alpha_0, \ldots, \alpha_{n-1}) : \alpha_0 < \cdots < \alpha_{n-1} \text{ in } I\}$ has just one element. If $(*)$ is false for all increasing sequences $\alpha_0 < \cdots < \alpha_{n-1}) < \beta_0 < \cdots \beta_{n-1}$ of elements of I, choose $\langle \alpha^\xi_i : \xi < \kappa, i < n\rangle$ so that $\alpha^\xi_i < \alpha^\eta_j$ if $\xi < \eta$ or $\xi = \eta$ and $i < j$. Then $\{t(\alpha^\xi_0, \ldots, \alpha^\xi_{n-1}) : \xi < \kappa\}$ has κ elements. This proves (3).

Now $S \subseteq \mathscr{P}^L(\lambda) \subseteq \mathscr{P}(\lambda)$ and $\lambda < \kappa$ and κ is strongly inaccessible, so $|S| < \kappa$. Hence by (3), for any Skolem term t for which $\{t(\alpha_0, \ldots, \alpha_{n-1}) : \alpha_0 < \cdots < \alpha_{n-1} \text{ in } I\}$ has κ elements, one of them, and hence by indiscernibility all of them, are not in S. Hence if one of them is in S, then $\{t(\alpha_0,\ldots,\alpha_{n-1}) : \alpha_0 < \cdots < \alpha_{n-1} \text{ in } I\}$ has only one element. It follows that $|S| \leq \lambda$, proving (2).

Now by Theorem 13.42 the transitive collapse of B is L_κ. Let π be the isomorphism of B with L_κ. Now clearly $\lambda \cup \{\lambda\} \subseteq B$, and so $\pi(X) = X$ for each $X \in B$ such that $X \subseteq \lambda$. In particular, $\forall X \in S[\pi(X) = X]$. Hence $\pi(S) = \mathscr{P}^L(\lambda) \cap L_\kappa = \mathscr{P}^L(\lambda)$ by (1). Since $|S| \leq \lambda$, this completes the proof. □

Corollary 17.42. *If there is a Ramsey cardinal, then $\mathscr{P}^L(\omega)$ is countable.* □

Corollary 17.43. *If there is a Ramsey cardinal, then $V \neq L$.* □

For α a limit ordinal, the *Erdös cardinal* η_α is the least κ such that $\kappa \to (\alpha)^{<\omega}$.

Lemma 17.44. *For κ an uncountable cardinal, κ is a Ramsey cardinal iff $\kappa = \eta_\kappa$.* □

Lemma 17.45. *If $\kappa \to (\alpha)^{<\omega}$, then $\kappa \to (\alpha)^{<\omega}_{2^\omega}$.*

Proof. Let $f : [\kappa]^{<\omega} \to {}^\omega\{0,1\}$. For each $n \in \omega$ let $f_n = f \restriction [\kappa]^n$, and for all $n, k \in \omega$ define for $x \in [\kappa]^n$

$$f_{nk}(x) = h(k), \quad \text{where} \quad h = f_n(x).$$

For m a positive integer there are unique n, k such that $m = 2^n \cdot (2k+1)$. Let $\pi(0) = 0$ and $\pi(m) = (n,k)$. Then $\pi : \omega \to \omega \times \omega$ is a bijection, and $\forall m[m \geq \pi(m)_0]$.

Now for each $m \in \omega$ and each $x \in [\kappa]^m$ define $g_m(x) = f_{nk}(x)$, where $\pi(m) = (n,k)$. Then there is an $H \subseteq \kappa$ of order type α which is homogeneous for all g_m. We claim that H is homogenous for f. Otherwise there is an $n \in \omega$ and $x, y \in [H]^n$ such that $f_n(x) \neq f_n(y)$. Say $f_n(x) = h$ and $f_n(y) = h'$; and say $h(k) \neq h'(k)$. Then $f_{nk}(x) \neq f_{nk}(y)$, hence $g_m(x) \neq g_m(y)$ with $\pi(m) = (n,k)$. This contradicts H being homogeneous for G_m. □

Lemma 17.46. *For every $\kappa < \eta_\alpha$, $\eta_\alpha \to (\alpha)^{<\omega}_\kappa$.*

Proof. Let $\kappa < \eta_\alpha$, and let $f : [\eta_\alpha]^{<\omega} \to \kappa$. Since $\kappa < \eta_\alpha$, there is a $g : [\kappa]^{<\omega} \to 2$ which has no homogeneous set of order type α. For each $n \in \omega$ let $f_n = f \restriction [\eta_\alpha]^n$ and $g_n = g \restriction [\eta_\alpha]^n$. Let $\mathfrak{A} = (V_{\eta_\alpha}, \in, f_n, g_n)_{n\in\omega}$.

By Lemmas 17.35 and 17.45, $\mathfrak{A}$ has a set H of indiscernibles of order type α. We claim

(1) H is homogeneous for f.

To prove (1) it suffices to show

(2) For each $n \in \omega$, $f_n(\{\alpha_0,\ldots,\alpha_{n-1}\}) = f_n(\{\beta_0,\ldots,\beta_{n-1}\})$ holds in $\mathfrak{A}$ for any increasing sequence $\alpha_0 < \cdots < \alpha_{n-1} < \beta_0 < \cdots < \beta_{n-1}$ of indiscernibles.

In fact, assume (2). Suppose that $\alpha_0 < \cdots < \alpha_{n-1}$ and $\beta_0 < \cdots < \beta_{n-1}$ are elements of H. Since α is a limit ordinal, choose $\beta'_0 < \cdots \beta'_{n-1}$ in H such that $\alpha_{n-1} < \beta'_0$ and $\beta_{n-1} < \beta'_0$. Then by (2).

$$f_n(\{\alpha_0,\ldots,\alpha_{n-1}\}) = f_n(\{\beta'_0,\ldots,\beta'_{n-1}\}) = f_n(\{\beta_0,\ldots,\beta_{n-1}\}),$$

as desired in (1).

Now to prove (2), suppose to the contrary that $n \in \omega$ and $\alpha_0 < \cdots < \alpha_{n-1} < \beta_0 < \cdots < \beta_{n-1}$ is an increasing sequence of indiscernibles such that $f_n(\{\alpha_0, \ldots, \alpha_{n-1}\}) \neq f_n(\{\beta_0, \ldots, \beta_{n-1}\})$. Hence by definition

(3) For every increasing sequence $\alpha_0 < \cdots < \alpha_{n-1} < \beta_0 < \cdots < \beta_{n-1}$ of indiscernibles we have $f_n(\{\alpha_0, \ldots, \alpha_{n-1}\}) \neq f_n(\{\beta_0, \ldots, \beta_{n-1}\})$.

We define a sequence $\langle \mu_\xi < \xi < \alpha \rangle$ such that each μ_ξ is an increasing n-sequence of elements of H and if $\xi < \eta$, then the last entry of μ_ξ is less than the first entry of μ_η. Suppose that μ_ξ has been constructed for all $\xi < \eta$, with $\eta < \alpha$. Let ρ be the concatenation of all μ_ξ for $\xi < \eta$.

Case 1. η is a successor ordinal $\nu + 1$. Write $\nu = \omega \cdot \sigma + m$. Then ρ has order type $n \cdot \omega \cdot \sigma + n \cdot m$.

Subcase 1.1. $\sigma = 0$. Then $\rho < \alpha$. Say $\rho + \tau = \alpha$. Then τ is a limit ordinal, and so we can define μ_η to satisfy the required conditions.

Subcase 1.9. $\sigma \neq 0$. Then the order type of ρ is $\omega \cdot \sigma + n \cdot m < \alpha$, and we can proceed as in Subcase 1.1.

Case 9. η is a limit ordinal. Write $\eta = \omega \cdot \sigma$. Then $n \cdot \eta = \eta$, so ρ has order type $\eta < \alpha$, and again we can proceed as in Subcase 1.1.

This finishes the construction of $\langle \mu_\xi < \xi < \alpha \rangle$.

Now for each $\xi < \alpha$ let $\gamma_\xi = f(\mu_\xi)$. If $\xi < \eta$ and $\mathfrak{A} \models f_n(\mu_\xi) > f_n(\mu_\eta)$, then since H is a set of indiscernibles we have $\forall \xi < \eta[\mathfrak{A} \models f_n(\mu_\xi) > f_n(\mu_\eta)]$. and so $\forall \xi < \eta[\gamma_\xi > \gamma_\eta]$, contradiction. It follows from (3) that $\forall \xi < \eta[\gamma_\xi < \gamma_\eta]$. Now we will get a contradiction by proving

(4) $G \stackrel{\text{def}}{=} \{\gamma_\xi : \xi < \alpha\}$ is homogeneous for g.

For, consider the formula

$$g_n(f(\mu_{\xi(0)}), \ldots, f(\mu_{\xi(k-1)})) = g_n(f(\mu_{\eta(0)}), \ldots, f(\mu_{\eta(k-1)})) \tag{5}$$

with $\xi_0 < \cdots < \xi(k-1) < \eta(0) < \cdots < \eta(k-1)$. By indiscernibility, either (5) holds for all $\xi_0 < \cdots < \xi(k-1) < \eta(0) < \cdots < \eta(k-1)$, or it fails for all $\xi_0 < \cdots < \xi(k-1) < \eta(0) < \cdots < \eta(k-1)$. It cannot fail for all $\xi_0 < \cdots < \xi(k-1) < \eta(0) < \cdots < \eta(k-1)$, since g takes only 2 values. So it holds for all $\xi_0 < \cdots < \xi(k-1) < \eta(0) < \cdots < \eta(k-1)$. Now (4) follows by the argument following (2) above. □

Theorem 17.47. *Every Erdös cardinal η_α is inaccessible.*

Proof. η_α is strong limit: Suppose that $\kappa < \eta_\alpha$. Now $2^\kappa \not\to (\alpha)^2_\kappa$, so $2^\kappa < \eta_\alpha$.

Suppose that η_α is singular; let $\kappa = \mathrm{cf}(\eta_\alpha)$. Let $\langle \lambda_\nu : \nu < \kappa \rangle$ is strictly increasing with supremum η_α. For each $\nu < \eta_\alpha$ let $f^\nu : [\lambda_\nu]^{<\omega} \to \{0, 1\}$ have no homogeneous set of order type α. For each $n \in \omega$ let $f^\nu_n = f^\nu \restriction [\lambda_\nu]^n$. Let $\mathfrak{A} = (V_{\eta_\alpha}, \in, \lambda_\nu, f^\nu_n)_{\nu<\kappa, n\in\omega}$. Now $\kappa < \eta_\alpha$ and η_α is strong limit, so $2^\kappa < \eta_\alpha$. Hence by Lemma 17.46, $\eta_\alpha \to (\alpha)^{<\omega}_{2^\kappa}$. Hence by Lemma 17.35, $\mathfrak{A}$ has a set of indiscernibles H of order type α. Choose $\nu < \kappa$ so the λ_ν

is greater than the least element of H. By indiscernibility, all elements of H are less than l_ν. Also by indiscernibility, since f^ν takes only two values, it follows that for each $n \in \omega$,

$$f_n^\nu(\{\alpha_0, \ldots, \alpha_{n-1}\}) = f_n^\nu(\{\beta_0, \ldots, \beta_{n-1}\})$$

for all increasing sequences $\alpha_0 < \cdots < \alpha_{n-1}$, $\beta_0 < \cdots < \beta_{n-1}$ of elements of H. So H is homogeneous for f^ν, contradiction. $\square$

Lemma 17.48. *If α and β are limit ordinals with $\alpha < \beta$, then $\eta_\alpha < \eta_\beta$.*

Proof. Clearly $\eta_\alpha \leq \eta_\beta$. Suppose that $\eta_\alpha = \eta_\beta$. For each $\xi < \eta_\alpha$ there is a $f^\xi : [\xi]^{<\omega} \to \{0,1\}$ such that ξ has no homogeneous subset of order type α. Define $g : [\eta_\beta]^{<\omega} \to \{0,1\}$ by

$$g(\{\xi_0, \ldots, \xi_{n-1}\}) = f^{\xi_{n-1}}(\{\xi_0, \ldots, \xi_{n-1}\}).$$

By the definition of η_β, let H be homogeneous for g with order type β. Thus $H \subseteq \eta_\beta$ and for each $n \in \omega$, g is constant on $[H]^n$. Then for each $\xi \in H$, each $n \in \omega$, and all $F, G \in [H \cap \xi]^n$, we have $g(F \cup \{\xi\}) = g(G \cup \{\xi\})$, and hence $f^\xi(F) = f^\xi(G)$. So $H \cap \xi$ is homogeneous for f_ξ. Hence the order type of each $H \cap \xi$ is less than α. Now $H \subseteq \eta_\beta$, so $H = \bigcup_{\xi < \eta_\beta}(H \cap \xi)$. If H has order type $> \alpha$, then if ξ is the α-th element of H, then $H \cap \xi$ has order type α, contradiction. So H has order type $\leq \alpha < \beta$, contradiction. $\square$

Lemma 17.49. *If S is stationary in κ and $S = A \cup B$ with $A \cap B = \emptyset$, then A is stationary or B is stationary.*

Proof. Suppose that A is not stationary. Let C be club such that $A \cap C = \emptyset$. Let D be any club. Then $\emptyset \neq S \cap C \cap D = B \cap C \cap D$. So B is stationary. $\square$

κ is *ineffable* iff for every system $\langle A_\alpha : \alpha < \kappa \rangle$ of sets $A_\alpha \subseteq \alpha$ there is an $A \subseteq \kappa$ such that $\{\alpha \in \kappa : A \cap \alpha = A_\alpha\}$ is stationary.

Lemma 17.50. *Every ineffable cardinal is weakly compact.*

Proof. Let $f : [\kappa]^2 \to \{0,1\}$ be a partition. For each $\alpha < \kappa$ let $A_\alpha = \{\xi < \alpha : f(\{\xi, \alpha\}) = 1\}$. Let $A \backslash \kappa$ be such that $S \stackrel{\text{def}}{=} \{\alpha < \kappa : A \cap \alpha = A_\alpha\}$ is stationary.

Case 1. $S \cap A$ is stationary. Let $\xi < \alpha$ be in $S \cap A$. Then $\xi \in A \cap \alpha = A_\alpha$, so $f(\{\xi, \alpha\}) = 1$. So $S \cap A$ is homogeneous.

Case 9. $S \backslash A$ is stationary. Let $\xi < \alpha$ be in $S \backslash A$. Then $\xi \notin A_\alpha$, so $f(\{\xi, \alpha\}) = 0$. So $S \backslash A$ is homogeneous. $\square$

Lemma 17.51. *Let M and N be transitive models of ZFC and $j : M \to N$ a non-trivial elementary embedding, with κ the least ordinal moved. Suppose that $\mathscr{P}^M(\kappa) = \mathscr{P}^N(\kappa)$. Then κ is ineffable.*

Proof. Let $\langle A_\alpha : \alpha < \kappa \rangle$ be a system of sets such that $\forall \alpha < \kappa [A_\alpha \subseteq \alpha]$. By Lemma 17.17 we have

(1) $\forall\alpha < \kappa[j(A_\alpha) = A_\alpha]$.

Now let $f = \langle A_\alpha : \alpha < \kappa\rangle$. Then f is a function with domain κ such that $\forall\alpha < \kappa[f(\alpha) \subseteq \alpha]$. Hence $j(f)$ is a function with domain $j(\kappa)$ such that $\forall\alpha < j(\kappa)[(j(f))(\alpha) \subseteq \alpha]$. In particular, $(j(f))(\kappa) \subseteq \kappa$. By hypothesis of the lemma, $(j(f))(\kappa) \in M$. Let $S = \{\alpha < \kappa : (j(f))(\kappa) \cap \alpha = A_\alpha\}$. We claim that S is stationary. For, let $C \subseteq \kappa$ be club. Then $j(C)$ is club in $j(\kappa)$. Note that $j(S) = \{\alpha < j(\kappa) : j((j(f))(\kappa)) \cap \alpha = A_\alpha\}$. By Lemma 17.14, $\kappa \in j(C)$. We claim

$(*)$ $j((j(f))(\kappa)) \cap \kappa = (j(f))(\kappa)$.

For, suppose that $\alpha \in j((j(f))(\kappa)) \cap \kappa$. Say $\alpha = j(\beta)$ with $\beta \in (j(f))(\kappa)$. Then $j(\beta) = \beta$; so $\alpha \in (j(f))(\kappa)$

Conversely, suppose that $\alpha \in (j(f))(\kappa)$. Then $\alpha \in \kappa$, so $j(\alpha) = \alpha$. Hence $\alpha \in j((j(f))(\kappa)) \cap \kappa$. So $(*)$ holds.

Now by $(*)$, $\kappa \in j(S)$. So $j(C) \cap j(S) \neq \emptyset$, hence $C \cap S \neq \emptyset$. □

Theorem 17.52. *If η_ω exists, then there is a weakly compact cardinal below η_ω.*

Proof. Let $\{h_\varphi : \varphi \text{ a formula}\}$ be a system of Skolem functions for V_{η_ω}, and let $\mathfrak{A} = (V_{\eta_\omega}, \in, h_\varphi^{\mathfrak{A}})_{\varphi \text{ a formula}}$. By definition, $\eta_\omega \to (\omega)^{<\omega}$. By Lemma 17.45, $\eta_\omega \to (\omega)^{<\omega}_{2^\omega}$. Then by Lemma 17.33, $\mathfrak{A}$ has a set I of indiscernibles of order type ω. Let $f : I \to I$ be order preserving and different from the identity. Let B be the closure of I under the Skolem functions. Now every element of B has the form $g(\overline{a})$ where g is a composition of $h_\psi^{\mathfrak{A}}$'s and $\overline{a} \subseteq I$. We extend f to B by defining $f(g(\overline{a})) = g(f \circ \overline{a})$. f is well-defined: Suppose that $g(\overline{a}) = g'(\overline{a}')$. Say g is m-ary and g' is n-ary. Let $\overline{c}$ be the increasing enumeration of $\mathrm{rng}(\overline{a}) \cup \mathrm{rng}(\overline{a}')$. Say $\forall i < m[a_i = c_{j(i)}]$ and $\forall i < n[a'_i = c_{k(i)}]$. Then let φ be the formula $g(v_{j(0)}, \ldots, v_{j(m-1)}) = g'(v_{k(0)}, \ldots, v_{k(n-1)})$. Then $\varphi(\overline{c})$, hence $\mathfrak{A} \models \varphi(\overline{c})$, hence $\mathfrak{A} \models \varphi(f \circ \overline{c})$, hence $g(f(c_{j(0)}), \ldots, f(c_{j(m-1)})) = g'(f(c_{k(0)}), \ldots, f(c_{k(n-1)}))$, hence $g(f \circ \overline{a}) = g'(f \circ \overline{a}')$, proving that f is well-defined.

(1) f is an elementary embedding.

In fact, first $f[B]$ is a substructure of B, since $g(f \circ \overline{a}) = f(g(\overline{a})$. Second, if $b \in B$ and $B \models \varphi(b, f \circ \overline{a})$, then $B \models \exists v\varphi(b, f \circ \overline{a})$, and so $B \models \varphi(h_\varphi^{\mathfrak{A}}(f \circ \overline{a}), f \circ \overline{a})$. Hence f is an elementary embedding by Tarski's criterion.

B is clearly extensional. Let e be the isomorphism of $(B, \in)$ onto a transitive set $(M, \in)$. Let $j = e \circ f \circ e^{-1}$. So j is a nontrivial elementary embedding of M into M. Since $B \preceq V_{\eta_\omega}$ and η_ω is inaccessible, V_{η_ω} is a model of ZFC and hence so are B and M. By Lemma 17.51, there is a weakly compact cardinal in M, and hence there is one in V_{η_ω}. □

Theorem 17.53. *If $\kappa \to (\omega)^{<\omega}$, then $L \models \kappa \to (\omega)^{<\omega}$.*

Proof. Assume that $\kappa \to (\omega)^{<\omega}$. Now κ is a cardinal in L, since if in L we have $\alpha < \kappa$ and $f : \alpha \to \kappa$ a bijection, then by absoluteness this holds, contradiction. First we claim:

(1) If $k \in \omega$, F is a finite constructible set, and $G = \{g : g$ is a one-one function with domain k and range included in $F\}$, then G is constructible.

In fact, G is finite and each member of G is constructible, so this is clear.

Now work in L. Suppose that $f : [\kappa]^{<\omega} \to \{0,1\}$. Let

$$\begin{aligned} T = \{t : \exists n \in \omega[&t \text{ is a function } \wedge \operatorname{dmn}(t) = n \wedge \operatorname{rng}(t) \subseteq \kappa \wedge \forall i, j < n \\ &[i < j \to t(i) < t(j)] \wedge \forall k \leq n \exists \varepsilon \in \{0,1\} \forall g \\ &[g \text{ is a one-one function } \wedge \operatorname{dmn}(g) = k \wedge \operatorname{rng}(g) \subseteq \operatorname{rng}(t) \to f(\operatorname{rng}(g)) = \varepsilon]\}. \end{aligned}$$

Now we return to the real world. Since $\kappa \to (\omega)^{<\omega}$, choose $H \subseteq \kappa$ of order type ω such that for each $n \in \omega$, f is constant on $[H]^n$. Define T as above, but now in the real world. Using (1) it follows that T is constructible. Now in the real world, $(T, \supseteq)$ is not well-founded. Since well-foundedness is an absolute property, it follows that in L $(T, \supseteq)$ is not well-founded, and this gives the desired homogeneous set. ☐

Suppose that $\mathscr{L}$ is a countable language with a distinguished unary relation symbol Q. An $\mathscr{L}$-structuer $\mathfrak{A} = (A, Q, \ldots)$ has *type* (κ, λ) iff $|A| = \kappa$ and $|Q| = \lambda$.

A cardinal $\kappa > \aleph_1$ is a *Rowbottom cardinal* iff for every uncountable $\lambda < \kappa$, every model of type (κ, λ) has an elementary submodel of type (κ, ω).

An infinite cardinal κ is a *Jónsson cardinal* iff every structure of size κ has a proper elementary substructure of size κ.

Proposition 17.54. *Every Rowbottom cardinal is a Jónsson cardinal.* ☐

Lemma 17.55. *Let κ be a Ramsey cardinal, and let λ be an infinite cardinal less than κ. Let $\mathscr{L}$ be a language of size $\leq \lambda$, and let $\mathfrak{A}$ be an $\mathscr{L}$-structure such that $\kappa \subseteq A$. Suppose that $P \in [A]^{<\kappa}$.*

Then $\mathfrak{A}$ has an elementary substructure $\mathfrak{B}$ such that $|B| = \kappa$ and $|B \cap P| \leq \lambda$.

If D is a normal κ-additive measure on κ, then we can find B so that $B \cap \kappa \in D$.

Proof. We expand $\mathfrak{A}$ to $\mathfrak{A}' = (A, \ldots, P, x, h_\varphi)_{x \in X, \varphi \in \mathrm{form}}$. Thus the language of $\mathscr{A}$ has size $\leq \lambda$. By Lemma 17.37, there is a set $I \subseteq \kappa$ of indiscernibles of size κ for $\mathscr{A}$. Let B be the elementary submodel of $\mathscr{A}$ generated by I.

(∗) $|P \cap B| \leq \lambda$.

This is proved as in the proof of Theorem 17.41.

The final statement follows by choosing $I \in D$, using Theorem 17.79. ☐

Proposition 17.56. *If U is a nonprincipal ultrafilter on ω, then $\mathrm{Ult}_{\omega U}$ is not well-founded.*

Proof. For each $k \in \omega$ define $f_k : \omega \to \omega$ by

$$f_k(n) = \begin{cases} n - k & \text{if } k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

If $k < l$ and $l \leq n$, then $n - l < n - k$, and so $\{n \in \omega : f_l(n) < f_k(n) \supseteq \omega \backslash l$, so that $f_l <^* f_k$. ☐

Lemma 17.57. *If U is an ultrafilter on κ, then the following conditions are equivalent:*

(i) U is not λ-complete.

(ii) There is a partition $\mathscr{P}$ of κ such that $|\mathscr{P}| < \lambda$ and $A \notin U$ for all $A \in \mathscr{P}$.

Proof. (ii)⇒(i): Assume (ii). Then $\forall A \in \mathscr{P}[\kappa\backslash A \in U]$, and $\bigcap_{A\in\mathscr{P}}(\kappa\backslash A) = \emptyset \notin U$. So (i) holds.

(i)⇒(ii): Assume that U is not λ-complete. Say $\mu < \lambda$, $A_\alpha \in U$ for all $\alpha < \mu$, but $\bigcap_{\alpha<\mu} A_\alpha \notin U$. Define

$$B_\alpha = \begin{cases} (\kappa\backslash A_0) \cup \bigcap_{\alpha<\mu} A_\alpha & \text{if } \alpha = 0, \\ (\kappa\backslash A_\alpha) \cap \bigcap_{\beta<\alpha} A_\beta & \text{if } \alpha \neq 0. \end{cases}$$

Clearly $\{B_\alpha : \alpha < \mu\}$ is a partition as desired in (i). □

Lemma 17.58. *If U is an ultrafilter on κ and U is not countably complete, then* $\mathrm{Ult}_{\kappa U}$ *is not well-founded.*

Proof. By Proposition 17.57 let $\langle A_n : n \in \omega\rangle$ be a partition of κ such that $\forall n \in \omega[\kappa\backslash A_n \in U]$. Define $f_k : \kappa \to \{0,1\}$ by

$$f_k(\alpha) = \begin{cases} n-k & \text{if } \alpha \in A_n \text{ and } k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Now for any k and n, if $k+1 \leq n$ and $\alpha \in A_n$, then $f_k(\alpha) = n-k > n-(k+1) = f_{k+1}(\alpha)$. So

$$\{\alpha : f_k(\alpha) > f_{k+1}(\alpha) \supseteq (A_{k+1} \cup A_{k+2} \cup \ldots) \in U.$$

So $[f_{k+1}] < [f_k]$. □

Lemma 17.59. *If Ult is well founded, then every ordinal is represented by a function $f : S \to$* **ON**.

Proof. Suppose that α is an ordinal. Say $\alpha = \pi([f])$, where f is a function with domain κ. Then Ult $\models [f]$ is an ordinal, so $\{\alpha < \kappa : f(\alpha) \text{ is an ordinal}\} \in U$. Define

$$g(\alpha) = \begin{cases} f(\alpha) & \text{if } f(\alpha) \text{ is an ordinal,} \\ 0 & \text{otherwise.} \end{cases}$$

Then $[f] = [g]$, as desired. □

Lemma 17.60. *If U is the principal ultrafilter $\{X \in S : x_0 \in S\}$, then $\pi([f]) = f(x_0)$ and $j(a) = a$ for all $a \in V$.*

Proof. Note that $[g] \in^* [f]$ iff $\{\alpha \in \kappa : g(\alpha) \in f(\alpha)\} \in U$ iff $g(x_0) \in f(x_0)$. Now we prove that $\pi([f]) = f(x_0)$ by induction on $[f]$ (o.k. since Ult_U is well-founded). $\pi([f]) = \{\pi([g]) : [g] \in^* [f]\} = \{g(x_0) : g(x_0) \in f(x_0)\} = f(x_0)$.

Finally, $j(a) = \pi([c_a]) = c_a(x_0) = a$. □

Lemma 17.61. *Let U be a nonprincipal σ-complete ultrafilter on S, and λ the largest cardinal such that U is λ-complete. Then $j_U(\lambda) > \lambda$.*

Proof. By Lemma 17.57 there is a partition $\langle X_\alpha : \alpha < \lambda\rangle$ of S into sets not in U. Note that some X_α's might be empty. Define $f : S \to \kappa$ by leting $f(s)$ be the $\alpha < \lambda$ such that $s \in X_\alpha$. Then

$$\begin{aligned}
\{x : \alpha < f(x)\} &= \{x : \exists\beta > \alpha[x \in X_\beta]\}\\
&= \bigcup_{\beta>\alpha}\{x : x \in X_\beta\}\\
&= \bigcup_{\beta>\alpha} X_\beta\\
&= S\backslash \bigcup_{\beta\le\alpha} X_\beta\\
&= \bigcap_{\beta\le\alpha}(S\backslash X_\beta) \in U.
\end{aligned}$$

Hence $\alpha < \pi([f])$. So $\lambda \le \pi([f])$. Now $[f] \in^* [c_\lambda]$, so $\pi([f]) < j(\lambda)$. Hence $\lambda < j(\lambda)$. □

Lemma 17.62. *If M is a transitive class and $j : V \to M$ is an elementary embedding, then $M = \bigcup_{\alpha\in\mathbf{ON}} j(V_\alpha)$.*

Proof. Assume that $j : V \to M$ is an elementary embedding, where M is a transitive class.

(1) If $x, y \in M$ and $M \models x \subseteq y$, then $x \subseteq y$.

For, $M \models \forall z[z \in x \to z \in y]$, so $x \subseteq y$.

(2) $x \subseteq y \to j(x) \subseteq j(y)$.

In fact, assume that $x \subseteq y$. Then $\forall z[z \in x \to z \in y]$, so $M \models \forall z[z \in j(x) \to z \in j(y)]$, hence $\forall z[z \in j(x) \to z \in j(y)]$, and (2) holds.

(3) $M \models \forall x[x \in j(V_{\alpha+1}) \leftrightarrow x \subseteq j(V_\alpha)]$.

In fact,

$$\begin{aligned}
&\forall x[x \in V_{\alpha+1} \leftrightarrow \forall z[z \in x \to z \in V_\alpha]], \quad \text{so}\\
M \models\ &\forall x[x \in j(V_{\alpha+1}) \leftrightarrow \forall z[z \in x \to z \in j(V_\alpha)]] \quad \text{hence}\\
M \models\ &\forall x[x \in j(V_{\alpha+1}) \leftrightarrow x \subseteq j(V_\alpha)].
\end{aligned}$$

(4) $\forall x[x \in j(V_{\alpha+1}) \leftrightarrow x \subseteq j(V_\alpha)]$.

(5) $\forall\alpha[V_\alpha \cap M \subseteq j(V_\alpha)]$.

We prove this by induction on α. It is clear for $\alpha = 0$. Now assume it for α, and let $x \in V_{\alpha+1} \cap M$. Thus $x \subseteq V_\alpha$. If $a \in x$, then $a \in V_\alpha \cap M$, so $a \in j(V_\alpha)$. Hence $x \subseteq j(V_\alpha)$, so by (6), $x \in j(V_{\alpha+1})$. Now suppose inductively that γ is a limit ordinal and $x \in V_\gamma \cap M$. Say $x \in V_\alpha$ with $\alpha < \gamma$. Then $x \in j(V_\alpha)$ by the inductive hypothesis. Then $x \in j(V_\gamma)$ by (2).

The lemma is clear from (5). □

Lemma 17.63. *If κ is measurable, then there is a normal ultrafilter D on κ such that* $\mathrm{Ult}_D \models \kappa$ *is not measurable.*

Proof. By Theorem 17.11 there is a normal ultrafilter on κ. Let D be a normal ultrafilter on κ such that $j_D(\kappa)$ is minimum. Let $M = \mathrm{Ult}_D(V)$. We claim that κ is not measurable in M. For, suppose that it is. Then by Theorem 17.11 there is a normal ultrafilter U on κ, with $U \in M$. Now by Lemma 17.17(v), $\mathscr{P}(\kappa) \subseteq M$. Hence $\mathrm{Ult}_U(\kappa) \in M$. By Lemma 17.9(iii), $j_U(\kappa) < (2^\kappa)^+)^M$. By Lemma 17.18(iii), $((2^\kappa)^+)^M \leq j_D(\kappa)$. This contradicts the minimality of D.

Lemma 17.64. *Let U be a nonprincipal κ-complete ultrafilter on κ. Then the following are equivalent:*

(i) Every club is in U.

(ii) There is no $f \in {}^\kappa\kappa$ such that $\pi([f])$ is nonconstant, increasing, and $\pi([f]) < \pi([d])$.

Proof. (i)⇒(ii): Suppose that (i) holds and $\pi([f])$ is a non-constant function with $\pi([f]) < \pi([d])$. Note that U consists of stationary sets. Thus $X = \{\alpha < \kappa : f(\alpha) < \alpha\} \in U$, so there is a stationary $Y \subseteq X$ on which f is constant, contradiction.

⇐: If there is a club C which is not in U, define $f(\alpha) = \sup(C \cap \alpha)$ for every $\alpha < \kappa$. Now $\kappa \backslash C \in U$. Suppose that $\alpha \in \kappa\backslash C$. If $\alpha = \beta + 1$, then $f(\alpha) \leq \beta < \alpha$. If α is limit and $f(\alpha) = \alpha$, then $\alpha \in C$, contradiction. So f is regressive on C. Hence $\pi([f]) < \pi([d])$. Clearly f is monotone. □

Lemma 17.65. *Recall that $h_*(U) = \{X \subseteq \kappa : h^{-1}[X] \in U\}$.*

Suppose that U is a κ-complete ultrafilter on κ and $h : \kappa \to \kappa$. Let $D = h_*(U)$. Define $k : \pi(\mathrm{Ult}_D(V)) \to \pi(\mathrm{Ult}_U(V))$ by $k(\pi([f]_D)) = \pi([f \circ h]_U)$. Then k is a well-defined elementary embedding.

Proof. If $\pi([f]_D) = \pi([f']_D)$, then $[f]_D = [f']_d$ and so $\{\alpha < \kappa : f(\alpha) = f'(\alpha)\} \in D$. Hence $h^{-1}[\{\alpha < \kappa : f(\alpha) = f'(\alpha)\}] \in U$, i.e., $\{\alpha < \kappa : f(h(\alpha)) = f'(h(\alpha))\} \in U$, so $[f \circ h]_D = [f' \circ h]_D$. So k is well-defined. Next,

$$\begin{aligned}
\mathrm{Ult}_D(V) \models \varphi(\pi([f^0]), \ldots, \pi([f^{m-1}])) \quad &\text{iff} \quad \{\alpha \in \kappa : \varphi(f^0(\alpha), \ldots, f^{m-1}(\alpha))\} \in D \\
&\text{iff} \quad h^{-1}[\{\alpha \in \kappa : \varphi(f^0(\alpha), \ldots, f^{m-1}(\alpha))\}] \in U \\
&\text{iff} \quad \{\alpha \in \kappa : \varphi(f^0(h(\alpha)), \ldots, f^{m-1}(h(\alpha))\} \in U \\
&\text{iff} \quad \mathrm{Ult}_D(V) \models \varphi(\pi([f^0 \circ h]), \ldots, \pi([f^{m-1} \circ h])) \\
&\text{iff} \quad \mathrm{Ult}_D(V) \models \varphi(k(\pi([f^0])), \ldots, k(\pi([f^{m-1}]))).
\end{aligned}$$

□

Lemma 17.66. *If D is a normal ultrafilter on κ and $\{\alpha < \kappa : 2^\alpha \leq \alpha^{++}\} \in D$, then $2^\kappa \leq \kappa^{++}$. More generally, if $\beta < \kappa$ and $\{\aleph_\alpha : 2^{\aleph_\alpha} \leq \aleph_{\alpha+\beta}\} \in D$, then $2^{\aleph_\kappa} \leq \aleph_{\kappa+\beta}$.*

Proof. Let $M = \mathrm{rng}(j_D)$. By induction,

(1) $\forall\alpha[\aleph_\alpha^M \le \aleph_\alpha]$.

(2) $\forall\alpha < \kappa[\aleph_\alpha^M = \aleph_\alpha]$.

For, if $\alpha < \kappa$ then $\aleph_\alpha < \kappa$, and so $j(\aleph_\alpha) = \aleph_\alpha$, and (2) holds.

(3) $\aleph_\kappa^M = \aleph_\kappa$.

(4) $\{\alpha < \kappa : \alpha \text{ limit}\} \in D$.

(5) $\{\alpha < \kappa : \alpha \text{ is an infinite cardinal}\} \in D$.

Otherwise $X \stackrel{\text{def}}{=} \{\alpha < \kappa : \alpha \text{ is limit but not a cardinal}\} \in D$. For each $\alpha \in X$ choose $\beta < \alpha$ and h_β a bijection from β onto α. Using a regressive function, there is a $Y \subseteq X$ with $Y \in D$ such that there is a β such that for each $\alpha \in Y$ h_β is a bijection from α onto α. Clearly this is impossible.

Now it is clear that the second statement of the exercise implies the first.

Now let $\varphi(x, y)$ be the formula $2^{\aleph_x} \le \aleph_{x+\beta}$. Then $\{\alpha < \kappa : 2^{\aleph_\alpha} \le \aleph_{\alpha+\beta} \in D$, so by Theorem 17.3, $(2^{\aleph_\kappa} \le \aleph_{\kappa+\beta})^M$. Now $2^\kappa \le (2^\kappa)^M$ by Theorem 17.18(iii), and clearly $(\aleph_{\kappa+\beta})^M \le \alpha_{\kappa+\beta}$. □

Lemma 17.67. *If D is a normal ultrafilter on κ and $\{\alpha < \kappa : 2^{\aleph_\alpha} < \aleph_{\alpha+\alpha}\} \in D$, then $2^{\aleph_\alpha} < \aleph_{\kappa+\kappa}$.*

Proof. $\{\alpha < \kappa : 2^{d(\aleph_\alpha)} < \aleph_{d(\aleph_\alpha)+d(\aleph_\alpha)}\} \in D$, so $M \models 2^\kappa < \aleph_{\kappa+\kappa}$. By (3) in the proof of Lemma 17.66, $(2^\kappa)^M = 2^\kappa$. By (1) in the proof of Lemma 17.66, $\aleph_{\kappa+\kappa}^M \le \aleph_{\kappa+\kappa}$. □

Lemma 17.68. *Assume that κ is measurable and $\lambda \stackrel{\text{def}}{=} \aleph_{\kappa+\kappa}$ is strong limit. Then $2^\lambda < \aleph_{(2^\kappa)^+}$.*

Proof. Let D be a normal κ-complete ultrafilter on κ. Then by Lemma 17.21, $2^\lambda < j(\lambda)$. Now with $\varphi(x, y)$ the formula $y = \aleph_x$ we have $\kappa = \{\alpha < \kappa : \varphi(\alpha + \alpha, \aleph_{\alpha+\alpha})\}$, and so $j(\lambda) = j(\aleph_{\kappa+\kappa}) = \aleph_{j(\kappa+\kappa)}$. Thus $j(\lambda) = \aleph_{j(\kappa+\kappa)}^M$.

(1) $\forall\beta < \kappa[j(\kappa + \beta) = j(\kappa) + \beta$.

We prove this by induction on β. It is clear for $\beta = 0$. Assuming it for β, by (2) on page 212, $j(\kappa + \beta + 1) = j(\kappa + \beta) + 1 = j(\kappa) + \beta + 1$. Now suppose it is true for all $\beta < \gamma$, with γ limit less than κ. Then $j(\kappa) + \gamma \le j(\kappa + \gamma)$. Suppose that $j(\kappa) + \gamma < j(\kappa + \gamma)$. Say $j(\kappa) + \gamma = j(a)$. Then $\{\alpha < \kappa : a(\alpha) < \kappa + \gamma\} \in D$. Now

$$\{\alpha < \kappa : a(\alpha) < \kappa + \gamma\} = \bigcup_{\beta<\gamma} \{\alpha < \kappa : a(\alpha) < \kappa + \beta\},$$

so, since D is κ-complete, there is a $\beta < \gamma$ such that $\{\alpha < \kappa : a(\alpha) < \kappa + \beta\} \in D$. Hence $j(\kappa) + \gamma = j(a) < j(\kappa) + \beta$, contradiction. Thus (1) holds.

(2) $j(\kappa + \kappa) = j(\kappa) + j(\kappa)$.

For, $c_{\kappa+\kappa} = c_\kappa + c_\kappa$, so $\{\alpha < \kappa : c_{\kappa+\kappa}(\alpha) = c_\delta(\alpha) + c_\kappa(\alpha)\} = \kappa \in D$, so $j(\kappa+\kappa) = j(\kappa)+j(\kappa)$.

Now by Lemma 17.9(iii), $j(\kappa) + j(\kappa) < (2^\kappa)^+$. Hence $2^\lambda < \aleph_{(2^\kappa)^+}$. □

Lemma 17.69. *If κ is measurable, λ is strong limit,* $\mathrm{cf}(\lambda) = \kappa$, *and $\lambda < \aleph_\lambda$, then $2^\lambda < \aleph_\lambda$.*

Proof. Let D be a normal κ-complete ultrafilter on κ. Say $\lambda = \aleph_\alpha$. Since j_D is an elementary embedding, $j_D(\aleph_\alpha) = \aleph^M_{j_D(\alpha)} \le \aleph_{j_D(\alpha)}$. Clearly $j_D(\alpha) < (\alpha^\kappa)^+$. Since $\lambda < \aleph_\lambda$, we have $\alpha < \lambda$. Also clearly $\kappa < \lambda$. Hence $(\alpha^\kappa)^+ < \lambda$. By Lemma 17.18,

$$2^\lambda < j_D(\lambda) = j_D(\aleph_\alpha) \le \aleph_{j_D(\alpha)} < \aleph_{(\alpha^\kappa)^+} < \aleph_\lambda.$$

Lemma 17.70. *If $\kappa = \lambda^+$, then the weak compactness theorem for $\mathscr{L}_{\kappa\omega}$ is false.*

Proof. Assume that $\kappa = \lambda^+$. Let $\mathscr{L}$ be the language with the following symbols:
Individual constants c_α for $\alpha \le \kappa$.
A binary relation $<$.
A ternary relation $\mathbf{R}$.

Let Σ be the following set of sentences:
(1) $<$ is a linear ordering.
(2) $c_\alpha < c_\beta$ for $\alpha < \beta \le \kappa$.
(3) For all x, y there is at most one z such that $\mathbf{R}(x, y, z)$.
(4) $\forall x, z[z < x \to \exists y \mathbf{R}(x, y, z)]$.
(5) $\forall x, y, z[\mathbf{R}(x, y, z) \to \bigvee_{\xi<\lambda}(y = c_\xi)]$.

(6) Σ does not have a model.

For, suppose that $\overline{A}$ is a model of Σ. For each $\xi < \kappa$ choose $\eta_\xi < \lambda$ such that $(c^{\overline{A}}_\kappa, c^{\overline{A}}_{\eta_\xi}, c_\xi) \in \mathbf{R}^{\overline{A}}$; this is possible by (4) and (5). Then there exist distinct $\xi, \xi' < \kappa$ such that $\eta_\xi = \eta_{\xi'}$. This contradicts (3).

(7) Every subset S of Σ of size $\le \lambda$ has a model.

In fact, let $S \in [\Sigma]^{\le\lambda}$. We may assume that $c_0 < c_\xi$ is a member of S for each $\xi \in \lambda\backslash\{0\}$. Let $A = \{\xi \le \kappa : c_\xi$ occurs in some member of $S\}$. Thus $\lambda \subseteq A$ by the previous assumption. Let $c^{\overline{A}}_\xi = \xi$ for each $\xi \in A$. Let $<^{\overline{A}} = <$ on A. For each $\xi \in A$ the set $B_\xi \stackrel{\text{def}}{=} \{\eta \in A : \eta < \xi\}$ has size $\le \lambda$, so we can let f_ξ be a bijection from some subset C_ξ of λ onto B_ξ. Let

$$\mathbf{R}^{\overline{A}} = \{(\xi, \eta, f_\xi(\eta)) : \xi \in A, \eta \in C_\xi, \},$$

Clearly this gives a model of S.

Lemma 17.71. *If κ is singular, then the weak compactness theorem for $\mathscr{L}_{\kappa\omega}$ is false.*

Proof. (Following Dickmann.) Suppose that κ is singular. Say $\langle\gamma_\xi : \xi < \mathrm{cf}(\kappa)\rangle$ is a strictly increasing sequence of cardinals with supremum κ. Let $\mathscr{L}$ be the language with individual constants c_α for $\alpha < \kappa$. Let Σ be the following set of sentences:

(1) $c_\alpha \neq c_\beta$ for $\alpha \neq \beta$.

(2) $\bigvee_{\xi<\mathrm{cf}(\kappa)} \forall x \bigvee_{\alpha<\gamma_\xi}(x = c_\alpha)$.

Then Σ does not have a model, since by (1) the model would be of size at least κ, while by (2) its size is $\leq \gamma_\xi$ for some $\xi < \mathrm{cf}(\kappa)$.

If S is a subset of Σ of size less than κ, let $A = \{\alpha < \kappa : c_\alpha$ occurs in some formula of type (1) which is in $S\}$. We may assume that $0 \in A$. So $|A| < \kappa$. Take $\xi < \mathrm{cf}(\kappa)$ such that $|A| < \gamma_\xi$. Let $f : A \to \gamma_\xi$ be an injection which is the identity on $A \cap \gamma_\xi$; this is possible since $|A| < \gamma_\xi$. We define for $\alpha < \kappa$,

$$c_\alpha^{\overline{A}} = \begin{cases} \alpha & \text{if } \alpha \in A, \\ f^{-1}(\alpha) & \text{if } \alpha \in \mathrm{rng}(f) \backslash A, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly the instances of (1) which are in S hold in $\overline{A}$. For (2), suppose that $\alpha \in A$. If $\alpha < \gamma_\xi$ then $\alpha = c_\alpha^{\overline{A}}$. Suppose that $\alpha \notin \gamma_\xi$. Then $f(\alpha) \notin A$, as otherwise $f(\alpha) \in A \cap \gamma_\xi$, hence $f(f(\alpha)) = f(\alpha)$, hence $f(\alpha) = \alpha$, contradicting $\alpha \notin \gamma_\xi$. It follows that $c_{f(\alpha)}^{\overline{A}} = \alpha$, as desired. So (2) holds. □

Lemma 17.72. *The least measurable cardinal is Σ_1^2-describable.*

Proof. Suppose that κ is the least measurable cardinal, and it is Σ_1^2-indescribable. Let σ be the sentence $\exists U[U$ is a κ-complete nonprincipal ultrafilter on $\kappa]$. This is a Σ_1^2 sentence holding in $(V_\kappa, \in, U)$: with $\mathscr{U}$ third order, X, Y, Z second order and x, y, λ first order,

$$\begin{aligned}
&\exists \mathscr{U}[\forall X \in \mathscr{U} \forall x \in X[x \text{ is an ordinal}] \wedge \exists X[X \in \mathscr{U}] \wedge \\
&\forall X \in \mathscr{U} \forall Y[\forall x[x \in X \to x \in Y] \wedge \forall y \in Y[y \text{ is an ordinal}] \to Y \in \mathscr{U}] \wedge \\
&\forall x, y[x \text{ is an ordinal } \wedge y = \{x\} \to y \notin \mathscr{U}] \wedge \\
&\forall X, Y, Z[\forall x[x \in X \leftrightarrow x \text{ is an ordinal }] \wedge \forall x \in X[x \in Y \vee x \in Z] \wedge \\
&\qquad \forall x[x \in Y \to x \notin Z] \to Y \in \mathscr{U} \vee Z \in \mathscr{U}] \\
&\forall \lambda[\lambda \text{ is a cardinal } \to \forall Z \forall \alpha[\alpha \in \lambda \to \forall X[\forall \beta[\beta \in X \leftrightarrow (\alpha, \beta) \in Z] \to X \in \mathscr{U}]] \to \\
&\qquad \forall X[\forall \beta[\beta \in X \leftrightarrow \forall \alpha[\alpha \in \lambda \wedge (\alpha, \beta) \in Z]] \to X \in \mathscr{U}]].
\end{aligned}$$

If κ is indescribable, then this holds in $(V_\alpha, \in, U \cap V_\alpha)$ for some $\alpha < \kappa$. So α is measurable, contradiction. □

Lemma 17.73. *If A is a complete BA, then for every $X \subseteq A$ there is a disjoint $Y \subseteq A$ such that $\sum X = \sum Y$.*

Proof. Given $X \subseteq A$, let Y be maximum such that Y is disjoint and $\forall y \in Y \exists x \in X[y \leq x]$. Clearly $\sum Y \leq \sum X$. Suppose that $\sum X \cdot -\sum Y \neq 0$. Then there is an $x \in X$ such that $x \dot{-} \sum Y \neq 0$. So $Y \subset Y \cup \{x \cdot -\sum Y\}$, contradiction. □

Lemma 17.74. *If κ is weakly compact, then there is no countably completely generated complete BA B of power κ.*

Proof. Suppose that B is a countably completely generated complete BA B of power κ.

(1) $\mathrm{sat}(B) = \kappa$.

In fact, suppose that $\mathrm{sat}(B) = \lambda < \kappa$. Then every disjoint subset of B is less than λ. Let X be a countable set completely generating B. Define $Y_0 = X$. If Y_α has been defined $(\alpha < \lambda)$, let

$$Y_{\alpha+1} = Y_\alpha \cup \{-x : x \in Y_\alpha\} \cup \{\sum Z : Z \subseteq Y_\alpha, |Z| < \lambda\}.$$

For α limit less than λ let $Y_\alpha = \bigcup_{\beta<\alpha} Y_\beta$. Then $\bigcup_{\alpha<\lambda} Y_\alpha = B$. By induction, $|Y_\alpha| \le \max(|X|, 2^\lambda)$ for all $\alpha < \lambda$. So $|B| \le \max(|X|, 2^\lambda)$, contradiction. So (1) holds.

We may assume that the universe of B is κ. Let $\mathscr{L}$ be the language with symbols $+, \cdot, -$ of obvious arity, R_0, R_1, unary relation symbols. Let σ be the following sentence, where U is a second order variable

(1) Axioms for BA. For example, $\forall x \in R_0 \forall y \in x[y \text{ is an ordinal}]$; $\forall x, y \in R_0 \exists z \in R_0 \forall w[w \in z \leftrightarrow w \in x + y]$.

(2) $R_1 \subseteq R_0$.

(3) $\forall x[\forall y \in x[y \text{ is an ordinal}] \to \exists z \in R_0[\forall y \in x[y \le z] \wedge [\forall w \in R_0[\forall y \in x[y \le w] \to z \le w]]]]]$. (Every subset of κ of size less than κ has a sum in B.)

(4) $\forall U[\forall x \in U \forall y \in x[y \in R_0 \wedge \forall x \in U[x \ne 0] \wedge \forall x, y \in U[x \cdot y = 0] \wedge \forall x \in R_0[\forall y \in U[y \le x] \to x = 1] \to \exists x \in R_0 \forall y[y \in x \leftrightarrow y \in U]]$. (Every disjoint subset of B has size less than κ.)

Then $(V_\kappa, \in, B, X) \models \sigma$. Hence there is an $\alpha < \kappa$ such that $(V_\alpha, \in, B \cap V_\alpha, X \cap V_\alpha \models \sigma$. Now α must be infinite, so $X \cap V_\alpha = X$. Hence $B \cap V_\alpha$ is a complete BA containing X of size $|\alpha| < \kappa$, contradiction.

Lemma 17.75. *η_ω is not weakly compact.*

Proof. Note that η_ω is the least κ such that

$$\forall F : [\kappa]^{<\omega} \to 2 \exists H \in [\kappa]^\omega \forall n[F \restriction [H]^n \text{ is constant.}]$$

Now it suffices to show that η_ω is Π^1_1-describable. Note that if $n \in \omega$ and $a \in [\kappa]^{<\omega}$, then $a \in V_\kappa$. In fact, $a \subseteq \beta$ for some $\beta < \kappa$, so $a \in V_{\beta+1} \subseteq V_\kappa$. Also, if $F : [\kappa]^{<\omega} \to 2$, then $F \subseteq V_\kappa$. In fact, if $(a, \delta) \in F$ then $a \in V_\kappa$ and $\delta \in V_\kappa$, so $(a, \delta) \in V_\kappa$. Also, clearly $[\kappa]^\omega \subseteq V_\kappa$.

Let σ say the following:

$$\forall F[(F : [\kappa]^{<\omega} \to 2) \to \exists H \in [\kappa]^\omega \forall n \exists \delta \in 2 \forall x \in [H]^n [F(x) = \delta]]].$$

Thus $(V_\kappa, \in, U) \models \sigma$. If κ is Π^1_1-indescribable, then $(V_\alpha, \in, U \cap v_\alpha) \models \sigma$, so $\eta_\omega \leq \alpha$, contradiction.

Lemma 17.76. *An infinite cardinal κ is a Jónsson cardinal iff for every $F : [\kappa]^{<\omega} \to \kappa$ there is a set $H \in [\kappa]^\kappa$ such that $f[[H]^{<\omega}] \neq \kappa$.*

Proof. $\Rightarrow$: Assume that κ is a Jónsson cardinal. Suppose that $F : [\kappa]^{<\omega} \to \kappa$. For each $n \in \omega$ define $F_n : {}^n\lambda \to \kappa$ by setting, for any $a \in {}^n\kappa$, $F_n(a) = F(\{a_i : i < n\})$. Let $\overline{A}$ be a proper elementary submodel of $(\kappa, F_0, F_1, \ldots)$. Then $F[[A]^{<\omega}]$ is a proper subset of κ. In fact, $F[[A]^{<\omega}] \subseteq A$.

$\Leftarrow$: Assume the indicated condition, and suppose that $\overline{A}$ is a model of size κ. Let $\langle h_n : n \in \omega\rangle$ be a system of Skolem functions for $\overline{A}$ closed under composition, each h_n n-ary. Define $F(x) = h_n(y)$, where y is an enumeration of x. □

Lemma 17.77. *If κ is a Rowbottom cardinal, then either κ is regular limit or* $\mathrm{cf}(\kappa) = \omega$.

Proof.

(1) $\kappa = \lambda^+$ is not Rowbottom if $\lambda > \aleph_0$.

For, for each $\alpha \in [\lambda, \kappa)$ let f_α be a bijection of α onto λ. Let $R = \{(\alpha, \beta, \gamma) : \alpha \in [\lambda, \kappa), \beta < \alpha$, and $f_\alpha(\beta) = \gamma\}$. Suppose that $\overline{B} \preceq \overline{A} \stackrel{\text{def}}{=} (\kappa, \lambda, <, R)$ and $|B| = \kappa$. Let α be the λ-th element of B. Then for each $\beta < \alpha$ with $\beta \in B$, $\overline{A} \models \exists!\gamma\mathbf{R}(\alpha, \beta, \gamma)$, and hence $\overline{B} \models \exists!\gamma\mathbf{R}(\alpha, \beta, \gamma)$. Hence $|B \cap \lambda| = \lambda > \aleph_0$.

(2) $\aleph_1$ is not Rowbottom.

This is true by definition.

(3) If κ is limit and $\mathrm{cf}(\kappa) > \omega$, then κ is not Rowbottom.

For, suppose that κ is limit and $\lambda \stackrel{\text{def}}{=} \mathrm{cf}(\kappa) > \omega$. Let $f : \lambda \to \kappa$ be strictly increasing with supremum κ. Let $\overline{A}(\kappa, \lambda, f)$. Suppose that $\overline{B} \preceq \overline{A}$ and $|B| = \kappa$. Say a has domain κ and range B and is strictly increasing. Now $\overline{A} \models \forall\alpha\exists\beta \in \lambda[\alpha < f(\beta)]$, so $\overline{B} \models \forall\alpha\exists\beta_\alpha \in B \cap \lambda[\alpha < f(\beta_\alpha)]$. Say $f(\beta_\alpha) = a_{\xi(\alpha)}$ for each $\alpha \in B$. If $|B \cap \lambda| < \lambda$, then there is an $\eta < \kappa$ such that $\xi_\alpha < \eta$ for all $\alpha < \kappa$. Then $a_\eta < f(\beta_{a_\eta}) = a_{\xi(a_\eta)} < a_\eta$, contradiction. □

Theorem 17.78. *If κ is measurable and D is a normal κ-complete ultrafilter on κ, $\lambda < \kappa$, m is a positive integer, and $f : [\kappa]^m \to \lambda$, then there is an $X \in D$ which is homogeneous for f.*

Proof. We prove this by induction on m. For $m = 1$ we have $f : \kappa \to \lambda$. Hence $\kappa = \bigcup_{\alpha<\lambda} f^{-1}[\{\alpha\}]$, so there is an $\alpha < \lambda$ such that $f^{-1}[\{\alpha\}] \in D$, as desired.

Now assume the result for $m > 0$ and let $f : [\kappa]^{m+1} \to \lambda$. For each $\sigma \in [\kappa]^m$ and $i < \lambda$ let $S'_{\sigma i} = \{\alpha : \alpha \in \kappa\backslash\sigma$ and $f(\sigma \cup \{\alpha\}) = i\}$.

(1) $\forall\sigma \in [\kappa]^m[\sigma \cup \bigcup_{i<\lambda} S'_{\sigma i} = \kappa]$.

(2) $\forall\sigma \in [\kappa]^m \forall i, i' < \lambda[i \neq i' \to S'_{\sigma i} \cap S'_{\sigma i'} = \emptyset]$.

(3) $\forall\sigma \in [\kappa]^m \exists! i_\sigma < \lambda[S'_{\sigma i_\sigma} \in D]$.

For each $\sigma \in [\kappa]^m$ let $S_\sigma = S'_{\sigma i_\sigma}$. For each $A \subseteq \kappa$ let

$$S^A = \begin{cases} \kappa\backslash A & \text{if } |A| < m, \\ \bigcap\{S_\sigma : \sigma \in [A]^m\} & \text{if } |A| \geq m. \end{cases}$$

(4) $\forall A \subseteq \kappa[A \cap S^A = \emptyset]$.

In fact, this is obvious if $|A| < m$, while if $|A| \geq m$ and $\alpha \in A \cap S^A$, then $\alpha \in S_\sigma = S'_{\sigma i_\sigma}$ for some $\sigma \in [A]^m$, contradiction.

(5) $\forall A \in [\kappa]^{<\kappa}[S^A \in D]$

Since $S_\sigma = S'_{\sigma i_\sigma} \in D$, this follows since D is κ-complete.

(6) If $A \subseteq A'$ and $m \leq |A|$, then $S^{A'} \subseteq S^A$.

We now define X and x with domain κ by induction. Let $X_0 = \kappa$ and $x_0 = 0$. If X_i and x_i have been defined, where $0 < i < \kappa$, let $B_{i+1} = \{x_j : j \leq i\}$ and $X_{i+1} = S^{B_{i+1}}\backslash(i+1)$. Then $X_{i+1} \in D$. Let x_{i+1} be the least element of X_{i+1}. For $i < \kappa$ limit, let $X_i = \bigcap_{j<i} X_j$ and let x_i be the least element of X_i.

(8) $\forall i < m[x_i = i$ and $B_i = i$ and $S^{B_i} = X_i = \{i, i+1, \ldots\}]$.

We prove this by induction. $x_0 = 0$ by definition, and $B_0 = 0$ and $S^{B_0} = \kappa$. Assume that the conditions hold for all $j < i$. Then $B_i = \{x_k : k < i\} = \{k : k < i\} = i$. $S^{B_i} = \kappa\backslash B_i = \{i, i+1, \ldots\}$ and $X_i = S^{B_i}\backslash i = \{i, i+1, \ldots\}$ and $x_i = i$. Thus (8) holds.

(9) If $i < i'$ then $X_i \supseteq X_{i'}$.

This is clear if $i' < m$ or $m \leq i$. Suppose that $m \leq i'$ and $i < m$. Then $X_i = \kappa\backslash i$ by (8). We have

$$X_{i'} \subseteq X_m = S^{B_m}\backslash m \subseteq \kappa\backslash m \subseteq \kappa\backslash i = X_i.$$

(10) $\bigcap_{i<\kappa} X_i = \emptyset$.

(11) $x_i \in X_i\backslash X_{i+1}$.

In fact, clearly $x_i \in X_i$. If $x_i \in X_{i+1}$ then $x_i \in S^{B_{i+1}} \cap B_{i+1}$, contradicting (4).

(12) If $i < i'$, then $x_i \neq x_{i'}$.

This is clear from (11) and the fact that the X_i's are decreasing.

(13) If $i_0 < \cdots < i_{m-1} < \kappa$ and $\sigma = \{x_{i_0}, \ldots, x_{i_{m-1}}\}$, then $\{x_j : i_{m-1} < j < \kappa\} \subseteq S_\sigma$, and hence $\forall j \in (i_{m-1}, \kappa)[f(\sigma \cup \{x_j\}) = i_\sigma$.

In fact, suppose that $i_{m-1} < j < \kappa$. Then $x_j \in X_j = S^{B_j}\backslash j \subseteq S^{B_i}\backslash i \subseteq S_\sigma$. And if $j \in (i_{m-1}, \kappa)$, then $x_j \in S_\sigma = S'_{\sigma i_\sigma}$ and so $f(\sigma \cup \{x_j\}) = i_\sigma$.

Now let $Y = \{x_i : i < \kappa\}$.

(14) $Y \in D$.

To prove this, define $g : \kappa \to \kappa$ as follows. For each $j < \kappa$ there is a unique $i(j) < \kappa$ such that $j \in X_{i(j)}\backslash X_{i(j)+1}$. We let $g(j) = x_{i(j)}$.

(15) $\forall j < \kappa[g(j) \leq j]$.

This is true since $x_j \in X_j$, using (9).

(16) $Y = \{j < \kappa : g(j) = j\}$.

In fact, suppose that $j \in Y$. Say $j = x_i$ with $i < \kappa$. By (11), $x_i \in X_i \backslash X_{i+1}$, so $i(j) = i$ and $g(j) = x_i = j$. Conversely, if $g(j) = j$, then $x_{i(j)} = j$, and so $j \in Y$. So (16) holds.

Now we prove (14) by contradiction. Suppose that $\kappa \backslash Y \in D$. Now $\kappa \backslash Y = \{j < \kappa : g(j) < j\}$ by (15) and (16), so by normality of D there is a $Z \subseteq \kappa \backslash Y$ with $Y' \in D$ such that g is constant on Z. Fix $k \in Z$. Now $Z \cap X_{i(k)+1} \in D$; choose $j \in Z \cap X_{i(k)+1}$. Now $j \in X_{i(j)} \backslash X_{i(j)+1}$. If $i(j) \leq i(k)$, then $j \in X_{i(j)+1}$, contradiction. Hence $i(k) < i(j)$. Now $x_{i(k)} = g(k) = g(j) = x_{i(j)}$, contradicting (12). So (14) holds.

Now define $f' : [\kappa]^m$ as follows. For any $\sigma \in [\kappa]^m$,

$$f'(\sigma) = \begin{cases} 0 & \text{if } \sigma \notin [Y]^m, \\ f(\sigma \cup \{x_{i_{n-1}+1}\}) & \text{if } \sigma = \{x_{i_0}, \ldots, x_{i_{m-1}}\}, \\ & \text{where } x_{i_0} < \cdots < x_{i_{m-1}}. \end{cases}$$

For each $\sigma \in [Y]^m$ let $\sigma = \{x_{i_0^\sigma}, \ldots, x_{i_{m-1}^\sigma}\}$. For each $\sigma \in [Y]^{m+1}$ let σ' be the first m members of σ. Now by the inductive hypothesis we have two cases.

Case 1. There is a $W \in D$ such that $\forall \sigma \in [W]^m[f'(\sigma) = 0]$. Then $W \cap Y \in D$ and for all $\sigma \in [W \cap Y]^m[f(\sigma \cup \{x_{i_{m-1}^\sigma+1}\}) = 0$. Then for all $\sigma \in [W \cap Y]^{m+1}$, by (13), $f(\sigma) = f(\sigma' \cup \{x_{i_{m-1}^{\sigma'}+1}\}) = 0$.

Case 9. There is a $W \in D$ such that $\forall \sigma \in [W]^m[f'(\sigma) = i]$, with $i > 0$. This is similar to Case 1. $\square$

Theorem 17.79. *Suppose that D is a normal κ-complete ultrafilter on a measurable cardinal κ, $\lambda < \kappa$, and $f : [\kappa]^{<\omega} \to \lambda$. Then there is an $X \in D$ such that $\forall n \in \omega[f \restriction [X]^n$ is constant].*

Proof. Take the intersection of sets given for each $m \in \omega$ by Theorem 17.78. $\square$

18. Large cardinals and L

Let λ be a limit ordinal. For each formula $\varphi(u, v_1, \ldots, v_n)$ the Skolem function h_φ^λ is defined by setting, for any $a_1, \ldots, a_n \in L_\lambda$, $h_\varphi^\lambda(a_1, \ldots, a_n)$ =the $<_{L_\lambda}$-least b such that $(L_\lambda, \in)] \models \varphi(b, a_1, \ldots, a_n)$ if there is such a b, $\emptyset$ otherwise.

Thus h_φ^λ is definable in $(L_\lambda, \in)$:

$$\begin{aligned}(L_\lambda, \in) \models & \forall u, v_1, \ldots, v_n)[h_\varphi^\lambda(v_1, \ldots, v_n) = v_{n+1} \leftrightarrow \\ & [\varphi(v_{n+1}, v_1, \ldots, v_n) \wedge \forall w[w \in v_{n+1} \rightarrow \neg\varphi(w, v_1, \ldots, v_n) \wedge]] \vee \\ & [[\forall w \neg\varphi(w, v_1, \ldots, v_n)] \wedge v_{n+1} = \emptyset]\end{aligned}$$

A *special model* is a model $\overline{A} = (A, E)$ elementarily equivalent to $(L_\lambda, \in)$ for some limit ordinal λ. For a special model $\overline{A}$ we let $\mathrm{Ord}^{\overline{A}} = \{a \in A : \overline{A} \models a \text{ is an ordinal}\}$, where v is an ordinal is the formula

$$\forall w \in v \forall x \in w[x \in v] \wedge \forall w \in v \forall x \in w \forall y \in x[y \in w].$$

Lemma 18.1. *If $\overline{A} = (A, E)$ is a special model, then $\mathrm{Ord}^{\overline{A}}$ is simply ordered by E.*

Proof. Say $\overline{A}$ is elementarily equivalent to $(L_\lambda, \in)$ with λ a limit ordinal. $(L_\lambda, \in) \models \forall x[x \notin x]$, so $\overline{A} \models \forall x[x \notin x]$, and so E is irreflexive on $\mathrm{Ord}^{\overline{A}}$. $(L_\lambda, \in) \models \in$ is transitive on ordinals, so E is transitive on $\mathrm{Ord}^{\overline{A}}$. $(L_\lambda, \in) \models$ any two ordinals are equal, or comparable under $\in$, so $\overline{A} \models$ and two elements of $\mathrm{Ord}^{\overline{A}}$ are equal or comparable under E. □

For $\overline{A}$ a special model we write $\alpha < \beta$ instead of $\alpha E \beta$ when $\alpha, \beta \in \mathrm{Ord}^{\overline{A}}$.

If $\overline{A}$ is a special model, then a set $I \subseteq \mathrm{Ord}^{\overline{A}}$ is a set of *indiscernibles* for $\overline{A}$ iff for every formula φ,

$$\overline{A} \models \varphi(x_1, \ldots, x_m) \quad \text{iff} \quad \overline{A} \models \varphi(y_1, \ldots, y_m)$$

whenever $x_1 < \cdots < x_m$ and $y_1 < \cdots < y_m$ are elements of I. If I is a set of indiscernibles for $\overline{A}$, we define

$$\Sigma(\overline{A}, I) = \{\varphi(v_1, \ldots, v_m) : \exists x_1 < \cdots < x_m \text{ in } I \text{ such that } \overline{A} \models \varphi(x_1, \ldots, x_m)\}.$$

Lemma 18.2. *If I is a set of indiscernibles for $\overline{A}$, then for every formula φ, $\varphi \in \Sigma(\overline{A}, I)$ or $\neg\varphi \in \Sigma(\overline{A}, I)$.*

Proof. Suppose that $\varphi \notin \Sigma(\overline{A}, I)$. Then for every $x_1 < \cdots < x_n$ in I, $\overline{A} \models \neg\varphi(x_1, \ldots, x_n)$. □

A set Σ of formulas is an *E.M.* set iff there exist a model $\overline{A}$ elementarily equivalent to some $(L_\lambda, \in)$ with λ limit and a set I of indisernibles for $\overline{A}$ such that $\Sigma = \Sigma(\overline{A}, I)$.

Lemma 18.3. *Suppose that κ is a Ramsey cardinal. Then $(L_\kappa, \in)$ has a set of indiscernibles of size κ.*

Proof. By Corollary 17.37. $\square$

If κ is a Ramsey cardinal, by Lemma 18.3 let I be a set of indiscernibles for $(L_\kappa, \in)$ of size κ, and let $\Sigma_\kappa^* = \Sigma((L_\kappa, \in), I)$.

Corollary 18.4. *If there is a Ramsey cardinal, then there is an E.M. set.* $\square$

Lemma 18.5. *If Σ is an E.M. set and α is an infinite ordinal, then there is a unique pair $(\overline{A}, I)$ satisfying the following conditionns:*

(i) $\overline{A} = (A, E)$ is elementarily equivalent to $(L_\gamma, \in)$ for some limit ordinal γ.
(ii) I is a set of indiscernibles for $\overline{A}$.
(iii) $\Sigma = \Sigma(\overline{A}, I)$.
(iv) I has order type α.
(v) $\overline{A}$ has definable Skolem functions.
(vi) $\overline{A}$ is the closure of I under Skolem functions.

Proof. First we prove uniqueness. So suppose that $(\overline{A}, I)$ and $(\overline{B}, J)$ both satisfy (i)–(vi). Say $\overline{A} = (A, E^{\overline{A}})$ and $\overline{B} = (B, E^{\overline{B}})$. By (iv), there is an order-isomorphism π from I onto J.

Claim. For terms t_1 and t_2 and $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_m$ in I,

$$t_1^{\mathfrak{A}}[x_1, \ldots, x_n] = t_2^{\mathfrak{A}}[y_1, \ldots, y_m] \quad \text{iff} \quad t_1^{\mathfrak{B}}[\pi(x_1), \ldots, \pi(x_n)] = t_2^{\mathfrak{B}}[\pi(y_1), \ldots, \pi(y_m)]$$
$$t_1^{\mathfrak{A}}[x_1, \ldots, x_n] E^{\mathfrak{A}} t_2^{\mathfrak{A}}[y_1, \ldots, y_m] \quad \text{iff} \quad t_1^{\mathfrak{B}}[\pi(x_1), \ldots, \pi(x_n)] E^{\mathfrak{B}} t_2^{\mathfrak{B}}[\pi(y_1), \ldots, \pi(y_m)]$$

Proof of claim. Let $z_1 < \cdots < z_p$ enumerate $\{x_i : 1 \le i \le n\} \cup \{y_i : 1 \le i \le m\}$ in increasing order. Say $x_i = z_{j(i)}$ for $1 \le i \le n$, and $y_i = z_{k(i)}$ for $1 \le i \le m$. Then

$$\begin{aligned}
&t_1^{\mathfrak{A}}[x_1, \ldots, x_n] = t_2^{\mathfrak{A}}[y_1, \ldots, y_m] \\
&\qquad \text{iff} \quad (t_1^{\mathfrak{A}}[z_{j(1)}, \ldots, z_{j(n)}] = t_2^{\mathfrak{A}}[z_{k(1)}, \ldots, z_{k(m)}]) \in \Sigma(\mathfrak{A}, I) = \Sigma(\mathfrak{B}, J) \\
&\qquad \text{iff} \quad \mathfrak{B} \models (t_1^{\mathfrak{B}}[\pi(z_{j(1)}), \ldots, \pi(z_{j(n)})] = t_2^{\mathfrak{B}}[\pi(z_{k(1)}), \ldots, \pi(z_{k(m))}]) \\
&\qquad \text{iff} \quad t_1^{\mathfrak{B}}[\pi(x_1), \ldots, \pi(x_n)] = t_2^{\mathfrak{B}}[\pi(y_1), \ldots, \pi(y_m)]
\end{aligned}$$

Similarly for E, so the claim holds.

From the claim it follows that $\pi(t^{\overline{A}}(x_1, \ldots, x_n)) = t^{\overline{B}}(\pi(x_1), \ldots, \pi(x_n))$ for $x_1 < \cdots < x_n$ is well-defined and gives an isomorphism from $\overline{A}$ onto $\overline{B}$ by (vi).

Now for existence we apply the compactness theorem. Since Σ is an E.M. set, let λ be a limit ordinal and $\overline{B} = (B, E)$ elementarily equivalent to $(L_\lambda, \in)$ with J a set of indiscernibles for $\overline{B}$ such that $\Sigma = \Sigma(\overline{A}, J)$. Expand the language for (B, E) by adding individual constants c_ξ for $\xi < \alpha$. Let Δ be the following set of sentences:

χ for each sentence χ holding in $(L_\lambda, \in)$

ψ for each ψ defining a Skolem function;

$c_\xi E c_\eta$ for $\xi < \eta < \alpha$;

$\varphi(c_{\xi_1}, \ldots, c_{\xi_n})$ for each φ in Σ, with $\xi_1 < \cdots < \xi_n$.

Clearly each finite subset of Δ has a model obtained from $\overline{B}$.

Let $\overline{C}$ be a model of Δ, with $I = \{c_\xi^{\overline{C}} : \xi < \alpha\}$. Let $\overline{A}$ be the closure of I in $\overline{C}$ under the Skolem functions. Clearly the conditions of the lemma hold. $\square$

If Σ is an E.M. set and α is an infinite ordinal, then the pair $(\overline{A}, I)$ given in Lemma 18.5 is called the (Σ, α)-*model*.

Lemma 18.6. *Let Σ be an E.M. set, and $\omega \le \alpha \le \beta$. Let $j : \alpha \to \beta$ be order preserving. Then j can be extended to an elementary embedding of the (Σ, α) model into the (Σ, β) model.*

Proof. Let $\overline{A}, \overline{B}$ be the (Σ, α) model, resp. the (Σ, β) model. By the claim in the proof of Lemma 18.5, j extends to an isomorphism from $\overline{A}$ into $\overline{B}$. Now we apply the Tarski criterion. Suppose that $\overline{B} \models \varphi(b, j(a_1), \ldots, j(a_n))$. Then $h^{\overline{B}}(j(a_1), \ldots, j(a_n)) = j(c)$ for some $c \in A$, so $\overline{B} \models \varphi(j(c), j(a_1), \ldots, j(a_n))$. $\square$

Lemma 18.7. *If Σ is an E.M set, then the following are equivalent:*

(i) For every limit ordinal α, the (Σ, α)-model is well-founded.

(ii) For some limit ordinal $\alpha \ge \omega_1$, the (Σ, α)-model is well-founded.

(iii) For every limit ordinal $\alpha < \omega_1$, the (Σ, α)-model is well-founded.

Proof. (i)$\Rightarrow$(ii): trivial.

(ii)$\Rightarrow$(iii): Assume (ii), and let α be as indicated. Let $(\overline{A}, I)$ be the (Σ, α)-model. Suppose that β is a limit ordinal less than ω_1. Let J consist of the first β elements of I, and let $\overline{B}$ be the closure of J under the Skolem functions. Then $(\overline{B}, J)$ is the (Σ, β)-model, and it is well-founded since $\overline{B}$ is a submodel of $\overline{A}$.

(iii)$\Rightarrow$(i): Assume (iii), but suppose that α is a limit ordinal such that the (Σ, α)-model is not well-founded. Let $(\overline{A}, I)$ be the (Σ, α)-model. Say $a_i \in A$ for all $i \in \omega$ and $a_1 E a_0$, $a_2 E a_1$, etc. Say $a_n = t_n^{\overline{A}}(x_{n0}, \ldots, x_{nm_n})$ with t_n a composition of Skolem functions. Let $J = \{x_{ni} : n \in \omega, i < m_n\}$. So J has order type some countable ordinal β. We may assume that β is a limit ordinal. Let $\overline{B}$ be the closure of J under Skolem functions. Then $(\overline{B}, J)$ is the (Σ, β) model, and it is not well-founded, contradicting (iii). $\square$

Corollary 18.8. *If there is a Ramsey cardinal κ, then for every limit ordinal α, the $(\Sigma_\kappa^*, \alpha)$-model is well-founded.*

Proof. By Lemmas 18.3 and 18.7. $\square$

A (Σ, α)-model $(\overline{A}, I)$ is *unbounded* iff I is unbounded in $\mathrm{Ord}^{\overline{A}}$.

Lemma 18.9. *For any E.M. set Σ the following are equivalent:*

(i) For every limit ordinal α, the (Σ, α)-model is unbounded.

(ii) For some limit ordinal α, the (Σ, α)-model is unbounded.

(iii) For every Skolem term $t(v_1, \ldots, v_n)$, Σ contains the formula

$$t(v_1, \ldots, v_n) \in \mathrm{Ord} \to t(v_1, \ldots, v_n) < v_{n+1}.$$

Proof. (i)⇒(ii): trivial.

(ii)⇒(iii): Assume that $(\overline{A}, I)$ is the (Σ, α)-model, with α a limit ordinal, and I is unbounded in $\mathrm{Ord}^{\overline{A}}$. To prove the condition in (iii) it suffices to take a Skolem term $t(v_1, \ldots, v_n)$ and an increasing sequence $x_1 < \cdots < x_{n+1}$ of elements of I and show that

$$(*) \qquad \overline{A} \models t(x_1, \ldots, x_n) \in \mathrm{Ord}^{\overline{A}} \to t(x_1, \ldots, x_n) < x_{n+1}.$$

If $t(x_1, \ldots, x_n) \notin \mathrm{Ord}^{\overline{A}}$, then $(*)$ vacuously holds. If $t(x_1, \ldots, x_n) \in \mathrm{Ord}^{\overline{A}}$, since I is unbounded in $\mathrm{Ord}^{\overline{A}}$ choose $y \in I$ such that $t(x_1, \ldots, x_n) < y$. Since I is a set of indiscernibles, $(*)$ follows.

(iii)⇒(i): Assume (iii), assume that α is a limit ordinal, and let $(\overline{A}, I)$ be the (Σ, α)-model. To prove that I is unbounded in $\mathrm{Ord}^{\overline{A}}$, let $y \in \mathrm{Ord}^{\overline{A}}$. There is a Skolem term t and $x_1 < \cdots < x_n$ in I such that $y = t(x_1, \ldots, x_n)$. If x_{n+1} is any element of I greater than x_n, then by (iii), $y = t(x_1, \ldots, x_n) < x_{n+1}$. □

An E.M. set Σ is *unbounded* iff Lemma 18.9(iii) holds.

Corollary 18.10. *If there is a Ramsey cardinal κ, then for every limit ordinal α, the $(\Sigma^*_\kappa, \alpha)$-model is well-founded and unbounded.* □

Let $\alpha > \omega$ be a limit ordinal, and let $(\overline{A}, I)$ be the (Σ, α)-model. Let $\langle i_\xi : \xi < \alpha \rangle$ enumerate I in increasing order. Then $(\overline{A}, I)$ is *remarkable* iff it is unbounded and every ordinal of $\overline{A}$ less than i_ω is in the Skolem closure of $\{i_n : n \in \omega\}$.

Lemma 18.11. *The following are equivalent for any unbounded E.M. set Σ:*

(i) For every limit ordinal $\alpha > \omega$ the (Σ, α)-model is remarkable.

(ii) For some limit ordinal $\alpha > \omega$ the (Σ, α)-model is remarkable.

(iii) For every Skolem term $t(v_1, \ldots, v_{m+n})$ the set Σ contains the formula

$$\begin{aligned} &t(v_1, \ldots, v_{m+n}) \in \mathrm{Ord} \wedge t(v_1, \ldots, v_{m+n}) < v_{m+1} \to \\ &t(v_1, \ldots, v_{m+n}) = t(v_1, \ldots, v_m, v_{m+n+1}, \ldots, v_{m+2n}). \end{aligned}$$

(iv) For every limit ordinal $\alpha > \omega$ and every limit ordinal $\gamma < \alpha$, if $(\overline{A}, I)$ is the (Σ, α)-model and $a \in \mathrm{Ord}^{\overline{A}}$ is less than i_γ, then a is in the Skolem closure of $\{i_\xi : \xi < \gamma\}$.

Proof. (i)⇒(ii): trivial.

(ii)⇒(iii): Assume (ii). Let α be a limit ordinal $> \omega$ and let $(\overline{A}, I)$ be the (Σ, α)-model; assume that $(\overline{A}, I)$ is remarkable. It suffices to show that for any Skolem term t,

$$(*) \qquad \begin{aligned} &t^{\overline{A}}(x_1, \ldots, x_m, y_1, \ldots, y_n) \in \mathrm{Ord}^{\overline{A}} \wedge t^{\overline{A}}(x_1, \ldots, x_m, y_1, \ldots, y_n) < y_1 \Rightarrow \\ &t^{\overline{A}}(x_1, \ldots, x_m, y_1, \ldots, y_n) = t^{\overline{A}}(x_1, \ldots, x_m, z_1, \ldots, z_n) \end{aligned}$$

for some sequence $x_1 < \cdots < x_m < y_1 < \cdots < y_n < z_1 < \cdots < z_n$ of elements of I. To prove $(*)$, let $x_1 < \cdots < x_m$ be the first m elements of I. Let y_1 be the ω-th element of I. Choose $y_2 < \cdots < y_n$ with $y_1 < y_2$. If $t^{\overline{A}}(x_1, \ldots, x_m, y_1, \ldots, y_n) \notin \mathrm{Ord}^{\overline{A}}$ or $t^{\overline{A}}(x_1, \ldots, x_m, y_1, \ldots, y_n) \geq y_1$, then $(*)$ holds for any $z_1 < \cdots < z_n$ such that $y_n < z_1$. Suppose that $t^{\overline{A}}(x_1, \ldots, x_m, y_1, \ldots, y_n) \in \mathrm{Ord}^{\overline{A}}$ and $t^{\overline{A}}(x_1, \ldots, x_m, y_1, \ldots, y_n) < y_1$. Then by remarkability, $t^{\overline{A}}(x_1, \ldots, x_m, y_1, \ldots, y_n)$ is in the closure of $\{i_n : n \in \omega\}$ under Skolem functions. Then there is a $k \in \omega$ with $m \leq k$ and a Skolem term s such that $t^{\overline{A}}(x_1, \ldots, x_m, y_1, \ldots, y_n) = s^{\overline{A}}(i_0, \ldots, i_k)$. By indiscernibility,

$$t^{\overline{A}}(x_1, \ldots, x_m, z_1, \ldots, z_n) = s^{\overline{A}}(i_0, \ldots, i_k).$$

Hence

$$t^{\overline{A}}(x_1, \ldots, x_m, y_1, \ldots, y_n) = t^{\overline{A}}(x_1, \ldots, x_m, z_1, \ldots, z_n),$$

as desired.

(iii)$\Rightarrow$(iv): Assume (iii), and suppose that $\alpha > \omega$ is a limit ordinal. Let $(\overline{A}, I)$ be the (Σ, α)-model. Also suppose that γ is a limit ordinal less than α, and $a \in \mathrm{Ord}^{\overline{A}}$ is less than i_γ. Since $\overline{A}$ is the Skolem closure of I, there is a Skolem term t and $x_1 < \cdots < x_m < y_1 < \cdots < y_n$ in I such that $y_1 = i_\gamma$ and $a = t^{\overline{A}}(x_1, \ldots, x_m, y_1, \ldots, y_n)$. Choose $w_1, \ldots, w_n$ and $z_1, \ldots, z_n$ in I so that

$$x_1 < \cdots x_n < w_1 < \cdots < w_n < y_1 < \cdots < y_n < z_1 < \cdots < z_n.$$

By (iii) we have

$$t^{\overline{A}}(x_1, \ldots, x_m, y_1, \ldots, y_n) = t^{\overline{A}}(x_1, \ldots, x_m, z_1, \ldots, z_n).$$

By indiscernibility it follows that

$$t^{\overline{A}}(x_1, \ldots, x_m, w_1, \ldots, w_n) = t^{\overline{A}}(x_1, \ldots, x_m, z_1, \ldots, z_n).$$

Hence $a = t^{\overline{A}}(x_1, \ldots, x_m, y_1, \ldots, y_n) = t^{\overline{A}}(x_1, \ldots, x_m, w_1, \ldots, w_n)$, so that a is in the Skolem closure of $\{i_\xi : \xi < \gamma\}$, as desired.

(iv)$\Rightarrow$(i): trivial $\square$

An E.M. set Σ is *remarkable* iff it is unbounded and contains the formulas 18.11(iii). It is *well-founded* if every (Σ, α)-model is well-founded.

Lemma 18.12. *Let $(\overline{A}, I)$ be a remarkable (Σ, α)-model and let $\gamma < \alpha$ be a limit ordinal. Let $J = \{i_\xi : \xi < \gamma\}$ and let $\overline{B}$ be the Skolem closure of J. Then $(\overline{B}, J)$ is the (Σ, J)-model, and $\mathrm{Ord}^{\overline{B}}$ is an initial segment of $\mathrm{Ord}^{\overline{A}}$.*

Proof. Clearly $(\overline{B}, J)$ is the (Σ, J)-model. Now suppose that $\beta \in \mathrm{Ord}^{\overline{B}}$, $\delta \in \mathrm{Ord}^{\overline{A}}$, and $\delta < \beta$. Say $\beta = t^{\overline{A}}(x_1, \ldots, x_m\}$ with $x_1 < \cdots < x_m$ in $\{i_\xi : \xi < \gamma\}$. Since $(\overline{A}, I)$ is unbounded, the formula

$$t(v_1, \ldots, v_m) \in \mathrm{Ord} \to t(v_1, \ldots, v_m) < v_{m+1}$$

is in Σ. Hence $\beta = t^{\overline{A}}(x_1, \ldots, x_m) < i_\gamma$. Hence $\delta < i_\gamma$, so by Lemma 9(iv), $\delta \in B$. Since $\overline{B}$ is an elementary substructure of $\overline{A}$, it follows that $\delta \in \mathrm{Ord}^{\overline{B}}$. □

Let $(\overline{A}, I)$ be the $\Sigma, \alpha)$-model. We say that I is *closed* in $\mathrm{Ord}^{\overline{A}}$ iff for every limit $\gamma < \alpha$, i_γ is the least upper bound in $\mathrm{Ord}^{\overline{A}}$ of $\{i_\xi : \xi < \gamma\}$.

Lemma 18.13. *Suppose that $(\overline{A}, I)$ is the (Σ, α)-model. If $(\overline{A}, I)$ is remarkable, then I is closed in* $\mathrm{Ord}^{\overline{A}}$.

Proof. Suppose that $\gamma < \alpha$ is a limit ordinal. Clearly i_γ is an upper bound for $\{i_\xi : \xi < \gamma\}$. Suppose that $a \in \mathrm{Ord}^{\overline{A}}$ is another upper bound, but $a < i_\gamma$. Then a is in the closure $\overline{B}$ of $J \overset{\text{def}}{=} \{i_\xi : \xi < \gamma\}$ under Skolem functions. Now $(\overline{B}, J)$ is unbounded, so there is a $\xi < \gamma$ such that $a < i_\xi$. But $a_\xi < a$, contradiction. □

Lemma 18.14. *If κ is an uncountable cardinal and there is a limit ordinal λ such that $(L_\lambda, \in)$ has a set I of indiscernibles of order type κ, then there is a limit ordinal $\gamma > \omega$ and a set $I \subseteq \gamma$ such that (L_γ, I) is remarkable.*

Proof. Let γ be the least limit ordinal greater than ω such that $(L_\gamma, \in)$ has a set I of indiscernibles of order type γ.

(1) L_γ is the closure of I under Skolem functions.

To prove (1), let A be the closure of I under Skolem functions. Thus $(A, \in)$ is an elementary substructure of $(L_\gamma, \in)$. By Theorem 13.42, $A = L_\alpha$ for some limit ordinal $\alpha \leq \gamma$. But $\gamma \leq \alpha$ since $I \subseteq A$. So $\alpha = \gamma$.

Now let α be such that α is a limit ordinal greater than ω such that $(L_\alpha, \in)$ has a set I of indiscernibles of order type α, L_α is the Skolem closure of I, and the ω-th element of I is minimum. We claim that (L_α, I) is remarkable. Note that (L_α, I) is the $(\Sigma(\overline{A}, I), \varphi)$-model.

Suppose that (L_α, I) is not remarkable. Then by Theorem 18.9(iii) there is a Skolem term $t(v_1, \ldots, v_{m+n})$ such that for any $x_1 < \cdots < x_m < y_1 < \cdots < y_n < z_1 < \cdots < z_n$ in I,

$$t(x_1, \ldots, x_m, y_1, \ldots, y_n) < y_1 \tag{2}$$

and

$$t(x_1, \ldots, x_m, y_1, \ldots, y_n) \neq t(x_1, \ldots, x_m, z_1, \ldots, z_n). \tag{3}$$

We now define $x_1 < \cdots < x_m$ to be the first m elements of I. $u^0_1 < \cdots < u^0_n$ is the next n elements of I. Suppose that $u^\beta_1 < \cdots < u^\beta_n$ in I has been defined for all $\beta < \delta$. If the sequences so far are unbounded in α, the construction stops. Otherwise, let $u^\delta_1 < \cdots < u^\delta_n$ be the next n elements of I. Thus the construction stops at some limit ordinal $\delta \leq \alpha$.

For each $\beta < \delta$ let $\varepsilon_\beta = t(x_1, \ldots, x_m, u^\beta_1, \ldots, u^\beta_n)$. Then by (2), each $\varepsilon_\beta \in \mathrm{Ord}^{L_\alpha} = \alpha$. Also, by (3) and indiscernibility, $\varepsilon_\beta \neq \varepsilon_\theta$ for distinct $\beta, \theta < \delta$. Also by indiscernibility, $\varepsilon_\beta < \varepsilon_\theta$ for $\beta < \theta < \delta$.

(4) $\{\varepsilon_\beta : \beta < \delta\}$ is a set of indiscernibles in $(L_\alpha, \in)$.

In fact, for any formula $\varphi(v_1, \ldots, v_k)$ and for $\varepsilon_{\beta_1} < \cdots < \varepsilon_{\beta_k}$ and $\varepsilon_{\theta_1} < \cdots < \varepsilon_{\theta_k}$,

$$
\begin{aligned}
&(L_\alpha, \in) \models \varphi(\varepsilon_{\beta_1}, \ldots, \varepsilon_{\beta_k}) \\
&\quad \text{iff} \quad (L_\alpha, \in) \models \varphi(t(x_1, \ldots, x_m, u_1^{\beta_1}, \ldots, u_n^{\beta_1})), \ldots t(x_1, \ldots, x_m, u_1^{\beta_k}, \ldots, u_n^{\beta_k})) \\
&\quad \text{iff} \quad (L_\alpha, \in) \models \varphi(t(x_1, \ldots, x_m, u_1^{\theta_1}, \ldots, u_n^{\theta_1})), \ldots t(x_1, \ldots, x_m, u_1^{\theta_k}, \ldots, u_n^{\theta_k})) \\
&\quad \text{iff} \quad (L_\alpha, \in) \models \varphi(\varepsilon_{\theta_1}, \ldots, \varepsilon_{\theta_k}).
\end{aligned}
$$

Now i_ω is the first element of u^{β_ω}, and by (2), $\varepsilon_{\beta_\omega} < i_\omega$. This contradicts the minimality of α. Hence (L_α, I) is remarkable. □

We let $\Sigma^{**} = \Sigma(L_\gamma, I)$ with (L_γ, I) as in Lemma 18.14.

Corollary 18.15. *If κ is an uncountable cardinal, γ is a limit ordinal, and $I \subseteq \gamma$ is a set of indiscernibles of order type κ, then $\Sigma(L_\gamma, I)$ is a well-founded remarkable E.M. set.* □

Lemma 18.16. *Assume that there is a well-founded remarkable E.M. set Σ and κ is an uncountable cardinal. Then the universe of the (Σ, κ)-model is isomorphic to L_κ.*

Proof. Let $(\overline{A}, I)$ be the (Σ, κ)-model. Then $\overline{A}$ is well-founded. Let $\overline{B}$ be the transitive collapse of $\overline{A}$ via a function π. Since $\overline{A}$ is elementarily equivalent to some L_γ with γ limit, so is $\overline{B}$. By Theorem 13.42′, $B = L_\delta$ for some limit ordinal $\delta \leq \gamma$. Now $\pi[I]$ is a set of indiscernibles of L_δ of size κ, so $\kappa \leq \delta$. Suppose that $\kappa < \delta$. Since I is unbounded in $\text{Ord}^{\overline{A}}$, $\pi[I]$ is unbounded in δ. Say $I = \{i_\xi : \xi < \kappa\}$. Then there is a $\xi < \kappa$ such that $\kappa < \pi(i_\xi)$. Now $(\overline{B}, \pi[I])$ is remarkable, so by Theorem 18.11(iv), κ is in the Skolem closure of $\{\pi(i_\eta) : \eta < \xi\}$. Now $\kappa \subseteq \{\pi(i_\eta) : \eta < \xi\}$ and $|\{\pi(i_\eta) : \eta < \xi\}| \leq \xi < \kappa$, contradiction. Hence $\kappa = \delta$. □

If Σ is a well-founded remarkable E.M. set and κ is an uncountable cardinal, let I_κ be the unique subset of κ such that (L_κ, I_κ) is the (Σ, κ)-model.

Lemma 18.17. *Suppose that there is a well-founded remarkable E.M. set. Suppose that $\kappa < \lambda$ are uncountable cardinals. Then $L_\kappa \preceq L_\lambda$, $I_\kappa = \kappa \cap I_\lambda$, and L_κ is the closure of I_κ under Skolem terms.*

Proof. I_λ has order type λ. Let J be the first κ elements of I_λ, and let $\overline{A}$ be the Skolem closure of J in L_λ. Then $(\overline{A}, J)$ is the (Σ, κ)-model, and by Lemma 18.12, $\text{Ord}^{\overline{A}}$ is an initial segment, say β, of λ. Since $\overline{A}$ is isomorphic to L_κ and the Mostowski collapse fixes ordinals, it follows that $\beta = \kappa$. By the uniqueness assertion in Lemma 18.55, $J = I_\kappa$. Clearly $I_\kappa = \kappa \cap I_\lambda$. Since $\overline{A}$ be the Skolem closure of J in L_λ, it is the closure of I_κ under Skolem terms.

By Lemma 18.6, $L_\kappa \preceq L_\lambda$. □

Theorem 18.18. *Assume that there is a well-founded remarkable E.M. set Σ.*

(i) If $\kappa < \lambda$ are uncountable cardinals, then $(L_\kappa, \in)$ is an elementary submodel of $(L_\lambda, \in)$.

(ii) There is a closed unbounded class I of ordinals containing all uncountable cardinals such that for every uncountable cardinal κ,

(a) $|I \cap \kappa| = \kappa$.

(b) $I \cap \kappa$ is a set of indiscernibles for $(L_\kappa, \in)$.

(c) Every $a \in L_\kappa$ is definable in $(L_\kappa, \in)$ from $I \cap \kappa$.

(d) L_κ is the closure of $I \cap \kappa$ under Skolem terms.

Proof. By Lemma 18.16, for each uncountable cardinal κ, the (Σ, κ) model is (L_κ, I_κ). By Lemma 18.17, $L_\kappa \preceq L_\lambda$ for $\kappa < \lambda$ uncountable cardinals, and $I_\kappa = \kappa \cap I_\lambda$. Let $J = \bigcup\{I_\kappa : \kappa \text{ an uncountable cardinal}\}$. Then for any uncountable cardinal κ, $J \cap \kappa = I_\kappa$, which has order type κ, and hence size κ. If κ is an uncountable cardinal, then I_κ is unbounded in κ, hence is in I_{κ^+} since I_{κ^+} is closed. Thus J contains all uncountable cardinals. $J \cap \kappa = I_\kappa$ is a set of indiscernibles for L_κ. By Lemma 18.5(vi), every element of L_κ is definable in $(L_\kappa, \in)$ from $I_\kappa = J \cap \kappa$. (d) holds by Lemma 18.17. $\square$

We say "$0^\sharp$ exists" if the conclusion of Theorem 18.8 holds. Moreover, we define

$$0^\sharp = \{\varphi : L_{\omega_1} \models \varphi(\aleph_1 \ldots, \aleph_m)\}.$$

Theorem 18.19. *There is at most one well-founded remarkable E.M. set.*

Proof. Suppose that Σ is a well-founded remarkable E.M. set. By the proof of Lemma 18.16, the universe of the $(\Sigma, \aleph_\omega)$-model is $L_{\aleph_\omega}$. Then

$$\varphi(v_1, \ldots, v_m) \in \Sigma \quad \text{iff} \quad L_{\aleph_\omega} \models \varphi(\aleph_1, \ldots, \aleph_m).$$

This shows that Σ is unique. $\square$

Now by Lemma 18.5, each I_κ for κ an uncountable cardinal is also uniquely determined.

Corollary 18.20. *If $0^\sharp$ exists, then for any $a_1, \ldots, a_m \in L$,*

$$L \models \varphi(a_1, \ldots, a_m) \quad \textit{iff} \quad \exists \textit{uncountable cardinal } \kappa[a_1, \ldots, a_m \in L_\kappa \wedge L_\kappa \models \varphi(a_1, \ldots, a_m)].$$

$\square$

Lemma 18.21. *Assume that $0^\sharp$ exists. If $\varphi(v_0, \ldots, v_{m-1})$ is a formula, and κ is an uncountable cardinal, then for all $x_0, \ldots, x_{n-1} \in L_\kappa$,*

$$L \models \varphi(x_0, \ldots, x_{m-1}) \quad \text{iff} \quad L_\kappa \models \varphi(x_0, \ldots, x_{m-1}).$$

Proof. Induction on φ. The inductive step involving $\exists v_n \varphi(v_0, \ldots, v_n)$ goes as follows. Assume that $a_0, \ldots, a_n \in L_\kappa$ and $L \models \exists v_n \varphi(a_0, \ldots, a_{n-1}, v_n)$. Say $b \in L$ and $L \models \varphi(a_0, \ldots, a_{n-1}, b)$. Let λ be a cardinal $> \kappa$ such that $b \in L_\lambda$. Then $L_\lambda \models \varphi(a_0, \ldots, a_{n-1}, b)$

by the inductive hypothesis, so $L_\lambda \models \exists v_n\varphi(a_0,\ldots,a_{n-1},v_n)$. Hence by Theorem 18.1(i), $L_\kappa \models \exists v_n\varphi(a_0,\ldots,a_{n-1},v_n)$. The other direction is clear. $\square$

Lemma 18.22. *Assume that $0^\sharp$ exists. Let I be as in Theorem 18.18. Then for any formula $\varphi(v_0,\ldots,v_{n-1}$ and all $\alpha_0<\cdots<\alpha_{n-1}$ and $\beta_0<\cdots<\beta_{n-1}$ in I,*

$$L\models\varphi(\alpha_0,\ldots,\alpha_{n-1})\quad\text{iff}\quad L\models\varphi(\beta_0,\ldots,\beta_{n-1}).$$

Proof. Suppose that $\alpha_1<\cdots<\alpha_n$ and $\beta_1<\cdots,\beta_n$ are sequences of elements of I, and $L\models\varphi[\alpha_1,\ldots,\alpha_n]$. Let κ be an uncountable cardinal such that $\alpha_i,\beta_i<\kappa$ for each $i<n$. By Lemma 18.21, $L_\kappa\models\varphi[\alpha_1,\ldots,\alpha_n]$. By Theorem 18.18(ii)(b), $L_\kappa\models\varphi[\beta_1,\ldots,\beta_n]$. Hence by Lemma 18.21 again, $L\models\varphi[\beta_1,\ldots,\beta_n]$. $\square$

Lemma 18.23. *Suppose that $0^\sharp$ exists Then every constructible set is definable from L.*

Proof. Let $a\in L$. Choose κ uncountable such that $a\in L_\kappa$. By Theorem 18.18(ii)(c), a is definable in $(L_\kappa,\in)$ from $I\cap\kappa$. Let $\varphi(x)$ be a formula such that $a=\{u\in L_\kappa:(L_\kappa,\in)\models\varphi[u]\}$. We claim that $a=\{u\in L:(L,\in)\models\varphi[u]\}$. For, suppose that $u\in a$. Then $(L_\kappa,\in)\models\varphi[u]$, so by Theorem 18.18(i), $(L_\lambda,\in)\models\varphi[u]$ for every uncountable cardinal λ such that $u\in L_\lambda$. Thus $(L,\in)\models\varphi[u]$. Conversely, suppose that $(L,\in)\models\varphi[u]$. By definition of $L\models$, in particular $(L_\kappa,\in)\models\varphi[u]$, so $u\in a$. $\square$

Lemma 18.24. *Suppose that $0^\sharp$ exists, with I as in Lemma 18.18. Then for any formula $\varphi(v_0,\ldots,v_{m-1})$ one of the following holds:*

(i) $L\models\varphi(\alpha_0,\ldots,\alpha_{n-1})$ for every increasing sequence $\alpha_0,\ldots,\alpha_{m-1}$ of members of I.

(i) $L\models\neg\varphi(\alpha_0,\ldots,\alpha_{n-1})$ for every increasing sequence $\alpha_0,\ldots,\alpha_{m-1}$ of members of I.

Proof. By Lemma 18.29. $\square$

Lemma 18.25. *Suppose $0^\sharp$ exists. Then every constructible set definable in L is countable.*

Proof. Let $x\in L$ be definable in $(L,\in)$ by $\varphi(u)$. Thus $x=\{u\in L:(L,\in)\models\varphi[u]\}$. So $(L,\in)\models\exists!v\forall u[u\in v\leftrightarrow\varphi(v)]$. Hence $(L_{\aleph_1},\in)\models\exists!v\forall u[u\in v\leftrightarrow\varphi(v)]$. Choose $y\in L_{\aleph_1}$ such that $(L_{\aleph_1},\in)\models\forall u[u\in y\leftrightarrow\varphi(v)]$. Hence by Theorem 18.18(i), $(L_\lambda,\in)\models\forall u[u\in y\leftrightarrow\varphi(v)]$ for all uncountable cardinals λ, so $(L,\in)\models\forall u[u\in y\leftrightarrow\varphi(v)]$. Hence $x=y$. $\square$

Lemma 18.26. *Suppose $0^\sharp$ exists. Then every uncountable cardinal (in V) is inaccessible in L.*

Proof. By Theorem 13.35, "κ is a regular cardinal" is Π_1. Hence $L\models[\aleph_1$ is regular$]$. Now by Theorem 18.18(ii), the uncountable cardinals are among the indiscernibles, so for every $\alpha\geq 1[L\models[\aleph_\alpha$ is regular$]$. Similarly, for every $\alpha\geq 1[L\models[\aleph_\alpha$ is a limit cardinal$]$. Hence $\alpha\geq 1[L\models[\aleph_\alpha$ is inaccessible$]$. $\square$

Lemma 18.27. *Suppose $0^\sharp$ exists. Then every uncountable cardinal (in V) is Mahlo in L.*

Proof. By Lemma 18.26 and Theorem 18.18, every member of I is inaccessible. Hence by Theorem 18.18, since $|I \cap \omega_1| = \omega_1$, $I \cap \omega_1$ is club in ω_1, and hence ω_1 is Mahlo in L. By indiscernibility the lemma follows. $\square$

Lemma 18.28. *Suppose $0^\sharp$ exists. Then for every $\alpha \geq \omega$, $|V_\alpha \cap L| \leq |\alpha|$.*

Proof. Let $\varphi(X, \alpha)$ express that $X = V^L_\alpha$. Thus $L \models \exists! X \varphi(X, \alpha)$, so by Lemma 19.21, $L_\kappa \models \exists! X \varphi(X, \alpha)$, where κ is the least cardinal greater than α. So there is an $X \in L_\kappa$ such that $L_\kappa \models \varphi(X, \alpha)$. Say $X \in L_\beta$ with $\alpha < \beta < \kappa$. Now $X = V^L_\alpha = V_\alpha \cap L$, so $V_\alpha \cap L \subseteq L_\beta$. Since $|\beta| = |\alpha|$, this gives the desired conclusion. $\square$

Lemma 18.29. *Suppose $0^\sharp$ exists. Then the set of constructible reals is countable.*

Proof. The reals are a subset of $V_{\omega+2}$. $\square$

Lemma 18.30. *Let $j : V \to P$ be a nontrivial elementary embedding. Suppose that M is a transitive model of ZFC containing all ordinals. Let $N = j[M] = \bigcup_{\alpha \in \mathrm{ON}} j[M \cap V_\alpha]$. Then N is a transitive model of ZF, and $j : M \to N$ is elementary.*

Proof. We prove that

$$M \models \varphi(a_0, \ldots, a_{m-1}) \quad \text{iff} \quad N \models \varphi(j(a_0), \ldots, j(a_{m-1}))$$

for all $a_0, \ldots, a_{m-1} \in M$ by induction on φ It is obvious for atomic formulas, and the induction steps using $\neg$, $\wedge$ are clear. Now suppose that $M \models \exists y \varphi(a_0, \ldots, a_{m-1}, y)$. Choose $b \in M$ so that $M \models \varphi(a_0, \ldots, a_{m-1}, b)$. By the inductive hypothesis, $N \models \varphi(j(a_0), \ldots, j(a_{m-1}), j(b))$. Hence $N \models \exists y \varphi(j(a_0), \ldots, j(a_{m-1}), y)$.

Conversely, suppose that $N \models \exists y \varphi(j(a_0), \ldots, j(a_{m-1}), y)$. Choose $b \in M$ so that $N \models \varphi(j(a_0), \ldots, j(a_{m-1}), j(b))$. Then $M \models \varphi(a_0, \ldots, a_{m-1}, b)$ by the inductive hypothesis, so $M \models \exists y \varphi(a_0, \ldots, a_{m-1}, y)$. $\square$

Lemma 18.31. *If $0^\sharp$ exists, then there is a nontrivial elementary embedding of L into L.*

Proof. Let I be as in Lemma 18.18. Let $\langle i_\alpha : \alpha \in \mathbf{ON} \rangle$ be the strictly increasing enumeration of I. We define $j : I \to I$ by setting

$$j(i_\alpha) = \begin{cases} i_{\alpha+1} & \text{if } \alpha \in \omega, \\ i_\alpha & \text{if } \omega \leq \alpha. \end{cases}$$

(1) If t_1, t_2 are Skolem terms and $\alpha_0 < \cdots < \alpha_{m-1}$ are members of I such that

$$t^L_1(\alpha_0, \ldots, \alpha_{m-1}) = t^L_2(\alpha_0, \ldots, \alpha_{m-1}),$$

then $t^L_1(j(\alpha_0), \ldots, j(\alpha_{m-1})) = t^L_2(j(\alpha_0), \ldots, j(\alpha_{m-1}))$.

This is true by the definition of indiscernibility. Now we define $j'(t^L(\alpha_0, \ldots, \alpha_{m-1}) = t^L(j(\alpha_0), \ldots, j(\alpha_{m-1}))$ with $\alpha_0 < \cdots < \alpha_{m-1}$ in L. By Lemma 18.18(ii)(d), j' has domain L. Now we show that j' is an elementary embedding of L into L by induction on φ.

The only nontrivial step is when $\varphi(v_0,\ldots,v_{m-1})$ is $\exists y\psi(v_0,\ldots,v_{m-1})$. So suppose that $a_0,\ldots,a_{m-1}\in L$ and $L\models\varphi(a_0,\ldots,a_{m-1})$. Let κ be an uncountable cardinal such that $a_0,\ldots,a_{m-1}\in L_\kappa$ and $L_\kappa\models\varphi(a_0,\ldots,a_{m-1})$; here we use Lemma 18.21. Say $b\in L_\kappa$ and $L_\kappa\models\psi(a_0,\ldots,a_{m-1},b)$. Then $L\models\psi(a_0,\ldots,a_{m-1},b)$, so by the inductive hypothesis, $L\models\psi(j'(a_0),\ldots,j'(a_{m-1},j'(b))$. Hence $L\models\varphi(j'(a_0),\ldots,j'(a_{m-1})$. The converse is similar. □

Lemma 18.32. *Let M be a transitive model of ZFC.*

(i) If $X,Y\in M$, then $X\cap Y\in M$.

(ii) If $X,Y\in M$ then $X\backslash Y\in M$.

(iii) If κ is a cardinal of M, $\alpha<\kappa$, and $X\in{}^\alpha\mathscr{P}^M(\kappa)\cap M$, then $\bigcap_{\xi<\alpha}X_\xi\in M$.

(iv) If $x\in M$ then $x\cup\{x\}\in M$.

Proof. (i): We have $\forall x\in M[x\in(X\cap Y)^M$ iff $x\in X$ and $x\in Y]$. Hence $\forall x[x\in(X\cap Y)^M$ iff $x\in X$ and $x\in Y]$. So $(X\cap Y)^M=X\cap Y$.

(ii) Suppose that $X,Y\in M$. Then $\forall z\in M[z\in(X\backslash Y)^M$ iff $z\in X$ and $z\notin Y]$ iff $\forall z[z\in(X\backslash Y)^M$ iff $z\in X$ and $z\notin Y]$. Hence $(X\backslash Y)=(X0Y)^M\in M$.

(iii) $\forall a\in M[a\in\bigcap_{\xi<\alpha}X_\xi)^M$ iff $\forall\xi<\alpha[a\in X_\xi]]$. Hence $\forall a[a\in\bigcap_{\xi<\alpha}X_\xi)^M$ iff $\forall\xi<\alpha[a\in X_\xi]]$. hence $\bigcap_{\xi<\alpha}X_\xi)=\bigcap_{\xi<\alpha}X_\xi)^M\in M$.

(iv): similarly. □

An *M-ultrafilter* on κ is a collection $D\subseteq\mathscr{P}^M(\kappa)$ such that

(i) $\kappa\in D$ and $\emptyset\notin D$.

(ii) If $X,Y\in D$, then $X\cap Y\in D$.

(iii) If $X\in D$, $X\subseteq Y\subseteq\kappa$, and $Y\in M$, then $Y\in D$.

(iv) For all $X\subseteq\kappa$ such that $X\in M$, either $X\in D$ or $(\kappa\backslash X)\in D$.

D is *κ-complete* iff $\forall\alpha<\kappa\forall X\in M\cap{}^\alpha D[\bigcap_{\xi<\alpha}X_\xi\in D]$. D is *normal* iff $\forall X\in D\forall f\in M[f$ is regressive on $X\to\exists Y\in D[f$ is constant on $Y]]$. If $j:M\to N$ is a nontrivial elementary embedding. By the proof of Lemma 17.9, j moves some ordinal. The least ordinal moved by j is called the *critical point* of j.

Lemma 18.33. *Let M and N be transitive models of ZFC each containing all ordinals, and $j:M\to N$ an elementary embedding with κ the least ordinal moved by j. Then κ is an uncountable regular cardinal, and if we let $D=\{X\in\mathscr{P}^M(\kappa):\kappa\in j(X)\}$, then D is a nonprincipal normal κ-complete ultrafilter in M on κ.*

Proof. $M\models\forall x[x\notin\emptyset]$, so $N\models\forall x[x\notin j(\emptyset)]$, so $j(\emptyset)=\emptyset$. Next, note that for any $x\in M$, also $x\cup\{x\}\in M$, by Lemma 18.32(iv). Also, for any $x\in M$, $M\models\forall y[y\in x\cup\{x\}\leftrightarrow y\in x\vee y=x]$, so $N\models\forall y[y\in j(x\cup\{x\})\leftrightarrow y\in j(x)\vee y=j(x)]$, so $j(x\cup\{x\})=j(x)\cup\{j(x)\}$. It follows that $j(n)=n$ for all $n\in\omega$. Next,

$$M\models\forall x\in\omega[x=\emptyset\vee\exists y\in x[x=y\cup\{y\}]]\wedge\forall x\in\omega[x\cup\{x\}\in\omega],$$

so

$$N\models\forall x\in j(\omega)[x=j(\emptyset)\vee\exists y\in x[x=y\cup\{y\}]]\wedge\forall x\in j(\omega)[x\cup\{x\}\in j(\omega)].$$

Hence by absoluteness,

$$\forall x \in j(\omega)[x = j(\emptyset) \vee \exists y \in x[x = y \cup \{y\}]] \wedge \forall x \in j(\omega)[x \cup \{x\} \in j(\omega)].$$

Hence $j(\omega) = \omega$.

Thus $\omega < \kappa$.

Since $\kappa < j(\kappa)$ we have $\kappa \in D$. $j(\emptyset) = \emptyset$, we have $\emptyset \notin D$. Suppose that $X \in D$ and $X \subseteq Y \in \mathscr{P}^M(\kappa)$. Then $\kappa \in j(X)$ and $j(X) \subseteq j(Y)$, so $\kappa \in j(Y)$. Suppose that $X, Y \in D$. So $\kappa \in j(X) \cap j(Y)$. Now $M \models \forall x[x \in X \cap Y \leftrightarrow x \in X \wedge x \in Y]$, so $N \models \forall x[x \in j(X \cap Y) \leftrightarrow x \in j(X) \wedge x \in j(Y)]$. So by absoluteness, $j(X \cap Y) = j(X) \cap j(Y)$. Hence $\kappa \in j(X \cap Y)$, so $X \cap Y \in D$. Next, for any $X \in \mathscr{P}^M(\kappa)$ we have $M \models \forall x[x \in \kappa\backslash X \leftrightarrow x \notin X]$, so $N \models \forall x[x \in j(\kappa\backslash X) \leftrightarrow x \notin j(X)]$. By absoluteness $j(\kappa\backslash X) = j(\kappa)\backslash j(X)$. Hence $\kappa \in j(X)$ or $\kappa \in j(\kappa\backslash X)$. Hence $X \in D$ or $\kappa\backslash X \in D$. So D is an ultrafilter on κ, in M.

D is nonprincipal. For, suppose that $\alpha < \kappa$. Now $M \models \forall x[x \in \{\alpha\} \leftrightarrow x = \alpha]$, so $N \models \forall x[x \in j(\{\alpha\}) \leftrightarrow x = j(\alpha)$. Thus by absoluteness, $\forall x[x \in j(\{\alpha\}) \leftrightarrow x = j(\alpha)$. So $j(\{\alpha\}) = \{j(\alpha)\} = \{\alpha\}$. Hence $\kappa \notin j(\{\alpha\})$; hence $\{\alpha\} \notin D$.

Next, D is κ-complete. For, suppose that $\gamma < \kappa$ and $X \in M \cap {}^\gamma D$. Thus $\kappa \in j(X_\xi)$ for all $\xi < \alpha$. Let $Y = \bigcap_{\xi<\gamma} X_\xi$. Now $j(X)$ is a function with domain $j(\gamma) = \gamma$.

(1) If $\xi < \gamma$, then $(j(X))_\xi = j(X_\xi)$.

In fact, $(\xi, X_\xi) \in X$, so $(j(\xi), j(X_\xi)) \in J(X)$, and (1) follows.

(2) $j(Y) = \bigcap_{\xi<\gamma} j(X_\xi)$.

In fact, $M \models \forall y[y \in Y \leftrightarrow \forall\xi < \gamma[y \in X_\xi]]$, so $N \models \forall y[y \in j(Y) \leftrightarrow \forall\xi < \gamma[y \in j(X_\xi)]]$, and (2) follows.

By (2), $\kappa \in j(Y)$. So $Y \in D$. This shows that D is κ-complete.

Now clearly κ is a limit ordinal. Since D is κ-complete, κ is an uncountable regular cardinal.

The normality of D is proved as for Theorem 17.11. $\square$

Now in M let $F = \{f \in M : \mathrm{dmn}(f) = \kappa\}$. We define for D an M-ultrafilter

$$f = *_M g \quad \text{iff} \quad f, g \in F \text{ and } \{\alpha \in \kappa : f(\alpha) = g(\alpha)\} \in D;$$
$$f \in^*_M g \quad \text{iff} \quad f, g \in F \text{ and } \{\alpha \in \kappa : f(\alpha) \in g(\alpha)\} \in D.$$

Then $=^*_M$ is an equivalence relation on F. For each $f \in F$ let, in M,

$$[f]_M = \{g \in M : f =^*_M g \text{ and } \forall h \in M[h =^*_M f \to \mathrm{rank}^M(g) \leq \mathrm{rank}^M h]\}.$$

Let $\mathrm{Ult}_D(M)$ be the collection of all equivalence classes, with $\in^M_D = \{(x,y) : \exists f, g[x = [f]_M,\ y = [g]_M,\ f \in^*_M g]\}$. Note that possibly $\mathrm{Ult}_D(M)$ is not well-founded. There could be $f_n \in F$ such that $[f_{n+1}] \in^*_D [f_n]$ for all n, while $\langle f_n : n \in \omega\rangle \notin M$, so that $\bigcap\{\alpha < \kappa : f_{n+1}(\alpha) \in f_n(\alpha)\} \notin M$.

Lemma 18.34.

$$\mathrm{Ult}_D(M) \models \varphi([f^0]_M, \ldots, [f^{m-1}]_M) \text{ iff } \{\alpha < \kappa : M \models \varphi(f^0(\alpha), \ldots, f^{m-1}(\alpha))\} \in D.$$

Proof. Induction on φ.

$$\begin{aligned}
\mathrm{Ult}_D(M) \models [f]=[g] \quad &\text{iff} \quad f =^* g \\
&\text{iff} \quad \{\alpha<\kappa : f(\alpha)=g(\alpha)\} \in D \\
&\text{similarly for } \in^*; \\
\mathrm{Ult}_D(M) \models \neg\varphi([f_1],\ldots,[f_n]) \quad &\text{iff} \quad \text{not}(\mathrm{Ult} \models \varphi([f_1],\ldots,[f_n])) \\
&\text{iff} \quad \text{not}(\{\alpha<\kappa M \models \varphi(f_1(\alpha),\ldots,f_n(\alpha))\} \in D \\
&\text{iff} \quad \kappa\backslash\{\alpha<\kappa : M \models \varphi(f_1(\alpha),\ldots,f_n(\alpha))\}) \in D \\
&\text{iff} \quad \{\alpha<\kappa : M \models \neg\varphi(f_1(\alpha),\ldots,f_n(\alpha))\}) \in D \\
&\text{similarly for } \in^* .
\end{aligned}$$

Now suppose that $\mathrm{Ult}_D(M) \models \exists u\varphi(u,[f_0],\ldots,[f_{n-1}])$. Choose g so that

$$\mathrm{Ult}_D(M) \models \varphi([g],[f_0],\ldots,[f_{n-1}]).$$

Then by the inductive hypothesis, $\{\alpha<\kappa : M \models \varphi(g(\alpha),f_0(\alpha),\ldots,f_{n-1}(\alpha))\} \in D$. Now

$$\{\alpha<\kappa : M \models \varphi(g(\alpha),f_0(\alpha),\ldots,f_{n-1}(\alpha))\} \subseteq \{\alpha<\kappa : M \models \exists u\varphi(u,f_0(\alpha),\ldots,f_{n-1}(\alpha))\},$$

so $\{\alpha<\kappa : M \models \exists u\varphi(u,f_0(\alpha),\ldots,f_{n-1}(\alpha))\} \in D$.

Conversely, suppose that $K \overset{\text{def}}{=} \{\alpha<\kappa : M \models \exists u\varphi(u,f_0(\alpha),\ldots,f_{n-1}(\alpha))\} \in D$. For each $\alpha \in K$ choose $g(\alpha)$ so that $M \models \varphi(g(\alpha),f_0(\alpha),\ldots,f_{n-1}(\alpha))$, and let $g(\alpha)$ be arbitrary for other α. Then by the inductive hypothesis, $\mathrm{Ult}_D(M) \models \varphi([g],[f_0],\ldots,[f_{n-1}])$, so $\mathrm{Ult} \models \exists u\varphi(u,[f_0],\ldots,[f_{n-1}])$. $\square$

For each $a \in M$ let $c_a : \kappa \to M$ be defined by $c_a(\alpha)=a$ for all $\alpha<\kappa$. Define $j_D(a)=[c_a]$.

Lemma 18.35. *j_D is an elementary embedding of M into $\mathrm{Ult}_D(M)$.*

Proof.

$$\begin{aligned}
M \models \varphi(a_0,\ldots,a_{m-1}) \quad &\text{iff} \quad \{\alpha<\kappa : M \models \varphi(c_{a_0}(\alpha),\ldots,c_{a_{m-1}}(\alpha))\} \in D \\
&\text{iff} \quad \mathrm{Ult}_D(M) \models \varphi([c_{a_0}],\ldots,[c_{a_{m-1}}]) \\
&\text{iff} \quad \mathrm{Ult}_D(M) \models \varphi(j_D(a_0),\ldots,j_D(a_{m-1})).
\end{aligned}$$ $\square$

Lemma 18.36. *If M and N are transitive models of ZF each containing all ordinals, and if $j : M \to N$ is an elementary embedding with κ the first ordinal moved, and if $D = \{X \in \mathscr{P}^M(\kappa) : \kappa \in j(X)\}$, then there is an elementary embedding k of $\mathrm{Ult}_D(M)$ into N such that $k \circ j_D = j$:*

xxx

Proof. We would like to define k as follows. Let u be any member of $\mathrm{Ult}_D(M)$. Choose f, a function with domain κ, such that $u=[f]$. Note that $M \models [f$ is a function with domain $\kappa]$, so $N \models [j(f)$ is a function with domain $j(\kappa)]$. Then define

$$k(u) = (j(f))(\kappa).$$

To show that this is possible, suppose that $f, g \in u$. Then $f =^* g$, and so $X \stackrel{\text{def}}{=} \{\alpha \in \kappa : f(\alpha) = g(\alpha)\} \in D$. Now

$$\forall x[x \in X \leftrightarrow x \in \kappa \text{ and } f(x) = g(x)],$$

so

$$\forall x[x \in j(X) \leftrightarrow x \in j(\kappa) \text{ and } (j(f))(x) = (j(g))(x)].$$

Now $X \in D$, so $\kappa \in j(X)$, so it follows that $(j(f))(\kappa) = (j(g))(\kappa)$. This shows that k is well-defined.

Next, to show that k is an elementary embedding, let $\varphi(v_0, \ldots, v_{n-1})$ be given. Suppose that $\mathrm{Ult}_D(M) \models \varphi([f_0], \ldots, [f_{n-1}])$. Then $\mathrm{Ult}_D(M) \models \varphi([f_0], \ldots, [f_{n-1}])$, so by the above, $X \stackrel{\text{def}}{=} \{\alpha \in \kappa : M \models \varphi(f_0(\alpha), \ldots, f_{n-1}(\alpha))\} \in D$. So $\kappa \in j(X)$. Now

$$\forall \alpha[\alpha \in X \leftrightarrow \alpha \in \kappa \wedge M \models \varphi(f_0(\alpha), \ldots, f_{n-1}(\alpha))],$$

and hence

$$\forall \alpha[\alpha \in j(X) \leftrightarrow \alpha \in j(\kappa) \wedge N \models \varphi((j(f_0))(\alpha), \ldots, (j(f_{n-1}))(\alpha))].$$

Hence $N \models \varphi((j(f_0))(\kappa), \ldots, (j(f_{n-1}))(\kappa))$. So $N \models \varphi(k([f_0]), \ldots, k([f_{n-1}]))$, as desired.

Finally, $k(j_D(a)) = k([c_a]) = (j(c_a))(\kappa)$. Now $\forall \alpha < \kappa(c_a^\kappa(\alpha) = a)$, so

$$\forall \alpha < j(\kappa)((j(c_a))(\alpha) = j(a)).$$

Since $\kappa < j(\kappa)$, we have $(j(c_a))(\kappa) = j(a)$, as desired. $\square$

Lemma 18.37. *If M and N are transitive models of ZF each containing all ordinals, and if $j : M \to N$ is an elementary embedding with κ the first ordinal moved, and if $D = \{X \in \mathscr{P}^M(\kappa) : \kappa \in j(X)\}$, then $\mathrm{Ult}_D(M)$ is well-founded.*

Proof. Suppose that $\cdots \in^* [f_{n+1}] \in^* [f_n] \in^* \cdots \in^* [f_0]$. Then by Lemma 18.36 $\cdots \in k([f_{n+1}]) \in k([f_n]) \in \cdots \in k([f_0])$, contradiction. $\square$

Lemma 18.38. *Let $j : L \to L$ be a non-trivial elementary embedding. Let $D = \{X \in \mathscr{P}^L(\gamma) : \gamma \in j(X)\}$, where γ is the critical point of j.*

(i) D is an L-ultrafilter.

(ii) $\mathrm{Ult}_D(L)$ is well-founded.

Now let π be the Mostowski collapsing function from $\mathrm{Ult}_D(L)$ onto a transitive class P.

(iii) $P = L$; so $\pi \circ j_D : L \to L$ is an elementary embedding.

(iv) $\forall \delta < \gamma[\pi(j_D(\delta)) = \delta]$.

Now let $d = \langle \xi : \xi < \gamma \rangle$.

(v) $\gamma \leq \pi([d])$.

(vi) $\pi([d]) < \pi([c_\gamma])$.

(vii) γ is the critical point of $\pi \circ j_D$.

Proof. (i) follows from Lemma 18.33. (ii) holds by Lemma 18.37. For (iii),

$$\begin{aligned}
&L \models \forall x \exists \alpha[\alpha \text{ is an ordinal} \wedge x \in L_\alpha]; \quad \text{hence}\\
&\mathrm{Ult}_D(L) \models \forall x \exists \alpha[\alpha \text{ is an ordinal} \wedge x \in L_\alpha]; \quad \text{hence}\\
&P \models \forall x \exists \alpha[\alpha \text{ is an ordinal} \wedge x \in L_\alpha];
\end{aligned}$$

It follows that $P = L$.

For (iv), suppose that $\delta < \gamma$ and $\delta < \pi(j_D(\delta))$, with δ minimum. Say $\pi([f]) = \delta$. Thus $\pi([f]) < \pi(j_D(\delta))$, so $[f] \in^* [c_\delta]$, hence $\{\xi < \gamma : f(\xi) < \delta\} \in D$. Now

$$\{\xi < \gamma : f(\xi) < \delta\} = \bigcup_{\varepsilon<\delta} \{\xi < \gamma : f(\xi) = \varepsilon\},$$

so, since D is γ-complete, there is a $\varepsilon < \delta$ such that $\{\xi < \gamma : f(\xi) = \varepsilon\} \in D$. Thus $[f] = [c_\varepsilon]$, so $\pi([f]) = \pi([c_\varepsilon]) = \pi(j_D(\varepsilon)) = \varepsilon$, contradiction. This proves (iv).

For (v), clearly γ is a limit ordinal. If $\delta < \gamma$, then $\{\xi < \gamma : \delta < d(\xi)\} = \gamma \backslash (\delta + 1) \in D$, so $[c_\delta] \in^* [d]$ and hence by (iv), $\delta = \pi([c_\delta]) < \pi([d])$. Now (v) follows.

For (vi), $\{\xi < \gamma : d(\xi) < \gamma\} = \gamma \in D$, so $[d] \in^* [c_\gamma]$, and hence (vi) follows.

Now (vii) follows from (iv), (v) and (vi). □

Lemma 18.39. *Suppose that $j : L \to L$ is a non-trivial elementary embedding, and let γ, D and π be as in Lemma 18.38. Suppose that κ is a limit cardinal such that $\mathrm{cf}(\kappa) > \gamma$. Then $\pi(j_D(\kappa)) = \kappa$.*

Proof.

(1) $\pi(j_D(\kappa)) = \bigcup_{\alpha<\kappa} \pi(j_D(\alpha))$.

In fact, if $\alpha < \kappa$, clearly $\pi(j_D(\alpha)) < \pi(j_D(\kappa))$. Now suppose that $\pi([f]) < \pi(j_D(\kappa))$. Thus $[f] \in^* [c_\kappa]$, so $K \stackrel{\text{def}}{=} \{\xi < \gamma : f(\xi) < \kappa\} \in D$. Let $\beta = \sup\{f(\xi) : \xi \in K\}$. Thus $\beta < \kappa$. Hence $K \subseteq \{\xi < \gamma : f(\xi) < \beta + 1\}$, so $\{\xi < \gamma : f(\xi) < \beta\} \in D$ and hence $[f] \in^* [c_{\beta+1}]$. Hence $\pi([f]) < \pi(j_D(\beta + 1))$, proving (1).

Now suppose that $\alpha < \kappa$. Then $\pi(j_D(\alpha)) = \pi([c_\alpha]) = \{\pi([f]) : [f] \in^* [c_\alpha]\}$. For each f with $[f] \in^* [c_\alpha]$ there is an $f' \in {}^\gamma\alpha$ such that $[f] = [f']$. It follows that $|\pi([c_\alpha])| \leq |({}^\gamma\alpha)^L| < \kappa$. Hence $\pi(j_D(\alpha)) < \kappa$. Now the lemma follows using (1). □

Lemma 18.40. *Suppose that $j : L \to L$ is an elementary embedding with γ as critical point and $j(\kappa) = \kappa$ for every limit cardinal κ such that $\mathrm{cf}(\kappa) > \gamma$. Define*

$$\begin{aligned}
U_0 &= \{\kappa : \kappa \textit{ is a limit cardinal } > \gamma\};\\
U_{\alpha+1} &= \{\kappa \in U_\alpha : |U_\alpha \cap \kappa| = \kappa\};\\
U_\lambda &= \bigcap_{\alpha<\lambda} U_\alpha \quad \textit{for } \lambda \textit{ limit.}
\end{aligned}$$

Then each U_α is a proper class and is δ-closed for each δ with $\mathrm{cf}(\delta) > \gamma$, i.e., if $\kappa \in {}^\delta U_\alpha$ is strictly increasing, then $\bigcup_{\xi<\delta} \kappa_\xi \in U_\alpha$.

Proof. Let $\langle \gamma^0_\xi : \xi \in \mathbf{ON}\rangle$ be the strictly increasing enumeration of $U_0 \stackrel{\text{def}}{=} \{\kappa : \mathrm{cf}(\kappa) > \gamma\}$. If $\langle \gamma^\alpha_\xi : \xi \in \mathbf{ON}\rangle$ has been defined and is strictly increasing, let $\langle \gamma^{\alpha+1}_\xi : \xi \in \mathbf{ON}\rangle$ be the strictly increasing enumeration of $U_{\alpha+1} \stackrel{\text{def}}{=} \{\kappa : \gamma^\alpha_\kappa = \kappa\}$. For λ limit let $U_\lambda = \bigcap_{\alpha<\lambda} U_\alpha$.

(1) $\forall \varepsilon \in \mathbf{ON} \exists \kappa \in U_\lambda[\varepsilon < \kappa]$.

In fact, for each $\alpha < \lambda$ choose ξ_α such that $\varepsilon < \gamma^\alpha_{\xi_\alpha}$. Let $\eta = \bigcup_{\alpha<\lambda} \xi_\alpha$. Then $\varepsilon < \bigcup_{\alpha<\lambda} \gamma^\alpha_\eta$, and for all $\beta < \lambda$,

$$\bigcup_{\alpha<\lambda} \gamma^\alpha_\eta \le \gamma^\beta_{\bigcup_{\alpha<\lambda} \gamma^\alpha_\eta} = \bigcup_{\alpha<\lambda} \gamma^\beta_{\gamma^\alpha_\eta} = \bigcup_{\beta\le\alpha<\lambda} \gamma^\beta_{\gamma^\alpha_\eta} = \bigcup_{\beta\le\alpha<\lambda} \gamma^\alpha_\eta \le \bigcup_{\alpha<\lambda} \gamma^\alpha_\eta$$

and so $\bigcup_{\alpha<\lambda} \gamma^\alpha_\eta \in U_\beta$. Thus (1) holds.

By (1), let $\langle \gamma^\lambda_\xi : \xi \in \mathbf{ON}\rangle$ enumerate U_λ.

(2) For all α and all δ such that $\mathrm{cf}(\delta) > \gamma$, for all $\kappa \in {}^\delta U_\alpha$, if κ is strictly increasing then $\bigcup_{\xi<\delta} \kappa_\xi \in U_\alpha$.

We prove (2) by induction on α. For $\alpha = 0$ we have $\mathrm{cf}(\bigcup_{\xi<\delta} \kappa_\xi) = \mathrm{cf}(\delta) > \gamma$, so $\bigcup_{\xi<\delta} \kappa_\xi \in U_0$. Now assume it for α and suppose that $\mathrm{cf}(\delta) > \gamma$ and $\kappa \in {}^\delta U_{\alpha+1}$ is is strictly increasing. For each $\xi < \delta$ we have $\gamma^\alpha_{\kappa_\xi} = \kappa_\xi$. Hence

$$\gamma^\alpha_{\bigcup_{\xi<\delta} \kappa_\xi} = \bigcup_{\xi<\delta} \gamma^\alpha_{\kappa_\xi} = \bigcup_{\xi<\delta} \kappa_\xi,$$

and so $\bigcup_{\xi<\delta} \kappa_\xi \in U_{\alpha+1}$. Finally, suppose that λ is limit and (2) holds for all $\alpha < \gamma$. Suppose that $\mathrm{cf}(\delta) > \gamma$ and $\kappa \in {}^\delta U_\lambda$ is strictly increasing. Then for all $\alpha < \lambda$, $\bigcup_{\xi<\delta} \kappa_\xi \in U_\alpha$, so $\bigcup_{\xi<\delta} \kappa_\xi \in U_\lambda$. Hence (2) holds. $\square$

Lemma 18.41. *Let j, γ, U_α be as in Lemma 18.40. Choose $\kappa \in U_{\omega_1}$. Then $i \stackrel{\text{def}}{=} j \restriction L_\kappa$ is an elementary embedding of L_κ into L_κ.*

Proof. Suppose that $x \in L_\kappa$. Then $\rho(x) < \kappa$, so $\rho(j(x)) < j(\kappa) = \kappa$. So i maps L_κ into L_κ.

For each formula φ let φ^* be obtained from φ by replacing $\exists x\psi$ by $\exists x[\rho(x) < y \wedge \psi$. Then for any $a_0, \ldots, a_{m-1} \in L_\kappa$,

$$\begin{aligned} L_\kappa \models \varphi(a_0, \ldots, a_{m-1}) \quad &\text{iff} \quad L \models \varphi^*(\kappa, a_0, \ldots, a_{m-1}) \\ &\text{iff} \quad L \models \varphi^*(\kappa, j(a_0), \ldots, j(a_{m-1})) \\ &\text{iff} \quad L_\kappa \models \varphi(j(a_0), \ldots, j(a_{m-1})). \end{aligned}$$

$\square$

Lemma 18.42. *Let j, γ, U_α be as in Lemma 18.40. Choose $\kappa \in U_{\omega_1}$. Let $X_\alpha = U_\alpha \cap \kappa$ and M_α be the closure of $\gamma \cup X_\alpha$ under Skolem functions, for each $\alpha < \omega_1$.*

(i) $\forall \alpha < \omega_1[M_\alpha \preceq L_\kappa]$.

Let π_α be the transitive collapse of M_α.

(ii) $\pi_\alpha[M_\alpha] = L_\kappa$.

(iii) π_α^{-1} is an elementary embedding of L_κ into L_κ.

For each $\alpha < \omega_1$ let $\gamma_\alpha = \pi_\alpha^{-1}(\gamma)$. Then for each $\alpha < \omega_1$.

(iv) γ_α is the least ordinal greater than γ in M_α; in particular, $\gamma \notin M_\alpha$.

(v) If $\alpha < \beta \in \omega_1$ and $x \in M_\beta$, then $\pi_\alpha^{-1}(x) = x$.

(vi) If $\alpha < \beta \in \omega_1$ then $\pi_\alpha^{-1}(\gamma_\beta) = \gamma_\beta$.

(vii) If $\alpha < \beta < \omega_1$, then $\gamma_\alpha < \gamma_\beta$.

Proof. (i) is clear. (ii) holds by Theorem 13.46′. (iii) holds by Lemma 18.41. For (iv), note that $\gamma \subseteq M_\alpha$ and hence $\pi_\alpha(\xi) = \xi$ for all $\xi < \gamma$. So $\pi_\alpha^{-1}(\xi) = \xi$ for each $\xi < \gamma$. Hence $\pi_\alpha^{-1}(\gamma)$ is the least ordinal in M_α greater or equal than γ. So it suffices to show that $\gamma \notin M_\alpha$. If $x \in M_\alpha$ then $x = t(a_0, \ldots, a_{m-1})$ with each $a_i \in \gamma \cup X_\alpha$. If $a_i < \gamma$ then $j(a_i) = a_i$ since γ is the least ordinal moved by j. If $a_i \in X_\alpha$ then $j(a_i) = a_i$ since $X_\alpha \subseteq U_\alpha \subseteq U_0$. So $j(x) = x$. Since $\gamma < j(\gamma)$, it follows that $\gamma \notin M_\alpha$.

For (v), suppose that $\alpha < \beta \in \omega_1$ and $x \in M_\beta$. Then $x = t(a_0, \ldots, a_{m-1})$ with each $a_i \in \gamma \cup X_\beta$. If $a_i < \gamma$ then $\pi_\alpha(a_i) = a_i$ since $\gamma \subseteq X_\alpha$. If $a_i \in X_\beta$ then $|X_\alpha \cap a_i| = a_i$ since $\alpha < \beta$.

(1) For all $\delta \in \gamma \cup X_\beta[\pi_\alpha(\delta) \leq \delta]$.

This is clear by induction: $\pi_\alpha(\delta) = \{\pi_\alpha(\varepsilon) : \varepsilon \in M_\alpha, \varepsilon \in \delta\} \subseteq \delta$.

It follows that $\pi_\alpha(a_i) \leq a_i$. Now $\pi_\alpha(a_i) = \{p_\alpha(x) : x \in M_\alpha \cap a_i\}$ and $|M_\alpha \cap a_i| \geq |X_\alpha \cap a_i| = a_i$, so $\pi_\alpha(a_i) = a_i$.

Now it follows that $\pi_\alpha(x) = x$.

(vi) follows from (v) since $\gamma_\beta \in M_\beta$.

(vii): $M_\alpha \subseteq M_\beta$, so by (iv), $\gamma_\alpha \leq \gamma_\beta$. Now $\gamma < \gamma_\alpha$, so $\pi_\alpha^{-1}(\gamma_\alpha) > \pi_\alpha(\gamma) = \gamma_\alpha$, while by (vi), $\pi_\alpha^{-1}(\gamma_\beta) = \gamma_\beta$. So $\gamma_\alpha < \gamma_\beta$. $\square$

Lemma 18.43. *Let j, γ, U_α be as in Lemma 18.40, and $\kappa, X_\alpha, M_\alpha, \pi_\alpha, \gamma_\alpha$ be as in Lemma 18.49.*

If $\alpha < \beta$, then there is an elementary embedding $i_{\alpha\beta} : L_\kappa \to L_\kappa$ such that $\forall \xi[\xi < \alpha \vee \beta < \xi[i_{\alpha\beta}(\gamma_\xi) = \gamma_\xi]]$ and $i_{\alpha\beta}(\gamma_\alpha) = \gamma_\beta$.

Proof. Let $M_{\alpha\beta}$ be the closure under Skolem terms in L_κ of $\gamma_\alpha \cup X_\beta$. Let $\pi_{\alpha\beta}$ be the transitive collapse of $M_{\alpha\beta}$ and let $i_{\alpha\beta} = \pi_{\alpha\beta}^{-1}$.

(1) $i_{\alpha,\beta}$ is an elementary embedding of L_κ into L_κ.

In fact, $M_{\alpha,\beta} \preceq L_\kappa$ and $|M_{\alpha,\beta}| = \kappa$, so by Theorem 13.42, $\pi_{\alpha,\beta}$ collapses $M_{\alpha,\beta}$ onto L_κ. Hence $i_{\alpha,\beta}$ maps L_κ onto $M_{\alpha,\beta}$. Hence

$$\begin{aligned} L_\kappa \models \varphi(\overline{x}) \quad &\text{iff} \quad M_{\alpha,\beta} \models \varphi(i_{\alpha,\beta} \circ \overline{x}) \\ &\text{iff} \quad L_\kappa \models \varphi(i_{\alpha,\beta} \circ \overline{x}) \end{aligned}$$

(2) $\pi_{\alpha,\beta} \restriction \gamma_\alpha$ is the identity.

This is clear since $\gamma_\alpha \subseteq M_{\alpha\beta}$.

(3) If $\eta \in X_{\beta+1}$, then $|X_\beta \cap \eta| = \eta$ and $i_{\alpha,\beta}(\eta) = \eta$.

For, suppose that $\eta \in X_{\beta+1}$. Now $X_{\beta+1} = U_{\beta+1} \cap \kappa$, and so $|X_\beta \cap \eta| = \eta$. By the argument around (1) in the proof of Lemma 18.42, $i_{\alpha,\beta}(\eta) = \eta$.

(4) For $\xi > \beta$, $\gamma_\xi \in M_{\beta+1}$ and so $i_{\alpha,\beta}(\gamma_\xi) = \gamma_\xi$.

For, by Lemma 18.24(i), $\gamma_\xi \in M_\xi$ and $M_\xi \subseteq M_{\beta+1}$ since $X_\xi \subseteq X_{\beta+1}$. So $i_{\alpha,\beta}(\gamma_\xi) = \gamma_\xi$ by the argument given.

Finally we show that $i_{\alpha\beta}(\gamma_\alpha) = \gamma_\beta$. Note that $\gamma < \gamma_\alpha$ by Lemma 18.42(iv), so $\gamma \cup X_\beta \subseteq \gamma_\alpha \cup X_\beta$, hence $M_\beta \subseteq M_{\alpha\beta}$. So $\gamma_\beta \in M_{\alpha\beta}$. Now let δ be the least ordinal in $M_{\alpha\beta}$ greater than or equal to γ_α. Then $\pi_{\alpha\beta}(\delta) = \{\pi_{\alpha\beta}(\varepsilon) : \varepsilon \in M_{\alpha\beta}, \varepsilon < \delta\}$. Now $\forall \varepsilon \in M_{\alpha\beta}[\varepsilon < \delta \leftrightarrow \varepsilon < \gamma_\alpha]$. Hence $\pi_{\alpha\beta}(\delta) = \{\pi_{\alpha\beta}(\varepsilon) : \varepsilon \in M_{\alpha\beta}, \varepsilon < \gamma_\alpha\} = \{\varepsilon : \varepsilon \in M_{\alpha\beta}, \varepsilon < \gamma_\alpha\} = \gamma_\alpha$. Hence $\delta = i_{\alpha\beta}(\gamma_\alpha)$. Since $\gamma_\beta \in M_{\alpha\beta}$, it follows that $i_{\alpha\beta}(\gamma_\alpha) \leq \gamma_\beta$.

(5) There is no $\delta \in M_{\alpha\beta}$ such that $\gamma_\alpha \leq \delta < \gamma_\beta$.

For, assume otherwise. Say $\gamma_\alpha \leq \delta < \gamma_\beta$ and $\delta = t(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_k)$ with each $\xi_i < \gamma_\alpha$ and each $\eta_i \in X_\beta$. Then

$$\begin{aligned}&(L_\kappa, \in) \models \exists \xi_1, \ldots, \xi_n < \gamma_\alpha \exists \eta_1, \ldots, \eta_k \in X_\beta\\ &[\gamma_\alpha \leq t(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_k) < \gamma_\beta]\end{aligned}$$

Now $\gamma_\alpha = \pi_\alpha^{-1}(\gamma)$, and $\forall i = 1, \ldots, k[\pi_\alpha^{-1}(\eta_i) = \eta_i]$, and $i_\alpha(\gamma_\beta) = \gamma_\beta$ by Lemma 18.49. Hence

$$\begin{aligned}&(L_\kappa, \in) \models \exists \xi_1, \ldots, \xi_n < \gamma \exists \eta_1, \ldots, \eta_k \in X_\beta\\ &[\gamma \leq t(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_k) < \gamma_\beta]\end{aligned}$$

It follows that $t(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_k) \in M_\beta$, contradicting Lemma 18.42(iv). □

Lemma 18.44. *Let j, γ, U_α be as in Lemma 18.40, $\kappa, X_\alpha, M_\alpha, \pi_\alpha, \gamma_\alpha$ be as in Lemma 18.42, and $i_{\alpha\beta}$ as in Lemma 18.43. Then $\{\gamma_\alpha : \alpha < \omega_1\}$ is a set of indiscernibles for $(L_\kappa, \in)$.*

Proof. Suppose that $\alpha_0 < \cdots < \alpha_{m-1} < \omega_1$ and $\beta_0 < \cdots \beta_{m-1} < \omega_1$. Choose $\delta_0 < \cdots < \delta_{m-1} < \omega_1$ so that $\alpha_{m-1}, \beta_{m-1} < \delta_0$. Then applying $i_{\alpha_{m-1}\delta_{m-1}}$,

$$L_\kappa \models \varphi(\gamma_{\alpha_0}, \ldots, \gamma_{\alpha_{m-1}} \quad \text{iff} \quad L_\kappa \models \varphi(\gamma_{\alpha_0}, \ldots, \gamma_{\alpha_{m-1}}, \gamma_{\delta_{m-1}}).$$

Then applying $i_{\alpha_{m-2}\delta_{m-2}}, \ldots, i_{\alpha_0\delta_0}$ we get

$$L_\kappa \models \varphi(\gamma_{\alpha_0}, \ldots, \gamma_{\alpha_{m-1}} \quad \text{iff} \quad L_\kappa \models \varphi(\gamma_{\delta_0}, \ldots, \gamma_{\delta_{m-1}}).$$

Doing the same with the β's we obtain

$$L_\kappa \models \varphi(\gamma_{\alpha_0}, \ldots, \gamma_{\alpha_{m-1}} \quad \text{iff} \quad L_\kappa \models \varphi(\gamma_{\beta_0}, \ldots, \gamma_{\beta_{m-1}}).$$ □

Theorem 18.45. *If there is a non-trivial elementary embedding of L into L, then $0^\sharp$ exists.* $\square$

Theorem 18.46. *Let κ be an uncountable cardinal and let $j : L_\kappa \to L_\kappa$ be an elementary embedding with critical point γ. Then $0^\sharp$ exists.*

Proof.

(1) If $X \in L$ and $X \subseteq \gamma$, then $X \in L_\kappa$.

For, suppose that $X \in L$ and $X \subseteq \gamma$. Let δ be a limit ordinal $> \kappa$ such that $X \in L_\delta$. Let M be an elementary substructure of L_δ such that $X \in M$, $\gamma \subseteq M$, and $|M| = |\gamma|$. Let N be the transitive collapse of M. Then by Theorem 13.42, $M = L_\varepsilon$ for some $\varepsilon \le \delta$. Since $|M| = |\gamma| < \kappa$, we have $X \in L_\kappa$, as desired in (1).

Now let $D = \{X : X \in L, X \subseteq \gamma, \gamma \in j(X)\}$. By the proof of Lemma 18.33, D is an L-ultrafilter. We claim that $\mathrm{Ult}_D(L)$ is well-founded. This will prove the theorem since it gives a non-trivial embedding of L into L.

So suppose that $[f_{n+1}] \in^* [f_n]$ for all $n \in \omega$, where each $f_n \in {}^\gamma L$, $f_n \in L$. Thus $\forall n \in \omega[\{\alpha < \gamma : f_{n+1}(\alpha) \in f_n(\alpha)\} \in D]$. Choose a limit ordinal $\theta < \kappa$ such that $\forall n \in \omega[f_n \in L_\theta]$. Let M be an elementary substructure of $(L_\theta, \in)$ such that $|M| = |\gamma|$, $\gamma \subseteq M$, and $\forall n \in \omega[f_n \in M]$. The transitive collapse of M has the form L_η with $\eta \le \theta$. Moreover, $\eta < \kappa$ since $|M| < \kappa$. Let π be the transitive collapse, and let $g_n = \pi(f_n)$ for all $n \in \omega$.

(2) $\forall \xi < \gamma \forall n \in \omega[g_{n+1}(\xi) \in g_n(\xi)$ iff $f_{n+1}(\xi) \in f_n(\xi)]$.

In fact suppose that $\xi < \gamma$ and $n \in \omega$. Then

$$\begin{aligned} g_n = \pi(f_n) &= \{\pi(x) : x \in f_n\} = \{\pi(\xi, f_n(\xi)) : \xi \in \gamma\} \\ &= \{\pi(\{\{\xi\}, \{\xi, f_n(\xi)\}\}) : \xi \in \gamma\} \\ &= \{\{\xi\}, \{\xi, \pi(f_n(\xi))\} : \xi \in \gamma\} \\ &= \{(\xi, \pi(f_n(\xi))) : \xi \in \gamma\}. \end{aligned}$$

Thus $g_n(\xi) = \pi(f_n(\xi))$. Hence

$$g_{n+1}(\xi) \in g_n(\xi) \quad \text{iff} \quad \pi(f_{n+1}(\xi)) \in \pi(f_n(\xi)) \quad \text{iff} \quad f_{n+1}(x) \in f_n(\xi).$$

Now each $g_n \in L_\eta \subseteq L_\kappa$.

Since $\{\alpha < \gamma : f_{n+1}(\alpha) \in f_n(\alpha)\} \in D]$, we also have $X \stackrel{\text{def}}{=} \{\alpha < \gamma : g_{n+1}(\alpha) \in g_n(\alpha)\} \in D]$. Hence $\gamma \in j(X)$, so $g_{n+1}(\gamma) < g_n(\gamma)$ for all $n \in \omega$, contradiction. $\square$

Theorem 18.47. *If there is a Jónsson cardinal, then $0^\sharp$ exists.*

Proof. Let κ be a Jónsson cardinal. Let A be a proper elementary substructure of $(L_\kappa, \in)$. Let B be the transitive collapse of A. By Theorem 13.42, $B = L_\kappa$. With π the transitive collapse function, π^{-1} is an non-trivial elementary embedding of L_κ into L_κ. $\square$

For Jensen's covering theorem see his origenal proof: *The fine stucture of the constructible universe.* Ann. Math. Logic 4 (1972), 229–308.

Chapter 19. Iterated ultrapowers and L[U]

Lemma 19.1. *Let κ be a measurable cardinal and let U be a nonprincipal κ-complete ultrafilter on κ. In $L[U]$, $\overline{U}$ is a κ-complete nonprincipal ultrafilter on κ.*

Proof. Recall from Theorem 13.60 that $\overline{U} = U \cap L[U]$. Suppose that $X \in \overline{U}$ and $X \subseteq Y \subseteq \kappa$, where $Y \in L[U]$. Then $Y \in U$, so $Y \in \overline{U}$.

Suppose that $X, Y \in \overline{U}$. Clearly $X \cap Y \in \overline{U}$.

Obviously $\kappa \in \overline{U}$ and $\emptyset \notin \overline{U}$. If $X \subseteq \kappa$ with $X \in L[U]$, then $X \in U$ or $(\kappa \backslash X) \in U$, hence $X \in \overline{U}$ or $(\kappa \backslash X) \in \overline{U}$.

If $\alpha \in \kappa$, then $\{\alpha\} \notin \overline{U}$.

If $\alpha < \kappa$ and $X_\xi \in \overline{U}$ for all $\xi < \alpha$, with $X \in L[U]$, then clearly $\bigcap_{\xi<\alpha} X_\alpha \in \overline{U}$. □

Lemma 19.2. *If $V = L[A]$ and $A \subseteq \mathscr{P}(\omega_\alpha)$, then $2^{\aleph_\alpha} = \aleph_{\alpha+1}$.*

Proof. Let $X \subseteq \omega_\alpha$. Let λ be a cardinal such that $A \in L_\lambda[A]$. Let M be an elementary submodel of $(L_\lambda[A], \in)$ such that $\omega_\alpha \subseteq M$, $A \in M$, $X \in M$. and $|M| = \aleph_\alpha$. Let π be the transitive collapse of M, and $N = \pi[M]$. Now if $Z \subseteq \omega_\alpha$ and $Z \in M$, then $\pi(Z) = \{\pi(\alpha) : \alpha \in Z\} = \{\alpha : \alpha \in Z\} = Z$. In particular, $\pi(X) = X$. Now $\pi(A \cap M) = \{\pi(x) : x \in A \cap M\} = \{\pi(x) : x \in A\} \cap N = A \cap N$. By Lemma 13.63 there is a limit ordinal γ such that $N = L_\gamma[\pi(A \cap M)]$. By the above, $\pi(A \cap M) = A \cap N$. So $N = L_\gamma[A]$. Clearly $\gamma < \aleph_{\alpha+1}$. So $X \in L_{\aleph_{\alpha+1}}$. Hence $2^{\aleph_\alpha} = \aleph_{\alpha+1}$. □

Theorem 19.3. *If $V = L[D]$ with D a normal ultrafilter on a measurable cardinal κ, then GCH holds.*

Proof. If $\lambda \geq \kappa$, then $D \subseteq \mathscr{P}(\lambda)$ and hence $2^\lambda = \lambda^+$ by Lemma 19.9. Now suppose that $\lambda < \kappa$ and $\lambda^+ < 2^\lambda$. Thus there is a set $X \subseteq \lambda$ which is the λ^+-th subset of λ in the well-order $<_{L[D]}$. Let α be the least ordinal such that $X \in L_\alpha[D]$. Now every subset of λ which precedes X is also in $L_\alpha[D]$, and hence $|\mathscr{P}(\lambda) \cap L_\alpha[D]| \geq \lambda^+$. Let η be limit $> \alpha$ such that $D, \kappa, X \in L_\eta[D]$. Since κ is strongly inaccessible we have $|\mathscr{P}(\lambda) \cap L_\eta[D]| < \kappa$. Note that $D \subseteq L_\eta[D]$. By Lemma 17.55 and its proof, $A \stackrel{\text{def}}{=} (L_\eta[D], \in, D, X, \alpha, \kappa)$ has an elementary substructure $(B, \in, D, X, \alpha, \kappa)$ such that $|B| = \kappa$, $|\mathscr{P}(\lambda) \cap L_\eta[D] \cap B| \leq \lambda$, $\lambda \subseteq B$, and $B \cap \kappa \in D$. Note that $D, X, \alpha, \kappa \in B$. With π the transitive collapse of B onto M we have $M = L_\gamma[\pi[B \cap D]]$ for some $\gamma \leq \kappa$ by Theorem 13.63. Note that $\pi[B \cap D] = \{p(W) : W \in B \cap D\} = \pi(D)$. So $M = L_\gamma[\pi(D)]$. We claim

(1) If β is an ordinal and $\beta \in B$, then $\pi(\beta) \leq \beta$.

In fact, $\pi(\beta) = \{\pi(\gamma) : \gamma < \beta \text{ and } \gamma \in B\}$, so (1) holds by induction.

(2) $\pi(\kappa) = \kappa$.

In fact, $L_\eta[\pi[B \cap D] \models [\kappa \text{ is an ordinal}]$, so $B \models [\kappa \text{ is an ordinal}]$, hence $M \models [\pi(\kappa) \text{ is an ordinal}]$. Now $\pi(\kappa) = \{\pi(\alpha) : \alpha < \kappa, \alpha \in B\} = \{\pi(\alpha) : \alpha \in \kappa \cap B\}$. Since $|B \cap \kappa| = \kappa$ because $B \cap \kappa \in D$, it follows that $\kappa \leq |\pi(\kappa)| \leq \pi(\kappa)$. Thus (1) yields (2).

(3) Let $Z = \{\xi \in B \cap \kappa : \pi(\xi) = \xi\}$. Then $Z \in D$

In fact, by (1), $\{\xi \in B \cap \kappa : \pi(\xi) \le \xi\} = B \cap \kappa \in D$, and $\{\xi \in B \cap \kappa : \pi(\xi) \le \xi\} = \{\xi \in B \cap \kappa : \pi(\xi) < \xi\} \cup \{\xi \in B \cap \kappa : \pi(\xi) = \xi\}$. Since π is one-one, Exercise 8.8 gives (3).

(4) If $X' \in \mathscr{P}(\kappa) \cap B$, then $X' \triangle \pi(X') \subseteq \kappa \backslash Z$.

For, suppose that $\xi \in (X' \triangle \pi(X')) \cap Z$. Then $\xi \in B \cap \kappa$ and $\pi(\xi) = \xi$.

Case 1. $\xi \in X' \backslash \pi(X')$. But $\xi = \pi(\xi) \in \pi(X')$, contradiction.

Case 9. $\xi \in \pi(X') \backslash X'$. Say $\xi = \pi(\eta)$ with $\eta \in X' \cap B$. Then $\pi(\xi) = \pi(\eta)$, so $\xi = \eta$, contradiction.

Thus (4) holds.

(5) For any $X' \in \mathscr{P}(\kappa) \cap B$, $X' \in D$ iff $\pi(X') \in D$.

This is immediate from (4).

(6) $\pi[B \cap D] = D \cap M$.

In fact, if $Y \in \pi[B \cap D]$, choose $X \in B \cap D$ such that $Y = \pi(X)$. By (5), $Y \in D$; so $Y \in D \cap M$. Conversely, suppose that $Y \in D \cap M$. Say $Y = \pi(X)$ with $X \in B$. Now $M \models \forall x \in \pi(X)[x \in \kappa]$, so $B \models \forall x \in X[x \in \kappa]$. So $X \in \mathscr{P}(\kappa) \in B$. Since $\pi(X) = Y \in D$, by (5) we have $X \in D$. Thus $Y \in \pi[B \cap D]$, and (6) holds.

As observed above, $\pi(D) = \pi[B \cap D]$. Hence $M = L_\gamma[D \cap M]$.

Now $D = \pi(D)$. For, $A \models D$ is an ultrafilter on κ, so $B \models D$ is an ultrafilter on κ, hence $M \models \pi(D)$ is an ultrafilter on κ, by (2). Now we claim that $\pi(D) \subseteq D$, and hence $\pi(D) = D$. For, suppose that $W \in \pi(D) \backslash D$. Now $\pi(D) = \pi[D \cap B]$ by the above, so there is an $S \in D \cap B$ such that $W = \pi(S)$. But by (5), $\pi(S) \in D$, contradiction. So $\pi(D) = D$ and hence

(7) $M = L_\gamma[D]$.

Now since $\lambda \subseteq B$ we have $\pi(X) = X$. By the minimality of α we have $\alpha \le \gamma$.

(8) $|\mathscr{P}(\lambda) \cap L_\gamma[D]| \le \lambda$.

This is true since clearly $\mathscr{P}(\lambda) \cap M = \mathscr{P}(\lambda) \cap B$, and $|\mathscr{P}(\lambda) \cap B| \le \lambda$.

Since $|\mathscr{P}(\lambda) \cap L_\alpha[D]| \ge \lambda^+$ and $\alpha \le \gamma$, this is a contradiction. □

Theorem 19.4. *If $V = L[D]$ where D is a normal κ-complete ultrafilter on k, then κ is the only measurable cardinal.*

Proof. Suppose to the contrary that also $\lambda \neq \kappa$ is a measurable cardinal. Let U be a nonprincipal λ-complete ultrafilter on λ. Ult_λ is the set of equivalence classes of functions with domain λ, $j_U : V \to \mathrm{Ult}_\lambda$ is the associated elementary embedding, and π is the transitive collapse function on Ult_λ. Let $M = \pi[\mathrm{Ult}_\lambda]$. We will prove that $M = V$, a contradiction since $U \notin M$ by Lemma 17.18(ii).

We claim

(1) For all α, $\pi(j_U(L_\alpha[D]) = L_\alpha[\pi(j_U(D)]$.

Induction on α. The inductive step from α to $\alpha+1$ is the only nontrivial thing.

$$\begin{aligned}
Y \in \pi(j_U(L_{\alpha+1}[D])) \quad &\text{iff} \quad \exists X \in L_{\alpha+1}[D][Y = \pi(j_U(X))] \\
&\text{iff} \quad \exists\varphi\exists\overline{a} \subseteq L_\alpha[D][Y = \pi(j_U(\{b \in L_\alpha[D] : \\
&\qquad (L_\alpha[D], \in, D \cap L_\alpha[D]) \models \varphi(b, \overline{a})\}))] \\
&\text{iff} \quad \exists\varphi\exists\overline{a} \subseteq L_\alpha[D][Y = \{b \in L_\alpha[j_U(D)] : (L_\alpha[j_U(D)], \in, j_U(D) \\
&\qquad \cap L_\alpha[j_U(D)]) \models \varphi(j_U(b), j_U \circ \overline{a})\} \\
&\text{iff} \quad Y \in L_{\alpha+1}[j_U(D)].
\end{aligned}$$

From (1) we get

(2) $M = L[\pi[j_U(D)]]$.

(3) If $\kappa < \lambda$, then $\pi(j_U(D)) = D$.

This follows from Lemma 17.17, since $D \in V_\lambda$.

Now if $\kappa < \lambda$ then by (2) and (3), $M = L[D] = V$, contradiction as above.

We now assume that $\lambda < \kappa$.

Since D is normal, by Theorem 17.19 the set $Z \stackrel{\text{def}}{=} \{\alpha < \kappa : \alpha$ is inaccessible and $\alpha > \lambda\}$ is in D.

(3) $\pi(j_U(D)) = D \cap M$.

Since $\pi(j_U(D))$ and $D \cap M$ are both ultrafilters on κ in M, it suffices to prove that $\pi(j_U(D)) \subseteq D$. So, let $X \in \pi(j_U(D))$. Say $X = \pi(j_U([f]))$. Thus $\pi([f]) \in \pi([c_D])$, so $[f] \in^* [c_D]$. Hence $K \stackrel{\text{def}}{=} \{\alpha < \lambda : f(\alpha) \in D\} \in U$. Let $Y = \bigcap_{\alpha \in K} f(\alpha)$. Thus $Y \in D$.

(4) $\pi(j_U(Y)) \subseteq X$.

For, suppose that $y \in \pi(j_U(Y))$. Say $y = \pi([g])$. Thus $[g] \in^* [c_Y]$, so $H \stackrel{\text{def}}{=} \{\alpha < \lambda : g(\alpha) \in Y\} \in U$. If $\alpha \in H$, then $g(\alpha) \in f(\beta)$ for all $\beta \in K$. So $H \cap K \subseteq \{\alpha < \lambda : g(\alpha) \in f(\alpha)\}$. Hence $[g] \in^* [f]$ and so $y \in X$.

Now if $\alpha \in Y \cap Z$, then $j_U(\alpha) = \alpha$ by Lemma 17.18(vi) and the definition of Z. Hence

$$X \supseteq \pi(j_U(Y)) \supseteq \pi(j_U[Y \cap Z]) = Y \cap Z \in D,$$

so $X \in D$.

So (3) holds. Hence $M = L[\pi(j_U(D)] = L[D \cap M] = L[D]$. (See the proof of Theorem 19.3.) $\square$

Now we develop Kunen's version of iterated ultrapowers.

Let M be a transitive model of ZFC and κ an infinite cardinal in the sense of M. Note that ${}^0\kappa = \{\emptyset\}$, and hence ${}^{{}^0\kappa}M = \{\{(\emptyset, x)\} : x \in M\}$. Thus ${}^{{}^0\kappa}M$ consists of all functions f such that $\text{dmn}(f) = \{\emptyset\}$ and $f(\emptyset) \in M$.

---

(1) If $f \in {}^{({}^\alpha\kappa)}M$, then a set $F \subseteq \alpha$ is a *support* of f iff $\forall s, t \in {}^\alpha\kappa[s \restriction F = t \restriction F \to f(s) = f(t)]$. $\text{Fn}_\alpha(M, \kappa)$ is the set of all $f \in {}^{({}^\alpha\kappa)}M$ with a finite support.

Proposition 19.5. *Suppose that $f \in \mathrm{Fn}_\alpha(M,\kappa)$ and $F \in [\alpha]^{<\omega}$. Then the following are equivalent:*

(i) F is a support of f.

(ii) There exist $n \in \omega$, an order preserving bijection $j : n \to F$, and $g : {}^n\kappa \to M$ such that $\forall s \in {}^\alpha\kappa[f(s) = g(s \circ j)]$.

Proof. (i)⇒(ii): Suppose that F is a support of f. Choose $n \in \omega$ and an order preserving bijection $j : n \to F$. Define $g : {}^n\kappa \to M$ as follows. For $t \in {}^n\kappa$, choose $s_t \in {}^\alpha\kappa$ such that $s_t \restriction F = t \circ j^{-1}$, and let $g(t) = f(s_t)$. Suppose that $u \in {}^\alpha\kappa$. Then $g(u \circ j) = f(s_{u\circ j})$. Now $s_{u\circ j} \restriction F = u \circ j \circ j^{-1} = u \restriction F$, so $f(s_{u\circ j}) = f(u)$, as desired.

(ii)⇒(i): Assume (ii). Suppose that $s, t \in {}^\alpha\kappa$ and $s \restriction F = t \restriction F$. Then $s \circ j = t \circ j$, so $f(s) = g(s \circ j) = g(t \circ j) = f(t)$. □

Proposition 19.6. *Suppose that $f \in \mathrm{Fn}_\alpha(M,\kappa)$. Then the collection of supports of f is a filter of subsets of α.*

Proof. Clearly if F is a support of f and $F \subseteq G \subseteq \alpha$ then G is a support of f. Now suppose that F and G are supports of f, $s, t \in {}^\alpha\kappa$, and $s \restriction (F \cap G) = t \restriction (F \cap G)$. For each $\xi < \alpha$ let

$$s'_\xi = \begin{cases} s_\xi & \text{if } \xi \in F, \\ t_\xi & \text{if } \xi \notin F. \end{cases}$$

Then $s \restriction F = s' \restriction F$ and $s' \restriction G = t \restriction G$, so $f(s) = f(s') = f(t)$. □

Note that $\mathscr{P}({}^\emptyset\kappa) = \{\emptyset, \{\emptyset\}\}$.

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(2) If $x \in \mathscr{P}^M({}^\alpha\kappa)$, then a subset F of α is a *support* of x iff $\forall s, t \in {}^\alpha\kappa[s \restriction F = t \restriction F \to [s \in x \leftrightarrow t \in x]$. $\mathscr{P}^M_\alpha(\kappa)$ is the set of all $x \in \mathscr{P}^M({}^\alpha\kappa)$ with a finite support.

Proposition 19.7. *Suppose that $x \in \mathscr{P}^M_\alpha(\kappa)$ and $F \in [\alpha]^{<\omega}$. Then the following are equivalent:*

(i) F is a support of x.

(ii) There exist an $n \in \omega$, a bijection $j : n \to F$, and a $y \subseteq {}^n\kappa$ such that for all $s \in {}^\alpha\kappa$, $s \in x$ iff $s \circ j \in y$.

Proof. (i)⇒(ii): Suppose that F is a support of x. Choose $n \in \omega$ and j a bijection from n onto F. We define $y \subseteq {}^n\kappa$ as follows. Let $u \in {}^n\kappa$. Choose $t_u \in {}^\alpha\kappa$ such that $t_u \circ j = u$. Then we put u in y iff $t_u \in x$. Suppose that $s \in {}^\alpha\kappa$. Then $s \circ j \in {}^n\kappa$, and $t_{s\circ j} \circ j = s \circ j$, hence $t_{s\circ j} \restriction F = s \restriction F$, so $s \in x$ iff $t_{s\circ j} \in x$ iff $s \circ j \in y$.

(ii)⇒(i): Assume (ii), and suppose that $s, t \in {}^\alpha\kappa$ with $s \restriction F = t \restriction F$. Then $s \circ j = t \circ j$. Hence $s \in x$ iff $s \circ j \in y$ iff $t \circ j \in y$ iff $t \in x$. □

Proposition 19.8. *Let $x \in \mathscr{P}^M_\alpha(\kappa)$. Then the collection of supports of x is a filter of subsets of α.*

Proof. Clearly if F is a support of x and $F \subseteq G \subseteq \alpha$ then G is a support of x. Now suppose that F and G are supports of x. For each $\xi < \alpha$ let

$$s'_\xi = \begin{cases} s_\xi & \text{if } \xi \in F, \\ t_\xi & \text{if } \xi \notin F. \end{cases}$$

Then $s \upharpoonright F = s' \upharpoonright F$ and $s' \upharpoonright G = t \upharpoonright G$, so $s \in x$ iff $s' \in x$ iff $t \in x$. $\square$

Theorem 19.9. *$\mathscr{P}^M_\alpha(\kappa)$ is a field of subsets of ${}^\alpha\kappa$.*

Proof. Clearly $\emptyset, {}^\alpha\kappa \in \mathscr{P}^M_\alpha(\kappa)$. If $x \in \mathscr{P}^M_\alpha(\kappa)$ with support F and $y \in \mathscr{P}^M_\alpha(\kappa)$ with support G, then $x \cup y$ has support $F \cup G$. If $x \in \mathscr{P}^M_\alpha(\kappa)$ has support F, then ${}^\alpha\kappa \backslash x$ has support F. $\square$

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(3) If $x \in \mathscr{P}^M_{\alpha+\beta}(\kappa)$, $f \in \mathrm{Fn}_{\alpha+\beta}(M,\kappa)$, and $s \in {}^\alpha\kappa$, we define

$$x_{(s)} = \{t \in {}^\beta\kappa : s^\frown t \in x\}.$$

and for all $t \in {}^\beta\kappa$,

$$f_{(s)}(t) = f(s^\frown t).$$

Note that for $\alpha = \beta = 0$ and $f \in \mathrm{Fn}_{0+0}(M,\kappa) = {}^{{}^0\kappa}M$ we have $f_{(\emptyset)}(\emptyset) = f(\emptyset)$. Also, $\mathscr{P}^M_{0+0}(\kappa) = \{\emptyset, \{\emptyset\}\}$, $\emptyset_{(\emptyset)} = \emptyset$, and $\{\emptyset\}_{(\emptyset)} = \{\emptyset\}$.

If $\alpha \neq 0 = \beta$ and $s \in {}^\alpha\kappa$, then $x_{(s)} = \begin{cases} \{\emptyset\} & \text{if } s \in x, \\ \emptyset & \text{if } s \notin x. \end{cases}$ Also, $f_{(s)} = \{(\emptyset, f(s))\}$.

If $\alpha = 0 \neq \beta$, then $x_{(\emptyset)} = x$ and $f_{(\emptyset)} = f$.

Proposition 19.10. *If n is finite, then $\mathscr{P}^M_n(\kappa) = \mathscr{P}^M({}^n\kappa)$.* $\square$

Proposition 19.11. *If n is finite, then $\mathrm{Fn}_n(M,\kappa) = {}^{({}^n\kappa)}M$.* $\square$

Now let j be a one-one order-preserving map from α into β.

(4) $j^{*M\kappa}_{\beta\alpha}$ is the function from ${}^\beta\kappa$ into ${}^\alpha\kappa$ defined by $(j^{*M\kappa}_{\beta\alpha}(s))(\gamma) = s(j(\gamma))$.

Thus $\forall s \in {}^\beta\kappa[j^{*M\kappa}_{\beta\alpha}(s) = s \circ j]$. If $\alpha = 0$ and $\beta = 0$, then $j = \emptyset$ and $\emptyset^{*M\kappa}_{00}(\emptyset) = \emptyset$. If $\alpha = 0$ and $\beta \neq 0$, then $j = \emptyset$, $\emptyset^{*M\kappa}_{\beta 0}$ maps ${}^\beta\kappa$ into $\{\emptyset\}$, and $\emptyset^{*M\kappa}_{\beta 0}(s) = \emptyset$ for all $s \in {}^\beta\kappa$. If $\alpha \neq 0$ and $\beta = 0$, then there does not exist a function $j : \alpha \to \beta$.

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(5) $j^{M\kappa}_{*\beta\alpha}$ is the function from $\mathrm{Fn}_\alpha(M,\kappa)$ to $\mathrm{Fn}_\beta(M,\kappa)$ defined by

$$\forall f \in \mathrm{Fn}_\alpha(M,\kappa) \forall s \in {}^\beta\kappa[(j^{M\kappa}_{*\beta\alpha}(f))(s) = f(j^{*M\kappa}_{\beta\alpha}(s))].$$

c^S_a is the function with domain S such that $c^S_a(s) = a$ for all $s \in S$.

Thus $\forall f \in \mathrm{Fn}_\alpha(M,\kappa)\forall s \in {}^\beta\kappa[(j^{M\kappa}_{*\beta\alpha}(f))(s) = f(s \circ j)]$. If $\alpha = \beta = 0$, then $j = \emptyset$ and $\forall f \in {}^{\{\emptyset\}}M[(\emptyset^{M\kappa}_{*00}(f)) = f]$. If $0 = \alpha \neq \beta$, then $j = \emptyset$ and $\forall f \in {}^{\{\emptyset\}}M\forall s \in {}^\beta\kappa[(\emptyset^{M\kappa}_{*\beta 0}(f))(s) = f(\emptyset)]$. Thus $\forall f \in {}^{\{\emptyset\}}M[\emptyset^{M\kappa}_{*\beta 0}(f) = c^{{}^\beta\kappa}_{f(\emptyset)}]$. Again, if $\alpha \neq 0 = \beta$, then there does not exist a function $j : \alpha \to \beta$.

Proposition 19.12. *$j^{M\kappa}_{*\beta\alpha}$ is well-defined, i.e., $\forall f \in \mathrm{Fn}_\alpha(M,\kappa)[j^{M\kappa}_{*\beta\alpha}(f) \in \mathrm{Fn}_\beta(M,\kappa)]$.*

Proof. Let F be a support for f. Then $j[F]$ is a support for $j_{*\beta\alpha}(f)$. In fact, suppose that $s,t \in {}^\beta\kappa$ and $s \restriction j[F] = t \restriction j[F]$. Then for any $\gamma \in F$ we have $(j^{*M\kappa}_{\beta\alpha}(s))(\gamma) = s(j(\gamma)) = t(j(\gamma)) = (j^{*M\kappa}_{\beta\alpha}(t))(\gamma)$. So $(j^{*M\kappa}_{\beta\alpha}(s)) \restriction F = (j^{*M\kappa}_{\beta\alpha}(t)) \restriction F$. Hence

$$((j^{M\kappa}_{*\beta\alpha})(f))(s) = f(j^{*M\kappa}_{\beta\alpha}(s)) = f(j^{*M\kappa}_{\beta\alpha}(t)) = ((j^{M\kappa}_{*\beta\alpha})(f))(t) \qquad \square$$

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(6) $j'^{M\kappa}_{*\beta\alpha}$ is the function from $\mathscr{P}^M_\alpha(\kappa)$ to $\mathscr{P}^M_\beta(\kappa)$ defined by

$$\forall x \in \mathscr{P}^M_\alpha(\kappa)[j'^{M\kappa}_{*\beta\alpha}(x) = \{s \in {}^\beta\kappa : j^{*\kappa}_{\beta\alpha}(s) \in x\}].$$

Thus

$$\forall x \in \mathscr{P}^M_\alpha(\kappa)[j'^{M\kappa}_{*\beta\alpha}(x) = \{s \in {}^\beta\kappa : s \circ j \in x\}].$$

If $\alpha = 0 = \beta$, then $j = \emptyset$ and $\emptyset'^{M\kappa}_{*00}(\emptyset) = \emptyset$, $\emptyset'^{M\kappa}_{*00}(\{\emptyset\}) = \{\emptyset\}$. If $\alpha = 0 \neq \beta$, then $j = \emptyset$ and $\emptyset'^{M\kappa}_{*M\beta 0}(\emptyset) = \emptyset$ and $\emptyset'^{M\kappa}_{*M\beta 0}(\{\emptyset\}) = {}^\beta\kappa$. For $\alpha \neq 0$ and $\beta = 0$ there is no function $j : \alpha \to \beta$.

Proposition 19.13. *$j'^{M\kappa}_{*\beta\alpha}$ is well-defined, i.e., $\forall x \in \mathscr{P}_\alpha(M,\kappa)[j'^{M\kappa}_{*\beta\alpha}(x) \in \mathscr{P}_\beta(M,\kappa)]$.*

Proof. Let F be a support for x. Then $j[F]$ is a support for $j'^{M\kappa}_{*\beta\alpha}(x)$. In fact, suppose that $s,t \in {}^\beta\kappa$ and $s \restriction j[F] = t \restriction j[F]$. Then for any $\gamma \in F$ we have $(j^{*M\kappa}_{\beta\alpha}(s))(\gamma) = s(j(\gamma)) = t(j(\gamma)) = (j^{*M\kappa}_{\beta\alpha}(t))(\gamma)$. So $(j^{*M\kappa}_{\beta\alpha}(s)) \restriction F = (j^{*M\kappa}_{\beta\alpha}(t)) \restriction F$. Hence

$$s \in j'^{M\kappa}_{*\beta\alpha}(x) \quad \text{iff} \quad j^{*M\kappa}_{\beta\alpha}(s) \in x \quad \text{iff} \quad j^{*M\kappa}_{\beta\alpha}(t) \in x \quad \text{iff} \quad t \in j'^{M\kappa}_{*\beta\alpha}(x). \qquad \square$$

For $\alpha \leq \beta$ e let $i^{M\kappa}_{\alpha\beta} = j^{M\kappa}_{*\beta\alpha}$ with j the identity on α.

Proposition 19.14. *Let $f \in \mathrm{Fn}_\beta(M,\kappa)$ with support F. Let $n = |F|$, let $k : n \to F$ be the order-preserving bijection. Thus k is an order preserving injection of n into β.*

Then there exist a $g \in \mathrm{Fn}_n(M,\kappa)$ such that $f = k^{M\kappa}_{*\beta n}(g)$.

Proof. Recall from Proposition 19.11 that $\mathrm{Fn}_n(M,\kappa) = {}^{{}^n\kappa}M$. Take any $t \in {}^n\kappa$. Let $s \in {}^\beta\kappa$ be such that $\forall\gamma < n[s(k(\gamma)) = t(\gamma)]$. Since k is one-one, this is well-defined. Let $g(t) = f(s)$. This does not depend on the particular s such that $\forall\gamma < n[s(k(\gamma)) = t(\gamma)]$. For, suppose also that $s' \in {}^\alpha\kappa$ and $\forall\gamma < n[s'(k(\gamma)) = t(\gamma)]$. Then $s \restriction F = s' \restriction F$, so $f(s) = f(s')$. Now take any $t \in {}^\beta\kappa$. Then

$$(k^{M\kappa}_{*\beta n}(g))(t) = g(k^{*M\kappa}_{\beta n}(t)) = g(t \circ k) = f(t). \qquad \square$$

Proposition 19.15. *Let $x \in \mathscr{P}_\beta(M,\kappa)$ with support F. Let $n = |F|$, let $k : n \to F$ be the order-preserving bijection. Thus k is an order preserving injection of n into β.*

Then there exist a $y \in \mathscr{P}_n(\kappa)$ such that $x = k'^{M\kappa}_{*\beta n}(y)$.

Proof. Recall from Proposition 19.10 that $\mathscr{P}^M_n(\kappa) = \mathscr{P}^M({}^n\kappa)$. Let $y = \{s \circ k : s \in x\}$. Thus $s \in x \to s \circ k \in y$. Conversely, if $s \circ k \in y$ then there is a $t \in x$ such that $s \circ k = t \circ k$. Hence $s \in x$ since F is a support for x. It follows that

$$k'^{M\kappa}_{*\beta n}(y) = \{s \in {}^\beta\kappa : k^{*M\kappa}_{\beta n}(s) \in y\} = \{s \in {}^\beta\kappa : s \circ k \in y\} = \{s \in {}^\beta\kappa : s \in x\} = x. \qquad \square$$

Proposition 19.16. *If $n \in \omega$, $j : n \to \alpha$ is an order-preserving injection, $y \in \mathscr{P}^M({}^n\kappa)$, and $x = j'^{M\kappa}_{*\alpha n}(y)$, then $\mathrm{rng}(j)$ is a support for x.*

Proof. Suppose $s, t \in {}^\alpha\kappa$ and $s \restriction \mathrm{rng}(j) = t \restriction \mathrm{rng}(j)$. Now $x = \{s \in {}^\alpha\kappa : s \circ j \in y\}$. Hence $s \in x$ iff $s \circ j \in y$ iff $t \circ j \in y$ iff $t \in x$. $\square$

Proposition 19.17. *If $f, g \in \mathrm{Fn}_\alpha(M,\kappa)$, then $\{s \in {}^\alpha\kappa : f(s) = g(s)\} \in \mathscr{P}^M_\alpha(\kappa)$.*

Proof. Choose n, j, f' so that $n \in \omega$, $j : n \to \alpha$ is injective and order preserving, $f' \in \mathrm{Fn}_n(M,\kappa)$, and $f = j'^{M\kappa}_{*\alpha n}(f')$.

(7) $\forall s \in {}^\alpha\kappa[f(s) = f'(s \circ j)]$.

In fact, $f(s) = (j_{*\alpha n}(f'))(s) = f'(j^*_{\alpha n}(s)) = f'(s \circ j)$.

Also choose m, k, g' so that $m \in \omega$, $k : m \to \alpha$ is injective and order preserving, $g' \in \mathrm{Fn}_m(\kappa) \cap M$, and $g = k_{*\alpha m}(g')$. Similarly to (7) we have

(8) $\forall s \in {}^\alpha\kappa[g(s) = g'(s \circ k)]$.

Now let p and $l : p \to \alpha$ with l injective and order preserving be such that $\mathrm{rng}(j) \cup \mathrm{rng}(k) \subseteq \mathrm{rng}(l)$. For each $\gamma < n$ choose $\delta_\gamma < p$ such that $j(\gamma) = l(\delta_\gamma)$, and for each $\gamma < m$ choose $\varepsilon_\gamma < p$ such that $k(\gamma) = l(\varepsilon_\gamma)$. Now let $y = \{s \in {}^p\kappa : f'(s \circ \delta) = g'(s \circ \varepsilon)\}$. Now

$$\begin{aligned} l'^{M\kappa}_{*\alpha p}(y) &= \{s \in {}^\alpha\kappa : l^{*M\kappa}_{\alpha p}(s) \in y\} = \{s \in {}^\alpha\kappa : s \circ l \in y\} \\ &= \{s \in {}^\alpha\kappa : f'(s \circ l \circ \delta) = g'(s \circ l \circ \varepsilon)\} \\ &= \{s \in {}^\alpha\kappa : f'(s \circ j) = g'(s \circ k)\} = \{s \in {}^\alpha\kappa : f(s) = g(s)\} = x. \qquad \square \end{aligned}$$

Proposition 19.18. *If $f, g \in \mathrm{Fn}_\alpha(M,\kappa)$, then $\{s \in {}^\alpha\kappa : f(s) \in g(s)\} \in \mathscr{P}^M_\alpha(\kappa)$.* $\square$

Proposition 19.19. $\mathscr{P}^M_0(\kappa) = \{\emptyset, \{\emptyset\}\}$.

Proof. By Proposition 19.9, $\mathscr{P}^M_0(\kappa) = \mathscr{P}^M({}^\emptyset\kappa)$. Now ${}^\emptyset\kappa = \{\emptyset\}$, so $\mathscr{P}^M_0(\kappa) = \{\emptyset, \{\emptyset\}\}$. Clearly $\mathscr{P}^M_0(\kappa) = \{\emptyset, \{\emptyset\}\}$. $\square$

Assume that $\mathscr{U}^{M\kappa}$ is an ultrafilter on κ, in M. Now we define inductively $\mathscr{U}^{M\kappa}_n \subseteq \mathscr{P}_n(\kappa)$. We define $\mathscr{U}^{M\kappa}_0 = \{\{\emptyset\}\}$ and $\mathscr{U}^{M\kappa}_1 = \mathscr{U}^{M\kappa}$.

Now assume that $\mathscr{U}_n^{M\kappa} \subseteq \mathscr{P}_n(\kappa)$ has been defined. where $n \geq 1$. We let

$$\mathscr{U}_{n+1}^{M\kappa} = \{x \in \mathscr{P}^M({}^{n+1}\kappa) : \{\xi : x_{(\xi)} \in \mathscr{U}_n^{M\kappa}\} \in \mathscr{U}^{M\kappa}\}.$$

Theorem 19.20. *Suppose that $\mathscr{U}^{M\kappa}$ is a nonprincipal κ-complete ultrafilter on κ, in M. Then for every positive integer n, $\mathscr{U}_n^{M\kappa}$ is a nonprincipal κ-complete ultrafilter on ${}^n\kappa$.*

Proof. We go by induction on n. It is given for $n = 1$. Now suppose that $n \geq 1$.

Nonprincipal: suppose that $\{s\} \in \mathscr{U}_{n+1}^{M\kappa}$. Then $\{s\} \in \mathscr{P}^M({}^{n+1}\kappa)$ and $\{\xi : \{s\}_{(\xi)} \in \mathscr{U}_n\} \in \mathscr{U}^{M\kappa}$. Now $\{\xi : \{s\}_{(\xi)} \in \mathscr{U}_n\} \neq \emptyset$; choose ξ such that $\{s\}_{(\xi)} \in \mathscr{U}_n^{M\kappa}$. But $\{s\}_{(\xi)} = \{t \in {}^n\kappa : \langle\xi\rangle^\frown t \in \{s\}\}$, so $\{s\}_{(\xi)}$ is a singleton, contradicting the inductive hypothesis.

Upward closed: suppose that $x \subseteq y \in \mathscr{P}^M({}^{n+1}\kappa)$ and $x \in \mathscr{U}_{n+1}^{M\kappa}$. Then $x \in \mathscr{P}^M({}^{n+1}\kappa)$ and $\{\xi : x_{(\xi)} \in \mathscr{U}_n^{M\kappa}\} \in \mathscr{U}^{M\kappa}$. We claim

(9) $\{\xi : x_{(\xi)} \in \mathscr{U}_n^{M\kappa}\} \subseteq \{\xi : y_{(\xi)} \in \mathscr{U}_n\}$.

For, suppose that $x_{(\xi)} \in \mathscr{U}_n^{M\kappa}$. Thus $\{t \in {}^n\kappa : \langle\xi\rangle^\frown t \in x\} \in \mathscr{U}_n^{M\kappa}$. Hence $\{t \in {}^n\kappa : \langle\xi\rangle^\frown t \in y\} \in \mathscr{U}_n^{M\kappa}$, i.e., $y_{(\xi)} \in \mathscr{U}_n^{M\kappa}$. So (9) holds.

By (9) and upward closure of $\mathscr{U}^{M\kappa}$ it follows that $y \in \mathscr{U}_{n+1}^{M\kappa}$.

Complement: suppose that $x \in \mathscr{P}^M({}^{n+1}\kappa)$ and $x \notin \mathscr{U}_{n+1}$. Then $\{\xi : x_{(\xi)} \in \mathscr{U}_n^{M\kappa}\} \notin \mathscr{U}^{M\kappa}$, so $A \stackrel{\text{def}}{=} \kappa\backslash\{\xi : x_{(\xi)} \in \mathscr{U}_n^{M\kappa}\} \in \mathscr{U}^{M\kappa}$. For any $\xi \in A$ we have $x_{(\xi)} \notin \mathscr{U}_n^{M\kappa}$, so by the inductive hypothesis, ${}^n\kappa\backslash x_{(\xi)} \in \mathscr{U}_n^{M\kappa}$. Now

(10) ${}^n\kappa\backslash x_{(\xi)} = ({}^{n+1}\kappa\backslash x)_{(x)}$.

In fact,

$${}^n\kappa\backslash x_{(\xi)} = \{t \in {}^n\kappa : \langle\xi\rangle^\frown t \notin x\} = \{t \in {}^n\kappa : \langle\xi\rangle^\frown t \in ({}^{n+1}\backslash x) = ({}^{n+1}\kappa\backslash x)_{(\xi)}.$$

So (10) holds.

It follows that $A \subseteq \{\xi : ({}^{n+1}\kappa\backslash x)_{(x)} \in \mathscr{U}_n\}$, so $\{\xi : ({}^{n+1}\kappa\backslash x)_{(x)} \in \mathscr{U}_n\} \in \mathscr{U}^{M\kappa}$. Hence ${}^{n+1}\kappa\backslash x \in \mathscr{U}_{n+1}^{M\kappa}$.

κ-complete: suppose that $\eta < \kappa$ and $x \in {}^\eta\mathscr{U}_{n+1}^{M\kappa}$. For each $\varphi < \eta$ we have $\{\xi : x_{\varphi(\xi)} \in \mathscr{U}_n^{M\kappa}\} \in \mathscr{U}^{M\kappa}$. Hence by κ-completeness of $\mathscr{U}^{M\kappa}$, $\bigcap_{\varphi<\eta}\{\xi : x_{\varphi(\xi)} \in \mathscr{U}_n^{M\kappa}\} \in \mathscr{U}^{M\kappa}$. Now if $\xi \in \bigcap_{\varphi<\mu}\{\xi : x_{\varphi(\xi)} \in \mathscr{U}_n^{M\kappa}\}$, then $\forall\varphi < \eta[x_{\varphi(\xi)} \in \mathscr{U}_n]$, so by the inductive hypothesis $\bigcap_{\varphi<\eta} x_{\varphi(\xi)} \in \mathscr{U}_n^{M\kappa}$. Thus $\{\xi < \kappa : \bigcap_{\varphi<\eta} x_{\varphi(\xi)} \in \mathscr{U}_n^{M\kappa}\} \in \mathscr{U}^{M\kappa}$. Now note that for any $\xi < \kappa$,

$$\begin{aligned}
\bigcap_{\varphi<\eta} x_{\varphi(\xi)} &= \bigcap_{\varphi<\eta}\{t \in {}^n\kappa : \langle\xi\rangle^\frown t \in x_\varphi\} \\
&= \left\{t \in {}^n\kappa : \langle\xi\rangle^\frown t \in \bigcap_{\varphi<\eta} x_\varphi\right\} \\
&= \left(\bigcap_{\varphi<\mu} x_\varphi\right)_{(\xi)}.
\end{aligned}$$

Hence $\bigcap_{\varphi<\eta} x_\varphi \in \mathscr{U}_{n+1}^{M\kappa}$. $\square$

Proposition 19.21. *For each positive integer n,*

$$\{\langle\xi_0,\ldots,\xi_{n-1}\rangle : \xi_0 < \cdots < \xi_{n-1} < \kappa\} \in \mathscr{U}_n.$$

Proof. First we prove

(11) For every positive integer n, $\forall\eta<\kappa[\{\langle\xi_0,\ldots,\xi_{n-1}\rangle : \eta < \xi_0 < \cdots < \xi_{n-1}\} \in \mathscr{U}_n^{M\kappa}]$.

We prove (11) by induction on n. It is true for $n = 1$. Now assume it for n. Let $y = \{\langle\xi_0,\ldots,\xi_n\rangle : \eta < \xi_0 < \cdots < \xi_n\}$. Then for any $\nu<\kappa$ we have

$$y_{(\nu)} = \{t \in {}^n\kappa : \langle\nu\rangle^\frown t \in y\} = \{\langle\varepsilon_0,\ldots\varepsilon_{n-1}\rangle : \eta < \nu < \varepsilon_0 < \cdots < \varepsilon_{n-1}\}.$$

Thus for $\eta<\nu$ this is

$$y_{(\nu)} = \{t \in {}^n\kappa : \langle\nu\rangle^\frown t \in y\} = \{\langle\varepsilon_0,\ldots\varepsilon_{n-1}\rangle : \nu < \varepsilon_0 < \cdots < \varepsilon_{n-1}\},$$

which is in $\mathscr{U}_n^{M\kappa}$ by the inductive hypothesis. Hence $\{\nu : \eta < \nu\} \subseteq \{\nu : y_{(\nu)} \in \mathscr{U}_n^{M\kappa}\}$, and so $\{\nu : y_{(\nu)} \in \mathscr{U}_n^{M\kappa}\} \in \mathscr{U}^{M\kappa}$. So $y \in \mathscr{U}_{n+1}^{M\kappa}$. This proves (11).

Now we prove the proposition itself by induction on n. It is clear for $n = 1$. Assume it for n. Let $x = \{\langle\xi_0,\ldots,\xi_n\rangle : \xi_0 < \cdots < \xi_n < \kappa\}$. For each $\eta < \kappa$ we have

$$x_{(\eta)} = \{t \in {}^n\kappa : \langle\eta\rangle^\frown t \in x\} = \{\langle\xi_0,\ldots,\xi_{n-1}\rangle : \eta < \xi_0 < \cdots < \xi_{n-1}\},$$

which is in $\mathscr{U}_n^{M\kappa}$ by (11). Hence $x \in \mathscr{U}_{n+1}^{M\kappa}$. $\square$

Proposition 19.22. *For $x \in \mathscr{P}^M({}^4\kappa)$, $x \in \mathscr{U}_4^{M\kappa}$ iff*

$$\{\xi_0 : \{\xi_1 : \{\xi_2 : x_{(\xi_0\xi_1\xi_2)} \in \mathscr{U}^{M\kappa}\} \in \mathscr{U}^{M\kappa}\} \in \mathscr{U}^{M\kappa}\} \in \mathscr{U}^{M\kappa}.$$

Proof.

$$\begin{aligned} x \in \mathscr{U}_4^{M\kappa} \quad &\text{iff} \quad \{\xi_0 : x_{(\xi_0)} \in \mathscr{U}_3^{M\kappa}\} \in \mathscr{U}^{M\kappa} \\ &\text{iff} \quad \{\xi_0 : \{\xi_1 : x_{(\xi_0\xi_1)} \in \mathscr{U}_2^{M\kappa}\} \in \mathscr{U}^{M\kappa}\} \in \mathscr{U}^{M\kappa} \\ &\text{iff} \quad \{\xi_0 : \{\xi_1 : \{\xi_2 : x_{(\xi_0\xi_1\xi_2)} \in \mathscr{U}^{M\kappa}\} \in \mathscr{U}^{M\kappa}\} \in \mathscr{U}^{M\kappa}\} \in \mathscr{U}^{M\kappa}. \end{aligned}$$ $\square$

Proposition 19.23. *If $m \geq 2$ and $x \in \mathscr{P}_m^M(\kappa)$, then $x \in \mathscr{U}_m^{M\kappa}$ iff*

$$\{\eta_0 : \{\eta_1 : \{\cdots : \{\eta_{m-2} : \{\eta_{m-1} : \langle\eta_0\eta_1\cdots\eta_{m-1}\rangle \in x\} \in \mathscr{U}^{M\kappa}\} \in \mathscr{U}^{M\kappa}\} \in \cdots \in \mathscr{U}^{M\kappa}.$$

Proof. We prove by induction on $i < m-1$ that $x \in \mathscr{U}_m^{M\kappa}$ is equivalent to

$$\{\eta_0 : \{\eta_1 : \{\cdots : \{\eta_i : x_{(\eta_0\eta_1\cdots\eta_i)} \in \mathscr{U}_{m-i-1}^{M\kappa}\} \in \mathscr{U}^{M\kappa}\} \in \cdots \in \mathscr{U}^{M\kappa}.$$

For $i = 0$, $x \in \mathscr{U}_m^{M\kappa}$ iff $\{\eta_0 : x_{(\eta_0)} \in \mathscr{U}_{m-1}\} \in \mathscr{U}^{M\kappa}$, as desired. Now assume it for i, with $i < m-2$. Then $x_{(\eta_0\eta_1\cdots\eta_i)} \in \mathscr{U}_{m-i-1}^{M\kappa}$ iff $\{\eta_{i+1} : x_{(\eta_0\eta_1\cdots\eta_{i+1})} \in \mathscr{U}_{m-i-2}^{M\kappa}\} \in \mathscr{U}^{M\kappa}$. This clearly gives our statement for $i+1$.

Now for $i = m-2$ we have that $x \in \mathscr{U}_m^{M\kappa}$ is equivalent to

$$\{\eta_0 : \{\eta_1 : \{\cdots : \{\eta_{m-2} : x_{(\eta_0\eta_1\cdots\eta_{m-2})} \in \mathscr{U}^{M\kappa}\} \in \mathscr{U}^{M\kappa}\} \in \cdots \in \mathscr{U}^{M\kappa}.$$

Finally, note that $x_{(\eta_0\eta_1\cdots\eta_{m-2})} = \{\eta_{m-1} : \langle\eta_0\eta_1\cdots\eta_{m-1}\rangle \in x\}$. □

Lemma 19.24. *If $m, n \in \omega$, $j : m \to n$ is one-one and order preserving, and $x \in \mathscr{P}_m^M(\kappa)$, then $x \in \mathscr{U}_m^{M\kappa}$ iff $j'^{M\kappa}_{*nm}(x) \in \mathscr{U}_n^{M\kappa}$.*

Proof. First we take $m = 0$. Then $j = \emptyset$, and $x \in \mathscr{U}_0^{M\kappa}$ iff $x = \{\emptyset\}$. Also, if $n = 0$, then $j'^{M\kappa}_{*00}(\{\emptyset\}) = \{s \in {}^0\kappa : s \circ \emptyset \in \{\emptyset\}\} = \{\emptyset\} \in \mathscr{U}_0$. If $n > 0$, then $j'^{M\kappa}_{*n0}(\{\emptyset\}) = \{s \in {}^n\kappa : s \circ \emptyset \in \{\emptyset\}\} = {}^n\kappa \in \mathscr{U}_n^{M\kappa}$.

Next we take $m = 1$. Let $y = j'^{M\kappa}_{*n1}(x)$. We may assume that $n > 1$. By Lemma 19.23 we have: $y \in \mathscr{U}_n^{M\kappa}$ iff

$$\{\eta_0 : \{\eta_1 : \{\cdots : \{\eta_{n-2} : \{\eta_{n-1} : \langle\eta_0\eta_1\cdots\eta_{n-1}\rangle \in y\} \in \mathscr{U}^{M\kappa}\} \in \mathscr{U}^{M\kappa}\} \in \cdots \in \mathscr{U}^{M\kappa}.$$

Now $y = j'^{M\kappa}_{*n1}(x) = \{s \in {}^n\kappa : s(j(0)) \in x\}$, so

$$\{\eta_{n-1} < \kappa : \langle\eta_0\cdots\eta_{n-1}\rangle \in y\} = \{\eta_{n-1} < \kappa : \eta_{j(0)} \in x\}.$$

Hence $y \in \mathscr{U}_n^{M\kappa}$ iff

$$\{\eta_0 : \{\eta_1 : \{\cdots : \{\eta_{n-2} : \{\eta_{n-1} < \kappa : \eta_{j(0)} \in x\} \in \mathscr{U}^{M\kappa}\} \in \mathscr{U}^{M\kappa}\} \in \cdots \in \mathscr{U}^{M\kappa}.$$

Case 1. $j(0) = n-1$. Then $y \in \mathscr{U}_n^{M\kappa}$ iff

$$\begin{aligned}
&\{\eta_0 : \{\eta_1 : \{\cdots : \{\eta_{n-2} : \{\eta_{n-1} < \kappa : \eta_{n-1} \in x\} \in \mathscr{U}^{M\kappa}\} \in \mathscr{U}^{M\kappa}\} \in \cdots \in \mathscr{U}^{M\kappa}\\
\text{iff}\quad &\{\eta_0 : \{\eta_1 : \{\cdots : \{\eta_{n-2} : x \in \mathscr{U}^{M\kappa}\} \in \mathscr{U}^{M\kappa}\} \in \cdots \in \mathscr{U}^{M\kappa}\\
\text{iff}\quad &x \in \mathscr{U}^{M\kappa}.
\end{aligned}$$

Case 9. $j(0) < n-1$. Then

$$x \subseteq \{\eta_{j(0)} : \{\eta_{j(0)+1} : \{\cdots : \{\eta_{n-1} : \eta_{j(0)} \in x\} \in \mathscr{U}^{M\kappa}\} \in \cdots \in \mathscr{U}^{M\kappa}.$$

Suppose that $x \in \mathscr{U}^{M\kappa}$. Then

$$\{\eta_{j(0)} : \{\eta_{j(0)+1} : \{\cdots : \{\eta_{n-1} : \eta_{j(0)} \in x\} \in \mathscr{U}^{M\kappa}\} \in \cdots \in \mathscr{U}^{M\kappa}.$$

Hence

$$\{\eta_{j(0)-1} : \{\eta_{j(0)} : \{\eta_{j(0)+1}\{\cdots : \{\eta_{n-1} : \eta_{j(0)} \in x\} \in \mathscr{U}^{M\kappa}\} \in \cdots \in \mathscr{U}^{M\kappa}\} = \kappa \in \mathscr{U}^{M\kappa},$$

etc. so $y \in \mathscr{U}_n^{M\kappa}$.

Suppose that $y \in \mathscr{U}_n^{M\kappa}$. Then

$$\{\eta_{j(0)} : \{\eta_{j(0)+1} : \{\cdots : \{\eta_{n-1} : \eta_{j(0)} \in x\} \in \mathscr{U}^{M\kappa}\} \in \cdots\} \in \mathscr{U}^{M\kappa}$$

and

$$\{\eta_{j(0)} : \{\eta_{j(0)+1} : \{\cdots : \{\eta_{n-1} : \eta_{j(0)} \in x\} \in \mathscr{U}^{M\kappa}\} \in \cdots\} \subseteq x,$$

so $x \in \mathscr{U}^{M\kappa}$.

This finishes the case $m = 1$. Now assume that $m > 1$. We may assume that $m < n$. Then by Proposition 19.23, $x \in \mathscr{U}_m^{M\kappa}$ iff

$$\{\eta_0 : \{\eta_1 : \{\cdots : \{\eta_{m-2} : \{\eta_{m-1} : \langle \eta_0 \cdots \eta_{m-1}\rangle \in x\} \in \mathscr{U}^{M\kappa}\} \in \mathscr{U}^{M\kappa}\} \in \cdots \in \mathscr{U}^{M\kappa}.$$

Let $y = j'^{M\kappa}_{*nm}(x)$. Then similarly, $y \in \mathscr{U}_n^{M\kappa}$ iff

$$\{\eta_0 : \{\eta_1 : \{\cdots : \{\eta_{n-2} : \{\eta_{n-1} : \langle \eta_0 \cdots \eta_{n-1}\rangle \in y\} \in \mathscr{U}^{M\kappa}\} \in \mathscr{U}^{M\kappa}\} \in \cdots \in \mathscr{U}^{M\kappa}.$$

Since $y = j'^{M\kappa}_{*nm}(x) = \{s \in {}^n\kappa : s \circ j \in x\}$, it follows that $y \in \mathscr{U}_n^{M\kappa}$ iff

$$\{\eta_0 : \{\eta_1 : \{\cdots : \{\eta_{n-2} : \{\eta_{n-1} : \langle \eta_{j(0)} \cdots \eta_{j(m-1)}\rangle \in x\} \in \mathscr{U}^{M\kappa}\} \in \mathscr{U}^{M\kappa}\} \in \cdots \in \mathscr{U}^{M\kappa}.$$

Let $i < n$ be maximum such that $i \notin \mathrm{rng}(j)$. Then

$$\{\eta_i : \{\eta_{i+1} : \{\cdots \{\eta_{n-2} : \{\eta_{n-1} : \langle \eta_{j(0)} \cdots \eta_{j(m-1)}\rangle \in x\} \in \mathscr{U}^{M\kappa}\} \in \mathscr{U}^{M\kappa}\} \in \cdots \in \mathscr{U}^{M\kappa}$$

iff

$$\{\eta_{i+1} : \{\cdots \{\eta_{n-2} : \{\eta_{n-1} : \langle \eta_{j(0)} \cdots \eta_{j(m-1)}\rangle \in x\} \in \mathscr{U}^{M\kappa}\} \in \mathscr{U}^{M\kappa}\} \in \cdots \in \mathscr{U}^{M\kappa}$$

Hence $y \in \mathscr{U}_n^{M\kappa}$ iff

$$\begin{aligned}&\{\eta_0 : \{\eta_1 : \{\cdots : \{\eta_{i-1} : \{\eta_{i+1} : \{\cdots \{\eta_{n-2} : \{\eta_{n-1} :\\ &\langle \eta_{j(0)} \cdots \eta_{j(m-1)}\rangle \in x\} \in \mathscr{U}^{M\kappa}\} \in \mathscr{U}^{M\kappa}\} \in \cdots \in \mathscr{U}^{M\kappa}.\end{aligned}$$

Continuing like this, it follows that $y \in \mathscr{U}_n^{M\kappa}$ iff

$$\{\eta_0 : \{\eta_1 : \{\cdots : \{\eta_{m-2} : \{\eta_{m-1} : \langle \eta_0 \cdots \eta_{m-1}\rangle \in x\} \in \mathscr{U}^{M\kappa}\} \in \mathscr{U}^{M\kappa}\} \in \cdots \in \mathscr{U}^{M\kappa}. \quad \square$$

Lemma 19.25. *If $x \in \mathscr{P}^M_{m+n}(\kappa)$, then $x \in \mathscr{U}_{m+n}$ iff $\{s \in {}^m\kappa : x_{(s)} \in \mathscr{U}_n^{M\kappa}\} \in \mathscr{U}_m^{M\kappa}$.*

Proof. By Proposition 19.23, $x \in \mathscr{U}_{m+n}$ iff

$$\begin{aligned}&\{\eta_0 : \{\eta_1 : \{\cdots : \{\eta_{m+n-2} : \{\eta_{m+n-1} : \langle \eta_0\eta_1 \cdots \eta_{m+n-1}\rangle \in x\}\\ &\in \mathscr{U}^{M\kappa}\} \in \mathscr{U}^{M\kappa}\} \in \cdots \in \mathscr{U}^{M\kappa}\}.\end{aligned}$$

Let $y = \{s \in {}^m\kappa : x_{(s)} \in \mathscr{U}_n^{M\kappa}\}$. Then $y \in \mathscr{U}_m^{M\kappa}$ iff

$$\{\eta_0 : \{\eta_1 : \{\cdots : \{\eta_{m-2} : \{\eta_{m-1} : \langle \eta_0\eta_1\cdots\eta_{m-1}\rangle \in y\} \in \mathscr{U}^{M\kappa}\} \in \mathscr{U}^{M\kappa}\} \in \cdots \in \mathscr{U}^{M\kappa}.$$

Now $\langle \eta_0\eta_1\cdots\eta_{m-1}\rangle \in y$ iff $x_{(\langle \eta_0\eta_1\cdots\eta_{m-1}\rangle)} \in \mathscr{U}_n^{M\kappa}$, and

$$x_{(\langle \eta_0\eta_1\cdots\eta_{m-1}\rangle)} = \{t \in {}^n\kappa : \langle \eta_0\eta_1\cdots\eta_{m-1}\rangle^\frown t \in x\}.$$

Hence $y \in \mathscr{U}_m^{M\kappa}$ iff

$$\begin{aligned}&\{\eta_0 : \{\eta_1 : \{\cdots : \{\eta_{m-2} : \{\eta_{m-1} : \{t \in {}^n\kappa : \langle \eta_0\eta_1\cdots\eta_{m-1}\rangle^\frown t \in x\}\\ &\in \mathscr{U}_n^{M\kappa}\} \in \mathscr{U}^{M\kappa}\} \in \cdots \in \mathscr{U}^{M\kappa}\} \in \mathscr{U}^{M\kappa}\} \in \mathscr{U}^{M\kappa}\}.\end{aligned}$$

Let $z = \{t \in {}^n\kappa : \langle \eta_0\eta_1\cdots\eta_{m-1}\rangle^\frown t \in x\}$. Then $z \in \mathscr{U}_n^{M\kappa}$ iff

$$\begin{aligned}&\{\xi_0 : \{\xi_1 : \{\cdots : \{\xi_{n-2} : \{\xi_{n-1} : \langle \xi_0\xi_1\cdots\xi_{n-1}\rangle \in z\} \in \mathscr{U}^{M\kappa}\} \in \mathscr{U}^{M\kappa}\} \in \cdots \in \mathscr{U}^{M\kappa}\\ &\text{iff}\quad \{\xi_0 : \{\xi_1 : \{\cdots : \{\xi_{n-2} : \{\xi_{n-1} : \langle \eta_0\eta_1\cdots\eta_{m-1}\xi_0\xi_1\cdots\xi_{n-1}\rangle \in x\}\\ &\in \mathscr{U}^{M\kappa}\} \in \cdots \in \mathscr{U}^{M\kappa}.\end{aligned}$$

It follows that $y \in \mathscr{U}_m^{M\kappa}$ iff

$$\begin{aligned}&\{\eta_0 : \{\eta_1 : \{\cdots : \{\eta_{m-2} : \{\eta_{m-1} : \{\xi_0 : \{\xi_1 : \{\cdots : \{\xi_{n-2} : \{\xi_{n-1} :\\ &\langle \eta_0\eta_1\cdots\eta_{m-1}\xi_0\xi_1\cdots\xi_{n-1}\rangle \in x\} \in \mathscr{U}^{M\kappa}\} \in \cdots \in \mathscr{U}^{M\kappa} \in \mathscr{U}^{M\kappa}\} \in \cdots \in \mathscr{U}^{M\kappa}.\end{aligned}$$

Thus $x \in \mathscr{U}_{m+n}^{M\kappa}$ iff $y \in \mathscr{U}_m^{M\kappa}$. □

We define for any ordinal α,

$$\begin{aligned}\mathscr{U}_\alpha^{M\kappa} = \{x \in \mathscr{P}_\alpha(\kappa) : \exists n \in \omega \exists j : n \to \alpha[&j \text{ is one-one and order preserving and}\\ &\exists y \in \mathscr{U}_n^{M\kappa}[x = j'^{M\kappa}_{*\alpha n}(y)]]\}.\end{aligned}$$

Proposition 19.26. *This definition is consistent with previous definitions, in the sense that for m finite, $\mathscr{U}_m^{M\kappa}$ satisfies the definition.*

Proof. Suppose that $x \in \mathscr{P}_m(M,\kappa)$, $n \in \omega$ $j : n \to m$ is one-one and order preserving, $y \in \mathscr{U}_n^{M\kappa}$, and $x = j'^{M\kappa}_{*mn}(y)$; we want to show that $x \in \mathscr{U}_m$. In fact, this is immediate from Lemma 19.

Conversely, suppose that $x \in \mathscr{U}_m^{M\kappa}$. Let j be the identity on m. Then $j'^{M\kappa}_{*mm}(x) = \{s \in {}^m\kappa : s \circ j \in x\} = x$, as desired. □

Proposition 19.27. *If $x \in \mathscr{P}_\alpha(M,\kappa)$, $n \in \omega$, $j : n \to \alpha$ is one-one and order preserving, $y \in \mathscr{P}_n(\kappa)$, and $x = j'^{M\kappa}_{*\alpha n}(y)$, then $x \in \mathscr{U}_\alpha^{M\kappa}$ iff $y \in \mathscr{U}_n^{M\kappa}$.*

Proof. We may assume that $\alpha > 0$.

$\Leftarrow$ holds by definition. For $\Rightarrow$, suppose that $x \in \mathscr{U}_\alpha^{M\kappa}$. Then there exist an $m \in \omega$, a $k : m \to \alpha$ which is one-one and order preserving and a $z \in \mathscr{U}_m^{M\kappa}$ such that $x = k'^{M\kappa}_{*\alpha m}(z)$. Now $\mathrm{rng}(j)$ and $\mathrm{rng}(k)$ are supports for x, so $F \stackrel{\mathrm{def}}{=} \mathrm{rng}(j) \cap \mathrm{rng}(k)$ is a support for x. For every $i < n$ such that $j(i) \in F$ there is a $u_i < m$ such that $j(i) = k(u(i))$. Hence there exist $p \in \omega$, an order-preserving bijection $l : p \to F$, and a $w \subseteq {}^p\kappa$ such that $\forall s \in {}^\alpha\kappa[s \in x \leftrightarrow s \circ l \in w]$. For each $i < p$ let $v(i) < n$ be such that $l(i) = j(v(i))$. Then $i < i'$ implies that $l(i) < l(i')$, hence $j(v(i)) < j(v(i'))$, hence $v(i) < v(i')$. Thus $l = j \circ v$ and $v : p \to n$ is one-one and order-preserving.

(12) $y = v'^{M\kappa}_{*np}(w)$.

In fact, for any $s \in {}^n\kappa$, choose $t_s \in {}^\alpha\kappa$ such that $t_s \circ j = s$. Then

$$s \circ v \in w \leftrightarrow t_s \circ j \circ v \in w \leftrightarrow t_s \circ l \in w \leftrightarrow t_s \in x \leftrightarrow t_s \circ j \in y \leftrightarrow s \in y,$$

proving (12).

For each $i < p$, $l(i) \in F$, so there is an $r_i < m$ such that $l(i) = k(r_i)$. Thus $l = k \circ r$.

(13) $z = r'^{M\kappa}_{*mp}(w)$.

In fact, for each $s \in {}^m\kappa$ choose $t_s \in {}^\alpha\kappa$ such that $t_s \circ k = s$. Then for any $s \in {}^m\kappa$,

$$s \circ r \in w \leftrightarrow t_s \circ k \circ r \in w \leftrightarrow t_s \circ l \in w \leftrightarrow t_s \in x \leftrightarrow t_s \circ k \in z \leftrightarrow s \in z,$$

proving (13).

Now $z \in \mathscr{U}_m^{M\kappa}$, so by (13), $w \in \mathscr{U}_p^{M\kappa}$. Then $y \in \mathscr{U}_n^{M\kappa}$. □

Lemma 19.28. *Let $j : \alpha \to \beta$ be injective and order preserving, and let $x \in \mathscr{P}_\alpha^M(\kappa)$. Then $x \in \mathscr{U}_\alpha^{M\kappa}$ iff $j'^{M\kappa}_{*\beta\alpha}(x) \in \mathscr{U}_\beta^{M\kappa}$.*

Proof. $\Rightarrow$: Assume that $x \in \mathscr{U}_\alpha^{M\kappa}$. Choose $n \in \omega$, $k : n \to \alpha$ which is one-one and order preserving, and $y \in \mathscr{U}_n^{M\kappa}$ such that $x = k'^{M\kappa}_{*\alpha n}(y)$. Then $j \circ k$ is injective and order preserving and $(j \circ k)'^{M\kappa}_{*\beta n}(y) = j'^{M\kappa}_{*\beta\alpha}(x)$. (Hence $j'^{M\kappa}_{*\beta\alpha}(x) \in \mathscr{U}_\beta^{M\kappa}$.) In fact,

$$\begin{aligned}(j \circ k)'^{M\kappa}_{*\beta n}(y) = \{s \in {}^\beta\kappa : s \circ j \circ k \in y\} &= \{s \in {}^\beta\kappa : s \circ j \in k'^{M\kappa}_{*\alpha n}(y)\}\\ &= \{s \in {}^\beta\kappa : s \circ j \in x\} = j'^{M\kappa}_{*\beta\alpha}(x).\end{aligned}$$

$\Leftarrow$: Assume that $j'^{M\kappa}_{*\beta\alpha}(x) \in \mathscr{U}_\beta^{M\kappa}$. Choose $n \in \omega$, $k : n \to \beta$ which is one-one and order preserving and $y \in \mathscr{U}_n^{M\kappa}$ such that $j'^{M\kappa}_{*\beta\alpha}(x) = k'^{M\kappa}_{*\beta n}(y)$. Thus $\forall s \in {}^\beta\kappa[s \circ j \in x \leftrightarrow s \circ k \in y]$. Let $F = \mathrm{rng}(j) \cap \mathrm{rng}(k)$.

(14) $\forall s, t \in {}^\beta\kappa[s \restriction F = t \restriction F \to [s \circ j \in x \leftrightarrow t \circ j \in x]$.

For, suppose that $s, t \in {}^\beta\kappa$ and $s \restriction F = t \restriction F$. Define $s'^{M\kappa} \in {}^\beta\kappa$ by

$$s'^{M\kappa}_\xi = \begin{cases} s_\xi & \text{if } \xi \in \mathrm{rng}(j),\\ t_\xi & \text{otherwise.}\end{cases}$$

Then $s \circ j = s'^{M\kappa} \circ j$ and $t \circ k = s'^{M\kappa} \circ k$, so

$$s \circ j \in x \leftrightarrow s'^{M\kappa} \circ j \in x \leftrightarrow s'^{M\kappa} \circ k \in y \leftrightarrow t \circ k \in y \leftrightarrow t \circ j \in x,$$

proving (14).

(15) $\forall s, t \in {}^{\beta}\kappa[s \restriction F = t \restriction F \to [s \circ k \in y \leftrightarrow t \circ k \in y]$.

This follows from (14) since $\forall s \in {}^{\beta}\kappa[s \circ j \in x \leftrightarrow s \circ k \in y]$.

(16) $k^{-1}[F]$ is a support for y.

For, suppose that $s, t \in {}^{\beta}\kappa$ and $s \restriction k^{-1}[F] = t \restriction k^{-1}[F]$. Define $s'^{M\kappa} \in {}^{\beta}\kappa$ by setting, for any $\xi < \beta$,

$$s'^{M\kappa}(\xi) = \begin{cases} s(k^{-1}(\xi)) & \text{if } x \in \text{rng}(k), \\ 0 & \text{otherwise.} \end{cases}$$

Then $s'^{M\kappa} \restriction F = t'^{M\kappa} \restriction F$. In fact, if $\xi \in F$, then $\xi \in \text{rng}(k)$ and so $s'^{M\kappa}(\xi) = s(k^{-1}(\xi)) = t(k^{-1}(\xi)) = t'^{M\kappa}(\xi)$. Hence $s'^{M\kappa} \circ k \in y$ iff $t'^{M\kappa} \circ k \in y$. Now for any $i < n$, $s'^{M\kappa}(k(i)) = s(i)$. Thus $s'^{M\kappa} \circ k = s$, and similarly $t'^{M\kappa} \circ k = t$, so $s \in y$ iff $t \in y$. This proves (16).

By (16) there exist an order preserving bijection c from p onto $k^{-1}[F]$ and a $w \subseteq {}^{p}\kappa$ such that $y = c'^{M\kappa}_{*np}(w)$. Hence $w \in \mathscr{U}_p^{M\kappa}$. Now suppose that $t \in {}^{\alpha}\kappa$. Choose $s \in {}^{\alpha}\kappa$ such that $t = s \circ j$. Then $t \in x$ iff $s \circ j \in x$ iff $s \circ k \in y$ iff $s \circ k \circ c \in w$. Hence $x = (k \circ c)'^{M\kappa}_{*\alpha p}(w)$, and it follows that $x \in \mathscr{U}_\alpha^{M\kappa}$. ☐

Lemma 19.29. *Suppose that $x \in \mathscr{P}_{\alpha+\beta}(M, \kappa)$. Then $x \in \mathscr{U}_{\alpha+\beta}^{M\kappa}$ iff $\{s \in {}^{\alpha}\kappa : x_{(s)} \in \mathscr{U}_\beta^{M\kappa}\} \in \mathscr{U}_\alpha^{M\kappa}$.*

Proof. $\Rightarrow$: Suppose that $x \in \mathscr{U}_{\alpha+\beta}^{M\kappa}$. Choose $n \in \omega$, $j : n \to \alpha + \beta$ one-one and order preserving and $y \in \mathscr{U}_n^{M\kappa}$ such that $x = j'^{M\kappa}_{*(\alpha+\beta)n}(y)$. Thus $x = \{s \in {}^{\alpha+\beta}\kappa : s \circ j \in y\}$.

Case 1. $\text{rng}(j) \subseteq \alpha$. Let $z = \{s \in {}^{\alpha}\kappa : s \circ j \in y\}$. Thus $z = j'^{M\kappa}_{*\alpha n}(y)$, so by definition, $z \in \mathscr{U}_\alpha^{M\kappa}$.

(17) $z \subseteq \{s \in {}^{\alpha}\kappa : x_{(s)} \in \mathscr{U}_\beta^{M\kappa}\}$.

In fact, suppose that $s \in z$. So $s \circ j \in y$. Now

$$x_{(s)} = \{t \in {}^{\beta}\kappa : s^\frown t \in x\} = \{t \in {}^{\beta}\kappa : (s^\frown t) \circ j \in y\} = \{t \in {}^{\beta}\kappa : s \circ j \in y\} = {}^{\beta}\kappa \in \mathscr{U}_\beta,$$

proving (17).

It follows that $\{s \in {}^{\alpha}\kappa : x_{(s)} \in \mathscr{U}_\beta^{M\kappa}\} \in \mathscr{U}_\alpha^{M\kappa}$. So $\Rightarrow$ is proved in this case.

Case 2. $\text{rng}(j) \subseteq (\alpha + \beta)\backslash\alpha$. Say $\forall i < n[j(i) = \alpha + j^*(i)]$, where $j^* : n \to \beta$. Let $z = \{s \in {}^{\beta}\kappa : s \circ j^* \in y\} = j^{*\prime}_{*\beta n}(y)$. So by definition, $z \in \mathscr{U}_\beta^{M\kappa}$.

(18) ${}^{\alpha}\kappa \subseteq \{s \in {}^{\alpha}\kappa : x_{(s)} \in \mathscr{U}_\beta^{M\kappa}\}$.

In fact, take any $s \in {}^{\alpha}\kappa$. Then

$$\begin{aligned} x_{(s)} &= \{t \in {}^{\beta}\kappa : s^\frown t \in x\} = \{t \in {}^{\beta}\kappa : (s^\frown t) \circ j \in y\} \\ &= \{t \in {}^{\beta}\kappa : t \circ j^* \in y\} = j^{*\prime}_{*\beta n}(y) \in \mathscr{U}_\beta^{M\kappa}. \end{aligned}$$

So (18) holds. Hence $\{s \in {}^{\alpha}\kappa : x_{(s)} \in \mathscr{U}_{\beta}^{M\kappa}\} \in \mathscr{U}_{\alpha}^{M\kappa}$, finishing this case of $\Rightarrow$.

Case 3. $\mathrm{rng}(j) \cap \alpha \neq \emptyset \neq \mathrm{rng}(j) \cap ((\alpha+\beta)\backslash\alpha)$. Then we can write $n = p + q$ where $j \upharpoonright p : p \to \alpha$ and $j \upharpoonright ((p+q)\backslash p) : ((p+q)\backslash p) \to (\alpha+\beta)\backslash\alpha$. Let $j^* = j \upharpoonright p$, and let $j^{**} : q \to \beta$ be such that $\forall i < q[j(p+i) = \alpha + j^{**}(i)]$. Then for every $s \in {}^{\alpha+\beta}\kappa$, if $i \in p$ then $s(j(i)) = s(j^*(i))$, while if $i < q$, then $s(j(p+i)) = \alpha + j^{**}(i)$. Thus $\forall s \in {}^{\alpha+\beta}\kappa[s \circ j = (s \circ j^*)^\frown(s \circ j^{**})$. Hence

(19) $x = \{s \in {}^{\alpha+\beta}\kappa : (s \circ j^*)^\frown(s \circ j^{**}) \in y\}$.

Now $y \in \mathscr{U}_{p+q}^{M\kappa}$, so $\{s \in {}^{p}\kappa : y_{(s)} \in \mathscr{U}_q\} \in \mathscr{U}_p^{M\kappa}$. If $s \in {}^{\alpha}\kappa$, then

$$x_{(s)} = \{t \in {}^{\beta}\kappa : s^\frown t \in x\} = \{t \in {}^{\beta}\kappa : (s^\frown t) \circ j \in y\} = \{t \in {}^{\beta}\kappa : (s \circ j^*)^\frown(t \circ j^{**}) \in y\}.$$

Now suppose that $s \in {}^{\alpha}\kappa$. Then $s \circ j^* \in {}^{p}\kappa$.

$$j^{**\prime}_{*\beta q}(y_{(s\circ j^*)}) = \{t \in {}^{\beta}\kappa : t \circ j^{**} \in y_{(s\circ j^*)}\} = \{t \in {}^{\beta}\kappa : (s \circ j^*)^\frown(t \circ j^{**}) \in y\} = x_{(s)}$$

Now suppose that $t \in {}^{p}\kappa$ and $y_{(t)} \in \mathscr{U}_q^{M\kappa}$. Then if $s \in {}^{\alpha}\kappa$ and $s \circ j^* = t$, then $x_{(s)} = j^{**\prime}_{*\beta q}(y_{(t)})$, and hence $x_{(s)} \in \mathscr{U}_{\beta}^{M\kappa}$. So $\{t \in {}^{p}\kappa : y_{(t)} \in \mathscr{U}_q^{M\kappa}\} \subseteq \{s \circ j^* : x_{(s)} \in \mathscr{U}_{\beta}^{M\kappa}\}$. It follows that $z \stackrel{\mathrm{def}}{=} \{s \circ j^* : x_{(s)} \in \mathscr{U}_{\beta}^{M\kappa}\} \in \mathscr{U}_p^{M\kappa}$. Now

$$j^{*\prime}_{*\alpha p}(z) = \{s \in {}^{\alpha}\kappa : s \circ j^* \in z\} = \{s \in {}^{\alpha}\kappa : x_{(s)} \in \mathscr{U}_{\beta}^{M\kappa}\},$$

so $\{s \in {}^{\alpha}\kappa : x_{(s)} \in \mathscr{U}_{\beta}^{M\kappa}\} \in \mathscr{U}_{\alpha}^{M\kappa}$.

This finishes the direction $\Rightarrow$ of the lemma.

$\Leftarrow$: Assume that $\{s \in {}^{\alpha}\kappa : x_{(s)} \in \mathscr{U}_{\beta}\} \in \mathscr{U}_{\alpha}^{M\kappa}$.

Case 1. $\alpha = 0$. Now $\mathscr{U}_0^{M\kappa} = \{\{\emptyset\}\}$. Hence $\{s \in {}^{\alpha}\kappa : x_{(s)} \in \mathscr{U}_{\beta}\} = \{\emptyset\}$. So $\forall s \in {}^{\alpha}\kappa[x_{(s)} \in \mathscr{U}_{\beta} \leftrightarrow s = \emptyset]$. Now $x_{(\emptyset)} = x$, so $x \in \mathscr{U}_{\beta}$.

Case 2. $\alpha \neq 0 = \beta$. Then $\mathscr{U}_{\beta}^{M\kappa} = \{\{\emptyset\}\}$. Let $A = \{s \in {}^{\alpha}\kappa : x_{(s)} \in \mathscr{U}_{\beta}^{M\kappa}\}$. So $A \in \mathscr{U}_{\alpha}$. If $s \in A$, then $x_{(s)} \neq \emptyset$, so $s \in x$. Thus $A \subseteq x$, so $x \in \mathscr{U}_{\alpha}^{M\kappa}$.

Case 3. $\alpha, \beta \neq \emptyset$. Since $\{s \in {}^{\alpha}\kappa : x_{(s)} \in \mathscr{U}_{\beta}^{M\kappa}\} \in \mathscr{U}_{\alpha}^{M\kappa}$, choose $n \in \omega$, $j : n \to \alpha$ which is one-one and order preserving and $y \in \mathscr{U}_n^{M\kappa}$ such that $\{s \in {}^{\alpha}\kappa : x_{(s)} \in \mathscr{U}_{\beta}\} = j'^{M\kappa}_{*\alpha n}(y)$. Let $F \subseteq \alpha+\beta$ be a finite support for x. Let $m = |\mathrm{rng}(j) \cup (F \cap \alpha)|$. Let $k : m \to \alpha$ be injective and order preserving, with $\mathrm{rng}(j) \subseteq \mathrm{rng}(k)$. Say $j_i = k_{l(i)}$ for all $i < n$. Thus $j = k \circ l$, and $l : n \to m$. If $i < i'^{M\kappa} < n$, then $j_i < j_{i'^{M\kappa}}$ and so $l(i) < l(i'^{M\kappa})$. By Lemma 23, $l'^{M\kappa}_{*mn}(y) \in \mathscr{U}_m^{M\kappa}$. Now

$$\begin{aligned} k'^{M\kappa}_{*\alpha m}(l'^{M\kappa}_{*mn}(y)) &= \{s \in {}^{\alpha}\kappa : s \circ k \in l'^{M\kappa}_{*mn}(y)\} = \{s \in {}^{\alpha}\kappa : s \circ k \circ l \in y\} \\ &= \{s \in {}^{\alpha}\kappa : s \circ j \in y\} = j'^{M\kappa}_{*\alpha n}(y) = \{s \in {}^{\alpha}\kappa : x_{(s)} \in \mathscr{U}_{\beta}^{M\kappa}\}. \end{aligned}$$

Let $y^* = l'^{M\kappa}_{*mn}(y)$. Let $F'^{M\kappa} = \{\xi < \beta : \alpha + \xi \in F\}$, $q = |F'^{M\kappa}|$, and $a : q \to F'^{M\kappa}$ injective and order preserving. For any $s \in {}^{\alpha}\kappa$ let $u_s = \{t \circ a : t \in x_{(s)}\}$. Then

(20) $x_{(s)} = a'^{M\kappa}_{*\beta q}(u_s)$.

In fact, $t \in x_{(s)}$ iff $t \circ a \in u_s$. Here $\Rightarrow$ is clear. For $\Leftarrow$, suppose that $t \circ a \in u_s$. Choose $t'^{M\kappa} \in x_{(s)}$ such that $t \circ a = t'^{M\kappa} \circ a$. Then $(s^\frown t) \restriction F = (s^\frown t'^{M\kappa}) \restriction F$, so $s^\frown t \in x$. So (20) holds.

By (20), $u_s \in \mathscr{U}_q^{M\kappa}$ for all s such that $x_{(s)} \in \mathscr{U}_\beta^{M\kappa}$. Now let

$$z = \{(s \circ k)^\frown(t \circ a) : s \in {}^\alpha\kappa,\ t \in {}^\beta\kappa,\ t \in x_{(s)}\}.$$

Clearly

(21) If $s \in {}^\alpha\kappa$, then $u_s \subseteq z_{(s \circ k)}$.

Now we claim

(22) $y^* \subseteq \{s \in {}^n\kappa : z_{(s)} \in \mathscr{U}_q^{M\kappa}\}$.

For, suppose that $s \in y^*$. Let $s^* \in {}^\alpha\kappa$ be such that $s = s^* \circ k$. Then $s^* \in k'^{M\kappa}_{*\alpha m}(y^*)$, so $x_{(s^*)} \in \mathscr{U}_\beta^{M\kappa}$. Hence $u_{s^*} \in \mathscr{U}_q^{M\kappa}$. Then $u_{s^*} \subseteq z_{(s^* \circ k)}$. This proves (22).

Now since $y^* \in \mathscr{U}_m^{M\kappa}$, it follows that $\{s \in {}^n\kappa : z_{(s)} \in \mathscr{U}_q\} \in \mathscr{U}_n^{M\kappa}$. Hence $z \in \mathscr{U}_{n+q}^{M\kappa}$. Now we let $b = k^\frown a$. We claim:

(23) $x = b'^{M\kappa}_{*(\alpha+\beta)(m+q)}(z)$.

In fact,

$$\begin{aligned}
b'^{M\kappa}_{(\alpha+\beta)(m+q)}(z) &= \{s \in {}^{\alpha+\beta}\kappa : s \circ b \in z\} \\
&= \{s \in {}^{\alpha+\beta}\kappa : (s \circ k)^\frown(s \circ a) \in z\} \\
&= \{s \in {}^{\alpha+\beta}\kappa : \exists s^* \in {}^\alpha\kappa \exists t^* \in {}^\beta\kappa \\
&\quad [(s \circ k)^\frown(s \circ a) = (s^* \circ k)^\frown(t^* \circ a) \wedge t^* \in x_{(s^*)}] \\
&= \{s \in {}^{\alpha+\beta}\kappa : \exists s^* \in {}^\alpha\kappa \exists t^* \in {}^\beta\kappa \\
&\quad [s \circ k = s^* \circ k \wedge s \circ a = t^* \circ a \wedge t^* \in x_{(s^*)}] \\
&= \{s \in {}^{\alpha+\beta}\kappa : \exists s^* \in {}^\alpha\kappa \exists t^* \in {}^\beta\kappa \\
&\quad [s \circ k = s^* \circ k \wedge s \circ a = t^* \circ a \wedge s^{*\frown} t^* \in x] \\
&= \{s \in {}^{\alpha+\beta}\kappa : \exists s^* \in {}^\alpha\kappa \exists t^* \in {}^\beta\kappa \\
&\quad [s = s^{*\frown} t^* \wedge s^{*\frown} t^* \in x].
\end{aligned}$$

The last step here holds because $\mathrm{rng}(k) \cup \mathrm{rng}(a)$ supports x. $\square$

Theorem 19.30. *For every ordinal α, $\mathscr{U}_\alpha^{M\kappa}$ is a nonprincipal ultrafilter on the Boolean algebra $\mathscr{P}_\alpha^M(\kappa)$.*

Proof. Nonprincipal: suppose that $z \in {}^\alpha\kappa$ and $\{z\} \in \mathscr{U}_\alpha^{M\kappa}$. Choose $n \in \omega$, $j : n \to \alpha$ one-one and order preserving, and $y \in \mathscr{U}_n^{M\kappa}$ so that $\{z\} = j'^{M\kappa}_{*\alpha n}(y)$. Thus $\{z\} = \{s \in {}^\alpha\kappa : s \circ j \in y\}$. Hence $z \circ j \in y$. If $t \in y$, choose $s \in {}^\alpha\kappa$ so that $s \circ j = t$. Then $s = z$. Thus $y = \{z \circ j\}$, contradicting Theorem 19.20.

Upwards closed: suppose that $x, y \in \mathscr{P}_\alpha(\kappa)$, $x \in \mathscr{U}_\alpha$, and $x \subseteq y$. Choose $n \in \omega$, $j : n \to \alpha$ one-one and order preserving, and $z \in \mathscr{U}_n^{M\kappa}$ so that $x = j'^{M\kappa}_{*\alpha n}(z)$. Thus

$x = \{s \in {}^\alpha\kappa : s \circ j \in z\}$. Let $F \in [\alpha]^{<\omega}$ be a support for y. Choose $m \in \omega$ with $n \leq m$ and $j^o : m \to \alpha$ injective and order preserving, so that $j \subseteq j^o$ and $F \subseteq \operatorname{rng}(j'^{M\kappa})$. Let $z'^{M\kappa} = \{s \in {}^m\kappa : s \restriction n \in z\}$. Then $x = j^{o\prime}_{*\alpha m}(z'^{M\kappa})$, since

$$j^{o\prime}_{*\alpha m}(z'^{M\kappa}) = \{s \in {}^\alpha : s \circ j^o \in z'^{M\kappa}\} = \{s \in {}^\alpha\kappa : s \circ j \in z\} = x.$$

By Proposition 22, $z'^{M\kappa} \in \mathscr{U}_m^{M\kappa}$.

Let $w = \{a \in {}^n\kappa : \exists s \in y[a = s \circ j^o]\}$. If $t \in z'^{M\kappa}$, choose $s \in {}^\alpha\kappa$ so that $s \circ j^o = t$. Then $s \in x$, so $s \in y$. Hence $t \in w$. So $z'^{M\kappa} \subseteq w$, and hence $w \in \mathscr{U}_m^{M\kappa}$. Now $y = \{s \in {}^\alpha\kappa : s \circ j^o \in w\}$. In fact, $\subseteq$ is clear. Now suppose that $s \in {}^\alpha\kappa$ and $s \circ j^o \in w$. Choose $s'^{M\kappa} \in y$ such that $s \circ j^o = s'^{M\kappa} \circ j^o$. Since $F \subseteq \operatorname{rng}(j^o)$, it follows that $s \in y$. So $y = \{s \in {}^\alpha\kappa : s \circ j^o \in w\}$. Hence $y \in \mathscr{U}_\alpha^{M\kappa}$.

Complement: suppose that $x \in \mathscr{P}_\alpha(\kappa)$ and $x \notin \mathscr{U}_\alpha$. Let $F \in [\alpha]^{<\omega}$ be a support for X, let $m = |F|$, and let $j : m \to F$ be an order preserving bijection. Let $y = \{s \circ j : s \in x\}$. Then $x = j'^{M\kappa}_{*\alpha m}(y)$. For, $j'^{M\kappa}_{*\alpha m}(y) = \{s \in {}^\alpha\kappa : s \circ j \in y\}$, and clearly $x \subseteq \{s \in {}^\alpha\kappa : s \circ j \in y\}$. If $s \in {}^\alpha\kappa$ and $s \circ j \in y$, choose $s'^{M\kappa} \in x$ such that $s \circ j = s'^{M\kappa} \circ j$. Since F is a support of x, it follows that $s \in x$. So $x = j'^{M\kappa}_{*\alpha m}(y)$. Now $x \notin \mathscr{P}_\alpha$, so $y \notin \mathscr{U}_n$. Hence ${}^n\kappa \backslash y \in \mathscr{P}_n$. We claim that

$$(*) \qquad {}^\alpha\kappa \backslash x = j'^{M\kappa}_{*\alpha m}({}^n\kappa \backslash y).$$

For, suppose that $s \in {}^\alpha\kappa \backslash x$. Then $s \notin j'^{M\kappa}_{*\alpha m}(y)$, i.e., $s \notin \{s'^{M\kappa} \in {}^\alpha\kappa : s'^{M\kappa} \circ j \in y\}$; so $s \circ j \notin y$, hence $s \circ j \in {}^n\kappa \backslash y$, so that $s \in j'^{M\kappa}_{*\alpha m}({}^n\kappa \backslash y)$. Conversely, if $s \in j'^{M\kappa}_{*\alpha m}({}^n\kappa \backslash y)$, then $s \circ j \notin y$, hence $s \notin x$. Thus $(*)$ holds, and so ${}^\alpha\kappa \backslash x \in \mathscr{U}_\alpha^{M\kappa}$.

Closure under $\cap$: Suppose that $x, y \in \mathscr{U}_\alpha^{M\kappa}$. Choose $m \in \omega$, $j : m \to \alpha$ one-one and order preserving, and $x'^{M\kappa} \in \mathscr{U}_m^{M\kappa}$ so that $x = j'^{M\kappa}_{*\alpha m}(x'^{M\kappa})$; and choose $n \in \omega$, $k : n \to \alpha$ one-one and order preserving, and $y'^{M\kappa} \in \mathscr{U}_n^{M\kappa}$ so that $y = j'^{M\kappa}_{*\alpha n}(y'^{M\kappa})$. Thus $x = \{s \in {}^\alpha\kappa : s \circ j \in x'^{M\kappa}\}$ and $y = \{s \in {}^\alpha\kappa : s \circ k \in y'^{M\kappa}\}$. So clearly $\operatorname{rng}(j)$ is a support of x and $\operatorname{rng}(k)$ is a support for y. Let p and l be such that $l : p \to \alpha$ is injective and order preserving with $\operatorname{rng}(j) \cup \operatorname{rng}(k) \subseteq \operatorname{rng}(l)$. Say $j(i) = l(j'^{M\kappa}(i))$ for all $i < m$ and $k(i) = l(k'^{M\kappa}(i))$ for all $i < n$. Now let $x'^{M\kappa} = \{s \in {}^p\kappa : s \circ j'^{M\kappa} \in x'^{M\kappa}\}$. Now

$$(**) \qquad x'^{M\kappa} = \{s \circ j'^{M\kappa} : s \in x'^{M\kappa}\}.$$

In fact, $\supseteq$ is clear. If $t \in x'^{M\kappa}$, choose $s \in {}^p\kappa$ such that $s \circ j'^{M\kappa} = t$. Then $s \in x'^{M\kappa}$, so $t \in \{s \circ j : s \in x'^{M\kappa}\}$.

Now $(**)$ says that $x'^{M\kappa} = j'^{M\kappa}_{*pm}(x'^{M\kappa}$, so $x'^{M\kappa} \in \mathscr{U}_p^{M\kappa}$ by Lemma 19. Similarly, let $y'^{M\kappa} = \{s \in {}^p\kappa : s \circ k'^{M\kappa} \in y'^{M\kappa}\}$. Then $y'^{M\kappa} \in \mathscr{U}_p$. Hence $x'^{M\kappa} \cap y'^{M\kappa} \in \mathscr{U}_p^{M\kappa}$ by Theorem 15. Now $x \cap y = l'^{M\kappa}_{*\alpha p}(x'^{M\kappa} \cap y'^{M\kappa}$. In fact, if $s \in x \cap y$, then $s \circ j \in x'^{M\kappa}$, hence $s \circ l \circ j'^{M\kappa} \in x'^{M\kappa}$, hence $s \circ l \in x'^{M\kappa}$. Similarly, $s \circ l \in y'^{M\kappa}$. This shows that $x \cap y \subseteq l'^{M\kappa}_{*\alpha p}(x'^{M\kappa} \cap y'^{M\kappa}$. Conversely, suppose that $s \in l'^{M\kappa}_{*\alpha p}(x'^{M\kappa} \cap y'^{M\kappa}$. Then $s \circ l \in x'^{M\kappa} \cap y'^{M\kappa}$, so $s \circ l \circ j'^{M\kappa}$, hence $s \circ j \in x'^{M\kappa}$, hence $s \in x$. Similarly $s \in y$. $\square$

For $f, g \in \operatorname{Fn}_\alpha(M, \kappa)$ we define

$$\begin{aligned}
&f \approx_\alpha^{M\kappa} g \quad \text{iff} \quad \{s \in {}^\alpha\kappa : f(s) = g(s)\} \in \mathscr{U}_\alpha^{M\kappa};\\
&[f]_\alpha^{M\kappa} = \{g : g \approx_\alpha^{M\kappa} f \wedge \forall h[h \approx_\alpha^{M\kappa} f \to \mathrm{rank}^M(g) \le \mathrm{rank}^M(h)]\}\\
&N_\alpha^{M\kappa} = \{[f]_\alpha^{M\kappa} : f \in \mathrm{Fn}_\alpha(M,\kappa)\};\\
&E_\alpha^{M\kappa} = \{([f]_\alpha^{M\kappa}, [g]_\alpha^{M\kappa}) : \{s \in {}^\alpha\kappa : f(s) \in g(s)\} \in \mathscr{U}_\alpha^{M\kappa}\};\\
&\mathrm{Ult}_\alpha(\mathscr{U}^{M\kappa}) = (N_\alpha^{M\kappa}, E_\alpha^{M\kappa}).
\end{aligned}$$

Note that for $f,g \in {}^0\kappa$ we have $f \approx^{M\kappa} g$ iff $f = g$. Hence for $f \in {}^0\kappa$ we have $[f]_0^{M\kappa} = \{f\}$. Hence $N_0^{M\kappa} = \{\{f\} : f$ is a function mapping $\{\emptyset\}$ into M. $E_0^{M\kappa} = \{(\{f\},\{g\}) : f(\emptyset) \in g(\emptyset)\}$.

Proposition 19.31. *$[f]_\alpha^{M\kappa} E_\alpha^{M\kappa} [g]_\alpha^{M\kappa}$ iff $\{s \in {}^\alpha\kappa : f(s) \in g(s)\} \in \mathscr{U}_\alpha^{M\kappa}$.*

Proof. $\Leftarrow$ is clear. $\Rightarrow$: Assume that $[f]_\alpha^{M\kappa} E_\alpha^{M\kappa} [g]_\alpha^{M\kappa}$. Then there exist $f', g' \in \mathrm{Fn}_\alpha(M,\kappa)$ such that $[f]_\alpha^{M\kappa} = [f']_\alpha^{M\kappa}$, $[g]^{M\kappa} = [g']_\alpha$, and $\{s \in {}^\alpha\kappa : f'(s) \in g'(s)\} \in \mathscr{U}_\alpha^{M\kappa}$. Now

$$\begin{aligned}
&\{s \in {}^\alpha\kappa : f(\alpha) = f'(\alpha)\} \cap \{s \in {}^\alpha\kappa : g(\alpha) = g'(\alpha)\}\\
&\cap \{s \in {}^\alpha\kappa : f'(s) \in g'(s)\} \subseteq \{{}^\alpha\kappa : f(s) \in g(s)\},
\end{aligned}$$

and the left side is in $\mathscr{U}_\alpha^{M\kappa}$; hence so is the right side. $\square$

Theorem 19.32. *$\forall\alpha[N_\alpha^{M\kappa}$ is well-founded$]$.*

Proof. Suppose that $\forall n \in \omega[[f_{n+1}]_\alpha^{M\kappa} E_\alpha^{M\kappa} [f_n]_\alpha^{M\kappa}]$. Let $x_n = \{s \in {}^\alpha\kappa : f_{n+1}(s) \in f_n(s)\}$. Suppose that $\gamma + \beta = \alpha$, $s \upharpoonright \gamma$ is defined, and $\forall n \in \omega[x_{n(s\upharpoonright\gamma)} \in \mathscr{U}_\beta^{M\kappa}]$. Note that this holds for $\gamma = 0$.

Case 1. $\gamma = \alpha$. Note that

$$\forall n \in \omega[x_{n(s\upharpoonright\alpha)} = \{s \in {}^0\kappa : s \upharpoonright \alpha \in x_n\} = \begin{cases} \{\emptyset\} & \text{if } s \upharpoonright \alpha \in x_n,\\ \emptyset & \text{otherwise.}\end{cases}$$

Now $\mathscr{U}_0^{M\kappa} = \{\{\emptyset\}\}$ and $x_{n(s\upharpoonright\gamma)} \in \mathscr{U}_0^{M\kappa}$, so $s \upharpoonright \alpha \in x_n$. This is true for all n. Hence $f_{n+1}(s \upharpoonright \alpha) \in f_n(s \upharpoonright \alpha)$ for all n, contradiction.

Case 9. $\gamma < \alpha$. So $\beta \ne 0$. Say $\beta = 1 + \delta$. Then by Lemma 38, for each $n \in \omega$, $\{\xi : (x_{n(s\upharpoonright\gamma)})_{(\xi)} \in \mathscr{U}_\delta^{M\kappa}\} \in \mathscr{U}^{M\kappa}$. Let $s(\gamma) \in \bigcap_{n\in\omega}\{\xi : (x_{n(s\upharpoonright\gamma)})_{(\xi)} \in \mathscr{U}_\delta\}$. Thus $\forall n \in \omega[(x_{n(s\upharpoonright\gamma)})_{s(\gamma)} \in \mathscr{U}_\delta^{M\kappa}]$. Now note that

$$\begin{aligned}
(x_{n(s\upharpoonright\gamma)})_{s(\gamma)} &= \{t \in {}^\delta\kappa : \langle s(\gamma)\rangle^\frown t \in x_{n(s\upharpoonright\gamma)}\}\\
&= \{t \in {}^\delta\kappa : (s \upharpoonright \gamma)^\frown\langle s(\gamma)\rangle^\frown t \in x_n\}\\
&= \{t \in {}^\delta\kappa : (s \upharpoonright (\gamma+1))^\frown t \in x_n\}\\
&= x_{n(s\upharpoonright(\gamma+1))}.
\end{aligned}$$

Now $\alpha = \gamma + 1 + \delta$, so the definition of s is complete. $\square$

Since $N_\alpha^{M\kappa}$ is well-founded, let $\pi_\alpha^{M\kappa}$ is the transitive collapse function, mapping $N_\alpha^{M\kappa}$ onto a transitive set $N_\alpha'^{M\kappa}$, and we set $\mathrm{Ult}_\alpha'^{M\kappa}(\mathscr{U}^{M\kappa}) = (N_\alpha'^{M\kappa}, \in)$. Then $N_0'^{M\kappa} = M$. In fact, for any $f \in \mathrm{Fn}_0(M, \kappa)$ we have

$$\begin{aligned}\pi_0^{M\kappa}([f]_0) &= \pi_0^{M\kappa}(\{f\}) = \{\pi_0^{M\kappa}(\{g\}) : \{g\}E_0^{M\kappa}\{f\}\} = \{\pi_0^{M\kappa}(\{g\}) : g(\emptyset) \in f(\emptyset)\} \\ &= \{g(\emptyset) : g(\emptyset) \in f(\emptyset)\} = f(\emptyset).\end{aligned}$$

Proposition 19.33. *$\pi^{M\kappa}([f]_\alpha^{M\kappa}) \in \pi^{M\kappa}([g]_\alpha^{M\kappa})$ iff $\{s \in {}^\alpha\kappa : f(s) \in g(s)\} \in \mathscr{U}_\alpha^{M\kappa}$.* $\square$

If $j : \alpha \to \beta$ injective and order preserving, we define $j_{*\beta\alpha}^{\star M\kappa} : N_\alpha'^{M\kappa} \to N_\beta'^{M\kappa}$ by $j_{*\beta\alpha}^{\star M\kappa}((\pi_\alpha^{M\kappa}[f]_\alpha^{M\kappa})) = \pi_\beta^{M\kappa}([j_{*\beta\alpha}^{M\kappa}(f)]_\beta^{M\kappa})$. Further, for $\alpha \le \beta$ and j the identity on α we define $i_{\beta\alpha}^{\star M\kappa} = j_{*\beta\alpha}^{\star M\kappa}$.

For $\alpha = \beta = 0$ we have $j = \emptyset$, and $\emptyset_{*00}^{\star M\kappa}$ is the identity on $N_0'^{M\kappa}$, since $\emptyset_{*00}^{\star M\kappa}(\pi_0^{M\kappa}([f]_0^{M\kappa})) = \pi_0^{M\kappa}([\emptyset_{*00}^{M\kappa}(f)]_0) = \pi_0^{M\kappa}([f]_0^{M\kappa})$. For $\alpha = 0 \neq \beta$ we have $j = \emptyset$ and $\emptyset_{*\beta 0}^{\star M\kappa}(\pi_0^{M\kappa}([f]_0^{M\kappa})) = \pi_\beta^{M\kappa}([\emptyset_{*\beta 0}^{M\kappa}(f)]_\beta) = \pi_\beta^{M\kappa}([c_{f(\emptyset)}^{\beta\kappa}])$. For $\alpha \neq 0 = \beta$ there is no function from α to β.

Proposition 19.34. *$j_{*\beta\alpha}^{\star M\kappa}([f]_\alpha^{M\kappa})$ is well-defined.*

Proof. Assume that $[f]_\alpha^{M\kappa} = [g]_\alpha^{M\kappa}$; we want to show that $[j_{*\beta\alpha}^{M\kappa}(f)]_\beta = [j_{*\beta\alpha}^{M\kappa}(g)]_\beta$. Thus our assumption is that $x \stackrel{\text{def}}{=} \{s \in {}^\alpha\kappa : f(s) = g(s)\} \in \mathscr{U}_\alpha^{M\kappa}$. Now

$$\{s \in {}^\beta\kappa : (j_{*\beta\alpha}^{M\kappa}(f))(s) = (j_{*\beta\alpha}^{M\kappa}(g))(s)\} = \{s \in {}^\beta\alpha : f(s \circ j) = g(s \circ j)\} = j_{*\beta\alpha}'^{M\kappa}(x).$$

It follows that $\{s \in {}^\beta\kappa : (j_{*\beta\alpha}^{M\kappa}(f))(s) = (j_{*\beta\alpha}^{M\kappa}(g))(s)\} \in \mathscr{U}_\beta$. $\square$

Theorem 19.35. *Let $\varphi(v_0, \ldots, v_{m-1})$ be a set-theoretic formula and let $f_0, \ldots, f_{m-1} \in \mathrm{Fn}_\alpha(M, \kappa)$. Then*

$$\begin{aligned}&\varphi^{N_\alpha'}(\pi_\alpha^{M\kappa}([f_0]_\alpha^{M\kappa}), \ldots, \pi_\alpha^{M\kappa}([f_{m-1}]_\alpha^{M\kappa})) \\ &\quad \textit{iff} \quad \{s \in {}^\alpha\kappa : \varphi^M(f_0(s), \ldots, f_{m-1}(s))\} \in \mathscr{U}_\alpha^{M\kappa}.\end{aligned}$$

Proof. Induction on φ.
Case 1. φ is $v_i \in v_j$. Then

$$\pi_\alpha^{M\kappa}([f_i]_\alpha^{M\kappa}) \in \pi_\alpha^{M\kappa}([f_j]_\alpha^{M\kappa}) \quad \text{iff} \quad \{s \in {}^\alpha\kappa : f_i(s) \in f_j(s)\} \in \mathscr{U}_\alpha^{M\kappa}.$$

Case 9. φ is $v_i = v_j$. Then

$$\pi_\alpha^{M\kappa}([f_i]_\alpha^{M\kappa}) = \pi_\alpha^{M\kappa}([f_j]_\alpha^{M\kappa}) \quad \text{iff} \quad \{s \in {}^\alpha\kappa : f_i(s) = f_j(s)\} \in \mathscr{U}_\alpha^{M\kappa}.$$

Case 3. $\neg$, $\wedge$. Clear.

Case 4. φ is $\exists v_i \psi(v_0, \ldots, v_{m-1})$. First suppose that

$$(\exists v_i \psi(\pi_\alpha^{M\kappa}([f_0]_\alpha^{M\kappa}), \ldots, \pi_\alpha^{M\kappa}([f_{i-1}]_\alpha^{M\kappa}), v_i, \pi_\alpha^{M\kappa}([f_{i+1}]_\alpha^{M\kappa}), \ldots, \pi_\alpha^{M\kappa}([f_{m-1}]_\alpha^{M\kappa}))^{N_\alpha}.$$

Choose $g \in \mathrm{Fn}_\alpha(M, \kappa)$ so that

$$\psi(\pi_\alpha^{M\kappa}([f_0]_\alpha^{M\kappa}), \ldots, \pi_\alpha^{M\kappa}([f_{i-1}]_\alpha^{M\kappa}), \pi_\alpha^{M\kappa}([g]^{M\kappa}), \pi_\alpha^{M\kappa}([f_{i+1}]_\alpha^{M\kappa}), \ldots, \pi_\alpha^{M\kappa}([f_{m-1}]_\alpha^{M\kappa}))^{N_\alpha}.$$

By the inductive hypothesis,

$$\{s \in {}^\alpha\kappa : (\psi(f_0(s), \ldots, f_{i-1}(s), g(s), f_{i+1}(s), \ldots, f_{m-1}(s)))^M\} \in \mathscr{U}_\alpha.$$

Clearly this set is contained in

$$x \stackrel{\text{def}}{=} \{s \in {}^\alpha\kappa : (\exists v_i \psi(f_0(s), \ldots, f_{i-1}(s), v_i, f_{i+1}(s), \ldots, f_{m-1}(s)))^M\}$$

Hence $x \in \mathscr{U}_\alpha^{M\kappa}$, as desired.

Second, suppose that $x \in \mathscr{U}_\alpha^{M\kappa}$. For each $s \in x$ choose $g(s)$ so that

$$(\psi^M(f_0(s), \ldots, f_{i-1}(s), g(s), f_{i+1}(s), \ldots, f_{m-1}(s))).$$

For $s \notin x$ let $g(s)$ be arbitrary. Then by the inductive hypothesis,

$$(\psi(\pi_\alpha^{M\kappa}([f_0]_\alpha^{M\kappa}), \ldots, \pi_\alpha^{M\kappa}([f_{i-1}]_\alpha^{M\kappa}, [g]_\alpha^{M\kappa}, \pi_\alpha^{M\kappa}([f_{i+1}]_\alpha^{M\kappa}), \ldots, \pi_\alpha^{M\kappa}([f_{m-1}]_\alpha^{M\kappa}))^{N'_\alpha},$$

so $\varphi^{(N'_\alpha)}(\pi_\alpha^{M\kappa}([f_0]_\alpha^{M\kappa}), \ldots, \pi_\alpha^{M\kappa}([f_{m-1}^{M\kappa}]_\alpha))$. □

Corollary 19.36. *$N'^{M\kappa}_\alpha$ is a model of ZFC.*

Theorem 19.37. *$j^{\star M\kappa}_{*\beta\alpha}$ is an elementary embedding.*

Proof. Assume that $f_0, \ldots, f_{m-1} \in \mathrm{Fn}_\alpha(M, \kappa)$. Let

$$x = \{s \in {}^\alpha\kappa : \varphi^M(f_0(s), \ldots, f_{n-1}(s))\}.$$

Note that

$$j'^{M\kappa}_{*\beta\alpha}(x) = \{s \in {}^\beta\kappa : s \circ j \in x\} = \{s \in {}^\beta\kappa : \varphi^M(f_0(s \circ j), \ldots, f_{n-1}(s \circ j))\}.$$

Hence

$$\begin{aligned}
&\varphi^{N'_\beta}(j^{\star M\kappa}_{*\beta\alpha}(\pi_\alpha^{M\kappa}([f_0]_\alpha^{M\kappa})), \ldots, j^{\star M\kappa}_{*\beta\alpha}(\pi_\alpha^{M\kappa}([f_{m-1}]_\alpha^{M\kappa}))\\
\text{iff}\quad &\varphi^{N'_\beta}(\pi_\beta^{M\kappa}([j^{M\kappa}_{*\beta\alpha}(f_0)]_\beta^{M\kappa}, \ldots, \pi_\beta^{M\kappa}([j_{*\beta\alpha}(f_{m-1})]_\beta^{M\kappa})\\
\text{iff}\quad &\{s \in {}^\beta\kappa : \varphi^M((j^{M\kappa}_{*\beta\alpha}(f_0))(s), \ldots, (j^{M\kappa}_{*\beta\alpha}(f_{m-1}))(s))\} \in \mathscr{U}_\beta^{M\kappa}\\
\text{iff}\quad &\{s \in {}^\beta\kappa : \varphi^M(f_0(s \circ j), \ldots, f_{n-1}(s \circ j))\} \in \mathscr{U}_\beta^{M\kappa}\\
\text{iff}\quad &j'^{M\kappa}_{*\beta\alpha}(x) \in \mathscr{U}_\beta^{M\kappa}\\
\text{iff}\quad &x \in \mathscr{U}_\alpha^{M\kappa} \quad \text{by Lemma 24}\\
\text{iff}\quad &\varphi^{N'_\alpha}(\pi_\alpha^{M\kappa}([f_0]_\alpha), \ldots, \pi_\alpha^{M\kappa}([f_{m-1}]_\alpha))
\end{aligned}$$

□

Corollary 19.38. *If $\alpha < \beta$ and $f, g \in \mathrm{Fn}_\alpha(M, \kappa)$, then*

$$[f]_\alpha^{M\kappa} E_\alpha^{M\kappa} [g]_\alpha^{M\kappa} \quad \textit{iff} \quad j_{*\beta\alpha}^{\star M\kappa}([f]_\alpha^{M\kappa}) E_\beta^{M\kappa} j_{*\beta\alpha}^{\star M\kappa}([g]_\alpha^{M\kappa}). \qquad \square$$

Proposition 19.39. *If $\alpha \le \beta \le \gamma$, then $i_{*\gamma\beta}^{\star M\kappa} \circ i_{*\beta\alpha}^{\star M\kappa} = i_{*\gamma\alpha}^{\star M\kappa}$.*

Proof. For any $f \in \mathrm{Fn}_\alpha(M, \kappa)$,

$$\begin{aligned} i_{*\gamma\beta}^{\star M\kappa}(i_{*\beta\alpha}^{\star M\kappa}(\pi_\alpha^{M\kappa}[f]_\alpha^{M\kappa})) &= i_{*\gamma\beta}^{\star M\kappa}(\pi_\beta^{M\kappa}([j_{*\beta\alpha}^{M\kappa}(f)]_\beta^{M\kappa}) \\ &= \pi_\gamma^{M\kappa}([j_{*\gamma\beta}^{M\kappa}(j_{*\beta\alpha}^{M\kappa}(f))]_\gamma^{M\kappa} \\ &= \pi_\gamma^{M\kappa}([j_{\gamma\beta}^{M\kappa}(f \circ j_{\beta\alpha})]_\gamma \\ &= \pi_\gamma^{M\kappa}([f \circ j_{\beta\alpha}^{M\kappa} \circ j_{\gamma\beta}^{M\kappa}]_\gamma \\ &= \pi_\gamma^{M\kappa}([f \circ j_{\gamma\alpha}]_\gamma^{M\kappa} = i_{*\gamma\alpha}^{\star M\kappa}([f]_\alpha^{M\kappa}). \qquad \square \end{aligned}$$

Now for each ordinal α we define $P_\alpha^{M\kappa}$ and $k_\alpha^{M\kappa}$ so that the following conditions hold for all β:

(24) $k_\beta^{M\kappa}$ is an isomorphism from $P_\beta^{M\kappa}$ onto $N_\beta'^{M\kappa}$.

(25) If $\alpha < \beta$, then $P_\alpha^{M\kappa} \subseteq P_\beta^{M\kappa}$.

(26) $\forall \alpha < \beta [i_{*\beta\alpha}^{\star M\kappa} \circ k_\alpha^{M\kappa} \subseteq k_\beta^{M\kappa}]$.

We let $P_0^{M\kappa} = N_0'^{M\kappa}$ and $k_0^{M\kappa}$ the identity on $P_0^{M\kappa}$. Clearly (24)–(26) hold for $\beta = 0$. If $P_\beta^{M\kappa}$ and $k_\beta^{M\kappa}$ have been defined so that (24)–(26) hold, then $i_{*(\beta+1)\beta}^{\star M\kappa} \circ k_\beta^{M\kappa}$ is an isomorphism from $P_\beta^{M\kappa}$ into $N_{\beta+1}'^{M\kappa}$, so there is a $P_{\beta+1}^{M\kappa}$ and $k_{\beta+1}^{M\kappa}$ such that $P_\beta^{M\kappa} \subseteq P_{\beta+1}^{M\kappa}$, $k_{\beta+1}^{M\kappa}$ is an isomorphism from $P_{\beta+1}^{M\kappa}$ onto $N_{\beta+1}'^{M\kappa}$, and $i_{*(\beta+1)\beta}^{\star M\kappa} \circ k_\beta^{M\kappa} \subseteq k_{\beta+1}^{M\kappa}$.

xxx

Clearly then (24) and (25) hold for $\beta+1$. Concerning (26), it clearly holds for $\alpha = \beta < \beta+1$. For $\alpha < \beta$,

$$k^{M\kappa}_\alpha \circ i^{\star M\kappa}_{*(\beta+1)\alpha} = k^{M\kappa}_\alpha \circ i^{\star M\kappa}_{*(\beta+1)\beta} \circ i^{\star M\kappa}_{*\beta\alpha} \subseteq k^{M\kappa}_\beta \circ i^{\star M\kappa}_{*\beta\alpha} \subseteq k^{M\kappa}_{\beta+1}.$$

Now suppose that γ is limit and $P^{M\kappa}_\beta, k^{M\kappa}_\beta$ have been defined for all $\beta < \gamma$ so that (24)–(26) hold. Then we let $P^{M\kappa}_\gamma = \bigcup_{\beta<\gamma} P^{M\kappa}_\beta$ and for any $x \in P^{M\kappa}_\gamma$, we take the least $\beta < \gamma$ such that $x \in P^{M\kappa}_\beta$ and define $k^{M\kappa}_\gamma(x) = i^{\star M\kappa}_{*\gamma\beta}(k^{M\kappa}_\beta(x))$.

(25) for γ is obvious. For (26), suppose that $\alpha < \gamma$ and $x \in P^{M\kappa}_\alpha$. Let β be minimum such that $x \in P^{M\kappa}_\beta$. Then $\beta \le \alpha$, and

$$k^{M\kappa}_\gamma(x) = i^{\star M\kappa}_{*\gamma\beta}(k^{M\kappa}_\beta(x)) = i^{\star M\kappa}_{*\gamma\alpha}(i^{\star M\kappa}_{*\alpha\beta}(k^{M\kappa}_\beta(x))) = i^{\star M\kappa}_{*\gamma\alpha}(k^{M\kappa}_\alpha(x)),$$

as desired.

To check (24) for $k^{M\kappa}_\gamma$, first suppose that $x, y \in P^{M\kappa}_\gamma$ and $x \ne y$. Say $x, y \in P^{M\kappa}_\beta$ with $\beta < \gamma$. Then

$$k^{M\kappa}_\gamma(x) = i^{\star M\kappa}_{*\gamma\alpha}(k^{M\kappa}_\alpha(x)) \ne i^{\star M\kappa}_{*\gamma\alpha}(k^{M\kappa}_\alpha(y)) = k^{M\kappa}_\gamma(y).$$

Thus $k^{M\kappa}_\gamma$ is one-one. To show that $k^{M\kappa}_\gamma$ maps onto $N'^{M\kappa}_\gamma$, suppose that $z \in N'^{M\kappa}_\gamma$. Say $z = [f]^{M\kappa}_\gamma$ with $f \in \mathrm{Fn}_\gamma(M,\kappa)$. Let $F \in [\gamma]^{<\omega}$ be a support for f. Say $F \subseteq \beta < \gamma$. Define $g \in \mathrm{Fn}_\beta(M,\kappa)$ by setting for any $s \in {}^\beta\kappa$, $g(s) = f(t)$ for any $t \in {}^\gamma\kappa$ such $s \subseteq t$. Then with $j : \beta \to \gamma$ the identity we have $\forall s \in {}^\gamma\kappa[f(s) = g(s \circ j)]$. Then

(27) $j^{\star M\kappa}_{*\gamma\beta}([g]_\beta) = [f]^{M\kappa}_\gamma = z$.

In fact, $j^{\star M\kappa}_{*\gamma\beta}([g]^{M\kappa}_\beta) = [j^{M\kappa}_{*\gamma\beta}(g)]_\gamma$, and for any $s \in {}^\gamma\kappa$, $(j^{M\kappa}_{*\gamma\beta}(g))(s) = g(s \circ j) = f(s)$. So (27) holds.

Now $[g]^{M\kappa}_\beta \in N_\beta$, so by (27), $j^{\star M\kappa}_{*\gamma\beta}(k^{M\kappa}_\alpha((k^{M\kappa}_\alpha)^{-1}([g]^{M\kappa}_\beta))) = z$. So k_γ maps onto N_γ.

To check that $k^{M\kappa}_\gamma$ preserves $\in$, suppose that $x, y \in P^{M\kappa}_\gamma$. Say $x, y \in P^{M\kappa}_\alpha$ with $\alpha < \gamma$. Then

$$\begin{aligned} x \in y \quad &\text{iff} \quad k^{M\kappa}_\alpha(x) \in k^{M\kappa}_\alpha(y) \\ &\text{iff} \quad j^{\star\kappa}_{*\gamma\alpha}(k^{M\kappa}_\alpha(x)) E^{M\kappa}_\gamma j^{\star M\kappa}_{*\gamma\alpha}(k^{M\kappa}_\alpha(y)) \\ &\text{iff} \quad k^{M\kappa}_\gamma(x) \in k^{M\kappa}_\gamma(y). \end{aligned}$$

This completes the construction of $P^{M\kappa}_\alpha$ and $k^{M\kappa}_\alpha$ for all ordinals α.

For each ordinal α let

$$\begin{aligned} \mathscr{U}^{(\alpha)M\kappa} = \{&x \in \mathscr{P}^{N'^{M\kappa}_\alpha}(i^{\star M\kappa}_{*\alpha 0}(\kappa)) : \\ &\exists f \in \mathrm{Fn}_\alpha(M,\kappa)[x = \pi_\alpha([f]^{M\kappa}_\alpha) \wedge \{s \in {}^\alpha\kappa : f(s) \in \mathscr{U}^{M\kappa}\} \in \mathscr{U}^{M\kappa}_\alpha]. \end{aligned}$$

Theorem 19.40. *In $N'^{M\kappa}_\alpha$, $\mathscr{U}^{(\alpha)M\kappa}$ is an ultrafilter on $\emptyset^{\star M\kappa}_{*\alpha 0}(\kappa)$.*

Proof. First we prove that $\emptyset^{\star M\kappa}_{*\alpha 0}(\kappa) \in \mathscr{U}^{(\alpha)M\kappa}$. By the argument before Proposition 29, $\kappa = \pi_0^{M\kappa}([c_\kappa^{{}^0\kappa}]_0)$. Now by the remarks after (5), $\emptyset^{M\kappa}_{*\alpha 0}(c_\kappa^{{}^0\kappa}) = c_\kappa^{{}^\alpha\kappa}$. Also, $\{s \in {}^\alpha\kappa : c_\kappa^{{}^\alpha\kappa}(s) = \kappa\} = {}^\alpha\kappa \in \mathscr{U}^{M\kappa}$, so $\pi_\alpha^{M\kappa}([c_\kappa^{{}^\alpha\kappa}]^{M\kappa} \in \mathscr{U}^{(\alpha)M\kappa}$. Hence

$$\emptyset^{\star M\kappa}_{*\alpha 0}(\kappa) = \emptyset^{\star M\kappa}_{*\alpha 0}(\pi_0^{M\kappa}([c_\kappa^{{}^0\kappa}]_0) = \pi_\alpha^{M\kappa}([\emptyset^{M\kappa}_{*\alpha 0}(c_\kappa^{{}^0\kappa})]_\alpha) = \pi_\alpha^{M\kappa}([c_\kappa^{{}^\alpha\kappa}]^{M\kappa}_\alpha) \in \mathscr{U}^{(\alpha)M\kappa}.$$

Now suppose that $x, y \in \mathscr{P}^{N'^{M\kappa}_\alpha}(i^{\star M\kappa}_{*\alpha 0}(\kappa))$, $x \subseteq y$, and $x \in \mathscr{U}^{(\alpha)M\kappa}$. Choose $f \in \mathrm{Fn}_\alpha(M,\kappa)$ such that $x = \pi_\alpha([f]^{M\kappa}_\alpha)$ and $\{s \in {}^\alpha\kappa : f(s) \in \mathscr{U}^{M\kappa}\} \in \mathscr{U}^{M\kappa}_\alpha]$. Choose $g \in \mathrm{Fn}_\alpha(M,\kappa)$ such that $y = \pi_\alpha([g]^{M\kappa}_\alpha)$. Then $x \subseteq y$ means that $N_\alpha \prime M\kappa \models \pi_\alpha([f]^{M\kappa}_\alpha) \subseteq \pi_\alpha([g]^{M\kappa}_\alpha)$, so by Theorem 31, $\{s \in {}^\alpha\kappa : f(s) \subseteq g(s)\} \in \mathscr{U}^{M\kappa}_\alpha$. Hence

$$\{s \in {}^\alpha\kappa : f(s) \subseteq g(s) \wedge f(s) \in \mathscr{U}^{M\kappa}\} \in \mathscr{U}^{M\kappa}_\alpha.$$

Now

$$\{s \in {}^\alpha\kappa : f(s) \subseteq g(s) \wedge f(s) \in \mathscr{U}^{M\kappa}\} \subseteq \{s \in {}^\alpha\kappa : g(s) \in \mathscr{U}^{M\kappa}\},$$

so it follows that $y \in \mathscr{U}^{(\alpha)M\kappa}$.

Now suppose that $x, y \in \mathscr{U}^{(\alpha)M\kappa}$. Choose f, g so that

$$\begin{aligned} &f \in \mathrm{Fn}_\alpha(M,\kappa),\ x = \pi_\alpha^{M\kappa}([f]^{M\kappa}_\alpha),\ \{s \in {}^\alpha\kappa : f(s) \in \mathscr{U}^{M\kappa}\} \in \mathscr{U}^{M\kappa}_\alpha, \\ &g \in \mathrm{Fn}_\alpha(M,\kappa),\ y = \pi_\alpha^{M\kappa}([g]_\alpha),\ \text{and } \{s \in {}^\alpha\kappa : f(s) \in \mathscr{U}^{M\kappa}\} \in \mathscr{U}^{M\kappa}_\alpha. \end{aligned}$$

For each $s \in {}^\alpha\kappa$ let $h(s) = f(s) \cap g(s)$. Then $\{s \in {}^\alpha\kappa : h(s) = f(s) \cap g(s)\} = {}^\alpha\kappa$, so $\pi_\alpha^{M\kappa}([h]^{M\kappa}_\alpha) = \pi_\alpha^{M\kappa}([f]^{M\kappa}_\alpha) \cap \pi_\alpha^{M\kappa}([g]^{M\kappa}_\alpha)$. Also,

$$\{s \in {}^\alpha\kappa : f(s) \in \mathscr{U}^{M\kappa}\} \cap \{s \in {}^\alpha\kappa : g(s) \in \mathscr{U}^{M\kappa} = \{s \in {}^\alpha\kappa : h(s) \in \mathscr{U}^{M\kappa}\}.$$

It follows that $x \cap y \in \mathscr{U}^{(\alpha)M\kappa}$.

For complements, suppose that $x \in \mathscr{P}^{N'_\alpha}(i^{\star\star}_{*\alpha 0}(\kappa))$ but $x \notin \mathscr{U}^{(\alpha)}$. Write $x = \pi_\alpha([f]_\alpha)$. Then $\{s \in {}^\alpha\kappa : f(s) \in \mathscr{U}\} \notin \mathscr{U}_\alpha$. Hence ${}^\alpha\kappa \backslash \{s \in {}^\alpha\kappa : f(s) \in \mathscr{U}\} \in \mathscr{U}_\alpha$. Now

$${}^\alpha\kappa \backslash \{s \in {}^\alpha\kappa : f(s) \in \mathscr{U}\} = \{s \in {}^\alpha\kappa : f(s) \notin \mathscr{U}\} = \{s \in {}^\alpha\kappa : (\kappa \backslash f(s)) \in \mathscr{U}\}.$$

For all $s \in {}^\alpha\kappa$ let $g(s) = \kappa \backslash f(s)$. Then $\pi_\alpha([g]_\alpha) = i^{\star\star}_{*\alpha 0}(\kappa) \backslash \pi_\alpha([f]_\alpha)$. In fact,

$$\begin{aligned} i^{\star\star}_{*\alpha 0} \backslash \pi_\alpha([f]_\alpha) &= \pi_\alpha([\emptyset_{*\alpha 0}(c^\kappa_\kappa)]_\alpha) \backslash \pi_\alpha([f]_\alpha) \\ &= \pi_\alpha([\emptyset_{*\alpha 0}(c^\kappa_\kappa)]_\alpha \backslash [f]_\alpha). \end{aligned}$$

Now for any $s \in {}^\alpha\kappa$, $c^\kappa_\kappa(s) \backslash f(s) = \kappa \backslash f(s) = g(s)$. So $\pi_\alpha([g]_\alpha) = i^{\star\star}_{*\alpha 0}(\kappa) \backslash \pi_\alpha([f]_\alpha)$. Hence $i^{\star\star}_{*\alpha 0}(\kappa) \backslash x \in \mathscr{U}^{(\alpha)}$. $\square$

20. Very large cardinals

Theorem 20.1. *Suppose that $\mathscr{L}_{\kappa\omega}$ is a first-order language, $\overline{A} = \langle \overline{A_i} : i \in I\rangle$ is a system of $\mathscr{L}$-structures, F is an κ-complete ultrafilter on I, and $a \in {}^{\omega}\prod_{i\in I} a_i$. The values of a will be denoted by $a^0, a^1, \ldots$. Let $\pi : \prod_{i\in I} A_i \to \prod_{i\in I} A_i/F$ be the natural mapping, taking each element of $\prod_{i\in I} A_i$ to its equivalence class under $\equiv_F^A$. For each $i \in I$ let $pr_i : \prod_{j\in I} A_j \to A_i$ be defined by setting $pr_i(x) = x_i$ for all $x \in \prod_{i\in I} A_i$. Suppose that φ is any formula of $\mathscr{L}$. Then*

$$\prod_{i\in I} \overline{A_i}/F \models \varphi[\pi \circ a] \text{ iff } \{i \in I : \overline{A_i} \models \varphi[\mathrm{pr}_i \circ a]\} \in F.$$

Proof. Because the situation and the notation are complicated, we are going to give the proof in full. For brevity let $\overline{B} = \prod_{i\in I} \overline{A_i}/F$. First we show

(1) For any term τ, $\tau^{\overline{B}}(\pi \circ a) = [\langle \tau^{\overline{A_i}}(\mathrm{pr}_i \circ a) : i \in I\rangle]_F$.

We prove (1) by induction on τ. For τ a variable v_k,

$$\tau^{\overline{B}}(\pi \circ a) = (\pi \circ a)(k) = [a^k]_F = [\langle a_i^k : i \in I\rangle]_F = [\langle v_k^{\overline{A_i}}(\mathrm{pr}_i \circ a) : i \in I\rangle]_F,$$

as desired. For τ an individual constant $\mathbf{k}$,

$$\mathbf{k}^{\overline{B}}(\pi \circ a) = \mathbf{k}^{\overline{B}} = [\langle \mathbf{k}^{\overline{A_i}} : i \in I\rangle]_F = [\langle \mathbf{k}^{\overline{A_i}}(\mathrm{pr}_i \circ a) : i \in I\rangle]_F.$$

The inductive step:

$$\begin{aligned}
(\mathbf{F}\sigma_0 \ldots \sigma_{m-1})^{\overline{B}}(\pi \circ a) &= \mathbf{F}^{\overline{B}}(\sigma_0^{\overline{B}}(\pi \circ a), \ldots, \sigma_{m-1}^{\overline{B}}(\pi \circ a)) \\
&= \mathbf{F}^{\overline{B}}([\langle \sigma_0^{\overline{A_i}}(\mathrm{pr}_i \circ a) : i \in I\rangle]_F, \ldots, [\langle \sigma_{m-1}^{\overline{A_i}}(\mathrm{pr}_i \circ a) : i \in I\rangle]_F) \\
&= [\langle \mathbf{F}^{\overline{A_i}}(\sigma_0^{\overline{A_i}}(\mathrm{pr}_i \circ a), \ldots, \sigma_{m-1}^{\overline{A_i}}(\mathrm{pr}_i \circ a)) : i \in I\rangle]_F \\
&= [\langle \tau^{\overline{A_i}}(\mathrm{pr}_i \circ a) : i \in I\rangle]_F,
\end{aligned}$$

as desired.

Now we begin the real proof of the theorem, proceding, of course, by induction on φ. Suppose that φ is $\sigma = \tau$. Then

$$\begin{aligned}
\overline{B} \models (\sigma = \tau)[\pi \circ a] &\text{ iff } \sigma^{\overline{B}}(\pi \circ a) = \tau^{\overline{B}}(\pi \circ a) \\
&\text{ iff } [\langle \sigma^{\overline{A_i}}(\mathrm{pr}_i \circ a) : i \in I\rangle]_F = [\langle \tau^{\overline{A_i}}(\mathrm{pr}_i \circ a) : i \in I\rangle]_F \\
&\text{ iff } \{i \in I : \sigma^{\overline{A_i}}(\mathrm{pr}_i \circ a) = \tau^{\overline{A_i}}(\mathrm{pr}_i \circ a)\} \in F \\
&\text{ iff } \{i \in I : \overline{A_i} \models (\sigma = \tau)[\mathrm{pr}_i \circ a]\} \in F,
\end{aligned}$$

as desired.

Now suppose that φ is $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$. Then

$$\begin{aligned}
\overline{B} \models \varphi[\pi \circ a] &\text{ iff } (\sigma_0^{\overline{B}}(\pi \circ a), \ldots, \sigma_{m-1}^{\overline{B}}(\pi \circ a)) \in \mathbf{R}^{\overline{B}} \\
&\text{ iff } ([\langle \sigma_0^{\overline{A_i}}(\mathrm{pr}_i \circ a) : i \in I\rangle]_F, \ldots, [\langle \sigma_{m-1}^{\overline{A_i}}(\mathrm{pr}_i \circ a) : i \in I\rangle]_F) \in \mathbf{R}^{\overline{B}} \\
&\text{ iff } \{i \in I : (\sigma_0^{\overline{A_i}}(\mathrm{pr}_i \circ a), \ldots, \sigma_{m-1}^{\overline{A_i}}(\mathrm{pr}_i \circ a)) \in \mathbf{R}^{\overline{A_i}}\} \in F \\
&\text{ iff } \{i \in I : \overline{A_i} \models \varphi[\mathrm{pr}_i \circ a]\} \in F,
\end{aligned}$$

as desired.

The inductive step when φ is $\neg\psi$:

$$\begin{aligned}
\overline{B} \models \varphi[\pi \circ a] &\text{ iff not}(\overline{B} \models \varphi[\pi \circ a]) \\
&\text{ iff not}(\{i \in I : \overline{A_i} \models \psi[\mathrm{pr}_i \circ a]\} \in F) \\
&\text{ iff } I \backslash \{i \in I : \overline{A_i} \models \psi[\mathrm{pr}_i \circ a]\} \in F \\
&\text{ iff } \{i \in I : \text{not}(\overline{A_i} \models \psi[\mathrm{pr}_i \circ a])\} \in F \\
&\text{ iff } \{i \in I : \overline{A_i} \models \varphi[\mathrm{pr}_i \circ a]\} \in F,
\end{aligned}$$

as desired.

The induction step for $\bigwedge$: Suppose that $\alpha < \kappa$.

$$\begin{aligned}
\overline{B} \models \bigwedge_{\xi<\alpha} \psi_\xi[\pi \circ a] \quad &\text{iff} \quad \forall \xi < \alpha[\overline{B} \models \psi_\xi[\pi \circ a]] \\
&\text{iff} \quad \forall \xi < \alpha[\{i \in I : \overline{A_i} \models \psi_\xi[\mathrm{pr}_i \circ a]\} \in F] \\
&\text{iff} \quad \bigcap_{\xi<\alpha} \{i \in I : \overline{A_i} \models \psi_\xi[\mathrm{pr} \circ a]\} \in F \\
&\text{iff} \quad \left[\left\{ i \in I : \overline{A_i} \models \bigwedge_{\xi<\alpha} \psi_\xi[\mathrm{pr}_i \circ a] \right\} \in F \right],
\end{aligned}$$

as desired.

It remains only to consider φ of the form $\exists v_k \psi$, which is the main case. We do each direction in the desired equivalence separately. First suppose that $\overline{B} \models \varphi[\pi \circ a]$. Choose $u \in \prod_{i \in I} A_i$ such that $\overline{B} \models \psi[(\pi \circ a)^k_{[u]_F}]$. Now $(\pi \circ a)^k_{[u]_F} = \pi \circ a^k_u$, so we can apply the induction hypothesis and get $\{i \in I : \overline{A_i} \models \psi[\mathrm{pr}_i \circ a^k_u]\} \in F$. But for each $i \in I$ we have $(\mathrm{pr}_i \circ a^k_u) = (\mathrm{pr}_i \circ a)^k_{u(i)}$, so

$$\{i \in I : \overline{A_i} \models \psi[\mathrm{pr}_i \circ a^k_u]\} \subseteq \{i \in I : \overline{A_i} \models \varphi[\mathrm{pr}_i \circ a]\},$$

and hence $\{i \in I : \overline{A_i} \models \varphi[\mathrm{pr}_i \circ a]\} \in F$. This finishes half of what we want.

Conversely, suppose that $\{i \in I : \overline{A_i} \models \varphi[\mathrm{pr}_i \circ a]\} \in F$. For each i in this set pick $u(i) \in A_i$ such that $\overline{A_i} \models \psi[(\mathrm{pr}_i \circ a)^k_{u(i)}]$ (using the axiom of choice). For other i's in I let $u(i)$ be any old element of A_i, just to fill out u to make it a member of $\prod_{i \in I} A_i$. Now

$(\mathrm{pr}_i \circ a)^k_{u(i)} = \mathrm{pr}_i \circ a^k_u$. Thus $\{i \in I : \overline{A_1} \models \psi[\mathrm{pr}_i \circ a^k_u]\} \in F$. By the inductive hypothesis it follows that $\overline{B} \models \psi[\pi \circ a^k_u]$. But $\pi \circ a^k_u = (\pi \circ a)^k_{[u]_F}$, so we finally get $\overline{B} \models \varphi[\pi \circ a]$. $\square$

Theorem 20.2. *For any regular cardinal κ the following are equivalent:*

(i) For any set S, every κ-complete filter on S can be extended to a κ-complete ultrafilter on S.

(ii) For any A such that $|A| \geq \kappa$, there is a fine ultrafilter on $[A]^{<\kappa}$.

(iii) For any set Σ of sentences of $L_{\kappa\omega}$, if every subset of Σ of size less than κ has a model, then Σ has a model.

Proof. (i)⇒(ii): obvious. (ii)⇒(iii): Assume (ii), and suppose that Σ is a set of sentences of $L_{\kappa\omega}$ such that every subset of Σ of size less than κ has a model. Let U be a fine ultrafilter on $[\Sigma]^{<\kappa}$. For each $S \in [\Sigma]^{,\kappa}$ let $\overline{A}_S$ be a model of S. Then for any sentence σ, by Theorem 20.1,

$$\prod_{S\in[\Sigma]^{<\kappa}} \overline{A}_S \models \sigma \quad \text{iff} \quad \{S \in [\Sigma]^{<\kappa} : \overline{A}_S \models \sigma\} \in U.$$

Since $\forall S \in [\Sigma]^{<\kappa}[\overline{A}_S \models \sigma]$, it follows that $\prod_{S\in[\Sigma]^{<\kappa}} \overline{A}_S \models \sigma$. Thus we have a model of Σ.

(iii)⇒(i): assume (iii), and let F be a κ-complete filter on a set S. Let $\mathscr{L}$ be the first-order language which has a unary relation symbol $\mathbf{R}_X$ for each $X \subseteq S$ and a constant symbol c. Let

$$\begin{aligned}\Sigma = \{&\sigma : \sigma \text{ is a sentence of } L_{\kappa\omega} \text{ which is true in } (S, X)_{X\subseteq S}\} \\ &\cup \{\mathbf{R}_X c : X \in F\} \cup \{\neg \mathbf{R}_\emptyset c\}.\end{aligned}$$

Let Σ' be a subset of Σ of size less than κ. Since F is κ-complete, let c be a member of X for each $\mathbf{R}_X$ occurring in Σ'. Then $(S, X, c)_{X\subseteq S}$ is a model of Σ'.

So, let $(A, \mathbf{R}^A_X, d)_{X\subseteq S}$ be a model of Σ. We define $U \subseteq \mathscr{P}(S)$ by

$$X \in U \quad \text{iff} \quad X \subset S \text{ and } d \in \mathbf{R}^A_X.$$

Then $S \in U$ since $S \in F$ and so $\mathbf{R}_S c \in \Sigma$. Also, $\emptyset \notin U$ since $\neg\mathbf{R}_\emptyset c \in \Sigma$. If $X \in U$ and $X \subseteq Y \subseteq S$, then $d \in \mathbf{R}^A_X$ and $\forall x[\mathbf{R}_X x \to \mathbf{R}_Y x] \in \Sigma$, so $d \in \mathbf{R}^A_Y$ and hence $Y \in U$.

Now suppose that $\alpha < \kappa$ and $X_\xi \in U$ for all $\xi < \alpha$. Now

$$\forall x \left[\bigwedge_{\xi<\alpha} \mathbf{R}_{X_\xi} x \to \mathbf{R}_{\bigcap_{\xi<\alpha} X_\xi} x\right]$$

is true in $(S, X)_{X\subseteq S}$ and hence is in Σ, and $\forall \xi < \alpha[d \in \mathbf{R}^A_{X_\xi}]$, so $d \in \mathbf{R}_{\bigcap_{\xi<\alpha} X_\xi}$, and hence $\bigcap_{\xi<\alpha} X_\xi \in U$. $\square$

A cardinal satisfying one of the conditions in Theorem 20.3 is *strongly compact.*

Corollary 20.3. *Every strongly compact cardinal is measurable.* □

Theorem 20.4. *If there is a strongly compact cardinal, then there is no set A such that $V = L[A]$.*

Proof. Assume that there is a strongly compact cardinal, and there is a set A such that $M = L[A]$. By Theorem 13.65 we may assume that A is a set of ordinals. Let κ be a strongly compact cardinal, and choose $\lambda \geq \kappa$ such that $A \subseteq \lambda$.

(1) $F \overset{\text{def}}{=} \{X \subseteq \lambda^+ : |\lambda^+ \backslash X| \leq \lambda\}$ is a κ-complete filter on λ^+.

First, F is a filter. For, suppose that $X \in F$ and $X \subseteq Y \subseteq \lambda^+$. Then $\lambda^+ \backslash Y \subseteq \lambda^+ \subseteq X$, so $|\lambda^+ \backslash Y| \leq \lambda$. Now suppose that $\eta < \kappa$ and $X_\xi \in F$ for all $\xi < \eta$. Thus $\forall \xi < \eta[|\lambda^+ \backslash X_\xi| \leq \lambda$. Hence

$$\begin{aligned} \left|\lambda^+ \backslash \bigcap_{\xi<\eta} X_\xi\right| &= \left|\bigcup_{\xi<\eta} (\lambda^+ \backslash X_\xi)\right| \\ &\leq \sum_{\xi<\eta} |\lambda^+ \backslash X_\xi| \\ &\leq \sum_{\xi<\eta} \sum_{\xi<\eta} \lambda = \lambda, \end{aligned}$$

as desired for (1).

Now let U be a κ-complete ultrafilter on λ^+ which extends F from (1).

(2) $\forall X \in U[|X| = \lambda^+]$.

For, suppose that $X \in U$ and $|X| \leq \lambda$. Let $Y = \lambda^+ \backslash X$. Thus $|\lambda^+ \backslash Y| = |X| \leq \lambda$, so $Y \in F \subseteq U$. But $X \cap Y = \emptyset$, contradiction.

Now we modify the basic construction of Ult, as follows. Let $\mathrm{Fcn}'(\lambda^+)$ consist of all functions with domain λ^+ which take on at most λ values. Let

$$\begin{aligned} f ='^* g \quad &\text{iff} \quad f, g \in \mathrm{Fcn}'(\lambda^+) \text{ and } \{x \in \lambda^+ : f(x) = g(x)\} \in U; \\ f \in'^* g \quad &\text{iff} \quad f, g \in \mathrm{Fcn}'(\lambda^+) \text{ and } \{x \in \lambda^+ : f(x) \in g(x)\} \in U. \end{aligned}$$

Clearly $=''^*$ is an equivalence relation on $\mathrm{Fcn}'(\lambda^+)$. We denote by $[f]'$ the Scott equivalence class of f:

$$[f]' = \{g : f =''^* g \text{ and } \forall h(h =''^* f \to \mathrm{rank}(g) \leq \mathrm{rank}(h))\}.$$

Then we define Ult' to be the collection of all equivalence classes, with $\in_{\mathrm{Ult}'} = \{([f]', [g]') : f \in'^* g\}$. We can write this as $\in_{\mathrm{Ult}'} = \{(x, y) : \exists f, g[x = [f]',\ y = [g]', \text{ and } f \in'^* g\}$.

(3) For $f_0, \ldots, f_{m-1} \in \mathrm{Fcn}'(\lambda^+)$ we have

$$\mathrm{Ult}' \models \varphi([f_0]', \ldots, [f_{m-1}]') \quad \text{iff} \quad \{\alpha < \lambda^+ : \varphi(f_0(\alpha), \ldots, f_{m-1}(\alpha))\} \in U.$$

To prove this we do the step involving $\exists$. Suppose that $\mathrm{Ult}' \models \exists u\varphi(u, [f_1]', \ldots, [f_n]')$. Choose $g \in \mathrm{Fcn}'(\lambda^+)$ so that $\mathrm{Ult} \models \varphi([g]', [f_1]', \ldots, [f_n]')$. Then by the inductive hypothesis, $\{\alpha < \lambda^+ : \varphi(g(\alpha), f_1(\alpha), \ldots, f_n(\alpha))\} \in U$. Now

$$\{\alpha < \lambda^+ : \varphi(g(\alpha), f_1(\alpha), \ldots, f_n(\alpha))\} \subseteq \{\alpha < \lambda^+ : \exists u\varphi(u, f_1(\alpha), \ldots, f_n(\alpha))\},$$

so $\{\alpha < \lambda^+ : \exists u\varphi(u, f_1(\alpha), \ldots, f_n(\alpha))\} \in U$.

Conversely, suppose that $K \stackrel{\text{def}}{=} \{\alpha < \lambda^+ : \exists u\varphi(u, f_1(\alpha), \ldots, f_n(\alpha))\} \in U$. Now each f_i has range of size at most λ, so there are at most λ formulas $\varphi(u, f_1(\alpha), \ldots, f_n(\alpha))$, Hence for each $\alpha \in K$ choose $g(\alpha)$ so that $\varphi(g(\alpha), f_1(\alpha), \ldots, f_n(\alpha))$, with g having range of size at most λ, and with $g(\alpha)$ arbitrary for other α. Then by the inductive hypothesis, $\mathrm{Ult}' \models \varphi([g]', [f_1]', \ldots, [f_n]')$, so $\mathrm{Ult}' \models \exists u\varphi(u, [f_1]', \ldots, [f_n]')$. This proves (3).

Now Ult' is clearly well-founded. Let N be its transitive collapse via the function π, and i the embedding of M into N given by $i(x) = \pi([c_x]')$ for all $x \in M$. As in the usual case, i is an elementary embedding of M into N.

For each $f \in \mathrm{Fcn}'(\lambda^+)$ let $k(\pi([f]')) = \pi'([f])$, where $\pi' : \mathrm{Ult} \to M$. Clearly k is well-defined.

(4) k is an elementary embedding of N into M.

We prove this by induction on φ.

(a) φ is $v_i = v_j$. Then $[f]' = [g]'$ iff $\{\alpha < \lambda^+ : f(\alpha) = (\alpha)\} \in U$ iff $[f] = [g]$.

(b) $\in, \neg, \wedge$ are similar.

(c) Suppose that $N \models \exists v_m\varphi([f_0]', \ldots, [f_{m-1}]', v_m)$. Choose $g \in \mathrm{Fcn}'(\lambda^+)$ such that $N \models \varphi([f_0]', \ldots, [f_{m-1}]', [g]')$. By the inductive assumption, $M \models \varphi([f_0], \ldots, [f_{m-1}], [g])$. Hence $M \models \exists v_m\varphi([f_0], \ldots, [f_{m-1}], v_m)$.

Conversely, suppose that $M \models \exists v_m\varphi([f_0], \ldots, [f_{m-1}], v_m)$. As in the proof of (3) we get the desired result. So (4) holds.

(5) With $j : M \to M$ the usual mapping, we have $k \circ i = j$.

For, $k(i(x)) = k(\pi([c_x]')) = \pi'([c_x]) = j(x)$.

(6) $i(\lambda^+) = \bigcup_{\alpha<\lambda^+} i(\alpha)$.

For, if $\alpha < \lambda^+$ then $i(\alpha) < i(\lambda^+)$. Suppose that $\mu < i(\lambda^+)$. Say $\mu = i([f]')$. Then $[f]' \in^* [c_{\lambda^+}]'$, so $\{\alpha < \lambda^+ : f(\alpha) < \lambda^+\} \in U$. Since $f \in \mathrm{Fcn}'(\lambda^+)$, there is a $\gamma < \lambda^+$ such that $\{\alpha < \lambda^+ : f(\alpha) < \gamma\} \in U$. Then $\mu < i(\gamma)$, proving (6).

(7) If $\xi < i(\lambda^+)$, then $k(\xi) = \xi$.

For, by the above, $\xi = i([f]')$ with $f : \lambda^+ \to \gamma$ for some $\gamma < \lambda^+$. Hence $[f]' = [f]$ and so $k(\xi) = \xi$.

(8) $i(A) = j(A)$.

For, $i(A) = \pi([c_A]') = \pi([c_A]) = j(A)$ since $c_A : l^+ \to A \subseteq \lambda$.

(9) If f is an elementary embedding of $L[A]$ onto a transitive class M, then $M = L[f[A]]$.

In fact, we prove by induction that $f[L_\alpha[A]] = L_\alpha[f[A]]$. The essential step is from α to $\alpha+1$. Suppose that $X \in L_{\alpha+1}[A]$. Say

$$X = \{b \in L_\alpha[A] : (L_\alpha[A], \in, A \cap L_\alpha[A]) \models \varphi(b, \overline{a}),$$

where $\overline{a} \subseteq L_\alpha[A]$. Then

$$f(b) \in f(X) \quad \text{iff} \quad L_\alpha[f[A]], \in, f[A] \cap L_\alpha[f[A]]) \models \varphi(f(b), f \circ \overline{a}),$$

so $f(X) \in L_{\alpha+1}[f[A]]$. This proves (9).

By (9) and (8) we have $M = L[j[A]] = L[i[A]] = N$.

Now if d is the identity on λ^+, then $\forall \gamma < \lambda^+[j([c_\gamma]) < j[d] < j(\lambda^+)$. But $i(\lambda^+) = \bigcup_{\gamma<\lambda^+} i(\gamma)$. So $i(\lambda^+) < j(\lambda^+)$. If $\xi < i(\lambda^+)$, then by (7), $j(\xi) = k(i(\xi)) = i(\xi)$. Now by elementarity, $N \models [i(\lambda^+)$ is the cardinal successor of $i(\lambda)]$, and $M \models [j(\lambda^+)$ is the cardinal successor of $j(\lambda)$. By (7), $i(\lambda) = j(\lambda)$. So this contradicts $M = N$. $\square$

Lemma 20.5. *Let κ be an inaccessible cardinal. Then there exists $\mathscr{G} \subseteq {}^\kappa\kappa$ such that $|\mathscr{G}| = 2^\kappa$ and $\forall H \in [\mathscr{G}]^{<\kappa} \forall \beta \in {}^H\kappa \exists \alpha < \kappa \forall g \in H[g(\alpha) = \beta_g]$.*

Proof.

(1) There is a family $\mathscr{A} \subseteq [\kappa]^\kappa$ such that $|\mathscr{A}| = 2^\kappa$ and $\forall A, B \in \mathscr{A}[A \neq B \to |A \cap B| < \kappa]$.

To prove this, for each $f \in {}^\kappa 2$ let $A_f = \{f \restriction \alpha : \alpha < \kappa\}$. Now $|\{{}^\alpha\kappa : \alpha < \kappa\}| = \kappa$, and $|A_f \cap A_g| < \kappa$ for all distinct $f, g \in {}^\kappa 2$. This proves (1).

Now for each $A \in \mathscr{A}$ write $A = \bigcup_{\alpha<\kappa} B_\alpha$ with each B_α of size κ, and $B_\alpha \cap B_\beta = \emptyset$ for $\alpha \neq \beta$. Define $f_A : A \to \kappa$ by setting $f_A(a) =$the $\beta < \kappa$ such that $a \in B_\beta$. Then for each $\beta < \kappa$, $|\{a \in A : f_A(a) = \beta\}| = \kappa$.

Let $\langle s_\alpha : \alpha < \kappa \rangle$ enumerate all subsets of κ of size less than κ.

For each $A \in \mathscr{A}$ define $g_A : \kappa \to \kappa$ by

$$g_A(\alpha) = \begin{cases} f_A(x) & \text{if } s_\alpha \cap A = \{x\}, \\ \emptyset & \text{otherwise.} \end{cases}$$

Let $\mathscr{G} = \{g_A : A \in \mathscr{A}\}$.

(2) If $A, B \in \mathscr{A}$ and $A \neq B$, then there is an $\alpha < \kappa$ such that $g_A(\alpha) \neq \emptyset$ and $g_B(\alpha) = \emptyset$.

In fact, choose $x \in A$ and let $s_\alpha = \{x\} \cup (B \backslash A)$.

From (2), we get $g_A \neq g_B$. So $|\mathscr{G}| = 2^\kappa$.

(3) If $\mathscr{H} \subseteq \mathscr{A}$, $|\mathscr{H}| < \kappa$, and $\{\beta_A : A \in \mathscr{H}\} \subseteq \kappa$, and $A \in \mathscr{H}$, then there is an $x_A \in A$ such that $\forall B \in \mathscr{H} \backslash \{A\}[x_A \notin B]$ and $f_A(x_A) = \beta_A$.

In fact, let $A' = \{a \in A : f_A(a) = \beta_A\}$. Then $|A'| = \kappa$. Now

$$A' \backslash \bigcup \{B : B \in \mathscr{H} \backslash \{A\}\} = A' \backslash \bigcup \{B \cap A : B \in \mathscr{H} \backslash \{A\}\},$$

so $|A' \backslash \bigcup \{B : B \in \mathscr{H} \backslash \{A\}\}| = \kappa$. Choose $x_A \in A' \backslash \bigcup \{B : B \in \mathscr{H} \backslash \{A\}\}$. Then (3) holds.

Now choose α so that $s_\alpha = \{x_A : A \in \mathscr{A}\}$. Note that $s_\alpha \cap A = \{x_A\}$ for all $A \in \mathscr{A}$. Hence for any $A \in \mathscr{A}$, $g_A(\alpha) = f_A(x_A) = \beta_A$. $\square$

Lemma 20.6. *Let κ be a strongly compact cardinal. Then for every $\delta < (2^\kappa)^+$ there is a κ-complete ultrafilter U on κ such that $\pi(j_U(\kappa)) > \delta$.*

Proof. Assume that κ is a strongly compact cardinal and $\delta < (2^\kappa)^+$. By Lemma 20.5 let $\mathscr{G} \subseteq {}^\kappa\kappa$ be such that $|\mathscr{G}| = |\delta|$ and $\forall H \in [\mathscr{G}]^{<\kappa}\forall\beta \in {}^H\kappa\exists\alpha < \kappa\forall g \in H[g(\alpha) = \beta_g]$. Say $\mathscr{G} = \{g_\alpha : \alpha \leq \delta\}$. For all $\alpha < \beta \leq \delta$ let $X_{\alpha\beta} = \{\xi < \kappa : g_\alpha(\xi) < g_\beta(\xi)\}$.

(1) Any collection of less than κ of the $X_{\alpha\beta}$'s has a nonempty intersection.

For, suppose that $\mathscr{A}$ is such a collection. Let $H = \{g_\alpha : \exists X_{\alpha\beta} \in \mathscr{A}$ or $\exists X_{\beta\alpha} \in \mathscr{A}\}$ and $H' = \{\alpha \leq \delta : g_\alpha \in H\}$. Thus $H \subseteq \mathscr{G}$ and $|H| < \kappa$. Let $\gamma : H' \to \kappa$ be strictly increasing, and define $\beta_{g_\alpha} = \gamma_\alpha$ for all $\alpha \in H'$. Choose $\xi < \kappa$ such that $g_\alpha(\xi) = \beta_\alpha$ for all $g_\alpha \in \mathscr{H}$. Then $\xi \in \bigcap\mathscr{A}$. In fact, if $X_{\alpha\beta} \in \mathscr{A}$ then $g_\alpha, g_\beta \in H$, and

$$g_\alpha(\xi) = \beta_{g_\alpha} = \gamma_\alpha < \gamma_\beta = \beta_{g_\beta} = g_\beta(\xi).$$

Thus (1) holds.

Let $\mathscr{X} = \{X_{\alpha\beta} : \alpha < \beta \leq \delta\}$. Then by (1),

$$F \stackrel{\text{def}}{=} \{Y \subseteq \kappa : \bigcap Z \subseteq Y \text{ for some } Z \in [\mathscr{X}]^{<\kappa}\}$$

is a κ-complete filter on κ. So there is a κ-complete ultrafilter $U \supseteq F$. If $\alpha < \beta \leq \delta$, then $[g_\alpha]_U < [g_\beta]_U < [d]_U$, and so $\delta < \pi(j_U(\kappa))$. $\square$

An ultrafilter U on a cardinal λ is *uniform* iff every member of U has size λ.

Lemma 20.7. *If κ is strongly compact and $\lambda > \kappa$ is a regular cardinal, then there exists a κ-complete uniform ultrafilter D on λ such that $\{\alpha < \lambda : \mathrm{cf}(\alpha) < \kappa\} \in D$.*

Proof. By Theorem 20.2 let U be a fine ultrafilter on $[\lambda]^{<\kappa}$.

(1) $\forall\alpha < \lambda[\{x \in [\lambda]^{<\kappa} : \alpha \in x\} \in U]$.

In fact, by the definition of fine ultrafilter on page 136, $\{x \in [\lambda]^{<\kappa} : \{\alpha\} \subseteq x\} \in U$.

Now let $\pi([f]_U) = \bigcup_{\gamma<\lambda} \pi(j_U(\gamma))$.

(2) $\{x \in [\lambda]^{<\kappa} : f(x) < \lambda\} \in U$.

In fact, let $g : [\lambda]^{<\kappa} \to \lambda$ be defined by $g(x) = \sup(x)$. If $\gamma < \lambda$, then $\{x \in [\lambda]^{<\kappa} : \gamma \leq g(x)\} \supseteq \{x \in [\lambda]^{<\kappa} : \gamma + 1 \in x\} \in U$. Hence $\pi(j_U(\gamma)) \leq \pi([g])$. Hence $\pi([f]) \leq \pi([g]) < \pi(j_U(\lambda))$, and (2) follows.

Let $X \in D$ iff $X \subseteq \lambda$ and $f^{-1}[X] \in U$. D is a filter: Suppose that $X \subseteq Y \subseteq \lambda$ and $X \in D$. Then $f^{-1}[X] \subseteq f^{-1}[Y]$, so $f^{-1}[Y] \in U$, hence $Y \in D$. D is a κ-complete filter: suppose that $\gamma < \kappa$ and $\forall\alpha < \gamma[X_\alpha \in D]$. Then $f^{-1}[\bigcap_{\alpha<\gamma} X_\alpha] = \bigcap_{\alpha<\gamma} f^{-1}[X_\alpha] \in U$, so $\bigcap_{\alpha<\gamma} X_\alpha \in D$. D is nonprincipal: suppose not; say $\alpha < \lambda$ and $\{\alpha\} \in D$. Thus $\{x \in [\lambda]^{<\kappa} : f(x) = \alpha\} = f^{-1}[\{\alpha\}] \in U$. So $[c_\alpha] =^* [f]$, contradicting $j_U(\alpha) < \pi([f])$.

If $d(\gamma) = \gamma$ for all $\gamma < \lambda$, then for each $\gamma < \lambda$, $j_D(\gamma) < \pi([d])$. For, let $X = \{\alpha < \lambda : \gamma < \alpha\}$. Then $f^{-1}[X] = \{x \in [\lambda]^{<\kappa} : \gamma < f(x)\} \in U$ since $j_U(\gamma) < \pi([f])$. Thus $X \in D$, and $X = \{\alpha < \lambda : c_\gamma(\alpha) < d(\alpha)\}$. Thus $[c_\gamma] \in^* [d]$, so $\pi(j_D(\gamma)) < \pi([d])$.

Now D is uniform. For, suppose that $X \in [\lambda]^{<\lambda}$ and $X \in D$. So $f^{-1}[X] \in U$. Choose $\gamma < \lambda$ such that $X \subseteq \gamma$. Then $f^{-1}[X] \subseteq f^{-1}[\gamma]$, so $f^{-1}[\gamma] \in U$. Thus $Y \stackrel{\text{def}}{=} \{x \in [\lambda]^{<\kappa} : f(x) < \gamma\} \in U$. Now $j_U(\gamma) < \pi([f])$, so $[c_\gamma] \in^* [f]$, hence $Z \stackrel{\text{def}}{=} \{x \in [\lambda]^{<\kappa} : \gamma < f(x)\} \in U$. Now $Y \cap Z \in U$, but $Y \cap Z = \emptyset$, contradiction.

Now let $X = \{\alpha < \lambda : \mathrm{cf}(\alpha) < \kappa\}$. We want to show that $X \in D$. Hence it suffices to show that $f^{-1}[X] = \{x \in [\lambda]^{<\kappa} : \mathrm{cf}(f(x)) < \kappa\} \in U$.

(3) $\{x \in [\lambda]^{<\kappa} : f(x) = \sup\{\alpha \in x : \alpha < f(x)\}\} \subseteq f^{-1}[X]$.

In fact, suppose that $f(x) = \sup\{\alpha \in x : \alpha < f(x)\}$. Since $|x| < \kappa$, it is clear that $\mathrm{cf}(f(x)) < \kappa$. So (3) holds.

Now clearly $\forall x \in [\lambda]^{<\kappa}[\sup\{\alpha \in x : \alpha < f(x)\} \subseteq f(x)]$. Now define h by setting, for each $x \in [\lambda]^{<\kappa}$, $h(x) = \sup\{\alpha \in x : \alpha < f(x)\}$.

(4) $\forall \gamma < \lambda[\pi(j_U(\gamma)) \le \pi([h])]$.

In fact, since $\pi(j_U(\gamma)) < \pi([f])$, by (1) we have $\{x \in [\lambda]^{<\kappa} : \gamma \in x \text{ and } \gamma < f(x)\} \in U$ Now $\{x \in [\lambda]^{<\kappa} : \gamma \in x \text{ and } \gamma < f(x)\} \subseteq \{x \in [\lambda]^{<\kappa} : \gamma \le h(x)\}$. Hence (4) holds.

By (4) we have $\pi([f]) \le \pi([h])$. Clearly also $\pi([h]) \le \pi([f])$, so $\{x \in \lambda]^{<\kappa} : f(x) = \sup\{\alpha \in x : \alpha < f(x)\} \in U$. Hence by (3), $f^{-1}[X] \in U$. $\square$

Lemma 20.8. *If κ is strongly compact and $\lambda > \kappa$ is regular, then there is a κ-complete nonprincipal ultrafilter D on λ and a collection $\{M_\alpha : \alpha < \lambda\}$ of subsets of λ such that*

(i) $\forall \alpha < \lambda[|M_\alpha| < \kappa]$.

(ii) $\forall \alpha < \lambda \forall \gamma < \lambda[\{\alpha \in \lambda : \gamma \in M_\alpha\} \in D]$.

Proof. Let D be the ultrafilter on λ constructed in the proof of Lemma 20.7.

(1) $\pi([d]) = \bigcup_{\gamma<\lambda} j_D(\gamma)$

In fact, by the proof of Lemma 20.7, $\pi(j_D(\gamma)) <^* \pi([d])$ for all $\gamma < \lambda$. Let $\pi([f]_U) = \bigcup_{\gamma<\lambda} \pi(j_U(\gamma))$. Let $\pi([g]) = \bigcup_{\gamma<\lambda} \pi(j_D(\gamma))$, and suppose that $\pi([g]) < \pi([d])$. Thus $X \stackrel{\text{def}}{=} \{\alpha < \lambda : g(\alpha) < \alpha\} \in D$. so $f^{-1}[X] = \{x \in [\lambda]^{<\kappa} : g(f(x)) < f(x)\} \in U$. Thus $\pi([g \circ f]) < \pi([f])$, so there is a $\gamma < \lambda$ such that $\pi([g \circ f]) < \pi([c_\gamma])$. Hence $Y \stackrel{\text{def}}{=} \{x \in [\lambda]^{<\kappa} : g(f(x)) < \gamma\} \in U$. Let $Z = \{\alpha < \gamma : g(\alpha) < \gamma\}$. Then $f^{-1}[Z] = Y \in U$, so $Z \in D$ and hence $\pi([g]) < \pi([\gamma])$, contradiction. It follows that $\pi([d]) = \bigcup_{\gamma<\lambda} j_D(\gamma)$.

Now by Lemma 20.7, $Y \stackrel{\text{def}}{=} \{\alpha < \lambda : \mathrm{cf}(\alpha) < \kappa\} \in D$. For each $\alpha \in Y$ let $A_\alpha \subseteq \alpha$ have size less than κ and be cofinal in α. For $\alpha \in \lambda \backslash Y$ let $A_\alpha = \emptyset$.

(2) $\pi([A])$ is cofinal in $\pi([d])$.

In fact, suppose that $\gamma < \pi([d])$. By the above, $j_D(\gamma) < \pi([d])$, so $K \stackrel{\text{def}}{=} \{\alpha < \lambda : \gamma < \alpha\} \in D$. Also, $S \stackrel{\text{def}}{=} \{\alpha < \lambda : A_\alpha \text{ is cofinal in } \alpha\} \in D$. For each $\alpha \in K \cap S$ choose $\delta_\alpha \in A_\alpha$ such that $\gamma < \delta_\alpha$. Then $\{\alpha < \lambda : \delta_\alpha \in A_\alpha\} \in D$, so $\pi([\delta]) \in \pi([A])$. Also, $\gamma \leq j_D(\gamma) < \pi([\delta])$. Since $\forall \alpha \in K \cap S[\delta_\alpha < \alpha]$, we have $\pi([\delta]) < \pi([d])$. So $\pi([A])$ is cofinal in $\pi([d])$.

(3) $\forall \eta < \lambda \exists \eta' \in (\eta, \lambda)[\pi([A]) \cap \{\xi : \pi(j_D(\eta)) \leq \xi < \pi(j_D(\eta')) \neq \emptyset]$.

This is clear by (1) and (2).

Now we define $\langle \eta_\gamma : \gamma < \lambda \rangle$. Let $\eta_0 = 0$ and for γ limit let $\eta_\gamma = \bigcup_{\delta<\gamma} \eta_\delta$. Having defined η_γ, let $\eta_{\gamma+1} \in (\eta_\gamma, \lambda)$ be such that $\pi([A]) \cap \{\xi : \pi(j_D(\eta_\gamma)) \leq \xi < \pi(j_D(\eta_{\gamma+1})) \neq \emptyset]$. For all $\gamma < \lambda$ let $I_\gamma = [\pi(j_D(\eta_\gamma)), \pi(j_D(\eta_{\gamma+1})))$. For each $\alpha < \lambda$ let $M_\alpha = \{\gamma < \lambda : I_\gamma \cap A_\alpha \neq \emptyset\}$. Then $\forall \gamma < \lambda[\{\alpha < \lambda : \gamma \in M_\alpha\} \in D]$. In fact, suppose that $\gamma < \lambda$. Then choose $j_D(\xi) \in I_\gamma \cap A$. So $j_D(\eta_\gamma) \leq j_D(\xi) < j_D(\eta_{\gamma+1})$. Now $j_D(\xi) \in j_D(A)$, so $\{\alpha < \lambda : \xi \in A_\alpha\} \in D$. Hence $\{\alpha < \lambda : \gamma \in M_\alpha\} \in D$. Also, each A_α intersects fewer than κ I_γ's, since $|A_\alpha| < \kappa$. $\square$

Lemma 20.9. *If κ is strongly compact and $\lambda > \kappa$ is regular, then there exists a collection $\{M_\alpha : \alpha < \lambda\} \subseteq [\lambda]^{<\kappa}$ such that*

$$[\lambda]^{<\kappa} = \bigcup_{\alpha<\lambda} \mathscr{P}(M_\alpha).$$

Hence $\lambda^{<\kappa} = \lambda$.

Proof. Let $\{M_\alpha : \alpha < \lambda\}$ be as in Lemma 20.8. If $x \in [\lambda]^{<\kappa}$ then $\{\alpha < \lambda : x \subseteq M_\alpha\} \in D$; so there is an $\alpha < \lambda$ such that $x \subseteq M_\alpha$. This proves $\subseteq$. Since $\forall \alpha < \lambda[|M_\alpha| < \kappa]$, $\supseteq$ holds. Since κ is inaccessible, $\forall \alpha < \lambda |\mathscr{P}(M_\alpha)| < \kappa$. Hence $\lambda^{<\kappa} = \lambda$. $\square$

Lemma 20.10. *Let κ be strongly compact and $\lambda \geq \kappa$ regular. Let U be a fine ultrafilter on $[\lambda]^{<\kappa}$, and $j_U : V \to M$ the associated elementary embedding. Then $2^\lambda < j_U(\kappa) \leq j_U(\lambda) < (2^\lambda)^+$.*

Proof.

(1) There is a bijection from $j_U(\lambda)$ to ${}^{[\lambda]^{<\kappa}}\lambda/U$.

In fact,

$$j_U(\lambda) = \pi([c_\lambda]) = \{\pi([x]) : [x] \in^* [c_\lambda]\} = \{\pi([x]) : \{a \in [\lambda]^{<\kappa} : x_a \in \lambda\} \in U\}.$$

Now given x with domain $[\lambda]^{<\kappa}$ such that $X \stackrel{\text{def}}{=} \{a \in [\lambda]^{<\kappa} : x_a \in \lambda\} \in U$, let x' have domain $[\lambda]^{<\kappa}$ extending $x \restriction X$, with $x'_a = 0$ for $a \notin X$. Then define $f(\pi([x])) = [x']$. Clearly f is the desired bijection.

Now by Lemma 20.9 it follows that $j_U(\lambda) < (2^\lambda)^+$.

Obviously $j_U(\kappa) \leq j_U(\lambda)$.

Now let $g(x) = \mathscr{P}(x)$ for all $x \in [\lambda]^{<\kappa}$. For each $x \in [\lambda]^{<\kappa}$ let $h(x)$ be a one-one function mapping $\mathscr{P}(x)$ onto some $\xi < \kappa$.

(1) $\pi([h])$ is a one-one function mapping $\pi([g])$ onto an ordinal $\theta < j_U(\kappa)$.

In fact, $\{x \in [\lambda]^{<\kappa} : h(x)$ is a one-one function mapping $g(x)$ onto an ordinal $< \kappa\} \in U$. Then (1) follows.

Now for any $A \subseteq \lambda$ and $x \in [\lambda]^{<\kappa}$ let $f_A(x) = A \cap x$.

(2) $\forall A \subseteq \lambda[\pi([f_A]) \in \pi([g])]$.

In fact, $\{x \in [\lambda]^{<\kappa} : A \cap x \in \mathscr{P}(x)\} = [\lambda]^{<\kappa}$, and (2) follows.

(3) If $A, B \subseteq \lambda$ and $A \neq B$, then $\pi([f_A]) \neq \pi([f_B])$.

For, say $\xi \in A \backslash B$. Then

$$\begin{aligned}\{x \in [\lambda]^{<\kappa} : \xi \in f_A(x) \backslash f_B(x)\} &= \{x \in [\lambda]^{<\kappa} : \xi \in A \cap x \backslash B \cap x\}\\ &= \{x \in [\lambda]^{<\kappa} : \{\xi\} \subseteq x\} \in U,\end{aligned}$$

and (3) follows.

Thus $\pi([h]) \circ \langle \pi([f_A]) : A \subseteq \lambda \rangle$ is a one-one mapping of $\mathscr{P}(\lambda)$ into θ. So $2^\lambda < j_U(\kappa)$. □

Lemma 20.11. *Let κ be strongly compact and $\lambda \geq \kappa$ regular. Let U be a fine ultrafilter on $[\lambda]^{<\kappa}$, and $j_U : V \to M$ the associated elementary embedding. Then $j_U(\lambda) > \sup\{j_U(\xi) : \xi < \lambda\}$.*

Proof. For any $x \in [\lambda]^{<\kappa}$ let

$$f(x) = \begin{cases} 0 & \text{if } x = \emptyset, \\ \sup(x) & \text{if } x \neq \emptyset. \end{cases}$$

Then for any $\xi < \lambda$, $\{x \in [\lambda]^{<\kappa} : \xi \in x\} \subseteq \{x \in [\lambda]^{<\kappa} : \xi \leq f(x)\}$ is in U, and so $j_U(\xi) < \pi([f])$. Clearly $\pi([f]) < j_U(\lambda)$, so $j_U(\lambda) > \sup\{j_U(\xi) : \xi < \lambda\}$. □

Lemma 20.12. *Let κ be strongly compact and $\lambda \geq \kappa$ regular. Let U be a fine ultrafilter on $[\lambda]^{<\kappa}$, and $j_U : V \to M$ the associated elementary embedding. Then*

$$\forall y \subseteq M[|y| \leq \lambda \to \exists z \in M[y \subseteq z \wedge |z| \leq 2^\lambda \wedge M \models |z| < j_U(\kappa)]].$$

Proof. Suppose that $y \subseteq M$ and $|y| \leq \lambda$. Say $y = \{a_\gamma : \gamma < \lambda\}$, and for each $\gamma < \lambda$, g_γ is a function with domain $[\lambda]^{<\kappa}$ such that $\pi([g_\gamma]) = a_\gamma$. For any $x \in [\lambda]^{<\kappa}$ let $s(x) = \{g_\gamma(x) : \gamma \in x\}$ and let $z = \pi([s])$.

(1) $y \subseteq z$.

For, let $\gamma < \lambda$. Then $\{x \in [\lambda]^{<\kappa} : \gamma \in x\} \subseteq \{x \in [\lambda]^{<\kappa}[g_\gamma(x) \in s_x\}$ and so $a_\gamma \in z$, proving (1).

(2) $M \models |z| < j_U(\kappa)$.

In fact, $\forall x \in [\lambda]^{<\kappa}[|s_x| \leq |x| < \kappa]$, so $M \models |z| < j_U(\kappa)$.

(3) $|z| \leq 2^\lambda$.

For, suppose that $\pi([h]) \in z$. Thus $Y \stackrel{\text{def}}{=} \{x \in [\lambda]^{<\kappa} : h(x) \in s_x\} \in U$. For $x \in Y$ say $h(x) = g_{\gamma_x}(x)$ with $\gamma_x \in x$. Define h' with domain $[\lambda]^{<\kappa}$ by

$$h'(x) = \begin{cases} h(x) & \text{if } x \in Y, \\ g_0(x) & \text{if } x \notin Y. \end{cases}$$

Then the function k such that $k([h]) = [h']$ for all h with $\pi([h]) \in z$ is well defined and one-one. For each $x \in [\lambda]^{<\kappa}$ we have $h'(x) \in \{g_\gamma(x) : \gamma < \lambda\}$, so $|z| \leq \lambda^\lambda = 2^\lambda$ by Lemma 20.9. □

Lemma 20.13. *Let κ be strongly compact and $\lambda \geq \kappa$ regular. Let U be a fine ultrafilter on $[\lambda]^{<\kappa}$, and $j_U : V \to M$ the associated elementary embedding. Then*

$$\forall y \subseteq M[y \subseteq \alpha \wedge |y| \leq \lambda \to \exists z \in M[y \subseteq z \subseteq \alpha \wedge |z| \leq 2^\lambda \wedge M \models |z| < j_U(\kappa)]].$$

Proof. Repeat the proof of Lemma 20.12, with z replaced by $z \cap \alpha$. □

Lemma 20.14. *Suppose that κ is strongly compact, λ and μ are regular, $\kappa \leq \lambda$, and $2^\mu < \lambda$. Then $\lambda^\mu = \lambda$.*

Proof. If $\mu < \kappa$ this is true by Lemma 20.9. So suppose that $\mu \geq \kappa$. Now λ is regular in M, and by Lemma 20.10, $2^\mu < j_U(\kappa) < (2^\mu)^+ \leq \lambda$. Since j_U is elementary, $j_U(\kappa)$ is strongly compact in M. By Lemma 20.9 applied in M to $j_U(\kappa)$, there is a bijection from $[\lambda]^{j_U(\kappa)}$ to λ.

For each $x \in [\lambda]^\mu$ in M we apply Lemma 20.13 with y, λ, α replaced by x, μ, λ to obtain $y_x \in M$ such that $x \subseteq y_x \subseteq \lambda$, $|y_x| \leq 2^\mu$, and $M \models |y_x| < j_U(\kappa)$. Now $y_x \in [\lambda]^{<j_U(\kappa)}$, so by the preceding paragraph there are at most λ y_x's. Hence it suffices to show that for each $z \in [\lambda]^{j_U(\kappa)}$, $|\{x \in [\lambda]^\mu : y_x = z\}| \leq \lambda$.

So, let $z \in [\lambda]^{j_U(\kappa)}$. Now for each $x \in [\lambda]^\mu$ with $y_x = z$ we have $x \in [y_x]^\mu = [z]^\mu$ and $|[z]^\mu| = |[y_x]^\mu \leq 2^\mu < \lambda$. Hence $|\{x \in [\lambda]^\mu : y_x = z\}| \leq \lambda$. □

Theorem 20.15. *If κ is strongly compact, then SCH holds above κ. Moreover,*

(i) $\forall \lambda > \kappa[\lambda^{\aleph_0} \leq \lambda^+]$.

(ii) If $\lambda > \kappa$ is singular strong limit, then $2^\lambda = \lambda^+$.

Proof. If $\lambda > \kappa$, then by Lemma 20.9,

$$\lambda^{<\kappa} \leq (\lambda^+)^{<\kappa} = \lambda^+.$$

Hence (i) holds. Now SCH above κ says that $\forall \lambda > \kappa[2^{\mathrm{cf}(\lambda)} < \lambda \to \lambda^{\mathrm{cf}(\lambda)} = \lambda^+]$. So suppose that $\lambda > \kappa$ and $2^{\mathrm{cf}(\lambda)} < \lambda$. Then $2^{\mathrm{cf}(\lambda)} < \lambda^+$, so by Lemma 20.14 $\lambda^{\mathrm{cf}(\lambda)} \leq (\lambda^+)^{\mathrm{cf}(\lambda)} = \lambda^+$, and hence $\lambda^{\mathrm{cf}(\lambda)} = \lambda^+$.

For (ii), suppose that $\lambda > \kappa$ is singular strong limit. Then $\mathrm{cf}(\lambda) < \lambda$ and hence $2^{\mathrm{cf}(\lambda)} < \lambda$. So by SCH, $\lambda^{\mathrm{cf}(\lambda)} = \lambda^+$. Now with $\langle \mu_\xi : \xi < \mathrm{cf}(\lambda) \rangle$ a strictly increasing sequence of cardinals with supremum λ we have

$$2^\lambda = 2^{\sum_{\xi < \mathrm{cf}(\lambda)} \mu_\xi} = \prod_{\xi < \mathrm{cf}(\lambda)} 2^{\mu_\xi} \leq \lambda^{\mathrm{cf}(\lambda)} = \lambda^+.$$ □

A fine ultrafilter U on $[\lambda]^{<\kappa}$ is *normal* iff $\forall f : [\lambda]^{<\kappa} \to \lambda[\{x \in [\lambda]^{<\kappa} : f(x) \in x\} \in U \to \exists Y \in U[f \restriction Y$ is constant]]. A cardinal κ is *supercompact* iff for every A with $|A| \geq \kappa$ there is a normal ultrafilter on $[A]^{<\kappa}$.

Lemma 20.16. *Suppose that $\lambda \geq \kappa$. Let U be a fine ultrafilter on $[\lambda]^{<\kappa}$ with j_U the associated elementary embedding. Let $d(x) = x$ for all $x \in [\lambda]^{<\kappa}$. Then $\forall X \subseteq [\lambda]^{<\kappa}[X \in U$ iff $\pi([d]) \in j_U(X)$.*

Proof.

$$\pi([d]) \in j_U(X) \quad \text{iff} \quad \pi([d]) \in \pi([c_X]) \quad \text{iff} \quad \{x \in [\lambda]^{<\kappa} : x \in X\} \in D \quad \text{iff} \quad X \in U. \quad \square$$

Lemma 20.17. *Let U be a normal ultrafilter on $[\lambda]^{<\kappa}$. Then*
(i) $\pi([d]) = \{j_U(\gamma) : \gamma < \lambda\} = j_U[\lambda]$.
(ii) $\forall X \subseteq [\lambda]^{<\kappa}[X \in U$ iff $j_U[\lambda] \in j_U(X)]$.

Proof. (i): If $\gamma < \lambda$, then $\{x \in [\lambda]^{<\kappa} : \gamma \in x\} \in U$, and hence $j_U(\gamma) \in \pi([d])$. If $\pi([f]) \in \pi([d])$, then $\{x \in [\lambda]^{<\kappa} : f(x) \in x\} \in U$, and so by normality there is a $Y \in U$ such that $f \restriction Y$ is constant, say with value γ. Thus $\pi([f]) = j_U(\gamma)$. So (i) holds.

(ii): Suppose that $X \subseteq [\lambda]^{<\kappa}$. Then $j_U[\lambda] \in j_U(X)$ iff $\pi([d] \in j_U(X)$ iff $X \in U$ by Lemma 20.16. $\square$

Lemma 20.18. *If U is a normal ultrafilter on $[\lambda]^{<\kappa}$, then*
(i) $\forall$ functions f, g with domain $[\lambda]^{<\kappa}$,

$$\pi([f]) = \pi([g]) \quad \textit{iff} \quad (j_U(f))(j_U[\lambda]) = (j_U([g]))(j_U[\lambda]).$$

(ii) $\forall$ functions f, g with domain $[\lambda]^{<\kappa}$,

$$\pi([f]) \in \pi([g]) \quad \textit{iff} \quad (j_U(f))(j_U[\lambda]) \in (j_U([g]))(j_U[\lambda]).$$

(iii) $\forall$ function f with domain $[\lambda]^{<\kappa}$, $\pi([f]) = (j_U(f))(j_U[\lambda])$.

Proof. (i): Let $X = \{x \in [\lambda]^{<\kappa} : f(x) = g(x)\}$. Then $\forall x[x \in X$ iff $f(x) = g(x)]$, so $\forall x[x \in j_U(X)$ iff $(j_U(f))(x) = (j_U(g))(x)]$. Hence

$$\begin{aligned}
[f] = [g] \quad & \text{iff} \quad \{x \in [\lambda]^{<\kappa} : f(x) = g(x)\} \in U \\
& \text{iff} \quad j_U[\lambda] \in j_U(\{x \in [\lambda]^{<\kappa} : f(x) = g(x)\}) \\
& \text{iff} \quad (j_U(f))(j_U[\lambda]) = (j_U(g))(j_U[\lambda]).
\end{aligned}$$

(ii): Let $X = \{x \in [\lambda]^{<\kappa} : f(x) \in g(x)\}$. Then $\forall x[x \in X$ iff $f(x) \in g(x)]$, so $\forall x[x \in j_U(X)$ iff $(j_U(f))(x) \in (j_U(g))(x)]$. Hence

$$\begin{aligned}
\pi([f]) \in \pi([g]) \quad & \text{iff} \quad \{x \in [\lambda]^{<\kappa} : f(x) \in g(x)\} \in U \\
& \text{iff} \quad j_U[\lambda] \in j_U(\{x \in [\lambda]^{<\kappa} : f(x) \in g(x)\}) \\
& \text{iff} \quad (j_U(f))(j_U[\lambda]) \in (j_U(g))(j_U[\lambda]).
\end{aligned}$$

(iii): Suppose that f is a function with domain$[\lambda]^{<\kappa}$. Then $\forall x \in [\lambda]^{<\kappa}\exists! y[(x,y) \in f]$. Hence $M \models \forall x \exists! y[(x,y) \in j_U(f)]$. So choose e so that $(j_U([d]), \pi([e]) \in j_U(f)$. Then $Y \stackrel{\text{def}}{=} \{x \in [\lambda]^{<\kappa} : (x, e_x) \in f\} \in U$. Also $Z \stackrel{\text{def}}{=} \{x \in [\lambda]^{<\kappa} : (x, f_x) \in f\} \in U$. Now $Y \cap Z \subseteq \{x \in [\lambda]^{<\kappa} : e_x = f(x)\}$, so this set is in U, and hence $\pi([e]) = \pi([f])$. Hence $(j_U(f))(j_U[\lambda]) = (j_U(f))(j_U[d]) = \pi([e]]) = \pi([f])$. $\square$

For any $x \in [\lambda]^{<\kappa}$ let

$$\kappa_x = x \cap \kappa,$$
$$\lambda_x = \text{o.t.}(x).$$

Lemma 20.19. *Let U is a normal ultrafilter on $[\lambda]^{<\kappa}$. For all $x \in [\lambda]^{<\kappa}$ let $f(x) = \text{o.t.}(x)$. Then $\pi([f]) = \lambda$.*

Proof. $\forall x \in [\lambda]^{<\kappa}[f(x) = \text{o.t.}(x)]$, so $M \models \forall x[(j_U(f))(x) = \text{o.t.}(x)]$. Hence $\pi([f]) = (j_U(f))(j_U[\lambda]) = \text{o.t.}(j_U[\lambda]) = \lambda$. $\square$

Lemma 20.20. *Let U is a normal ultrafilter on $[\lambda]^{<\kappa}$, and let $j_U : V \to M$ be the associated elementary embedding. Then*

(i) $\forall \gamma < \kappa[j_U(\gamma) = \gamma]$.

(ii) $\lambda < j_U(\kappa)$.

(iii) ${}^\lambda M \subseteq M$.

Proof. (i): We prove this by induction. Suppose that it is true for all $\gamma < \beta$, with $\beta < \kappa$, while $\beta < j(\beta)$. Say $\beta = \pi([g])$. Thus $\pi([g]) < \pi([c_\beta])$. Hence $\{x \in [\lambda]^{<\kappa} : g(x) < \beta\} \in U$. Now $\{x \in [\lambda]^{<\kappa} : g(x) < \beta\} = \bigcup_{\gamma<\beta}\{x \in [\lambda]^{<\kappa} : g(x) = \gamma\}$, so by the κ-completeness of U there is a $\gamma < \beta$ such that $\{x \in [\lambda]^{<\kappa} : g(x) = \gamma\} \in U$. Hence $\beta = \pi([g]) = \pi(c_\gamma]) = \gamma$, contradiction.

(ii): Let f be as in Lemma 20.19. Then $\forall x \in [\lambda]^{<\kappa}[f(x) < \kappa]$, so $\lambda = \pi([f]) < j_U(\kappa)$.

(iii): It suffices to show that $\forall a \in {}^\lambda M[\text{rng}(a) \in M]$.

For, suppose we have shown this, and now let $a \in {}^\lambda M$. Define $b(\alpha) = (\alpha, a(\alpha))$ for all $\alpha < \lambda$. Then $b \in {}^\lambda M$. If we know that $\text{rng}(b) \in M$, then we can define $a(\alpha)$ to be the second coordinate of the member of $\text{rng}(b)$ with first coordinate α.

So now assume that $a \in {}^\lambda M$. For each $\alpha < \lambda$ let f_α have domain $[\lambda]^{<\kappa}$ with $\pi([f_\alpha]) = a_\alpha$. For each $x \in [\lambda]^{<\kappa}$ let $f(x) = \{f_\alpha(x) : \alpha \in x\}$. We claim that $\pi([f]) = \{a_\alpha : \alpha < \lambda\}$.

If $\alpha < \lambda$, then $\{x \in [\lambda]^{<\kappa} : \alpha \in x\} \in U$. Now $\{x \in [\lambda]^{<\kappa} : \alpha \in x\} \subseteq \{x \in [\lambda]^{<\kappa} : f_\alpha(x) \in f(x\}$, so $a_\alpha = \pi([f_\alpha]) \in \pi([f])$. If $\pi([g]) \in \pi([f])$, then $Y \stackrel{\text{def}}{=} \{x \in [\lambda]^{<\kappa} : g(x) \in f(x)\} \in U$. For each $x \in Y$ choose $\alpha_x \in x$ so that $g(x) = f_{\alpha_x}(x)$. For $x \notin Y$ let $\alpha_x = 0$. Thus $Y \subseteq \{x \in [\lambda]^{<\kappa} : \alpha_x \in x\}$. Hence by normality there exist a $Z \in U$ and a $\gamma < \lambda$ such that $\forall x \in Z[\alpha_x = \gamma]$. Then for all $x \in Y \cap Z[g(x) = f_\gamma(x)]$. Hence $\pi([g]) = \pi([f_\gamma]) = a_\gamma$. This proves the claim. $\square$

Lemma 20.21. *Suppose that $\lambda \geq \kappa$ and there is an elementary embedding $j : V \to M$ such that the following conditions hold:*

(i) $\forall \gamma < \kappa[j(\gamma) = \gamma]$.

(ii) $\lambda < j(\kappa)$.
(iii) ${}^{\lambda}M \subseteq M$.

Then there is a normal ultrafilter on $[\lambda]^{<\kappa}$.

Proof. Assume the hypotheses. Then by (iii), $\langle j(\gamma) : \gamma < \lambda\rangle \in M$, so also $\{j(\gamma) : \gamma < \lambda\} \in M$. Now we define

$$\text{(1)} \qquad X \in U \quad \text{iff} \quad X \subseteq [\lambda]^{<\kappa} \text{ and } j[\lambda] \in j(X)$$

We claim that U is as desired. To prove this takes several steps.

(2) $[\lambda]^{<\kappa} \in U$.

In fact, $\forall x[x \in [\lambda]^{<\kappa}$ iff $x \subseteq \lambda$ and $|x| < \kappa]$, so $\forall x[x \in j([\lambda]^{<\kappa})$ iff $x \subseteq j(\lambda)$ and $|x| < j(\kappa)]$. Now $\forall \gamma < \lambda[j(\gamma) < j(\lambda)]$, so $j[\lambda] \subseteq j(\lambda)$. Also $|j[\lambda]| = \lambda < j(\kappa)$ by (ii). Hence $j[\lambda] \in j([\lambda]^{<\kappa})$, as desired in (2).

Closure upwards: suppose that $X \in U$ and $X \subseteq Y$. We need:

(3) $X \subseteq Y$ implies that $j(X) \subseteq j(Y)$.

In fact, $\forall x \in X[x \in Y]$, so $\forall x \in j(X)[x \in j(Y)]$. Now upwards closure is clear.

We show closure under $\cap$ as part of the proof of κ-completeness. Suppose that $\gamma < \kappa$ and $X_\alpha \in U$ for all $\alpha < \gamma$. Thus $\forall \alpha < \gamma[j[\lambda] \subseteq j(X_\alpha)]$. Now $\forall x[x \in \bigcap_{\alpha<\gamma} X_\alpha \leftrightarrow \forall \alpha < \gamma[x \in X_\alpha]]$. Since $j(\gamma) = \gamma$, it follows that $\forall x[x \in j(\bigcap_{\alpha<\gamma} X_\alpha) \leftrightarrow \forall \alpha < \gamma[x \in j(X_\alpha)]]$. That is, $j(\bigcap_{\alpha<\gamma} X_\alpha) = \bigcap_{\alpha<\gamma} j(X_\alpha)$. Hence $\bigcap_{\alpha<\gamma} X_\alpha \in U$.

Clearly $\emptyset \notin U$.

(4) $j(X \cup Y) = j(X) \cup j(Y)$.

In fact, $\forall x[x \in X \cup Y \leftrightarrow x \in X \vee x \in Y]$, so $\forall x[x \in j(X \cup Y) \leftrightarrow x \in j(X) \vee x \in j(Y)]$.

From (4) it follows that for any $X \subseteq [\lambda]^{<\kappa}$, $X \in U$ or $[\lambda]^{<\kappa}\backslash X \in U$. Thus U is an ultrafilter.

(5) $\forall \alpha \in \lambda[\{x \in [\lambda]^{<\kappa} : \alpha \in x\} \in U$.

In fact, let $X = \{x \in [\lambda]^{<\kappa} : \alpha \in x\}$. Then $\forall x \in [\lambda]^{<\kappa}[x \in X$ iff $\alpha \in x]$, so $\forall x \in j([\lambda]^{<\kappa})[x \in j(X)$ iff $j(\alpha) \in x]$, Now $j(\alpha) \in j[\lambda]$, so $j[\lambda] \in j(X)$, and so $X \in U$.

(6) If $P \in [\lambda]^{<\kappa}$, then $\hat{P} \in U$.

For, $\hat{P} = \{Q \in [\lambda]^{<\kappa} : P \subseteq Q\} = \bigcap_{\alpha \in P}\{Q \in [\lambda]^{<\kappa} : \alpha \in Q\} \in U$.

From (6) it follows that U is a fine ultrafilter on $[\lambda]^{<\omega}$.

(7) U is normal.

For, suppose that $f : [\lambda]^{<\kappa} \to \lambda$ and $X \stackrel{\text{def}}{=} \{x \in [\lambda]^{<\kappa} : f(x) \in x\} \in U$. Now $\forall x[x \in X$ iff $f(x) \in x]$, so $\forall x \in j(X)[(j(f))(x) \in x]$. Now by (iii), $j[\lambda] \in M$, so $(j(f))(j[\lambda]) \in j[\lambda]$. Say $(j(f))(j[\lambda]) = j(\gamma)$ with $\gamma < \lambda$. Let $Y = \{x \in [\lambda]^{<\kappa} : f(x) = \gamma\}$. Then $\forall x[x \in Y$ iff $f(x) = \gamma]$, so by (i), $\forall x[x \in j(Y)$ iff $(j(f))(x) = j(\gamma)]$. Since $(j(f))(j[\lambda])j(\gamma)$, it follows that $j[\lambda] \in j(Y)$. Hence $Y \in U$. $\square$

Lemma 20.22. *Let κ, λ, j, U be as in Lemma 20.21 and its proof. Thus $j : V \to M$ and $j_U : V \to N$ are elementary embeddings. Now define for any f with domain $[\lambda]^{<\kappa}$, $k(\pi([f]_U)) = (j(f))(j[\lambda])$. Then $k : N \to M$ is a well-defined elementary embedding, and $j = k \circ j_U$. Moreover, $\forall \alpha \leq \lambda[k(\alpha) = \alpha]$.*

Proof. k is well-defined and one-one by Lemma 20.18. Elementarity:

$$N \models \varphi([f^0]_U, \ldots, [f^{m-1}_U) \quad \text{iff} \quad X \stackrel{\text{def}}{=} \{x \in [\lambda]^{<\kappa} : \varphi(f^0(x), \ldots, f^{m-1}(x)\} \in U.$$

Now

$$\forall x \in [\lambda]^{<\kappa}[x \in X \text{ iff } \varphi(f^0(x), \ldots, f^{m-1}(x))],$$

so

$$\forall x \in j([\lambda]^{<\kappa}) \quad x \in j([\lambda]^{<\kappa} \text{ and } M \models \varphi((j(f^0))(x), \ldots, j(f^{m-1})(x)).$$

Now the definition of U in the proof of Lemma 20.21 yields $j[\lambda] \in j([\lambda]^{<\kappa})$, so it follows that

$$M \models \varphi((j(f^0))(j[\lambda]), \ldots, (j(f^{m-1})(j[\lambda]))), \quad \text{i.e.} \quad M \models \varphi(k([f^0]_U), \ldots, k([f^{m-1}]_U)).$$

This shows that k is an elementary embedding.

Now if $a \in V$, then $k(j_U(a)) = k(\pi([c_a])) = (j(c_a))(j[\lambda])$. Now $\forall x \in [\lambda]^{<\kappa}[c_a(x) = a]$, so $\forall x \in j([\lambda]^{<\kappa})[(j(c_a))(x) = j(a)]$. Since $j[\lambda] \in j([\lambda]^{<\kappa})$, it follows that $(j(c_a))(j[\lambda]) = j(a)$. This shows that $k \circ j_U = j$.

Now take any $\alpha \leq \lambda$, and for each $x \in [\lambda]^{<\kappa}$ let $f(x) = \text{o.t.}(x \cap \alpha)$. Thus $\forall x \in j_U([\lambda]^{<\kappa}[(j_U(f))(x) = \text{o.t.}(x \cap j_U(\alpha))]$. Hence by Lemma 20.18, $\pi([f]_U) = (j_U(f))(j_U[\lambda]) = \text{o.t.}(j_U[\lambda] \cap j_U(\alpha)) = \alpha$. Then

$$k(\alpha) = k(\pi([f]_U) = (j(f))(j[\lambda]) = \text{o.t.}(j[\lambda] \cap j(\alpha) = \alpha \qquad \square$$

A cardinal κ is λ-*supercompact* iff it satisfies the conditions of Lemma 20.21.

Lemma 20.23. *If $\lambda \geq \kappa$ and κ is λ-supercompact, then κ is measurable.*

Proof. This follows from the proof ofLemma 17.9. $\square$

Lemma 20.24. *If κ is λ-supercompact and $\forall \mu < \kappa[2^\mu = \mu^+]$, then $\forall \mu \leq \lambda[2^\mu = \mu^+]$.*

Proof. Let j satisfy the conditions in Lemma 20.21.

(1) If $X \subseteq M$ and $|X| \leq \lambda$, then $X \in M$.

For, let $f : \lambda \to X$ be onto. Then $f \in {}^\lambda M$, so $f \in M$. Hence $X = \text{rng}(f) \in M$.

Now $\forall \mu < \kappa[2^\mu = \mu^+]$, so $\forall \mu < j(\kappa)[2^\mu = \mu^+]$. Thus $(2^\mu)^M = (\mu^+)^M$. By (1), if $\mu \leq \lambda$ then $\mathscr{P}^M(\mu) = \mathscr{P}(\mu)$. Hence $2^\mu = (2^\mu)^M = (\mu^+)^M = \alpha^+$. $\square$

Lemma 20.25. *If κ is supercompact, then there is a normal ultrafilter D on κ such that $\{\mu < \lambda : \mu \text{ is measurable}\} \in D$.*

Proof. Let $\lambda = 2^\kappa$. Let j satisfy the conditions of Lemma 20.21. Define $D = \{X \subseteq \kappa : \kappa \in j(X)\}$.

(1) If $\gamma < \kappa$ and X has domain γ, then $j(\bigcap_{\alpha<\gamma} X_\alpha) = \bigcap_{\alpha<\gamma} j(X_\alpha)$.

In fact, $\forall x[x \in \bigcap_{\alpha<\gamma} X_\alpha \leftrightarrow \forall \alpha < \gamma[x \in X_\alpha]]$, so $\forall x[x \in j(\bigcap_{\alpha<\gamma} X_\alpha) \leftrightarrow \forall \alpha < j(\gamma)[x \in j(X_\alpha)]]$. Since $j(\gamma) = \gamma$, (1) follows.

By (1), D is κ-complete. Clearly D is closed upwards.

(2) $j(X \cup Y) = j(X) \cup j(Y)$.

For, $\forall x[x \in X \cup Y \leftrightarrow x \in X \vee x \in Y]$, so $\forall x[x \in j(X \cup Y) \leftrightarrow x \in j(X) \vee x \in j(Y)]$.

It follows that D is an ultrafilter.

(3) $\forall \alpha < \kappa[j(\{\alpha\}) = \{j(\alpha)\} = \{\alpha\}$.

For, $\forall x[x \in \{\alpha\} \leftrightarrow x = \alpha]$, so $\forall x[x \in j(\{\alpha\}) \leftrightarrow x = j(\alpha)$.

Now for any $\alpha < \kappa$, $\kappa \in j(\kappa \backslash \{\alpha\})$ or $\kappa \in j(\{\alpha\})$. By (3), $\kappa \in j(\kappa \backslash \{\alpha\})$. Thus D is nonprincipal.

Now let k be the elementary embedding of Ult_D into M given in the proof of Lemma 17.10: for any f with domain κ, $k(\pi([f]_D)) = (j(f))(\kappa)$.

(4) $\forall \gamma < \kappa[\pi(j_D(\gamma)) = \gamma]$.

For, $\gamma \leq \pi(j_D(\gamma)) \leq k(\pi(j_D(\gamma))) = j(\gamma) = \gamma$.

(5) $k(\kappa) = \kappa$.

For, choose $f \in {}^\kappa\kappa$ so that $\pi([f]_D) = \kappa$. Let $d(\alpha) = \alpha$ for all $\alpha < \kappa$. Then $\pi([f]_D) \leq \pi([d]_D)$. In fact, for any $\gamma < \kappa$ let $X = \{\alpha < \kappa : \gamma < d(\alpha)\}$. Then $j(X) = \{\alpha < j(\kappa) : \gamma < \alpha\}$, and so $\kappa \in j(X)$. Hence $X \in D$, and so $j_D(\gamma) < \pi([d]_D)$. Since this is true for any $\gamma < \kappa$, we have $\pi([f]_D) = \kappa \leq \pi([d]_D)$. Hence $Y \overset{\text{def}}{=} \{\alpha < \kappa : f(\alpha) \leq d(\alpha)\} \in D$. Now $j(Y) = \{\alpha < j(\kappa) : (j(f))(\alpha) \leq \alpha\}$, so $(j(f))(\kappa) \leq \kappa$. Thus $\kappa \leq k(\kappa) = k(\pi([f]_D)) = (j(f))(\kappa) \leq \kappa$. So (5) holds.

Now if $X \subseteq \kappa$, let $f : \lambda \to X$ be a surjection. Then $f \in {}^\lambda M$, so $f \in M$, and hence $X = \mathrm{rng}(f) \in M$. Similarly, every subset of M of size at most λ is in M. Also, if $U \subseteq \mathscr{P}(\kappa)$, then $|U| \leq 2^\kappa$ and so there is a surjection g from λ onto U. Each member of U is a subset of κ and so is in M by the above. It follows that $U \in M$. Now κ is measurable, so

$$\begin{aligned}\exists E[&E \subseteq \mathscr{P}(\kappa) \wedge \emptyset \notin E \wedge \kappa \in E\\ &\wedge \forall X \in E \forall Y \subseteq \kappa[X \subseteq Y \to Y \in E]\\ &\wedge \forall \alpha < \kappa[\{\alpha\} \notin E]\\ &\wedge \forall \alpha < \kappa \forall X \in {}^\alpha E\left[\bigcap_{\xi<\alpha} X_\xi \in E\right].\end{aligned}$$

By the above, $E \in M$ and hence $M \models E$ is a κ-complete nonprincipal ultrafilter on κ. So $M \models \kappa$ is measurable. Hence $\pi(\mathrm{Ult}_D) \models \kappa$ is measurable. Hence the lemma follows. $\square$

Lemma 20.26. *Let κ be a measurable cardinal such that there are κ strongly compact cardinals below κ. Then κ is strongly compact.*

Proof. Let κ be a measurable cardinal.

(1) If $A \subseteq \kappa$ has size κ, then there is a κ-complete ultrafilter U on κ such that $A \in U$.

For, let D be any κ-complete ultrafilter on κ, and let $f : \kappa \to A$ be a bijection. Define $U = \{X \subseteq \kappa : f^{-1}[X] \in D\}$. Clearly U is a κ-complete ultrafilter on κ and $A \in U$. U is nonprincipal, since $f^{-1}[\{\alpha\}]$ is either empty (if $\alpha \notin A$), or a singleton.

Let $|A| \geq \kappa$; we show that there is a fine ultrafilter on $[A]^{<\kappa}$.

Let $C = \{\alpha < \kappa : \alpha$ is strongly compact$\}$, and by (1) let F be a nonprincipal κ-complete ultrafilter on κ such that $C \in F$. For each $\alpha \in C$ let W_α be a fine measure on $[A]^{<\alpha}$. Then we define

$$X \in U \quad \text{iff} \quad X \subseteq [A]^{<\kappa} \text{ and } \{\alpha \in C : X \cap [A]^{<\alpha} \in W_\alpha\} \in F.$$

Suppose that $X \in U$ and $X \subseteq Y \subseteq [A]^{<\kappa}$. Let $\alpha \in C$ with $X \cap [A]^{<\alpha} \in W_\alpha$. Then $Y \cap [A]^{<\alpha} \in W_\alpha$. So $Y \in U$.

Suppose that $\beta < \kappa$ and $X_\xi \in U$ for all $\xi < \beta$. Then for all $\xi < \beta$, $Y_\xi \stackrel{\text{def}}{=} \{\alpha \in C : X_\xi \cap [A]^{<\alpha} \in W_\alpha\} \in F$. Hence

$$\begin{aligned}\bigcap_{\xi<\beta} Y_\xi &= \{\alpha \in C : \forall \xi < \beta[X_\xi \cap [A]^{<\alpha} \in W_\alpha]\}\\ &= \left\{\alpha \in C : \bigcap_{\xi<\beta} X_\xi \cap [A]^{<\alpha} \in W_\alpha\right\} \in F.\end{aligned}$$

Hence $\bigcap_{\xi<\beta} X_\xi \in U$. Clearly $\emptyset \notin U$. If $X \subseteq [A]^{<\kappa}$ and $X \notin U$, then $\{\alpha \in C : X \cap [A]^{<\alpha} \in W_\alpha\} \notin F$, hence $\{\alpha \in C : X \cap [A]^{<\alpha} \notin W_\alpha\} \in F$, hence $\{\alpha \in C : ([A]^{<\kappa}\backslash X) \cap [A]^{<\alpha} \in W_\alpha\} \in F$, hence $([A]^{<\kappa}\backslash X) \in U$. So U is a κ-complete ultrafilter. Now suppose that $P \in [A]^{<\kappa}$. Say $|P| < \gamma < \kappa$. Then for any $\alpha \in C$ with $\gamma < \alpha$, $\hat{P} \cap [A]^{<\alpha}$ is $\hat{P}$ in the sense of $[A]^{<\alpha}$, and hence $\hat{P} \cap [A]^{<\alpha} \in W_\alpha$. Hence $\hat{P} \in U$. $\square$

Proposition 20.27. *If there is a measurable cardinal which is a limit of strongly compact cardinals, then the least such is strongly compact but not supercompact.*

Proof. Let κ be measurable which is the least measurable cardinal which is a limit of strongly compact cardinals. By Lemma 20.26, κ is strongly compact. Suppose that κ is supercompact. Let $\lambda = 2^\kappa$. By the definition of supercompact and Lemma 20.20 let $j_U : V \to M$ be an elementary embedding such that κ is the first ordinal moved and ${}^\lambda M \subseteq M$. If $\alpha < \kappa$ is strongly compact, then $M \models j_U(\alpha)$ is strongly compact. Since $j_U\alpha) = \alpha$, it follows that $M \models \alpha$ is strongly compact. So $M \models \kappa$ is a limit of strongly compact cardinals. Now κ is measurable in M. Since $\kappa < j_U(\kappa)$, this a contradiction. $\square$

Lemma 20.28. *If U is a normal ultrafilter on $[\lambda]^{<\kappa}$, then j_U is the identity on V_κ.*

Proof. Suppose that $y \in V_\kappa$ and $j_U(z) = z$ for all $z \in y$, but $j_U(y) \neq y$.

Case 1. There is a $z \in y \backslash j_U(y)$, Then $z = j_U(z) \in j_U(y)$, contradiction.

Case 9. There is a $z \in j_U(y) \backslash y$. Now $j_U(y) = \pi([y])$. Say $z = \pi([g])$. Then $\{x \in [\lambda]^{<\kappa} : g(x) \in y\} \in U$. Since U is κ complete and $|y| < \kappa$, (Since $V_\kappa = H(\kappa)$ because κ is inaccessible), there is an $w \in y$ such that $\{x \in [\lambda]^{<\kappa} : g(x) = w\} \in U$. Then $z = w \in y$, contradiction. □

Theorem 20.29. *Let κ be supercompact. Then there exists $f : \kappa \to V_\kappa$ such that $\forall x \forall \lambda \geq \kappa[\lambda \geq |\text{t.c.}(x)| \to \exists U[U$ is a normal ultrafilter on $[\lambda]^{<\kappa}$ such that $(j_U(f))(\kappa) = x]]$.*

Proof. Let κ be supercompact, and suppose that the conclusion is false. Then for each $f : \kappa \to V_\kappa$ here exist x, λ such that $\lambda \geq \kappa$, $|\text{t.c.}(x)| \leq \lambda$, and for every U which is a normal ultrafilter on $[\lambda]^{<\kappa}$ we have $(j_U(f))(\kappa) \neq x$. For every $f : \kappa \to V_\kappa$ let $\lambda_f \geq \kappa$ be minimum such that there is an x such that $|\text{t.c.}(x)| \leq \lambda$, and for every U which is a normal ultrafilter on $[\lambda]^{<\kappa}$ we have $(j_U(f))(\kappa) \neq x$. Let ν be greater than all λ_f's and let $j : V \to M$ be an elementary embedding satisfying the conditions of Lemma 20.21 with λ replaced by ν.

Let $\varphi(g, \delta)$ be the statement that for some cardinal α, $g : \alpha \to V_\alpha$ and δ is the least cardinal $\geq \alpha$ such that there is an x with $|\text{t.c.}(x)| \leq \delta$ for which there is no normal ultrafilter U on $[\delta]^{<\alpha}$ with $(j_U(g))(\alpha) = x$. Let λ_g denote this δ.

(1) $\forall f : \kappa \to V_\kappa[M \models \varphi(f, \lambda_f)]$.

In fact, let $f : \kappa \to V_\kappa$. Now $\kappa \in M$, and by Lemma 17.17, $V_\kappa \in M$. Hence $f \in {}^\kappa M$ and hence $f \in M$ since ${}^\nu M \subseteq M$. Now $M \models \varphi(f, \lambda_f)]$ by absoluteness.

Let A be the set of all $\alpha < \kappa$ such that $\forall g : \alpha \to V_\alpha[\varphi(g, \lambda_g)]$.

(2) $\kappa \in j(A)$.

In fact, $\forall \alpha[\alpha \in A$ iff $\alpha < \kappa$ and $\forall g : \alpha \to V_\alpha[\varphi(g, \lambda_g)]$, so $\forall \alpha[\alpha \in j(A)$ iff $\alpha < j(\kappa)$ and $\forall g : \alpha \to V_\alpha[\varphi(g, \lambda_g)]$. Since $\kappa < j(\kappa)$ and $\forall g : \kappa \to V_\kappa[\varphi(g, \lambda_g)]$, (2) follows.

Now we define $f : \kappa \to V_\kappa$ by recursion. If $\alpha \in A$, let $f(\alpha)$ be an x_α such that x_α witnesses $\varphi(f \restriction \alpha, \lambda_{f\restriction\alpha})$. For $\alpha \in \kappa \backslash A$ let $f(\alpha) = \emptyset$.

Let $x = (j(f))(\kappa)$.

(3) t.c.$(x) \leq \lambda_f$ and there is no normal ultrafilter U on $[\lambda_f]^{,\kappa}$ with $(j_U(f))(\kappa) = x$.

For,

$$\forall \alpha \in A[f \restriction \alpha : \alpha \to V_\alpha \wedge |\text{t.c.}(f(\alpha))| \leq \lambda_{f\restriction\alpha} \wedge \varphi(f \restriction \alpha, \lambda_{f\restriction\alpha}).$$

Hence

$$\forall \alpha \in j(A)[j(f \restriction \alpha) : \alpha \to V_\alpha \wedge |\text{t.c.}((j(f)(\alpha))| \leq \lambda_{f\restriction\alpha} \wedge \varphi(f \restriction \alpha, \lambda_{f\restriction\alpha}).$$

By (2) it follows that

$$j(f) : \kappa \to V_\kappa \wedge |\text{t.c.}((j(f))(\kappa)| \leq \lambda_f \wedge \varphi(f, \lambda_f).$$

Thus (3) holds.

Let $U = \{X \in [\lambda]^{<\kappa} : j[\lambda] \in j(X)\}$.

(4) U is a normal ultrafilter on $[\lambda]^{<\kappa}$.

For, see the proof of Lemma 20.21.

Now let k be given by Lemma 20.29. Then

(5) $k(x) = x$.

For, since $|\text{t.c.}(x)| \leq \lambda_f$, there is a bijection of x onto some $\beta \leq \lambda_f$. Since $k(\beta) = \beta$ by Lemma 20.22, it follows that $k(x) = x$.

It follows that

$$j_U(\kappa) = k^{-1}((j(f))(\kappa)) = k^{-1}(x) = x.$$

This contradicts (3). $\square$

21. Large cardinals and forcing

Theorem 21.1. *Let κ be a measurable cardinal in M, and let P be a poset such that $|P| < \kappa$. Then κ is measurable in $M[G]$.*

Proof. We work in a ctm M. Assume that κ is a measurable cardinal in M, P is a forcing order, $P \subseteq B(P)$, and $|P| < \kappa$. So wlog $B(P) \in M_\kappa$. Let $j : M \to N$ be the elementary embedding determined by κ. $j(B) = B$ by Lemma 17.17. So $B \in N$. Assume that G is generic over M.

(1) G is generic over N.

$N \in M$, so $N \subseteq M$. Hence (1) is clear.

(2) $j(V_\alpha^B) = V_\alpha^B$ and $j(\dot{y}) = \dot{y}$ for any B-name $\dot{y}$.

This holds by an easy induction.

(3) If $\dot{x}_M^G = \dot{y}_M^G$, then $\dot{x}_N^G = \dot{y}_N^G$.

In fact, suppose that $\dot{x}_M^G = \dot{y}_M^G$, and choose $p \in G$ so that $p \Vdash_M \dot{x} = \dot{y}$. Then $j(p) \Vdash_N j(\dot{x}) = j(\dot{y})$. Now $j(p) = p$, so $p \Vdash_N j(\dot{x}) = j(\dot{y})$. Hence $\dot{x}_N^G = \dot{y}_N^G$. So (3) holds.

Let $x \in M[G]$. Say $\dot{x}$ is a name such that $\dot{x}_M^G = x$. Then we define $j'(x) = \dot{x}_N^G$.

Now to show that j' is elementary, suppose that $M[G] \models \varphi(x_0, \ldots)$. So $M[G] \models \varphi(\dot{x}_0^G, \ldots)$, so there is a $p \in G$ such that $p \Vdash_M \varphi(\dot{x}_0, \ldots)$. Applying j, we get $p \Vdash_N \varphi(\dot{x}_0), \ldots)$. Hence $N[G] \models \varphi(j'(x_0), \ldots)$.

If $x \in M$, then $j(x) = j(\check{x}_M^G) = \check{x}_N^G = j'(x)$.

$\forall \gamma < \kappa[j(\gamma) = \gamma]$, so $\forall \gamma < \kappa[j'(\gamma) = \gamma]$. Also, $\kappa < j(\kappa) = j'(\kappa$. ☐

Theorem 21.2. *Let κ be an infinite cardinal and P a forcing poset such that $|P| < \kappa$. Let G be a V-generic filter on P.*

Then κ is inaccessible in V iff it is inaccessible in $V[G]$.

Proof. Assume that κ is inaccessible in V. Then κ is regular in $V[G]$. Then κ is inaccessible by Lemma 14.50 of these notes.

On the other hand, suppose that κ is inaccessible in $V[G]$. Now "regular cardinal" is Π_1, so κ is regular in M. Now suppose that $\mu < \kappa$ and $f \in M$ is a one-one function mapping κ into $\mathscr{P}(\mu)$. Then $f \in M[G]$ and $\forall \alpha < \kappa[f(\alpha) \subseteq \mu]$, so by absoluteness this holds in $M[G]$, contradiction. ☐

Theorem 21.3. *Let κ be an infinite cardinal and P a forcing poset such that $|P| < \kappa$. Let G be a V-generic filter on P.*

Then κ is Mahlo in V iff it is Mahlo in $V[G]$.

Proof. First let κ be Mahlo in V. Then it is inaccessible in $V[G]$ by Theorem 21.9. Now if α is regular and $\alpha > |P|$, then α is regular in $V[G]$. Clearly if α is regular in $V[G]$ then it is regular in V. Now $S \stackrel{\text{def}}{=} \{\alpha : \alpha < \kappa \text{ is regular and } |P| < \alpha\}$ is stationary in κ, in the sense of V, by the definition of Mahlo.

(1) If $C \subseteq \kappa$ is club in $V[G]$, then there is a $C' \subseteq C$ which is club in V.

In fact, suppose that $C \subseteq \kappa$ is club in $V[G]$. Let $\dot{C}$ be a name such that $\dot{C}^G = C$. For all $p \in P$ let $C_p = \{\alpha < \kappa : p \Vdash \check{\alpha} \in \dot{C}\}$.

(2) $\exists p \in G[C_p$ is unbounded].

In fact, suppose not. For each $p \in G$ let β_p be a bound for C_p, and let $\gamma < \kappa$ be $> \beta_p$ for each $p \in G$. C is unbounded, so choose $\alpha \in C$ with $\gamma < \alpha$. Then there is a $p \in G$ such that $p \Vdash \check{\alpha} \in \dot{C}$. So $\alpha \in C_p$, hence $\alpha < \beta_p < \gamma < \alpha$, contradiction. Thus (2) holds.

Now if $\alpha \in C_p$, then $p \Vdash \check{\alpha} \in \dot{C}$, so $\alpha \in C$. Hence $C_p \subseteq C$, so the closure of C_p is a subset of C. This proves (1).

It follows that $S \cap C \neq \emptyset$, since $S \cap C' \neq \emptyset$. Thus S is stationary in $V[G]$, so κ is Mahlo in $V[G]$.

Now suppose that κ is Mahlo in $V[G]$. Then $S \stackrel{\text{def}}{=} \{\alpha < \kappa : \alpha \text{ regular}\}$ is stationary in κ, in $V[G]$. Now for $|P| < \alpha < \kappa$, α is regular in V iff α is regular in $V[G]$. Suppose in V that $C \subseteq \kappa$ is club. Then it is clearly club in $V[G]$, so $S \cap C \neq \emptyset$. Hence κ is Mahlo in V. □

Theorem 21.4. *Let κ be an infinite cardinal and P a forcing poset such that $|P| < \kappa$. Let G be a V-generic filter on P.*

Then κ is weakly compact in V iff it is weakly compact in $V[G]$.

Proof. First suppose that κ is weakly compact in V. Suppose that $f : [\kappa]^2 \to 2$ in $V[G]$. Let $\dot{f}$ be a name such that $\dot{f}^G = f$. Then there is a $p \in G$ such that $p \Vdash \dot{f} : [\kappa]^2 \to 2$. Define $H : [\kappa]^2 \to B(P)$ by setting $H(x) = ||\dot{f}(\check{x}) = 0||$. By Theorem 17.23 we have $\kappa \to [\kappa]^2_{|B(P)|^+}$, so there exist in V a $K \in [\kappa]^\kappa$ and a $b \in B(P)$ such that $H(x) = b$ for all $x \in K$. Thus $\forall x \in K[||\dot{f}(\check{x}) = 0|| = b]$.

Case 1. $b \in G$. Then $\forall x \in K[f(x) = 0]$.

Case 9. $-b \in G$. Thus

$$\forall x \in K[-b = -||\dot{f}(\check{x}) = 0|| = ||\neg(\dot{f}(\check{x}) = 0)|| = ||\dot{f}(\check{x}) = 1||],$$

so $\forall x \in K[f(x) = 1$.

Now conversely suppose that κ is weakly compact in $V[G]$. Suppose that $f \in V$ maps $[\kappa]^2$ into 9. Then there exist in $V[G]$ a set $H \in [\kappa]^\kappa$ and an $\varepsilon \in 2$ such that $f[[H]^2] = \{\varepsilon\}$. Say $\dot{H}_G = H$. For all $p \in P$ let $C_p = \{\alpha < \kappa : p \Vdash \check{\alpha} \in \dot{H}\}$.

(1) There is a $p \in G$ such that C_p is unbounded in κ.

For, suppose not. For each $p \in G$ let γ_p be a bound for C_p. Since $|P| < \kappa$, choose $\delta < \kappa$ such that $\gamma_p < \delta$ for all $p \in G$. Choose $\varphi \in H$ such that $\delta < \varphi$. Then there is a $p \in G$ such that $p \Vdash \check{\varphi} \in \dot{H}$. Thus $\varphi \in C_p$. Hence $\delta < \varphi < \gamma_p < \delta$, contradiction. So (1) holds.

Now for all $\alpha \in C_p$ we have $\alpha \in H$. So $f[[C_p]^2] = \{\varepsilon\}$, as desired. □

Theorem 21.5. *Let κ be an infinite cardinal and P a forcing poset such that $|P| < \kappa$. Let G be a V-generic filter on P.*

Then κ is Ramsey in V iff it is Ramsey in $V[G]$.

Proof. First suppose that κ is Ramsey in V. Suppose that $f : \bigcup_{m\in\omega}[\kappa]^m \to 2$ in $V[G]$. Let $\dot{f}$ be a name such that $\dot{f}^G = f$. Then there is a $p \in G$ such that $p \Vdash \dot{f} : \bigcup_{m\in\omega}[\kappa]^m \to 2$. Define $H : \bigcup_{m\in\omega}[\kappa]^m \to B(P)$ by setting $H(x) = ||\dot{f}(\check{x}) = 0||$. By Theorem 17.23 we have $\kappa \to \bigcup_{m\in\omega}[\kappa]^m_{|B(P)|^+}$, so there exist in V a $K \in [\kappa]^\kappa$ and a $b \in {}^\omega B(P)$ such that $H(x) = b_m$ for all $x \in [K]^m$. Thus $\forall x \in [H]^m[||\dot{f}(\check{x}) = 0|| = b_m]$.

Case 1. $b_m \in G$. Then $\forall x \in [K]^m[f(x) = 0]$.

Case 9. $-b_m \in G$. Thus

$$\forall x \in [K]^m[-b = -||\dot{f}(\check{x}) = 0|| = ||\neg(\dot{f}(\check{x}) = 0)|| = ||\dot{f}(\check{x}) = 1||],$$

so $\forall x \in [K]^m[f(x) = 1$. Thus κ is Ramsey in $V[G]$.

Now conversely suppose that κ is Ramsey in $V[G]$. Suppose that $f \in V$ maps $[\kappa]^2$ into 9. Then there exist in $V[G]$ a set $K \in [\kappa]^\kappa$ and an $\varepsilon \in 2$ such that $f[[K]^2] = \{\varepsilon\}$. Say $\dot{K}_G = K$. For all $p \in P$ let $C_p = \{\alpha < \kappa : p \Vdash \check{\alpha} \in \dot{K}\}$.

(1) There is a $p \in G$ such that C_p is unbounded in κ.

For, suppose not. For each $p \in G$ let γ_p be a bound for C_p. Since $|P| < \kappa$, choose $\delta < \kappa$ such that $\gamma_p < \delta$ for all $p \in G$. Choose $\varphi \in K$ such that $\delta < \varphi$. Then there is a $p \in G$ such that $p \Vdash \check{\varphi} \in \dot{K}$. Thus $\varphi \in C_p$. Hence $\delta < \varphi < \gamma_p < \delta$, contradiction. So (1) holds.

Now for all $\alpha \in C_p$ we have $\alpha \in K$. So C_p has size κ and is homogeneous for f. $\square$

Lemma 21.6. *Let κ be uncountable and regular, and let $\nu < \kappa$. Suppose that I is a κ-complete ν-saturated ideal on κ containing all singletons. Then either κ is measurable or $\kappa \le 2^\nu$.*

Proof. Clearly all subsets of size less than κ are in I.

(1) If $Z \in [\kappa]^\kappa$ and $\mathscr{P}(Z) \cap I$ is a maximal ideal on Z, then it is κ-complete and contains all singletons, so that κ is measurable.

For, if $\alpha \in Z$ then $\{\alpha\} \in \mathscr{P}(Z) \cap I$. If $\gamma < \kappa$ and $X_\alpha \in \mathscr{P}(Z) \cap I$ for all $\alpha < \gamma$, then $\bigcup_{\alpha<\gamma} X_\alpha \in \mathscr{P}(Z) \cap I$.

Assume

(2) For every $Z \in [\kappa]^\kappa$, the set $\mathscr{P}(Z) \cap I$ is not a maximal ideal on Z.

We construct a tree T of subsets of κ by recursion. Each member of T is not in I. The 0th level of T is κ. If $X \in T$ is at level α, then by (2) there is a subset Y of X such that $Y, X\backslash Y \notin I$; we let Y and $X\backslash Y$ be the successors of X in the tree. If α is a limit ordinal, then the α-th level of T consists of all intersections $\bigcap_{\xi<\alpha} X_\xi$ such that each X_ξ is at level ξ and $\bigcap_{\xi<\alpha} X_\xi \notin I$; if there are no such intersections, then the height of T has been reached.

Each branch of T has length $< \nu$, since if $\langle X_\xi : \xi < \alpha\rangle$ is a branch, with each X_ξ at level ξ, then $\langle X_\xi \backslash X_{\xi+1} : \xi < \alpha\rangle$ is a pairwise disjoint system of elements of $\mathscr{P}(\kappa)\backslash I$ so $\alpha < \nu$ by the ν-saturation of I.

The levels of T have size $< \nu$ for the same reason, since a level consists of pairwise disjoint sets each of which is not in I.

It follows that T has at most $\prod_{\alpha<\nu}\nu = 2^\nu$ branches. Let $\langle b_\xi : \xi < 2^\nu\rangle$ enumerate all of the branches of T such that $\bigcap b_\xi \neq \emptyset$, and let $Z_\xi = \bigcap b_\xi$.

(3) $\kappa = \bigcup_{\xi<2^\nu} b_\xi$.

In fact, suppose that $\alpha \in \kappa$. Say $\alpha \in C_\eta$ with C_η at level η, for all $\eta < \rho$. Then $\alpha \in \bigcap_{\eta<\xi} C_\rho$, and $\langle C_\eta : \eta < \rho\rangle$ is a branch, and hence α is in some b_ξ. Clearly $\langle Z_\xi : \xi < 2^\nu\rangle$ is a partition of κ into at most 2^ν nonempty sets, all in I. Since I is κ-complete and $X = \bigcup_{\xi<2^\nu} Z_\xi$, it follows that $\kappa \le 2^\nu$. □

Theorem 21.7. *Let κ be an infinite cardinal and P a forcing poset such that $|P| < \kappa$. Let G be a V-generic filter on P.*

Then κ is measurable in V iff it is measurable in $V[G]$.

Proof. $\Rightarrow$ is given by Theorem 21.1. Now assume that κ is measurable in $V[G]$. Let U be a nonprincipal κ-complete ultrafilter on κ in $V[G]$. Let $J = \{X \subseteq \kappa : (\kappa\backslash X) \in U\}$. So J is a nonprincipal κ-complete maximal ideal on κ in $V[G]$. Let $\dot J$ be such that $\dot J^G = J$. Choose $p \in G$ such that $p \Vdash \dot J$ is a nonprincipal maximal ideal on κ. Let $I = \{X \subseteq \kappa : p \Vdash \check X \in \dot J\}$.

(1) I is a κ-complete ideal containing all singletons.

In fact, if $X \in I$ and $Y \subseteq X$, then $p \Vdash \check Y \subseteq \check X$, so $p \Vdash \check Y \in \dot J$, and hence $Y \in I$. If $\alpha < \kappa$ and $X_\xi \in I$ for all $\xi < \alpha$, then $\forall \xi < \alpha[p \Vdash \check X_\xi \in \dot J]$, and so $p \Vdash \bigcup_{\xi<\alpha} X_\xi \in \dot J$; hence $\bigcup_{\xi<\alpha} \check X_\xi \in I$. For any $\alpha < \kappa$, $p \Vdash \{\alpha\}\check{} \in \dot J$, so $\{\alpha\} \in I$. Thus (1) holds.

(2) I is $|P|^+$-saturated.

Suppose that $\langle [X_\xi] : \xi < |P|^+\rangle$ is a system of pairwise disjoint nonzero members of $\mathscr{P}(\kappa)/I$. Thus $\forall \xi < |P|^+[X_\xi \notin I]$ and $\forall \xi, \eta < |P|^+[\xi \neq \eta \to X_\xi \cap X_\eta \in I]$. So $\forall \xi < |P|^+[p \nVdash \check X_\xi \in \dot J]$ and $\forall \xi, \eta < |P|^+[\xi \neq \eta \to p \Vdash (X_\xi \cap X_\eta)\check{} \in J]$. Thus $\forall \xi < |P|^+(e(p) \cdot -||\check X_\xi \in \dot J||) \neq 0)$, and $\forall \xi, \eta < |P|^+(e(p) \cdot -||(X_\xi \cap X_\eta)\check{} \in \dot J||) = 0$. Now clearly for $\xi \neq \eta$,

$$(e(p) \cdot -||\check X_\xi \in \dot J||) \cdot (e(p) \cdot -||\check X_\eta \in \dot J||) \le (e(p) \cdot -||(X_\xi \cap X_\eta)\check{} \in \dot J||),$$

so this is a contradiction. Hence (2) holds.

Now by Lemma 21.6, κ is measurable in V. □

Theorem 21.8. *Let κ be an infinite cardinal and P a forcing poset such that $|P| < \kappa$. Let G be a V-generic filter on P.*

If κ is strongly compact in V then it is strongly compact in $V[G]$.

Proof. Assume that κ is strongly compact in V. Let $\lambda \ge \kappa$. We want to find a fine ultrafilter on $[\lambda]^{<\kappa}$ in $V[G]$.

Let U be a fine ultrafilter on $[\lambda]^{<\kappa}$ in V, and let j_U be the canonical elementary embedding of V into V^κ_U. Let $d(Z) = Z$ for all $Z \in [\lambda]^{<\kappa}$. By Lemma 20.16 we have:

(1) $X \in U$ iff $\pi([d]) \in j_U(X)$.

(2) $j_U[\lambda] \subseteq \pi([d])$.

This holds by the proof of Lemma 20.17.

Now we may assume wlog that $P \in V_\kappa$. Then by Lemma 17.17, $\forall p \in P[j_U(p) = p]$, $j_U(P) = P$, and $j_U(B(P)) = B(P)$. Clearly G is generic over V_U^κ. Now for any $x \in V[G]$, let $\dot{x}$ be a name such that $\dot{x}_{GV} = x$, and define

$$j'(x) = \dot{x}_{GV_U^\kappa}.$$

As in the proof of Theorem 21.1, j' is well-defined and is an elementary embedding of $V[G]$ into $V_U^\kappa[G]$. Now we define in $V[G]$, for any $X \in [\lambda]^{<\kappa}$,

$$X \in W \quad \text{iff} \quad \pi([d]) \in \dot{X}_{GV_U^\kappa}.$$

We claim that W is a fine ultrafilter on $[\lambda]^{<\kappa}$ in $V[G]$. Closure upwards: suppose that $X \in W$ and $X \subseteq Y \in [\lambda]^{<\kappa}$. Thus $\pi([d]) \in \dot{X}_{GV_U^\kappa}$. Now $\dot{X}_{GV} = X \subseteq Y = \dot{Y}_{GV}$. Choose $p \in G$ so that $p \Vdash \dot{X} \subseteq \dot{Y}$. Then $\pi([d]) \in \dot{X}_{GV_U^\kappa} \subseteq \dot{Y}_{GV_U^\kappa}$. So $Y \in W$. Now suppose that $H \in [\lambda]^{<\kappa}$ in $V[G]$. We want to show that $T \stackrel{\text{def}}{=} \{K \in [\lambda]^{<\kappa} : H \subseteq K\} \in W$. Now $\forall K \in [\lambda]^{<\kappa}[K \in T \text{ iff } H \subseteq K]$, so $\forall K[K \in j_U(T) \text{ iff } j_U(H) \subseteq K]$. Now $j_U(H) = \{j_U(\alpha) : \alpha \in H\} \subseteq j_U[\lambda] \subseteq \pi([d])$. Thus $\pi([d]) \in j_U(T)$, so $T \in W$. □

Theorem 21.9. *Let κ be a measurable cardinal. Then there is a generic extension in which $\mathrm{cf}(\kappa) = \omega$ and no cardinals are collapsed.*

Proof. Let κ be a measurable cardinal and let D be a normal ultrafilter on κ. Let P consist of all pairs (s, A) such that $s \in [\kappa]^{<\omega}$ and $A \in D$. We define $(s, A) \preceq (t, B)$ iff

(i) $t = s \cap \alpha$ for some α.
(ii) $A \subseteq B$.
(iii) $s \backslash t \subseteq B$.

(1) P is a forcing poset with greatest element $(\emptyset, \kappa)$.

For, clearly $(s, A) \preceq (\emptyset, \kappa)$ for all $(s, A) \in P$. Clearly $(s, A) \preceq (s, A)$. Suppose that $(u, C) \preceq (s, A) \preceq (t, B)$. So $C \subseteq A \subseteq B$. Say $t = s \cap \alpha$ and $s = u \cap \beta$. Note that $t \subseteq s \subseteq u$.

Case 1. $\alpha \leq \beta$. Then $u \cap \alpha = (u \cap \beta) \cap \alpha = s \cap \alpha = t$, as desired. Also, $u \backslash t = (u \backslash s) \cup (s \backslash t) \subseteq A \cup B = B$, as desired.

Case 9. $\beta < \alpha$. We have $s \subseteq \beta$ and so $t = s \cap \alpha = s$. Also, $t = s \cap \beta = u \cap \beta$, as desired. And $u \backslash t = u \backslash s \subseteq A \subseteq B$, as desired.

(2) If $(s, A), (s, B) \in P$, then $(s, A \cap B) \in P$ and $(s, A \cap B) \leq (s, A), (s, B)$.

In fact, clearly $A \cap B \in D$, so this is clear.

(3) If (s, A) and (t, B) are compatible, then s is an initial segment of t or t is an initial segment of s.

In fact, suppose that (s, A) and (t, B) are compatible. Say $(u, C) \leq (s, A), (t, B)$. Then s is an initial segment of u and t is an initial segment of u, so the conclusion is clear.

(4) All cardinals and cofinalities greater than κ are preserved.

In fact, P satisfies the κ^+-cc by (1), so this holds by Proposition 14.64.

(5) $\mathrm{cf}(\kappa) = \omega$ in $M[G]$.

In fact, by (2), $\bigcup_{(s,A)\in G} s$ is a subset of κ, and if $\alpha < \bigcup_{(s,A)\in G} s$, then there is an $(s, A) \in G$ such that $\alpha \in s$ and so $\alpha \cap \bigcup_{(s,A)\in G} s$ is finite. It follows that $\bigcup_{(s,A)\in G} s$ is countable. If $\alpha < \kappa$, then let $E = \{(s, A) \in P : \exists\beta \in s[\alpha < \beta]\}$. Then E is dense. For, suppose that $(t, B) \in P$. Now D is nonprincipal and κ-complete, so it is uniform. Thus $|B| = \kappa$. Choose $\beta \in B$ with β greater than α and each member of t. Let $s = t \cup \{\beta\}$. Then $(s, B) \in E$ and $(s, B) \leq (t, B)$, as desired. Since E is dense, $\bigcup_{(s,A)\in G} s$ is cofinal in κ. Thus (5) holds.

(6) Let σ be a sentence in the forcing language, and let (s_0, A_0) be a member of P. Then there is an $A \subseteq A_0$ such that $(s_0, A) \Vdash \sigma$ or $(s_0, A) \Vdash \neg\sigma$.

Let $A_1 = A_0\backslash(\max(s_0) + 1)$. Let

$$\begin{aligned}
S^+ &= \{t \in [\kappa]^{<\omega} : s_0 < t \wedge \exists A \subseteq A_1[A \in D \wedge (s_0 \cup t, A) \Vdash \sigma]\};\\
S^- &= \{t \in [\kappa]^{<\omega} : s_0 < t \wedge \exists A \subseteq A_1[A \in D \wedge (s_0 \cup t, A) \Vdash \neg\sigma]\};\\
T &= [\kappa]^{<\omega}\backslash(S^+ \cup S^-).
\end{aligned}$$

By Theorem 10.22 there is an $A \in D$ such that for every n, $[A]^n \subseteq S^+$ or $[A]^n \subseteq S^-$ or $[A]^n \subseteq T$. Then $(s_0, A) \Vdash \sigma$ or $(s_0, A) \Vdash \neg\sigma$. Otherwise there exist $(t, A') \leq (s_0, A)$ such that $(t, A') \Vdash \neg\sigma$ and $(u, A'') \leq (s_0, A)$ such that $(u, A'') \Vdash \sigma$. Now $t\backslash s_0 \subseteq A$ and $u\backslash s_0 \subseteq A$. Say $|t\backslash s_0| = m$ and $|u\backslash s_0| = n$ with $m \leq n$. Let t' end-extend t using elements of A'. Then $(s_0 \cup t', A') \leq (t, A')$, so $(s_0 \cup t', A') \Vdash \neg\sigma$. So $(s_0 \cup t', A_1 \cap A') \Vdash \neg\sigma$. Also $(u, A'' \cap A_1) \Vdash \sigma$. Then $t'\backslash s_0 \in S^-$ and $u\backslash s_0 \in S^+$, contradiction.

Now suppose in $M[G]$ that λ is a cardinal less than κ and $X \subseteq \lambda$. We want to show that $X \in M$. Let $\dot{X}$ be a name and $p_0 \in G$ such that $p_0 \Vdash \dot{X} \subseteq \check{\lambda}$. We claim

(7) $\forall p \leq p_0 \exists q \leq p \exists Z \subseteq \lambda[Z \in M$ and $q \Vdash \dot{X} = \check{Z}$.

To prove (7), suppose that $p \leq p_0$. Say $p = (s_0, A)$. By (5), for each $\alpha \in \lambda$ choose $A_\alpha \in D$ with $A_\alpha \subseteq A$ such that $(s_0, A_\alpha) \Vdash \check{\alpha} \in \dot{X}$ or $(s_0, A_\alpha) \Vdash \check{\alpha} \notin \dot{X}$. Let $B = \bigcap_{\alpha<\lambda} A_\alpha$. Then $B \in D$. Let $q = (s_0, B)$. Let $Z = \{\alpha < \lambda : q \Vdash \check{\alpha} \in \dot{X}\}$. We claim that $q \Vdash \dot{X} = \check{Z}$. For, suppose that $q \in H$ generic. If $\alpha \in Z$, then $\alpha \in \dot{X}^H$. Suppose that $\alpha \in \dot{X}^H$. Choose $r \in H$ so that $r \Vdash \check{\alpha} \in \dot{X}$. Say $s \in H$ with $s \leq q, r$. If $q \Vdash \check{\alpha} \notin \dot{X}$, then also $s \Vdash \check{\alpha} \notin \dot{X}$. But $s \leq r$ implies that $s \Vdash \check{\alpha} \in \dot{X}$. It follows that $q \Vdash \check{\alpha} \in \dot{X}$, and so $\alpha \in Z$. So the claim holds. This proves (7).

By (7), the set $D \stackrel{\text{def}}{=} \{q : \exists Z \in M[Z \subseteq \lambda$ and $q \Vdash \dot{X} \in \check{Z}]\}$ is dense below p_0. Choose $q \in D \cap G$. Then choose $Z \in M$ with $Z \subseteq \lambda$ such that $q \Vdash \dot{X} \in \check{Z}]\}$. Thus $X = Z$.

We have shown

(8) If $\lambda < \kappa$ and $A \subseteq \lambda$ in $M[G]$, then $A \in M$.

Now suppose that $\alpha < \lambda < \kappa$ and f is a function with domain α and range cofinal in λ, in $M[G]$. Let $B = \{(x, f(\xi)) : x < \alpha\}$. Let F be the order isomorphism of $\lambda \times \lambda$ onto λ. Let $A = F[B]$. Then $A \in M$ by the above, so also $B \in M$, hence $f \in M$

Hence a regular cardinal less than κ in M is also a regular cardinal in $M[G]$. It follows that M and $M[G]$ have the same cardinals. $\square$

22. Saturated ideals

Let κ be an uncountable regular cardinal. Let I be a κ-complete ideal on κ. Let $B = \mathscr{P}(\kappa)/I$. B is λ-*saturated* iff every disjoint subset of B has size less than λ. $\mathrm{sat}(B)$ is the least λ such that B is λ-saturated. I is λ-*saturated* iff B is, and $\mathrm{sat}(I) = \mathrm{sat}(B)$. A subset A of κ is an *atom* of I iff $A \notin I$ and there do not exist $B, C \notin I$ with $B \cap C = \emptyset$ and $A = B \cup C$.

Proposition 22.1. *If* $\mathrm{sat}(I)$ *is finite, then* κ *is the union of finitely many atoms of* I.

Proof. Assume that $\lambda \stackrel{\text{def}}{=} \mathrm{sat}(I)$ is finite.

Case 1. $I = \mathscr{P}(\kappa)$. Then $|B| = 1$ and I does not have any atoms. κ is the union of 0-many atoms of I.

Case 9. I is proper and B is finite. Then B is atomic. Let the atoms of B be $[A_0], \ldots, [A_{m-1}]$. We may assume that $\langle A_i : i < m\rangle$ is a partition of κ.

Case 3. I is proper and B is infinite. But every infinite BA has an infinite disjoint subset. So this case does not occur. □

Proposition 22.2. *If* $\mathrm{sat}(I)$ *is infinite, then it is uncountable and regular.*

Proof. Assume that $\mathrm{sat}(I)$ is infinite. Every infinite BA has an infinite disjoint subset, so $\mathrm{sat}(I)$ is uncountable. It is regular by the Erdös, Tarski theorem. □

Proposition 22.3. *If* $\lambda \le \kappa$, *then* I *is* λ*-saturated iff there is no disjoint collection* W *of size* λ *of subsets of* κ *such that* $\forall X \in W[X \notin I]$.

Proof. Suppose that $\lambda \le \kappa$. $\Rightarrow$ is clear. $\Leftarrow$: Suppose that I is not λ-saturated. Let $\langle A_\alpha : \alpha < \lambda\rangle$ be a system of subsets of κ such that $\forall \alpha < \lambda[A_\alpha \notin I]$ and $\forall \alpha, \beta < \lambda[\alpha \ne \beta \to A_\alpha \cap A_\beta \in I]$. For each $\beta < \lambda$ let $C_\beta = A_\beta \backslash \bigcup_{\alpha<\beta} A_\alpha$. Clearly the C_β's are pairwise disjoint. Suppose that $\beta < \lambda$ and $C_\beta \in I$. Then

$$A_\beta = C_\beta \cup \bigcup_{\alpha<\beta} (A_\beta \cap A_\alpha) \in I,$$

contradiction. □

Proposition 22.4. I *is* $(2^\kappa)^+$*-saturated.* □

Proposition 22.5. *If* I *is atomless, then* $\mathrm{sat}(I)$ *is a regular cardinal, and* $\aleph_1 \le \mathrm{sat}(I) \le (2^\kappa)^+$. □

Proposition 22.6. *If* μ *is a nontrivial* κ*-additive real-valued measure on* κ, *then the ideal* I_μ *of measure 0 sets is a* σ*-saturated* κ*-complete ideal on* κ. (see page 128 for the definition of σ-saturated ideal.)

Proof. $\forall \alpha \in \kappa[\{\alpha\} \in I]$ by the definition of nontrivial measure. If W is an uncountable disjoint collection of sets each of positive measure, then $W = \bigcup_{n\in\omega}\{X \in W :$

$\frac{1}{n+1} \leq \mu(X)\}$, so there is an $n \in \omega$ such that $\{X \in W : \frac{1}{n+1} \leq \mu(X)\}$ is uncountable, contradiction. $\square$

Lemma 22.7. *If I is atomless, then $\kappa \leq 2^{\aleph_0}$.*

Proof. In the proof in these notes of Lemma 10.9(ii), (1) must hold, as otherwise Z is an atom. Namely, suppose that $\mathscr{P}(Z) \cap I$ is a maximal ideal in Z. If $Z = W' \cup W''$ with $W', W'' \notin I$ and $W' \cap W'' = \emptyset$, we contradict $\mathscr{P}(Z) \cap I$ being a maximal ideal. So (1) holds. Hence by the formulation in that proof, $\kappa \leq 2^{\aleph_0}$. $\square$

Lemma 22.8. *If I is a κ-complete σ-saturated ideal on κ and $2^\omega < \kappa$, then for every $X \notin I$ there is an atom $A \subseteq X$.*

Proof. Suppose that $X \notin I$ and there is no atom below X. Let $I' = \mathscr{P}(X) \cap I$. Then I' is a κ-complete σ-saturated ideal on X. If $Z \subseteq X$ and $\mathscr{P}(Z) \cap I'$ is a maximal ideal on Z, note that $\mathscr{P}(Z) \cap I' = \mathscr{P}(Z) \cap I$, so that Z is an atom below X in the sense of I, contradiction. It follows that $\kappa = |X| \leq 2^\omega$, contradiction. $\square$

Lemma 22.9. *Let I be a κ-complete σ-saturated ideal on κ. If $2^{\aleph_0} < \kappa$, then there is a countable disjoint collection W of atoms of I such that $\kappa = \bigcup_{A \in W} A$.*

Proof. Let W be a maximal disjoint collection of atoms of I. Since I is σ-saturated, W is countable. Let $B = \bigcup_{X \in W} X$. If $\kappa \backslash B \notin I$, we can use Lemma 22.8 to extend W, contradiction. So $\kappa \backslash B \in I$. We can add $\kappa \backslash B$ to any member of W to obtain the desired set. $\square$

A real-valued measure μ is *normal* iff I_μ is normal.

Lemma 22.10. *If I is a σ-saturated κ-complete ideal on an uncountable cardinal κ, then there is a function $f : \kappa \to \kappa$ such that*

$$J \stackrel{\text{def}}{=} f_*(I) = \{X \subseteq \kappa : f^{-1}[X] \in I\}$$

is a normal σ-saturated κ-complete ideal on κ.

Proof. Clearly every bounded subset of κ is in I. Let S be a set of positive measure. A function $f : S \to \kappa$ is *unbounded* iff there is no $\gamma < \kappa$ and no $T \subseteq S$ of positive measure such that $\forall \alpha \in T[f(\alpha) < \gamma]$. Thus f is unbounded iff

$$\forall \gamma < \kappa \forall T \subseteq S[T \notin I \to \exists \alpha \in T[\gamma \leq f(\alpha)]]. \tag{1}$$

Equivalently, f is unbounded iff

$$\forall \gamma < \kappa \forall T \subseteq S[T \notin I \to \exists \alpha \in T[\gamma < f(\alpha)]]. \tag{2}$$

Let $\mathscr{F}$ be the set of all functions g mapping into κ defined on a set of positive measure and unbounded on its domain. We define $g < h$ iff $g, h \in \mathscr{F}$, $\text{dmn}(g) \subseteq \text{dmn}(h)$ and $\forall \alpha \in \text{dmn}(g)[g(\alpha) < h(\alpha)]$. We call $g \in \mathscr{F}$ *minimal* iff there is no $h \in \mathscr{F}$ such that $h < g$.

(3) $\mathscr{F} \neq \emptyset$.

In fact, let $f(\alpha) = \alpha$ for all $\alpha \in \kappa$. To show that f is unbounded on κ, suppose that $\gamma < \kappa$ and $T \subseteq \kappa$ is not in I. Then T is unbounded, so there is an $\alpha \in T \backslash (\gamma + 1)$. Then $\gamma < \alpha = f(\alpha)$. Thus $f \in \mathscr{F}$.

(4) There is a minimal $g \in \mathscr{F}$.

For, suppose not. Let $g \in \mathscr{F}$ be arbitrary. Let W be a maximal collection of elements of $\mathscr{F}$ such that $\forall h \in W[h < g]$ and $\forall h_1, h_2 \in W[h_1 \neq h_2 \rightarrow \mathrm{dmn}(h_1) \cap \mathrm{dmn}(h_2) \in I]$. Since I is σ-saturated, W is countable.

(5) $(\mathrm{dmn}(g) \backslash \bigcup_{h \in W} \mathrm{dmn}(h)) \in I$.

For, suppose not. Let $g' = g \restriction (\mathrm{dmn}(g) \backslash \bigcup_{h \in W} \mathrm{dmn}(h))$. We claim that g' is unbounded on its domain. For, suppose that $T \subseteq \mathrm{dmn}(g')$, $T \notin I$, and $\gamma < \kappa$. Choose $\alpha \in T$ with $\gamma \leq g(\alpha)$. Clearly $\alpha \in \mathrm{dmn}(g')$. So $g' \in \mathscr{F}$. Take $k \in \mathscr{F}$ with $k < g'$. This contradicts the maximality of W.

Let $f = \bigcup_{h \in W} h$. Then clearly f is defined on a set of positive measure. Now suppose that $\gamma < \kappa$ and $J \subseteq \mathrm{dmn}(f)$ is not in I. Then $J = \bigcup_{h \in W} (J \cap \mathrm{dmn}(h))$. Since $\{\mathrm{dmn}(h) : h \in W\}$ is countable, by the κ-completeness of I it follows that there is a $h \in W$ such that $J \cap \mathrm{dmn}(h) \notin I$. Since h is unbounded, there is an $\alpha \in J \cap \mathrm{dmn}(h)$ such that $\gamma < h(\alpha) \leq f(\alpha)$. This shows that $f \in \mathscr{F}$.

Repeating this construction we obtain $g_0 > g_1 > g_2 > \cdots$, each $g_i \in \mathscr{F}$ and $\mathrm{dmn}(g_i) \backslash \mathrm{dmn}(g_{i+1}) \in I$ for each i. Since κ is uncountable and I is κ-complete, it follows that $\bigcup_{i \in \omega} (\mathrm{dmn}(g_i) \backslash \mathrm{dmn}(g_{i+1})) \in I$. Now

$$\mathrm{dmn}(g_0) = \bigcup_{i \in \omega} (\mathrm{dmn}(g_i) \backslash \mathrm{dmn}(g_{i+1})) \cup \bigcap_{i \in \omega} \mathrm{dmn}(g_i).$$

Hence $\bigcap_{i \in \omega} \mathrm{dmn}(g_i) \notin I$. Taking any $\alpha \in \bigcap_{i \in \omega} \mathrm{dmn}(g_i)$, we have $g_0(\alpha) > g_1(\alpha) > \cdots$, contradiction. This proves (4).

By the same argument, for every $h \in \mathscr{F}$ there is a minimal $g \in \mathscr{F}$ such that $g < h$ or $g = h$. Let W be a maximal collection of minimal members of $\mathscr{F}$ such that $\mathrm{dmn}(g_1) \cap \mathrm{dmn}(g_2)$ has measure 0 for all distinct $g_1, g_2 \in W$.

(6) $\left(\kappa \backslash \bigcup_{g \in W} \mathrm{dmn}(g)\right) \in I$.

In fact, otherwise let $D = \kappa \backslash \bigcup_{g \in W} \mathrm{dmn}(g)$, and let h have domain D with $h(\alpha) = \alpha$ for all $\alpha \in D$. Then h is unbounded on D, for if $\gamma < \kappa$ and $T \subseteq D$ with $T \notin I$, choose $\alpha \in T$ with $\gamma < \alpha$. Then $\gamma < h(\alpha)$, as desired. Thus $h \in \mathscr{F}$. There is a minimal $k < h$, or $k = h$. Hence $W \cup \{h\}$ contradicts the maximality of W. So (6) holds.

Let $f = \bigcup W$. Then f is unbounded. For, suppose that $\gamma < \kappa$, $T \subseteq \mathrm{dmn}(f)$, and $T \notin I$. Now $T = \bigcup_{g \in W} (\mathrm{dmn}(g) \cap T)$ and W is countable, so there is a $g \in W$ such that $(\mathrm{dmn}(g) \cap T) \notin I$. Hence there is an $\alpha \in \mathrm{dmn}(g) \cap T$ such that $\gamma \leq g(\alpha) = f(\alpha)$. Clearly f is defined on a set of positive measure, so $f \in \mathscr{F}$. In fact, $\mathrm{dmn}(f)$ has measure 1.

(7) if $h \in \mathscr{F}$, then $\{\alpha \in \mathrm{dmn}(h) \cap \mathrm{dmn}(f) : h(\alpha) < f(\alpha)\} \in I$.

In fact, suppose not. Let $k = h \restriction \{\alpha \in \mathrm{dmn}(h) \cap \mathrm{dmn}(f) : h(\alpha) < f(\alpha)\}$. To show that k is unbounded on its domain, suppose that $\gamma < \kappa$ and $T \subseteq \mathrm{dmn}(k)$ with $T \notin I$. Choose $\alpha \in T$ such that $\gamma \le h(\alpha)$. Then $\gamma \le k(\alpha)$, as desired. Let $k' \in \mathscr{F}$ be minimal such that $k' < k$ or $k' = k$.

(8) $\exists g \in W[\mathrm{dmn}(k') \cap \mathrm{dmn}(g) \notin I]$.

In fact, otherwise $\mathrm{dmn}(k') \cap \bigcup_{g\in W} \mathrm{dmn}(g) \in I$, and so by (6), $\mathrm{dmn}(k') \in I$, contradiction.

We choose g as in (8). Then for any $\alpha \in \mathrm{dmn}(k') \cap \mathrm{dmn}(g)$ we have $k'(\alpha) \le k(\alpha) = h(\alpha) < f(\alpha) = g(\alpha)$. Let $l = k' \restriction \mathrm{dmn}(k') \cap \mathrm{dmn}(g)$. Then clearly $l \in \mathscr{F}$ and $l < g$, contradicting the minimality of g. This proves (7).

Now let $f' = f \cup \{(\alpha, 0) : \alpha \in (\kappa \backslash \bigcup_{g\in W} \mathrm{dmn}(g))\}$. Then f' is unbounded. For, suppose that $T \subseteq \kappa$, $\gamma < \kappa$, and $T \notin I$. Clearly then $T \cap \bigcup_{g\in W} \mathrm{dmn}(g) \notin I$. Now $T = \bigcup_{g\in W}(\mathrm{dmn}(g) \cap T)$ and W is countable, so there is a $g \in W$ such that $(\mathrm{dmn}(g) \cap T) \notin I$. Hence there is an $\alpha \in \mathrm{dmn}(g) \cap T$ such that $\gamma \le g(\alpha) = f(\alpha) = f'(\alpha)$. So f' is unbounded, and hence $f' \in \mathscr{F}$.

(9) if $h \in \mathscr{F}$, then $\{\alpha \in \mathrm{dmn}(h) : h(\alpha) < f'(\alpha)\} \in I$.

In fact, $\{\alpha \in \mathrm{dmn}(h) : h(\alpha) < f'(\alpha)\} = \{\alpha \in \mathrm{dmn}(h) \cap \mathrm{dmn}(f) : h(\alpha) < f(\alpha)\}$, so (9) follows from (7).

Now let $J = \{X : f'^{-1}[X] \in I\})$ for any $X \subseteq \kappa$. If $Y \subseteq X \in J$, then $f'^{-1}[Y] \subseteq f'^{-1}[X] \in I$, so $Y \in J$. If $\alpha < \kappa$ and $X \in {}^{\alpha}J$, then

$$f'^{-1}\left[\bigcup_{\xi<\alpha} X_\xi\right] = \bigcup_{\xi<\alpha} f'^{-1}[X_\xi] \in I,$$

so $\bigcup_{\xi<\alpha} X_\xi \in J$. Thus J is a κ-complete ideal on κ. If $\gamma < \kappa$, then $f'^{-1}[\{\gamma\}] \in I$ since f' is unbounded, so $\{\gamma\} \in J$.

Now suppose that $\langle X_i : i \in K\rangle$ is a system of pairwise disjoint sets not in J. Then $\langle f'^{-1}[X_i] : i \in K\rangle$ is a system of pairwise disjoint sets not in I, so K is countable. Thus J is σ-saturated.

To show that J is normal, suppose that $S \notin J$ and g is a function with domain S such that $g(\alpha) < \alpha$ for all $\alpha \in S$. Thus if we let $T \stackrel{\mathrm{def}}{=} f'^{-1}[S]$, then $T \notin I$. Define $\mathrm{dmn}(g') = T$, $g'(\alpha) = g(f'(\alpha))$. Then $\forall \alpha \in T[g'(\alpha) < f'(\alpha)]$. From (9) it follows that $g' \notin \mathscr{F}$. Thus there exist $\gamma < \kappa$ and $T' \subseteq T$ such that $T' \notin I$ and $\forall \alpha \in T'[g'(\alpha) < \gamma]$. Now

$$T' = \bigcup_{\delta<\gamma} \{\alpha \in T' : g'(\alpha) = \delta\}.$$

Since $T' \notin I$ and I is κ-complete, it follows that there is a $\delta < \gamma$ such that $T'' \stackrel{\mathrm{def}}{=} \{\alpha \in T' : g'(\alpha) = \delta\}$ is not in I. Thus $g' \restriction T''$ is constant. Let $S' = f[T'']$. Then $T'' \subseteq f^{-1}[S']$ and $T'' \notin I$, so $S' \notin J$. Take any $\alpha \in S'$. Say $\alpha = f(\beta)$ with $\beta \in T''$. Then $g(\alpha) = g(f(\beta)) = g'(\beta) = \delta$.

This completes the proof. $\square$

Lemma 22.11. *If μ is a κ-additive real-valued measure on κ, then there is a function $f : \kappa \to \kappa$ such that*

$$\nu(X) = \mu(f^{-1}[X]) \quad \forall X \subseteq \kappa$$

defines a normal κ-additive real-valued measure on κ.

Proof. Clearly every bounded subset of κ has measure 0.. Let S be a set of positive measure. A function $f : S \to \kappa$ is *unbounded* iff there is no $\gamma < \kappa$ and no $T \subseteq S$ of positive measure such that $\forall \alpha \in T[f(\alpha) < \gamma]$. Thus f is unbounded iff

$$\forall \gamma < \kappa \forall T \subseteq S[\mu(T) > 0 \to \exists \alpha \in T[\gamma \leq f(\alpha)]]. \tag{1}$$

Equivalently, f is unbounded iff

$$\forall \gamma < \kappa \forall T \subseteq S[\mu(T) > 0 \to \exists \alpha \in T[\gamma < f(\alpha)]]. \tag{2}$$

Let $\mathscr{F}$ be the set of all functions g mapping into κ defined on a set of positive measure and unbounded on its domain. We define $g < h$ iff $\mathrm{dmn}(g) \subseteq \mathrm{dmn}(h)$ and $\forall \alpha \in \mathrm{dmn}(g)[g(\alpha) < h(\alpha)]$. We call $g \in \mathscr{F}$ *minimal* iff there is no $h \in \mathscr{F}$ such that $h < g$.

(3) $\mathscr{F} \neq \emptyset$.

In fact, let $f(\alpha) = \alpha$ for all $\alpha \in \kappa$. To show that f is unbounded on κ, suppose that $\gamma < \kappa$ and $T \subseteq \kappa$ has positive measure. Then T is unbounded, so there is an $\alpha \in T \backslash (\gamma + 1)$. Then $\gamma < \alpha = f(\alpha)$. Thus $f \in \mathscr{F}$.

(4) There is a minimal $g \in \mathscr{F}$.

For, suppose not. Let $g \in \mathscr{F}$ be arbitrary. Let W be a maximal collection of elements of $\mathscr{F}$ such that $\forall h \in W[h < g]$ and $\forall h_1, h_2 \in W[h_1 \neq h_2 \to \mathrm{dmn}(h_1) \cap \mathrm{dmn}(h_2) = \emptyset]$. Since $\forall k \in \mathscr{F}[\mu(\mathrm{dmn}(k)) > 0]$, W is countable.

(5) $\mu((\mathrm{dmn}(g) \backslash \bigcup_{h \in W} \mathrm{dmn}(h)) = 0$.

For, suppose not. Let $g' = g \restriction (\mathrm{dmn}(g) \backslash \bigcup_{h \in W} \mathrm{dmn}(h))$. Clearly $g' \in \mathscr{F}$. Take $k \in \mathscr{F}$ with $k < g'$. This contradicts the maximality of W.

Let $f = \bigcup_{h \in W} h$. Then clearly f is defined on a set of positive measure. Now suppose that $\gamma < \kappa$ and $J \subseteq \mathrm{dmn}(f)$ has positive measure. Then $J = \bigcup_{h \in W}(J \cap \mathrm{dmn}(h))$. Since $\{\mathrm{dmn}(h) : h \in W\}$ is countable, by the κ-completeness of μ it follows that there is a $h \in W$ such that $\mu(J \cap \mathrm{dmn}(h)) > 0$. Since h is unbounded, there is an $\alpha \in J \cap \mathrm{dmn}(h)$ such that $\gamma < h(\alpha) \leq f(\alpha)$. This shows that $f \in \mathscr{F}$.

Repeating this construction we obtain $g_0 > g_1 > g_2 > \cdots$, each $g_i \in \mathscr{F}$ and $\mu(\mathrm{dmn}(g_i) \backslash \mathrm{dmn}(g_{i+1})) = 0$ for each i. Since κ is uncountable and μ is κ-complete, it follows that $\mu(\bigcup_{i \in \omega}(\mathrm{dmn}(g_i) \backslash \mathrm{dmn}(g_{i+1}))) = 0$. Now

$$\mathrm{dmn}(g_0) = \bigcup_{i \in \omega}(\mathrm{dmn}(g_i) \backslash \mathrm{dmn}(g_{i+1})) \cup \bigcap_{i \in \omega} \mathrm{dmn}(g_i).$$

Hence $\mu(\bigcap_{i \in \omega} \mathrm{dmn}(g_i)) > 0$. Taking any $\alpha \in \bigcap_{i \in \omega} \mathrm{dmn}(g_i)$, we have $g_0(\alpha) > g_1(\alpha) > \cdots$, contradiction. This proves (4).

By the same argument, for every $h \in \mathscr{F}$ there is a minimal $g \in \mathscr{F}$ such that $g < h$ or $g = h$. Let W be a maximal collection of minimal members of $\mathscr{F}$ such that $\mathrm{dmn}(g_1) \cap \mathrm{dmn}(g_2) = \emptyset$ for all distinct $g_1, g_2 \in W$.

(6) $\mu\left(\kappa \backslash \bigcup_{g \in W} \mathrm{dmn}(g)\right) = 0$.

In fact, otherwise let $D = \kappa \backslash \bigcup_{g \in W} \mathrm{dmn}(g)$, and let h have domain D with $h(\alpha) = \alpha$ for all $\alpha \in D$. Then h is unbounded on D, for if $\gamma < \kappa$ and $T \subseteq D$ with $\mu(T) > 0$, choose $\alpha \in T$ with $\gamma < \alpha$. Then $\gamma < h(\alpha)$, as desired. Thus $h \in \mathscr{F}$. There is a minimal $k < h$, or $k = h$. Hence $W \cup \{h\}$ contradicts the maximality of W. So (6) holds.

Let $f = \bigcup W$. Then f is unbounded. For, suppose that $\gamma < \kappa$, $T \subseteq \mathrm{dmn}(f)$, and $\mu(T) > 0$. Now $T = \bigcup_{g \in W} (\mathrm{dmn}(g) \cap T)$ and W is countable, so there is a $g \in W$ such that $\mu(\mathrm{dmn}(g) \cap T) > 0$. Hence there is an $\alpha \in \mathrm{dmn}(g) \cap T$ such that $\gamma \leq g(\alpha) = f(\alpha)$. Clearly f is defined on a set of positive measure, so $f \in \mathscr{F}$. In fact, $\mathrm{dmn}(f)$ has measure 1.

(7) if $h \in \mathscr{F}$, then $\mu(\{\alpha \in \mathrm{dmn}(h) \cap \mathrm{dmn}(f) : h(\alpha) < f(\alpha)\}) = 0$.

In fact, suppose not. Let $k = h \restriction \{\alpha \in \mathrm{dmn}(h) \cap \mathrm{dmn}(f) : h(\alpha) < f(\alpha)\}$. Clearly $k \in \mathscr{F}$. Let $k' \in \mathscr{F}$ be minimal such that $k' < k$ or $k' = k$.

(8) $\exists g \in W[\mu(\mathrm{dmn}(k') \cap \mathrm{dmn}(g)) > 0]$.

In fact, otherwise $\mu(\mathrm{dmn}(k') \cap \bigcup_{g \in W} \mathrm{dmn}(g)) = 0$, and so by (6), $\mu(\mathrm{dmn}(k')) = 0$, contradiction.

We choose g as in (8). Then for any $\alpha \in \mathrm{dmn}(k') \cap \mathrm{dmn}(g)$ we have $k'(\alpha) \leq k(\alpha) = h(\alpha) < f(\alpha) = g(\alpha)$. Let $l = k' \restriction \mathrm{dmn}(k') \cap \mathrm{dmn}(g)$. Then clearly $l \in \mathscr{F}$ and $l < g$, contradicting the minimality of g. This proves (7).

Now let $f' = f \cup \{(\alpha, 0) : \alpha \in (\kappa \backslash \bigcup_{g \in W} \mathrm{dmn}(g))\}$. Then f' is unbounded. For, suppose that $T \subseteq \kappa$, $\gamma < \kappa$, and $\mu(T) > 0$. Clearly then $\mu(T \cap \bigcup_{g \in W} \mathrm{dmn}(g)) > 0$. Now $T = \bigcup_{g \in W} (\mathrm{dmn}(g) \cap T)$ and W is countable, so there is a $g \in W$ such that $(\mathrm{dmn}(g) \cap T) \notin I$. Hence there is an $\alpha \in \mathrm{dmn}(g) \cap T$ such that $\gamma \leq g(\alpha) = f(\alpha) = f'(\alpha)$. So f' is unbounded, and hence $f' \in \mathscr{F}$.

(9) if $h \in \mathscr{F}$, then $\mu(\{\alpha \in \mathrm{dmn}(h) : h(\alpha) < f'(\alpha)\}) = 0$.

In fact, $\{\alpha \in \mathrm{dmn}(h) : h(\alpha) < f'(\alpha)\} = \{\alpha \in \mathrm{dmn}(h) \cap \mathrm{dmn}(f) : h(\alpha) < f(\alpha)\}$, so (9) follows from (7).

Now let $\nu(X) = \mu(f'^{-1}[X])$ for any $X \subseteq \kappa$. Clearly $\nu(\emptyset) = \emptyset$ and $\nu(\kappa) = 1$. If $X \subseteq Y$, then $\nu(X) \leq \nu(Y)$. For each $\alpha \in \kappa$, $\nu(\{\alpha\}) = \mu(f'^{-1}[\{\alpha\}] = \mu(\{X : f'(X) = \alpha\})$. Now $f' \in \mathscr{F}$, so f' is unbounded. If $\mu(\{X : f'(X) = \alpha\}) > 0$, then there is an X with $f'(X) = \alpha$ but also $f'(X) > \alpha$, contradiction. So $\nu(\{\alpha\}) = 0$. If $\alpha < \kappa$ and $\langle X_\xi : \xi \in \alpha \rangle$ is a system of pairwise disjoint subsets of κ, then so is $\langle f'^{-1}[X_\xi] : \xi \in \alpha \rangle$, and so

$$\nu\left(\bigcup_{\xi \in \alpha} X_n\right) = \mu\left(f'^{-1}\left[\bigcup_{\xi \in \alpha} X_\xi\right]\right)$$

$$
\begin{aligned}
&= \mu\left(\bigcup_{\xi\in\alpha} f'^{-1}[X_\xi]\right)\\
&= \sum_{\xi\in\alpha} \mu(f'^{-1}[X_\xi])\\
&= \sum_{\xi\in\alpha} \nu(X_\xi).
\end{aligned}
$$

Thus ν is a κ-additive measure on κ.

To show that ν is normal, let $I_\nu = \{X \subseteq \kappa : \nu(X) = 0\}$; we want to show that I_ν is normal. Suppose that $S \notin I_\nu$ and g is a function with domain S such that $g(\alpha) < \alpha$ for all $\alpha \in S$. Thus $T \stackrel{\text{def}}{=} f'^{-1}[S]$ has positive μ-measure. Define $\mathrm{dmn}(g') = T$, $g'(\alpha) = g(f'(\alpha))$. Then $\forall\alpha \in T[g'(\alpha) < f'(\alpha)]$. From (9) it follows that $g' \notin \mathscr{F}$. Thus there exist $\gamma < \kappa$ and $T' \subseteq T$ such that $\mu(T') > 0$ and $\forall\alpha \in T'[g'(\alpha) < \gamma]$. Now

$$
T' = \bigcup_{\delta<\gamma} \{\alpha \in T' : g'(\alpha) = \delta\}.
$$

Since $\mu(T') > 0$ and μ is κ-complete, it follows that there is a $\delta < \gamma$ such that $T'' \stackrel{\text{def}}{=} \{\alpha \in T' : g'(\alpha) = \delta\}$ has positive μ-measure. Thus $g' \restriction T''$ is constant. Let $S' = f[T'']$. Then $T'' \subseteq f^{-1}[S']$ and $\mu(T'') > 0$, so $S' \notin I_\nu$. Take any $\alpha \in S'$. Say $\alpha = f(\beta)$ with $\beta \in T''$. Then $g(\alpha) = g(f(\beta)) = g'(\beta) = \delta$.

This completes the proof. □

Lemma 22.12. *Let I be a normal σ-saturated κ-complete ideal on κ. Suppose that S is a set of postive measure and $f : S \to \kappa$ is regressive. Then $\exists\gamma < \kappa[\{\alpha \in S : f(\alpha) \geq \gamma\} \in I]$.*

Proof. By Exercise 8.8, for every $X \subseteq S$ of positive measure there exists $Y \subseteq X$ of positive measure such that f is constant on Y. Let W be a maximal disjoint family of sets $X \subseteq S$ of positive measure such that f is constant on X. Let $T = \bigcup_{X\in W} X$. The family W is countable, and so there is a $\gamma < \kappa$ such that $\forall\alpha \in T[f(\alpha) < \gamma]$. We claim that $A \stackrel{\text{def}}{=} \{\alpha \in S : f(\alpha) \geq \gamma\} \in I$. Suppose not. Now f is regressive on A, so it is constant on some U of positive measure. By maximality of W there is an $X \in W$ such that $X \cap U \neq \emptyset$. If $X \cap U \in I$, then $U\backslash X \notin I$, and the maximality of W is contradicted. So $X \cap U \notin I$. But then the constant value taken on X is the same as the constant value taken on U, contradicting the choice of γ. □

Lemma 22.13. *If κ carries a σ-saturated κ-additive ideal, then κ is weakly Mahlo.*

Proof. By Corollary 10.15, κ is regular limit. By Lemma 22.10 let I be a normal σ-saturated κ-additive ideal on κ.

(1) If $C \subseteq \kappa$ is club, then $\kappa\backslash C \in I$.

For, let $F = \{X \subseteq \kappa : \kappa\backslash X \in I\}$. Thus F is a normal filter on κ. Since I contains all $\alpha < \kappa$, F contains all final segments of κ. Hence by Lemma 8.11, F contains all clubs. So (1) holds.

(2) It suffices to show $\kappa\backslash\text{reg} \in I$, where reg is the set of all regular cardinals less than κ.

In fact, if this is true and C is club in κ, then by (1),

$$\kappa\backslash(\text{reg} \cap C) = (\kappa\backslash\text{reg}) \cup (\kappa\backslash C) \in I,$$

and hence $\text{reg} \cap C \neq \emptyset$. So (2) holds.

So, suppose that $\kappa\backslash\text{reg} \notin I$.

(3) $D \overset{\text{def}}{=} \{\alpha < \kappa : \alpha \text{ is limit and } \text{cf}(\alpha) < \alpha\} \notin I$.

In fact, suppose that $D \in I$. Now $E \overset{\text{def}}{=} \{\alpha < \kappa : \alpha \text{ is not a cardinal}\} \in I$ by (1), so $D \cup E \in I$. Now $\kappa\backslash(D \cup E) = \text{reg}$, so this contradicts our supposition.

Now by the normality of I and exercise 8.8 there exist a $T \subseteq D$ with $T \notin I$ and a $\lambda < \kappa$ such that $\text{cf}(\alpha) = \lambda$ for all $\alpha \in T$. For each $\alpha \in T$ let $\langle \beta^\alpha_\nu : \nu < \lambda\rangle$ be a strictly increasing sequence with supremum α. For each $\nu < \lambda$ the function $\langle \beta^\alpha_\nu : \alpha \in T\rangle$ is regressive on T, and so by Lemma 22.12 there is a $\gamma_\nu < \kappa$ such that $\{\alpha \in T : \beta^\alpha_\nu \geq \gamma_\nu\} \in I$. Let $\delta = \sup\{\gamma_\nu : \nu \in \lambda\}$. Since $\lambda < \kappa$ and I is κ-complete, it follows that $\bigcup_{\nu\in\lambda}\{\alpha \in T : \beta^\alpha_\nu \geq \gamma_\nu\} \in I$. Now

$$\forall \nu < \lambda[\{\alpha \in T : \beta^\alpha_\nu \geq \delta\} \subseteq \bigcup_{\nu\in\lambda}\{\alpha \in T : \beta^\alpha_\nu \geq \gamma_\nu\}]. \tag{4}$$

In fact, if $\nu < \lambda$, $\alpha \in T$, and $\beta^\alpha_\nu \geq \delta$, then $\beta^\alpha_\nu \geq \gamma_\nu$. So (4) holds.

Therefore, $\forall \nu < \lambda[\{\alpha \in T : \beta^\alpha_\nu \geq \delta\} \in I]$. Hence $\bigcup_{\nu<\lambda}\{\alpha \in T : \beta^\alpha_\nu \geq \delta\} \in I$. Clearly $\{\alpha \in T : \alpha \geq \delta\} \subseteq \bigcup_{\nu<\lambda}\{\alpha \in T : \beta^\alpha_\nu \geq \delta\}$, so $\{\alpha \in T : \alpha \geq \delta\} \in I$. Now $T \subseteq \delta \cup \{\alpha \in T : \alpha \geq \delta\}$, so $T \in I$, contradiction. □

Lemma 22.14. *Let I be a normal σ-saturated κ-complete ideal on κ. Assume that $\gamma < \kappa$ and $f : [\kappa]^{<\omega} \to \gamma$. Then there is an $H \subseteq \kappa$ with $\kappa\backslash H \in I$ such that $f[[H]^{<\omega}]$ is countable.*

Proof. It suffices to show that for each positive n there is an $H_n \subseteq \kappa$ with $\kappa\backslash H_n \in I$ such that $f[[H_n]^n]$ is countable. Then $\kappa\backslash\bigcap_{n\in\omega\backslash 1} H_n = \bigcup_{n\in\omega\backslash 1}(\kappa\backslash H_n) \in I$, and for all $m \in \omega$, $f[[\bigcap_{n\in\omega} H_n]^m] \subseteq f[[H_m]^m]$ is countable.

We prove this by induction on n. For $n = 1$, suppose that $f : \kappa \to \gamma$.

(1) There is an $H \subseteq \kappa$ with $H \notin I$ such that f is constant on H.

In fact, f is regressive on $\kappa\backslash\gamma$ and $\gamma \in I$, so by exercise 8.8 there is an $H \subseteq \kappa\backslash\gamma$ such that $H \notin I$ and f is constant on H. So (1) holds.

Let W be a maximal collection of pairwise disjoint family of subsets of κ, each not in I, and such that f is constant on each of them. Then W is countable by σ-saturation, so $f[\bigcup W]$ is countable. If $\kappa\backslash\bigcup W \notin I$, then we could get a $Y \subseteq \kappa\backslash\bigcup W$ with f constant on Y and $Y \notin I$, contradicting the maximality of W. So $\kappa\backslash\bigcup W \in I$. This proves our statement for $n = 1$.

Now we assume the statement for n. Suppose that $f : [\kappa]^{n+1} \to \gamma$. For each $\alpha < \kappa$ define $f_\alpha : [\kappa\backslash\{\alpha\}]^n \to \gamma$ by setting $f_\alpha(x) = f(x \cup \{\alpha\})$. By the inductive hypothesis, for each $\alpha < \kappa$ we get X_α with $\kappa\backslash X_\alpha \in I$ and $f_\alpha[[X_\alpha]^n]$ countable. For each $\alpha < \kappa$ let $A_\alpha = f_\alpha[[X_\alpha]^n]$. Let X be the diagonal intersection

$$X = \left\{ \alpha < \kappa : \alpha \in \bigcap_{\xi<\alpha} X_\xi \right\}.$$

Now each X_α is in the filter dual to I, so this diagonal intersection is in that filter also. Thus $\kappa\backslash X \in I$. Now if $\alpha < \alpha_1 < \cdots < \alpha_n$ with each $\alpha_i \in X$, then $\{\alpha_1, \ldots, \alpha_n\} \in [X_\alpha]^n$. Hence $f(\{\alpha, \alpha_1, \ldots, \alpha_n\}) = f_\alpha(\{\alpha_1, \ldots, \alpha_n\}) \in A_\alpha$.

For each $\alpha < \kappa$ let $A_\alpha = \{a_{\alpha n} : n \in \omega\}$. For each $n \in \omega$ define $g_n : X \to \gamma$ by $g_n(\alpha) = a_{\alpha n}$. Then there is a set $H_n \subseteq X$ such that $\kappa\backslash H_n \in I$ and $g_n[H_n]$ is countable. Let $H = \bigcap_{n\in\omega} H_n$. Then $\kappa\backslash H \in I$ and $\bigcup\{A_\alpha : \alpha \in H\} = \bigcup_{n\in\omega} g_n[H]$ is countable. Hence $f[[H]^{n+1}]$ is countable. □

Now suppose that M is a ctm of ZFC. Let κ be an uncountable regular cardinal in M, and let I be an ideal on κ. Let $P = \{X \subseteq \kappa : X \in M, X \notin I\}$ with $\subseteq$ as the ordering. Suppose that G is P-generic over M. We now use the notion of M-ultrafilter defined before Lemma 18.33.

Lemma 22.15. *G is an M-ultrafilter on κ.*

Proof. We only need to show that if $X \subseteq \kappa$ with $X \in M$ then $X \in G$ or $(\kappa\backslash X) \in G$. Let $D = \{Y \subseteq \kappa : Y \in M$ and $y \subseteq X$ or $Y \subseteq (\kappa\backslash X)$. Then D is dense in P, since if $Z \in P$, then $Z \notin I$ and hence $Z \cap X \notin I$ or $Z\backslash X \notin I$.

Lemma 22.16. *If $X \in I$, then $(\kappa\backslash X) \in G$.*

Proof. Let $D = \{Y \subseteq \kappa : Y \cap X = \emptyset\}$. We claim that D is dense in P. For, suppose that $Z \in P$. Now $(Z\backslash X) \in P$, since if $(Z\backslash X) \in I$ then $Z \subseteq X \cup (Z\backslash X) \in I$, hence $Z \in I$, contradiction. Now $(Z\backslash X) \in M$ and $(Z\backslash X) \cap X = \emptyset$. So D is dense. Choose $Y \in D \cap G$. Then $Y \subseteq (\kappa\backslash X)$, so $(\kappa\backslash X) \in G$. □

Lemma 22.17. *Let G be P-generic over M. Then the following are equivalent:*

(i) G is κ-complete.

(ii) If $\alpha < \kappa$ and $X \in M \cap {}^\alpha\mathscr{P}(\kappa)$ is such that X is a partition of κ, then there is a $\xi < \alpha$ such that $X_\xi \in G$.

Proof. (i)⇒(ii): assume (i), and suppose that $\alpha < \kappa$ and $X \in M \cap {}^\alpha\mathscr{P}(\kappa)$ is such that X is a partition of κ. Suppose also that $\kappa\backslash X_\xi \in G$ for all $\xi < \alpha$. Clearly $\langle \kappa\backslash X_\xi : \xi < \kappa\rangle \in M$, so $\emptyset = \bigcap_{\xi<\alpha}(\kappa\backslash X_\xi) \in G$, contradiction.

(ii)⇒(i): Assume (ii), and suppose that $\alpha < \kappa$ and $X \in M \cap {}^\alpha G$. For all $\xi < \alpha$ let $Y_\xi = (\kappa\backslash X_\xi) \cap \bigcap_{\eta<\xi} X_\eta$, and let $Y_\alpha = \bigcap_{\xi<\alpha} X_\xi$. Then $Y \in M$ is a partition of κ, so by (ii) we have two cases.

Case 1. There is a $\xi < \alpha$ such that $Y_\xi \in G$. Then also $(\kappa\backslash X_\xi) \in G$, contradiction.

Case 9. $Y_\alpha \in G$. This is as desired. □

Lemma 22.18. *If I is κ-complete in M, then G is a κ-complete M-ultrafilter.*

Proof. Suppose that $\langle X_\xi : \xi < \gamma\rangle \in M$ is a partition of κ with $\gamma < \kappa$. Let $D = \{Y \subseteq \kappa : \exists \xi < \gamma[Y \subseteq X_\xi]\}$. Then D is dense in P. For, suppose that $Y \in P$. If $\forall \xi < \gamma[Y \cap X_\xi \in I]$, then by κ-completeness of I, $Y = \sum_{\xi<\gamma}(Y \cap X_\xi) \in I$, contradiction. So $Y \cap X_\xi \notin I$ for some $\xi < \gamma$, showing that D is dense. □

Lemma 22.19. *If I is normal, then G is normal.*

Proof. Assume that I is normal. To show that G is normal, suppose that $f : X \to \kappa$ is regressive, with $X \in G$. Let $D = \{Y \subseteq X : Y \in M, f \text{ is constant on } Y\}$. We claim that D is dense below X. For, suppose that $Z \subseteq X$ with $Z \in P$. Then $Z \notin I$ and $f \restriction Z$ is regressive. Hence by normality of I there is a $W \subseteq Z$ with $W \notin I$ and with f constant on W. This proves that D is dense. Choose $Y \in D \cap G$. Thus G is normal. □

We now assume that $\forall \alpha < \kappa[\{\alpha\} \in I]/$

Lemma 22.20. *G is nonprincipal.*

Proof. If $\{\alpha\} \in G$, then $\{\alpha\} \in P$, so $\{\alpha\} \notin I$, contradiction. □

Lemma 22.21. *If I is atomless, then $G \notin M$.*

Proof. Suppose that I is atomless and $G \in M$. Now $\forall X \notin I \exists Y, Z \subseteq X[Y \cap Z = \emptyset \wedge X = Y \cup Z \wedge Y, Z \notin I]$. Let $D = \{X \in P : X \notin G\}$. Then D is dense. For suppose that $X \in P$. So $X \notin I$, and hence there are disjoint Y, Z such that $Y, Z \notin I$ and $X = Y \cup Z$. Hence $Y \notin G$ or $Z \notin G$, as desired. So D is dense, and hence $D \cap G \neq \emptyset$, contradiction. □

Lemma 22.22. *If I is a maximal ideal, then $G = \{X \subseteq \kappa : X \in M$ and $(\kappa \backslash X) \in I$; hence $G \in M$.*

Proof. $\supseteq$ holds by Lemma 22.16. Now suppose that $(\kappa\backslash X) \notin I$. Then $X \in I$, and so by Lemma 22.16, $(\kappa\backslash X) \in G$; hence $X \notin G$. □

We now carry through, in $M[G]$, the ultraproduct construction. So we let S be the set of all functions in M with domain κ, and define

$$\begin{aligned}
f =^* g \quad &\text{iff} \quad \{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in G;\\
f \in^* g \quad &\text{iff} \quad \{\alpha < \kappa : f(\alpha) \in g(\alpha)\} \in G;\\
[f]_G &= \{g : f =^* g \text{ and } \forall h[f =^* h \to \operatorname{rank}(g) \leq \operatorname{rank}(h)];\\
&\quad (\text{rank in the sense of } M)\\
\operatorname{Ult}_G(M) &= \text{collection of all } [f]_G;\\
\in_{\operatorname{Ult}_B(M)} &= \{([f]_G, [g]_G) : f \in^* g\}.
\end{aligned}$$

Theorem 22.23. *If $f_0, \ldots, f_{m-1} \in M$ are functions with domain κ, then*

$$\operatorname{Ult}_G(M) \models \varphi([f_0]_G, \ldots, [f_{m-1}]_G) \quad \textit{iff} \quad \{\alpha < \kappa : M \models \varphi(f_0(\alpha), \ldots, f_{m-1}(\alpha))\} \in G.$$

Proof. Proof by induction on φ:

φ is $v_i \in v_j$: $\text{Ult}_G(M) \models [f_i] \in [f_j]$ iff $\{\alpha \in \kappa : f_i(\alpha) \in f_j(\alpha)\} \in G$.

φ is $v_i = v_j$: $\text{Ult}_G(M) \models [f_i] = [f_j]$ iff $\{\alpha \in \kappa : f_i(\alpha) = f_j(\alpha)\} \in G$.

φ is $\neg\psi$ or $\psi \to \chi$: clear.

φ is $\exists v_i \psi$. First suppose that $\text{Ult}_G(M) \models \varphi([f_1], \ldots, [f_m])$. Choose a function g defined on κ such that

$$\text{Ult}_G(M) \models \psi([f_1], \ldots, [f_{i-1}].[g], [f_{i+1}], \ldots, [f_m]).$$

Then by the inductive hypothesis,

$$\{\alpha \in \kappa : M \models \psi(f_1(\alpha), \ldots, f_{i-1}(\alpha), g(\alpha), \ldots, f_{m-1}(\alpha))\} \in G.$$

Since

$$\begin{aligned}&\{\alpha \in \kappa : M \models \psi(f_1(\alpha), \ldots, f_{i-1}(\alpha), g(\alpha), \ldots, f_{m-1}(\alpha))\}\\ &\subseteq \{\alpha \in \kappa : M \models \varphi(f_1(\alpha), \ldots, f_{i-1}(\alpha), v_i, \ldots, f_{m-1}(\alpha))\},\end{aligned}$$

it follows that

$$\{\alpha \in \kappa : M \models \varphi(f_1(\alpha), \ldots, f_{i-1}(\alpha), v_i, \ldots, f_{m-1}(\alpha))\} \in G.$$

Second, suppose that

$$\{\alpha \in \kappa : M \models \varphi(f_1(\alpha), \ldots, f_{i-1}(\alpha), v_i, \ldots, f_{m-1}(\alpha))\} \in G.$$

By the axiom of choice, let g have domain κ such that

$$\begin{aligned}&\{\alpha \in \kappa : \varphi(f_1(\alpha), \ldots, f_{i-1}(\alpha), v_i, \ldots, f_{m-1}(\alpha))\}\\ &\subseteq \{\alpha \in \kappa : M \models \psi(f_1(\alpha), \ldots, f_{i-1}(\alpha), g(\alpha), \ldots, f_{m-1}(\alpha))\}.\end{aligned}$$

It follows that

$$\alpha \in \kappa : M \models \psi(f_1(\alpha), \ldots, f_{i-1}(\alpha), g(\alpha), \ldots, f_{m-1}(\alpha))\} \in G,$$

and hence

$$\alpha \in \kappa : M \models \varphi(f_1(\alpha), \ldots, f_{i-1}(\alpha), v_i, \ldots, f_{m-1}(\alpha))\} \in G. \qquad \square$$

For each $x \in M$, c_x is the function with domain κ and constant value x. Then we define $j_G : M \to \text{Ult}_G(M)$ by $j_G(x) = [c_x]_G$.

Theorem 22.24. *j_G is an elementary embedding.*

Proof.

$$\begin{aligned}M \models \varphi(x_0, \ldots, x_{m-1}) \quad &\text{iff} \quad \forall \alpha < \kappa[M \models \varphi(c_{x_0}(\alpha), \ldots, c_{x_{m-1}}(\alpha))\\ &\text{iff} \quad \text{Ult}_G(M) \models \varphi([c_{x_0}]_G, \ldots, [c_{x_{m-1}}]_G)\end{aligned} \qquad \square$$

Note that $\mathrm{Ult}_G(M)$ is not necessarily well-founded. If $[f_0]_G \ni \cdots \ni [f_n]_G \ni \ldots$, it is possible that f itself is not in M.

We now let $N = \mathrm{Ult}_G(M)$.

Lemma 22.25. *Let* $\mathrm{Ord}(X)$ *be the statement that* x *is transitive and* $\forall y \in x[y$ *is transitive*$]$. *Then* $\{x \in N : N \models \mathrm{Ord}(x)\}$ *is linearly ordered by* $\in^*$.

Proof.

$\{\alpha < \kappa : M \models f(\alpha)$ is an ordinal $\rightarrow f(\alpha) \notin f(\alpha)\} = \kappa$,
so by Theorem 22.23, $\mathrm{Ult}_G(M) \models [f]_G$ is an ordinal $\rightarrow [f]_G \notin^* [f]_G$;
$\{\alpha < \kappa : M \models f(\alpha), g(\alpha), h(\alpha)$ are ordinals and
$f(\alpha) \in g(\alpha) \in h(\alpha) \rightarrow f(\alpha) \in h(\alpha)\} = \kappa$,
so by Theorem 22.23, $\mathrm{Ult}_G(M) \models [f]_G, [g]_G, [h]_G$ are ordinals and
$[f]_G \in^* [g]_G \in^* [h]_G \rightarrow [f]_G \in^* [h]_G$;
$\{\alpha < \kappa : M \models f(\alpha), g(\alpha)$ are ordinals $\rightarrow$
$f(\alpha) \in g(\alpha)$ or $f(\alpha) = g(\alpha)$ or $g(\alpha) \in f(\alpha)$,
so by Theorem 22.23, $\mathrm{Ult}_G(M) \models [f]_G, [g]_G$ are ordinals
$\rightarrow [f]_G \in^* [g]_G$ or $[f]_G = [g]_G$ or $[g]_G \in^* [f]_G$. □

Lemma 22.26. $j_G(\gamma) = \gamma$ *for all* $g < \kappa$.

Proof. Induction on γ. Suppose that $j(\delta) = \delta$ for all $\delta < \gamma$, while $\gamma < j(\gamma)$. Say $\gamma = j(\varepsilon)$. Then $[c_\varepsilon] \in^* [c_\gamma]$, so $\{\alpha < \kappa : \varepsilon < \gamma\} \in G$. So $\varepsilon < \gamma$. Hence by the inductive hypothesis $\gamma = j(\varepsilon) = \varepsilon < \gamma$, contradiction. □

Lemma 22.27. *Let* $d(\alpha) = \alpha$ *for all* $\alpha < \kappa$. *Then* $\kappa \le [d]$.

Proof. If $\gamma < \kappa$, then $\{\alpha < \kappa : \gamma < d(\alpha)\} = \kappa \backslash (\gamma + 1)$ and $\gamma + 1 \in I$, so by Lemma 22.16, $\{\alpha < \kappa : \gamma < d(\alpha) \in G$. Hence $[c_\gamma] < [d]$. □

Lemma 22.28. $[d] < j_G(\kappa)$ *and so* $\kappa < j_G(\kappa)$.

Proof. $\{\alpha < \kappa : \alpha < \kappa\} = \kappa$, so $[d] < j_G(\kappa)$. Hence $\kappa < j_G(\kappa)$ by Lemma 22.27. □

Lemma 22.22. *If* I *is normal, then* $[d] = \kappa$.

Proof. See the proof of Theorem 17.19. □

Let $S \notin I$. An *I-partition* of S is a maximal collection W of subsets of S, each not in I, such that $X \cap Y \in I$ for distinct $X, Y \in W$. Thus W is an I-partition of S iff $\forall X \in W[[X] \neq 0]$ and $\{[X] : X \in W\}$ is a partition of $[S]$. An I-partition W_1 is a *refinement* of an I-partition W_2, $W_1 \le W_2$, iff $\forall X \in W_1 \exists Y \in W_2[X \subseteq Y]$.

An ideal I is *precipitous* iff for every $S \notin I$ and every system

$$W_0 \ge W_1 \ge \cdots \ge W_n \ge \cdots$$

of I-partitions of S there are

$$X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n \supseteq \cdots$$

such that $\forall n \in \omega[X_n \in W_n]$ and $\bigcap_{n\in\omega} X_n \neq \emptyset$.

Theorem 22.30. *If I is κ^+-saturated, then I is precipitous.*

Proof. Suppose that $S \notin I$ and

$$W_0 \geq W_1 \geq \cdots \geq W_n \geq \cdots$$

is a system of I-partitions of S. For each $n \in \omega$ let $\delta_n = |W_n|$. Since I is a κ^+-saturated, each $\delta_n \leq \kappa$. Write $W_n = \{X^n_i : i < \delta_n\}$ without repetitions for each $n \in \omega$. Now we will construct Y^n_i for $n \in \omega$ and $i < \delta_n$ so that

(1) If $n \in \omega$ and $i, j < \delta_n$ with $i \neq j$, then $Y^n_i \cap Y^n_j = \emptyset$.

(2) $\forall n \in \omega \forall i < \delta_n[Y^n_i \subseteq X^n_i$ and $X^n_i \backslash Y^n_i \in I]$.

(3) With $Z_n = \{Y^n_i : i < \delta_n\}$, Z_n is an I-partition and $Z_n \geq Z_{n+1}$.

(4) If $n > 0$ and $X^n \subseteq X^{n-1}_k$, then $Y^n_i \subseteq Y^{n-1}_k$.

For each $i < \delta_0$ let $Y^0_i = X^0_i \backslash \bigcup_{j<i} X^0_j$. Clearly (1) holds, and $Y^0_i \subseteq X^0_i$. Also, $X^0_0 \backslash Y^0_0 = \emptyset \in I$, and for $i > 0$, $X^0_i \backslash Y^0_i = \bigcap_{j\leq i} X^0_j \in I$. Hence $[Y^0_i] = [X^0_i]$. Hence Z_0 is an I-partition of S.

Now if Y^{n-1}_i has been defined for all $i < \delta_{n-1}$, for $i < \delta_n$ we define

$$Y^n_i = \left(X^n_i \backslash \bigcup_{j<i} X^n_j \right) \cap Y^{n-1}_k,$$

where k is the unique element such that $X^n_i \subseteq X^{n-1}_k$.

Clearly (1) holds. Clearly $Y^n_i \subseteq X^n_i$. Now

$$\begin{aligned} X^n_i \backslash Y^n_i &= \left(X^n_i \cap \bigcup_{j<i} X^n_j \right) \cup (X^n_i \backslash Y^{n-1}_k) \\ &\subseteq \left(X^n_i \cap \bigcup_{j<i} X^n_j \right) \cup (X^{n-1}_k \backslash Y^{n-1}_k) \in I \end{aligned}$$

So (2) holds. Clearly (3) and (4) hold.

For each $n \in \omega$ let $S_n = \bigcup Z_n$. Since Z_n is an I-partition, $S \backslash S_n \in I$. Hence $S \backslash \bigcap_{n\in\omega} S_n = \bigcup_{n\in\omega}(S \backslash S_n) \in I$; so $\bigcap_{n\in\omega} S_n \neq \emptyset$. Fix $\alpha \in \bigcap_{n\in\omega} S_n$. For each $n \in \omega$ there is an $i_n < \delta_n$ such that $\alpha \in Y^n_{i_n}$. Let $U_n = X^n_{i_n}$. Thus $\alpha \in U_n$ and $U_n \in W_n$ for all $n \in \omega$. We finish the proof by showing that $U_n \subseteq U_{n-1}$ for all $n \in \omega$.

(4) $\forall n \in \omega[Y^n_{i_n} \subseteq Y^{n-1}_{i_{n-1}}]$.

In fact, by construction $Y^n_{i_n} \subseteq Y^{n-1}_k$ where $X^n_{i_n} \subseteq X^{n-1}_k$. Hence $\alpha \in Y^{n-1}_k$, so by (1), $k = i_{n-1}$, giving (4).

Now $Y^n_{i_n} \subseteq X^n_{i_n} \cap X^{n-1}_{i_{n-1}}$, so $X^n_{i_n} \cap X^{n-1}_{i_{n-1}} \notin I$. Since $W_n \leq W_{n-1}$, it follows that $X^n_{i_n} \subseteq X^{n-1}_{i_{n-1}}$, i.e., $U_n \subseteq U_{n-1}$. □

Let I be an ideal on κ and $S \notin I$. A *functional* on S is a collection F of functions such that $W_F \stackrel{\text{def}}{=} \{\text{dmn}(f) : f \in F\}$ is an I-partition of κ and $\forall f, g \in F[f \neq g \to \text{dmn}(f) \neq \text{dmn}(g)]$. F is *ordinal-valued* iff $\forall f \in F[\text{rng}(f) \subseteq \mathbf{On}]$. Let F and G be two ordinal valued functionals on S. Then $F < G$ iff the following two conditions hold:

(1) $W_F \leq W_G$.

(2) For all $f \in F$ and $g \in G$, if $\text{dmn}(f) \subseteq \text{dmn}(g)$, then $\forall \alpha \in \text{dmn}(f)[f(\alpha) < g(\alpha)]$.

Theorem 22.31. *I is precipitous iff there do not exist a set $S \notin I$ and a sequence*

$$F_0 > F_1 > \cdots > F_n > \cdots$$

of functionals on S.

Proof. $\Rightarrow$: Suppose that $S \notin I$ and

$$F_0 > F_1 > \cdots > F_n > \cdots$$

is a sequence of functionals on S, while I is precipitous. Let

$$X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n \supseteq \cdots$$

be such that $\forall n \in \omega[X_n \in W_{F_n}]$ and $\bigcap_{n\in\omega} X_n \neq \emptyset$. For each $n \in \omega$ let f_n be a function such that $f_n \in W_{F_n}$. Take any $\alpha \in \bigcap_{n\in\omega} X_n$. Then

$$f_0(\alpha) > f_1(\alpha) > \cdots > f_n(\alpha) > \cdots,$$

contradiction.

$\Leftarrow$: Assume that I is not precipitous. Then there is an $S \notin I$ and a sequence

$$W_0 \geq W_1 \geq \cdots \geq W_n \geq \cdots$$

of I-partitions of S such that there is no sequence

$$X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n \supseteq \cdots$$

such that $\forall n \in \omega[X_n \in W_n]$ and $\bigcap_{n\in\omega} X_n \neq \emptyset$. We want to construct a sequence

$$F_0 > F_1 > \cdots > F_n > \cdots$$

of functionals on S.

Let $\alpha \in \prod_{Y \notin I} Y$. Now for each $n \in \omega$ and $X \in W_n$ we define a finite set $F(n, X)$ by recursion on n. For each $X \in W_0$ let $F(0, X) = \{\alpha_X\}$. If $F(n, Y)$ has been defined for all $Y \in W_n$, and $X \in W_{n+1}$, choose $Y \in W_n$ such that $X \subseteq Y$. Note that $X \backslash F(n, Y) \notin I$, since $X \notin I$ and $F(n, Y)$ is finite. Let $F(n+1, X) = F(n, Y) \cup \{\alpha_{X \backslash F(n,Y)}\}$. Now for each $n \in \omega$ let $W'_n = \{X \backslash F(n, X) : X \in W_n\}$.

(1) $\forall n \in \omega \forall X' \in W'_{n+1} \exists Y' \in W'_n [X' \subset Y]'$.

In fact, let $n \in \omega$ and $X' \in W'_{n+1}$. Say $X' = X \backslash F(n+1, X)$ with $X \in W_{n+1}$. Say $X \subseteq Y$ with $Y \in W_n$. Then

$$X' = X \backslash F(n+1, X) = (X \backslash F(n, Y)) \backslash \{\alpha_{X \backslash F(n,Y)}\} \subseteq (Y \backslash F(n, Y)) \in W'_n;$$

moreover, $\alpha(X \backslash F(n, Y)) \in (X \backslash F(n, Y)) \subseteq (Y \backslash F(n, Y)) \stackrel{\text{def}}{=} Y'$. Now

$$X \backslash F(n+1, X) = X \backslash (F(n, Y) \cup \{\alpha_{X \backslash F(n,Y)}\}).$$

Hence $\alpha(X \backslash F(n, Y)) \notin X \backslash F(N+1, X) = X'$. So $X' \subset Y'$. Thus (1) holds.

Let $T = \bigcup_{n \in \omega} W'_n$.

(2) Every sequence $Z_0 \supset Z_1 \supset \cdots \supset Z_n \supset \cdots$ of members of T has empty intersection..

In fact, for each $n \in \omega$ choose m_n such that $Z_n \in W'_{m_n}$.

(3) If $n < p$, then $m_n < m_p$.

In fact, suppose that $n < p$ and $m_p \leq m_n$. Now $Z_n \in W'_{m_n}$, so there is a $U \in W'_{m_p}$ such that $Z_n \subseteq U$. Also $Z_p \in W'_{m_p}$ and $Z_p \subseteq Z_n$, so $Z_p = U$ and $Z_n = Z_p$, contradiction. So (3) holds.

Now let $Z'_0 \in W'_0$ be such that $Z_0 \subseteq Z'_0$. If $Z'_n \in W'_n$ has been defined, note that $n \leq m_n < m_{n+1}$, so choose $Z'_{n+1} \in W'_{n+1}$ such that $Z_{n+1} \subseteq Z'_{n+1}$. Say $U \in W'_n$ with $Z'_{n+1} \subseteq U$. Now $Z_{n+1} \subseteq Z'_{n+1} \subseteq U$ and $Z_{n+1} \subseteq Z_n \subseteq Z'_n \in W'_n$. So $U \cap Z'_n \notin I$. Hence $U = Z'_n$. So $Z'_{n+1} \subseteq Z'_n$.

It now follows from the initial assumption of $\Leftarrow$ that $\bigcap_{n \in \omega} Z'_n = \emptyset$. Since $\forall n \in \omega [Z_n \subseteq Z'_n$, we also have $\bigcap_{n \in \omega} Z_n = \emptyset$. This proves (2).

Now for each $z \in S$ the set $T_z = \{X \in T : z \in X\}$ is well-founded under $\subset$, by (2). Let ρ_z be the associated rank function: for any $X \in T_z$ let $\rho_z(X) = \{\rho_z(Y) \cup \{\rho_z(Y)\} : Y \in T_z$ and $Y \subset X\}$. Thus $z \in Y \subset X$ implies that $\rho_z(Y) < \rho_z(X)$. For each $X \in T$ we define a function f_X with domain X by setting, for all $z \in X$, $f_X(z) = \rho_z(X)$. For each $n \in \omega$ let $F_n = \{f_X : X \in W_n\}$.

(4) $\forall n \in \omega [F_n$ is an ordinal-valued functional on $S]$.

For, suppose that $n \in \omega$. Then $W_{F_n} = \{\text{dmn}(f_X) : X \in W_n\} = W_n$ is an I-partition of S. F_n is ordinal-valued by Proposition 19.16 of full. If $X, Y \in W_n$ and $X \neq Y$, then $\text{dmn}(f_X) = X \neq Y = \text{dmn}(f_Y)$.

(5) $F_0 > F_1 > \cdots > F_n > \cdots$.

In fact, if $n \in \omega$, then $W_{F_n} = W_n \geq W_{n+1} = W_{F_{n+1}}$. Suppose that $Y \in W_{n+1}$, $X \in W_n$, and $Y \subset X$. For $\alpha \in Y$ we have $f_Y(\alpha) = \rho_\alpha(Y) < \rho_\alpha(X) = f_X(\alpha)$. $\square$

Theorem 22.32. $[\kappa]^{<\kappa}$ *is not precipitous.*

Proof. Let $I = [\kappa]^{<\kappa}$. Note that $X \notin I$ iff $|X| = \kappa$. For each $X \notin I$ let f_X be the order-preserving function mapping X onto κ.

(1) For all $X \notin I$ there is a $Y \subseteq X$ with $Y \notin I$ such that $f_Y(\alpha) < f_X(\alpha)$ for all $\alpha \in Y$.

In fact, let $Y = \{f(\alpha) - 1 : \alpha \in X, f(\alpha) \text{ a successor ordinal}\}$.

It follows that for every $X \notin I$ there is an I-partition W_X of X such that $\forall Y \in W_X \forall \alpha \in Y[f_Y(\alpha) < f_X(\alpha)]$.

Now define $W_0 = \{\kappa\}$, and if W_n has been defined, let $W_{n+1} = \bigcup_{X \in W_n} W_X$. For each n let $F_n = \{f_X : X \in W_n\}$. Clearly each F_n is a functional on κ and $F_0 > F_1 > \cdots > F_n > \cdots$. $\square$

Theorem 22.33. *I is precipitous iff* Ult_G *is well-founded.*

Proof. $\Leftarrow$: Assume that Ult_G is well-founded. Suppose that $S \notin I$ and

$$W_0 \geq W_1 \geq \cdots \geq W_n \geq \cdots$$

is a system of I-partitions of S.

(1) $\forall n \in \omega \exists a_n \in W_n \cap G$.

In fact, let $D = \{b \notin I : \exists c \in W_n[b \leq c]\}$. Then D is dense below S. In fact, if $c \subseteq S$ with $c \notin I$, then there is a $d \in W_n$ such that $c \cap d \notin I$. Then $c \cap d \in D$, as desired. Choose $b \in D \cap G$. Then $b \leq a_n \in G$ for some a_n, proving (1).

If $m < n \in \omega$, then $a_m \cap a_n \in G$, and so $a_n \subseteq a_m$.

(2) With d the identity on κ, $\forall A \in \mathscr{P}(\kappa) \backslash I[A \in G \text{ iff } [d]Ej(A)]$.

In fact, $A = \{\alpha : \alpha \in A\} = \{\alpha : d(\alpha) \in c_A(\alpha)\} \notin I$, so $[d]E[c_A] = j(A)$. So (2) holds.

It follows that $\pi([d]) \in \pi(j(a_n))$ for all n. Hence

$$N \models \exists a \left[a \text{ is a function with domain } \omega \text{ and } \forall n \in \omega[a_n \in W_n] \wedge \bigcap_{n \in \omega} a_n \neq \emptyset \right].$$

By elementarity this holds in the universe, as desired.

$\Rightarrow$: Suppose that Ult_G is not well-founded. Then there exist an $S \in G$ and names $\dot{f}_n$ for $n \in \omega$ such that $\forall n \in \omega[S \Vdash \dot{f}_n : \kappa \to M$, $S \Vdash \dot{f}_n \in M$, and $S \Vdash \dot{f}_{n+1} E \dot{f}_n]$. Then by Lemma 14.42 we have

(1) $\forall n \in \omega \forall q \leq S \exists b \in M \exists r \leq q[r \Vdash \dot{f}_n = \check{b}]$.

For each $n \in \omega$ let $A_n = \{r : \exists b[r \Vdash \dot{f}_n = \check{b}]\}$. By (1), A_n is dense below S. Let $W_0 \subseteq A_0$ be maximal pairwise disjoint. For each $r \in W_0$ choose g_r^0 such that $r \Vdash \dot{f}_0 = \check{g}_r^0$. Having defined W_n, let $W_{n+1} \subseteq A_{n+1}$ be maximal pairwise disjoint such that $\forall q \in W_{n+1} \exists r \in W_n[q \leq r]$. Then for each $r \in W_{n+1}$ choose g_r^{n+1} such that $r \Vdash \dot{f}_{n+1} = \check{g}_r^{n+1}$.

(2) If $r \in W_{n+1}$, $s \in W_n$, and $r \subseteq s$, then $\forall \alpha \in \kappa[g_r^{n+1}(\alpha) \in g_r^n(\alpha)]$.

In fact, suppose that $r \in W_{n+1}$, $s \in W_n$, and $r \subseteq s$. Since $r \leq S$, we have $r \Vdash \dot{f}_{n+1} E \dot{f}_n$. But also $r \Vdash \dot{f}_{n+1} = \check{g}_r^{n+1}$ and $r \Vdash \dot{f}_n = \check{g}_r^n$. So $r \Vdash \check{g}_r^{n+1} E \check{g}_r^n$, so $\forall \alpha \in \kappa[g_r^{n+1}(\alpha) \in g_r^n(\alpha)]$.

Now I is not precipitious, since a nonempty intersection of X_n's would yield a decreasing $\in$-chain. □

Lemma 22.34. *If κ is an uncountable cardinal and I is a proper κ-saturated κ-complete ideal on κ containing all singletons, then there is a function $f : \kappa \to \kappa$ such that*

$$J \stackrel{\text{def}}{=} \{X \subseteq \kappa : f^{-1}[X] \in I\}$$

is a proper normal κ-saturated κ-complete ideal on κ containing all singletons.

Proof. Clearly every bounded subset of κ is in I. Let $S \notin I$. A function $f : S \to \kappa$ is *unbounded* iff there is no $\gamma < \kappa$ and no $T \subseteq S$ with $T \notin I$ such that $\forall \alpha \in T[f(\alpha) < \gamma]$. Thus f is unbounded iff

$$\forall \gamma < \kappa \forall T \subseteq S[T \notin I \to \exists \alpha \in T[\gamma \leq f(\alpha)]]. \tag{1}$$

Equivalently, f is unbounded iff

$$\forall \gamma < \kappa \forall T \subseteq S[T \notin I \to \exists \alpha \in T[\gamma < f(\alpha)]]. \tag{2}$$

Let $\mathscr{F}$ be the set of all functions g mapping into κ defined on a set not in I and unbounded on its domain. We define $g < h$ iff $\mathrm{dmn}(g) \subseteq \mathrm{dmn}(h)$ and $\forall \alpha \in \mathrm{dmn}(g)[g(\alpha) < h(\alpha)]$. We call $g \in \mathscr{F}$ *minimal* iff there is no $h \in \mathscr{F}$ such that $h < g$.

(3) $\mathscr{F} \neq \emptyset$.

In fact, let $f(\alpha) = \alpha$ for all $\alpha \in \kappa$. To show that f is unbounded on κ, suppose that $\gamma < \kappa$ and $T \subseteq \kappa$ is not in I. Then T is unbounded, so there is an $\alpha \in T \backslash (\gamma + 1)$. Then $\gamma < \alpha = f(\alpha)$. Thus $f \in \mathscr{F}$.

(4) There is a minimal $g \in \mathscr{F}$.

For, suppose not. Let $g \in \mathscr{F}$ be arbitrary. Let W be a maximal collection of elements of $\mathscr{F}$ such that $\forall h \in W[h < g]$ and $\forall h_1, h_2 \in W[h_1 \neq h_2 \to \mathrm{dmn}(h_1) \cap \mathrm{dmn}(h_2) = \emptyset]$. Since I is κ-saturated, we have $|W| < \kappa$.

(5) $(\mathrm{dmn}(g) \backslash \bigcup_{h \in W} \mathrm{dmn}(h)) \in I$.

For, suppose not. Let $g' = g \restriction (\mathrm{dmn}(g) \backslash \bigcup_{h \in W} \mathrm{dmn}(h))$. Clearly $g' \in \mathscr{F}$. Take $k \in \mathscr{F}$ with $k < g'$. This contradicts the maximality of W.

Let $f = \bigcup_{h \in W} h$. Then clearly f is defined on a set not in I. Now suppose that $\gamma < \kappa$ and $J \subseteq \mathrm{dmn}(f)$ is not in I. Then $J = \bigcup_{h \in W}(J \cap \mathrm{dmn}(h))$. Since $\{\mathrm{dmn}(h) : h \in W\}$ has size less than κ, by the κ-completeness of I it follows that there is a $h \in W$ such that $J \cap \mathrm{dmn}(h) \notin I$. Since h is unbounded, there is an $\alpha \in J \cap \mathrm{dmn}(h)$ such that $\gamma < h(\alpha) \le f(\alpha)$. This shows that $f \in \mathscr{F}$.

Repeating this construction we obtain $g_0 > g_1 > g_2 > \cdots$, each $g_i \in \mathscr{F}$ and $\mathrm{dmn}(g_i)\backslash\mathrm{dmn}(g_{i+1}) \in I$ for each i. Since κ is uncountable and I is κ-complete, it follows that $\bigcup_{i\in\omega}(\mathrm{dmn}(g_i)\backslash\mathrm{dmn}(g_{i+1})) \in I$. Now

$$\mathrm{dmn}(g_0) = \bigcup_{i\in\omega}(\mathrm{dmn}(g_i)\backslash\mathrm{dmn}(g_{i+1})) \cup \bigcap_{i\in\omega}\mathrm{dmn}(g_i).$$

Hence $\bigcap_{i\in\omega}\mathrm{dmn}(g_i) \notin I$. Taking any $\alpha \in \bigcap_{i\in\omega}\mathrm{dmn}(g_i)$, we have $g_0(\alpha) > g_1(\alpha) > \cdots$, contradiction. This proves (4).

By the same argument, for every $h \in \mathscr{F}$ there is a minimal $g \in \mathscr{F}$ such that $g < h$ or $g = h$. Let W be a maximal collection of minimal members of $\mathscr{F}$ such that $\mathrm{dmn}(g_1) \cap \mathrm{dmn}(g_2) = \emptyset$ for all distinct $g_1, g_2 \in W$.

(6) $\left(\kappa\backslash\bigcup_{g\in W}\mathrm{dmn}(g)\right) \in I$.

In fact, otherwise let $D = \kappa\backslash\bigcup_{g\in W}\mathrm{dmn}(g)$, and let h have domain D with $h(\alpha) = \alpha$ for all $\alpha \in D$. Then h is unbounded on D, for if $\gamma < \kappa$ and $T \subseteq D$ with $T \notin I$, choose $\alpha \in T$ with $\gamma < \alpha$. Then $\gamma < h(\alpha)$, as desired. Thus $h \in \mathscr{F}$. There is a minimal $k < h$, or $k = h$. Hence $W \cup \{h\}$ contradicts the maximality of W. So (6) holds.

Let $f = \bigcup W$. Then f is unbounded. For, suppose that $\gamma < \kappa$, $T \subseteq \mathrm{dmn}(f)$, and $T \notin I$. Now $T = \bigcup_{g\in W}(\mathrm{dmn}(g) \cap T)$ and W has size less than κ, so there is a $g \in W$ such that $(\mathrm{dmn}(g) \cap T) \notin I$. Hence there is an $\alpha \in \mathrm{dmn}(g) \cap T$ such that $\gamma \le g(\alpha) = f(\alpha)$. Clearly f is defined on a set not in I, so $f \in \mathscr{F}$. In fact, $\kappa\backslash\mathrm{dmn}(f) \in I$.

(7) if $h \in \mathscr{F}$, then $\{\alpha \in \mathrm{dmn}(h) \cap \mathrm{dmn}(f) : h(\alpha) < f(\alpha)\} \in I$.

In fact, suppose not. Let $k = h \restriction \{\alpha \in \mathrm{dmn}(h) \cap \mathrm{dmn}(f) : h(\alpha) < f(\alpha)\}$. Now $k \in \mathscr{F}$. For, k is defined on a set not in I. To show that it is unbounded on its domain, suppose that $\gamma < \kappa$ and $T \subseteq \mathrm{dmn}(k)$ with $T \notin I$. Choose $\alpha \in T$ such that $\gamma \le h(\alpha)$. Then $\gamma \le k(\alpha)$. So $k \in \mathscr{F}$. Let $k' \in \mathscr{F}$ be minimal such that $k' < k$ or $k' = k$.

(8) $\exists g \in W[\mathrm{dmn}(k') \cap \mathrm{dmn}(g) \notin I]$.

In fact, otherwise $\mathrm{dmn}(k') \cap \bigcup_{g\in W}\mathrm{dmn}(g) \in I$, and so by (6), $\mathrm{dmn}(k') \in I$, contradiction.

We choose g as in (8). Then for any $\alpha \in \mathrm{dmn}(k') \cap \mathrm{dmn}(g)$ we have $k'(\alpha) \le k(\alpha) = h(\alpha) < f(\alpha) = g(\alpha)$. Let $l = k' \restriction \mathrm{dmn}(k') \cap \mathrm{dmn}(g)$. Then clearly $l \in \mathscr{F}$ and $l < g$, contradicting the minimality of g. This proves (7).

Now let $f' = f \cup \{(\alpha, 0) : \alpha \in (\kappa\backslash\bigcup_{g\in W}\mathrm{dmn}(g))\}$. Then f' is unbounded. For, suppose that $T \subseteq \kappa$, $\gamma < \kappa$, and $T \notin I$. Clearly then $T \cap \bigcup_{g\in W}\mathrm{dmn}(g) \notin I$. Now $T = \bigcup_{g\in W}(\mathrm{dmn}(g) \cap T)$ and $|W| < \kappa$, so there is a $g \in W$ such that $(\mathrm{dmn}(g) \cap T) \notin I$. Hence there is an $\alpha \in \mathrm{dmn}(g) \cap T$ such that $\gamma \le g(\alpha) = f(\alpha) = f'(\alpha)$. So f' is unbounded, and hence $f' \in \mathscr{F}$.

(9) if $h \in \mathscr{F}$, then $\{\alpha \in \mathrm{dmn}(h) : h(\alpha) < f'(\alpha)\} \in I$.

In fact, $\{\alpha \in \mathrm{dmn}(h) : h(\alpha) < f'(\alpha)\} = \{\alpha \in \mathrm{dmn}(h) \cap \mathrm{dmn}(f) : h(\alpha) < f(\alpha)\}$, so (9) follows from (7).

Now let $J = \{X : f'^{-1}[X] \in I\})$ for any $X \subseteq \kappa$. If $Y \subseteq X \in J$, then $f'^{-1}[Y] \subseteq f'^{-1}[X] \in I$, so $Y \in J$. If $\alpha < \kappa$ and $X \in {}^{\alpha}J$, then

$$f'^{-1}\left[\bigcup_{\xi<\alpha} X_\xi\right] = \bigcup_{\xi<\alpha} f'^{-1}[X_\xi] \in I,$$

so $\bigcup_{\xi<\alpha} X_\xi \in J$. Thus J is a κ-complete ideal on κ. If $\gamma < \kappa$, then $f'^{-1}[\{\gamma\}] \in I$ since f' is unbounded, so $\{\gamma\} \in J$.

Now suppose that $\langle X_i : i \in k\rangle$ is a system of pairwise disjoint sets not in J. Then $\langle f'^{-1}[X_i] : i \in K\rangle$ is a system of pairwise disjoint sets not in I, so $|K| < \kappa$. Thus J is κ-saturated.

To show that J is normal, suppose that $S \notin J$ and g is a function with domain S such that $g(\alpha) < \alpha$ for all $\alpha \in S$. Thus if we let $T \stackrel{\text{def}}{=} f'^{-1}[S]$, then $T \notin I$. Define $\operatorname{dmn}(g') = T$, $g'(\alpha) = g(f'(\alpha))$. Then $\forall\alpha \in T[g'(\alpha) < f'(\alpha)]$. From (9) it follows that $g' \notin \mathscr{F}$. Thus there exist $\gamma < \kappa$ and $T' \subseteq T$ such that $T' \notin I$ and $\forall\alpha \in T'[g'(\alpha) < \gamma]$. Now

$$T' = \bigcup_{\delta<\gamma} \{\alpha \in T' : g'(\alpha) = \delta\}.$$

Since $T' \notin I$ and I is κ-complete, it follows that there is a $\delta < \gamma$ such that $T'' \stackrel{\text{def}}{=} \{\alpha \in T' : g'(\alpha) = \delta\}$ is not in I. Thus $g' \restriction T''$ is constant. Let $S' = f[T'']$. Then $T'' \subseteq f^{-1}[S']$ and $T'' \notin I$, so $S' \notin J$. Take any $\alpha \in S'$. Say $\alpha = f(\beta)$ with $\beta \in T''$. Then $g(\alpha) = g(f(\beta)) = g'(\beta) = \delta$.

This completes the proof. $\square$

Lemma 22.35. *Let I be a normal κ-saturated κ-complete ideal on κ. Suppose that $S \subseteq \kappa$, $S \notin I$, and $f : S \to \kappa$ is regressive. Then there is a $\gamma < \kappa$ such that $\{\alpha \in S : \gamma \le f(\alpha)\} \in I$.*

Proof. By exercise 8.8, for every $X \subseteq \kappa$ with $X \notin I$ there is a $Y \subseteq X$ such that $Y \notin I$ and f is constant on Y. Let W be a maximal disjoint family of subsets of S with each $X \in W$ not in I and f constant on X. Let $T = \bigcup_{X\in W} X$. Since $|W| < \kappa$, there is a $\gamma < \kappa$ such that $f(\alpha) < \gamma$ for all $\alpha \in T$. Suppose that $A \stackrel{\text{def}}{=} \{\alpha \in S : f(\alpha) \ge \gamma\} \notin I$. Then there is a $Y \subseteq A$ such that $Y \notin I$ and f is constant on Y. By the maximality of W, there is an $X \in W$ such that $X \cap Y \neq \emptyset$. Take any $\alpha \in X \cap Y$. Then $f(\alpha) < \gamma$ since $\alpha \in X \in W$, but $f(\alpha) \ge \gamma$ since $\alpha \in Y \subseteq A$, contradiction. It follows that $A \in I$, and hence $\{\alpha \in S : f(\alpha) < \gamma\} = \kappa\backslash A$. $\square$

Lemma 22.36. *If κ is an uncountable cardinal and there is a proper κ-saturated κ-complete ideal I on κ, with every singleton in I, then κ is regular limit.*

Proof. By Lemma 10.13, $\kappa > \omega_1$. By a theorem of Tarski, κ is regular. Suppose that $\kappa = \lambda^+$. For each $\xi < \kappa$ let $f_\xi : \lambda \to \kappa$ be such that $\xi \subseteq \operatorname{rng}(f_\xi)$. For each $\alpha < \kappa$ and $\eta < \lambda$ let

$$A_{\alpha\eta} = \{\xi < \kappa : f_\xi(\eta) = \alpha\}.$$

Clearly

(1) If $\alpha, \beta < \kappa$ with $\alpha \neq \beta$, and $\eta < \lambda$, then $A_{\alpha\eta} \cap A_{\beta\eta} = \emptyset$.

(2) $\forall \alpha < \kappa[|\kappa \backslash \bigcup_{\eta<\lambda} A_{\alpha\eta}| \leq \lambda$.

In fact, suppose that $\alpha < \kappa$ and $\xi \in \kappa \backslash \bigcup_{\eta<\lambda} A_{\alpha\eta}$. Then $\forall \eta < \lambda[f_\xi(\eta) \neq \alpha]$, i.e., $\alpha \notin \operatorname{rng}(f_\xi)$. Hence $\xi \leq \alpha$. This proves (2).

By (2) we have

(3) $\forall \alpha < \kappa[\kappa \backslash \bigcup_{\eta<\lambda} A_{\alpha\eta} \in I]$.

(4) $\forall \alpha < \kappa \exists \eta < \lambda[A_{\alpha\eta} \notin I]$.

In fact, otherwise there is an $\alpha < \kappa$ such that $\forall \eta < \lambda[A_{\alpha\eta} \in I]$, hence by the κ-completeness of I, $\bigcup_{\eta<\lambda} A_{\alpha\eta} \in I$. By (3), $\kappa \in I$, contradiction.

By (4) it follows that there exist a $\Gamma \in [\kappa]^\kappa$ and an $\eta < \lambda$ such that $\forall \alpha \in \Gamma[A_{\alpha\eta} \notin I]$. This contradicts I being κ-saturated. $\square$

Lemma 22.37. *Let κ be an uncountable regular cardinal. Let $M[G]$ be a generic extension by a κ-cc forcing. Then every club $C \subseteq \kappa$ in $M[G]$ has a club subset $D \in M$.*

Proof. Assume the hypotheses, and suppose that $C \subseteq \kappa$ is club in $M[G]$. Say P is the k-cc forcing mentioned. Then there is a name $\dot{C}$ such that $\mathbb{1} \Vdash [\dot{C}$ is a club subset of $\kappa]$ and $\dot{C}^G = C$. Let

$$D = \{\alpha < \kappa : ||\check{\alpha} \in \dot{C}|| = 1\}.$$

(1) $D \subseteq C$

In fact, suppose that $\alpha \in D$. Then $\mathbb{1} \Vdash \check{\alpha} \in \dot{C}$. So $\alpha \in \dot{C}^G = C$, proving (1).

(2) D is closed.

For, let β be a limit ordinal $< \kappa$, and suppose that $D \cap \beta$ is unbounded in β. We claim that $C \cap \beta$ is unbounded in β. For, suppose that $\gamma < \beta$. Choose $\alpha \in D \cap \beta$ with $\alpha \in [\gamma, \beta)$. Then $||\check{\alpha} \in \dot{C}|| = 1$. So $\alpha \in D$, as desired in (2).

(3) D is unbounded.

For, let $\alpha < \kappa$. Then $\mathbb{1} \Vdash \exists \beta < \check{\kappa}[\alpha < \beta \wedge \beta \in \dot{C}]$. Hence by Lemma 14.31,

$$\forall p \in P \exists q \leq p \exists \beta < \kappa[q \Vdash [\check{\alpha} \in \check{\beta} \wedge \check{\beta} \in \dot{C}].$$

Hence there is a maximal incompatible set W of members of P each $< p$ such that for all $q \in W$ there is an ordinal $\beta_q < \kappa$ such that $q \Vdash [\check{\alpha} \in \check{\beta}_q \wedge \check{\beta}_q \in \dot{C}]$. Now $|W| < \kappa$, so we can choose $\gamma < \kappa$ such that $\beta_q < \gamma$ for all $q \in W$.

(4) $\forall p \in P[p \Vdash \exists \beta \in \dot{C}[\check{\alpha} < \beta < \check{\gamma}]$.

In fact, let $p \in P$. Suppose that $p \in H$ generic. Then there is a $q \in W$ such that $q \in H$. Then $\alpha < \beta_q < \gamma$ and $\beta_q \in H$. So (4) holds.

Repeating this argument, we get $\alpha_1 < \alpha_2 < \cdots$ such that for all $p \in P$ and all $n \in \omega$ we have $p \Vdash \exists \beta \in \dot{C}[\check{\alpha}_n < \beta < \check{\alpha}_{n+1}]$. Let $\gamma = \sup_{n\in\omega} \alpha_n$. Clearly $\forall p[p \Vdash \check{\alpha} < \check{\gamma} \wedge \check{\gamma} \in \dot{C}]$

This proves (3). $\square$

Lemma 22.38. *Let κ be an uncountable regular cardinal. Let $M[G]$ be a generic extension by a κ-cc forcing. Suppose that $S \in M$ is stationary in κ. Then S is stationary in κ in the sense of $M[G]$.*

Proof. Let C be club in κ, with $C \in M[G]$. By Lemma 22.37 let $D \subseteq C$ be club in κ with $D \in M$. Then $\emptyset \neq S \cap D \subseteq S \cap C$. $\square$

Lemma 22.39. *Suppose that I is a proper normal κ-saturated κ-complete ideal on κ containing all singletons. Also suppose that $S \subseteq \kappa$ is stationary. Then $\{\alpha < \kappa : \alpha$ limit, $S \cap \alpha$ is not stationary in $\alpha\} \in I$.*

Proof. Suppose not. Thus $\{\alpha < \kappa : \alpha \text{ limit}, S \cap \alpha \text{ is not stationary in } \alpha\} \notin I$. Since $\omega + 1 \in I$, we have $X \stackrel{\text{def}}{=} \{\alpha < \kappa : \alpha \text{ limit}, S \cap \alpha \text{ is not stationary in } \alpha\} \backslash (\omega + 1) \notin I$. Let G be a generic set over $\{Y \subseteq \kappa : Y \notin I\}$ with $X \in G$. By Lemmas 22.18 and 22.22, G is κ-complete and normal. By Theorems 22.30 and 22.33,, Ult_G is well-founded. Let $N = \pi(\mathrm{Ult}_G)$. By Theorem 22.22, $\kappa = \pi([d])$. Now

$$\begin{aligned}&\{\alpha < \kappa : \alpha \text{ limit}, \mathrm{cf}(\alpha) > \omega, \exists C[C \subseteq \alpha \wedge \forall \beta < \alpha\\ &[\beta \text{ limit } \wedge \forall \gamma < \beta \exists \delta \in C[\gamma < \delta < \beta] \rightarrow \beta \in C] \wedge \forall \beta < \alpha\\ &\exists \gamma \in C[\beta < \gamma < \alpha] \rightarrow \alpha \in C \wedge \forall \beta \in C[\beta \notin S]\} \in G\end{aligned}$$

Hence

$$\begin{aligned}N \models &\exists C[C \subseteq \kappa \wedge \forall \beta < \kappa\\ &\quad[\beta \text{ limit } \wedge \forall \gamma < \beta \exists \delta \in C[\gamma < \delta < \beta] \rightarrow \beta \in C] \wedge \forall \beta < \kappa\\ &\quad\exists \gamma \in C[\beta < \gamma < \kappa] \rightarrow \alpha \in C \wedge \forall \beta \in C[\beta \notin j(S)]\end{aligned}$$

Thus $j(S) \cap \kappa$ is not stationary in κ. Now

(1) $S = j(S) \cap \kappa$.

In fact, clearly $S \subseteq j(S) \cap \kappa$. If $\alpha \in j(S) \cap \kappa$, then $\alpha = j(\alpha) \in j(S)$ and so $\alpha \in S$. So (1) holds.

Thus S is not stationary in κ, contradiction. $\square$

Lemma 22.40. *If I is normal and $X \notin I$, then X is stationary.*

Proof. Suppose that I is normal, $X \notin I$, and X is not stationary. Let C be club such that $X \cap C = \emptyset$. Then $X \subseteq \kappa \backslash C$. Let F be the dual of I. Then by Lemma 8.11, $C \in F$. So $\kappa \backslash C \in I$, so $X \in I$, contradiction. $\square$

Theorem 22.41. *If I is a normal κ-saturated ideal and $\kappa \backslash X \in I$, then $\kappa \backslash M(X) \in I$, where*

$$M(X) = \{\alpha < \kappa : \mathrm{cf}(\alpha) > \omega \textit{ and } X \cap \alpha \textit{ is stationary in } \alpha\}.$$

Proof. By the lemma, X is stationary. Then by Lemma 22.40, $\kappa \backslash M(X) \in I$. $\square$

Theorem 22.42. *If κ is an uncountable regular cardinal and there is a κ-saturated κ-complete ideal on κ, then κ is weakly Mahlo.*

Proof.

(1) $\{\alpha < \kappa : \alpha \text{ is singular}\} \in I$.

For, assume otherwise. Let $X = \{\alpha < \kappa : \alpha \text{ is singular}\}$; so by assumption $X \notin I$. Let $X \in G$ generic and $N = \pi(\mathrm{Ult}_G)$. Hence $N \models \kappa$ is singular. Now G is κ-saturated, so by Proposition 14.64, κ is a regular cardinal in $M[G]$. Since $N \subseteq M[G]$, κ is regular in N, contradiction. This proves (1).

By (1), $\{\alpha < \kappa : \alpha \text{ is regular}\} \notin I$. By the lemma above, $\{\alpha < \kappa : \alpha \text{ is regular}\}$ is stationary. $\square$

Theorem 22.43. *If κ is an uncountable regular cardinal and there is a κ-saturated κ-complete ideal on κ, then $\{\alpha < \kappa : \alpha$ is weakly Mahlo$\}$ is stationary.*

Proof. By the above, there is a normal κ-saturated κ-complete ideal on κ. Let $X = \{\alpha < \kappa : \alpha \text{ is regular}\}$. Hence $\kappa \backslash X \in I$ and $\kappa \backslash (X \cap M(X)) \in I$. By the lemma, $X \cap M(X)$ is stationary. $\square$

23. The nonstationary ideal

Lemma 23.1. *The following principle is equivalent to $\diamondsuit$:*

There is a sequence $\langle S_\alpha : \alpha < \omega_1\rangle$ such that for each $X \subseteq \omega_1$ the set $\{\alpha < \omega_1 : X \cap \alpha \in S_\alpha\}$ is stationary.

Proof. First assume $\diamondsuit$. Let $\langle S_\alpha : \alpha < \omega_1\rangle$ be a $\diamondsuit$-sequence, and let $S'_\alpha = \{S_\alpha\}$ for all $\alpha < \omega_1$. Clearly $\langle S'_\alpha : \alpha < \omega_1\rangle$ is a sequence as in the lemma.

Now let $\langle S_\alpha : \alpha < \omega_1\rangle$ be a sequence as in the lemma.

(1) There is a bijection f of ω_1 onto $\omega_1 \times \omega$ such that for all limit ordinals $\alpha < \omega_1$, $f[\alpha] = \alpha \times \omega$.

We construct $f \restriction \alpha$ for α limit less than ω_1 by induction. Let $f \restriction \omega$ be a bijection from ω onto $\omega \times \omega$. If α is limit less than ω_1 and $f \restriction \alpha$ has been constructed, extend $f \restriction \alpha$ to $f \restriction (\alpha + \omega)$ by defining

$$f(\alpha + m) = (\alpha + p, q), \quad \text{where} \quad f(m) = (p, q).$$

If $f \restriction \alpha$ has been defined for all $\alpha < \beta$, where β is a limit of limits, let $f \restriction \beta = \bigcup\{f \restriction \alpha : \alpha$ limit less than $\beta\}$. This proves (1).

For $S_\alpha \neq \emptyset$ write $S_\alpha = \{B^n_\alpha : n \in \omega\}$. Define

$$A^n_\alpha = \begin{cases} \{\delta < \omega_1 : (\delta, n) \in f[B^n_\alpha]\} & \text{if } S_\alpha \neq \emptyset, \\ \emptyset & \text{otherwise.} \end{cases}$$

(2) $\exists n[\langle A^n_\alpha \cap \alpha : \alpha < \omega_1\rangle$ is a $\diamondsuit$-sequence].

Suppose not. Then for all $n \in \omega$ there exist an $X_n \subseteq \omega_1$ and a club C_n such that $X_n \cap \alpha \neq A^n_\alpha \cap \alpha$ for every $\alpha \in C_n$. Let

$X = \{f^{-1}(\delta, n) : n \in \omega, \delta \in X_n\}$;
$D = \{\alpha \in \bigcap_{n\in\omega} D_n : \alpha \text{ limit}\}$.

Then D is club. Choose $\alpha \in D$ such that $X \cap \alpha \in S_\alpha$. Say $X \cap \alpha = B^n_\alpha$. Then

(3) $A^n_\alpha \cap \alpha = X_n \cap \alpha$.

In fact, if $\delta \in A^n_\alpha$, then $(\delta, n) \in f[B^n_\alpha] = f[X \cap \alpha]$; say $(\delta, n) = f(\beta)$ with $\beta \in X \cap \alpha$. Say $\beta = f^{-1}(\varepsilon, m)$ with $\varepsilon \in X_m$. Then $(\delta, n) = f(\beta) = (\varepsilon, m)$, so $\delta = \varepsilon$ and $n = m$. Thus $\delta \in X_n$. Conversely, if $\delta \in X_n \cap \alpha$, then $f^{-1}(\delta, n) \in X$. Now $(\delta, n) \in \alpha \times \omega = f[\alpha]$, so $f^{-1}(\delta, n) \in \alpha$. Hence $\delta \in A^n_\alpha$. Thus (3) holds. This contradicts $\alpha \in D_n$. $\square$

If κ is regular and E is a stationary subset of κ, then $\diamondsuit(\kappa, E)$ is the following statement:

There is a sequence $\langle S_\alpha : \alpha \in E\rangle$ of sets such that:

(i) $\forall \alpha \in E[S_\alpha \subseteq \alpha]$.

(ii) $\forall X \subseteq \kappa[\{\alpha \in E : X \cap \alpha = S_\alpha\}$ is stationary in $\kappa]$.

Lemma 23.2. *If E is a stationary subset of κ^+, then $\diamondsuit(\kappa, E)$ iff there is a sequence $\langle S_\alpha : \alpha \in E\rangle$ such that $\forall \alpha \in E[|S_\alpha| \leq \kappa]$ and $\forall X \subseteq \kappa^+[\{\alpha \in E : X \cap \alpha \in S_\alpha\}$ is stationary in $\kappa^+]$.*

Proof. $\Rightarrow$: let $\langle S_\alpha : \alpha \in E\rangle$ be as in the definition of $\diamondsuit(\kappa, E)$, and for each $\alpha \in E$ let $S'_\alpha = \{S_\alpha\}$.

$\Leftarrow$: assume the indicated condition.

(1) There is a bijection f of κ^+ onto $\kappa^+ \times \kappa$ such that for all ordinals $\alpha < \kappa^+$ with $\mathrm{cf}(\alpha) = \kappa$, $f[\alpha] = \alpha \times \kappa$.

We construct $f \restriction \alpha$ for $\mathrm{cf}(\alpha) = \kappa$ and $\alpha < \kappa^+$ by induction. Let $f \restriction \kappa$ be a bijection from κ onto $\kappa \times \kappa$. If $\mathrm{cf}(\alpha) = \kappa$ and $f \restriction \alpha$ has been constructed, extend $f \restriction \alpha$ to $f \restriction (\alpha + \kappa)$ by defining

$$f(\alpha + \beta) = (\alpha + \rho, \sigma), \quad \text{where} \quad f(\beta) = (\rho, \sigma).$$

If $f \restriction \alpha$ has been defined for all $\alpha < \beta$, where β is a limit of limits, let $f \restriction \beta = \bigcup\{f \restriction \alpha : \alpha$ limit less than $\beta\}$. This proves (1).

For $S_\alpha \neq \emptyset$ write $S_\alpha = \{B^\beta_\alpha : \beta \in \kappa\}$. Define for $\beta \in \kappa$

$$A^\beta_\alpha = \begin{cases} \{\delta < \kappa^+ : (\delta, \beta) \in f[B^\beta_\alpha]\} & \text{if } S_\alpha \neq \emptyset, \\ \emptyset & \text{otherwise.} \end{cases}$$

(2) $\exists\beta \in \kappa[\langle A^\beta_\alpha \cap \alpha : \alpha < \kappa^+\rangle$ is a $\diamondsuit(\kappa^+, E)$-sequence$]$.

Suppose not. Then for all $\beta \in \kappa$ there exist an $X_\beta \subseteq \kappa^+$ and a club C_β in κ^+ such that $X_\beta \cap \alpha \neq A^\beta_\alpha \cap \alpha$ for every $\alpha \in D_\beta$. Let

$X = \{f^{-1}(\delta, \beta) : \beta \in \kappa, \delta \in X_\beta\}$;
$D = \{\alpha \in \bigcap_{\beta \in \kappa} D_\beta : \mathrm{cf}(\alpha) = \kappa\}$.

Then D is club. Choose $\alpha \in D$ such that $X \cap \alpha \in S_\alpha$. Say $X \cap \alpha = B^\beta_\alpha$. Then

(3) $A^\beta_\alpha \cap \alpha = X_\beta \cap \alpha$.

In fact, if $\delta \in A^\beta_\alpha$, then $(\delta, \beta) \in f[B^\beta_\alpha] = f[X \cap \alpha]$; say $(\delta, \beta) = f(\gamma)$ with $\gamma \in X \cap \alpha$. Say $\gamma = f^{-1}(\varepsilon, \varphi)$ with $\varepsilon \in X_\varphi$. Then $(\delta, \beta) = f(\gamma) = (\varepsilon, \varphi)$, so $\delta = \varepsilon$ and $\beta = \varphi$. Thus $\delta \in X_\beta$. Conversely, if $\delta \in X_\beta \cap \alpha$, then $f^{-1}(\delta, \beta) \in X$. Now $(\delta, \beta) \in \alpha \times \kappa$ and $f[\alpha] = \alpha \times \kappa$, so $f^{-1}(\delta, \beta) \in \alpha$. Hence $\delta \in A^\beta_\alpha$. Thus (3) holds. This contradicts $\alpha \in D_\beta$. $\square$

Theorem 23.3. *Suppose that λ is regular. $\kappa^\lambda = \kappa$, and $2^\kappa = \kappa^+$. Then $\diamondsuit(\kappa^+, E^{\kappa^+}_\lambda)$ holds.*

Proof.

(1) There are κ^+ bounded subsets of κ^+.

In fact, let X be the collection of all bounded subsets of κ^+. Then $X = \{\mathrm{rng}(f) : \exists\alpha < \kappa^+[f \in {}^\alpha\kappa^+]\}$. For each $\alpha < \kappa^+$,

$$|{}^\alpha\kappa^+| = (\kappa^+)^{|\alpha|} \leq (\kappa^+)^\kappa = \left|\bigcup_{\beta<\kappa^+} {}^\alpha\beta\right| \leq \sum_{\beta<\kappa^+} 2^\kappa = \kappa^+.$$

So (1) holds.

Let $\langle x_\alpha : \alpha < \kappa^+ \rangle$ enumerate all the bounded subsets of κ^+. For each $\alpha \in E_\lambda^{\kappa^+}$ let

$$S_\alpha = \{Y \subseteq \alpha : \exists Z \in [\{x_\beta : \beta < \alpha\}]^{\leq \lambda}[Y = \bigcup Z]\}.$$

(2) $\forall \alpha \in E_\lambda^{\kappa^+}[|S_\alpha| \leq \kappa]$.

In fact,

$$|\{x_\beta : \beta < \alpha\}]^{\leq \lambda}\}| = \sum_{\mu \leq \lambda} (\kappa^+)^\mu \leq \lambda \cdot (\kappa^+)^\lambda = \kappa^\lambda = \kappa,$$

using (5.22).

We claim that $\langle S_\alpha : \alpha \in E_\lambda^{\kappa^+} \rangle$ satisfies the condition of Lemma 23.9. Suppose that $X \subseteq \kappa^+$. Let

$$C = \{\alpha < \kappa^+ : \forall \beta < \alpha \exists \gamma < \alpha[X \cap \beta = x_\gamma]\}.$$

(3) C is club in κ^+.

For, to show that C is closed in κ^+, suppose that $\alpha < \kappa^+$ is limit and $C \cap \alpha$ is unbounded in α. Suppose that $\beta < \alpha$. Choose $\delta \in C \cap \alpha$ with $\beta < \delta$. Then there is a $\gamma < \delta$ such that $X \cap \beta = x_\gamma$. This shows that $\alpha \in C$.

To show that C is unbounded in κ^+, suppose that $\delta < \kappa^+$. Let $\varepsilon_0 = \delta + 1$. If $\varepsilon_n < \kappa^+$ has been defined, for each $\beta < \varepsilon_n$ let γ_β be such that $X \cap \beta = x_{\gamma_\beta}$. Then let $\varepsilon_{n+1} = \sup\{\gamma_\beta + 1 : \beta < \varepsilon_n\}$. Let $\varepsilon_\omega = \sup_{n \in \omega} \varepsilon_n$. If $\beta < \varepsilon_\omega$, say $\beta < \varepsilon_n$. Then $X \cap \beta = x_{\gamma_\beta}$. We have $\gamma_\beta < \varepsilon_{n+1} \leq \varepsilon_\omega$. So $\varepsilon_\omega \in C$, as desired. This proves (3).

Let $\alpha \in E_\lambda^{\kappa^+}$. We claim that $X \cap \alpha \in S_\alpha$. (This finishes the proof.) Let $Z \subseteq \alpha$ be cofinal in α with $|Z| = \lambda$. For each $\beta \in Z$ let $\gamma(\beta)$ be such that $X \cap \beta = x_{\gamma(\beta)}$. Then

(4) $$X \cap \alpha = \bigcup_{\beta \in Z} x_{\gamma(\beta)}$$

In fact, if $\delta \in X \cap \alpha$, choose $\beta \in Z$ such that $\delta < \beta$. Then $\delta \in X \cap \beta = x_{\gamma(\beta)}$. This proves $\subseteq$ in (4).

If $\delta \in x_{\gamma(\beta)}$ with $\beta \in Z$, then $\delta \in X \cap \beta \subseteq X \cap \alpha$. This proves (4).

It follows that $X \cap \alpha \in S_\alpha$. $\square$

Lemma 23.4. *Let κ be a regular uncountable cardinal and let P be $< \kappa$-closed. Then every stationary $S \subseteq \kappa$ in M remains stationary in $M[G]$.*

Proof. Let $C \subseteq \kappa$ be club in $M[G]$. Let $\dot{C}$ be a name such that $\dot{C}^G = C$. Choose $p \in G$ such that $p \Vdash \dot{C}$ is club.

(*) $\{q \in P : q \leq p$ and $\exists \gamma \in S[q \Vdash \check{\gamma} \in \dot{C}]\}$ is dense below p.

We construct $\langle r_\alpha : \alpha < \kappa \rangle$ and $\langle \delta_\alpha : \alpha < \kappa \rangle$ be recursion, so that $p = r_0 \geq r_1 \geq \cdots$, $r_\alpha \Vdash \check{\delta}_\alpha \in \dot{C}$, and $\langle \delta_\alpha : \alpha < \kappa \rangle$ is strictly increasing. Now $p \Vdash \exists \varepsilon[\varepsilon$ is an ordinal and $\varepsilon \in \dot{C}]$. Thus $e(p) \leq \sum_{x \in M^P} ||x$ is an ordinal and $x \in \dot{C}||$, so by Lemma 14.38, $e(p) \leq \sum_{\alpha \in \mathbf{ON}} ||\check{\alpha} \in \dot{C}||$. Hence there is an $r_0 \leq p$ and a δ_0 such that $r_0 \Vdash \check{\delta}_0 \in \dot{C}$. Now assume that r_α and δ_α have been defined, so that $r_\alpha \Vdash \check{\delta}_\alpha \in \dot{C}$. Then $r_\alpha \Vdash \exists x[x$ is an ordinal and $\check{\delta}_\alpha < x$ and $x \in \dot{c}]$. Then by Lemma 14.38 we get $r_{\alpha+1} \le r_\alpha$ and $\delta_{\alpha+1} > \delta_\alpha$ such that $r_{\alpha+1} \Vdash \check{\delta}_{\alpha+1} \in \dot{C}$. For α limit we get $r_\alpha \le r_\beta$ for all $\beta < \alpha$. Let $\delta_\alpha = \sup_{\beta<\alpha} \delta_\beta$. Then $r_\alpha \Vdash \check{\delta}_\alpha \in \dot{C}$. This completes the construction. Choose $\varepsilon \in \operatorname{rng}(\delta) \cap S$. Then there is an α such that $r_\alpha \Vdash \check{\varepsilon} \in \dot{C}$. This proves $(*)$. Hence $S \cap C \neq \emptyset$. $\square$

Theorem 23.5. *Let S be a stationary subset of ω_1. Then there is a forcing poset P_S such that if G is P_S-generic over M, then $\bigcup G$ is a club which is $\subseteq S$, $\aleph_1$ is preserved, and if X is a countable set with $X \in M[G]$, then $X \in M$.*

Proof. Let P_S consist of all closed bounded subsets of S, with $p \le q$ iff $\exists \alpha < \omega[q = p \cap \alpha]$.

(1) $p \le q$ iff $q \subseteq p$ and $\forall \xi \in p \backslash q \forall \eta \in q[\eta \le \xi]$.

In fact, suppose that $p \le q$. Choose $\alpha < \omega_1$ such that $q = p \cap \alpha$. Suppose that $\xi \in p \backslash q$ and $\eta \in q$. Then $\alpha \le \xi$ and $\eta < \alpha$, so $\eta < \xi$.

Conversely, suppose that $q \subseteq p$ and $\forall \xi \in p \backslash q \forall \eta \in q[\eta \le \xi]$. Let $\alpha = \sup_{\eta \in q}(\eta + 1)$. If $\eta \in q$, then $\eta \in p \cap \alpha$. Conversely, if $\eta \in p \cap \alpha$ then $\eta \in q$. So (1) holds.

Now let G be P_S-generic over M. Clearly $\bigcup G \subseteq S$.

(2) If $p, q \in G$, then $p \le q$ or $q \le p$.

In fact, suppose that $p, q \in G$. Choose $r \in G$ with $r \le p, q$. Say $\alpha, \beta < \omega_1$ with $p = r \cap \alpha$ and $q = r \cap \beta$. Wlog $\alpha \le \beta$. Then $q \cap \alpha = r \cap \beta \cap \alpha = r \cap \alpha = p$, so $p \le q$. This proves (2).

(3) If $p \le q \le r$ then $p \le r$.

For, assume that $p \le q \le r$. Say $\alpha, \beta \in \omega_1$ with $q = p \cap \alpha$ and $r = q \cap \beta$.

Case 1. $\alpha \le \beta$. Then $r = q \cap \beta = p \cap \alpha \cap \beta = p \cap \alpha$ and so $p \le r$.

Case 9. $\beta < \alpha$. Then $r = q \cap \beta = p \cap \alpha \cap \beta = p \cap \beta$, so $p \le r$.

(4) $\forall \alpha < \omega_1[\{p \in P_S : p \neq \emptyset$ and $\alpha \le \max(p)\}$ is dense in P_S.

In fact, suppose that $\alpha < \omega_1$ and $q \in P_S$. Choose $\beta < \omega_1$ such that $\alpha < \beta$ and $\forall \gamma \in q[\gamma < \beta]$. Then let $p = q \cup \{\beta\}$. Then $p < q$, $p \neq \emptyset$, and $\alpha < \max(p)\}$ is dense in P_S, so (4) holds.

Now for any $\alpha < \omega_1$ choose $p \in G$ such that $p \neq \emptyset$ and $\alpha \le \max(p)\}$. This shows that $\bigcup G$ is unbounded in ω_1.

(5) Every closed and bounded subset of ω_1 has a maximum element.

This is clear.

(6) $\bigcup G$ is closed.

For, suppose that α is a limit ordinal less than ω_1 and $\bigcup G \cap \alpha$ is unbounded in α. Let $q \in G$ with $\alpha \le \max(q)$. Then $q \cap (\alpha + 1)$ is closed and bounded, and $q \le q \cap (\alpha + 1)$. So $q \cap (\alpha + 1) \in G$. By (2), each $r \in G \cap \alpha$ is contained in q, so by (5), $\alpha \in q \cap (\alpha + 1)$ and hence $\alpha \in G$.

Thus $\bigcup G$ is club and $\bigcup G \subseteq S$.

(7) P_S is ω-closed.

For, suppose that $p_0 > p_1 > \cdots$. Let $Q = \bigcup_{n\in\omega} P_n \cup \{\alpha\}$, where $\alpha = \bigcup_{n\in\omega} \max(P_n)$. Clearly $q < p_n$ for all n.

By Lemma 15.10, P_S is ω-distributive. Hence by Theorem 15.8, if $X \in M[G]$ is countable, then $X \in M$. Then $\aleph_1$ is preserved. $\square$

Lemma 23.6. *Let κ be regular uncountable, $\alpha < \kappa$, $\mathrm{cf}(\alpha) > \omega$, $S \subseteq \kappa$ stationary, and $\forall\beta \in S[\mathrm{cf}(\beta) \geq \mathrm{cf}(\alpha)]$. Then S does not reflect at α, i.e., $S \cap \alpha$ is not stationary in α.*

Proof. Let C be a club of α with order type $\mathrm{cf}(\alpha)$. Let β be an ordinal and $f : \beta \to \alpha$ a normal function with $\mathrm{rng}(f) = C$. Then $\beta = \mathrm{cf}(\alpha)$, and for each $\xi < \beta$ we have $\mathrm{cf}(f(\xi)) = \mathrm{cf}(\xi) \leq \xi < \beta = \mathrm{cf}(\alpha)$. Hence $C \cap S \cap \alpha = \emptyset$. $\square$

Corollary 23.7. $\forall\alpha < \omega_2[E^{\omega_2}_{\omega_1}$ *does not reflect in* $\alpha]$.

Proof. For $\mathrm{cf}(\alpha) > \omega$, in Lemma 23.6 take $\kappa = \omega_2$ and $S = E^{\omega_2}_{\omega_1}$. For $\mathrm{cf}(\alpha) = \omega$, let C be club in α of order type ω with each member of C a successor ordinal. $\square$

Lemma 23.8. *Suppose that κ is regular and uncountable, $\alpha < \kappa$, $\lambda < \mathrm{cf}(\alpha)$ regular. Then E^κ_λ reflects at α.*

Proof. Let $C \subseteq \alpha$ be club. Let $\delta : \beta \to \alpha$ be a normal function with $C = \mathrm{rng}(\delta$. Then $\mathrm{cf}(\alpha) \leq |\beta| \leq \beta$. Then $cf(delta_\lambda) = \lambda$, so $\delta_\lambda \in C \cap E^\kappa_\lambda$. $\square$

Corollary 23.9. $E^{\omega_2}_\omega$ *refects at each* $\alpha \in E^{\omega_2}_{\omega_1}$.

Proof. In Lemma 23.8 take $\kappa = \omega_2$ and $\lambda = \omega$. $\square$

24. Cardinal arithmetic

Cartesian products are denoted by $\prod_{i\in A} A_i$, while the product of cardinals κ_i is denoted by $\prod^c_{i\in I}\kappa_i$.

Theorem 24.1. *Let κ and λ be uncountable regular cardinals such that $\forall\delta<\lambda[\delta^\kappa<\lambda]$. Assume that $\langle\mu_\alpha:\alpha<\kappa\rangle$ is a sequence of cardinals such that $\forall\alpha<\kappa[\prod^c_{\beta<\alpha}\mu_\beta<\aleph_\lambda]$.*

Then $\prod^c_{\alpha<\kappa}\mu_\alpha<\aleph_\lambda$.

Corollary 24.2. *Let κ and λ be uncountable regular cardinals such that $\forall\delta<\lambda[\delta^\kappa<\lambda]$. Suppose that $\forall\sigma<\kappa[\tau^\sigma<\aleph_\lambda]$.*

Then $\tau^\kappa<\aleph_\lambda$.

Proof. Assume the hypotheses. For each $\alpha<\kappa$ let $\mu_\alpha=\tau$. Then for any $\alpha<\kappa$, $\prod^c_{\beta<\alpha}\mu_\beta=\tau^{|\alpha|}<\aleph_\lambda$. Hence by Theorem 1, $\tau^\kappa=\prod^c_{\alpha<\kappa}\mu_\alpha<\aleph_\lambda$. □

Corollary 24.3. *Let κ and λ be uncountable regular cardinals such that $\forall\delta<\lambda[\delta^\kappa<\lambda]$. Suppose that τ is a cardinal such that $\mathrm{cf}(\tau)=\kappa$, and $\forall\sigma<\tau[2^\sigma<\aleph_\lambda]$.*

Then $2^\tau<\aleph_\lambda$.

Proof. Clearly

(1) There is a sequence $\langle\nu_\xi:\xi<\kappa\rangle$ of cardinals such that $\forall\xi,\eta[\xi<\eta<\kappa\to\nu_\xi\le\nu_\eta<\tau]$ and $\sum_{\xi<\kappa}\nu_\xi=\tau$.

Now for each $\xi<\kappa$ let $\mu_\xi=2^{\nu_\xi}$. Now suppose that $\alpha<\kappa$. Let $\sigma=\sum_{\beta<\alpha}\nu_\beta$. Then

$$\sigma\le|\beta|\cdot\nu_\beta=\begin{cases}|\beta| & \text{if } \kappa=\tau,\\ <\tau & \text{if } \kappa<\tau.\end{cases}$$

Hence

$$\prod^c_{\beta<\alpha}\mu_\beta=\prod^c_{\beta<\alpha}2^{\nu_\beta}=2^\sigma<\aleph_\lambda,$$

Hence by Theorem 1,

$$2^\tau=2^{\sum_{\xi<\kappa}\nu_\xi}=\prod^c_{\xi<\kappa}2^{\nu_\xi}=\prod^c_{\xi<\kappa}\mu_\xi<\aleph_\lambda.$$ □

Corollary 24.4. *Let κ be an uncountable regular cardinal and let ρ and τ be cardinals such that $2\le\rho$ and $\forall\sigma<\kappa[\tau^\sigma<\aleph_{(\rho^\kappa)^+}]$*

Then $\tau^\kappa<\aleph_{(\rho^\kappa)^+}$.

Proof. Let $\lambda=(\rho^\kappa)^+$. Then for all $\delta<\lambda$, $\delta^\kappa\le(\rho^\kappa)^\kappa=\rho^\kappa>\lambda$. Also, $\forall\sigma<\kappa[\tau^\sigma<\aleph_{(\rho^\kappa)^+}=\aleph_\lambda$. Hence by Corollary 2, $\tau^\kappa<\aleph_\lambda=\aleph_{(\rho^\kappa)^+}$. □

Corollary 24.5. *Suppose that ρ and τ are cardinals, $\rho\ge2$, $\mathrm{cf}(\tau)=\kappa>\omega$, and $\forall\sigma<\tau[2^\sigma<\aleph_{(\rho^\kappa)^+}]$.*

Then $2^\tau < \aleph_{(\rho^\kappa)^+}$.

Proof. Let $\lambda = (\rho^\kappa)^+$. If $\delta < \lambda$, then $\delta \le \rho^\kappa$ and so $\delta^\kappa < \lambda$. If $\sigma < \tau$, then $2^\sigma < \aleph_\lambda$. Hence by Corollary 3, $2^\tau < \aleph_{(\rho^\kappa)^+}$. $\square$

Corollary 24.6. *Let ξ be an ordinal with* $\mathrm{cf}(\xi) > \omega$. *Assume that* $\forall\alpha < \xi[2^{\aleph_\alpha} < \aleph_{(|\xi|^{\mathrm{cf}(\xi)})^+}]$.

Then $2^{\aleph_\xi} < \aleph_{(|\xi|^{\mathrm{cf}(\xi)})^+}$.

Proof. Let $\rho = |\xi|$, $\tau = \aleph_\xi$, and $\kappa = \mathrm{cf}(\xi)$. Then $|\xi| \ge 2$ and $\mathrm{cf}(\tau) = \mathrm{cf}(\xi) = \kappa > \omega$

(1) $\forall\sigma < \tau[2^\sigma < \aleph_{(\rho^\kappa)^+}]$.

In fact, suppose that $\sigma < \tau$.

Case 1. $\sigma < \omega$. Obiously then $2^\sigma < \aleph_{(\rho^\kappa)^+}$.

Case 9. $\sigma = \aleph_\alpha$ for some α. Then $\alpha < \xi$, so

$$2^\sigma = 2^{\aleph_\alpha} < \aleph_{(|\xi|^{\mathrm{cf}(\xi)})^+} = \aleph_{(\rho^\kappa)^+}.$$

Thus $\forall\sigma < \tau[2^\sigma < \aleph_{(\rho^\kappa)^+}]$. It follows from Corollary 5 that $2^{\aleph_\xi} = 2^\tau < \aleph_{(\rho^\kappa)^+} = \aleph_{(|\xi|^{\mathrm{cf}(\xi)})^+}$. $\square$

Corollary 24.7. *If $\aleph_\alpha$ is strong limit singular with* $\mathrm{cf}(\alpha) > \omega$, *then* $2^{\aleph_\alpha} < \aleph_{(2^{|\alpha|})^+}$.

Proof. Assume that $\aleph_\alpha$ is strong limit singular with $\mathrm{cf}(\alpha) > \omega$. Let $\rho = 2$, $\kappa = |\alpha|^+$, and $\tau = \aleph_\alpha$. If $\sigma < \tau$, then $2^\sigma < \aleph_\alpha \le \aleph_{(2^{|\alpha|})^+}$. Hence by Corollary 6, $2^{\aleph_\alpha} < \aleph_{(2^{|\alpha|})^+}$. $\square$

Corollary 24.8. *Let ξ be an ordinal with* $\mathrm{cf}(\xi) > \omega$. *Assume* $\forall\sigma < \mathrm{cf}(\xi)\forall\alpha < \xi[\aleph_\alpha^\sigma < \aleph_{(|\xi|^{\mathrm{cf}(\xi)})^+}]$.

Then $\aleph_\xi^{\mathrm{cf}(\xi)} < \aleph_{(|\xi|^{\mathrm{cf}(\xi)})^+}]$.

Proof. Let $\rho = |\xi|$, $\kappa = \mathrm{cf}(\xi)$, and $\tau = \aleph_\xi$. Then $\rho \ge 2$ and for all $\sigma < \kappa$,

$$\begin{aligned}\tau^\sigma = \aleph_\xi^\sigma = |{}^\sigma\aleph_\xi| = \left|{}^\sigma\left(\bigcup_{\alpha<\xi}\aleph_\alpha\right)\right| = \left|\bigcup_{\alpha<\xi}({}^\sigma\aleph_\alpha)\right| \le \sum_{\alpha<\xi}\aleph_\alpha^\sigma \\ \le |\xi| \cdot \aleph_{|\xi|^{\mathrm{cf}(\xi)}} = \aleph_{|\xi|^{\mathrm{cf}(\xi)}} < \aleph_{(|\xi|^{\mathrm{cf}(\xi)})^+} = \aleph_{(\rho^\kappa)^+}\end{aligned}$$

Hence by Corollary 4, $\aleph_\xi^{\mathrm{cf}(\xi)} = \tau^\kappa < \aleph_{(\rho^\kappa)^+} = \aleph_{(|\xi|^{\mathrm{cf}(\xi)})^+}$. $\square$

Corollary 24.9. *If* $\forall\alpha < \omega_1[2^{\aleph_\alpha} < \aleph_{(2^{\aleph_1})^+}]$, *then* $2^{\aleph_{\omega_1}} < \aleph_{(2^{\aleph_1})^+}$.

Proof. Take $\xi = \omega_1$ in Corollary 6. $\square$

Corollary 24.10. *If* $\forall\alpha < \omega_1[\aleph_\alpha^\omega < \aleph_{(2^{\aleph_1})^+}]$, *then* $\aleph_{\omega_1}^{\aleph_1} < \aleph_{(2^{\aleph_1})^+}$.

Proof. Take $\xi = \omega_1$ in Corollary 8. $\square$

If $A = \langle A_\alpha : \alpha < \kappa\rangle$ is a system of sets, an *almost disjoint transversal for* A, a.d.t., is a set $F \subseteq \prod_{\alpha<\kappa} A_\alpha$ such that $\forall f, g \in F[f \neq g \to |\{\alpha < \kappa : f(\alpha) = g(\alpha)\}| < \kappa]$.

Lemma 24.11. *Let $\langle \kappa_\alpha : \alpha < \lambda\rangle$ be a system of cardinals, with λ a cardinal. For each $\alpha < \lambda$ let $A_\alpha = \prod_{\beta<\alpha} \kappa_\beta$. Then there is an a.d.t. F for A with $|F| = \prod^c_{\alpha<\lambda} \kappa_\alpha$.*

Proof. Let $\tau = \prod^c_{\alpha<\lambda} \kappa_\alpha$, and let $\langle g_\xi : \xi < \tau\rangle$ enumerate $\prod_{\alpha<\lambda} \kappa_\alpha$ without repetitions. For each $\xi < \tau$ and $\alpha < \lambda$ let $f_\xi(\alpha) = g_\xi \restriction \alpha$. Thus $f_\xi \in \prod_{\alpha<\lambda} A_\alpha$ for each $\xi < \tau$. If $\xi, \eta < \tau$ and $\xi \neq \eta$, then $g_\xi \neq g_\eta$; choose β minimum so that $g_\xi(\beta) \neq g_\eta(\beta)$. Then for any $\alpha < \lambda$, $f_\xi(\alpha) = f_\eta(\alpha)$ iff $g_\xi \restriction \alpha = g_\eta \restriction \alpha$ iff $\alpha \leq \beta$. So $|\{\alpha < \lambda : f_\xi(\alpha) = f_\eta(\alpha)\}| < \kappa$. Hence $\mathrm{rng}(f)$ is the required a.d.t. $\square$

Lemma 24.12. *Let λ be an uncountable regular cardinals, and κ a cardinal. Assume that $\forall\delta < \lambda[\delta^\kappa < \lambda]$. Let $A = \langle A_\alpha : \alpha < \kappa\rangle$ be a system of sets such that $\forall\alpha < \kappa[|A_\alpha| < \aleph_\lambda]$. Suppose that F is an a.d.t. for A,*

Then $|F| < \aleph_\lambda$.

Proof of the theorem. Assume the hypotheses, and for all $\alpha < \kappa$ let $A_\alpha = \prod_{\beta<\alpha} \mu_\beta$. By Lemma 11 there is an a.d.t. F for A with $|F| = \prod^c_{\alpha<\lambda} \mu_\alpha$. Now for each $\alpha < \kappa$, $|A_\alpha| = \prod^c_{\beta<\alpha} \mu_\beta < \alpha_\lambda$. Then by Lemma 12, $\prod^c_{\alpha<\lambda} \mu_\alpha = |F| < \aleph_\lambda$. $\square$

Proof of Lemma 24.12 First note:

(1) If $|A_\alpha| = |B_\alpha|$ for all $\alpha < \kappa$, and let τ be a cardinal. Then (there is an a.d.t. F for A with $|F| = \tau$) iff (there is an a.d.t. G for B with $|G| = \tau$).

In fact, assume that $|A_\alpha| = |B_\alpha|$ for all $\alpha < \kappa$. By symmetry it suffices to assume that F is an a.d.t. for A and find an a.d.t. G for B such that $|F| = |G|$. For each $\alpha < \kappa$ let $f_\alpha : A_\alpha \to B_\alpha$ be a bijection. For each $g \in F$ define $g' \in \prod_{\alpha<\kappa} B_\alpha$ by setting $g'(\alpha) = f_\alpha(g(\alpha))$. Let $G = \{g' : g \in F\}$. If $g, h \in F$ and $g \neq h$, then

$$g(\alpha) = h(\alpha) \quad \text{iff} \quad f_\alpha(g(\alpha)) = f_\alpha(h(\alpha)) \quad \text{iff} \quad g'(\alpha) = h'(\alpha).$$

It follows that $|F| = |G|$ and G is an a.d.t. for B. So (1) holds.

Let ${}^\kappa\mathbf{ON}$ be the class of ordinal-valued functions with domain κ. For $\varphi, \psi \in {}^\kappa\mathbf{ON}$ define $\varphi \prec \psi$ iff $|\{\alpha < \kappa : \varphi(\alpha) \geq \psi(\alpha)\}| < \kappa$.

(2) $\prec$ is a well-founded partial order on ${}^\kappa\mathbf{ON}$.

In fact, clearly $\prec$ is irreflexive. Suppose that $\varphi \prec \psi \prec \theta$. Then

$$\{\alpha < \kappa : \varphi(\alpha) < \psi(\alpha)\} \cap \{\alpha < \kappa : \psi(\alpha) < \theta(\alpha)\} \subseteq \{\alpha < \kappa : \varphi(\alpha) < \theta(\alpha)\},$$

so

$$\{\alpha < \kappa : \varphi(\alpha) \geq \theta(\alpha)\} \subseteq \{\alpha < \kappa : \varphi(\alpha) \geq \psi(\alpha)\} \cup \{\alpha < \kappa :: \psi(\alpha) \geq \theta(\alpha)\},$$

and hence $\varphi \prec \theta$.

Now suppose that $\cdots \varphi_{n+1} \prec \varphi_n \prec \cdots \prec \varphi_0$. For all $n \in \omega$ let $X_n = \{\alpha < \kappa : \varphi_{n+1}(\alpha) \geq \varphi_n(\alpha)\}$. Then $|X_n| < \kappa$ for all $n \in \omega$. Then $Y \stackrel{\text{def}}{=} \bigcup_{n\in\omega} X_n$ has size less than κ. Choose $\alpha \in \kappa\backslash Y$. Then $\forall n[\varphi_{n+1}(\alpha) < \varphi_n(\alpha)]$, contradiction. So (2) holds.

For each $\varphi \in {}^\kappa\mathbf{ON}$ let

$$T(\varphi) = \sup\{|F| : F \text{ is an a.d.t. for } \varphi\}.$$

(3) It suffices to show that $\forall\varphi \in {}^\kappa\lambda[T(\aleph \circ \varphi) < \aleph_\lambda]$.

In fact, assume the statement in (3), and suppose that $\forall\delta < \lambda[\delta^\kappa < \lambda]$, $A = \langle A_\alpha : \alpha < \kappa\rangle$, $\forall\alpha < \kappa[|A_\alpha| < \aleph_\lambda]$, and F is an a.d.t. for A. For each $\alpha < \kappa$ let

$$A'_\alpha = \begin{cases} A_\alpha & \text{if } A_\alpha \text{ is infinite,} \\ B_\alpha & \text{with } A_\alpha \subseteq B_\alpha \text{ and } |B_\alpha| = \omega \text{ otherwise.} \end{cases}$$

Clearly F is an a.d.t. for B. Now let $\varphi \in {}^\kappa\lambda$ be such that $|B_\alpha| = \aleph_{\varphi(\alpha)}$ for all $\alpha < \kappa$. By (1) there is an a.d.t. G for $\aleph \circ \varphi$ such that $|F| = |G|$. Thus by (3), $|F| < \aleph_\lambda$.

Now we prove (3) by contradiction: suppose it does not hold, and let $\varphi \in {}^\kappa\lambda$ be minimal such that $\aleph_\lambda \leq T(\aleph \circ \varphi)$. We define

$$I = \{X \subseteq \kappa : \exists\psi \in {}^\kappa\lambda[\forall\alpha \in X[\psi(\alpha) < \varphi(\alpha) \text{ or } \psi(\alpha) = 0] \text{ and } T(\aleph \circ \psi) \geq \aleph_\lambda]\}.$$

Obviously

(4) If $Y \subseteq X \in I$ then $Y \in I$.

(5) $[\kappa]^{<\kappa} \subseteq I$.

In fact, let $X \in [\kappa]^{<\kappa}$. For any $\alpha \in \kappa$ define

$$\psi(\alpha) = \begin{cases} 0 & \text{if } \alpha \in X, \\ \varphi(\alpha) & \text{if } \alpha \notin X. \end{cases}$$

We claim that ψ shows that $X \in I$. For this it suffices to show that $T(\aleph \circ \psi) \geq \aleph_\lambda$. Let F be an a.d.t. for $\aleph \circ \varphi$ such that $|F| \geq \aleph_\lambda$. For each $f \in F$ define $f' \in \prod_{\alpha<\kappa} \aleph_{\psi(\alpha)}$ by setting for any $\alpha \in \kappa$

$$f'(\alpha) = \begin{cases} 0 & \text{if } \alpha \in X, \\ f(\alpha) & \text{if } \alpha \notin X. \end{cases}$$

If $f, g \in F$ and $f' = g'$, then $\{\alpha < \kappa : f(\alpha) = g(\alpha)\} \supseteq (\kappa\backslash X)$ and $\kappa\backslash X$ has size κ, so $f = g$. Let $G = \{f' : f \in F\}$. Thus $|G| = |F|$. If $f, g \in F$ and $f \neq g$, then

$$\{\alpha < \kappa : f'(\alpha) = g'(\alpha)\} = X \cup \{\alpha \in \kappa\backslash X : f(\alpha) = g(\alpha)\},$$

and this set has size less than κ. It follows that G is an a.d.t. for $\aleph \circ \psi$ of size $\geq \aleph_\lambda$, proving (5).

(6) I is κ-complete.

For, suppose that $0 < \delta < \kappa$ and $X_\mu \in I$ for all $\mu < \delta$. Say that for all $\mu < \delta$ we have $\psi_\mu \in {}^\kappa\lambda$ such that

$$\forall \alpha \in X_\mu[\psi_\mu(\alpha) < \varphi(\alpha) \text{ or } \psi_\mu(\alpha) = 0] \text{ and } T(\aleph \circ \psi_\mu) \geq \aleph_\lambda.$$

For all $\alpha < \kappa$ let $\chi(\alpha) = \min_{\mu<\delta} \psi_\mu(\alpha)$.

(7) There is a system $\langle S_\mu : \mu < \delta\rangle$ of pairwise disjoint subsets of κ such that $\bigcup_{\mu<\delta} S_\mu = \kappa$ and $\forall \mu < \delta \forall \alpha \in S_\mu[\chi(\alpha) = \psi_\mu(\alpha)]$.

In fact, for each $\mu < \delta$ let $S'_\mu = \{\alpha < \kappa : \chi(\alpha) = \psi_\mu(\alpha)\}$. Then let $S_\mu = S'_\mu \backslash \bigcup_{\nu<\mu} S'_\nu$. Clearly $\langle S_\mu : \mu < \delta\rangle$ is a pairwise disjoint system of subsets of κ. For any $\alpha \in \kappa$ let μ be minimum such that $\alpha \in S'_\mu$. Then $\alpha \in S_\mu$. So $\bigcup_{\mu<\delta} S_\mu = \kappa$. Clearly $\forall \mu < \delta \forall \alpha \in S_\mu[\chi(\alpha) = \psi_\mu(\alpha)]$. Thus (7) holds.

(8) If τ is a cardinal and $\forall \mu < \delta[\langle f_{\mu\xi} : \xi < \tau\rangle$ is an a.d.t. for $\aleph \circ \psi_\mu]$, and $\forall \xi < \tau[h_\xi = \bigcup_{\mu<\delta}(f_{\mu\xi} \restriction S_\mu)]$, then $F \stackrel{\text{def}}{=} \{h_\xi : \xi < \tau\}$ is an a.d.t. for $\aleph \circ \chi$.

In fact, if $\xi < \tau$ then $\forall \mu < \delta[h_\xi \restriction S_\mu \in \prod_{\alpha \in S_\mu} \aleph_{\psi_\mu}]$, and hence $h_\xi \in \prod_{\alpha\in\kappa} \aleph_{\chi(\alpha)}$. If $\xi, \eta < \tau$ and $\xi \neq \eta$, then

$$\{\alpha < \kappa : h_\xi(\alpha) = h_\eta(\alpha)\} = \bigcup_{\mu<\delta} \{\alpha \in S_\mu : f_{\mu\xi}(\alpha) = f_{\mu,\eta}(\alpha)\}.$$

Since $\delta < \kappa$ and $\forall \mu < \delta[|\{\alpha \in S_\mu : f_{\mu\xi}(\alpha) = f_{\mu,\eta}(\alpha)\}| < \kappa$ and κ is regular, we have $|\{\alpha < \kappa : h_\xi(\alpha) = h_\eta(\alpha)\}| < \kappa$. This proves (8).

Now clearly $\forall \alpha \in \bigcup_{\alpha<\delta} X_\mu[\chi(\alpha) < \varphi(\alpha)$ or $\chi(\alpha) = 0]$. Also, for each $\mu < \delta$ we have an a.d.t. G_μ for $\aleph \circ \psi_\mu$ with $|G_\mu| \geq \aleph_\lambda$. Choose $\mu < \delta$ with $|G_\mu|$ minimum, and now for any $\nu < \delta$ let G'_ν be a subset of G_ν of size $|G_\mu|$. Say $|G_\mu| = \tau$. Write $G'_\nu = \{f_{\mu\xi} : \xi < \tau\}$. Then by (8) we get an a.d.t. F for $\aleph \circ \chi$ of size $|G_\mu$. Hence $\bigcup_{\mu<\delta} X_\mu \in I$, proving (6).

Now let

$$\begin{aligned}
X_0 &= \{\alpha < \kappa : \varphi(\alpha) = 0\};\\
X_1 &= \{\alpha < \kappa : \varphi(\alpha) \text{ is a limit ordinal}\};\\
X_2 &= \{\alpha < \kappa : \varphi(\alpha) \text{ is a successor ordinal}\}.
\end{aligned}$$

(9) $|X_0| < \kappa$ (Hence $X_0 \in I$.)

For, suppose that $|X_0| = \kappa$. Then we claim

(10) $T(\aleph \circ \varphi) \leq \aleph_0^\kappa$.

For, suppose that F is an a.d.t for $\aleph \circ \varphi$ and $|F| > \aleph_0^\kappa$. Then there exist distinct $f, g \in F$ such that $f \restriction X = g \restriction X$. So $|\{\alpha < \kappa : f(\alpha) = g(\alpha)\}| = \kappa$, contradiction. So (10) holds.

But by an assumption of the lemma, $\forall \delta < \lambda[\delta^\kappa < \lambda]$. So $\aleph_0^\kappa < \lambda \leq \aleph_\lambda$. Hence (10) contradicts the choice of φ. Hence (9) holds.

(11) I is a proper ideal.

Suppose to the contrary that $\kappa \in I$. Choose $\psi \in {}^\kappa\lambda$ so that $\forall\alpha \in \kappa[\psi(\alpha) < \varphi(\alpha)$ or $\psi(\alpha) = 0]$ and $T(\aleph \circ \psi) \geq \aleph_\lambda]\}$. Then $\psi \prec \varphi$, since $\{\alpha < \kappa : \psi(\alpha) \geq \varphi(\alpha)\} = \{\alpha < \kappa : \varphi(\alpha) = 0\} = X_0$. Since $T(\aleph \circ \psi) \geq \aleph_\lambda$, this contradicts the minimality of φ. So (11) holds.

(12) $X_1 \in I$.

To prove this, first note that since λ is uncountable and regular and $\kappa < \lambda$, there is an ordinal $\rho < \lambda$ such that $\varphi \in {}^\kappa\rho$. Let Q be the set of all functions $\chi \in {}^\kappa\lambda$ such that $\forall\alpha \in X_1[\psi(\alpha) < \varphi(\alpha)]$ and $\forall\alpha \in (\kappa\backslash X_1)[\psi(\alpha) = 0]$. Then $|Q| \leq |\rho|^\kappa < \lambda$. Now since $T(\aleph\circ\varphi) \geq \aleph_\lambda$, for each $\mu < \lambda$ there is an a.d.t. F_μ for $\aleph\circ\varphi$ such that $|F_\mu| > \aleph_\mu$. For each $\mu < \lambda$ and $\psi \in Q$ let $F^\psi_\mu = F_\mu \cap \prod_{\alpha<\kappa} \aleph_{\psi(\alpha)}$.

(13) $\forall\mu < \lambda\forall\psi \in Q[F^\psi_\mu$ is an a.d.t. for $\aleph \circ \psi]$.

In fact, $F^\psi_\mu \subseteq \prod_{\alpha<\kappa} \aleph_{\psi(\alpha)}$. Suppose that $f, g \in F^\psi_\mu$ and $f \neq g$. Since $F^\psi_\mu \subseteq F_\mu$ it follows that $|\{\alpha < \kappa : f(\alpha) = g(\alpha)\}| < \kappa$. So (13) holds.

(14) $F_\mu = \bigcup_{\psi\in Q} F^\psi_\mu$.

In fact, $\supseteq$ is clear. Now if $f \in F_\mu$, then for all $\alpha \in X_1$, $\varphi(\alpha)$ is a limit ordinal, and $f(\alpha) \in \aleph_{\varphi(\alpha)}$, so there is a $\psi(\alpha) < \varphi(\alpha)$ such that $f(\alpha) \in \aleph_{\psi(\alpha)}$; and let $\psi(\alpha) = 0$ for $\alpha \in \kappa\backslash X_1$. Then $\psi \in Q$ and $f \in F^\psi_\mu$. This proves (14).

Now if $|Q| \leq \mu < \lambda$, then $|F_\mu| > \aleph_\mu$, and hence by (14) there is a $\psi_\mu \in Q$ such that $|F^{\psi_\mu}_\mu| > \aleph_\mu$. Now $\lambda\backslash|Q| = \bigcup_{\chi\in Q}\{\mu \in \lambda\backslash|Q| : \psi_\mu = \chi\}$ and λ is regular, so there is a $\chi \in Q$ such that $|\{\mu \in \lambda\backslash|Q| : \psi_\mu = \chi\}| = \lambda$. Thus for every $\mu < \lambda$ choose $\mu' \in \lambda\backslash|Q|$ such that $\mu < \mu'$ and $\psi_{\mu'} = \chi$. Then $F^\chi_{\mu'} = F_{\mu'} \cap \prod_{\alpha<\kappa} \aleph_{\chi(\alpha)}$ has size $> \aleph_{\mu'}$. Hence $T(\aleph \circ \chi) \geq \aleph_\lambda$. This proves (12).

For each $X \subseteq X_2$ define $\psi_X \in {}^\kappa\lambda$ as follows: for any $\alpha \in \kappa$ let

$$\psi_X(\alpha) = \begin{cases} \varphi(\alpha) - 1 & \text{if } \alpha \in X, \\ \varphi(\alpha) & \text{if } \alpha \notin X. \end{cases}$$

(15) For all $X \in \mathscr{P}(X_2)\backslash I$ there is a $\rho(X) < \lambda$ such that $T(\aleph \circ \psi_X) \leq \aleph_{\rho(X)}$.

In fact, clearly $\forall\alpha \in X[\psi_X(\alpha) < \varphi(\alpha)]$. Since $X \notin I$, it follows that $T(\aleph \circ \psi_X) < \aleph_\lambda$, so (15) follows.

Now clearly

(16) There is a $\rho < \lambda$ such that $2^\kappa \leq \aleph_\rho$ and $\forall X \in \mathscr{P}(X_2)\backslash I[\rho(X) \leq \rho]$.

Now let F be an a.d.t. for $\aleph \circ \varphi$ such that $|F| > \aleph_{\rho+1}$. For all $f \in F$ and $X \in \mathscr{P}(X_2)\backslash I$ let $H_X(f) = \{g \in F : \forall\alpha \in X[g(\alpha) < f(\alpha)]\}$. Now for all $f \in F$ and $X \in \mathscr{P}(X_2)\backslash I$ and all $\alpha < \kappa$ let

$$A^{fX}_\alpha = \begin{cases} f(\alpha) & \text{if } \alpha \in X, \\ \aleph_{\varphi(\alpha)} & \text{if } \alpha \notin X. \end{cases}$$

(17) For all $f \in F$ and $X \in \mathscr{P}(X_2)\backslash I$, $H_X(f)$ is an a.d.t. for A^{fX}.

In fact, if $g \in H_X(f)$, then $\alpha \in X \to [g(\alpha) < f(\alpha) = A^{fX}_\alpha]$ and $\alpha \in \kappa\backslash X \to [g(\alpha) \in \aleph_{\varphi(\alpha)} = A^{fX}_\alpha]$. Thus $H_X(f) \subseteq \prod_{\alpha<\kappa} A^{fX}_\alpha$. Now suppose that $g, h \in H_X(f)$ with $g \neq h$. Then $g, h \in F$, and hence $|\{\alpha < \kappa : g(\alpha) = h(\alpha)\}| < \kappa$. so (17) holds.

(18) $\forall f \in F \forall X \in \mathscr{P}(X_2)\backslash I \forall \alpha < \kappa[|A^{fX}_\alpha| \leq \aleph_{\psi_X(\alpha)}]$.

In fact, assume that $f \in F$, $X \in \mathscr{P}(X_2)\backslash I$, and $\alpha < \kappa$. If $\alpha \in X$, then $A^{fX}_\alpha = f(\alpha)$, and $f(\alpha) \in \aleph_{\varphi(\alpha)}$. Since $X \subseteq X_2$, we have $\varphi(\alpha) = (\varphi(\alpha) - 1) + 1$, and hence $|A^{fX}_\alpha| = |f(\alpha)| \leq \aleph_{\psi_X(\alpha)}$. If $\alpha \notin X$, then $A^{fX}_\alpha = \aleph_{f(\alpha)}$ and so $|A^{fX}_\alpha| = \aleph_{f(\alpha)} = \aleph_{\psi_X(\alpha)}$. So (18) holds.

(19) For all $f \in F$ and $X \in \mathscr{P}(X_2)\backslash I$ there is an a.d.t. G^{fX} for $\aleph \circ \psi_X$ such that $|G^{fX}| = |H_X(f)|$.

Assume that $f \in F$ and $X \in \mathscr{P}(X_2)\backslash I$. By (18) for each $\alpha < \kappa$ let h_α be an injection of A^{fX}_α into $\aleph_{\psi_X(\alpha)}$. By (1) and (17), there is an a.d.t. G^{fX} for $\langle h_\alpha[A^{fx}_\alpha] : \alpha < \kappa$ such that $|G^{fX}| = |H_X(f)|$. Clearly G^{fX} is an a.d.t. for $\aleph \circ \psi_X$. So (19) holds.

Thus

$$\forall f \in F \forall X \in \mathscr{P}(X_2)\backslash I[|H_X(f)| = |G^{fX}| \leq T(\aleph \circ \psi_X) \leq \aleph_{\rho(X)} \leq \aleph_\rho.$$

Now for any $f \in F$ let $H(f) = \bigcup\{H_X(f) : X \in \mathscr{P}(X_2)\backslash I\}$. Then for any $f \in F$, $|H(f)| \leq 2^\kappa \cdot \aleph_\rho = \aleph_\rho$. Recall that $F > \aleph_{\rho+1}$. Let $G \subseteq F$ with $|G| = \aleph_{\rho+1}$.

(20) $(F\backslash G)\backslash \bigcup_{g\in G} H(g) \neq \emptyset$.

In fact, $|F| > \aleph_{\rho+1}$, $|G| = \aleph_{\rho+1}$, and $\forall g \in H[|H(g)| \leq \aleph_\rho]$. So (20) is clear.

We choose $f_0 \in (F\backslash G)\backslash \bigcup_{g\in G} H(g)$. Clearly $G\backslash H(f_0) \neq \emptyset$; we choose $g_0 \in G\backslash H(f_0)$. Clearly

(21) $f_0, g_0 \in F$, $f_0 \neq g_0$, $f_0 \notin H(g_0)$ and $g_0 \notin H(f_0)$.

(22) $\{\alpha < \kappa : f_0(\alpha) = g_0(\alpha)\} \in I$.

This holds since $f, g \in F$ and $f \neq g$, by (5).

(23) $\{\alpha \in X_2 : f_0(\alpha) < g_0(\alpha)\} \in I$.

In fact, let $X = \{\alpha \in X_2 : f_0(\alpha) < g_0(\alpha)\}$. Then $f \in H_X(g)$. If $X \notin I$, then $f \in H(g)$, contradicting (21). So (23) holds. Similarly,

(24) $\{\alpha \in X_2 : g_0(\alpha) < f_0(\alpha)\} \in I$.

Now

$$\begin{aligned}\kappa = X_0 \cup X_1 \cup \{\alpha < \kappa : f_0(\alpha) = g_0(\alpha)\}\\ \cup \{\alpha \in X_2 : f_0(\alpha) < g_0(\alpha)\} \cup \{\alpha \in X_2 : g_0(\alpha) < f_0(\alpha)\},\end{aligned}$$

and all the sets on the right are in I, by (9), (12), (22), (23), and (24). This contradicts (6) and (11). $\square$

PCF

We follow rather closely the chapter of Abraham and Magidor in the Handbook of Set Theory.

24a. Cofinality of posets

We begin the study of possible cofinalities of partially ordered sets—the PCF theory. In this chapter we develop some combinatorial principles needed for the main results.

Ordinal-valued functions and their orderings

A *filter* on a set A is a collection F of subsets of A with the following properties:

(1) $A \in F$.
(2) If $X \in F$ and $X \subseteq Y \subseteq A$, then $Y \in F$.
(3) If $X, Y \in F$ then $X \cap Y \in F$.

A filter F is *proper* iff $F \neq \mathscr{P}(A)$.

Suppose that F is a filter on a set A and $R \subseteq \mathbf{On} \times \mathbf{On}$. Then for functions $f, g \in {}^A\mathbf{On}$ we define

$$f \, R_F \, g \quad \text{iff} \quad \{i \in A : f(i) \, R \, g(i)\} \in F.$$

The most important cases of this notion that we will deal with are $f <_F g$, $f \leq_F g$, and and $f =_F g$. Thus

$$\begin{aligned} f <_F g \quad &\text{iff} \quad \{i \in A : f(i) < g(i)\} \in F;\\ f \leq_F g \quad &\text{iff} \quad \{i \in A : f(i) \leq g(i)\} \in F;\\ f =_F g \quad &\text{iff} \quad \{i \in A : f(i) = g(i)\} \in F. \end{aligned}$$

Sometimes we use this notation for ideals rather than filters, using the duality between ideals and filters, which we now describe. An *ideal* on a set A is a collection I of subsets of A such that the following conditions hold:

(4) $\emptyset \in I$
(5) If $X \subseteq Y \in I$ then $X \in I$.
(6) If $X, Y \in I$ then $X \cup Y \in I$.

An ideal I is *proper* iff $I \neq \mathscr{P}(A)$.

If F is a filter on A, let $F' = \{X \subseteq A : A \backslash X \in F\}$. Then F' is an ideal on A. If I is an ideal on A, let $I^* = \{X \subseteq A : A \backslash X \in I\}$. Then I^* is a filter on A. If F is a filter on A, then $F'^* = F$. If I is an ideal on A, then $I^{*\prime} = I$.

Now if I is an ideal on A, then

$$\begin{aligned} f \, R_I \, g \quad &\text{iff} \quad \{i \in A : \neg(f(i) \, R_I \, g(i))\} \in I;\\ f <_I g \quad &\text{iff} \quad \{i \in A : f(i) \geq g(i)\} \in I;\\ f \leq_I g \quad &\text{iff} \quad \{i \in A : f(i) > g(i)\} \in I;\\ f =_I g \quad &\text{iff} \quad \{i \in A : f(i) \neq g(i)\} \in I. \end{aligned}$$

Some more notation: $R_I(f, g) = \{i \in I : f(i) \, R \, g(i)\}$. In particular, $<_I (f, g) = \{i \in I : f(i) < g(i)\}$ and $\leq_I (f, g) = \{i \in I : f(i) \leq g(i)\}$.

The following trivial proposition is nevertheless important in what follows.

Proposition 24a.1. *Let F be a proper filter on A. Then*

(i) $<_F$ is irreflexive and transitive.
(ii) $\leq_F$ is reflexive on ${}^A\mathbf{On}$, and it is transitive.
(iii) $f \leq_F g <_F h$ implies that $f <_F h$.
(iv) $f <_F g \leq_F h$ implies that $f <_F h$.
(v) $f <_F g$ or $f =_F g$ implies $f \leq_F g$.
(vi) If $f =_F g$, then $g \leq_F f$.
(vii) If $f \leq_F g \leq_F f$, then $f =_F g$. □

Some care must be taken in working with these notions. The following examples illustrate this.

(1) An example with $f \leq_F g$ but neither $f <_F g$ nor $f =_F g$ nor $f = g$: Let $A = \omega$, $F = \{A\}$, and define $f, g \in {}^\omega\omega$ by setting $f(n) = n$ for all n and

$$g(n) = \begin{cases} n & \text{if } n \text{ is even,} \\ n+1 & \text{if } n \text{ is odd.} \end{cases}$$

(2) An example where $f =_F g$ but neither $f <_F g$ nor $f = g$: Let $A = \omega$ and let F consist of all subsets of ω that contain all even natural numbers. Define f and g by

$$f(n) = \begin{cases} n & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd;} \end{cases} \quad g(n) = \begin{cases} n & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Products and reduced products

In the preceding section we were considering ordering-type relations on the proper classes ${}^A\mathbf{On}$. Now we restrict ourselves to sets. Suppose that $h \in {}^A\mathbf{On}$. We specialize the general notion by considering $\prod_{a\in A} h(a) \subseteq {}^A\mathbf{On}$. To eliminate trivialities, we usually assume that $h(a)$ is a limit ordinal for every $a \in A$; then we call h *non-trivial.*

Proposition 24a.2. *If F is a proper filter on A, $g, h \in {}^A\mathbf{On}$, h is non-trivial, and $g <_F h$, then there is a $k \in \prod_{a\in A} h(a)$ such that $g =_F k$.*

Proof. For any $a \in A$ let

$$k(a) = \begin{cases} g(a) & \text{if } g(a) < h(a), \\ 0 & \text{otherwise.} \end{cases}$$

Thus $k \in \prod_{a\in A} h(a)$. Moreover,

$$\{a \in A : g(a) = k(a) \supseteq \{a \in A : g(a) < h(a)\} \in F,$$

so $g =_F k$. □

We will frequently consider the structure $(\prod_{a\in A} h(a), <_F, \leq_F)$ in what follows. For most considerations it is equivalent to consider the associated *reduced product*, which we define as follows. Note that $=_F$ is an equivalence relation on the set $\prod_{a\in A} h(a)$. We define the underlying set of the reduced product to be the collection of all equivalence classes under $=_F$; it is denoted by $\prod_{a\in A} h(a)/F$. Further, we define, for $x, y \in \prod_{a\in A} h(a)/F$,

$$x <_F y \quad \text{iff} \quad \exists f, g \in \prod A[x = [f],\ y = [g], \text{ and } f <_F g];$$
$$x \leq_F y \quad \text{iff} \quad \exists f, g \in \prod A[x = [f],\ y = [g], \text{ and } f \leq_F g].$$

Here $[h]$ denotes the equivalence class of $h \in \prod A$ under $=_F$.

Proposition 24a.3. *Suppose that $h \in {}^A\mathbf{On}$ is nontrivial, and $f, g \in \prod_{a\in A} h(a)$. Then*
(i) $[f] <_F [g]$ iff $f <_F g$.
(ii) $[f] \leq_F [g]$ iff $f \leq_F g$.

Proof. (i): The direction $\Leftarrow$ is obvious. Now suppose that $[f] <_F [g]$. Then there are $f', g' \in \prod A$ such that $[f] = [f']$, $[g] = [g']$, and $f' <_F g'$. Hence

$$\begin{aligned}&\{\kappa \in A : f(\kappa) = f'(\kappa)\} \cap \{\kappa \in A : g(\kappa) = g'(\kappa)\} \cap \{\kappa \in A : f'(\kappa) < g'(\kappa)\}\\ &\quad \subseteq \{\kappa \in A : f(\kappa) < g(\kappa)\},\end{aligned}$$

and it follows that $\{\kappa \in A : f(\kappa) < g(\kappa)\} \in F$, and so $f <_F g$.
(ii): similarly. ☐

A filter F on A is an *ultrafilter* iff F is proper, and is maximal under all the proper filters on A. Equivalently, F is proper, and for any $X \subseteq A$, either $X \in F$ or $A\backslash X \in F$. The dual notion to an ultrafilter is a maximal ideal.

If F is an ultrafilter on A, then $\prod_{a\in A} h(a)/F$ is an *ultraproduct* of h.

Proposition 24a.4. *If $h \in^A \mathbf{On}$ is nontrivial and F is an ultrafilter on A, then $<_F$ is a linear order on $\prod_{a\in A} h(a)/F$, and $[f] \leq_F [g]$ iff $[f] <_F [g]$ or $[f] = [g]$.*

Proof. By Proposition 24a.1(iii) and Proposition 24a.3, $<_F$ is transitive. Also, from Proposition 24a.3 it is clear that $<_F$ is irreflexive. Now suppose that $f, g \in \prod A$; we want to show that $[f]$ and $[g]$ are comparable. Assume that $[f] \neq [g]$. Thus $\{\kappa \in A : f(\kappa) = g(\kappa)\} \notin F$, so $\{\kappa \in A : f(\kappa) \neq g(\kappa)\} \in F$. Since

$$\{\kappa \in A : f(\kappa) \neq g(\kappa)\} = \{\kappa \in A : f(\kappa) < g(\kappa)\} \cup \{\kappa \in A : g(\kappa) < f(\kappa)\},$$

it follows that $[f] < [g]$ or $[g] < [f]$.

Thus $<_F$ is a linear order on $\prod A/F$.

Next,

$$\{\kappa \in A : f(\kappa) \leq g(\kappa)\} = \{\kappa \in A : f(\kappa) = g(\kappa)\} \cup \{\kappa \in A : f(\kappa) < g(\kappa)\},$$

so it follows by Proposition 24a.3 that $[f] \leq_F [g]$ iff $[f] = [g]$ or $[f] <_F [g]$. ☐

Basic cofinality notions

In this section we allow partial orders P to be proper classes. We may speak of a partial ordering P if the relation $<_P$ is clear from the context. Recall the essential equivalence of the notion of a partial ordering with the "$\leq$" version; see the easy exercise E13.15.

A *double ordering* is a system $(P, \leq_P, <_P, =_P)$ such that the following conditions hold (cf. Proposition 24a.1):

(i) $<_P$ is irreflexive and transitive.

(ii) $\leq_P$ is reflexive on P, and it is transitive.

(iii) $f \leq_P g <_P h$ implies that $f <_P h$.

(iv) $f <_P g \leq_P h$ implies that $f <_P h$.

(v) $f <_P g$ or $f =_P g$ implies $f \leq_P g$.

(vi) If $f =_P g$, then $g \leq_P f$.

(vii) If $f \leq_P g \leq_P f$, then $f =_P g$.

Proposition 24a.5. *For any set A any proper filter F on A, and any $P \subseteq {}^A\mathbf{On}$ the system $(P, \leq_F, <_F, =_F)$ is a double ordering.* □

Proposition 24a.6. *Let $h \in {}^A\mathbf{On}$, with h taking only limit ordinal values, and let F be a proper filter on A. Then $(\prod_{a\in A} h(a)/F, \leq_F, <_F, =)$ is a double ordering.* □

We now give some general definitions, applying to any double ordering $(P, \leq_P, <_P)$ unless otherwise indicated.

• A subclass $X \subseteq P$ is *cofinal* in P iff $\forall p \in P \exists q \in X(p \leq_P q)$. By the condition (3) above, this is equivalent to saying that X is cofinal in P iff $\forall p \in P \exists q \in X(p <_P q)$.

• Since clearly P itself is cofinal in P, we can make the basic definition of the cofinality $\mathrm{cf}(P)$ of P, for a *set* P:

$$\mathrm{cf}(P) = \min\{|X| : X \text{ is cofinal in } P\}.$$

Note that $\mathrm{cf}(P)$ can be singular. For, let $A = \omega$, $h(a) = \omega_a$ for all $a \in \omega$, $I = \{\emptyset\}$, and $Y = \prod_{a\in A} h(a)$.. Suppose that X is cofinal in $\prod_{a\in A} h(a)$. Take any $a \in \omega$; we show that $\omega_a \leq |X|$. We define a one-one sequence $\langle f_\alpha : \alpha < \omega_i\rangle$ of elements of X by recursion. Suppose that f_β has been defined for all $\beta < \alpha$. Let k be the member of $\prod_{a\in A} h(a)$ such that $k(b) = 0$ for all $b \neq a$, while $k(a) \in \omega_a \backslash \{f_\beta(a) : \beta < \alpha\}$. Choose $f_\alpha \in X$ such that $k <_I f_\alpha$.

• A sequence $\langle p_\xi : \xi < \lambda\rangle$ of elements of P is *$<_P$-increasing* iff $\forall \xi, \eta < \lambda(\xi < \eta \rightarrow p_\xi <_P p_\eta)$. Similarly for *$\leq_P$-increasing.*

• Suppose that P is a double order and is a set. We say that P *has true cofinality* iff P has a linearly ordered subset which is cofinal.

Proposition 24a.7. *Suppose that a set P is a double order, and $\langle p_\alpha : \alpha < \lambda\rangle$ is strictly increasing in the sense of P, is cofinal in P, and λ is regular. Then P has true cofinality, and its cofinality is λ.*

Proof. Obviously P has true cofinality. If X is a subset of P of size less than λ, for each $q \in X$ choose $\alpha_q < \lambda$ such that $q < p_{\alpha_q}$. Let $\beta = \sup_{q\in X} \alpha_q$. Then $\beta < \lambda$ since λ is regular. For any $q \in X$ we have $q < p_\beta$. This argument shows that $\mathrm{cf}(P) = \lambda$. □

Proposition 24a.8. *Suppose that P is a double ordering, P a set, and P has true cofinality. Then:*

(i) $\mathrm{cf}(P)$ *is regular.*

(ii) $\mathrm{cf}(P)$ *is the least size of a linearly ordered subset which is cofinal in P.*

(iii) There is a $<_P$-increasing, cofinal sequence in P of length $\mathrm{cf}(P)$.

Proof. Let X be a linearly ordered subset of P which is cofinal in P, and let $\{y_\alpha : \alpha < \mathrm{cf}(P)\}$ be a subset of P which is cofinal in P; we do not assume that $\langle y_\alpha : \alpha < \mathrm{cf}(P)\rangle$ is $<_P$- or $\leq_P$-increasing.

(iii): We define a sequence $\langle x_\alpha : \alpha < \mathrm{cf}(P)\rangle$ by recursion. Let x_0 be any element of X. If x_α has been defined, let $x_{\alpha+1} \in X$ be such that $x_\alpha, y_\alpha < x_{\alpha+1}$; it exists since X is cofinal, using condition (3). Now suppose that $\alpha < \mathrm{cf}(P)$ is limit and x_β has been defined for all $\beta < \alpha$. Then $\{x_\beta : \beta < \alpha\}$ is not cofinal in P, so there is a $z \in P$ such that $z \not\leq x_\beta$ for all $\beta < \alpha$. Choose $x_\alpha \in X$ so that $z < x_\alpha$. Since X is linearly ordered, we must then have $x_\beta < x_\alpha$ for all $\beta < \alpha$. This finishes the construction. Since $y_\alpha < x_{\alpha+1}$ for all $\alpha < \mathrm{cf}(P)$, it follows that $\{x_\alpha : \xi < \mathrm{cf}(P)\}$ is cofinal in P. So (iii) holds.

(i): Suppose that $\mathrm{cf}(P)$ is singular, and let $\langle \beta_\xi : \xi < \mathrm{cf}(\mathrm{cf}(P))\rangle$ be a strictly increasing sequence cofinal in $\mathrm{cf}(P)$. With $\langle x_\alpha : \alpha < \mathrm{cf}(P)\rangle$ as in (iii), it is then clear that $\{x_{\beta_\xi} : \xi < \mathrm{cf}(\mathrm{cf}(P))\}$ is cofinal in P, contradiction (since $\mathrm{cf}(\mathrm{cf}(P)) < \mathrm{cf}(P)$ because $\mathrm{cf}(P)$ is singular).

(ii): By (iii), there is a linearly ordered subset of P of size $\mathrm{cf}(P)$ which is cofinal in P; by the definition of cofinality, there cannot be one of smaller size. □

For P with true cofinality, the cardinal $\mathrm{cf}(P)$ is called the *true cofinality* of P, and is denoted by $\mathrm{tcf}(P)$. We write $\mathrm{tcf}(P) = \lambda$ to mean that P has true cofinality, and it is equal to λ.

• P is λ-*directed* iff for any subset Q of P such that $|Q| < \lambda$ there is a $p \in P$ such that $q \leq_P p$ for all $q \in Q$; equivalently, there is a $p \in P$ such that $q <_P p$ for all $q \in Q$.

Proposition 24a.9. (Pouzet) *Assume that P is a double ordering which is a set. For any infinite cardinal λ, we have* $\mathrm{tcf}(P) = \lambda$ *iff the following two conditions hold:*

(i) P has a cofinal subset of size λ.

(ii) P is λ-directed.

Proof. $\Rightarrow$ is clear, remembering that λ is regular. Now assume that (i) and (ii) hold, and let X be a cofinal subset of P of size λ.

First we show that λ is regular. Suppose that it is singular. Write $X = \bigcup_{\alpha<\mathrm{cf}(\lambda)} Y_\alpha$ with $|Y_\alpha| < \lambda$ for each $\alpha < \mathrm{cf}(\lambda)$. Let p_α be an upper bound for Y_α for each $\alpha < \mathrm{cf}(\lambda)$, and let q be an upper bound for $\{p_\alpha : \alpha < \mathrm{cf}(\lambda)\}$. Choose $r > q$. Then choose $s \in X$ with $r \leq s$. Say $s \in Y_\alpha$. Then $s \leq p_\alpha \leq q < r \leq s$, contradiction.

So, λ is regular. Let $X = \{r_\alpha : \alpha < \lambda\}$. Now we define a sequence $\langle p_\alpha : \alpha < \lambda\rangle$ by recursion. Having defined p_β for all $\beta < \alpha$, by (ii) let p_α be such that $p_\beta < p_\alpha$ for all $\beta < \alpha$, and $r_\beta < p_\alpha$ for all $\beta < \alpha$. Clearly this sequence shows that $\mathrm{tcf}(P, <_P) = \lambda$. □

Proposition 24a.10. *Let P be a set. If G is a cofinal subset of P, then* $\mathrm{cf}(P) = \mathrm{cf}(G)$. *Moreover,* $\mathrm{tcf}(P) = \mathrm{tcf}(G)$, *in the sense that if one of them exists then so does the other, and they are equal.* (That is what we mean in the future too when we assert the equality of true cofinalities.)

Proof. Let H be a cofinal subset of P of size $\mathrm{cf}(P)$. For each $p \in H$ choose $q_p \in G$ such that $p \leq_P q_p$. Then $\{q_p : p \in H\}$ is cofinal in G. In fact, if $r \in G$, choose $p \in H$ such that $r \leq_P p$. Then $r \leq_P q_p$, as desired. This shows that $\mathrm{cf}(G) \leq \mathrm{cf}(P)$.

Now suppose that K is a cofinal subset of G. Then it is also cofinal in P. For, if $p \in P$ choose $q \in G$ such that $p \leq_P q$, and then choose $r \in K$ such that $q \leq_P r$. So $p \leq_P r$, as desired. This shows the other inequality.

For the true cofinality, we apply Theorem 24a.9. So suppose that P has true cofinality λ. By Theorem 24a.9 and the first part of this proof, G has a cofinal subset of size λ, since cofinality is the same as true cofinality when the latter exists. Now suppose that $X \subseteq G$ is of size $< \lambda$. Choose an upper bound p for it in P. Then choose $q \in G$ such that $p \leq_P q$. So q is an upper bound for X, as desired. Thus since Theorem 24a.9(i) and 24a.9(ii) hold for G, it follows from that theorem that $\mathrm{tcf}(G) = \lambda$.

The other implication, that the existence of $\mathrm{tcf}(G, <)$ implies that of $\mathrm{tcf}(P, <)$ and their equality, is even easier, since a sequence cofinal in G is also cofinal in P. □

• A sequence $\langle p_\xi : \xi < \lambda\rangle$ of elements of P is *persistently cofinal* iff

$$\forall h \in P \exists \xi_0 < \lambda \forall \xi(\xi_0 \leq \xi < \lambda \Rightarrow h <_P p_\xi).$$

Proposition 24a.11. *(i) If $\langle p_\xi : \xi < \lambda\rangle$ is $<_P$-increasing and cofinal in P, then it is persistently cofinal.*

(ii) If $\langle p_\xi : \xi < \lambda\rangle$ and $\langle p'_\xi : \xi < \lambda\rangle$ are two sequences of members of P, $\langle p_\xi : \xi < \lambda\rangle$ is persistently cofinal in P, and $p_\xi \leq_P p'_\xi$ for all $\xi < \lambda$, then also $\langle p'_\xi : \xi < \lambda\rangle$ is persistently cofinal in P. □

• If $X \subseteq P$, then an *upper bound* for X is an element $p \in P$ such that $q \leq_P p$ for all $q \in X$.

• If $X \subseteq P$, then a *least upper bound* for X is an upper bound a for X such that $a \leq_P a'$ for every upper bound a' for X. So if a and b are least upper bounds for X, then $a \leq_P b \leq_P a$.

It is possible here to have $a \neq b$. For example, let $A = \omega$, $h(a) = \omega + \omega$ for all $a \in \omega$, $f_n(m) = m + n$ for all $m, n \in \omega$, $I = \{Y \subseteq \omega : \text{each member of } Y \text{ is odd}\}$. $X = \{f_n : n \in \omega\}$. We consider the double order $(\prod_{a\in\omega} h(a), \leq_I, <_I)$. Let

$$g(m) = \begin{cases} \omega & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd} \end{cases} \qquad h(m) = \begin{cases} \omega & \text{if } m \text{ is even,} \\ 1 & \text{if } m \text{ is odd} \end{cases}$$

Then g and h are least upper bounds for X, while $g \neq h$.

• If $X \subseteq P$, then a *minimal upper bound* for X is an upper bound a for X such that if b is an upper bound for X and $b \leq_P a$, then $a \leq_P b$.

Proposition 24a.12. *If $X \subseteq P$ and a is a least upper bound for X, then a is a minimal upper bound for X.* □

Now we come to an ordering notion which is basic for pcf theory.

• If $X \subseteq P$ and for every $x \in X$ there is an $x' \in X$ such that $x <_P x'$, then an element $a \in P$ is an *exact upper bound* of X provided

(1) a is a least upper bound for X, and

(2) X is cofinal in $\{p \in P : p <_P a\}$.

Note that under the hypothesis here, $a \notin X$, and hence $x <_F a$ for all $x \in X$ by (1).

Here is an example of a set X with a least upper bound but no exact upper bound. Let $A = \omega$, $h(a) = \omega + \omega$ for all $a \in \omega$, and for $m, n \in \omega$,

$$f_n(m) = \begin{cases} n & \text{if } m \neq n, \\ 0 & \text{if } m = n, \end{cases}$$

$X = \{f_n : n \in \omega\}$, $I = \{\emptyset\}$. We consider the double order $(\prod_{a\in\omega} h(a), \leq_I, <_I)$. Then a least upper bound for X is the function a such that $a(m) = \omega$ for all $m \in \omega$, but X does not have an exact upper bound.

Ordinal-valued functions and exact upper bounds

In this section we give some simple facts about exact upper bounds in the case of most interest to us—the partial ordering of ordinal-valued functions.

First we note that the rough equivalence between products and reduced products continues to hold for the cofinality notions introduced above. We state this for the most important properties above:

Proposition 24a.13. *Suppose that $h \in {}^A\mathbf{On}$, and h takes only limit ordinal values. Then*

(i) If $X \subseteq \prod_{a\in A} h(a)$, then X is cofinal in $(\prod_{a\in A} h(a), <_I, \leq_I)$ iff $\{[f] : f \in X\}$ is cofinal in $(\prod_{a\in A} h(a)/I, <_I, \leq_I)$.

(ii) $\mathrm{cf}(\prod_{a\in A} h(a), <_I, \leq_I) = \mathrm{cf}(\prod_{a\in A} h(a)/I, <_I, \leq_I)$.

(iii) $\mathrm{tcf}(\prod_{a\in A} h(a), <_I, \leq_I) = \mathrm{tcf}(\prod_{a\in A} h(a)/I, <_I, \leq_I)$.

(iv) If $X \subseteq \prod_{a\in A} h(a)$ and $f \in \prod_{a\in A} h(a)$, then f is an exact upper bound for X iff $[f]$ is an exact upper bound for $\{[g] : g \in X\}$.

Proof. (i) is immediate from Proposition 24a.3. For (ii), if X is cofinal in the system $(\prod_{a\in A} h(a), <_I, \leq_I)$, then clearly $\{[f] : f \in X\}$ is cofinal in $(\prod_{a\in A} h(a)/I, <_I, \leq_I)$, by Proposition 24a.3 again; so $\geq$ holds. Now suppose that $\{[f] : f \in Y\}$ is cofinal in $(\prod_{a\in A} h(a)/I, <_I, \leq_I)$. Given $g \in \prod_{a\in A} h(a)$, choose $f \in Y$ such that $[g] <_I [f]$. Then $g <_I f$. So Y is cofinal in $(\prod_{a\in A} h(a), <_I, \leq_I)$, and $\leq$ holds.

(iii) and (iv) are proved similarly. □

The following obvious proposition will be useful.

Proposition 24a.14. *Suppose that $F \cup \{f, g\} \subseteq {}^A\mathbf{On}$, I is an ideal on A, and $f =_I g$. Suppose that f is an upper bound, least upper bound, minimal upper bound, or exact upper bound for F under $\leq_I$. Then also g is an upper bound, least upper bound, minimal upper bound, or exact upper bound for F under $\leq_I$, respectively.*

Here is our simplest existence theorem for exact upper bounds.

• If X is a collection of members of ${}^A\mathbf{On}$, then $\sup X \in {}^A\mathbf{On}$ is defined by

$$(\sup X)(a) = \sup\{f(a) : f \in X\}.$$

Proposition 24a.15. *Suppose that $\lambda > |A|$ is a regular cardinal, and $f = \langle f_\xi : \xi < \lambda \rangle$ is an increasing sequence of members of ${}^A\mathbf{On}$ in the partial ordering $<$ of everywhere dominance. (That is, $f < g$ iff $f(a) < g(a)$ for all $a \in A$.) Then* $\sup f$ *is an exact upper bound for f, and* $\mathrm{cf}((\sup f)(a)) = \lambda$ *for every $a \in A$.*

Proof. For brevity let $h = \sup f$. Then clearly h is an upper bound for f. Now suppose that $f_\xi \leq g \in {}^A\mathbf{On}$ for all $\xi < \lambda$. Then for any $a \in A$ we have $h(a) = \sup_{\xi<\lambda} f_\xi(a) \leq g(a)$, so $h \leq g$. Thus h is a least upper bound for f. Now suppose that $k \in {}^A\mathbf{On}$ and $k < h$. Then for every $a \in A$ we have $k(a) < h(a)$, and hence there is a $\xi_a < \lambda$ such that $k(a) < f_{\xi_a}(a)$. Let $\eta = \sup_{a\in A} \xi_a$. So $\eta < \lambda$ since λ is regular and greater than $|A|$. Clearly $k < f_\eta$, as desired. □

The next proposition gives equivalent definitions of least upper bounds for our special partial order.

Proposition 24a.16. *Suppose that I is a proper ideal on A, $F \subseteq {}^A\mathbf{On}$, and $f \in {}^A\mathbf{On}$. Then the following conditions are equivalent.*

(i) f is a least upper bound of F under $\leq_I$.

(ii) f is an upper bound of F under $\leq_I$, and for any $f' \in {}^A\mathbf{On}$, if f' is an upper bound of F under $\leq_I$ and $f' \leq_I f$, then $f =_I f'$.

(iii) f is a minimal upper bound of F under $\leq_I$.

Proof. (i)⇒(ii): Assume (i) and the hypotheses of (ii). Hence $f \leq_I f'$, so $f =_I f'$ by Proposition 24a.1(vii).

(ii)⇒(iii): Assume (ii), and suppose that $g \in {}^A\mathbf{On}$ is an upper bound for F and $g \leq_I f$. Then $g =_I f$ by (ii), so $f \leq_I g$.

(iii)⇒(i): Assume (iii). Let $g \in {}^A\mathbf{On}$ be any upper bound for F. Define $h(a) = \min(f(a), g(a))$ for all $a \in A$. Then h is an upper bound for F, since if $k \in F$, then $\{a \in A : k(a) > f(a)\} \in I$ and also $\{a \in A : k(a) > g(a)\} \in I$, and

$$\{a \in A : k(a) > \min(f(a), g(a))\} \subseteq \{a \in A : k(a) > f(a)\} \cup \{a \in A : k(a) > g(a)\} \in I,$$

so $k \leq_I h$. Also, clearly $h \leq_I f$. So by (iii), $f \leq_I h$, and hence $f \leq_I g$, as desired. $\square$

In the next proposition we see that in the definition of exact upper bound we can weaken the condition (1), under a mild restriction on the set in question.

Proposition 24a.17. *Suppose that F is a nonempty set of functions in ${}^A\mathbf{On}$ and $\forall f \in F \exists f' \in F[f <_I f']$. Suppose that h is an upper bound of F, and $\forall g \in^A \mathbf{On}$, if $g <_I h$ then there is an $f \in F$ such that $g <_I f$. Then h is an exact upper bound for F.*

Proof. First note that $\{a \in A : h(a) = 0\} \in I$. In fact, choose $f \in F$. Then $f <_I h$, and so $\{a \in A : h(a) = 0\} \subseteq \{a \in A : f(a) \geq h(a)\} \in I$, as desired.

Now we show that h is a least upper bound for F. Let k be any upper bound. Let

$$l(a) = \begin{cases} k(a) & \text{if } k(a) < h(a), \\ 0 & \text{otherwise.} \end{cases}$$

Since $\{a \in A : l(a) \geq h(a)\} \subseteq \{a \in A : h(a) = 0\}$, it follows by the above that $\{a \in A : l(a) \geq h(a)\} \in I$, and so $l <_I h$. So by assumption, choose $f \in F$ such that $l <_I f$. Now $f \leq_I k$, so $l <_I k$ and hence

$$\{a \in A : k(a) < h(a)\} \subseteq \{a \in A : l(a) \geq k(a)\} \in I,$$

so $h \leq_I k$, as desired.

For the other property in the definition of exact upper bound, suppose that $g <_I h$. Then by assumption there is an $f \in F$ such that $g <_I f$, as desired. $\square$

Corollary 24a.18. *If $h \in {}^A\mathbf{On}$ is non trivial and $F \subseteq \prod_{a\in A} h(a)$, then h is an exact upper bound of F with respect to an ideal I on A iff F is cofinal in $\prod_{a\in A} h(a)$.* $\square$

In the next proposition we use the standard notation I^+ for $A\backslash I$. The proposition shows that exact upper bounds restrict to smaller sets A.

Proposition 24a.19. *Suppose that F is a nonempty subset of ${}^A\mathbf{On}$, I is a proper ideal on A, h is an exact upper bound for F with respect to I, and $\forall f \in F \exists f' \in F(f <_I f')$. Also, suppose that $A_0 \in I^+$. Then:*

(i) $J \stackrel{\text{def}}{=} I \cap \mathscr{P}(A_0)$ is a proper ideal on A_0.

(ii) For any $f, f' \in {}^A\mathbf{On}$, if $f <_I f'$ then $f \restriction A_0 <_J f' \restriction A_0$.

(iii) $h \restriction A_0$ is an exact upper bound for $\{f \restriction A_0 : f \in F\}$.

(i) is clear. Assume the hypotheses of (ii). Then

$$\{a \in A_0 : f'(a) \leq f(a)\} \subseteq \{a \in A : f'(a) \leq f(a)\} \in I,$$

and so $f \restriction A_0 <_J f' \restriction A_0$.

For (iii), by (ii) we see that $h \restriction A_0$ is an upper bound for $\{f \restriction A_0 : f \in F\}$. To see that it is an exact upper bound, we will apply Proposition 24a.18. So, suppose that $k <_J h \restriction A_0$. Fix $f \in F$. Now define $g \in {}^A\mathbf{On}$ by setting

$$g(a) = \begin{cases} f(a) & \text{if } a \in A\backslash A_0, \\ k(a) & \text{if } a \in A_0. \end{cases}$$

Then

$$\{a \in A : g(a) \geq h(a)\} \subseteq \{a \in A : f(a) \geq h(a)\} \cup \{a \in A_0 : k(a) \geq h(a)\} \in I,$$

so $g <_I h$. Hence there is an $l \in F$ such that $g <_I l$. Hence

$$\{a \in A_0 : k(a) \geq l(a)\} \subseteq \{a \in A : g(a) \geq l(a)\} \in I,$$

so $k <_J l$, as desired. $\square$

Next, increasing the ideal maintains exact upper bounds:

Proposition 24a.20. *Suppose that F is a nonempty subset of ${}^A\mathbf{On}$, I is a proper ideal on A, h is an exact upper bound for F with respect to I, and $\forall f \in F \exists f' \in F(f <_I f')$.*

Let J be a proper ideal on A such that $I \subseteq J$. Then h is an exact upper bound for F with respect to J.

Proof. We will apply Proposition 24a.17. Note that h is clearly an upper bound for F with respect to J. Now suppose that $g <_J h$. Let $f \in F$. Define g' by

$$g'(a) = \begin{cases} g(a) & \text{if } g(a) < h(a), \\ f(a) & \text{otherwise.} \end{cases}$$

Then $\{a \in A : g'(a) \geq h(a)\} \subseteq \{a \in A : f(a) \geq h(a)\} \in I$, since $f <_I h$. So $g' <_I h$. Hence by the exactness of h there is a $k \in F$ such that $g' <_I k$. So

$$\begin{aligned} \{a : g(a) \geq k(a)\} &\subseteq \{a \in A : h(a) > g(a) \geq k(a)\} \cup \{a \in A : h(a) \leq g(a)\} \\ &\subseteq \{a \in A : g'(a) \geq k(a)\} \cup \{a \in A : h(a) \leq g(a)\}, \end{aligned}$$

and this union is in J since the first set is in I and the second one is in J. Hence $g <_J k$, as desired. $\square$

Again we turn from the general case of proper classes ${}^A\mathbf{On}$ to the sets $\prod_{a \in A} h(a)$, where $h \in^A \mathbf{On}$ has only limit ordinal values. We prove some results which show that under a weak hypothesis we can restrict attention to $\prod A$ for A a nonempty set of infinite regular cardinals instead of $\prod_{a \in A} h(a)$, as far as cofinality notions are concerned. Here $\prod A$ consists of all choice functions f with domain A; $f(a) \in a$ for all $a \in A$.

Proposition 24a.21. *Suppose that $h \in {}^A\mathbf{On}$ and $h(a)$ is a limit ordinal for every $a \in A$. For each $a \in A$, let $S(a) \subseteq h(a)$ be cofinal in $h(a)$ with order type* $\mathrm{cf}(h(a))$. *Suppose that I is a proper ideal on A. Then*

(i) $\mathrm{cf}(\prod_{a \in A} h(a), <_I) = \mathrm{cf}(\prod_{a \in A} S(a), <_I)$ *and*

(ii) $\mathrm{tcf}(\prod_{a \in A} h(a), <_I) = \mathrm{tcf}(\prod_{a \in A} S(a), <_I)$.

Proof. For each $f \in \prod h$ define $g_f \in \prod_{a \in A} S(a)$ by setting

$$g_f(a) = \text{ least } \alpha \in S(a) \text{ such that } f(a) \leq \alpha.$$

We prove (i): suppose that $X \subseteq \prod h$ and X is cofinal in $(\prod h, <_I)$; we show that $\{g_f : f \in X\}$ is cofinal in $\mathrm{cf}(\prod_{a\in A} S(a), <_I)$, and this will prove $\geq$. So, let $k \in \prod_{a\in A} S(a)$. Thus $k \in \prod h$, so there is an $f \in X$ such that $k <_I f$. Since $f \leq g_f$, it follows that $k <_I g_f$, as desired. Conversely, suppose that $Y \subseteq \prod_{a\in A} S(a)$ and Y is cofinal in $(\prod_{a\in A} S(a), <_I)$; we show that also Y is cofinal in $\prod h$, and this will prove $\leq$ of the claim. Let $f \in \prod h$. Then $f \leq g_f$, and there is a $k \in Y$ such that $g_f <_I k$; so $f <_I k$, as desired.

This finishes the proof of (i).

For (ii), first suppose that $\mathrm{tcf}(\prod h, <_I)$ exists; call it λ. Thus λ is an infinite regular cardinal. Let $\langle f_i : i < \lambda\rangle$ be a $<_I$-increasing cofinal sequence in $\prod h$. We claim that $g_{f_i} \leq g_{f_j}$ if $i < j < \lambda$. In fact, if $a \in A$ and $f_i(a) < f_j(a)$, then $f_i(a) < f_j(a) \leq g_{f_j}(a) \in S(a)$, and so by the definition of g_{f_i} we get $g_{f_i}(a) \leq g_{f_j}(a)$. This implies that $g_{f_i} \leq_I g_{f_j}$. Now $\mathrm{cf}(\prod h, <_I) = \lambda$, so for any $B \in [\lambda]^{<\lambda}$ there is a $j < \lambda$ such that $g_{f_i} <_I f_j \leq g_{f_j}$. It follows that we can take a subsequence of $\langle g_{f_i} : i < \lambda\rangle$ which is strictly increasing modulo I; it is also clearly cofinal, and hence $\lambda = \mathrm{tcf}(\prod_{a\in A} S(a), <_I)$.

Conversely, suppose that $\mathrm{tcf}(\prod_{a\in A} S(a), <_I)$ exists; call it λ. Let $\langle f_i : i < \lambda\rangle$ be a $<_I$-increasing cofinal sequence in $\prod_{a\in A} S(a)$. Then it is also a sequence showing that $\mathrm{tcf}(\prod h, <_I)$ exists and equals $\mathrm{tcf}(\prod_{a\in A} S(a), <_I)$. $\square$

Proposition 24a.22. *Suppose that $\langle L_a : a \in A\rangle$ and $\langle M_a : a \in A\rangle$ are systems of linearly ordered sets such that each L_a and M_a has no last element. Suppose that L_a is isomorphic to M_a for all $a \in A$. Let I be any ideal on A. Then*

$$\left(\prod_{a\in A} L_a, <_I, \leq_I\right) \cong \left(\prod_{a\in A} M_a, <_I, \leq_I\right). \qquad \square$$

Putting the last two propositions together, we see that to determine cofinality and true cofinality of $(\prod h, <_I, \leq_I)$, where $h \in {}^A\mathbf{On}$ and $h(a)$ is a limit ordinal for all $a \in A$, it suffices to take the case in which each $h(a)$ is an infinite regular cardinal. (One passes from $h(a)$ to $S(a)$ and then to $\mathrm{cf}(h(a))$.) We can still make a further reduction, given in the following useful lemma.

Lemma 24a.23. (Rudin-Keisler) *Suppose that c maps the set A into the class of regular cardinals, and $B = \{c(a) : a \in A\}$ is its range. For any ideal I over A, define its Rudin-Keisler projection J on B by*

$$X \in J \quad \textit{iff} \quad X \subseteq B \textit{ and } c^{-1}[X] \in I.$$

Then J is an ideal on B, and there is an isomorphism h of $\prod B/J$ into $\prod_{a\in A} c(a)/I$ such that for any $e \in \prod B$ we have $h(e/J) = \langle e(c(a)) : a \in A\rangle/I$.

If $|A| < \min(B)$, then the range of h is cofinal in $\prod_{a\in A} c(a)/I$, and we have
(i) $\mathrm{cf}(\prod B/J) = \mathrm{cf}(\prod_{a\in A} c(a)/I$ *and*
(ii) $\mathrm{tcf}(\prod B/J) = \mathrm{tcf}(\prod_{a\in A} c(a)/I)$.

Proof. Clearly J is an ideal. Next, for any $e \in \prod B$ let $\overline{e} = \langle e(c(a)) : a \in A \rangle$. Then for any $e_1, e_2 \in \prod B$ we have

$$\begin{aligned} e_1 =_J e_2 \quad &\text{iff} \quad \{b \in B : e_1(b) \neq e_2(b)\} \in J \\ &\text{iff} \quad c^{-1}[\{b \in B : e_1(b) \neq e_2(b)\}] \in I \\ &\text{iff} \quad \{a \in A : e_1(c(a)) \neq e_2(c(a))\} \in I \\ &\text{iff} \quad \overline{e_1} =_I \overline{e_2}. \end{aligned}$$

This shows that h exists as indicated and is one-one. Similarly, h preserves $<_I$ in each direction. So the first part of the lemma holds.

Now suppose that $|A| < \min(B)$. Let G be the range of h. By Proposition 24a.11, (i) and (ii) follow from G being cofinal in $\prod_{a \in A} c(a)/I$. Let $g \in \prod_{a \in A} c(a)$. Define $e \in \prod B$ by setting, for any $b \in B$,

$$e(b) = \sup\{g(a) : a \in A \text{ and } c(a) = b\}.$$

The additional supposition implies that $e \in \prod B$. Now note that $\{a \in A : g(a) > e(c(a))\} = \emptyset \in I$, so that $g/I \leq h(e/J)$, as desired. $\square$

According to these last propositions, the calculation of true cofinalities for partial orders of the form $(\prod_{a \in A} h(a), <_I)$, with $h \in {}^A\mathbf{On}$ and $h(a)$ a limit ordinal for every $a \in A$, and with $|A| < \min(\text{cf}(h(a))$, reduces to the calculation of true cofinalities of partial orders of the form $(\prod B, <_J)$ with B a set of regular cardinals with $|B| < \min(B)$.

Lemma 24a.24. *If $(P_i, <_i)$ is a partial order with true cofinality λ_i for each $i \in I$ and D is an ultrafilter on I, then $\text{tcf}(\prod_{i \in I} \lambda_i/D) = \text{tcf}(\prod_{i \in I} P_i/D)$.*

Proof. Note that $\prod_{i \in I} \lambda_i/D$ is a linear order, and so its true cofinality μ exists and equals its cofinality. So the lemma is asserting that the ultraproduct $\prod_{i \in I} P_i/D$ has μ as true cofinality.

Let $\langle g_\xi : \xi < \mu \rangle$ be a sequence of members of $\prod_{i \in I} \lambda_i$ such that $\langle g_\xi/D : \xi < \mu \rangle$ is strictly increasing and cofinal in $\prod_{i \in I} \lambda_i/D$. For each $i \in I$ let $\langle f_{\xi,i} : \xi < \lambda_i \rangle$ be strictly increasing and cofinal in $(P_i, <_i)$. For each $\xi < \mu$ define $h_\xi \in \prod_{i \in I} P_i$ by setting $h_\xi(i) = f_{g_\xi(i),i}$. We claim that $\langle h_\xi/D : \xi < \mu \rangle$ is strictly increasing and cofinal in $\prod_{i \in I} P_i/D$ (as desired).

To prove this, first suppose that $\xi < \eta < \mu$. Then

$$\{i \in I : h_\xi(i) < h_\eta(i)\} = \{i \in I : f_{g_\xi(i),i} <_i f_{g_\eta(i),i}\} = \{i \in I : g_\xi(i) < g_\eta(i)\} \in D;$$

so $h_\xi/D < h_\eta/D$.

Now suppose that $k \in \prod_{i \in I} P_i$; we want to find $\xi < \mu$ such that $k/D < h_\xi/D$. Define $l \in \prod_{i \in I} \lambda_i$ by letting $l(i)$ be the least $\xi < \mu$ such that $k(i) < f_{\xi,i}$. Choose $\xi < \mu$ such that $l/D < g_\xi/D$. Now if $l(i) < g_\xi(i)$, then $k(i) < f_{l(i),i} <_i f_{g_\xi(i),i} = h_\xi(i)$. So $k/D < h_\xi/D$. $\square$

Existence of exact upper bounds

We introduce several notions leading up to an existence theorem for exact upper bounds: projections, strongly increasing sequences, a partition property, and the bounding projection property.

We start with the important notion of **projections**. By a *projection framework* we mean a triple (A, I, S) consisting of a nonempty set A, an ideal I on A, and a sequence $\langle S_a : a \in A\rangle$ of nonempty sets of ordinals. Suppose that we are given such a framework. We define $\sup S$ in the natural way: it is a function with domain A, and $(\sup S)(a) = \sup(S_a)$ for every $a \in A$. Thus $\sup S \in {}^A\mathbf{On}$. Now suppose also that we have a function $f \in {}^A\mathbf{On}$. Then we define the *projection* of f onto $\prod_{a\in A} S_a$, denoted by $f^+ = \mathrm{proj}(f, S)$, by setting, for any $a \in A$,

$$f^+(a) = \begin{cases} \min(S_a \backslash f(a)) & \text{if } f(a) < \sup(S_a), \\ \min(S_a) & \text{otherwise.} \end{cases}$$

Thus

$$f^+(a) = \begin{cases} f(a) & \text{if } f(a) \in S_a \text{ and } f(a) \text{ is not} \\ & \text{the largest element of } S_a, \\ \text{least } x \in S_a \text{ such that } f(a) < x & \text{if } f(a) \notin S_a \text{ and } f(a) < \sup(S_a), \\ \min(S_a) & \text{if } \sup(S_a) \le f(a). \end{cases}$$

Proposition 24a.25. *Let a projection framework be given, with the notation above.*

(i) If $f \in {}^A\mathbf{On}$, then $f^+ \in \prod_{a\in A} S_a$.

(ii) If $f_1, f_2 \in {}^A\mathbf{On}$ and $f_1 =_I f_2$, then $f_1^+ =_I f_2^+$.

(iii) If $f \in {}^A\mathbf{On}$ and $f <_I \sup S$, then $f \le_I f^+$, and for every $g \in \prod_{a\in A} S_a$, if $f \le_I g$ then $f^+ \le_I g$.

Proof. (i) and (ii) are clear. For (iii), suppose that $f \in {}^A\mathbf{On}$ and $f <_I \sup S$. Then if $f(a) > f^+(a)$ we must have $f(a) \ge \sup(S_a)$. Hence $f \le_I f^+$. Now suppose that $g \in \prod_{a\in A} S_a$ and $f \le_I g$. If $f(a) \le g(a)$ and $f(a) < \sup(S_a)$, then $f^+(a) \le g(a)$. Hence

$$\{a \in A : g(a) < f^+(a)\} \subseteq \{a \in A : f(a) > g(a)\} \cup \{a \in A : f(a) \ge \sup(S_a)\} \in I,$$

so $f^+ \le_I g$. □

Another important notion in discussing exact upper bounds is as follows. Let I be an ideal over A, L a set of ordinals, and $f = \langle f_\xi : \xi \in L\rangle$ a sequence of members of ${}^A\mathbf{On}$. Then we say that f is *strongly increasing under I* iff there is a system $\langle Z_\xi : \xi \in L\rangle$ of members of I such that

$$\forall \xi, \eta \in L[\xi < \eta \Rightarrow \forall a \in A\backslash(Z_\xi \cup Z_\eta)[f_\xi(a) < f_\eta(a)]].$$

Under the same assumptions we say that f is *very strongly increasing under I* iff there is a system $\langle Z_\xi : \xi \in L\rangle$ of members of I such that

$$\forall \xi, \eta \in L[\xi < \eta \Rightarrow \forall a \in A\backslash Z_\eta[f_\xi(a) < f_\eta(a)].$$

Proposition 24a.26. *Under the above assumptions, f is very strongly increasing under I iff for every $\xi \in L$ we have*

$$(*) \qquad \sup\{f_\alpha + 1 : \alpha \in L \cap \xi\} \leq_I f_\xi.$$

Proof. $\Rightarrow$: suppose that f is very strongly increasing under I, with sets Z_ξ as indicated. Let $\xi \in L$. Suppose that $a \in A \backslash Z_\xi$. Then for any $\alpha \in L \cap \xi$ we have $f_\alpha(a) < f_\xi(a)$, and so $\sup\{f_\alpha(a)+1 : \alpha \in L \cap \xi\} \leq f_\xi(a)$; it follows that $(*)$ holds.

$\Leftarrow$: suppose that $(*)$ holds for each $\xi \in L$. For each $\xi \in L$ let

$$Z_\xi = \{a \in A : \sup\{f_\alpha(a)+1 : \alpha \in L \cap \xi\} > f_\xi(a)\};$$

it follows that $Z_\xi \in I$. Now suppose that $\alpha \in L$ and $\alpha < \xi$. Suppose that $a \in A\backslash Z_\xi$. Then $f_\alpha(a) < f_\alpha(a) + 1 \leq \sup\{f_\beta(a) + 1 : \beta \in L \cap \xi\} \leq f_\xi(a)$, as desired. □

Lemma 24a.27. (The sandwich argument) *Suppose that $h = \langle h_\xi : \xi \in L\rangle$ is strongly increasing under I, L has no largest element, and ξ' is the successor in L of ξ for every $\xi \in L$. Also suppose that $f_\xi \in {}^A\mathbf{On}$ is such that*

$$h_\xi <_I f_\xi \leq_I h_{\xi'} \text{ for every } \xi \in L.$$

Then $\langle f_\xi : \xi \in L\rangle$ is also strongly increasing under I.

Proof. Let $\langle Z_\xi : \xi \in L\rangle$ testify that h is strongly increasing under I. For every $\xi \in L$ let

$$W_\xi = \{a \in A : h_\xi(a) \geq f_\xi(a) \text{ or } f_\xi(a) > h_{\xi'}(a)\}.$$

Thus by hypothesis we have $W_\xi \in I$. Let $Z^\xi = W_\xi \cup Z_\xi \cup Z_{\xi'}$ for every $\xi \in L$; so $Z_\xi \in I$. Then if $\xi_1 < \xi_2$, both in L, and if $a \in A\backslash(Z^{\xi_1} \cup Z^{\xi_2})$, then

$$f_{\xi_1}(a) \leq h_{\xi_1'}(a) \leq h_{\xi_2}(a) < f_{\xi_2}(a);$$

these three inequalities hold because $a \in A\backslash W_{\xi_1}$, $a \in A\backslash(Z_{\xi_1'} \cup Z_{\xi_2})$, and $a \in A\backslash W_{\xi_2}$ respectively. □

Now we give a proposition connecting the notion of strongly increasing sequence with the existence of exact upper bounds.

Proposition 24a.28. *Let I be a proper ideal over A, let $\lambda > |A|$ be a regular cardinal, and let $f = \langle f_\xi : \xi < \lambda\rangle$ be a $<_I$ increasing sequence of functions in ${}^A\mathbf{On}$. Then the following conditions are equivalent:*

(i) f has a strongly increasing subsequence of length λ under I.

(ii) f has an exact upper bound h such that $\{a \in A : \mathrm{cf}(h(a)) \neq \lambda\} \in I$.

(iii) f has an exact upper bound h such that $\mathrm{cf}(h(a)) = \lambda$ for all $a \in A$.

(iv) There is a sequence $g = \langle g_\xi : \xi < \lambda\rangle$ such that $g_\xi < g_\eta$ (everywhere) for $\xi < \eta$, and f is cofinally equivalent to g, in the sense that $\forall\xi < \lambda\exists\eta < \lambda(f_\xi <_I g_\eta)$ and $\forall\xi < \lambda\exists\eta < \lambda(g_\xi <_I f_\eta)$.

Proof. (i)⇒(ii): Let $\langle \eta(\xi) : \xi < \lambda \rangle$ be a strictly increasing sequence of ordinals less than λ, thus with supremum λ since λ is regular, and assume that $\langle f_{\eta(\xi)} : \xi < \lambda \rangle$ is strongly increasing under I. Hence for each $\xi < \lambda$ let $Z_\xi \in I$ be chosen correspondingly. We define for each $a \in A$

$$h(a) = \sup\{f_{\eta(\xi)}(a) : \xi < \lambda, a \notin Z_\xi\}.$$

To see that h is an exact upper bound for f, we are going to apply Proposition 24a.17. If $f_{\eta(\xi)}(a) > h(a)$, then $a \in Z_\xi \in I$. Hence $f_{\eta(\xi)} \leq_I h$ for each $\xi < \lambda$. Then for any $\xi < \lambda$ we have $f_\xi \leq_I f_{\eta(\xi)} \leq_I h$, so h bounds every f_ξ. Now suppose that $d <_I h$. Let $M = \{a \in A : d(a) \geq h(a)\}$; so $M \in I$. For each $a \in A\backslash M$ we have $d(a) < h(a)$, and so there is a $\xi_a < \lambda$ such that $d(a) < f_{\eta(\xi_a)}(a)$ and $a \notin Z_{\xi_a}$. Since $|A| < \lambda$ and λ is regular, the ordinal $\rho \stackrel{\text{def}}{=} \sup_{a \in A\backslash M} \xi_a$ is less than λ. We claim that $d <_I f_{\eta(\rho)}$. In fact, suppose that $a \in A\backslash(M \cup Z_\rho)$. Then $a \in A\backslash(Z_{\xi_a} \cup Z_\rho)$, and so $d(a) < f_{\eta(\xi_a)}(a) \leq f_{\eta(\rho)}(a)$. Thus $d <_I f_{\eta(\rho)}$, as claimed. Now it follows easily from Proposition 24a.17 that h is an exact upper bound for f.

For the final portion of (ii), it suffices to show

(1) There is a $W \in I$ such that $\mathrm{cf}(h(a)) = \lambda$ for all $a \in A\backslash W$.

In fact, let

$$W = \{a \in A : \exists \xi_a < \lambda \forall \xi' \in [\xi_a, \lambda)[a \in Z_{\xi'}]\}.$$

Since $|A| < \lambda$, the ordinal $\rho \stackrel{\text{def}}{=} \sup_{a \in W} \xi_a$ is less than λ. Clearly $W \subseteq Z_\rho$, so $W \in I$. For $a \in A\backslash W$ we have $\forall \xi < \lambda \exists \xi' \in [\xi, \lambda)[a \notin Z_{\xi'}]$. This gives an increasing sequence $\langle \sigma_\nu : \nu < \lambda \rangle$ of ordinals less than λ such that $a \notin Z_{\sigma_\nu}$ for all $\nu < \lambda$. By the strong increasing property it follows that $f_{\eta(\sigma_0)}(a) < f_{\eta(\sigma_1)}(a) < \cdots$, and so $h(a)$ has cofinality λ. This proves (1), and with it, (ii).

(ii)⇒(iii): Let $W = \{a \in A : \mathrm{cf}(h(a)) \neq \lambda\}$; so $W \in I$ by (ii). Since I is a proper ideal, choose $a_0 \in A\backslash W$, and define

$$h'(a) = \begin{cases} h(a) & \text{if } a \in A\backslash W, \\ h(a_0) & \text{if } a \in W. \end{cases}$$

Then $h =_I h'$, and it follows that h' satisfies the properties needed.

(iii)⇒(iv): For each $a \in A$, let $\langle \mu^a_\xi : \xi < \lambda \rangle$ be a strictly increasing sequence of ordinals with supremum $h(a)$. Define $g_\xi(a) = \mu^a_\xi$ for all $a \in A$ and $\xi < \lambda$. Clearly $g_\xi < g_\eta$ if $\xi < \eta$. Now suppose that $\xi < \lambda$. Then $f_\xi <_I h$. For each $a \in A$ such that $f_\xi(a) < h(a)$ choose $\rho_a < \lambda$ such that $f_\xi(a) < \mu^a_{\rho_a}$. Since $|A| < \lambda$, choose $\eta < \lambda$ such that $\rho_a < \eta$ for all $a \in A$. Then for any $a \in A$ such that $f_\xi(a) < h(a)$ we have $f_\xi(a) < \mu^a_\eta = g_\eta(a)$. Hence $f_\xi <_I g_\eta$, which is half of what is desired in (iv).

Now suppose that $\xi < \lambda$. Then $g_\xi < h$, so by the exactness of h, there is an $\eta < \lambda$ such that $g_\xi <_I f_\eta$, as desired.

(iv)⇒(i): Assume (iv). Define strictly increasing continuous sequences $\langle \eta(\xi) : \xi < \lambda \rangle$ and $\langle \rho(\xi) : \xi < \lambda \rangle$ of ordinals less than λ as follows. Let $\eta(0) = 0$, and choose $\rho(0)$ so that $g_0 <_I f_{\rho(0)}$. If $\eta(\xi)$ and $\rho(\xi)$ have been defined, choose $\eta(\xi + 1) > \eta(\xi)$ such that

$f_{\rho(\xi)} \leq_I g_{\eta(\xi+1)}$, and choose $\rho(\xi+1) > \rho(\xi)$ such that $g_{\eta(\xi+1)} <_I f_{\rho(\xi+1)}$. Thus for every $\xi < \lambda$ we have

$$g_{\eta(\xi)} <_I f_{\rho(\xi)} \leq_I g_{\eta(\xi+1)}.$$

since obviously $\langle g_{\eta(\xi)} : \xi < \lambda \rangle$ is strongly increasing under I, Lemma 24a.27 gives (i). $\square$

The notion of a strongly increasing sequence is clarified by giving an example of a sequence such that no subsequence is strongly increasing. This example depends on the following well-known lemma.

Lemma 24a.29. *If κ is a regular cardinal and I is the ideal $[\kappa]^{<\kappa}$ on κ, then there is a sequence $f \stackrel{\text{def}}{=} \langle f_\xi : \xi < \kappa^+ \rangle$ of members of ${}^\kappa\kappa$ such that $f_\xi <_I f_\eta$ whenever $\xi < \eta < \kappa$.*

Proof. We construct the sequence by recursion. Let $f_0(\alpha) = 0$ for all $\alpha < \kappa$. If f_ξ has been defined, let $f_{\xi+1}(\alpha) = f_\xi(\alpha) + 1$ for all $\alpha < \kappa$. Now suppose that $\xi < \kappa$ is a limit ordinal, and f_η has been defined for every $\eta < \xi$. Let $\langle \eta(\beta) : \beta < \gamma \rangle$ be a strictly increasing sequence of ordinals with supremum ξ, where $\gamma = \mathrm{cf}(\xi)$. Thus $\gamma \leq \kappa$. Define

$$f_\xi(\alpha) = (\sup_{\beta \leq \alpha} f_{\eta(\beta)}(\alpha)) + 1.$$

The sequence constructed this way is as desired. For example, if ξ is a limit ordinal as above, then for each $\rho < \kappa$ we have $\{\alpha < \kappa : f_{\eta(\rho)}(\alpha) \geq f_\xi(\alpha)\} \subseteq \rho$, and so $f_{\eta(\rho)} <_I f_\xi$. $\square$

Now let $A = \kappa$ and let I and f be as in the lemma. Suppose that f has a strongly increasing subsequence of length κ^+ under I. Then by proposition 24a.28, f has an exact upper bound h such that $\mathrm{cf}(h(\alpha)) = \kappa^+$ for all $\alpha < \kappa$. Now the function k with domain κ taking the constant value κ is clearly an upper bound for f. Hence $h \leq_I k$. Hence there is an $\alpha < \kappa$ such that $h(\alpha) \leq k(\alpha) = \kappa$, contradiction.

A further fact along these lines is as follows.

Lemma 24a.30. *Suppose that $I = [\omega]^{<\omega}$ and $f \stackrel{\text{def}}{=} \langle f_\xi : \xi < \lambda \rangle$ is a $<_I$-increasing sequence of members of ${}^\omega\omega$ which has an exact upper bound h, where λ is an infinite cardinal. Then $\langle f_\xi : \xi < \lambda \rangle$ is a scale, i.e., for any $g \in {}^\omega\omega$ there is a $\xi < \lambda$ such that $g <_I f_\xi$.*

Proof. Let $k(m) = \omega$ for all $m < \omega$. Then k is an upper bound for f under $<_I$, and so $h \leq_I k$. Letting $h'(m) = \min(h(m), k(m))$ for all $m \in \omega$, we thus get $h =_I h'$. So by Proposition 24a.14, h' is also an exact upper bound for f. Hence we may assume that $h(m) \leq \omega$ for every $m < \omega$. Now we claim

(1) $\exists n < \omega \forall p \geq n (0 < h(p))$.

In fact, the set $\{p \in \omega : f_0(p) \geq h(p)\}$ is in I, so there is an n such that $f_0(p) < h(p)$ for all $p \geq n$, as desired in (1).

Let n_0 be as in (1).

(2) $M \stackrel{\text{def}}{=} \{p \in \omega : h(p) \neq \omega\}$ is finite.

For, suppose that M is infinite. Define

$$l(p) = \begin{cases} h(p) - 1 & \text{if } 0 < h(p) < \omega, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $l <_I h$. For, $\{p : l(p) \geq h(p)\} \subseteq \{p : h(p) = 0\} \in I$. So our claim holds. Now by exactness, choose $\xi < \kappa$ such that $l <_I f_\xi$. Then we can choose $p \in M$ such that $l(p) < f_\xi(p) < h(p)$, contradiction.

Thus M is finite. Hence we may assume that $h(p) = \omega$ for all p, and the desired conclusion of the lemma follows. □

Now there is a model M of ZFC in which there are no scales (see for example Blass [∞]), and yet it is easy to see that there is a sequence $f \stackrel{\text{def}}{=} \langle f_\xi : \xi < \omega_1 \rangle$ which is $<_I$-increasing. Hence by Lemma 24a.30, this sequence does not have an exact upper bound.

Another fact which helps the intuition on exact upper bounds is as follows.

Lemma 24a.31. *Let κ be a regular cardinal, and let $I = [\kappa]^{<\kappa}$. For each $\xi < \kappa$ let $f_\xi \in {}^\kappa\kappa$ be defined by $f_\xi(\alpha) = \xi$ for all $\alpha < \kappa$. Thus $f \stackrel{\text{def}}{=} \langle f_\xi : \xi < \kappa \rangle$ is increasing everywhere. Claim: f does not have a least upper bound under $<_I$. (Hence it does not have an exact upper bound.)*

Proof. Suppose that h is an upper bound for f under $<_I$. We find another upper bound k for f under $<_I$ such that h is not $\leq_I k$. First we claim

(1) $\forall \alpha < \kappa \exists \beta < \kappa \forall \gamma \geq \beta (\alpha \leq h(\gamma))$.

In fact, otherwise we get a $\xi < \kappa$ such that for all $\beta < \kappa$ there is a $\gamma > \beta$ such that $\xi > h(\gamma)$. But then $|\{\alpha < \kappa : f_\xi(\alpha) > h(\alpha)\}| = \kappa$, contradiction.

By (1) there is a strictly increasing sequence $\langle \beta_\alpha : \alpha < \kappa \rangle$ of ordinals less than κ such that for all $\alpha < \kappa$ and all $\gamma \geq \beta_\alpha$ we have $\alpha < h(\gamma)$. Now we define $k \in {}^\kappa\kappa$ by setting, for each $\gamma < \kappa$,

$$k(\gamma) = \begin{cases} \alpha & \text{if } \beta_{\alpha+1} \leq \gamma < \beta_{\alpha+2}, \\ h(\gamma) & \text{otherwise.} \end{cases}$$

To see that k is an upper bound for f under $<_I$, take any $\xi < \kappa$. If $\beta_{\xi+1} \leq \gamma$, then $h(\gamma) \geq \xi + 1$, and hence $k(\gamma) \geq \xi = f_\xi(\gamma)$, as desired. For each $\xi < \kappa$ we have $k(\beta_{\xi+1}) = \xi < h(\beta_{\xi+1})$, so h is not $\leq_I k$. □

Now we define a partition property. Suppose that I is an ideal over a set A, λ is an uncountable regular cardinal $> |A|$, $f = \langle f_\xi : \xi < \lambda \rangle$ is a $<_I$-increasing sequence of members of ${}^A\mathbf{On}$, and κ is a regular cardinal such that $|A| < \kappa \leq \lambda$. The following property of these things is denoted by $(*)_\kappa$:

$(*)_\kappa$ For all unbounded $X \subseteq \lambda$ there is an $X_0 \subseteq X$ of order type κ such that $\langle f_\xi : \xi \in X_0 \rangle$ is strongly increasing under I.

Proposition 24a.32. *Assume the above notation, with $\kappa < \lambda$. Then $(*)_\kappa$ holds iff the set*

$$\{\delta < \lambda : \operatorname{cf}(\delta) = \kappa \textit{ and } \langle f_\xi : \xi \in X_0 \rangle \textit{ is strongly increasing under } I$$
$$\textit{for some unbounded } X_0 \subseteq \delta\}$$

is stationary in λ.

Proof. Let S be the indicated set of ordinals δ.

$\Rightarrow$: Assume $(*)_\kappa$ and suppose that $C \subseteq \lambda$ is a club. Choose $C_0 \subseteq C$ of order type κ such that $\langle f_\xi : \xi \in C_0\rangle$ is strongly increasing under I. Let $\delta = \sup(C_0)$. Clearly $\delta \in C \cap S$.

$\Leftarrow$: Assume that S is stationary in λ, and suppose that $X \subseteq \lambda$ is unbounded. Define

$$C = \{\alpha \in \lambda : \alpha \text{ is a limit ordinal and } X \cap \alpha \text{ is unbounded in } \alpha\}.$$

We check that C is club in λ. For closure, suppose that $\alpha < \lambda$ is a limit ordinal and $C \cap \alpha$ is unbounded in α; we want to show that $\alpha \in C$. So, we need to show that $X \cap \alpha$ is unbounded in α. To this end, take any $\beta < \alpha$; we want to find $\gamma \in X \cap \alpha$ such that $\beta < \gamma$. Since $C \cap \alpha$ is unbounded in α, choose $\delta \in C \cap \alpha$ such that $\beta < \delta$. By the definition of C we have that $X \cap \delta$ is unbounded in δ. So we can choose $\gamma \in X \cap \delta$ such that $\beta < \gamma$. Since $\gamma < \delta < \alpha$, γ is as desired. So, indeed, C is closed.

To show that C is unbounded in λ, take any $\beta < \lambda$; we want to find an $\alpha \in C$ such that $\beta < \alpha$. Since X is unbounded in λ, we can choose a sequence $\gamma_0 < \gamma_1 < \cdots$ of elements of X with $\beta < \gamma_0$. Now λ is uncountable and regular, so $\sup_{n\in\omega} \gamma_n < \lambda$, and it is the member of C we need.

Now choose $\delta \in C \cap S$. This gives us an unbounded set X_0 in δ such that $\langle f_\xi : \xi \in X_0\rangle$ is strongly increasing under I. Now also $X \cap \delta$ is unbounded, since $\delta \in C$. Hence we can define by induction two increasing sequences $\langle \eta(\xi) : \xi < \kappa\rangle$ and $\langle \nu(\xi) : \xi < \kappa\rangle$ such that each $\eta(\xi)$ is in X_0, each $\nu(\xi)$ is in X, and $\eta(\xi) < \nu(\xi) \leq \eta(\xi + 1)$ for all $\xi < \kappa$. It follows by the sandwich argument, Lemma 24a.28, that $X_1 \stackrel{\text{def}}{=} \{\nu(\xi) : \xi < \kappa\}$ is a subset of X as desired in $(*)_\kappa$. $\square$

Finally, we introduce the bounding projection property.

Suppose that $f = \langle f_\xi : \xi < \lambda\rangle$ is a $<_I$-increasing sequence of functions in ${}^A\mathbf{On}$, with λ a regular cardinal $> |A|$. Also suppose that κ is a regular cardinal and $|A| < \kappa \leq \lambda$.

We say that f has the *bounding projection property for* κ iff whenever $\langle S(a) : a \in A\rangle$ is a system of nonempty sets of ordinals such that each $|S(a)| < \kappa$ and for each $\xi < \lambda$ we have $f_\xi <_I \sup(S(a))$, then for some $\xi < \lambda$, the function $\operatorname{proj}(f_\xi, \langle S(a) : a \in A\rangle)$ $<_I$-bounds f.

We need the following simple result.

Proposition 24a.33. *Suppose that* $f = \langle f_\xi : \xi < \lambda\rangle$ *is a* $<_I$*-increasing sequence of functions in* $\mathbf{On}^A$*, with* λ *a regular cardinal* $> |A|$*. Also suppose that* κ *is a regular cardinal and* $|A| < \kappa \leq \lambda$*. Assume that* f *has the bounding projection property for* κ*.*

Also suppose that $f' = \langle f'_\xi : \xi < \lambda\rangle$ *is a sequence of functions in* $\mathbf{On}^A$*, and* $f_\xi =_I f'_\xi$ *for every* $\xi < \lambda$*.*

Then f' *has the bounding projection property for* κ*.*

Proof. Clearly f' is $<_I$-increasing, so that the setup for the bounding projection property holds. Now suppose that $\langle S(a) : a \in A\rangle$ is a system of nonempty sets of ordinals such that each $|S(a)| < \kappa$ and for each $\xi < \lambda$ we have $f'_\xi <_I \sup(S)$. Then the same is true for f, so by the bounding projection property for f we can choose $\xi < \lambda$ such that the function $\operatorname{proj}(f_\xi, \langle S(a) : a \in A\rangle)$ $<_I$-bounds f. Now suppose that $\eta < \lambda$. Then

$f_\eta \le_I \operatorname{proj}(f_\xi, \langle S(a) : a \in A\rangle)$. Hence $f'_\eta \le_I \operatorname{proj}(f_\xi, \langle S(a) : a \in A\rangle)$, and $\operatorname{proj}(f_\xi, \langle S(a) : a \in A\rangle) = \operatorname{proj}(f'_\xi, \langle S(a) : a \in A\rangle)$, as desired. □

The following proposition shows that we can weaken the bounded projection property somewhat, by replacing "$<_I$" by "$<$ (everywhere)".

Proposition 24a.34. *Suppose that $f = \langle f_\xi : \xi < \lambda\rangle$ is a $<_I$-increasing sequence of functions in $\mathbf{On}^A$, with λ a regular cardinal $> |A|$. Also suppose that κ is a regular cardinal and $|A| < \kappa \le \lambda$. Then the following conditions are equivalent:*

(i) f has the bounding projection property for κ.

(ii) If $\langle S(a) : a \in A\rangle$ is a system of nonempty sets of ordinals such that each $|S(a)| < \kappa$ and for each $\xi < \lambda$ we have $f_\xi < \sup(S)$ (everywhere), then for some $\xi < \lambda$, the function proj$(f_\xi, \langle S(a) : a \in A\rangle)$ $<_I$-bounds f.

Proof. Obviously (i)⇒(ii). Now assume that (ii) holds, and suppose that $\langle S(a) : a \in A\rangle$ is a system of sets of ordinals such that each $|S(a)| < \kappa$ and for each $\xi < \lambda$ we have $f_\xi <_I \sup(S)$. Now for each $a \in A$ let

$$\gamma(a) = \begin{cases} \sup\{f_\xi(a)+1 : \xi < \lambda \text{ and } f_\xi(a) \ge \sup(S(a))\} & \text{if this set is nonempty,} \\ \sup(S(a))+1 & \text{otherwise;} \end{cases}$$

$$S'(a) = S(a) \cup \{\gamma(a)\}.$$

Note that $f_\xi < \sup(S')$ everywhere. Hence by (ii), there is a $\xi < \lambda$ such that the function $\operatorname{proj}(f_\xi, \langle S'(a) : a \in A\rangle)$ $<_I$-bounds f. Now let $\eta < \lambda$. If $f_\xi(a) < \sup(S(a))$ and $f_\eta(a) < (\operatorname{proj}(f_\xi, \langle S'(a) : a \in A\rangle))(a)$, then

$$\begin{aligned}(\operatorname{proj}(f_\xi, \langle S'(a) : a \in A\rangle))(a) &= \min(S'(a)\backslash f_\xi(a)) \\ &= \min(S(a)\backslash f_\xi(a)) \\ &= (\operatorname{proj}(f_\xi, \langle S(a) : a \in A\rangle))(a).\end{aligned}$$

Hence $f_\eta <_I \operatorname{proj}(f_\xi, \langle S(a) : a \in A\rangle)$, as desired. □

Lemma 24a.35. (Bounding projection lemma) *Suppose that I is an ideal over A, $\lambda > |A|$ is a regular cardinal, $f = \langle f_\xi : \xi < \lambda\rangle$ is a $<_I$-increasing sequence satisfying $(*)_\kappa$ for a regular cardinal κ such that $|A| < \kappa \le \lambda$. Then f has the bounding projection property for κ.*

Proof. Assume the hypothesis of the lemma and of the bounding projection property for κ. For every $\xi < \lambda$ let

$$f^+_\xi = \operatorname{proj}(f_\xi, S).$$

Suppose that the conclusion of the bounding projection property fails. Then for every $\xi < \lambda$, the function f^+_ξ is not a bound for f, and so there is a $\xi' < \lambda$ such that $f_{\xi'} \not\le_I f^+_\xi$. Since $f_\xi \le_I f^+_\xi$, we must have $\xi < \xi'$. Clearly for any $\xi'' \ge \xi'$ we have $f_{\xi''} \not\le_I f^+_\xi$. Thus for every $\xi'' \ge \xi'$ we have $<(f^+_\xi, f_{\xi''}) \in I^+$. Now we define a sequence $\langle \xi(\mu) : \mu < \lambda\rangle$ of elements of λ by recursion. Let $\xi(0) = 0$. Suppose that $\xi(\mu)$ has been defined. Choose $\xi(\mu+1) > \xi(\mu)$ so that $<(f^+_{\xi(\mu)}, f_{\xi''}) \in I^+$ for every $\xi'' \geq \xi(\mu+1)$. If ν is limit and $\xi(\mu)$ has been defined for all $\mu < \nu$, let $\xi(\nu) = \sup_{\mu<\nu} \xi(\mu)$. Then let X be the range of this sequence. Thus

$$\text{if } \xi, \xi' \in X \text{ and } \xi < \xi', \text{ then } <(f^+_\xi, f_{\xi'}) \in I^+.$$

Since $(*)_\kappa$ holds, there is a subset $X_0 \subseteq X$ of order type κ such that $\langle f_\xi : \xi \in X_0\rangle$ is strongly increasing under I. Let $\langle Z_\xi : \xi \in X_0\rangle$ be as in the definition of strongly increasing under I.

For every $\xi \in X_0$, let ξ' be the successor of ξ in X_0. Note that

$$<(f^+_\xi, f_{\xi'})\backslash(Z_\xi \cup Z_{\xi'} \cup \{a \in A : f_\xi(a) \geq \sup(S(a))\}) \in I^+,$$

and hence it is nonempty. So, choose

$$a_\xi \in <(f^+_\xi, f_{\xi'})\backslash(Z_\xi \cup Z_{\xi'} \cup \{a \in A : f_\xi(a) \geq \sup(S(a))\}).$$

Note that this implies that $f^+_\xi(a_\xi) \in S(a_\xi)$. Since $\kappa > |A|$, we can find a single $a \in A$ such that $a = a_\xi$ for all ξ in a subset X_1 of X_0 of size κ. Now for $\xi_1 < \xi_2$ with both in X_1, we have

$$f^+_{\xi_1}(a) < f_{\xi'_1}(a) \leq f_{\xi_2}(a) \leq f^+_{\xi_2}(a).$$

[The first inequality is a consequence of $a = a_{\xi_1} \in <(f^+_{\xi_1}, f_{\xi'_1})$, the second follows from $\xi'_1 \leq \xi_2$ and the fact that

$$a = a_{\xi_1} = a_{\xi_2} \in A\backslash(Z_{\xi'_1} \cup Z_{\xi_2}),$$

and the third is true by the definition of $f^+_{\xi_2}$.]

Thus $\langle f^+_\xi(a) : \xi \in X_1\rangle$ is a strictly increasing sequence of members of $S(a)$. This contradicts our assumption that $|S(a)| < \kappa$. $\square$

The next lemma reduces the problem of finding an exact upper bound to that of finding a least upper bound.

Lemma 24a.36. *Suppose that I is a proper ideal over A, $\lambda \geq |A|^+$ is a regular cardinal, and $f = \langle f_\xi : \xi \in \lambda\rangle$ is a $<_I$-increasing sequence of functions in ${}^A\mathbf{On}$ satisfying the bounding projection property for $|A|^+$. Suppose that h is a least upper bound for f. Then h is an exact upper bound.*

Proof. Assume the hypotheses, and suppose that $g <_I h$; we want to find $\xi < \lambda$ such that $g <_I f_\xi$. By increasing h on a subset of A in the ideal, we may assume that $g < h$ everywhere. Define $S_a = \{g(a), h(a)\}$ for every $a \in A$. By the bounding projection property we get a $\xi < \lambda$ such that $f^+_\xi \stackrel{\text{def}}{=} \text{proj}(f_\xi, \langle S_a : a \in A\rangle)$ is an upper bound for f. We shall prove that $g <_I f_\xi$, as required.

Since h is a least upper bound, it follows that $h \leq_I f^+_\xi$. Thus $M \stackrel{\text{def}}{=} \{a \in A : h(a) > f^+_\xi(a)\} \in I$. Also, the set $N \stackrel{\text{def}}{=} \{a \in A : f_\xi(a) \geq \sup S_a\}$ is in I. Suppose that

$a \in A\backslash(M \cup N)$. Then $g(a) < h(a) \leq f^+_\xi(a) = \min(S_a\backslash f_\xi(a))$, and this implies that $g(a) < f_\xi(a)$. So $g <_I f_\xi$, as desired. $\square$

Here is our first existence theorem for exact upper bounds.

Theorem 24a.37. (Existence of exact upper bounds) *Suppose that I is a proper ideal over A, $\lambda > |A|^+$ is a regular cardinal, and $f = \langle f_\xi : \xi \in \lambda\rangle$ is a $<_I$-increasing sequence of functions in ${}^A\mathbf{On}$ that satisfies the bounding projection property for $|A|^+$. Then f has an exact upper bound.*

Proof. Assume the hypotheses. By Lemma 24a.36 it suffices to show that f has a least upper bound, and to do this we will apply Proposition 24a.16(ii). Suppose that f does not have a least upper bound. Since it obviously has an upper bound, this means, by Proposition 24a.16(ii):

(1) For every upper bound $h \in {}^A\mathbf{On}$ for f there is another upper bound h' for f such that $h' \leq_I h$ and $\{a \in A : h'(a) < h(a)\} \in I^+$.

In fact, Proposition 24a.16(ii) says that there is another upper bound h' for f such that $h' \leq_I h$ and it is not true that $h =_I h'$. Hence $\{a \in A : h(a) < h'(a)\} \in I$ and $\{a \in A : h(a) \neq h'(a)\} \in I^+$. So

$$\{a \in A : h(a) \neq h'(a)\}\backslash\{a \in A : h(a) < h'(a)\} \in I^+ \quad \text{and}$$
$$\{a \in A : h(a) \neq h'(a)\}\backslash\{a \in A : h(a) < h'(a)\} = \{a \in A : h'(a) < h(a)\},$$

so (1) follows.

Now we shall define by induction on $\alpha < |A|^+$ a sequence $S^\alpha = \langle S^\alpha(a) : a \in A\rangle$ of sets of ordinals satisfying the following conditions:

(2) $0 < |S^\alpha(a)| \leq |A|$ for each $a \in A$;

(3) $f_\xi(a) < \sup S^\alpha(a)$ for all $\xi \in \lambda$ and $a \in A$;

(4) If $\alpha < \beta$, then $S^\alpha(a) \subseteq S^\beta(a)$, and if δ is a limit ordinal, then $S^\delta(a) = \bigcup_{\alpha<\delta} S^\alpha(a)$.

We also define sequences $\langle h_\alpha : \alpha < |A|^+\rangle$ and $\langle h'_\alpha : \alpha < |A|^+\rangle$ of functions and $\langle \xi(\alpha) : \alpha < |A|^+\rangle$ of ordinals.

The definition of S^α for α limit is fixed by (4), and the conditions (2)–(4) continue to hold. To define S^0, pick any function k that bounds f (everywhere) and define $S^0(a) = \{k(a)\}$ for all $a \in A$; so (2)–(4) hold.

Suppose that $S^\alpha = \langle S^\alpha(a) : a \in A\rangle$ has been defined, satisfying (2)–(4); we define $S^{\alpha+1}$. By the bounding projection property for $|A|^+$, there is a $\xi(\alpha) < \lambda$ such that $h_\alpha \stackrel{\text{def}}{=} \text{proj}(f_{\xi(\alpha)}, S^\alpha)$ is an upper bound for f under $<_I$. Then

(5) if $\xi(\alpha) \leq \xi' < \lambda$, then $h_\alpha =_I \text{proj}(f_{\xi'}, S^\alpha)$.

In fact, recall that $h_\alpha(a) = \min(S^\alpha(a)\backslash f_{\xi(\alpha)}(a))$ for every $a \in A$, using (3). Now suppose that $\xi(\alpha) < \xi' < \lambda$. Let $M = \{a \in A : f_{\xi(\alpha)}(a) \geq f_{\xi'}(a)\}$. So $M \in I$. For any $a \in A\backslash M$ we have $f_{\xi(\alpha)}(a) < f_{\xi'}(a)$, and hence

$$\min(S^\alpha(a)\backslash f_{\xi(\alpha)}(a)) \leq \min(S^\alpha(a)\backslash f_{\xi'}(a));$$

it follows that $h_\alpha \leq_I \text{proj}(f_{\xi'}, S^\alpha)$. For the other direction, recall that h_α is an upper bound for f under $<_I$. So $f_{\xi'} \leq_I h_\alpha$. If a is any element of A such that $f_{\xi'}(a) \leq h_\alpha(a)$ then, since $h_\alpha(a) \in S^\alpha(a)$, we get $\min(S^\alpha(a)\backslash f_{\xi'}(a)) \leq h_\alpha(a)$. Thus $\text{proj}(f_{\xi'}, S^\alpha) \leq_I h_\alpha$.

This checks (5).

Now we apply (1) to get an upper bound h'_α for f such that $h'_\alpha \leq_I h_\alpha$ and $<(h'_\alpha, h_\alpha) \in I^+$. We now define $S^{\alpha+1}(a) = S^\alpha(a) \cup \{h'_\alpha(a)\}$ for any $a \in A$.

(6) If $\xi(\alpha) \leq \xi < \lambda$, then $\text{proj}(f_\xi, S^{\alpha+1}) =_I h'_\alpha$.

For, we have $f_\xi \leq_I h'_\alpha$ and, by (5), $h_\alpha =_I \text{proj}(f_\xi, S^\alpha)$. If $a \in A$ is such that $f_\xi(a) \leq h'_\alpha(a)$, $h'_\alpha(a) \leq h_\alpha(a)$, and $h_\alpha(a) = \text{proj}(f_\xi, S^\alpha)(a)$, then $\min(S^\alpha(a)\backslash f_\xi(a)) = h_\alpha(a) \geq h'_\alpha(a) \geq f_\xi(a)$, and hence

$$\text{proj}(f_\xi, S^{\alpha+1})(a) = \min(S^{\alpha+1}(a)\backslash f_\xi(a)) = h'_\alpha(a).$$

It follows that $\text{proj}(f_\xi, S^{\alpha+1}) =_I h'_\alpha$, as desired in (6).

Now since $|A|^+ < \lambda$, let $\xi < \lambda$ be greater than each $\xi(\alpha)$ for $\alpha < |A|^+$. Define $H_\alpha = \text{proj}(f_\xi, S^\alpha)$ for each $\alpha < |A|^+$. Since $\xi > \xi(\alpha)$, we have $H_\alpha =_I h_\alpha$ by (5). Note that $H_{\alpha+1} = \text{proj}(f_\xi, S^{\alpha+1}) =_I h'_\alpha$; so $<(H_{\alpha+1}, H_\alpha) \in I^+$. Now clearly by the construction we have $S^{\alpha_1}(a) \subseteq S^{\alpha_2}(a)$ for all $a \in A$ when $\alpha_1 < \alpha_2 < |A|^+$. Hence we get

(7) if $\alpha_1 < \alpha_2 < |A|^+$, then $H_{\alpha_2} \leq H_{\alpha_1}$, and $<(H_{\alpha_2}, H_{\alpha_1}) \in I^+$.

Now for every $\alpha < |A|^+$ pick $a_\alpha \in A$ such that $H_{\alpha+1}(a_\alpha) < H_\alpha(a_\alpha)$. We have $a_\alpha = a_\beta$ for all α, β in some subset of $|A|^+$ of size $|A|^+$, and this gives an infinite decreasing sequence of ordinals, contradiction. □

Lemma 24a.38. *Suppose that I is a proper ideal over A, $\lambda \geq |A|^+$ is a regular cardinal, $f = \langle f_\xi : \xi < \lambda\rangle$ is a $<_I$-increasing sequence of functions in ${}^A\mathbf{On}$, $|A|^+ \leq \kappa \leq \lambda$, f satisfies the bounding projection property for κ, and g is an exact upper bound for f. Then*

$$\{a \in A : g(a) \textit{ is non-limit, or } \text{cf}(g(a)) < \kappa\} \in I.$$

Proof. Let $P = \{a \in A : g(a) \text{ is non-limit, or } \text{cf}(g(a)) < \kappa\}$. If $a \in P$ and $g(a)$ is a limit ordinal, choose $S(a) \subseteq g(a)$ cofinal in $g(a)$ and of order type $< \kappa$. If $g(a) = 0$ let $S(a) = \{0\}$, and if $g(a) = \beta + 1$ for some β let $S(a) = \{\beta\}$. Finally, if $g(a)$ is limit but is not in P, let $S(a) = \{g(a)\}$.

Now for any $\xi < \lambda$ let

$$\begin{aligned} N_\xi &= \{a \in A : f_\xi(a) \geq f_{\xi+1}(a)\} \quad \text{and} \\ Q_\xi &= \{a \in A : f_{\xi+1}(a) \geq g(a)\}. \end{aligned}$$

Then clearly

(∗) If $a \in A\backslash(N_\xi \cup Q_\xi)$, then $f_\xi(a) < \sup(S(a))$.

It follows that $\{a \in A : f_\xi(a) \geq \sup S(a)\} \subseteq N_\xi \cup Q_\xi \in I$. Hence the hypothesis of the bounding projection property holds. Applying it, we get $\xi < \lambda$ such that $f^+_\xi \stackrel{\text{def}}{=}$

$\text{proj}(f_\xi, \langle S(a) : a \in A\rangle)$ $<_I$-bounds f. Since g is a least upper bound for f, we get $g \leq_I f_\xi^+$, and hence $M \stackrel{\text{def}}{=} \{a \in A : f_\xi^+(a) < g(a)\} \in I$. By $(*)$, for any $a \in P\backslash(N_\xi \cup Q_\xi)$ we have $f_\xi^+(a) = \min(S(a)\backslash f_\xi(a)) < g(a)$. This shows that $P\backslash(N_\xi \cup Q_\xi) \subseteq M$, hence $P \subseteq N_\xi \cup Q_\xi \cup M \in I$, so $P \in I$, as desired. $\square$

Now we give our main theorem on the existence of exact upper bounds.

Theorem 24a.39. *Suppose that I is a proper ideal over A, $\lambda > |A|^+$ is a regular cardinal, $f = \langle f_\xi : \xi < \lambda\rangle$ is a $<_I$-increasing sequence of functions in ${}^A\mathbf{On}$, and $|A|^+ \leq \kappa$. Then the following are equivalent:*

(i) $(*)_\kappa$ holds for f.

(ii) f satisfies the bounding projection property for κ.

(iii) f has an exact upper bound g such that

$$\{a \in A : g(a) \textit{ is non-limit, or } \operatorname{cf}(g(a)) < \kappa\} \in I.$$

Proof. (i)⇒(ii): By the bounding projection lemma, Lemma 24a.35.

(ii)⇒(iii): Since the bounding projection property for κ clearly implies the bounding projection property for $|A|^+$, this implication is true by Theorem 24a.37 and Lemma 24a.38.

(iii)⇒(i): Assume (iii). By modifying g on a set in the ideal we may assume that $g(a)$ is a limit ordinal and $\operatorname{cf}(g(a)) \geq \kappa$ for all $a \in A$. Choose a club $S(a) \subseteq g(a)$ of order type $\operatorname{cf}(g(a))$. Thus the order type of $S(a)$ is $\geq \kappa$. We prove that $(*)_\kappa$ holds. So, assume that $X \subseteq \lambda$ is unbounded; we want to find $X_0 \subseteq X$ of order type κ over which f is strongly increasing under I. To do this, we intend to define by induction on $\alpha < \kappa$ a function $h_\alpha \in \prod S$ and an index $\xi(\alpha) \in X$ such that

(1) $h_\alpha <_I f_{\xi(\alpha)} \leq_I h_{\alpha+1}$.

(2) The sequence $\langle h_\alpha : \alpha < \kappa\rangle$ is $<$-increasing (increasing everywhere; and hence it certainly is strongly increasing under I).

(3) $\langle \xi(\alpha) : \alpha < \kappa\rangle$ is strictly increasing.

After we have done this, the sandwich argument (Lemma 24a.27) shows that $\langle f_{\xi(\alpha)} : \alpha < \kappa\rangle$ is strongly increasing under I and of order type κ, giving the desired result.

The functions h_α are defined as follows.

$h_0 \in \prod S$ is arbitrary.

For a limit ordinal $\delta < \kappa$ let $h_\delta = \sup_{\alpha<\delta} h_\alpha$.

Having defined h_α, we define $h_{\alpha+1}$ as follows. Since g is an exact upper bound and $h_\alpha < g$, choose $\xi(\alpha)$ greater than all $\xi(\beta)$ for $\beta < \alpha$ such that $h_\alpha <_I f_{\xi(\alpha)}$. Also, since $f_\xi <_I g$ for all $\xi < \lambda$, the projections $f_\xi^+ = \text{proj}(f, S)$ are defined. We define

$$h_{\alpha+1}(a) = \begin{cases} \max(h_\alpha(a), f^+_{\xi(\alpha)}(a)) + 1 & \text{if } f_{\xi(\alpha)}(a) < g(a), \\ h_\alpha(a) + 1 & \text{if } f_{\xi(\alpha)}(a) \geq g(a). \end{cases}$$

Thus we have

$$h_\alpha <_I f_{\xi(\alpha)} \le_I h_{\alpha+1}, \text{ for every } \alpha.$$

So conditions (1)–(3) hold. $\square$

Now we apply some infinite combinatorics to get information about $(*)_\kappa$.

Theorem 24a.40. (Club guessing) *Suppose that κ is a regular cardinal, λ is a cardinal such that* $\mathrm{cf}(\lambda) \ge \kappa^{++}$*, and $S^\lambda_\kappa = \{\delta \in \lambda : \mathrm{cf}(\delta) = \kappa\}$. Then there is a sequence $\langle C_\delta : \delta \in S^\lambda_\kappa\rangle$ such that:*

(i) For every $\delta \in S^\lambda_\kappa$ the set $C_\delta \subseteq \delta$ is club, of order type κ.

(ii) For every club $D \subseteq \lambda$ there is a $\delta \in D \cap S^\lambda_\kappa$ such that $C_\delta \subseteq D$.

The sequence $\langle C_\delta : \delta \in S^\lambda_\kappa\rangle$ is called a *club guessing sequence* for S^λ_κ.

Proof. First we take the case of uncountable κ. Fix a sequence $C' = \langle C'_\delta : \delta \in S^\lambda_\kappa\rangle$ such that $C'_\delta \subseteq \delta$ is club in δ of order type κ, for every $\delta \in S^\lambda_\kappa$. For any club E of λ, let

$$C' \restriction E = \langle C'_\delta \cap E : \delta \in S^\lambda_\kappa \cap E'\rangle,$$

where $E' = \{\delta \in E : E \cap \delta \text{ is unbounded in } \delta\}$. Clearly E' is also club in λ. Also note that $C'_\delta \cap E$ is club in δ for each $\delta \in S^\lambda_\kappa \cap E'$. We claim:

(1) There is a club E of λ such that for every club D of λ there is a $\delta \in D \cap E' \cap S^\lambda_\kappa$ such that $C'_\delta \cap E \subseteq D$.

Note that if we prove (1), then the theorem follows by defining $C_\delta = C'_\delta \cap E$ for all $\delta \in E' \cap S^\lambda_\kappa$, and $C_\delta = C'_\delta$ for $\delta \in S^\kappa_\lambda \backslash E'$.

Assume that (1) is false. Hence for every club $E \subseteq \lambda$ there is a club $D_E \subseteq \lambda$ such that for every $\delta \in D_E \cap E' \cap S^\lambda_\kappa$ we have

$$C'_\delta \cap E \not\subseteq D_E.$$

We now define a sequence $\langle E^\alpha : \alpha < \kappa^+\rangle$ of clubs of λ decreasing under inclusion, by induction on α:

(2) $E^0 = \lambda$.

(3) If $\gamma < \kappa^+$ is a limit ordinal and E^α has been defined for all $\alpha < \gamma$, we set $E^\gamma = \bigcap_{\alpha<\gamma} E^\alpha$. Since $\gamma < \kappa^+ < \mathrm{cf}(\lambda)$, E^γ is club in λ.

(4) If E^α has been defined, let $E^{\alpha+1}$ be the set of all limit points of $E^\alpha \cap D_{E^\alpha}$, i.e., the set of all $\varepsilon < \lambda$ such that $E^\alpha \cap D_{E^\alpha} \cap \varepsilon$ is unbounded in ε.

This defines the sequence. Let $E = \bigcap_{\alpha<\kappa^+} E^\alpha$. Then E is club in λ. Take any $\delta \in S^\lambda_\kappa \cap E$. Since $|C'_\delta| = \kappa$ and the sequence $\langle E^\alpha : \alpha < \kappa^+\rangle$ is decreasing, there is an $\alpha < \kappa^+$ such that $C'_\delta \cap E = C'_\delta \cap E^\alpha$. So $C'_\delta \cap E^\alpha = C'_\delta \cap E^{\alpha+1}$. Hence $C'_\delta \cap E^\alpha \subseteq D_{E^\alpha}$, contradiction.

Thus the case κ uncountable has been finished.

Now we take the case $\kappa = \omega$. For $S = S^\lambda_{\aleph_0}$ fix $C = \langle C_\delta : \delta \in S\rangle$ so that C_δ is club in δ with order type ω. We denote the n-th element of C_δ by $C_\delta(n)$. For any club $E \subseteq \lambda$ and any $\delta \in S \cap E'$ we define

$$C^E_\delta = \{\max(E \cap (C_\delta(n) + 1)) : n \in \omega\},$$

where again E' is the set of limit points of members of E. This set is cofinal in δ. In fact, given $\alpha < \delta$, there is a $\beta \in E \cap \delta$ such that $\alpha < \beta$ since $\delta \in E'$, and there is an $n \in \omega$ such that $\beta < C_\delta(n)$. Then $\alpha < \max(E \cap (C_\delta(n) + 1))$, as desired. There may be repetitions in the description of C^E_δ, but $\max(E \cap (C_\delta(n) + 1)) \leq \max(E \cap (C_\delta(m) + 1))$ if $n < m$, so C^E_δ has order type ω. We claim

(5) There is a closed unbounded $E \subseteq \lambda$ such that for every club $D \subseteq \lambda$ there is a $\delta \in D \cap S \cap E'$ such that $C^E_\delta \subseteq D$. [This proves the club guessing property.]

Suppose that (5) fails. Thus for every closed unbounded $E \subseteq \lambda$ there exist a club $D_E \subseteq \lambda$ such that for every $\delta \in D_E \cap S \cap E'$ we have $C^E_\delta \not\subseteq D$. Then we construct a descending sequence E^α of clubs in λ as in the case $\kappa > \omega$, for $\alpha < \omega_1$. Thus for each $\alpha < \omega_1$ and each $\delta \in D_{E^\alpha} \cap S \cap (E^\alpha)'$ we have $C^{E^\alpha}_\delta \not\subseteq D_{E^\alpha}$. Let $E = \bigcap_{\alpha<\omega_1} E^\alpha$. Take any $\delta \in S \cap E$. For $n \in \omega$ and $\alpha < \beta$ we have

$$E^\alpha \cap (C_\delta(n) + 1) \supseteq E^\beta \cap (C_\delta(n) + 1),$$

and so $\max(E^\alpha \cap (C_\delta(n)+1)) \geq \max(E^\beta \cap (C_\delta(n)+1))$; it follows that there is an $\alpha_n < \omega_1$ such that $\max(E^\beta \cap (C_\delta(n) + 1)) = \max(E^{\alpha_n} \cap (C_\delta(n) + 1))$ for all $\beta > \alpha_n$. Choose γ greater than all α_n. Thus

(6) For all $\varepsilon > \gamma$ and all $n \in \omega$ we have $\max(E^\varepsilon \cap (C_\delta(n) + 1)) = \max(E^\gamma \cap (C_\delta(n) + 1))$.

But there is a $\rho \in C^{E^\gamma}_\delta \backslash D_{E^\gamma}$; say that $\rho = \max(E^\gamma \cap (C_\delta(n) + 1))$. Then $\rho = \max(E^{\gamma+1} \cap (C_\delta(n) + 1)) \in E^{\gamma+1} = (E^\gamma \cap D_{E^\gamma})' \in D_{E^\gamma}$, contradiction. $\square$

Lemma 24a.41. *Suppose that:*

(i) I is an ideal over A.

(ii) κ and λ are regular cardinals such that $|A| < \kappa$ and $\kappa^{++} < \lambda$.

(iii) $f = \langle f_\xi : \xi < \lambda\rangle$ is a sequence of length λ of functions in ${}^A\mathbf{On}$ that is $<_I$-increasing and satisfies the following condition:

For every $\delta < \lambda$ with $\mathrm{cf}(\delta) = \kappa^{++}$ there is a club $E_\delta \subseteq \delta$ such that for some $\delta' \geq \delta$ with $\delta' < \lambda$,

$$(\star) \qquad \sup\{f_\alpha : \alpha \in E_\delta\} \leq_I f_{\delta'}.$$

Under these assumptions, $(*)_\kappa$ holds for f.

Proof. Assume the hypotheses. Let $S = S^{\kappa^{++}}_\kappa$; so S is stationary in κ^{++}. By Theorem 24a.40, let $\langle C_\delta : \delta \in S\rangle$ be a club guessing sequence for S; thus

(1) For every $\delta \in S$, the set $C_\delta \subseteq \delta$ is a club of order type κ.

(2) For every club $D \subseteq \kappa^{++}$ there is a $\delta \in D \cap S$ such that $C_\delta \subseteq D$.

Now let $U \subseteq \lambda$ be unbounded; we want to find $X_0 \subseteq U$ of order type κ such that $\langle f_\xi : \xi \in X_0 \rangle$ is strongly increasing under I. To do this we first define an increasing continuous sequence $\langle \xi(i) : i < \kappa^{++} \rangle \in {}^{\kappa^{++}}\lambda$ recursively.

Let $\xi(0) = 0$. For i limit, let $\xi(i) = \sup_{k<i} \xi(k)$.

Now suppose for some $i < \kappa^{++}$ that $\xi(k)$ has been defined for every $k \leq i$; we define $\xi(i+1)$. For each $\alpha \in S$ we define

$$h_\alpha = \sup\{f_\eta : \eta \in \xi[C_\alpha \cap (i+1)]\} \quad \text{and}$$
$$\sigma_\alpha = \begin{cases} \text{least } \sigma \in (\xi(i), \lambda) \text{ such that } h_\alpha \leq_I f_\sigma & \text{if there is such a } \sigma, \\ \xi(i)+1 & \text{otherwise.} \end{cases}$$

Now we let $\xi(i+1)$ be the least member of U which is greater than $\sup\{\sigma_\alpha : \alpha \in S\}$. It follows that

(3) If $\alpha \in S$ and the first case in the definition of σ_α holds, then $h_\alpha <_I f_{\xi(i+1)}$.

Now the set $F \stackrel{\text{def}}{=} \{\xi(k) : k \in \kappa^{++}\}$ is closed, and has order type κ^{++}. Let $\delta = \sup F$. Then F is a club of δ, and $\mathrm{cf}(\delta) = \kappa^{++}$. Hence by the hypothesis (iii) of the lemma, there is a club $E_\delta \subseteq \delta$ and a $\delta' \in [\delta, \lambda)$ such that $(\star)$ in the lemma holds. Note that $F \cap E_\delta$ is club in δ.

Let $D = \xi^{-1}[F \cap E_\delta]$. Since ξ is strictly increasing and continuous, it follows that D is club in κ^{++}. Hence by (2) there is an $\alpha \in D \cap S$ such that $C_\alpha \subseteq D$. Hence

$$\overline{C}_\alpha \stackrel{\text{def}}{=} \xi[C_\alpha] \subseteq F \cap E_\delta$$

is club in $\xi(\alpha)$ of order type κ. Then by $(\star)$ we have

$$\sup\{f_\rho : \rho \in \overline{C}_\alpha\} \leq_I f_{\delta'}.$$

Now

(4) For every $\rho < \rho'$ both in $\overline{C}_\alpha$, we have $\sup\{f_\zeta : \zeta \in \overline{C}_\alpha \cap (\rho+1)\} <_I f_{\rho'}$.

To prove this, note that there is an $i < \kappa^{++}$ such that $\rho = \xi(i)$. Now follow the definition of $\xi(i+1)$. There C_α was considered (among all other closed unbounded sets in the guessing sequence), and h_α was formed at that stage. Now

$$h_\alpha = \sup\{f_\eta : \eta \in \xi[C_\alpha \cap (i+1)]\} \leq \sup\{f_\eta : \eta \in \xi[C_\alpha]\} = \sup\{f_\eta : \eta \in \overline{C}_\alpha\} \leq_I f_{\delta'},$$

so the first case in the definition of σ_α holds. Thus by (3), $h_\alpha <_I f_{\xi(i+1)}$. Clearly $\xi(i+1) \leq \rho'$, so (4) follows.

Now let $\langle \eta(\nu) : \nu < \kappa \rangle$ be the strictly increasing enumeration of $\overline{C}_\alpha$, and set

$$X_0 = \{\eta(\omega \cdot \rho + 2m + 1) : \rho < \kappa, 0 < m < \omega\},$$
$$X_1 = \{\eta(\omega \cdot \rho + 2m) : \rho < \kappa, 0 < m < \omega\},$$

and for each $\beta \in X_1$ let $f'_\beta = \sup\{f_\sigma + 1 : \sigma \in X_0 \cap \beta\}$. Then for $\beta < \beta'$, both in X_1, we have $f'_\beta < f'_{\beta'}$. Now suppose that $\zeta \in X_0$; say $\zeta = \eta(\omega \cdot \rho + 2m + 1)$ with $\rho < \kappa$ and $0 < m < \omega$. Then

$$\begin{aligned} f'_{\eta(\omega\cdot\rho+2m)} &= \sup\{f_\sigma + 1 : \sigma \in X_0 \cap \eta(\omega \cdot \rho + 2m)\} <_I f_\zeta \quad \text{by (4)} \\ &\leq \sup\{f_\sigma + 1 : \sigma \in X_0 \cap \eta(\omega \cdot \rho + 2m + 2)\} \\ &= f'_{\eta(\omega\cdot\rho+2m+2)}. \end{aligned}$$

Hence by Proposition 24a.27, $\langle f_\zeta : \zeta \in X_0\rangle$ is very strongly increasing under I. □

Now we need a purely combinatorial proposition.

Proposition 24a.42. *Suppose that κ and λ are regular cardinals, and $\kappa^{++} < \lambda$. Suppose that F is a function with domain contained in $[\lambda]^{<\kappa}$ and range contained in λ. Suppose that for every $\delta \in S^\lambda_{\kappa^{++}}$ there is a closed unbounded set $E_\delta \subseteq \delta$ such that $[E_\delta]^{<\kappa} \subseteq \mathrm{dmn}(F)$. Then the following set is stationary:*

$$\begin{aligned} \{\alpha \in S^\lambda_\kappa : \ &\text{there is a closed unbounded } D \subseteq \alpha \text{ such that for any } a, b \in D \\ &\text{with } a < b,\ \{d \in D : d \leq a\} \in \mathrm{dmn}(F) \text{ and } F(\{d \in D : d \leq a\}) < b\} \end{aligned}$$

Proof. We follow the proof of Theorem 24a.41 closely. Call the indicated set T. Let U be a closed unbounded subset of λ. We want to find a member of $T \cap U$.

Let $S = S^{\kappa^{++}}_\kappa$; so S is stationary in κ^{++}. By Theorem 24a.40, let $\langle C_\delta : \delta \in S\rangle$ be a club guessing sequence for S; thus

(1) For every $\delta \in S$, the set $C_\delta \subseteq \delta$ is a club of order type κ.

(2) For every club $D \subseteq \kappa^{++}$ there is a $\delta \in D \cap S$ such that $C_\delta \subseteq D$.

We define an increasing continuous sequence $\langle \xi(i) : i < \kappa^{++}\rangle \in {}^{\kappa^{++}}\lambda$ recursively.

Let $\xi(0)$ be the least member of U. For i limit, let $\xi(i) = \sup_{k<i} \xi(k)$.

Now suppose for some $i < \kappa^{++}$ that $\xi(k)$ has been defined for every $k \leq i$; we define $\xi(i+1)$. For each $\alpha \in S$ we consider two possibilities. If $\xi[C_\alpha \cap (i+1)] \in \mathrm{dmn}(F)$, we let σ_α be any ordinal greater than both $\xi(i)$ and $F(\xi[C_\alpha \cap (i+1)])$. Otherwise, we let $\sigma_\alpha = \xi(i) + 1$. Since $|S| < \lambda$, we can let $\xi(i+1)$ be the least member of U greater than all σ_α for $\alpha \in S$. Hence

(3) If $\alpha \in S$ and the first case in the definition of σ_α holds, then $\xi[C_\alpha \cap (i+1)] \in \mathrm{dmn}(F)$ and $F(\xi[C_\alpha \cap (i+1)]) < \xi(i+1)$.

Now the set $G = \mathrm{rng}(\xi)$ is closed and has order type κ^{++}. Let $\delta = \sup(G)$. Hence by the hypothesis of the proposition, there is a closed unbounded set $E_\delta \subseteq \delta$ such that $[E_\delta]^{<\kappa} \subseteq \mathrm{dmn}(F)$. Note that $G \cap E_\delta$ is also closed unbounded in δ.

Let $H = \xi^{-1}[G \cap E_\delta]$. Thus H is club in κ^{++}. Hence by (2) there is an $\alpha \in H \cap S$ such that $C_\alpha \subseteq H$. Hence $\overline{C}_\alpha \stackrel{\text{def}}{=} \xi[C_\alpha] \subseteq G \cap E_\delta$ is club in $\xi(\alpha)$ of order type κ. We claim that $\overline{C}_\alpha$ is as desired in the proposition. For, suppose that $a, b \in \overline{C}_\alpha$ and $a < b$. Write

$a = \xi(i)$. Then $\{d \in \overline{C}_\alpha : d \le a\} = \xi[C_\alpha \cap (i+1)] \subseteq E_\delta$, and so (3) gives the desired conclusion. $\square$

Next we give a condition under which $(*)_\kappa$ holds.

Lemma 24a.43. *Suppose that I is a proper ideal over a set A of regular cardinals such that $|A| < \min(A)$. Assume that $\lambda > |A|$ is a regular cardinal such that $(\prod A, <_I)$ is λ-directed, and $\langle g_\xi : \xi < \lambda\rangle$ is a sequence of members of $\prod A$.*

Then there is a $<_I$-increasing sequence $f = \langle f_\xi : \xi < \lambda\rangle$ of length λ in $\prod A$ such that:

(i) $g_\xi < f_{\xi+1}$ for every $\xi < \lambda$.

(ii) $(*)_\kappa$ holds for f, for every regular cardinal κ such that $\kappa^{++} < \lambda$ and $\{a \in A : a \le \kappa^{++}\} \in I$.

Proof. Let f_0 be any member of $\prod A$. At successor stages, if f_ξ is defined, let $f_{\xi+1}$ be any function in $\prod A$ that $<$-extends f_ξ and g_ξ.

At limit stages δ, there are three cases. In the first case, $\mathrm{cf}(\delta) \le |A|$. Fix some $E_\delta \subseteq \delta$ club of order type $\mathrm{cf}(\delta)$, and define

$$f_\delta = \sup\{f_i : i \in E_\delta\}.$$

For any $a \in A$ we have $\mathrm{cf}(\delta) \le |A| < \min(A) \le a$, and so $f_\delta(a) < a$. Thus $f_\delta \in \prod A$.

In the second case, $\mathrm{cf}(\delta) = \kappa^{++}$, where κ is regular, $|A| < \kappa$, and $\{a \in A : a \le \kappa^{++}\} \in I$. Then we define f'_δ as in the first case. Then for any $a \in A$ with $a > \kappa^{++}$ we have $f'_\delta(a) < a$, and so $\{a \in A : a \le f'_\delta(a)\} \in I$, and we can modify f'_δ on this set which is in I to obtain our desired f_δ.

In the third case, neither of the first two cases holds. Then we let f_δ be any $\le_I$-upper bound of $\{f_\xi : \xi < \delta\}$; it exists by the λ-directedness assumption.

This completes the construction. Obviously (i) holds. For (ii), suppose that κ is a regular cardinal such that $\kappa^{++} < \lambda$ and $\{a \in A : a \le \kappa^{++}\} \in I$. If $|A| < \kappa$, the desired conclusion follows by Lemma 24a.41. In case $\kappa \le |A|$, note that $\langle f_\xi : \xi < \kappa\rangle$ is $<$-increasing, and so is certainly strongly increasing under I. $\square$

Now we apply these results to the determination of true cofinality for some important concrete partial orders.

Notation. *For any set X of cardinals, let*

$$X^{(+)} = \{\alpha^+ : \alpha \in X\}.$$

Theorem 24a.44. (Representation of μ^+ as a true cofinality, I) *Suppose that μ is a singular cardinal with uncountable cofinality. Then there is a club C in μ such that C has order type $\mathrm{cf}(\mu)$, every element of C is greater than $\mathrm{cf}(\mu)$, and*

$$\mu^+ = \mathrm{tcf}\left(\prod C^{(+)}, <_{J^{\mathrm{bd}}}\right),$$

where J^{bd} is the ideal of all bounded subsets of $C^{(+)}$.

Proof. Let C_0 be any closed unbounded set of limit cardinals less than μ such that $|C_0| = \mathrm{cf}(\mu)$ and all cardinals in C_0 are above $\mathrm{cf}(\mu)$. Then

(1) all members of C_0 which are limit points of C_0 are singular.

In fact, suppose on the contrary that $\kappa \in C_0$, κ is a limit point of C_0, and κ is regular. Thus $C_0 \cap \kappa$ is unbounded in κ, so $|C_0 \cap \kappa| = \kappa$. But $\mathrm{cf}(\mu) < \kappa$ and $|C_0| = \mathrm{cf}\mu$, contradiction. So (1) holds. Hence wlog every member of C_0 is singular.

Now we claim

(2) $(\prod C_0^{(+)}, <_{J^{\mathrm{bd}}})$ is μ-directed.

In fact, suppose that $F \subseteq \prod C_0^{(+)}$ and $|F| < \mu$. For $a \in C_0^{(+)}$ with $|F| < a$ let $h(a) = \sup_{f\in F} f(a)$; so $h(a) \in a$. For $a \in C_0^{(+)}$ with $a \leq |F|$ let $h(a) = 0$. Clearly $f \leq_{J^{\mathrm{bd}}} h$ for all $f \in F$. So (2) holds.

(3) $(\prod C_0^{(+)}, <_{J^{\mathrm{bd}}})$ is μ^+-directed.

In fact, by (2) it suffices to find a bound for a subset F of $\prod C_0^{(+)}$ such that $|F| = \mu$. Write $F = \bigcup_{\alpha<\mathrm{cf}(\mu)} G_\alpha$, with $|G_\alpha| < \mu$ for each $\alpha < \mathrm{cf}(\mu)$. By (2), each G_α has an upper bound k_α under $<_{J^{\mathrm{bd}}}$. Then $\{k_\alpha : \alpha < \mathrm{cf}(\mu)\}$ has an upper bound h under $<_{J^{\mathrm{bd}}}$. Clearly h is an upper bound for F.

Now we are going to apply Lemma 24a.43 to J^{bd}, $C_0^{(+)}$, and μ^+ in place of I, A, and λ; and with anything for g. Clearly the hypotheses hold, so we get a $<_{J^{\mathrm{bd}}}$-increasing sequence $f = \langle f_\xi : \xi < \mu^+\rangle$ in $\prod C_0^{(+)}$ such that $(*)_\kappa$ holds for f and the bounding projection property holds for κ, for every regular cardinal $\kappa < \mu$. It also follows that the bounding projection property holds for $|A|^+$, and hence by 24a.37, f has an exact upper bound h. Then by Lemma 24a.38, for every regular $\kappa < \mu$ we have

$$(\star) \qquad \{a \in C_0^{(+)} : h(a) \text{ is non-limit, or } \mathrm{cf}(h(a)) < \kappa\} \in J^{\mathrm{bd}}.$$

Now the identity function k on $C_0^{(+)}$ is obviously is an upper bound for f, so $h \leq_{J^{\mathrm{bd}}} k$. By modifying h on a set in J^{bd} we may assume that $h(a) \leq a$ for all $a \in C_0^{(+)}$. Now we claim

$(\star\star)$ The set $C_1 \stackrel{\mathrm{def}}{=} \{\alpha \in C_0 : h(\alpha^+) = \alpha^+\}$ contains a club of μ.

Assume otherwise. Then for every club K, $K \cap (\mu\backslash C_1) \neq 0$. This means that $\mu\backslash C_1$ is stationary, and hence $S \stackrel{\mathrm{def}}{=} C_0\backslash C_1$ is stationary. For each $\alpha \in S$ we have $h(\alpha^+) < \alpha^+$. Hence $\mathrm{cf}(h(\alpha^+)) < \alpha$ since α is singular. Hence by Fodor's theorem $\langle \mathrm{cf}(h(\alpha^+)) : \alpha \in C_0\rangle$ is bounded by some $\kappa < \mu$ on a stationary subset of S. This contradicts $(\star)$.

Thus $(\star\star)$ holds, and so there is a club $C \subseteq C_0$ such that $h(\alpha^+) = \alpha^+$ for all $\alpha \in C$. Now $\langle f_\xi \restriction C^{(+)} : \xi < \mu^+\rangle$ is $<_{J^{\mathrm{bd}}}$-increasing. We claim that it is cofinal in $(\prod C^{(+)}, <_{J^{\mathrm{bd}}})$. For, suppose that $g \in \prod C^{(+)}$. Let g' be the extension of g to $\prod C_0^{(+)}$ such that $g'(a) = 0$ for any $a \in C_0\backslash C$. Then $g' <_{J^{\mathrm{bd}}} h$, and so there is a $\xi < \mu^+$ such that $g' <_{J^{\mathrm{bd}}} f_\xi$. So $g <_{J^{\mathrm{bd}}} f_\xi \restriction C^{(+)}$, as desired. This shows that $\mu^+ = \mathrm{tcf}(\prod C^{(+)}, <_{J^{\mathrm{bd}}})$. $\square$

Theorem 24a.45. (Representation of μ^+ as a true cofinality, II) *If μ is a singular cardinal of countable cofinality, then there is an unbounded set $D \subseteq \mu$ of regular cardinals such that*

$$\mu^+ = \mathrm{tcf}\left(\prod D, <_{J^{\mathrm{bd}}}\right).$$

Proof. Let C_0 be a set of uncountable regular cardinals with supremum μ, of order type ω.

(1) $\prod C_0/J^{\mathrm{bd}}$ is μ-directed.

For, let $X \subseteq \prod C_0$ with $|X| < \mu$. For each $a \in C_0$ such that $|X| < a$, let $h(a) = \sup\{f(a) : f \in X\}$, and extend h to all of C_0 in any way. Clearly $h \in \prod C_0$ and it is an upper bound in the $<_{J^{\mathrm{bd}}}$ sense for X.

From (1) it is clear that $\prod C_0/J^{\mathrm{bd}}$ is also μ^+-directed. By Lemma 24a.43 we then get a $<_{J^{\mathrm{bd}}}$-increasing sequence $\langle f_\xi : \xi < \mu^+\rangle$ which satisfies $(*)_\kappa$ for every regular $\kappa < \mu^+$. By Theorems 24a.37 and 24a.38 f has an exact upper bound h such that $\{a \in C_0 : h(a)$ is non-limit or $\mathrm{cf}(h(a)) < \kappa\} \in J^{\mathrm{bd}}$ for every regular $\kappa < \mu^+$. We may assume that $h(a) \le a$ for all $a \in C_0$, since the identity function is clearly an upper bound for f; and we may assume that each $h(a)$ is a limit ordinal of uncountable cofinality since $\{a \in C_0 : \mathrm{cf}(h(a)) < \omega_1\} \in J^{\mathrm{bd}}$.

(2) $\mathrm{tcf}\left(\prod_{a\in C_0} \mathrm{cf}(h(a)), <_{J^{\mathrm{bd}}}\right) = \mu^+$.

To prove this, for each $a \in C_0$ let D_a be club in $h(a)$ of order type $\mathrm{cf}(h(a))$, and let $\langle \eta_{a\xi} : \xi < \mathrm{cf}(h(a))\rangle$ be the strictly increasing enumeration of D_a. For each $\xi < \mu^+$ we define $f'_\xi \in \prod_{a\in C_0} \mathrm{cf}(h(a))$ as follows. Since $f_\xi <_{J^{\mathrm{bd}}} h$, the set $\{a \in C_0 : f_\xi(a) \ge h(a)\}$ is bounded, so choose $a_0 \in C_0$ such that for all $b \in C_0$ with $a_0 \le b$ we have $f_\xi(b) < h(b)$. For such a b we define $f'_\xi(b)$ to be the least ν such that $f_\xi(b) < \eta_{b\nu}$. Then we extend f'_α in any way to a member of $\prod_{a\in C_0} \mathrm{cf}(h(a)))$.

(3) $\xi < \sigma < \mu^+$ implies that $f'_\xi \le_{J^{\mathrm{bd}}} f'_\sigma$.

This is clear by the definitions.

Now for each $l \in \prod_{a\in C_0} \mathrm{cf}(h(a)))$ define $k_l \in \prod C_0$ by setting $k_l(a) = \eta_{al(a)}$ for all a. So $k_l < h$. Since h is an exact upper bound for f, choose $\xi < \mu^+$ such that $k_l <_{J^{\mathrm{bd}}} f_\xi$. Choose a such that $k_l(b) < f_\xi(b) < h(b)$ for all $b \ge a$. Then for all $b \ge a$, $\eta_{bl(b)} < \eta_{bf'_\xi(b)}$, and hence $l(b) < f'_\xi(b)$. This proves that $l <_{J^{\mathrm{bd}}} f'_\xi$. This proves the following statement.

(4) $\{f'_\xi : \xi < \mu^+\}$ is cofinal in $\left(\prod_{a\in C_0} \mathrm{cf}(h(a)), <_{J^{\mathrm{bd}}}\right)$.

Now (3) and (4) yield (2).

Now let $B = \{\mathrm{cf}(h(a)) : a \in C_0\}$. Define

$$X \in J \text{ iff } X \subseteq B \text{ and } h^{-1}[\mathrm{cf}^{-1}[X]] \in J^{\mathrm{bd}}.$$

By Lemma 24a.24 we get $\mathrm{tcf}(\prod B/J) = \mu^+$. It suffices now to show that J is the ideal of bounded subsets of B. Suppose that $X \in J$, and choose $a \in C_0$ such that $h^{-1}[\mathrm{cf}^{-1}[X]] \subseteq \{b \in C_0 : b < a\}$. Thus $X \subseteq \{b \in A : \mathrm{cf}(h(b)) < a\} \in J^{\mathrm{bd}}$, so X is bounded. Conversely, if X is bounded, choose $a \in B$ such that $X \subseteq \{b \in B : b \le a\}$. Now

$$\begin{aligned} h^{-1}[\mathrm{cf}^{-1}[X]] &= \{b \in C_0 : \mathrm{cf}(h(b)) \in X\} \\ &\subseteq \{b \in C_0 : \mathrm{cf}(h(b)) \le a\}, \end{aligned}$$

and this is bounded by the choice of h. $\square$

Proposition 24a.46. (The trichotomy theorem) *Suppose that $\lambda > |A|^+$ is a regular cardinal and $f = \langle f_\xi : \xi < \lambda\rangle$ is a $<_I$-increasing sequence. Consider the following properties of f and a regular cardinal κ such that $|A| < \kappa \leq \lambda$:*

$\mathbf{Bad}_\kappa$*: There exist:*

(a) sets S_a of ordinals for $a \in A$ such that $f_\alpha <_I \sup S$ for all $\alpha < \lambda$ and $|S_a| < \kappa$; and

(b) an ultrafilter D over A extending the dual of I

such that for every $\alpha < \lambda$ there is a $\beta < \lambda$ such that proj$(f_\alpha, S) <_D f_\beta$.

Ugly *There exists a function $g \in {}^A$ Ord such that, defining $t_\alpha = \{a \in A : g(a) < f_\alpha(a)\}$, the sequence $\langle t_\alpha : \alpha < \lambda\rangle$ does not stabilize modulo I. That is, for every α there is a $\beta > \alpha$ in λ such that $t_\beta \backslash t_\alpha \in I^+$. Note here that $\langle t_\alpha : \alpha < \lambda\rangle$ is $\subseteq_I$-increasing.*

$\mathbf{Good}_\kappa$ *There exists an exact upper bound g for f such that $\mathrm{cf}(g(a)) \geq \kappa$ for every $a \in A$.*

Then the assertion of this theorem is that the bounding projection property for κ is equivalent to $\neg\mathbf{Bad}_\kappa \wedge \neg\mathbf{Ugly}$. Hence if neither $\mathbf{Bad}_\kappa$ nor **Ugly***, then $\mathbf{Good}_\kappa$.*

Proof. We use the abbreviation f_α^+ for $\mathrm{proj}(f_\alpha, S)$.

First assume the bounding projection property for κ. Suppose that $\mathbf{Bad}_\kappa$ holds, and assume the notation of it. Choose $\alpha < \lambda$ such that f_α^+ is a $<_I$-upper bound for f. Choose $\beta > \alpha$ as in the definition of $\mathbf{Bad}_\kappa$. Then $f_\beta <_D f_\alpha^+ <_D f_\beta$, contradiction.

To prove $\neg$**Ugly**, suppose that g is as in the definition of **Ugly**. By 9.13, let h be an exact upper bound for f, and for each $a \in A$ let $S(a) = \{g(a), h(a)\}$. Thus $f_\alpha <_I \sup S$ for each $\alpha < \lambda$. By the bounding projection property, choose $\alpha < \lambda$ such that f_α^+ $<_I$-bounds f. Take any $\beta > \alpha$. Then $f_\beta <_I f_\alpha^+$, so $\{a : f_\beta(a) \geq f_\alpha^+(a)\} \in I$. Now

$$t_\beta \backslash t_\alpha = \{a : f_\alpha(a) \leq g(a) < f_\beta(a)\} \subseteq \{a : f_\beta(a) \geq f_\alpha^+(a)\} \in I,$$

contradiction.

Conversely, assume $\neg\mathbf{Bad}_\kappa$ and $\neg$**Ugly**, but also suppose that the bounding projection property for κ fails to hold. By the last supposition we get the hypothesis of the bounding projection property, but there is no $\xi < \lambda$ such that f_ξ^+ bounds f. For all $\xi, \alpha < \lambda$ let $t_\alpha^\xi = \{a \in A : f_\xi^+(a) < f_\alpha(a)\}$.

(1) For every $\xi < \lambda$ there is a $\beta_\xi > \xi$ such that $t_{\beta_\xi}^\xi \in I^+$ and for all $\gamma > \beta_\xi$ we have $t_\gamma^\xi \backslash t_{\beta_\xi}^\xi \in I$.

In fact, since f_ξ^+ does not bound f, we can choose $\beta_\xi > \xi$ such that $f_{\beta_\xi} \not\leq_I f_\xi^+$; and since $\neg$**Ugly**, we can choose $\delta_\xi > \xi$ such that for all $\gamma > \delta_\xi$ we have $t_\gamma^\xi \backslash t_{\delta_\xi}^\xi \in I$. We may assume that $\beta_\xi = \delta_\xi$, and this gives the desired conclusion of (1).

By (1) we can define strictly increasing sequences $\langle \xi(\nu) : \nu < \lambda\rangle$ and $\langle \beta(\nu) : \nu < \lambda\rangle$ such that for all $\nu < \lambda$, $t_{\beta(\nu)}^{\xi(\nu)} \in I^+$, $\xi(\nu) < \beta(\nu)$, $\beta(\nu) < \xi(\rho)$ if $\nu < \rho < \lambda$, and $t_\gamma^{\xi(\nu)} \backslash t_{\beta(\nu)}^{\xi(\nu)} \in I$ for all $\gamma > \beta(\nu)$.

(2) If $\nu < \rho < \lambda$, then

$$t^{\xi(\rho)}_{\beta(\rho)} \subseteq \left(t^{\xi(\nu)}_{\beta(\nu)} \cap t^{\xi(\rho)}_{\beta(\rho)}\right) \cup \left(t^{\xi(\nu)}_{\beta(\rho)} \backslash t^{\xi(\nu)}_{\beta(\nu)}\right) \cup \left\{a \in A : f^+_{\xi(\rho)}(a) < f^+_{\xi(\nu)}(a)\right\}.$$

To prove this, suppose that a is not a member of the right side. Then the following conditions hold:

(3) $f_{\beta(\nu)}(a) \leq f^+_{\xi(\nu)}(a)$ or $f_{\beta(\rho)}(a) \leq f^+_{\xi(\rho)}(a)$.

(4) $f_{\beta(\rho)}(a) \leq f^+_{\xi(\nu)}(a)$ or $f^+_{\xi(\nu)}(a) < f_{\beta(\nu)}(a)$.

(5) $f^+_{\xi(\nu)}(a) \leq f^+_{\xi(\rho)}(a)$.

Clearly then, $f_{\beta(\rho)}(a) \leq f^+_{\xi(\rho)}(a)$, which shows that a is not in the left side. So (2) holds.

(6) If $\nu_1 < \cdots < \nu_m < \lambda$, then $t^{\xi(\nu_1)}_{\beta(\nu_1)} \cap \ldots \cap t^{\xi(\nu_m)}_{\beta(\nu_m)} \in I^+$.

We prove this by induction on m. It is clear for $m = 1$. Assume it for m, and suppose that $\nu_1 < \cdots < \nu_{m+1}$. Then by (2),

$$\begin{aligned} t^{\xi(\nu_2)}_{\beta(\nu_2)} \cap \ldots \cap t^{\xi(\nu_m)}_{\beta(\nu_m)} \subseteq & \left(t^{\xi(\nu_1)}_{\beta(\nu_1)} \cap \ldots \cap t^{\xi(\nu_{m+1})}_{\beta(\nu_{m+1})}\right) \\ & \cup \left(t^{\xi(\nu_1)}_{\beta(\nu_2)} \backslash t^{\xi(\nu_1)}_{\beta(\nu_1)}\right) \cup \left\{a \in A : f^+_{\xi(\nu_2)}(a) < f^+_{\xi(\nu_1)}(a)\right\}, \end{aligned}$$

and the last two sets are in I, so our conclusion follows by the inductive hypothesis.

By (6), the set $I^* \cup \{t^{\xi(\nu)}_{\beta(\nu)} : \nu < \lambda\}$ has fip, and hence is contained in an ultrafilter D.

By $\neg\mathbf{Bad}_\kappa$, choose $\alpha < \lambda$ such that f^+_α is a $<_D$-bound for f. Take ν with $\alpha < \xi(\nu)$. Now $t^{\xi(\nu)}_{\beta(\nu)} \in D$, so $f^+_{\xi(\nu)} <_D f_{\beta(\nu)}$. Thus $f^+_\alpha \leq_D f^+_{\xi(\nu)} <_D f_{\beta(\nu)} <_D f^+_\alpha$, contradiction.

The final assertion of the theorem follows by 9.15. ☐

Proposition 24a.47. *Suppose that λ is a regular cardinal, A is an infinite set such that $\forall \mu < \lambda(\mu^{|A|} < \lambda$, and $\langle f_\alpha : \alpha < \lambda\rangle$ is a system of members of ^{A}Ord.*

Then there is a stationary subset E of λ such that for all $\alpha, \beta \in E$, if $\alpha < \beta$ then $f_\alpha \leq f_\beta$. Moreover, for all $a \in A$, either $\langle f_\alpha(a) : \alpha \in E\rangle$ is a constant sequence, or it is strictly increasing.

If in addition I is a proper ideal on A and $\langle f_\alpha : \alpha < \lambda\rangle$ is $<_I$-increasing, then $(*)_\lambda$ holds.

Proof. For each $a \in A$ fix $\gamma_a > \sup\{f_\alpha(a) : \alpha < \lambda\}$. For $\alpha < \lambda$ and $a \in A$ let $S^\alpha(a) = \{f_\beta(a) : \beta < \alpha\} \cup \{\gamma_a\}$. For any $\alpha < \lambda$ and $a \in A$, let $g_\alpha(a) = \min(S^\alpha \backslash f_\alpha(a))$. Thus either $g_\alpha(a) = \gamma_a$ or $g_\alpha(a) = f_\beta(a)$ for some $\beta < \alpha$. In the second case, choose such a β; call it β_a.

Let $T = \{\delta < \lambda : \mathrm{cf}(\delta) = |A|^+\}$. Suppose that $\alpha \in T$. Since $\mathrm{cf}(\alpha) = |A|^+$, we can choose $\mu_\alpha < \alpha$ such that $\beta_a < \mu_\alpha$ for all $a \in A$ for which β_a is defined. Hence $g_\alpha = \mathrm{proj}(f_\alpha, S^{\mu_\alpha})$. By Fodor's theorem we may assume that $\mu = \mu_\alpha$ is fixed on a stationary subset T' of T. Since S^μ has size $\mu < \lambda$, $\mu^{|A|} < \lambda$, and g_α maps A into a set of size at most μ, we may assume that $g = g_\alpha$ is fixed for all α in a stationary subset T'' of T'. Now

$$T'' = \bigcup_{h \in {}^A 2} \{\alpha \in T'' : \forall a \in A(f_\alpha(a) < g(a) \leftrightarrow h(a) = 1\}.$$

Since $2^{|A|} < \lambda$, it follows that there is an $h \in {}^A 2$ such that

$$E \stackrel{\text{def}}{=} \{\alpha \in T'' : \forall a \in A(f_\alpha(a) < g(a) \leftrightarrow h(a) = 1\}$$

is stationary. Suppose that $\alpha, \beta \in E$, $\alpha < \beta$, $a \in A$, and $f_\beta(a) < f_\alpha(a)$. Then

$$f_\alpha(a) \leq \min(S^\alpha(a) \backslash f_\alpha(a)) = g(a) = \min(S^\beta(a) \backslash f_\beta(a)) \leq f_\alpha(a),$$

and so $f_\alpha(a) = g(a)$. It follows that $h(a) = 0$. But $f_\beta(a) < f_\alpha(a) = g(a)$, contradiction.

So we have proved that if $\alpha, \beta \in E$ and $\alpha < \beta$, then $f_\alpha \leq f_\beta$.

We claim that also for each $a \in A$, either $f_\alpha(a) = g(a)$ for all $\alpha \in E$, or $f_\alpha(a) < f_\beta(a)$ for all $\alpha, \beta \in E$ such that $\alpha < \beta$. Otherwise, there is an $\alpha \in E$ with $f_\alpha(a) < g(a)$ and there are $\beta, \delta \in E$ with $\beta < \delta$ and $f_\beta(a) = f_\delta(a)$. Then $g(a) = \min(S^\delta(a) \backslash f_\delta(a)) = f_\delta(a)$, since $f_\beta(a) \in S^\delta(a)$. But then $h(a) = 0$, contradicting $f_\alpha(a) < g(a)$.

For the last statement of the theorem, assume that $f_\alpha <_I f_\beta$ for all $\alpha < \beta < \lambda$. Now if $\alpha, \beta \in E$ and $\alpha < \beta$, then $\{a \in A : f_\alpha(a) \geq f_\beta(a)\} \in I$. Since $f_\alpha \leq f_\beta$, this means that $\{a \in A : f_\alpha(a) = f_\beta(a)\} \in I$. But this set is $B \stackrel{\text{def}}{=} \{a \in A : f_\delta(a) = f_\varepsilon(a)$ for all $\delta, \varepsilon \in E\}$. For $a \notin B$ and $\alpha < \beta$, both in E, we have $f_\alpha(a) < f_\beta(a)$. So $\langle f_\alpha : \alpha \in E\rangle$ is strongly increasing mod I. By 9.6 it follows then that $\langle f_\alpha : \alpha < \lambda\rangle$ has an exact upper bound h such that $\text{cf}(h(a)) = \lambda$ for all $a \in A$. Hence by 9.15, $(*)_\lambda$ holds for $\langle f_\alpha : \alpha < \lambda\rangle$. $\square$

Basic properties of PCF

For any set A of regular cardinals define

$$\text{pcf}(A) = \left\{\text{cf}\left(\prod A/D\right) : D \text{ is an ultrafilter on } A\right\}.$$

By definition, $\text{pcf}(\emptyset) = \emptyset$. We begin with a very easy proposition which will be used a lot in what follows.

Proposition 24b.1. *Let A and B be sets of regular cardinals.*

(i) $A \subseteq \text{pcf}(A)$.

(ii) If $A \subseteq B$, then $\text{pcf}(A) \subseteq \text{pcf}(B)$.

(iii) $\text{pcf}(A \cup B) = \text{pcf}(A) \cup \text{pcf}(B)$.

(iv) If $B \subseteq A$, then $\text{pcf}(A) \backslash \text{pcf}(B) \subseteq \text{pcf}(A \backslash B)$.

(v) If A is finite, then $\text{pcf}(A) = A$.

(vi) If $B \subseteq A$, B is finite, and A is infinite, then $\text{pcf}(A) = \text{pcf}(A \backslash B) \cup B$.

(vii) $\min(A) = \min(\text{pcf}(A))$.

(viii) If A is infinite, then the first ω members of A are the same as the first ω members of $\text{pcf}(A)$.

Proof. (i): For each $a \in A$, the principal ultrafilter with $\{a\}$ as a member shows that $a \in \mathrm{pcf}(A)$.

(ii): Any ultrafilter F on A can be extended to an ultrafilter G on B. The mapping $[f] \mapsto [f]$ is easily seen to be an isomorphism of $\prod A/F$ onto $\prod B/G$. Note here that $[f]$ is used in two senses, one for an element of $\prod A/F$, where each member of $[f]$ is in $\prod A$, and the other for an element of $\prod B/G$, with members in the larger set $\prod B$.

(iii): $\supseteq$ holds by (ii). Now suppose that D is an ultrafilter on $A \cup B$. Then $A \in D$ or $B \in D$, and this proves $\subseteq$.

(iv): Suppose that $B \subseteq A$ and $\lambda \in \mathrm{pcf}(A) \backslash \mathrm{pcf}(B)$. Let D be an ultrafilter on A such that $\lambda = \mathrm{cf}(\prod A/D)$. Then $B \notin D$, as otherwise $\lambda \in \mathrm{pcf}(B)$. So $A \backslash B \in D$, and so $\lambda \in \mathrm{pcf}(A \backslash B)$.

(v): If A is finite, then every ultrafilter on A is principal.

(vi): We have

$$\begin{aligned}\mathrm{pcf}(A) &= \mathrm{pcf}(A\backslash B) \cup \mathrm{pcf}(B) \quad \text{by (iii)}\\ &= \mathrm{pcf}(A\backslash B) \cup B \quad \text{by (v)}\end{aligned}$$

(vii): Let $a = \min(A)$. Thus $a \in \mathrm{pcf}(A)$ by (i). Suppose that $\lambda \in \mathrm{pcf}(A)$ with $\lambda < a$; we want to get a contradiction. Say $\langle [g_\xi] : \xi < \lambda \rangle$ is strictly increasing and cofinal in $\prod A/D$. Now define $h \in \prod A$ as follows: for any $b \in A$, $h(b) = \sup\{g_\xi(b) + 1 : \xi < \lambda\}$. Thus $[g_\xi] < [h]$ for all $\xi < \lambda$, contradiction.

(viii): Suppose that $\lambda \in \mathrm{pcf}(A) \backslash A$. Suppose that $\lambda \cap A$ is finite, and let $a = \min(A \backslash \lambda)$. So $\lambda \leq a$, and if $b \in A \cap a$ then $b < \lambda$. Thus $A \cap \lambda = A \cap a$. Hence $\lambda \in \mathrm{pcf}(A) = \mathrm{pcf}(A \backslash a) \cup (A \cap \lambda)$ by (vi), and so $a \leq \lambda$ by (vii). So $\lambda = a$, contradiction. Thus $\lambda \cap A$ is infinite, and this proves (viii). $\square$

The following result gives a connection with earlier material; of course there will be more connections shortly.

Proposition 24b.2. *If A is a collection of regular cardinals, F is a proper filter on A, and $\lambda = \mathrm{tcf}(\prod A/F)$, then $\lambda \in \mathrm{pcf}(A)$.*

Proof. Let $\langle f_\xi : \xi < \lambda \rangle$ be a $<_F$-increasing cofinal sequence in $\prod A/F$. Let D be any ultrafilter containing F. Then clearly $\langle f_\xi : \xi < \lambda \rangle$ is a $<_D$-increasing cofinal sequence in $\prod A/D$. $\square$

Definitions. A set A is *progressive* iff A is an infinite set of regular cardinals and $|A| < \min(A)$.

If $\alpha < \beta$ are ordinals, then $(\alpha, \beta)_{\mathrm{reg}}$ is the set of all regular cardinals κ such that $\alpha < \kappa < \beta$. Similarly for $[\alpha, \beta)_{\mathrm{reg}}$, etc. All such sets are called *intervals of regular cardinals.*

Proposition 24b.3. *Assume that A is a progressive set, then*

(i) Every infinite subset of A is progressive.

(ii) If α is an ordinal and $A \cap \alpha$ is unbounded in α, then α is a singular cardinal.

(iii) If A is an infinite interval of regular cardinals, then A does not have any weak inaccessible as a member, except possibly its first element. Moreover, there is a singular cardinal λ such that $A \cap \lambda$ is unbounded in λ and $A \backslash \lambda$ is finite.

Proof. (i): Obvious.

(ii): Obviously α is a cardinal. Now $A\cap\alpha$ is cofinal in α and $|A\cap\alpha| \le |A| < \min(A) < \alpha$. Hence α is singular.

(iii): If $\kappa \in A$, then by (ii), $A \cap \kappa$ cannot be unbounded in κ; hence κ is a successor cardinal, or is the first element of A. For the second assertion of (iii), let $\sup(A) = \aleph_{\alpha+n}$ with α a limit ordinal. Since A is an infinite interval of regular cardinals, it follows that $A\cap\aleph_\alpha$ is unbounded in $\aleph_\alpha$, and hence by (ii), $\aleph_\alpha$ is singular. Hence the desired conclusion follows. $\square$

Theorem 24b.4. (Directed set theorem) *Suppose that A is a progressive set, and λ is a regular cardinal such that $\sup(A) < \lambda$. Suppose that I is a proper ideal over A containing all proper initial segments of A and such that $(\prod A, <_I)$ is λ-directed. Then there exist a set A' of regular cardinals and a proper ideal J over A' such that the following conditions hold:*

(i) $A' \subseteq [\min(A), \sup(A))$ and A' is cofinal in $\sup(A)$.

(ii) $|A'| \le |A|$.

(iii) J contains all bounded subsets of A'.

(iv) $\lambda = \operatorname{tcf}(\prod A', <_J)$.

Proof. First we note:

$(*)$ A does not have a largest element.

For, suppose that a is the largest element of A. Note that then $I = \mathscr{P}(A\backslash\{a\})$. For each $\xi < a$ define $f_\xi \in \prod A$ by setting

$$f_\xi(b) = \begin{cases} 0 & \text{if } b \ne a, \\ \xi & \text{if } b = a. \end{cases}$$

Since $a < \lambda$, choose $g \in \prod A$ such that $f_\xi <_I g$ for all $\xi \in a$. Thus $\{b \in A : f_\xi(b) \ge g(b)\} \in I$, so $f_\xi(a) < g(a)$ for all $\xi < a$. This is clearly impossible. So $(*)$ holds.

Now by Lemma 24a.43 there is a $<_I$-increasing sequence $f = \langle f_\xi : \xi < \lambda\rangle$ in $\prod A$ which satisfies $(*)_\kappa$ for every $\kappa \in A$. Hence by 24a.37–24a.39, f has an exact upper bound $h \in {}^A\mathbf{On}$ such that

$$\{a \in A : h(a) \text{ is non-limit or } \operatorname{cf}(h(a)) < \kappa\} \in I \tag{1}$$

for every $\kappa \in A$. Now the identity function k on A is clearly an upper bound for f, so $h \le_I k$; and by (1), $\{a \in A : h(a) \text{ is non-limit or } \operatorname{cf}(h(a)) < \min(A)\} \in I$. Hence by changing h on a set in the ideal we may assume that

$$\min(A) \le \operatorname{cf}(h(a)) \le a \quad \text{for all } a \in A. \tag{2}$$

Now f shows that $(\prod h, <_I)$ has true cofinality λ. Let $A' = \{\operatorname{cf}(h(a)) : a \in A\}$. By Lemma 24a.23 there is a proper ideal J on A' such that $(\prod A', <_J)$ has true cofinality λ; namely,

$$X \in J \quad \text{iff} \quad X \subseteq A' \text{ and } h^{-1}[\operatorname{cf}^{-1}[X]] \in I.$$

Clearly (ii) and (iv) hold. By (2) we have $A' \subseteq [\min(A), \sup(A))$. Now to show that A' is cofinal in $\sup(A)$, suppose that $\kappa \in A$; we find $\mu \in A'$ such that $\kappa \leq \mu$. In fact, $\{a \in A : \mathrm{cf}(h(a)) < \kappa\} \in I$ by (1). Let $X = \{b \in A' : b < \kappa\}$. Then

$$h^{-1}[\mathrm{cf}^{-1}[X]] = \{a \in A : \mathrm{cf}(h(a)) < \kappa\} \in I,$$

and so $X \in J$. Taking any $\mu \in A' \backslash X$ we get $\kappa \leq \mu$. Thus (i) holds. Finally, for (iii), suppose that $\mu \in J$; we want to show that $Y \stackrel{\text{def}}{=} \{b \in A' : b < \mu\} \in J$. By (i), choose $\kappa \in A$ such that $\mu \leq \kappa$. Then $Y \subseteq \{b \in A' : b < \kappa\}$, and by the argument just given, the latter set is in J. So (iii) holds. $\square$

Corollary 24b.5. *Suppose that A is progressive, is an interval of regular cardinals, and λ is a regular cardinal $> \sup(A)$. Assume that I is a proper ideal over A such that $(\prod A, <_I)$ is λ-directed. Then $\lambda \in \mathrm{pcf}(A)$.*

Proof. We may assume that I contains all proper initial segments of A. For, suppose that this is not true. Then there is a proper initial segment B of A such that $B \notin I$. With $a \in A \backslash B$ we then have $B \subseteq A \cap a$, and so $A \cap a \notin I$. Let a be the smallest element of A such that $A \cap a \notin I$. Then $J \stackrel{\text{def}}{=} I \cap \mathscr{P}(A \cap a)$ is a proper ideal that contains all proper initial segments of $A \cap a$. we claim that $(\prod(A \cap a), J)$ is λ-directed. For, suppose that $X \subseteq \prod(A \cap a)$ with $|X| < \lambda$. For each $g \in X$ let $g^+ \in \prod A$ be such that $g^+ \supseteq g$ and $g^+(b) = 0$ for all $b \in A \backslash a$. Choose $f \in \prod A$ such that $g^+ \leq_I f$ for all $g \in X$. So if $g \in X$ we have

$$\{b \in A \cap a : g(b) > f(b)\} = \{b \in A : g^+(b) > f(b)\} \in I \cap \mathscr{P}(A \cap a),$$

and so $g \leq_J (f \restriction (A \cap a)$ for all $g \in X$, as desired.

Now the corollary follows from the theorem. $\square$

The ideal $J_{<\lambda}$

Let A be a set of regular cardinals. We define

$$J_{<\lambda}[A] = \{X \subseteq A : \mathrm{pcf}(X) \subseteq \lambda\}.$$

In words, $X \in J_{<\lambda}[A]$ iff X is a subset of A such that for any ultrafilter D over A, if $X \in D$, then $\mathrm{cf}(\prod A, <_D) < \lambda$. Thus X "forces" the cofinalities of ultraproducts to be below λ.

Clearly $J_{<\lambda}[A]$ is an ideal of A. If $\lambda < \min(A)$, then $J_{<\lambda}[A] = \{\emptyset\}$ by 24b.1(vii). If $\lambda < \mu$, then $J_{<\lambda}[A] \subseteq J_{<\mu}[A]$. If $\lambda \notin \mathrm{pcf}(A)$, then $J_{<\lambda}[A] = J_{<\lambda^+}[A]$. If λ is greater than each member of $\mathrm{pcf}(A)$, then $J_{<\lambda}[A]$ is the improper ideal $\mathscr{P}(A)$. If $\lambda \in \mathrm{pcf}(A)$, then $A \notin J_{<\lambda}[A]$.

If A is clear from the context, we simply write $J_{<\lambda}$.

If I and J are ideals on a set A, then $I + J$ is the smallest ideal on A which contains $I \cup J$; it consists of all X such that $X \subseteq Y \cup Z$ for some $Y \in I$ and $Z \in J$.

Lemma 24b.6. *If A is an infinite set of regular cardinals and B is a finite subset of A, then for any cardinal λ we have*

$$J_{<\lambda}[A] = J_{<\lambda}[A\backslash B] + \mathscr{P}(B \cap \lambda).$$

Proof. Let $X \in J_{<\lambda}[A]$. Thus $\mathrm{pcf}(X) \subseteq \lambda$. Using 24b.1(vi) we have $\mathrm{pcf}(X) = \mathrm{pcf}(X\backslash B) \cup (X \cap B)$, so $X\backslash B \in J_{<\lambda}[A\backslash B]$ and $X \cap B \subseteq B \cap \lambda$, and it follows that $X \in J_{<\lambda}[A\backslash B] + \mathscr{P}(B \cap \lambda)$.

Now suppose that $X \in J_{<\lambda}[A\backslash B] + \mathscr{P}(B \cap \lambda)$. Then there is a $Y \in J_{<\lambda}[A\backslash B]$ such that $X \subseteq Y \cup (B \cap \lambda)$. Hence by 24b.1(vi) again, $\mathrm{pcf}(X) \subseteq \mathrm{pcf}(Y) \cup (B \cap \lambda) \subseteq \lambda$, so $X \in J_{<\lambda}[A]$. □

Recall that for any ideal on a set Y, $I^* = \{a \subseteq Y : Y\backslash a \in I\}$ is the filter corresponding to I.

Proposition 24b.7. *If A is a collection of regular cardinals and λ is a cardinal, then*

$$J^*_{<\lambda}[A] = \bigcap \left\{ D : D \textit{ is an ultrafilter and } \mathrm{cf}\left(\prod A/D\right) \geq \lambda \right\}.$$

The intersection is to be understood as being equal to $\mathscr{P}(A)$ if there is no ultrafilter D such that $\mathrm{cf}(\prod A/D) \geq \lambda$.

Proof. Note that for any $X \subseteq A$, $X \in J^*_{<\lambda}[A]$ iff $A\backslash X \in J_{<\lambda}[A]$ iff $\mathrm{pcf}(A\backslash X) \subseteq \lambda$. Now suppose that $X \in J^*_{<\lambda}[A]$ and D is an ultrafilter such that $\mathrm{cf}(\prod A/D) \geq \lambda$. If $X \notin D$, then $A\backslash X \in D$ and hence $\mathrm{pcf}(A\backslash X) \not\subseteq \lambda$, contradiction. Thus X is in the indicated intersection.

If X is in the indicated intersection, we want to show that $A\backslash X \subseteq \lambda$. To this end, suppose that D is an ultrafilter such that $A\backslash X \in D$, and to get a contradiction suppose that $\mathrm{cf}(\prod A/D) \geq \lambda$. Then $X \in D$ by assumption, contradiction.

Note that the argument gives the desired result in case there are no ultrafilters D as indicated in the intersection; in this case, $\mathrm{pcf}(A\backslash X) \subseteq \lambda$ for every $X \subseteq A$, and so $J^*_{<\lambda}[A] = \mathscr{P}(A)$. □

Theorem 24b.8. (λ-directedness) *Assume that A is progressive. Then for every cardinal λ, the partial order $(\prod A, <_{J_{<\lambda}[A]})$ is λ-directed.*

Proof. We may assume that there are infinitely many members of A less than λ. For, suppose not. Let $F \subseteq \prod A$ with $|F| < \lambda$. We define $g \in \prod A$ by setting, for any $a \in A$,

$$g(a) = \begin{cases} \sup\{f(a) : f \in F\} & \text{if } |F| < a, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $f \leq g \bmod J_{<\lambda}[A]$ for all $f \in F$. For, if $f(a) > g(a)$, then $\lambda > |F| \geq a$; thus $\{a : f(a) > g(a)\} \subseteq \lambda \cap A$. Now $\mathrm{pcf}(\lambda \cap A) = \lambda \cap A \subseteq \lambda$, so $\{a : f(a) > g(a)\} \in J_{<\lambda}[A]$.

So, we make the indicated assumption. By this assumption, the set $B \stackrel{\text{def}}{=} A \cap \{|A|^+, |A|^{++}, |A|^{+++}, |A|^{++++}\} \subseteq \lambda$. Suppose that we have shown that $(\prod(A\backslash B), J_{<\lambda}(A\backslash B))$

is λ-directed. Now let $Y \subseteq \prod A$ with $|Y| < \lambda$. Choose $g \in \prod(A\backslash B)$ such that $f \restriction (A\backslash B) <_{J_{<\lambda}[A\backslash B]} g$ for all $f \in Y$. Let $g^+ \in \prod A$ be an extension of g. Then

$$\begin{aligned}\{a : f(a) > g^+(a)\} &= \{a \in A\backslash B : f(a) > g(a)\} \cup \{a \in B : f(a) > g^+(a)\} \\ &\in J_{<\lambda}[A\backslash B] + \mathscr{P}(B \cap \lambda) \\ &= J_{<\lambda}[A] \quad \text{by Lemma 24b.6.}\end{aligned}$$

Thus g^+ is an upper bound for Y mod $J_{<\lambda}[A]$.

Hence we may assume that $|A|^{+3} < \min(A)$.

Now we prove by induction on the cardinal λ_0 that if $\lambda_0 < \lambda$ and $F = \{f_i : i < \lambda_0\} \subseteq \prod A$ is a family of functions of size λ_0, then F has an upper bound in $(\prod A, <_{J_{<\lambda}})$. So, we assume that this is true for all cardinals less than λ_0. If $\lambda_0 < \min(A)$, then $\sup(F)$ is as desired. So, assume that $\min(A) \leq \lambda_0$.

First suppose that λ_0 is singular. Let $\langle \alpha_i : i < \mathrm{cf}(\lambda_0)\rangle$ be increasing and cofinal in λ_0, each α_i a cardinal. By the inductive hypothesis, let g_i be a bound for $\{f_\xi : \xi < \alpha_i\}$ for each $i < \mathrm{cf}\lambda_0$, and then let h be a bound for $\{g_i : i < \mathrm{cf}\lambda_0\}$. Clearly h is a bound for F.

So assume that λ_0 is regular. We are now going to define a $<_{J_{<\lambda}}$-increasing sequence $\langle f'_\xi : \xi < \lambda_0\rangle$ which satisfies $(*)_\kappa$, with $\kappa = |A|^+$, and such that $f_i \leq f'_i$ for all $i < \lambda_0$. To do this choose, for every $\delta \in S^{\lambda_0}_{\kappa^{++}}$ a club $E_\delta \subseteq \delta$ of order type κ^{++}. Now for such a δ we define

$$f'_\delta = \sup(\{f'_j : j \in E_\delta\} \cup \{f_\delta\}).$$

For ordinals $\delta < \lambda_0$ of cofinality $\neq \kappa^{++}$ we apply the inductive hypothesis to get f'_δ such that $f'_\xi <_{J_{<\lambda}} f'_\delta$ for every $\xi < \delta$ and also $f_\delta <_{J_{<\lambda}} f'_\delta$.

This finishes the construction. By Lemma 24a.41, $(*)_{|A|^+}$ holds for f, and hence by Theorem 24a.39, f has an exact upper bound $g \in {}^A\mathbf{On}$ with respect to $<_{J_{<\lambda}}$. The identity function on A is an upper bound for f, so we may assume that $g(a) \leq a$ for all $a \in A$. Now we shall prove that $B \stackrel{\text{def}}{=} \{a \in A : g(a) = a\} \in J_{<\lambda}[A]$, so a further modification of g yields the desired upper bound for f.

To get a contradiction, suppose that $B \notin J_{<\lambda}[A]$. Hence $\mathrm{pcf}(B) \not\subseteq \lambda$, and so there is an ultrafilter D over A such that $B \in D$ and $\mathrm{cf}(\prod A/D) \geq \lambda$. Clearly $D \cap J_{<\lambda}[A] = \emptyset$, as otherwise $\mathrm{cf}(\prod A/D) < \lambda$. Now f has length $\lambda_0 < \lambda$, and so it is bounded in $\prod A/D$; say that $f_i <_D h \in \prod A$ for all $i < \lambda_0$. Thus $h(a) < a = g(a)$ for all $a \in B$. Now we define $h' \in \prod A$ by

$$h'(a) = \begin{cases} h(a) & \text{if } a \in B, \\ 0 & \text{otherwise.}\end{cases}$$

Then $h' <_{J_{<\lambda}} g$, since

$$\{a \in A : h'(a) \geq g(a)\} = \{a \in A : g(a) = 0\} \subseteq \{a \in A : f_0(a) \geq g(a)\} \in J_{<\lambda}.$$

Hence by the exactness of g it follows that $h' <_{J_{<\lambda}} f_i$ for some $i < \lambda_0$. But $B \in D$ and hence $h =_D h'$. So $h <_D f_i$, contradiction. $\square$

Corollary 24b.9. *Suppose that A is progressive, D is an ultrafilter over A, and λ is a cardinal. Then:*

(i) $\mathrm{cf}(\prod A/D) < \lambda$ *iff* $J_{<\lambda}[A] \cap D \neq \emptyset$.
(ii) $\mathrm{cf}(\prod A/D) = \lambda$ *iff* $J_{<\lambda^+} \cap D \neq \emptyset = J_{<\lambda} \cap D$.
(iii) $\mathrm{cf}(\prod A/D) = \lambda$ *iff* λ^+ *is the first cardinal* μ *such that* $J_{<\mu} \cap D \neq \emptyset$.

Proof. (i): $\Rightarrow$: Assuming that $J_{<\lambda}[A] \cap D = \emptyset$, the fact from Theorem 24b.8 that $<_{J_{<\lambda}}$ is λ-directed implies that also $\prod A/D$ is λ-directed, and hence $\mathrm{cf}(\prod A/D) \geq \lambda$.

$\Leftarrow$: Assume that $J_{<\lambda}[A] \cap D \neq \emptyset$. Choose $X \in J_{<\lambda} \cap D$. Then by definition, $\mathrm{pcf}(A) \subseteq \lambda$, and hence $\mathrm{cf}(\prod A/D) < \lambda$.

(ii): Immediate from (i).

(iii): Immediate from (ii). $\square$

We now give two important theorems about pcf.

Theorem 24b.10. *If* A *is progressive, then* $|\mathrm{pcf}(A)| \leq 2^{|A|}$.

Proof. By Corollary 24b.9, for each $\lambda \in \mathrm{pcf}(A)$ we can select an element $f(\lambda) \in J_{<\lambda^+} \backslash J_{<\lambda}$. Clearly f is a one-one function from $\mathrm{pcf}(A)$ into $\mathscr{P}(A)$. $\square$

Notation. We write $J_{\leq\lambda}$ in place of $J_{<\lambda^+}$.

Theorem 24b.11. (The max pcf theorem) *If* A *is progressive, then* $\mathrm{pcf}(A)$ *has a largest element.*

Proof. Let
$$I = \bigcup_{\lambda \in \mathrm{pcf}(A)} J_{<\lambda}[A].$$
Now clearly each ideal $J_{<\lambda}$ is proper (since for example $\{\lambda\} \notin J_{<\lambda}$), so I is also proper. Extend the dual of I to an ultrafilter D, and let $\mu = \mathrm{cf}(\prod A/D)$. Then for each $\lambda \in \mathrm{pcf}(A)$ we have $J_{<\lambda} \cap D = \emptyset$ since $I \cap D = \emptyset$, and by Corollary 24b.9 this means that $\mu \geq \lambda$. $\square$

Corollary 24b.12. *Suppose that* A *is progressive. If* λ *is a limit cardinal, then*
$$J_{<\lambda}[A] = \bigcup_{\theta<\lambda} J_{\leq\theta}[A].$$

Proof. The inclusion $\supseteq$ is clear. Now suppose that $X \in J_{<\lambda}[A]$. Thus $\mathrm{pcf}(X) \subseteq \lambda$. Let μ be the largest element of $\mathrm{pcf}(X)$. Then $\mu \in \lambda$, and $\mathrm{pcf}(X) \subseteq \mu^+$, so $X \in J_{<\mu^+}$, and the latter is a subset of the right side. $\square$

Theorem 24b.13. (The interval theorem) *If* A *is a progressive interval of regular cardinals, then* $\mathrm{pcf}(A)$ *is an interval of regular cardinals.*

Proof. Let $\mu = \sup(A)$. By 24b.3(iii) and 24b.1(vi) we may assume that μ is singular. By Theorem 24b.11 let $\lambda_0 = \max(\mathrm{pcf}(A))$. Thus we want to show that every regular cardinal λ in (μ, λ_0) is in $\mathrm{pcf}(A)$. By Theorem 24b.8, the partial order $(\prod A, <_{J_{<\lambda}})$ is λ-directed. Clearly $J_{<\lambda}$ is a proper ideal, so $\lambda \in \mathrm{pcf}(A)$ by Corollary 24b.5. $\square$

Definition. If κ is a cardinal $\leq |A|$, then we define

$$\operatorname{pcf}_\kappa(A) = \bigcup\{\operatorname{pcf}(X) : X \subseteq A \text{ and } |X| = \kappa\}.$$

Theorem 24b.14. *If A is an interval of regular cardinals and $\kappa < \min(A)$, then $\operatorname{pcf}_\kappa(A)$ is an interval of regular cardinals.*

Note here that we do not assume that A is progressive.

Proof. Let $\lambda_0 = \sup \operatorname{pcf}_\kappa(A)$. Note that each subset X of A of cardinality κ is progressive, and so $\max(\operatorname{pcf}(X))$ exists by Theorem 24b.11. Thus

$$\lambda_0 = \sup\{\max(\operatorname{pcf}(X)) : X \subseteq A \text{ and } |X| = \kappa\}.$$

To prove the theorem it suffices to take any regular cardinal λ such that $\min(A) < \lambda < \lambda_0$ and show that $\lambda \in \operatorname{pcf}_\kappa(A)$. In fact, this will show that $\operatorname{pcf}_\kappa(A)$ is an interval of regular cardinals, whether or not λ_0 is regular. Since $\lambda < \lambda_0$, there is an $X \subseteq A$ of size κ such that $\lambda \leq \max(\operatorname{pcf}(X))$. Hence $X \notin J_{<\lambda}[X]$. If there is a proper initial segment Y of X which is not in $J_{<\lambda}[X]$, we can choose the smallest $a \in X$ such that $X \cap a \notin J_{<\lambda}[X]$ and work with $X \cap a$ rather than X. So we may assume that every proper initial segment of X is in $J_{<\lambda}[X]$. If $\lambda \in A$, clearly $\lambda \in \operatorname{pcf}_\kappa(A)$. So we may assume that $\lambda \notin A$. If $\lambda < \sup(X)$, then $\lambda \in A$, contradiction. If $\lambda = \sup(X)$, then $\lambda = \sup(A)$ since $\lambda \notin A$, and this contradicts Proposition 24b.3(ii). So $\sup(X) < \lambda$. Since $J_{<\lambda}[X]$ is λ-directed by Theorem 24b.8, we can apply 24b.4 to obtain $\lambda \in \operatorname{pcf}(X)$, and hence $\lambda \in \operatorname{pcf}_\kappa(A)$, as desired. $\square$

Another of the central results of pcf theory is as follows.

Theorem 24b.15. (Closure theorem.) *Suppose that A is progressive, $B \subseteq \operatorname{pcf}(A)$, and B is progressive. Then $\operatorname{pcf}(B) \subseteq \operatorname{pcf}(A)$. In particular, if $\operatorname{pcf}(A)$ itself is progressive, then $\operatorname{pcf}(\operatorname{pcf}(A)) = \operatorname{pcf}(A)$.*

Proof. Suppose that $\mu \in \operatorname{pcf}(B)$, and let E be an ultrafilter on B such that $\mu = \operatorname{cf}(\prod B/E)$. For every $b \in B$ fix an ultrafilter D_b on A such that $b = \operatorname{cf}(\prod A/D_b)$. Define F by

$$X \in F \quad \text{iff} \quad X \subseteq A \text{ and } \{b \in B : X \in D_b\} \in E.$$

It is straightforward to check that F is an ultrafilter on A. The rest of the proof consists in showing that $\mu = \operatorname{cf}(\prod A/F)$.

By Proposition 24b.22 we have

$$\mu = \operatorname{cf}\left(\prod_{b \in B}\left(\prod A/D_b\right)/E\right).$$

Hence it suffices by Proposition 24b.10 to show that $\prod A/F$ is isomorphic to a cofinal subset of this iterated ultraproduct. To do this, we consider the Cartesian product $B \times A$ and define

$$H \in P \quad \text{iff} \quad H \subseteq B \times A \text{ and } \{b \in B : \{a \in A : (b,a) \in H\} \in D_b\} \in E.$$

Again it is straightforward to check that P is an ultrafilter over $B \times A$. Let $r(b,a) = a$ for any $(b,a) \in B \times A$. Then

$$(*) \qquad \left(\prod_{(b,a)\in B\times A} a\right)/P \cong \prod_{b\in B}\left(\prod A/D_b\right)/E.$$

To prove $(*)$, for any $f \in \prod_{\langle b,a\rangle\in B\times A} a$ we define $f' \in \prod_{b\in B}(\prod A/D_b)$ by setting

$$f'(b) = \langle f(b,a) : a \in A\rangle/D_b.$$

Then for any $f, g \in \prod_{\langle b,a\rangle\in B\times A} a$ we have

$$\begin{aligned} f =_P g \quad &\text{iff} \quad \{(b,a) : f(b,a) = g(b,a)\} \in P \\ &\text{iff} \quad \{b : \{a : f(b,a) = g(b,a)\} \in D_b\} \in E \\ &\text{iff} \quad \{b : f'(b) = g'(b)\} \in E \\ &\text{iff} \quad f' =_E g'. \end{aligned}$$

Hence we can define $k(f/P) = f'/E$, and we get a one-one function. To show that it is a surjection, suppose that $h \in \prod_{b\in B}(\prod A/D_b)$. For each $b \in B$ write $h(b) = h'_b/D_b$ with $h'_b \in \prod A$. Then define $f(b,a) = h'_b(a)$. Then

$$f'(b) = \langle f(b,a) : a \in A\rangle/D_b = \langle h'_b(a) : a \in A\rangle/D_b = h'_b/D_b = h(b),$$

as desired. Finally, k preserves order, since

$$\begin{aligned} f/P < g/P \quad &\text{iff} \quad \{(b,a) : f(b,a) < g(b,a)\} \in P \\ &\text{iff} \quad \{b : \{a : f(b,a) < g(b,a)\} \in D_b\} \in E \\ &\text{iff} \quad \{b : f'(b) < g'(b)\} \in E \\ &\text{iff} \quad k(f/P) < k(g/P). \end{aligned}$$

So $(*)$ holds.

Now we apply Lemma 24b.23, with $r, B \times A, A, P$ in place of c, A, B, I respectively. Then F is the Rudin-Keisler projection on A, since for any $X \subseteq A$,

$$\begin{aligned} X \in F \quad &\text{iff} \quad \{b \in B : X \in D_b\} \in E \\ &\text{iff} \quad \{b \in B : \{a \in A : r(b,a) \in X\} \in D_b\} \in E \\ &\text{iff} \quad \{b \in B : \{a \in A : (b,a) \in r^{-1}[X]\} \in D_b\} \in E \\ &\text{iff} \quad r^{-1}[X] \in P. \end{aligned}$$

Thus by Lemma 24b.23 we get an isomorphism h of $\prod A/F$ into $\prod_{(b,a)\in B\times A} a/P$ such that $h(e/F) = \langle e(r(b,a)) : (b,a) \in B \times A\rangle/P$ for any $e \in \prod A$. So now it suffices now to show that the range of h is cofinal in $\prod_{(b,a)\in B\times A} a/P$. Let $g \in \prod_{(b,a)\in B\times A} a$. For every

$b \in B$ define $g_b \in \prod A$ by $g_b(a) = g(b, a)$. Let $\lambda = \min(B)$. Since B is progressive, we have $|B| < \lambda$. Hence by the λ-directness of $\prod A/J_{<\lambda}[A]$ (Theorem 24b.8), there is a function $k \in \prod A$ such that $g_b <_{J_{<\lambda}} k$ for each $b \in B$. Now $\lambda \leq b$ for all $b \in B$, so $J_{<\lambda} \cap D_b = \emptyset$, and so $g_b <_{D_b} k$. It follows that $g/P <_P h(k/D)$. In fact, let $H = \{(b, a) : g(b, a) < k(r(b, a))\}$. Then

$$\{b \in B : \{a \in A : (b, a) \in H\} \in D_b\} = \{b \in B : \{a \in A : g_b(a) < k(a)\} \in D_b\} = B \in E,$$

as desired. □

Generators for $J_{<\lambda}$

If I is an ideal on a set A and $B \subseteq A$, then $I + B$ is the ideal generated by $I \cup \{B\}$; that is, it is the intersection of all ideals J on A such that $I \cup \{B\} \subseteq J$.

Proposition 24b.16. *Suppose that I is an ideal on A and $B, X \subseteq A$. Then the following conditions are equivalent:*

(i) $X \in I + B$.

(ii) There is a $Y \in I$ such that $X \subseteq Y \cup B$.

(iii) $X \backslash B \in I$.

Proof. Clearly (ii)⇒(i). The set

$$\{Z \subseteq A : \exists Y \in I[Z \subseteq Y \cup B]\}$$

is clearly an ideal containing $I \cup \{B\}$, so (i)⇒(ii). If Y is as in (ii), then $X \backslash B \subseteq Y$, and hence $X \backslash B \in I$; so (ii)⇒(iii). If $X \backslash B \in I$, then $X \subseteq (X \backslash B) \cup B$, so X satisfies the condition of (ii). So (iii)⇒(ii). □

The following easy lemma will be useful later.

Lemma 24b.17. *Suppose that A is progressive and $B \subseteq A$.*

(i) $\mathscr{P}(B) \cap J_{<\lambda}[A] = J_{<\lambda}[B]$.

(ii) If $f, g \in \prod A$ and $f <_{J_{<\lambda}[A]} g$, then $(f \restriction B) <_{J_{<\lambda}[B]} (g \restriction B)$.

Proof. (i): Suppose that $X \in \mathscr{P}(B) \cap J_{<\lambda}[A]$ and $X \in D$, an ultrafilter on B. Extend D to an ultrafilter E on A. Then $\prod B/D \cong \prod A/E$, and $\mathrm{cf}(\prod A/E) < \lambda$. So $X \in J_{<\lambda}[B]$. The converse is proved similarly.

(ii): Assume that $f, g \in \prod A$ and $f <_{J_{<\lambda}[A]} g$. Then

$$\{a \in B : g(b) \leq f(b)\} \in \mathscr{P}(B) \cap J_{<\lambda}[A] = J_{<\lambda}[B]$$

by (i), as desired. □

Definitions. If there is a set X such that $J_{\leq\lambda}[A] = J_{<\lambda} + X$, then we say that λ is *normal.*

Let A be a set of regular cardinals, and λ a cardinal. A subset $B \subseteq A$ is a λ-*generator over A* iff $J_{\leq\lambda}[A] = J_{<\lambda}[A] + B$. We omit the qualifier "over A" if A is understood from the context.

Suppose that $\lambda \in \mathrm{pcf}(A)$. A *universal sequence for* λ is a sequence $f = \langle f_\xi : \xi < \lambda \rangle$ which is $<_{J_{<\lambda}[A]}$-increasing, and for every ultrafilter D over A such that $\mathrm{cf}(\prod A/D) = \lambda$, the sequence f is cofinal in $\prod A/D$.

Theorem 24b.18. (Universal sequences) *Suppose that A is progressive. Then every $\lambda \in \mathrm{pcf}(A)$ has a universal sequence.*

Proof. First,

(1) We may assume that $|A|^+ < \min(A) < \lambda$.

In fact, suppose that we have proved the theorem under the assumption (1), and now take the general situation. Recall from Proposition 3.19(vii) that $\min(A) \le \lambda$. If $\lambda = \min(A)$, define $f_\xi \in \prod A$, for $\xi < \lambda$, by $f_\xi(a) = \xi$ for all $a \in A$. Thus f is $<$-increasing, hence $<_{J_{<\lambda}[A]}$-increasing. Suppose that D is an ultrafilter on A such that $\mathrm{cf}(\prod A/D) = \lambda$. Then $\{\min(A)\} \in D$, as otherwise $A\backslash\{\min(A)\} \in D$ and hence $\mathrm{cf}(\prod A/D) > \lambda$ by Proposition 24b.1(vii). Thus for any $g \in \prod A$, let $\xi = g(\min(A)) + 1$. Then $\{a \in A : g(a) < f_\xi(a)\} \supseteq \{\min(A)\} \in D$, so $[g] < [f_\xi]$. Hence $\langle [f_\xi] : \xi < \lambda\rangle$ is cofinal in $\prod A/D$.

Now suppose that $\min(A) < \lambda$. Let $a_0 = \min A$. Let $A' = A\backslash\{a_0\}$. If D is an ultrafilter such that $\lambda = \mathrm{cf}(\prod A/D)$, then $A' \in D$ since $a_0 < \lambda$, hence $\{a_0\} \notin D$. It follows that $\lambda \in \mathrm{pcf}(A')$. Clearly $|A'|^+ < \min A' \le \lambda$. Hence by assumption we get a system $\langle f_\xi : \xi < \lambda\rangle$ of members of $\prod A'$ which is increasing in $<_{J_{<\lambda}[A']}$ such that for every ultrafilter D over A' such that $\lambda = \mathrm{cf}(\prod A'/D)$, f is cofinal in $\prod A'/D$. Extend each f_ξ to $g_\xi \in \prod A$ by setting $g_\xi(a_0) = 0$. If $\xi < \eta < \lambda$, then

$$\{a \in A : g_\xi(a) \ge g_\eta(a)\} \subseteq \{a \in A' : f_\xi(a) \ge f_\eta(a)\} \cup \{a_0\},$$

and $\{a \in A' : f_\xi(a) \ge f_\eta(a)\} \in J_{<\lambda}[A'] \subseteq J_{<\lambda}[A]$ and also $\{a_0\} \in J_{<\lambda}[A]$ since $a_0 < \lambda$, so $g_\xi <_{J_{<\lambda}} g_\eta$. Now let D be an ultrafilter over A such that $\lambda = \mathrm{cf}(\prod A/D)$. As above, $A' \in D$; let $D' = D \cap \mathscr{P}(A')$. Then $\lambda = \mathrm{cf}(\prod A'/D')$. To show that g is cofinal in $\prod A/D$, take any $h \in \prod A$. Choose $\xi < \lambda$ such that $(h \restriction A')/D' < f_\xi/D'$. Then

$$\{a \in A : h(a) \ge g_\xi(a)\} \supseteq \{a \in A' : h(a) \ge f_\xi(a)\},$$

so $h/D < g_\xi/D$, as desired.

Thus we can make the assumption as in (1). Suppose that there is no universal sequence for λ. Thus

(2) For every $<_{J_{<\lambda}}$-increasing sequence $f = \langle f_\xi : \xi < \lambda\rangle$ there is an ultrafilter D over A such that $\mathrm{cf}(\prod A/D) = \lambda$ but f is not cofinal in $\prod A/D$.

We are now going to construct a $<_{J_{<\lambda}}$-increasing sequence $f^\alpha = \langle f^\alpha_\xi : \xi < \lambda\rangle$ for each $\alpha < |A|^+$. We use the fact that $\prod A/J_{<\lambda}$ is λ-directed (Theorem 24b.8).

Using this directedness, we start with any $<_{J_{<\lambda}}$-increasing sequence $f^0 = \langle f^0_\xi : \xi < \lambda\rangle$.

For δ limit $< |A|^+$ we are going to define f^δ_ξ by induction on ξ so that the following conditions hold:

(3) $f^\delta_i <_{J_{<\lambda}} f^\delta_\xi$ for $i < \xi$,

(4) $\sup\{f^\alpha_\xi : \alpha < \delta\} \le f^\delta_\xi$.

Suppose that f^δ_i has been defined for all $i < \xi$. By λ-directedness, choose g such that $f^\delta_i <_{J_{<\lambda}} g$ for all $i < \xi$. Now for any $a \in A$ we have $\sup\{f^\alpha_\xi(a) : \alpha < \delta\} < a$, since $\delta < |A|^+ < \min A \le a$. Hence we can define

$$f^\delta_\xi(a) = \max\{g(a), \sup\{f^\alpha_\xi(a) : \alpha < \delta\}\}.$$

Clearly the conditions (3), (4) hold.

Now suppose that f^α has been defined and is $<_{J_{<\lambda}}$-increasing; we define $f^{\alpha+1}$. By (2), choose an ultrafilter D_α over A such that

(5) $\mathrm{cf}(\prod A/D_\alpha) = \lambda$;

(6) The sequence f^α is bounded in $<_{D_\alpha}$.

By (6), choose $f^{\alpha+1}_0$ which bounds f^α in $<_{D_\alpha}$; in addition, $f^{\alpha+1}_0 \ge f^\alpha_0$. Let $\langle h_\xi/D_\alpha : \xi < \lambda\rangle$ be strictly increasing and cofinal in $\prod A/D_\alpha$. Now we define $f^{\alpha+1}_\xi$ by induction on ξ when $\xi > 0$. First, by λ-directness, choose k such that $f^{\alpha+1}_i <_{J_{<\lambda}} k$ for all $i < \xi$. Then for any $a \in A$ let

$$f^{\alpha+1}_\xi(a) = \max(k(a), h_\xi(a), f^\alpha_\xi(a)).$$

Then the following conditions hold:

(7) $f^{\alpha+1}$ is strictly increasing and cofinal in $\prod A/D_\alpha$;

(8) $f^{\alpha+1}_i \ge f^\alpha_i$ for every $i < \lambda$.

This finishes the construction. Clearly we then have

(9) If $i < \lambda$ and $\alpha_1 < \alpha_2 < |A|^+$, then $f^{\alpha_1}_i \le f^{\alpha_2}_i$.

(10) f^α is bounded in $\prod A/D_\alpha$ by $f^{\alpha+1}_0$.

(11) $f^{\alpha+1}$ is cofinal in $\prod A/D_\alpha$.

Now let $h = \sup_{\alpha<|A|^+} f^\alpha_0$. Then $h \in \prod A$, since $|A|^+ < \min(A)$. By (11), for each $\alpha < |A|^+$ choose $i_\alpha < \lambda$ such that $h <_{D_\alpha} f^{\alpha+1}_{i_\alpha}$. Since $\lambda > |A|^+$ is regular, we can choose $i < \lambda$ such that $i_\alpha < i$ for all $\alpha < |A|^+$. Now define

$$A^\alpha = \{a \in A : h(a) \le f^\alpha(a)\}.$$

By (9) we have $A^\alpha \subseteq A^\beta$ for $\alpha < \beta < |A|^+$. We are going to get a contradiction by showing that $A^\alpha \subset A^{\alpha+1}$ for every $\alpha < |A|^+$.

In fact, this follows from the following two statements.

(12) $A^\alpha \notin D_\alpha$.

This holds because $f^\alpha_i <_{D_\alpha} f^{\alpha+1}_i \le h$.

(13) $A^{\alpha+1} \in D_\alpha$.

This holds because $h <_{D_\alpha} f^{\alpha+1}_i$ by the choice of i and (7). □

Proposition 24b.19. *If A is a set of regular cardinals, λ is the largest member of* $\mathrm{pcf}(A)$*, and $\langle f_\xi : \xi < \lambda\rangle$ is universal for λ, then it is cofinal in $(\prod A, J_{<\lambda})$.*

Proof. Assume the hypotheses. Fix $g \in \prod A$; we want to find $\xi < \lambda$ such that $g <_{J_{<\lambda}} f_\xi$. Suppose that no such ξ exists. Then, we claim, the set

$$J^*_{<\lambda} \cup \{\{a \in A : g(a) \geq f_\xi(a)\} : \xi < \lambda\} \tag{1}$$

has fip. For, suppose that it does not have fip. Then there is a finite nonempty subset F of λ such that

$$\bigcup_{\xi \in F} \{a \in A : g(a) < f_\xi(a)\} : \xi < \lambda\} \in J^*_{<\lambda}. \tag{2}$$

Let η be the largest member of F. Note that the set

$$\{a \in A : f_\xi(a) < f_\rho(a) \text{ for all } \xi, \rho \in F \text{ such that } \xi < \rho\}$$

is also a member of $J^*_{<\lambda}$; intersecting this set with the set of (2), we get a member of $J^*_{<\lambda}$ which is a subset of $\{a \in A : g(a) < f_\eta(a)\}$, so that $g <_{J_{<\lambda}} f_\eta$, contradiction.

Thus the set (1) has fip. Let D be an ultrafilter containing it. Then $\mathrm{cf}(\prod A/D) = \lambda$, so by hypothesis there is a $\xi < \lambda$ such that $g <_D f_\xi$. Thus $\{a \in A : g(a) < f_\xi(a)\} \in D$. But also $\{a \in A : g(a) \geq f_\xi(a)\} \in D$, contradiction. $\square$

Theorem 24b.20. *If A is progressive, then* $\mathrm{cf}(\prod A, <) = \max(\mathrm{pcf}(A))$*. In particular,* $\mathrm{cf}(\prod A, <)$ *is regular.*

Proof. First we prove $\geq$. Let $\lambda = \max(\mathrm{pcf}(A))$, and let D be an ultrafilter on A such that $\lambda = \mathrm{cf}(\prod A/D)$. Now for any $f, g \in \prod A$, if $f < g$ then $f <_D g$. Hence any cofinal set in $(\prod A, <)$ is also cofinal in $(\prod A, <_D)$, and so $\lambda = \mathrm{cf}(\prod A, <_D) \leq \mathrm{cf}(\prod A, <)$.

To prove $\leq$, we exhibit a cofinal subset of $(\prod A, <)$ of size λ. For every $\mu \in \mathrm{pcf}(A)$ fix a universal sequence $f^\mu = \langle f^\mu_i : i < \mu\rangle$ for μ, by Theorem 24b.18. Let F be the set of all functions of the form

$$\sup\{f^{\mu_1}_{i_1}, f^{\mu_2}_{i_2}, \ldots, f^{\mu_n}_{i_n}\},$$

where $\mu_1, \mu_2, \ldots, \mu_n$ is a finite sequence of members of $\mathrm{pcf}(A)$, possibly with repetitions, and $i_k < \mu_k$ for each $k = 1, \ldots, n$. We claim that F is cofinal in $(\prod A, <)$; this will complete the proof.

To prove this claim, let $g \in \prod A$. Let

$$I = \{>(f, g) : f \in F\}.$$

(Recall that $>(f, g) = \{a \in A : f(a) > g(a)\}$.) Now I is closed under unions, since

$$>(f_1, g) \cup >(f_2, g) = >(\sup(f_1, f_2), g).$$

If $A \in I$, then $A = >(f, g)$ for some $f \in F$, as desired. So, suppose that $A \notin I$. Now $J \stackrel{\text{def}}{=} \{A \backslash X : X \in I\}$ has fip since I is closed under unions, and so this set can be extended to an ultrafilter D over A. Let $\mu = \text{cf}(\prod A/D)$. Then f^μ is cofinal in $(\prod A, <_D)$ since it is universal for μ. But $f_i^\mu \leq_I g$ for all $i < \mu$, since $f_i^\mu \in F$ and so $>(f_i^\mu, g) \in I$. This is a contradiction. $\square$

Note that Theorem 24b.20 is not talking about true cofinality. In fact, clearly any increasing sequence of elements of $\prod A$ under $<$ must have order type at most $\min(A)$, and so true cofinality does not exist if A has more than one element.

Lemma 24b.21. *Suppose that A is progressive, $\lambda \in \text{pcf}(A)$, and $f' = \langle f'_\xi : \xi < \lambda \rangle$ is a universal sequence for λ. Suppose that $f = \langle f_\xi : \xi < \lambda \rangle$ is $<_{J_{<\lambda}}$-increasing, and for every $\xi' < \lambda$ there is a $\xi < \lambda$ such that $f'_{\xi'} \leq_{J_{<\lambda}} f_\xi$. Then f is universal for λ.*

Proof. This is clear, since for any ultrafilter D over A such that $\text{cf}(\prod A/D) = \lambda$ we have $D \cap J_{<\lambda} = \emptyset$, and hence $f'_{\xi'} \leq_{J_{<\lambda}} f_\xi$ implies that $f'_{\xi'} \leq_D f_\xi$. $\square$

For the next result, note that if A is progressive, then $|A| < \min(A)$, and hence $|A|^+ \leq \min(A)$. So $A \cap |A|^+ = \emptyset \in J_{<\lambda}$ for any λ. So if μ is an ordinal and $A \cap \mu \notin J_{<\lambda}$, then $|A|^+ < \mu$.

Lemma 24b.22. *Suppose that A is a progressive set of regular cardinals and $\lambda \in \text{pcf}(A)$.*

(i) Let μ be the least ordinal such that $A \cap \mu \notin J_{<\lambda}[A]$. Then there is a universal sequence for λ that satisfies $(*)_\kappa$ for every regular cardinal κ such that $\kappa < \mu$.

(ii) There is a universal sequence for λ that satisfies $(*)_{|A|^+}$.

Proof. First note that (ii) follows from (i) by the remark preceding this lemma. Now we prove (i). Note by the minimality of μ that either $\mu = \rho + 1$ for some $\rho \in A$, or μ is a limit cardinal and $A \cap \mu$ is unbounded in μ.

(1) $\mu \leq \lambda + 1$.

For, let D be an ultrafilter such that $\lambda = \text{cf}(\prod A/D)$. Then $A \cap (\lambda + 1) \in D$, as otherwise $\{a \in A : \lambda < a\} \in D$, and so $\text{cf}(\prod A/D) > \lambda$ by 24b.1(vii), contradiction. Thus $\lambda \in \text{pcf}(A \cap (\lambda + 1))$, and hence $\text{pcf}((A \cap (\lambda + 1)) \not\subseteq \lambda$, proving (1).

(2) $\mu \neq \lambda$.

For, $|A| < \min(A) \leq \lambda$, so $A \cap \lambda$ is bounded in λ because λ is regular. Hence $\mu \neq \lambda$ by an initial remark of this proof.

Now we can complete the proof for the case in which μ is $\rho + 1$ for some $\rho \in A$. In this case, actually $\rho = \lambda$. For, we have $A \cap \rho \in J_{<\lambda}[A]$ while $A \cap (\rho + 1) \notin J_{<\lambda}[A]$. Let D be an ultrafilter on A such that $A \cap (\rho + 1) \in D$ and $\text{cf}(\prod A/D) \geq \lambda$. Then $A \cap \rho \notin D$, since $A \cap \rho \in J_{<\lambda}[A]$, so $\{\rho\} \in D$, and so $\rho \geq \lambda$. By (1) we then have $\rho = \lambda$.

Now define, for $\xi < \lambda$ and $a \in A$,

$$f_\xi(a) = \begin{cases} 0 & \text{if } a < \lambda, \\ \xi & \text{if } \lambda \leq a. \end{cases}$$

Thus $f_\xi \in \prod A$. The sequence $\langle f_\xi : \xi < \lambda \rangle$ is $<_{J_{<\lambda}[A]}$-increasing, since if $\xi < \eta < \lambda$ then $\{a \in A : f_\xi(a) \geq f_\eta(a)\} \subseteq A \cap \lambda \in J_{<\lambda}[A]$. It is also universal for λ. For, suppose that D is an ultrafilter on A such that $\mathrm{cf}(\prod A/D) = \lambda$. Suppose that $g \in \prod A$. Now $|A| < \min(A) \le \lambda$, so $\xi \stackrel{\text{def}}{=} (\sup_{a\in A} g(a)) + 1$ is less than λ. Now $\{a \in A : g(a) < f_\xi(a)\} = A \in D$, so $g <_D f_\xi$, as desired. Finally, $\langle f_\xi : \xi < \lambda\rangle$ satisfies $(*)_\lambda$, since it is itself strongly increasing under $J_{<\lambda}[A]$. In fact, if $\xi < \eta < \lambda$ and $a \in A\backslash\lambda$, then $f_\xi(a) = \xi < \eta = f_\eta(a)$, and $A \cap \lambda \in J_{<\lambda}[A]$.

Hence the case remains in which $\mu < \lambda$ and $A \cap \mu$ is unbounded in μ. Let $\langle f'_\xi : \xi < \lambda\rangle$ be any universal sequence for λ. We now apply Lemma 24b.43 with I replaced by $J_{<\lambda}[A]$. (Recall that $(\prod A, I_{<\lambda}[A]$ is λ-directed by Theorem 24b.8.) This gives us a $<_{J_{<\lambda}[A]}$-increasing sequence $f = \langle f_\xi : \xi < \lambda\rangle$ such that $f'_\xi < f_{\xi+1}$ for every $\xi < \lambda$, and $(*)_\kappa$ holds for f, for every regular cardinal κ such that $\kappa^{++} < \lambda$ and $\{a \in A : a \le \kappa^{++}\} \in J_{<\lambda}[A]$. Clearly then f is universal for λ. If κ is a regular cardinal less than μ, then $\kappa^{++} < \mu < \lambda$, and $\{a \in A : a \le \kappa^{++}\} \subseteq J_{<\lambda}[A]$ by the minimality of μ, so the conclusion of the lemma holds. $\square$

Lemma 24b.23. *Suppose that A is a progressive set of regular cardinals, $B \subseteq A$, and λ is a regular cardinal. Then the following conditions are equivalent:*

(i) $J_{\le\lambda}[A] = J_{<\lambda}[A] + B$.

(ii) $B \in J_{\le\lambda}[A]$, and for every ultrafilter D on A, if $\mathrm{cf}(\prod A/D) = \lambda$, then $B \in D$.

Proof. (i)⇒(ii): Assume (i). Obviously, then, $B \in J_{\le\lambda}[A]$. Now suppose that D is an ultrafilter on A and $\mathrm{cf}(\prod A/D) = \lambda$. By Corollary 24b.9(ii) we have $J_{\le\lambda}[A] \cap D \neq \emptyset = J_{<\lambda}[A] \cap D$. Choose $X \in J_{\le\lambda}[A] \cap D$. Then by Proposition 24b.16, $X\backslash B \in J_{<\lambda}[A]$, so since $J_{<\lambda}[A] \cap D = \emptyset$, we get $B \in D$.

(ii)⇒(i): $\supseteq$ is clear. Now suppose that $X \in J_{\le\lambda}[A]$. If $X \subseteq B$, then obviously $X \in J_{<\lambda}[A] + B$. Suppose that $X \not\subseteq B$, and let D be any ultrafilter such that $X\backslash B \in D$. Then $\mathrm{cf}(\prod A/D) \le \lambda$ since $\mathrm{pcf}(X) \subseteq \lambda^+$, and so $\mathrm{cf}(\prod A/D) < \lambda$ by the second assumption in (ii). This shows that $\mathrm{pcf}(X\backslash B) \subseteq \lambda$, so $X\backslash B \in J_{<\lambda}[A]$, and hence $X \in J_{<\lambda}[A] + B$ by Proposition 24b.16. $\square$

Theorem 24b.24. *If A is progressive, then every member of* $\mathrm{pcf}(A)$ *has a generator.*

Proof. First suppose that we have shown the theorem if $|A|^+ < \min(A)$. We show how it follows when $|A|^+ = \min(A)$. The least member of $\mathrm{pcf}(A)$ is $|A|^+$ by 24b.1(vii). We have $J_{<|A|^+}[A] = \{\emptyset\}$ and $J_{\le|A|^+}[A] = \{\emptyset, \{|A|^+\}\} = J_{<|A|^+}[A] + |A|^+$, so $|A|^+$ is a $|A|^+$-generator. Now suppose that $\lambda \in \mathrm{pcf}(A)$ with $\lambda > |A|^+$. Let $A' = A\backslash\{|A|^+\}$. By 24b.1(vi) we also have $\lambda \in \mathrm{pcf}(A')$. By the supposed result there is a $b \subseteq A'$ such that $J_{\le\lambda}[A'] = J_{<\lambda}[A'] + b$. Hence, applying Lemma 24b.6 to λ^+ and $\{|A|^+\}$,

$$\begin{aligned} J_{\le\lambda}[A] &= J_{\le\lambda}[A'] + \{|A|^+\} \\ &= J_{<\lambda}[A'] + b + \{|A|^+\} \\ &= J_{<\lambda}[A] + b, \end{aligned}$$

as desired.

Thus we assume henceforth that $|A|^+ < \min(A)$. Suppose that $\lambda \in \mathrm{pcf}(A)$. First we take the case $\lambda = |A|^{++}$. Hence by Lemma 24b.1(vii) we have $\lambda \in A$. Clearly

$$J_{\le\lambda}[A] = \{\emptyset, \{\lambda\}\} = \{\emptyset\} + \{\lambda\} = J_{<\lambda}[A] + \{\lambda\},$$

so λ has a generator in this case. So henceforth we assume that $|A|^{++} < \lambda$.

By Lemma 24b.22, there is a universal sequence $f = \langle f_\xi : \xi < \lambda \rangle$ for λ such that $(*)_{|A|^+}$ holds. Hence by Lemma 8.40, f has an exact upper bound h with respect to $<_{J_{<\lambda}}$. Since h is a least upper bound for f and the identity function on A is an upper bound for f, we may assume that $h(a) \leq a$ for all $a \in A$. We now define

$$B = \{a \in A : h(a) = a\}.$$

Thus we can finish the proof by showing that

$$(\star) \qquad J_{\leq\lambda}[A] = J_{<\lambda}[A] + B$$

First we show that $B \in J_{\leq\lambda}[A]$, i.e., that $\operatorname{pcf}(B) \subseteq \lambda^+$. Let D be any ultrafilter over A having B as an element; we want to show that $\operatorname{cf}(\prod A/D) \leq \lambda$. If $D \cap J_{<\lambda} \neq \emptyset$, then $\operatorname{cf}(\prod A/D) < \lambda$ by the definition of $J_{<\lambda}$. Suppose that $D \cap J_{<\lambda} = \emptyset$. Now since f is $<_{J_{<\lambda}}$-increasing and $D \cap J_{<\lambda} = \emptyset$, the sequence f is also $<_D$-increasing. It is also cofinal; for let $g \in \prod A$. Define

$$g'(a) = \begin{cases} g(a) & \text{if } a \in B, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{a \in A : g'(a) \geq h(a)\} \subseteq \{a \in A : h(a) = 0\} \subseteq \{a \in A : f_0(a) \geq h(a)\} \in J_{<\lambda}$. So $g' <_{J_{<\lambda}} h$. Since h is an exact upper bound for f, there is hence a $\xi < \lambda$ such that $g' <_{J_{<\lambda}} f_\xi$. Hence $g' <_D f_\xi$, and clearly $g =_D g'$, so $g <_D f_\xi$. This proves that $\operatorname{cf}(\prod A/D) = \lambda$. So we have proved $\supseteq$ in $(\star)$.

For $\subseteq$, we argue by contradiction and suppose that there is an $X \in J_{\leq\lambda}$ such that $X \notin J_{<\lambda}[A] + B$. Hence (by Proposition 24b.16), $X \backslash B \notin J_{<\lambda}$. Hence $J^*_{<\lambda} \cup \{X \backslash B\}$ has fip, so we extend it to an ultrafilter D. Since $D \cap J_{<\lambda} = \emptyset$, we have $\operatorname{cf}(\prod A/D) \geq \lambda$. But also $X \in D$ since $X \backslash B \in D$, and $X \in J_{\leq\lambda}$, so $\operatorname{cf}(\prod A/D) = \lambda$. By the universality of f it follows that f is cofinal in $\operatorname{cf}(\prod A/D)$. But $A \backslash B \in D$, so $\{a \in A : h(a) < a\} \in D$, and so there is a $\xi < \lambda$ such that $h <_D f_\xi$. This contradicts the fact that h is an upper bound of f under $<_{J_{<\lambda}}$. $\square$

Now we state some important properties of generators.

Lemma 24b.25. *Suppose that A is progressive, $\lambda \in \operatorname{pcf}(A)$, and $B \subseteq A$.*

(i) If B is a λ-generator, D is an ultrafilter on A, and $\operatorname{cf}(\prod A/D) = \lambda$, then $B \in D$.

(ii) If B is a λ-generator, then $\lambda \notin \operatorname{pcf}(A \backslash B)$.

(iii) If $B \in J_{\leq\lambda}$ and $\lambda \notin \operatorname{pcf}(A \backslash B)$, then B is a λ-generator.

(iv) If $\lambda = \max(\operatorname{pcf}(A))$, then A is a λ-generator on A.

(v) If B is a λ-generator, then the restrictions to B of any universal sequence for λ are cofinal in $(\prod B, <_{J_{<\lambda}[B]})$.

(vi) If B is a λ-generator, then $\operatorname{tcf}(\prod B, <_{J_{<\lambda}[B]}) = \lambda$.

(vii) If B is a λ-generator on A, then $\lambda = \max(\operatorname{pcf}(B))$.

(viii) If B is a λ-generator on A and D is an ultrafilter on A, then $\operatorname{cf}(\prod A/D) = \lambda$ iff $B \in D$ and $D \cap J_{<\lambda} = \emptyset$.

(ix) If B is a λ-generator on A and $B =_{J_{<\lambda}} C$, then C is a λ-generator on A. [Here $X =_I Y$ means that the symmetric difference of X and Y is in I, for any ideal I.]

(x) If B is a λ-generator, then so is $B \cap (\lambda+1)$.
(xi) If B and C are λ-generators, then $B =_{J_{<\lambda}} C$.
(xii) If $\lambda = \max(\mathrm{pcf}(A))$ and B is a λ-generator, then $A \backslash B \in J_{<\lambda}$.

Proof. (i): By Corollary 24b.9(ii), choose $C \in J_{\leq\lambda} \cap D$. Hence $C \subseteq X \cup B$ for some $X \in J_{<\lambda}$. By Corollary 24b.9(ii) again, $J_{<\lambda} \cap D = \emptyset$, so $X \notin D$. Thus $C \backslash X \subseteq B$ and $C \backslash X \in D$, so $B \in D$.

(ii): Clear by (i).

(iii): Assume the hypothesis. We need to show that every member C of $J_{\leq\lambda}$ is a member of $J_{<\lambda} + B$. Now $\mathrm{pcf}(C) \subseteq \lambda^+$. Hence $\mathrm{pcf}(C \backslash B) \subseteq \lambda$, so $C \backslash B \in J_{<\lambda}$, and the desired conclusion follows from Proposition 24b.16.

(iv): By (iii).

(v): Suppose not. Let $f = \langle f_\xi : \xi < \lambda \rangle$ be a universal sequence for λ such that there is an $h \in \prod B$ such that h is not bounded by any $f_\xi \restriction B$. Thus $\leq (f_\xi \restriction B, h) \notin J_{<\lambda}[B]$ for all $\xi < \lambda$. Now suppose that $\xi < \eta < \lambda$. Then

$$\begin{aligned} \leq (f_\eta \restriction B, h) \backslash (\leq (f_\xi \restriction B, h)) &= \{a \in B : f_\eta(a) \leq h(a) < f_\xi(a)\} \\ &\subseteq \{a \in A : f_\eta(a) < f_\xi(a)\} \in J_{<\lambda}[A]. \end{aligned}$$

Hence by Lemma 24b.17(i) we have $\leq (f_\eta \restriction B, h) \backslash (\leq (f_\xi \restriction B, h)) \in J_{<\lambda}[B]$. It follows that if N is a finite subset of λ with largest element less than η, then

$$(*) \qquad (\leq (f_\eta \restriction B, h)) \backslash \bigcap_{\xi \in N} (\leq (f_\xi \restriction B, h)) \in J_{<\lambda}[B].$$

We claim now that

$$M \stackrel{\text{def}}{=} \{\leq (f_\xi \restriction B, h) : \xi < \lambda\} \cup (J_{<\lambda}[B])^*$$

has fip. Otherwise, there is a finite subset N of λ and a set $C \in J_{<\lambda}[B]$ such that

$$\left(\bigcap_{\xi \in N} \leq (f_\xi \restriction B, h) \right) \cap (B \backslash C) = \emptyset;$$

hence if ξ is the largest member of N we get $\leq (f_\xi \restriction B, h) \in J_{<\lambda}[B]$ by $(*)$, contradiction. So we extend the set M to an ultrafilter D on B, then to an ultrafilter E on A. Note that $B \in E$. Also, $E \cap J_{<\lambda}[A] = \emptyset$. In fact, if $X \in E \cap J_{<\lambda}[A]$, then $X \cap B \in J_{<\lambda}[A]$, so $X \cap B \in D \cap J_{<\lambda}[B]$ by Lemma 24b.17(i). But $D \cap J_{<\lambda}[B] = \emptyset$ by construction. Now $B \in E \cap J_{\leq\lambda}[A]$, so $\mathrm{cf}(\prod A/E) = \lambda$, and h bounds all f_ξ in this ultraproduct, contradicting the universality of f.

(vi): By Lemma 24b.17 and (v).

(vii): By (i) we have $\lambda \in \mathrm{pcf}(B)$. Now $B \in J_{\leq\lambda}[A]$, so $\mathrm{pcf}(B) \subseteq \lambda^+$. The desired conclusion follows.

(viii): For $\Rightarrow$, suppose that $\mathrm{cf}(\prod A/D) = \lambda$. Then $B \in D$ by (i), and obviously $D \cap J_{<\lambda} = \emptyset$. For $\Leftarrow$, assume that $B \in D$ and $D \cap J_{<\lambda} = \emptyset$. Now $B \in J_{\leq\lambda}$, so $\mathrm{cf}(\prod A/D) = \lambda$ by Corollary 24b.9(ii).

(ix): We have $B \in J_{\leq\lambda}$ and $C = (C\backslash B) \cup (C \cap B)$, so $C \in J_{\leq\lambda}$. Suppose that $\lambda \in \mathrm{pcf}(A\backslash C)$. Let D be an ultrafilter on A such that $\mathrm{cf}(\prod A/D) = \lambda$ and $A\backslash C \in D$. Now $B \in D$ by (i), so $B\backslash C \in D$. This contradicts $B\backslash C \in J_{<\lambda}$. So $\lambda \notin \mathrm{pcf}(A\backslash C)$. Hence C is a λ-generator, by (iii).

(x): Let $B' = B \cap (\lambda + 1)$. Clearly $B' \in J_{\leq\lambda}$. Suppose that $\lambda \in \mathrm{pcf}(A\backslash B')$. Say $\lambda = \mathrm{cf}(\prod A/D)$ with $A\backslash B' \in D$. Also $A \cap (\lambda+1) \in D$, since $A\backslash(\lambda+1) \in D$ would imply that $\mathrm{cf}(\prod A/D) > \lambda$ by Proposition 24b.1(vii). Since clearly

$$(A\backslash B') \cap (A \cap (\lambda+1)) \subseteq A\backslash B,$$

this yields $A\backslash B \in D$, contradicting (ii). Therefore, $\lambda \notin \mathrm{pcf}(A\backslash B')$. So B' is a λ-generator, by (iii).

(xi): This is clear from Proposition 24b.16.

(xii): Clear by (iv) and (xi). □

Lemma 24b.26. *Suppose that A is a progressive set, F is a proper filter over A, and λ is a cardinal. Then the following are equivalent.*

(i) $\mathrm{tcf}(\prod A/F) = \lambda$.

(ii) $\lambda \in \mathrm{pcf}(A)$, *$F$ has a λ-generator on A as an element, and* $J^*_{<\lambda} \subseteq F$.

(iii) $\mathrm{cf}(\prod A/D) = \lambda$ *for every ultrafilter D extending F.*

Proof. (i)⇒(iii): obvious.

(iii)⇒(ii): Obviously $\lambda \in \mathrm{pcf}(A)$. Let B be a λ-generator on A. Suppose that $B \notin F$. Then there is an ultrafilter D on A such that $A\backslash B \in D$ and D extends F. Then $\mathrm{cf}(\prod A/D) = \lambda$ by (iii), and this contradicts Lemma 24b.25(i).

(ii)⇒(i): Let $B \in F$ be a λ-generator. By Lemma 24b.25(vi) we have $\mathrm{tcf}(\prod B/J_{<\lambda}) = \lambda$, and hence $\mathrm{tcf}(\prod A/F) = \lambda$ since $B \in F$ and $J^*_{<\lambda} \subseteq F$. □

Proposition 24b.27. *Suppose that A is a progressive set of regular cardinals, and λ is any cardinal. Then the following conditions are equivalent:*

(i) $\lambda = \max(\mathrm{pcf}(A))$.

(ii) $\lambda = \mathrm{tcf}(\prod A/J_{<\lambda}[A])$.

(iii) $\lambda = \mathrm{cf}(\prod A/J_{<\lambda}[A])$.

Proof. (i)⇒(ii): By Lemma 24b.25(iv),(vi).

(ii)⇒(iii): Obvious.

(iii)⇒(ii): Assume (iii). Let $\mu = \max(\mathrm{pcf}(A))$. By (i)⇒(iii) we have $\lambda = \mu$. □

Lemma 24b.28. *Suppose that A is progressive, $A_0 \subseteq A$, and $\lambda \in \mathrm{pcf}(A_0)$. Suppose that B is a λ-generator on A. Then $B \cap A_0$ is a λ-generator on A_0.*

Proof. Since $B \in J_{\leq\lambda}[A]$, we have $\mathrm{pcf}(B) \subseteq \lambda^+$ and hence $\mathrm{pcf}(B \cap A_0) \subseteq \lambda^+$ and so $B \cap A_0 \in J_{\leq\lambda}[A_0]$. If $\lambda \in \mathrm{pcf}(A_0\backslash B)$, then also $\lambda \in \mathrm{pcf}(A\backslash B)$, and this contradicts Lemma 24b.25(ii). Hence $\lambda \notin \mathrm{pcf}(A_0\backslash B)$, and hence $B \cap A_0$ is a λ-generator for A_0 by Lemma 24b.25(iii). □

Definition. If A is progressive, a *generating sequence for A* is a sequence $\langle B_\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ such that B_λ is a λ-generator on A for each $\lambda \in \mathrm{pcf}(A)$.

Theorem 24b.29. *Suppose that A is progressive, $\langle B_\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ is a generating sequence for A, and $X \subseteq A$. Then there is a finite subset N of* $\mathrm{pcf}(X)$ *such that $X \subseteq \bigcup_{\mu\in N} B_\mu$.*

Proof. We show that for all $X \subseteq A$, if $\lambda = \max(\mathrm{pcf}(X))$, then there is a finite subset N as indicated, using induction on λ. So, suppose that this is true for every cardinal $\mu < \lambda$, and now suppose that $X \subseteq A$ and $\max(\mathrm{pcf}(X)) = \lambda$. Then $\lambda \notin \mathrm{pcf}(X\backslash B_\lambda)$ by Lemma 24b.25(ii), and so $\mathrm{pcf}(X\backslash B_\lambda) \subseteq \lambda$. Hence $\max(\mathrm{pcf}(X\backslash B_\lambda)) < \lambda$. Hence by the inductive hypothesis there is a finite subset N of $\mathrm{pcf}(X\backslash B_\lambda)$ such that $X\backslash B_\lambda \subseteq \bigcup_{\mu\in N} B_\mu$. Hence

$$X \subseteq B_\lambda \cup \bigcup_{\mu\in N} B_\mu,$$

and $\{\lambda\} \cup N \subseteq \mathrm{pcf}(X)$. $\square$

Corollary 24b.30. *Suppose that A is progressive, $\langle B_\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ is a generating sequence for A, and $X \subseteq A$. Suppose that λ is any infinite cardinal. Then $X \in J_{<\lambda}[A]$ iff $X \subseteq \bigcup_{\mu\in N} B_\mu$ for some finite subset N of $\lambda \cap \mathrm{pcf}(A)$.*

Proof. $\Rightarrow$: Assume that $X \in J_{<\lambda}[A]$. Thus $\mathrm{pcf}(X) \subseteq \lambda$, and Theorem 24b.29 gives the desired conclusion.

$\Leftarrow$: Assume that a set N is given as indicated. Suppose that $\rho \in \mathrm{pcf}(X)$. Say $\rho = \mathrm{cf}(\prod A/D)$ with $X \in D$. Then $B_\mu \in D$ for some $\mu \in N$. By the definition of generator, $B_\mu \in J_{\leq\mu}[A]$, and hence $\rho \leq \mu < \lambda$. Thus we have shown that $\mathrm{pcf}(X) \subseteq \lambda$, so $X \in J_{<\lambda}[A]$. $\square$

Lemma 24b.31. *Suppose that A is progressive and $\langle B_\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ is a generating sequence for A. Suppose that D is an ultrafilter on A. Then there is a $\lambda \in \mathrm{pcf}(A)$ such that $B_\lambda \in D$, and if λ is minimum with this property, then $\lambda = \mathrm{cf}(\prod A/D)$.*

Proof. Let $\mu = \mathrm{cf}(\prod A/D)$. Then $\mu \in \mathrm{pcf}(A)$ and $B_\mu \in D$ by Lemma 24b.25(i). Suppose that $B_\lambda \in D$ with $\lambda < \mu$. Now $B_\lambda \in J_{\leq\lambda} \subseteq J_{<\mu}$, contradicting Lemma 24b.25(viii), applied to μ. $\square$

Lemma 24b.32. *If A is progressive and also* $\mathrm{pcf}(A)$ *is progressive, and if $\lambda \in \mathrm{pcf}(A)$ and B is a λ-generator for A, then* $\mathrm{pcf}(B)$ *is a λ-generator for* $\mathrm{pcf}(A)$.

Proof. Note by Theorem 24b.15 that $\mathrm{pcf}(\mathrm{pcf}(B)) = \mathrm{pcf}(B)$ and $\mathrm{pcf}(\mathrm{pcf}(A\backslash B)) = \mathrm{pcf}(A\backslash B)$. Since $B \in J_{\leq\lambda}[A]$, we have $\mathrm{pcf}(B) \subseteq \lambda^+$, and hence $\mathrm{pcf}(\mathrm{pcf}(B)) \subseteq \lambda^+$ and so $\mathrm{pcf}(B) \in J_{\leq\lambda}[\mathrm{pcf}(A)]$. Now suppose that $\lambda \in \mathrm{pcf}(\mathrm{pcf}(A)\backslash\mathrm{pcf}(B))$. Then by Lemma 24b.1(iv) we have $\lambda \in \mathrm{pcf}(\mathrm{pcf}(A\backslash B)) = \mathrm{pcf}(A\backslash B)$, contradicting Lemma 24b.25(ii). So $\lambda \notin \mathrm{pcf}(\mathrm{pcf}(A)\backslash\mathrm{pcf}(B))$. It now follows by Lemma 24b.25(iii) that $\mathrm{pcf}(B)$ is a λ-generator for $\mathrm{pcf}(A)$. $\square$

The following result is relevant to Theorem 24a.44. Let μ be a singular cardinal, C a club of μ, and suppose that $X \in J_{<\mu}[C^{(+)}]$. Now $\mathrm{pcf}(X)$ has a maximal element, and so there is an $\alpha < \mu$ such that $X \subseteq \mathrm{pcf}(X) \subseteq \alpha$. Thus $J_{<\mu}[C^{(+)}] \subseteq J^{\mathrm{bd}}$.

Lemma 24b.33. *If μ is a singular cardinal of uncountable cofinality, then there is a club $C \subseteq \mu$ such that* $\mathrm{tcf}(\prod C^{(+)}/J_{<\mu}[C^{(+)}]) = \mu^+$*.*

Proof. Let C_0 be a club in μ such that such that $\mu^+ = \mathrm{tcf}(\prod C_0^{(+)}/J^{bd})$, by Theorem 24b.44. Let $C_1 \subseteq C_0$ be such that the order type of C_1 is $\mathrm{cf}(\mu)$, C_1 is cofinal in μ, and $\forall\kappa \in C_1[\mathrm{cf}(\mu) < \kappa]$. Hence $C_1^{(+)}$ is progressive. Now $\mu^+ \in \mathrm{pcf}(C_1^{(+)})$ by Lemma 24b.26. Let B be a μ^+-generator for $C_1^{(+)}$. Define $C = \{\delta \in C_1 : \delta^+ \in B\}$. Now $C_1\backslash C$ is bounded. Otherwise, let $X = C_1^{(+)}\backslash B = (C_1\backslash C)^{(+)}$. So X is unbounded, and hence clearly $\mu^+ = \mathrm{tcf}(\prod X/J^{\mathrm{bd}})$. Hence $\mu^+ \in \mathrm{pcf}(X)$. This contradicts Lemma 24b.25(ii).

So, choose $\varepsilon < \mu$ such that $C_1\backslash C \subseteq \varepsilon$. Hence $C_1\backslash\varepsilon \subseteq C\backslash\varepsilon \subseteq C_1\backslash\varepsilon$, so $C_1\backslash\varepsilon = C\backslash\varepsilon$. Clearly $\mu^+ = \mathrm{tcf}(\prod(C_1\backslash\varepsilon)^{(+)}/J^{\mathrm{bd}})$, so $\mu^+ \in \mathrm{pcf}((C_1\backslash\varepsilon)^{(+)})$. We claim that $\mathrm{tcf}(\prod(C_1\backslash\varepsilon)^{(+)}/J_{<\mu^+}[(C_1\backslash\varepsilon)^{(+)}]) = \mu^+$. To show this, we apply Lemma 24b.26. Suppose that D is any ultrafilter on $(C_1\backslash\varepsilon)^{(+)}$ such that $J_{<\mu^+}[(C_1\backslash\varepsilon)^{(+)}] \cap D = \emptyset$. Now by Lemma 24b.28, $B \cap (C_1\backslash\varepsilon)^{(+)}$ is a μ^+-generator for $(C_1\backslash\varepsilon)^{(+)}$. Note that $C^+ \subseteq B$. Now $B \cap (C_1\backslash\varepsilon)^{(+)} = B \cap (C\backslash\varepsilon)^{(+)} = (C\backslash\varepsilon)^{(+)}$. It follows by Lemma 24b.25(viii) that $\mathrm{cf}(\prod(C_1\backslash\varepsilon)^{(+)}/D) = \mu^+$. This proves that $\mathrm{tcf}(\prod(C_0\backslash\varepsilon)^{(+)}/J_{<\mu^+}[(C_1\backslash\varepsilon)^{(+)}]) = \mu^+$. Now we claim that $J_{<\mu^+}[(C_1\backslash\varepsilon)^{(+)}] = J_{<\mu}[(C_1\backslash\varepsilon)^{(+)}]$. For, suppose that $X \in J_{<\mu^+}[(C_1\backslash\varepsilon)^{(+)}]$. So $\mathrm{pcf}(X) \subseteq \mu^+$. Since X is progressive (because $C_1\backslash\varepsilon)^{(+)}$ is), we have $\max(\mathrm{pcf}(X)) < \mu$, hence $\mathrm{pcf}(X) \subseteq \mu$. □

By essentially the same proof as for Lemma 24b.24a we get

Lemma 24b.34. *If μ is a singular cardinal of countable cofinality, then there is an unbounded subset C of μ consisting of regular cardinals such that* $\mathrm{tcf}(\prod C/J_{<\mu}[C]) = \mu^+$*.*

Proof. Let C_0 be an unbounded collection of regular cardinals in μ such that $\mu^+ = \mathrm{tcf}(\prod C_0/J^{bd})$, by Theorem 24b.45. Let $C_1 \subseteq C_0$ be such that the order type of C_1 is $\mathrm{cf}(\mu)$, C_1 is cofinal in μ, and $\forall\kappa \in C_1[\omega < \kappa]$. Hence C_1 is progressive. Now $\mu^+ \in \mathrm{pcf}(C_1)$ by Lemma 24b.26. Let B be a μ^+-generator for C_1. Define $C = B \cap C_1$. Now $C_1\backslash C$ is bounded. Otherwise, let $X = C_1\backslash B = C_1\backslash C$. So X is unbounded, and hence clearly $\mu^+ = \mathrm{tcf}(\prod X/J^{\mathrm{bd}})$. Hence $\mu^+ \in \mathrm{pcf}(X)$. This contradicts Lemma 24b.25(ii).

So, choose $\varepsilon < \mu$ such that $C_1\backslash C \subseteq \varepsilon$. Hence $C_1\backslash\varepsilon \subseteq C\backslash\varepsilon \subseteq C_1\backslash\varepsilon$, so $C_1\backslash\varepsilon = C\backslash\varepsilon$. Clearly $\mu^+ = \mathrm{tcf}(\prod(C_1\backslash\varepsilon)/J^{\mathrm{bd}})$, so $\mu^+ \in \mathrm{pcf}(C_1\backslash\varepsilon)$. We claim that

$$\mathrm{tcf}(\prod(C_1\backslash\varepsilon)/J_{<\mu^+}[C_1\backslash\varepsilon] = \mu^+.$$

To show this, we apply Lemma 24b.26. Suppose that D is any ultrafilter on $C_1\backslash\varepsilon$ such that $J_{<\mu^+}[C_1\backslash\varepsilon] \cap D = \emptyset$. Now by Lemma 24b.28, $B \cap (C_1\backslash\varepsilon)$ is a μ^+-generator for $C_1\backslash\varepsilon$. Note that $C \subseteq B$. Now $B \cap (C_1\backslash\varepsilon) = B \cap (C\backslash\varepsilon) = (C\backslash\varepsilon)$. It follows by Lemma 24b.25(viii) that $\mathrm{cf}(\prod(C_1\backslash\varepsilon)/D) = \mu^+$. This proves that $\mathrm{tcf}(\prod(C_0\backslash\varepsilon)/J_{<\mu^+}[C_1\backslash\varepsilon]) = \mu^+$. Now we claim that $J_{<\mu^+}[C_1\backslash\varepsilon] = J_{<\mu}[C_1\backslash\varepsilon]$. For, suppose that $X \in J_{<\mu^+}[C_1\backslash\varepsilon]$. So $\mathrm{pcf}(X) \subseteq \mu^+$. Since X is progressive (because $C_1\backslash\varepsilon$ is), we have $\max(\mathrm{pcf}(X)) < \mu$, hence $\mathrm{pcf}(X) \subseteq \mu$. □

Proposition 24b.35. *Suppose that F is a proper filter over a progressive set A of regular cardinals. Define*

$$\mathrm{pcf}_F(A) = \left\{\mathrm{cf}\left(\prod A/D\right) : D \textit{ is an ultrafilter extending } F\right\}.$$

Then:

(i) $\max(\mathrm{pcf}_F(A))$ *exists.*

(ii) $\mathrm{cf}(\prod A/F) = \max(\mathrm{pcf}_F(A))$.

(iii) If $B \subseteq \mathrm{pcf}_F(A)$ *is progressive, then* $\mathrm{pcf}(B) \subseteq \mathrm{pcf}_F(A)$.

(iv) If A is a progressive interval of regular cardinals with no largest element, and

$$F = \{X \subseteq A : A\backslash X \text{ is bounded}\}$$

is the filter of co-bounded subsets of A, then $\mathrm{pcf}_F(A)$ *is an interval of regular cardinals.*

Proof. (i): Clearly $\mathrm{pcf}_F(A) \subseteq \mathrm{pcf}(A)$, and so if $\lambda = \max(\mathrm{pcf}(A))$, then $A \in F \cap J_{<\lambda^+}[A]$. Hence we can choose μ minimum such that $F \cap J_{<\mu}[A] \neq \emptyset$. By Corollary 24b.12, μ is not a limit cardinal; write $\mu = \lambda^+$. Then $F \cap J_{<\lambda} = \emptyset$, and so $F \cup J^*_{<\lambda}$ has fip; let D be an ultrafilter containing this set. Then $D \cap J_{\leq\lambda} \supseteq F \cap J_{\leq\lambda} \neq \emptyset$, while $D \cap J_{<\lambda} = \emptyset$. Hence $\mathrm{cf}(\prod A/D) = \lambda$ by Corollary 24b.9. On the other hand, since $F \cap J_{\leq\lambda}[A] \neq \emptyset$, any ultrafilter E containing F must be such that $\mathrm{cf}(\prod A/E) \leq \lambda$.

(ii): Cf. the proof of Theorem 24b.20. Let $\lambda = \max(\mathrm{pcf}_F(A))$, and let D be an ultrafilter extending F such that $\lambda = \mathrm{cf}(\prod A/D)$. Let $\langle f_\alpha : \alpha < \lambda\rangle$ be strictly increasing and cofinal mod D. Now if $g < h$ mod F, then also $g < h$ mod D. So a cofinal subset of $\prod A$ mod F is also a cofinal subset mod D, so $\lambda \leq \mathrm{cf}(\prod A/F)$. Hence it suffices to exhibit a cofinal subset of $\prod A$ mod F of size λ. For every $\mu \in \mathrm{pcf}_F(A)$ fix a universal sequence $f^\mu = \langle f^\mu_i : i < \mu\rangle$ for μ, by Theorem 24b.18. Let G be the set of all functions of the form

$$\sup\{f^{\mu_1}_{i_1}, f^{\mu_2}_{i_2}, \ldots, f^{\mu_n}_{i_n}\},$$

where $\mu_1, \mu_2, \ldots, \mu_n$ is a finite sequence of members of $\mathrm{pcf}_F(A)$, possibly with repetitions, and $i_k < \mu_k$ for each $k = 1, \ldots, n$. We claim that G is cofinal in $(\prod A, <_F)$; this will complete the proof of (ii).

To prove this claim, let $g \in \prod A$. Suppose that $g \not< f$ mod F for all $f \in G$. Then, we claim, the set

$$(*) \qquad F \cup \{\{a \in A : f(a) \leq g(a)\} : f \in G\}$$

has fip. For, suppose not. Then there is a finite subset G' of G such that $\bigcup_{g\in G'}\{a \in A : g(a) < f(a)\} \in F$. Let $h = \sup_{f\in G'} f$. Then $g < h$ mod F and $h \in G$, contradiction. Thus $(*)$ has fip, and we let D be an ultrafilter containing it. Let $\mu = \mathrm{cf}(\prod A/D)$. Then $\mu \in \mathrm{pcf}_F(A)$, and $f \leq g$ mod D for all $f \in G$. Since the members of a universal sequence for μ are in G, this is a contradiction. This completes the proof of (ii).

For (iii), we look at the proof of Theorem 24b.15. Let F' be the ultrafilter named F at the beginning of that proof. Since $B \subseteq \mathrm{pcf}_F(A)$, each $b \in B$ is in $\mathrm{pcf}_F(A)$, and hence the ultrafilters D_b can be taken to extend F. Hence $F \subseteq F'$, and so $\mu \in \mathrm{pcf}_F(A)$, as desired in (iii).

Finally, we prove (iv). Let $\lambda_0 = \min(\mathrm{pcf}_F(A))$ and $\lambda_1 = \max(\mathrm{pcf}_F(A))$, and suppose that μ is a regular cardinal such that $\lambda_0 < \mu < \lambda_1$. Let D be an ultrafilter such that $F \subseteq D$ and $\mathrm{cf}(\prod A/D) = \lambda_1$. Then by Corollary 24b.9(ii), $D \cap J_{<\lambda_1} = \emptyset$, so $J^*_{\lambda_1} \subseteq D$. Thus

$F \cup J^*_{<\mu} \subseteq F \cup J^*_{<\lambda_1} \subseteq D$, so $F \cup J^+_{<\mu}$ generates a proper filter G. Since $(\prod A, <_{J_{<\mu}})$ is μ-directed by Theorem 24b.8, so is $(\prod A, <_G)$. Note that if $a \in A$, then $\{b \in A : a < b\} \in F$. It follows that $\sup(A) \leq \lambda_0 < \mu$. Hence we can apply Theorem 24b.4 and get a subset A' of A (since A is an interval of regular cardinals) and a proper ideal K over A' such that A' is cofinal in A, K contains all proper initial segments of A', and $\mathrm{tcf}(\prod A, <_K) = \mu$. Let $\langle f_\alpha : \alpha < \mu\rangle$ be strictly increasing and cofinal mod K. Extend K^* to a filter L on A, and extend each function f_α to a function f^+_α on A. Then clearly $\langle f^+_\alpha : \alpha < \mu\rangle$ is strictly increasing and cofinal mod L, and L contains F. This shows that $\mu \in \mathrm{pcf}_F(A)$. $\square$

24c. Main cofinality theorems

The sets H_Ψ

We will shortly give several proofs involving the important general idea of making elementary chains inside the sets H_Ψ. Recall that H_Ψ, for an infinite cardinal Ψ, is the collection of all sets hereditarily of size less than Ψ, i.e., with transitive closure of size less than Ψ. We consider H_Ψ as a structure with $\in$ together with a well-ordering $<^*$ of it, possibly with other relations or functions, and consider elementary substructures of such structures.

Recall that A is an *elementary substructure* of B iff A is a subset of B, and for every formula $\varphi(x_0, \ldots, x_{m-1})$ and all $a_0, \ldots, a_{m-1} \in A$, $A \models \varphi(a_0, \ldots, a_{m-1})$ iff $B \models \varphi(a_0, \ldots, a_{m-1})$.

The basic downward Löwenheim-Skolem theorem will be used a lot. This theorem depends on the following lemma.

Lemma 24c.1. (Tarski) *Suppose that A and B are first-order structures in the same language, with A a substructure of B. Then the following conditions are equivalent:*

(i) A is an elementary substructure of B.

(ii) For every formula of the form $\exists y\varphi(x_0, \ldots, x_{m-1}, y)$ and all $a_0, \ldots, a_{m-1} \in A$, if $B \models \exists y\varphi(a_0, \ldots, a_{m-1}, y)$ then there is a $b \in A$ such that $B \models \varphi(a_0, \ldots, a_{m-1}, b)$.

Proof. (i)⇒(ii): Assume (i) and the hypotheses of (ii). Then by (i) we see that $A \models \exists y\varphi(a_0, \ldots, a_{m-1}, y)$, so we can choose $b \in A$ such that $A \models \varphi(a_0, \ldots, a_{m-1}, b)$. Hence $B \models \varphi(a_0, \ldots, a_{m-1}, b)$, as desired.

(ii)⇒(i): Assume (ii). We show that for any formula $\varphi(x_0, \ldots, x_{m-1})$ and any elements $a_0, \ldots, a_{m-1} \in A$, $A \models \varphi(a_0, \ldots, a_{m-1})$ iff $B \models \varphi(a_0, \ldots, a_{m-1})$, by induction on φ. It is true for φ atomic by our assumption that A is a substructure of B. The induction steps involving $\neg$ and $\vee$ are clear. Now suppose that $A \models \exists y\varphi(a_0, \ldots, a_{m-1}, y)$, with $a_0, \ldots, a_{m-1} \in A$. Choose $b \in A$ such that $A \models \varphi(a_0, \ldots, a_{m-1}, b)$. By the inductive assumption, $B \models \varphi(a_0, \ldots, a_{m-1}, b)$. Hence $B \models \exists y\varphi(a_0, \ldots, a_{m-1}, y)$, as desired.

Conversely, suppose that $B \models \exists y\varphi(a_0, \ldots, a_{m-1}, y)$. By (ii), choose $b \in A$ such that $B \models \varphi(a_0, \ldots, a_{m-1}, b)$. By the inductive assumption, $A \models \varphi(a_0, \ldots, a_{m-1}, b)$. Hence $A \models \exists y\varphi(a_0, \ldots, a_{m-1}, y)$, as desired. $\square$

Theorem 24c.2. *Suppose that A is an L-structure, X is a subset of A, κ is an infinite cardinal, and κ is $\geq$ both $|X|$ and the number of formulas of $\mathscr{L}$, while $\kappa \leq |A|$. Then A has an elementary substructure B such that $X \subseteq B$ and $|B| = \kappa$.*

Proof. Let a well-order $\prec$ of A be given. We define $\langle C_n : n \in \omega\rangle$ by recursion. Let C_0 be a subset of A of size κ with $X \subseteq C_0$. Now suppose that C_n has been defined. Let M_n be the collection of all pairs of the form $(\exists y\varphi(x_0, \ldots, x_{m-1}, y), a)$ such that a is a sequence of elements of C_n of length m. For each such pair we define $f(\exists y\varphi(x_0, \ldots, x_{m-1}, y), a)$ to be the $\prec$-least element b of A such that $A \models \varphi(a_0, \ldots, a_{m-1}, b)$, if there is such an element, and otherwise let it be the least element of C_n. Then we define

$$C_{n+1} = C_n \cup \{f(\exists y\varphi(x_0, \ldots, x_{m-1}, y), a) : (\exists y\varphi(x_0, \ldots, x_{m-1}, y), a) \in M_n\}.$$

Finally, let $B = \bigcup_{n\in\omega} C_n$.

By induction it is clear that $|C_n| = \kappa$ for all $n \in \omega$, and so also $|B| = \kappa$.

Now to show that B is an elementary substructure of A we apply Lemma 24c.1. First we show that B is a substructure of A; this amounts to showing that B is closed under each fundamental operation F^A. Say F is m-ary, and $b_0, \ldots, b_{m-1} \in B$. Then there is an n such that $b_0, \ldots, b_{m-1} \in C_n$. Now $(\exists y[Fx_0 \ldots x_{m-1} = y], \langle b_0, \ldots, b_{m-1}\rangle) \in M_n$. Let $c = F^A(b_0, \ldots, b_{m-1})$; so $f((\exists y[Fx_0 \ldots x_{m-1} = y], \langle b_0, \ldots, b_{m-1}\rangle) = c \in C_{n+1} \subseteq B$.

Now suppose that we are given a formula of the form $\exists y\varphi(x_0, \ldots, x_{m-1}, y)$ and elements $a_0, \ldots, a_{m-1}$ of B, and $A \models \exists y\varphi(a_0, \ldots, a_{m-1}, y)$. Clearly there is an $n \in \omega$ such that $a_0, \ldots, a_{m-1} \in C_n$. Then $(\exists y\varphi(x_0, \ldots, x_{m-1}, y), a) \in M_n$, and

$$f(\exists y\varphi(x_0, \ldots, x_{m-1}, y), a)$$

is an element b of $C_{n+1} \subseteq B$ such that $A \models \varphi(a_0, \ldots, a_{m-1}, b)$. This is as desired in Lemma 24c.1. $\square$

Given an elementary substructure A of a set H_Ψ, we will frequently use an argument of the following kind. A set theoretic formula holds in the real world, and involves only sets in A. By absoluteness, it holds in H_Ψ, and hence it holds in A. Thus we can transfer a statement to A even though A may not be transitive; and the procedure can be reversed.

To carry this out, we need some facts about transitive closures first of all.

Lemma 24c.3. *(i) If $X \subseteq A$, then $\operatorname{tr\,cl}(X) \subseteq \operatorname{tr\,cl}(A)$.*

(ii) $\operatorname{tr\,cl}(\mathscr{P}(A)) = \mathscr{P}(A) \cup \operatorname{tr\,cl}(A)$.

(iii) If $\operatorname{tr\,cl}(A)$ is infinite, then $|\operatorname{tr\,cl}(\mathscr{P}(A))| \le 2^{|\operatorname{tr\,cl}(A)|}$.

(iv) $\operatorname{tr\,cl}(A \cup B) = \operatorname{tr\,cl}(A) \cup \operatorname{tr\,cl}(B)$.

(v) $\operatorname{tr\,cl}(A \times B) = (A \times B) \cup \{\{a\} : a \in A\} \cup \{\{a, b\} : a \in A, b \in B\} \cup \operatorname{tr\,cl}(A) \cup \operatorname{tr\,cl}(B)$.

(vi) If $\operatorname{tr\,cl}(A)$ or $\operatorname{tr\,cl}(B)$ is infinite, then $|\operatorname{tr\,cl}(A \times B)| \le \max(\operatorname{tr\,cl}(A), \operatorname{tr\,cl}(B)$.

(vii) $\operatorname{tr\,cl}({}^AB) \subseteq ({}^AB) \cup \operatorname{tr\,cl}(A \times B)$.

(viii) If $\operatorname{tr\,cl}(A)$ or $\operatorname{tr\,cl}(B)$ is infinite, then $|\operatorname{tr\,cl}({}^AB)| \le 2^{\max(|\operatorname{tr\,cl}(A)|, |\operatorname{tr\,cl}(A)|)}$.

(ix) If $\operatorname{tr\,cl}(A)$ is infinite, then $|\operatorname{tr\,cl}(\prod A)| \le 2^{|\operatorname{tr\,cl}(A)|}$.

(x) If $\operatorname{tr\,cl}(A)$ or $\operatorname{tr\,cl}(B)$ is infinite, then $|\operatorname{tr\,cl}({}^A(\prod B))| \le 2^{2^{\max(|\operatorname{tr\,cl}(A)|, |\operatorname{tr\,cl}(B)|)}}$.

(xi) If A is an infinite set of regular cardinals, then $|\operatorname{tr\,cl}(\operatorname{pcf}(A))| \le 2^{|\operatorname{tr\,cl}(A)|}$.

Proof. (i)–(viii) are clear. For (ix), note that $\prod A \subseteq {}^A\bigcup A$, so (ix) follow from (viii). For (x),

$$|\operatorname{tr\,cl}\left({}^A\left(\prod B\right)\right)| \le 2^{\max\left(|\operatorname{tr\,cl}(A), |\operatorname{tr\,cl}(\prod B)|\right)} \quad \text{by (viii)}$$

$$
\begin{aligned}
&\leq 2^{\max(|\mathrm{tr\,cl}(A),2^{|\mathrm{tr\,cl}(B)|})}\\
&\leq 2^{2^{\max(|\mathrm{tr\,cl}(A)|,|\mathrm{tr\,cl}(B)|)}}.
\end{aligned}
$$

Finally, for (xi), note that $\mathrm{tr\,cl}(\mathrm{pcf}(A)) = \mathrm{pcf}(A) \cup \bigcup \mathrm{pcf}(A)$. Now $|\mathrm{pcf}(A)| \leq 2^{|A|} \leq 2^{|\mathrm{tr\,cl}(A)|}$ by Theorem 24b.10. □

We also need the fact that some rather complicated formulas and functions are absolute for sets H_Ψ. Note that H_Ψ is transitive. Many of the indicated formulas are not absolute for H_Ψ in general, but only under the assumptions given that Ψ is much larger than the sets in question.

Lemma 24c.4. *Suppose that Ψ is an uncountable regular cardinal. Then the following formulas (as detailed in the proof) are absolute for H_Ψ.*

(i) $B = \mathscr{P}(A)$.

(ii) "D is an ultrafilter on A".

(iii) κ is a cardinal.

(iv) κ is a regular cardinal.

(v) "κ and λ are cardinals, and $\lambda = \kappa^+$".

(vi) $\kappa = |A|$.

(vii) $B = \prod A$.

(viii) $A = {}^BC$.

(ix) "A is infinite", if Ψ is uncountable.

(x) "A is an infinite set of regular cardinals and D is an ultrafilter on A and λ is a regular cardinal and $f \in {}^\lambda \prod A$ and f is strictly increasing and cofinal modulo D", provided that $2^{|\mathrm{tr\,cl}(A)|} < \Psi$.

(xi) "A is an infinite set of regular cardinals, and $B = \mathrm{pcf}(A)$", if $2^{|\mathrm{tr\,cl}(A)|} < \Psi$.

(xii) "A is an infinite set of regular cardinals and $f = \langle J_{<\lambda}[A] : \lambda \in \mathrm{pcf}(A)\rangle$", provided that $2^{|\mathrm{tr\,cl}(A)|} < \Psi$.

(xiii) "A is an infinite set of regular cardinals and $B = \langle B_\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ and $\forall\lambda \in \mathrm{pcf}(A)(B_\lambda$ is a λ-generator)", if $2^{2^{|\mathrm{tr\,cl}(A)|}} < \Psi$.

Proof. Absoluteness follows by easy arguments upon producing suitable formulas, as follows.

(i): Suppose that $A, B \in H_\Psi$. We may take the formula $B = \mathscr{P}(A)$ to be

$$\forall x \in B[\forall y \in x(y \in A)] \wedge \forall x[\forall y \in x(y \in A) \to x \in B].$$

The first part is obviously absolute for H_Ψ. If the second part holds in V it clearly holds in H_Ψ. Now suppose that the second part holds in H_Ψ. Suppose that $x \subseteq A$. Hence $x \in H_\Psi$ and it follows that $x \in B$.

(ii): Assume that $A, D \in H_\Psi$. We can take the statement "D is an ultrafilter on A" to be the following statement:

$$
\begin{aligned}
&\forall X \in D(X \subseteq A) \wedge A \in D \wedge \forall X, Y \in D(X \cap Y \in D) \wedge \emptyset \notin D\\
&\wedge \forall Y \forall X \in D[X \subseteq Y \wedge Y \subseteq A \to Y \in D] \wedge \forall Y[Y \subseteq A \to Y \in D \vee (A \backslash Y) \in D].
\end{aligned}
$$

Again this is absolute because $Y \subseteq A$ implies that $Y \in H_\Psi$.

(iii): Suppose that $\kappa \in H_\Psi$. Then

$$\begin{aligned}\kappa \text{ is a cardinal} \quad \text{iff} \quad & \kappa \text{ is an ordinal and } \forall f[f \text{ is a function and} \\ & \operatorname{dmn}(f) = \kappa \text{ and } \operatorname{rng}(f) \in \kappa \to f \text{ is not one-to-one}].\end{aligned}$$

Note here that if f is a function with $\operatorname{dmn}(f) = \kappa$ and $\operatorname{rng}(f) \subseteq \kappa$, then $f \subseteq \kappa \times \kappa$, and hence $f \in H_\Psi$.

(iv): Assume that $\kappa \in H_\Psi$. Then

$$\begin{aligned}\kappa \text{ is a regular cardinal} \quad \text{iff} \quad & \kappa \text{ is a cardinal, } 1 < \kappa, \text{ and } \forall f[f \text{ is a function} \\ & \text{and } \operatorname{dmn}(f) \in \kappa \text{ and } \operatorname{rng}(f) \subseteq \kappa \text{ and} \\ & \forall \alpha, \beta \in \operatorname{dmn}(f)(\alpha < \beta \to f(\alpha) < f(\beta)) \\ & \to \exists \gamma < \kappa \forall \alpha \in \operatorname{dmn}(f)(f(\alpha) \in \gamma)].\end{aligned}$$

(v): Assume that $\kappa, \lambda \in H_\Psi$. Then ($\kappa$ and λ are cardinals and $\lambda = \kappa^+$) iff

$$\begin{aligned}&\kappa \text{ is a cardinal and } \lambda \text{ is a cardinal and } \kappa < \lambda \\ &\text{and } \forall \alpha < \lambda[\kappa < \alpha \to \exists f[f \text{ is a function and } \operatorname{dmn}(f) = \kappa \\ &\text{and } \operatorname{rng}(f) = \alpha \text{ and } f \text{ is one-one and } \operatorname{rng}(f) = \alpha]].\end{aligned}$$

(vi): Suppose that $\kappa, A \in H_\Psi$. Then

$$\begin{aligned}\kappa = |A| \quad \text{iff} \quad & \kappa \text{ is a cardinal and } \exists f[f \text{ is a function} \\ & \text{and } \operatorname{dmn}(f) = \kappa \text{ and } \operatorname{rng}(f) = A \text{ and } f \text{ is one-to-one}]\end{aligned}$$

(vii): Assume that $A, B \in H_\Psi$. Then

$$\begin{aligned}B = \prod A \quad \text{iff} \quad & \forall f \in B[f \text{ is a function and } \operatorname{dmn}(f) = A \text{ and} \\ & \forall x \in A[f(x) \in x]] \text{ and } \forall f[f \text{ is a function and} \\ & \operatorname{dmn}(f) = A \text{ and } \forall x \in A[f(x) \in x] \to f \in B].\end{aligned}$$

Note that if f is a function with domain A and $f(x) \in x$ for all $x \in A$, then $f \subseteq A \times \bigcup A$, and hence $f \in H_\Psi$.

(viii): Suppose that $A, B, C \in H_\Psi$. Then

$$\begin{aligned}A = {}^B C \quad \text{iff} \quad & \forall f \in A[f \text{ is a function and } \operatorname{dmn}(f) = B \\ & \text{and } \operatorname{rng}(f) \subseteq C] \text{ and } \forall f[f \text{ is a function} \\ & \text{and } \operatorname{dmn}(f) = B \text{ and } \operatorname{rng}(f) \subseteq C \to f \in A].\end{aligned}$$

(ix): "A is infinite" iff $\exists f(f$ is a one-one function, $\operatorname{dmn}(f) = \omega$, and $\operatorname{rng}(f) \subseteq A)$.

(x): Suppose that $A, D, \lambda, f \in H_\Psi$, and $2^{|\mathrm{tr\,cl}(A)|} < \Psi$. Then $\prod A \in H_\Psi$ by Lemma 24c.3(ix). Now

$$\begin{aligned}&A \text{ is an infinite set of regular cardinals and } D \text{ is an ultrafilter on } A\\&\text{and } \lambda \text{ is a regular cardinal and } f \in {}^\lambda\prod A \text{ and } f \text{ is strictly}\\&\text{increasing and cofinal modulo } D\end{aligned}$$

iff

$$\begin{aligned}&A \text{ is infinite and } \forall x \in A[x \text{ is a regular cardinal] and } D \text{ is an ultrafilter on } A \text{ and}\\&\lambda \text{ is a regular cardinal and } \exists B\Big[B = \prod A \text{ and } f \text{ is a function}\\&\qquad \text{and } \mathrm{dmn}(f) = \lambda \text{ and } \mathrm{rng}(f) \subseteq B \text{ and}\\&\qquad \forall \xi, \eta < \lambda \forall X \subseteq A[\forall a \in A[a \in X \Leftrightarrow f_\xi(a) < f_\eta(a)] \to X \in D]\\&\qquad \text{and } \forall g \in B \exists \xi < \lambda \forall X \subseteq A[\forall a \in A[a \in X \Leftrightarrow g(a) < f_\xi(a)] \to X \in D]\Big].\end{aligned}$$

(xi): Assume that $2^{|\mathrm{tr\,cl}(A)|)} < \Psi$, and $A, B \in H_\Psi$. Let $\varphi(A, D, \lambda, f)$ be the statement of (x). Note:

(1) If $\varphi(A, D, \lambda, f)$, then $D, \lambda, f \in H_\Psi$, and $\max(\lambda, |\mathrm{tr\,cl}(A)|) \leq 2^{|\mathrm{tr\,cl}(A)|}$.

In fact, $D \subseteq \mathscr{P}(A)$, so $\mathrm{tr\,cl}(D) \subseteq \mathrm{tr\,cl}(\mathscr{P}(A)) = \mathscr{P}(A) \cup \mathrm{tr\,cl}(A)$, and so $|\mathrm{tr\,cl}(D)| < \Psi$ by Lemma 24c.3(iii); so $D \in H_\Psi$. Now f is a one-one function from λ into $\prod A$, so $\lambda \leq |\prod A| < \Psi$, and hence $\lambda \in H_\Psi$ and $\max(\lambda, |\mathrm{tr\,cl}(A)|) \leq 2^{|\mathrm{tr\,cl}(A)|}$. Finally, $f \subseteq \lambda \times \prod A$, so it follows that $f \in H_\Psi$.

Thus (1) holds. Hence the following equivalence shows the absoluteness of the statement in (xi):

$$A \text{ is an infinite set of regular cardinals and } B = \mathrm{pcf}(A)$$

iff

$$\begin{aligned}&A \text{ is infinite, and } \forall \mu \in A(\mu \text{ is a regular cardinal}) \wedge \forall \lambda \in B \exists D \exists f \varphi(A, D, \lambda, f)\\&\wedge \forall D \forall \lambda \forall f[\varphi(A, D, \lambda, f) \to \lambda \in B].\end{aligned}$$

(xii): Assume that $2^{|\mathrm{tr\,cl}(A)|)} < \Psi$. By Lemma 24c.3(xi) we have $\mathrm{pcf}(A) \in H_\Psi$. Hence

$$A \text{ is an infinite set of regular cardinals } \wedge f = \langle J_{<\lambda}[A] : \lambda \in \mathrm{pcf}(A)\rangle$$

iff

A is infinite and $\forall\kappa\in A(\kappa$ is a regular cardinal and

f is a function and $\exists B[B=\mathrm{pcf}(A)\wedge B=\mathrm{dmn}(f)]$

$\forall\lambda\in\mathrm{dmn}(f)\forall X\subseteq A[A\in f(\lambda)$ iff $\exists C[C=\mathrm{pcf}(X)\wedge C\subseteq\lambda]]$

(xiii): Assume that $2^{2^{|\mathrm{tr\,cl}(A)|}}<\Psi$, and $A,B\in H_\Psi$. Note as above that $\mathrm{pcf}(A)\in H_\Psi$. Note that for any cardinal λ we have $J_{<\lambda}[A]\subseteq\mathscr{P}(A)$ and, with f as in (xi), $f\subseteq\mathrm{pcf}(A)\times\mathscr{P}(\mathscr{P}(A))$; so $f\in H_\Psi$. Let $\varphi(f,A)$ be the formula of (xii). Thus

A is a set of regular cardinals and $B=\langle B_\lambda:\lambda\in\mathrm{pcf}(A)\rangle$

and $\forall\lambda\in\mathrm{pcf}(A)(B_\lambda$ is a λ-generator)

iff

B is a function and $\exists C[C=\mathrm{pcf}(A)\wedge C=\mathrm{dmn}(B)]\wedge\exists f[\varphi(f,A)\wedge$

$\forall\lambda\in\mathrm{dmn}(B)\forall\mu\in\mathrm{dmn}(B)[\lambda$ is a cardinal and μ is a cardinal and

$\mu=\lambda^+\to B_\lambda\in f(\mu)\wedge\forall X\subseteq A[X\in f(\mu)$ iff $X\backslash B_\lambda\in f(\lambda)]]]$ $\square$

Now we turn to the consideration of elementary substructures of H_Ψ. The following lemma gives basic facts used below.

Lemma 24c.5. *Suppose that Ψ is an uncountable cardinal, and N is an elementary substructure of H_Ψ (under $\in$ and a well-order of H_Ψ).*

(i) For every ordinal α, $\alpha\in N$ iff $\alpha+1\in N$.

(ii) $\omega\subseteq N$.

(iii) If $a\in N$, then $\{a\}\in N$.

(iv) If $a,b\in N$, then $\{a,b\},(a,b)\in N$.

(v) If $A,B\in N$, then $A\times B\in N$.

(vi) If $A\in N$ then $\bigcup A\in N$.

(vii) If $f\in N$ is a function, then $\mathrm{dmn}(f),\mathrm{rng}(f)\in N$.

(viii) If $f\in N$ is a function and $a\in N\cap\mathrm{dmn}(f)$, then $f(a)\in N$.

(ix) If $X,Y\in N$, $X\subseteq N$, and $|Y|\le|X|$, then $Y\subseteq N$.

(x) If $X\in N$ and $X\neq\emptyset$, then $X\cap N\neq\emptyset$.

(xi) $\mathscr{P}(A)\in N$ if $A\in N$ and $2^{|\mathrm{tr\,cl}(A)|}<\Psi$.

(xii) If ρ is an infinite ordinal, $|\rho|^+<\Psi$, and $\rho\in N$, then $|\rho|\in N$ and $|\rho|^+\in N$.

(xiii) If $A\in N$, then $\prod A\in N$ if $2^{|\mathrm{tr\,cl}(A)|}<\Psi$.

(xiv) If $A\in N$, A is a set of regular cardinals, and $A\subseteq H_\Psi$, then $\mathrm{pcf}(A)\in N$ if $2^{|\mathrm{tr\,cl}(A)|}<\Psi$.

(xv) If $A\in N$, A is a set of regular cardinals, then $\langle J_{<\lambda}[A]:\lambda\in\mathrm{pcf}(A)\rangle\in N$ if $2^{2^{|\mathrm{tr\,cl}(A)|}}<\Psi$.

(xvi) If $A\in N$ and A is a set of regular cardinals, then there is a function $\langle B_\lambda:\lambda\in\mathrm{pcf}(A)\rangle\in N$, where for each $\lambda\in\mathrm{pcf}(A)$, the set B_λ is a λ-generator, if $2^{2^{|\mathrm{tr\,cl}(A)|}}<\Psi$.

Proof. (i): Let α be an ordinal, and suppose that $\alpha \in N$. Then $\alpha \in H_\Psi$, and hence $\alpha \cup \{\alpha\} \in H_\Psi$. By absoluteness, $H_\Psi \models \exists x(x = \alpha \cup \{\alpha\})$, so $N \models \exists x(x = \alpha \cup \{\alpha\})$. Choose $b \in N$ such that $N \models b = \alpha \cup \{\alpha\}$. Then $H_\Psi \models b = \alpha \cup \{\alpha\}$, so by absoluteness, $b = \alpha \cup \{\alpha\}$. This proves that $\alpha \cup \{\alpha\} \in N$.

The method used in proving (i) can be used in the other parts; so it suffices in most other cases just to indicate a formula which can be used.

(ii): An easy induction, using the formulas $\exists x \forall y \in x(y \neq y)$ and $\exists x[a \subseteq x \wedge a \in x \wedge \forall y \in x[y \in a \vee y = a]]$.

(iii): Use the formula $\exists x[\forall y \in x(y = a) \wedge a \in x]$.

(iv): Similar to (ii).

(v): Use the formula

$$\exists C[\forall a \in A \forall b \in B[(a, b) \in C] \wedge \forall x \in C \exists a \in A \exists b \in B[x = (a, b)]].$$

(vi): Use the formula $\exists B[\forall x \in A[x \subseteq B] \wedge \forall y \in B \exists x \in A(y \in x)]$.

(vii): Use the formula $\exists A[\forall x \forall y[(x, y) \in f \to x \in A] \wedge \forall x \in A \exists y[(x, y) \in f]]$. Note that this formula is absolute for H_Ψ for example $(x, y) \in f \in H_\Psi$ implies that $x, y \in H_\Psi$.

(viii): Use the formula $\exists x[(a, x) \in f]$.

(ix): Let f be a function mapping X onto Y (assuming, as we may, that $Y \neq \emptyset$). Then $f \in H_\Psi$, so by the above method, we get another function $g \in N$ which maps X onto Y. Now (viii) gives the conclusion of (ix).

(x): Use the formula $\exists x \in X[x = x]$.

(xi): $\mathscr{P}(A) \in H_\Psi$ by Lemma 24c.3(iii). Hence we can use the formula

$$\exists B[\forall x \in B(x \subseteq A) \wedge \forall x[x \subseteq A \to x \in B]].$$

(xii): Assume that ρ is an infinite ordinal and $\rho \in N$. Then

$$H_\Psi \models \exists \alpha \leq \rho[(\exists f : \rho \to \alpha,\ \text{a bijection}) \wedge \forall \beta \leq \rho[(\exists g : \rho \to \beta,\ \text{a bijection}) \to \alpha \leq \beta]].$$

Hence by the standard argument, there are $\alpha, f \in N$ such that

$$H_\Psi \models f : \rho \to \alpha \text{ is a bijection } \wedge \forall \beta \leq \rho[(\exists g : \rho \to \beta,\ \text{a bijection}) \to \alpha \leq \beta].$$

Clearly then $\alpha = |\rho|$.

For $|\rho|^+$, use the formula

$$\begin{aligned}\exists \beta \exists \Gamma \Big[&\forall \gamma \in \Gamma \exists f[f \text{ is a bijection from } \rho \text{ onto } \gamma] \\ &\wedge \forall \gamma \forall f[f \text{ is a bijection from } \rho \text{ onto } \gamma \to \gamma \in \Gamma] \\ &\wedge \beta = \bigcup \Gamma \Big].\end{aligned}$$

(xiii): Note that $\prod A \in H_\Psi$ by Lemma 24c.3(ix). Then use the formula

$$\begin{aligned}\exists B \Big[&\forall f \in B(f \text{ is a function } \wedge \operatorname{dmn}(f) = A \wedge \forall a \in A(f(a) \in a)) \\ &\wedge \forall f[f \text{ is a function } \wedge \operatorname{dmn}(f) = A \wedge \forall a \in A(f(a) \in a) \to f \in B] \Big].\end{aligned}$$

(xiv): $\text{pcf}(A) \in H_\Psi$ by Lemma 24c.3(xi), so by Lemma 24c.4(xi) we can use the formula $\exists B[B = \text{pcf}(A)]$.

(xv): We have $\text{pcf}(A) \in H_\Psi$ and $\mathscr{P}(\mathscr{P}(H_\Psi))$ by Lemma 24c.3(iii),(xi). It follows that $\langle J_{<\lambda}[A] : \lambda \in \text{pcf}(A)\rangle \in H_\Psi$. Hence by Lemma 24c.4(xii) we can use the formula $\exists f[f = \langle J_{<\lambda}[A] : \lambda \in \text{pcf}(A)\rangle]$.

(xvi): By Lemma 24c.3(iii),(xi) and Lemma 24c.4(xiii) we can use the formula

$$\exists B[B : \text{pcf}(A) \to \mathscr{P}(A) \wedge \forall \lambda \in \text{pcf}(A)[B_\lambda \text{ is a } \lambda \text{ generator for } A]]. \qquad \square$$

Definition. Let κ be a regular cardinal. An elementary substructure N of H_Ψ is κ-*presentable* iff there is an increasing and continuous chain $\langle N_\alpha : \alpha < \kappa\rangle$ of elementary substructures of H_Ψ such that:

(1) $|N| = \kappa$ and $\kappa + 1 \subseteq N$.

(2) $N = \bigcup_{\alpha<\kappa} N_\alpha$.

(3) For every $\alpha < \kappa$, the function $\langle N_\beta : \beta \leq \alpha\rangle$ is a member of $N_{\alpha+1}$.

It is obvious how to construct a κ-presentable substructure of H_Ψ.

Lemma 24c.6. *If N is a κ-presentable substructure of H_Ψ, with notation as above, and if $\alpha < \kappa$, then $\alpha + \omega \subseteq N_\alpha \in N_{\alpha+1}$.*

Proof. First we show that $\alpha \subseteq N_\alpha$ for all $\alpha < \kappa$, by induction. It is trivial for $\alpha = 0$, and the successor step is immediate from the induction hypothesis and Lemma 24c.5(vii). The limit step is clear.

Now it follows that $\alpha + \omega \subseteq N_\alpha$ by an inductive argument using Lemma 24c.5(i). Finally, $N_\alpha \in N_{\alpha+1}$ by (3) and Lemma 24c.5(viii). $\square$

For any set M, we let $\overline{M}$ be the set of all ordinals α such that $\alpha \in M$ or $M \cap \alpha$ is unbounded in α.

Lemma 24c.7. *If N is a κ-presentable substructure of H_Ψ, with notation as above, then*

(i) If $\alpha < \kappa$, then $\overline{N_\alpha} \subseteq N$.

(ii) If $\kappa < \alpha \in \overline{N}\backslash N$, then α is a limit ordinal and $\text{cf}(\alpha) = \kappa$, *and in fact there is a closed unbounded subset E of α such that $E \subseteq N$ and E has order type κ.*

Proof. First we consider (i). Suppose that $\gamma \in \overline{N}_\alpha$. We may assume that $\gamma \notin N_\alpha$.

Case 1. $\gamma = \sup(N_\alpha \cap \mathbf{On})$. Then

$$H_\Psi \models \exists\gamma'[\forall\delta(\delta \in N_\alpha \to \delta \leq \gamma') \wedge \forall\varepsilon[\forall\delta(\delta \in N_\alpha \to \delta \leq \varepsilon) \to \gamma' \leq \varepsilon]];$$

in fact, our given γ is the unique γ' for which this holds. Hence this statement holds in N, as desired.

Case 9. $\exists\theta \in N_\alpha(\gamma < \theta)$. We may assume that θ is minimum with this property. Now for any $\beta \in N_\alpha$ we can let $\rho(\beta)$ be the supremum of all ordinals in N_α which are less than β. So $\rho(\theta) = \gamma$. By absoluteness we get

$$\begin{aligned} H_\Psi \models &\forall\beta \in N_\alpha \exists\rho[\forall\varepsilon \in N_\alpha(\varepsilon < \beta \to \varepsilon < \rho) \\ &\wedge \forall\chi[\forall\varepsilon \in N_\alpha(\varepsilon < \beta \to \varepsilon < \chi) \to \rho \leq \chi]]; \end{aligned}$$

Hence N models this formula too; applying it to θ in place of β, we get $\rho \in N$ such that

$$\begin{aligned} N \models & \forall \varepsilon \in N_\alpha(\varepsilon < \theta \to \varepsilon < \rho) \\ & \wedge \forall \chi[\forall \varepsilon \in N_\alpha(\varepsilon < \theta \to \varepsilon < \chi) \to \rho \leq \chi]. \end{aligned}$$

Thus $\gamma = \rho \in N$, as desired. This proves (i).

For (ii), suppose that $\kappa < \alpha \in \overline{N}\backslash N$. Let $E = \{\sup(\alpha \cap N_\xi) : \xi < \kappa\}$. Note that if $\xi < \kappa$, then by (i), $\sup(\alpha \cap N_\xi) \in N$. So $E \subseteq N$. It is clearly closed in α. It is unbounded, since for any $\beta \in \alpha \cap N$ there is a $\xi < \kappa$ such that $\beta \in N_\xi$, and so $\beta \leq \sup(\alpha \cap N_\xi) \in N$. □

For any set N we define the *characteristic function* of N; it is defined for each regular cardinal μ as follows:

$$\mathrm{Ch}_N(\mu) = \sup(N \cap \mu).$$

Proposition 24c.8. *Let κ be a regular cardinal, let N be a κ-presentable substructure of H_Ψ, and let μ be a regular cardinal.*

(i) If $\mu \leq \kappa$, then $\mathrm{Ch}_N(\mu) = \mu \in N$.

(ii) If $\kappa < \mu$, then $\mathrm{Ch}_N(\mu) \notin N$, $\mathrm{Ch}_N(\mu) < \mu$, and $\mathrm{Ch}_N(\mu)$ has cofinality κ.

(iii) For every $\alpha \in \overline{N} \cap \mu$ we have $\alpha \leq \mathrm{Ch}_N(\mu)$.

Proof. (i): True since $\kappa + 1 \subseteq N$.

(ii): Since $|N| = \kappa < \mu$ and μ is regular, we must have $\mathrm{Ch}_N(\mu) \notin N$ and $\mathrm{Ch}_N(\mu) < \mu$. Then $\mathrm{Ch}_N(\mu)$ has cofinality κ by Lemma 24c.7.

(iii): clear. □

Theorem 24c.9. *Suppose that M and N are elementary substructures of H_Ψ and $\kappa < \mu$ are cardinals, with $\mu < \Psi$.*

(i) If $M \cap \kappa \subseteq N \cap \kappa$ and $\sup(M \cap \nu^+) = \sup(M \cap N \cap \nu^+)$ for every successor cardinal $\nu^+ \leq \mu$ such that $\nu^+ \in M$, then $M \cap \mu \subseteq N \cap \mu$.

(ii) If M and N are both κ-presentable and if $\sup(M \cap \nu^+) = \sup(N \cap \nu^+)$ for every successor cardinal $\nu^+ \leq \mu$ such that $\nu^+ \in M$, then $M \cap \mu = N \cap \mu$.

Proof. (i): Assume the hypothesis. We prove by induction on cardinals δ in the interval $[\kappa, \mu]$ that $M \cap \delta \subseteq N \cap \delta$. This is given for $\delta = \kappa$. If, inductively, δ is a limit cardinal, then the desired conclusion is clear. So assume now that δ is a cardinal, $\kappa \leq \delta < \mu$, and $M \cap \delta \subseteq N \cap \delta$. If $\delta^+ \notin M$, then by Lemma 24c.5(xii), $[\delta, \delta^+] \cap M = \emptyset$, so the desired conclusion is immediate from the inductive hypothesis. So, assume that $\delta^+ \in M$. Then the hypothesis of (i) implies that there are ordinals in $[\delta, \delta^+]$ which are in $M \cap N$, and hence by Lemma 24c.5(xii) again, $\delta^+ \in N$. Now to show that $M \cap [\delta, \delta^+] \subseteq N \cap [\delta, \delta^+]$, take any ordinal $\gamma \in M \cap [\delta, \delta^+]$. We may assume that $\gamma < \delta^+$. Since $\sup(M \cap \delta^+) = \sup(M \cap N \cap \delta^+)$ by assumption, we can choose $\beta \in M \cap N \cap \delta^+$ such that $\gamma < \beta$. Let f be the $<^*$-smallest bijection from β to δ. So $f \in M \cap N$. Since $\gamma \in M$, we also have $f(\gamma) \in M$ by Lemma 24c.5(viii). Now $f(\gamma) < \delta$, so by the inductive assumption that $M \cap \delta \subseteq N \cap \delta$, we have $f(\gamma) \in N$. Since $f \in N$, so is f^{-1}, and $f^{-1}(f(\gamma)) = \gamma \in N$, as desired. This finishes the proof of (i).

(ii): Assume the hypothesis. Now we want to check the hypothesis of (i). By the definition of κ-presentable we have $\kappa = M \cap \kappa = N \cap \kappa$. Now suppose that ν is a cardinal and $\nu^+ \leq \mu$ with $\nu^+ \in M$. We may assume that $\kappa < \nu^+$. Let $\gamma = \mathrm{Ch}_M(\nu^+)$; this is the same as $\mathrm{Ch}_N(\nu^+)$ by the hypothesis of (ii). By Lemma 24c.8 we have $\gamma \notin M \cup N$; hence by Lemma 24c.7 there are clubs P, Q in γ such that $P \subseteq M$ and $Q \subseteq N$. Hence $\sup(M \cap \nu^+) = \sup(M \cap \nu^+) = \sup(M \cap N \cap \nu^+)$. This verifies the hypothesis of (i) for the pair M, N and also for the pair N, M. So our conclusion follows. $\square$

Minimally obedient sequences

Suppose that A is progressive, $\lambda \in \mathrm{pcf}(A)$, and B is a λ-generator for A. A sequence $\langle f_\xi : \xi < \lambda \rangle$ of members of $\prod A$ is called *persistently cofinal* for λ, B provided that $\langle (f_\xi \restriction B) : \xi < \lambda \rangle$ is persistently cofinal in $(\prod B, <_{J_{<\lambda}[B]})$. Recall that this means that for all $h \in \prod B$ there is a $\xi_0 < \lambda$ such that for all ξ, if $\xi_0 \leq \xi < \lambda$, then $h <_{J_{<\lambda}[B]} (f_\xi \restriction B)$.

Lemma 24c.10. *Suppose that A is progressive, $\lambda \in \mathrm{pcf}(A)$, and B and C are λ-generators for A. A sequence $\langle f_\xi : \xi < \lambda \rangle$ of members of $\prod A$ is persistently cofinal for λ, B iff it is persistently cofinal for λ, C.*

Proof. Suppose that $\langle f_\xi : \xi < \lambda \rangle$ is persistently cofinal for λ, B, and suppose that $h \in \prod C$. Let $k \in \prod B$ be any function such that $h \restriction (B \cap C) = k \restriction (B \cap C)$. Choose $\xi_0 < \lambda$ such that for all $\xi \in [\xi_0, \lambda)$ we have $k <_{J_{<\lambda}[B]} (f_\xi \restriction B)$. Then for any $\xi \in [\xi_0, \lambda)$ we have

$$\begin{aligned}\{a \in C : h(a) \geq f_\xi(a)\} &= \{a \in B \cap C : h(a) \geq f_\xi(a)\} \cup \{a \in C \backslash B : h(a) \geq f_\xi(a)\} \\ &\subseteq \{a \in B : k(a) \geq f_\xi(a)\} \cup (C \backslash B);\end{aligned}$$

Now $(C \backslash B) \in J_{<\lambda}[A]$ by Lemma 24b.25(xi), so $h <_{J_{<\lambda}[C]} (f_\xi \restriction C)$. By symmetry the lemma follows. $\square$

Because of this lemma we say that f is persistently cofinal for λ iff it is persistently cofinal for λ, B for some λ-generator B.

Lemma 24c.11. *Suppose that A is progressive, $\lambda \in \mathrm{pcf}(A)$, and $f \stackrel{\mathrm{def}}{=} \langle f_\xi : \xi < \lambda \rangle$ is universal for λ. Then f is persistently cofinal for λ.*

Proof. Let B be a λ-generator. Then by Lemma 24b.25(vii), λ is the largest member of $\mathrm{pcf}(B)$. By Lemma 24b.17, $\langle (f_\xi \restriction B) : \xi < \lambda \rangle$ is strictly increasing under $<_{J_{<\lambda}[B]}$, and by Lemma 24b.25(v) it is cofinal in $(\prod B, <_{J_{<\lambda}[B]})$. By Proposition 24b.11, it is thus persistently cofinal in $(\prod B, <_{J_{<\lambda}[B]})$. $\square$

Lemma 24c.12. *Suppose that A is progressive, $\lambda \in \mathrm{pcf}(A)$, and $A \in N$, where N is a κ-presentable elementary substructure of H_Ψ, with $|A| < \kappa < \min(A)$ and $2^{|\mathrm{tr\,cl}(A)|} < \Psi$. Suppose that $f = \langle f_\xi : \xi < \lambda \rangle$ is a sequence of functions in $\prod A$.*

Then for every $\xi < \lambda$ there is an $\alpha < \kappa$ such that for any $a \in A$,

$$f_\xi(a) < \mathrm{Ch}_N(a) \quad \text{iff} \quad f_\xi(a) < \mathrm{Ch}_{N_\alpha}(a).$$

Proof.

$$
\begin{aligned}
\mathrm{Ch}_N(a) &= \sup(N \cap a) \\
&= \bigcup (N \cap a) \\
&= \bigcup \left(a \cap \bigcup_{\alpha<\kappa} N_\alpha \right) \\
&= \bigcup_{\alpha<\kappa} \bigcup (N_\alpha \cap a) \\
&= \bigcup_{\alpha<\kappa} \mathrm{Ch}_{N_\alpha}(a).
\end{aligned}
$$

Hence for every $a \in A$ for which $f_\xi(a) < \mathrm{Ch}_N(a)$, there is an $\alpha_a < \kappa$ such that $f_\xi(a) < Ch_{N_{\alpha_a}}(a)$. Hence the existence of α as indicated follows. $\square$

Lemma 24c.13. *Suppose that A is progressive, κ is regular, $\lambda \in \mathrm{pcf}(A)$, and $A, \lambda \in N$, where N is a κ-presentable elementary substructure of H_Ψ, with $|A| < \kappa < \min(A)$ and Ψ is big. Suppose that $f = \langle f_\xi : \xi < \lambda \rangle \in N$ is a sequence of functions in $\prod A$ which is persistently cofinal in λ. Then for every $\xi \geq \mathrm{Ch}_N(\lambda)$ the set*

$$\{a \in A : \mathrm{Ch}_N(a) \leq f_\xi(a)\}$$

is a λ-generator for A.

Proof. Assume the hypothesis, including $\xi \geq \mathrm{Ch}_N(\lambda)$. Let α be as in Lemma 24c.19. We are going to apply Lemma 24b.25(ix). Since $A, f, \lambda \in N$, we may assume that $A, f, \lambda \in N_0$, by renumbering the elementary chain if necessary. Now $\kappa \subseteq N$, and $|A| < \kappa$, so we easily see that there is a bijection $f \in N$ mapping an ordinal $\alpha < \kappa$ onto A; hence $A \subseteq N$ by Lemma 24c.5(viii), and so $A \subseteq N_\beta$ for some $\beta < \kappa$. We may assume that $A \subseteq N_0$. By Lemma 24c.5(xvi),(viii), there is a λ-generator B which is in N_0.

Now the sequence f is persistently cofinal in $\prod B/J_{<\lambda}$, and hence

$$
\begin{aligned}
H_\Psi &\models \forall h \in \prod B \exists \eta < \lambda \forall \rho \geq \eta [h \restriction B <_{J_{<\lambda}} f_\rho \restriction B]; \quad \text{hence} \\
N &\models \forall h \in \prod B \exists \eta < \lambda \forall \rho \geq \eta [h \restriction B <_{J_{<\lambda}} f_\rho \restriction B];
\end{aligned}
$$

Hence for every $h \in N$, if $h \in \prod B$ then there is an $\eta < \lambda$ with $\eta \in N$ such that $N \models \forall \rho \geq \eta [h \restriction B <_{J_{<\lambda}} f_\varphi \restriction B]$; going up, we see that really for every $h \in N \cap \prod A$ there is an $\eta_h \in N \cap \lambda$ such that for all ρ with $\rho \geq \eta_h$ we have $h \restriction B <_{J_{<\lambda}} f_\rho \restriction B$. Since ξ, as given in the statement of the Lemma, is $\geq$ each member of $N \cap \lambda$, hence $\geq \eta_h$ for each $h \in N \cap \prod A$, we see that

(1) $$h \restriction B <_{J_{<\lambda}} f_\xi \restriction B \text{ for every } h \in N \cap \prod A.$$

Now we can apply (1) to $h = \mathrm{Ch}_{N_\alpha}$, since this function is clearly in N. So $\mathrm{Ch}_{N\alpha} \restriction B <_{J_{<\lambda}[B]} f_\xi \restriction B$. Hence by the choice of α (see Lemma 24c.12)

(2) $$\mathrm{Ch}_N \restriction B \leq_{J_{<\lambda}[B]} f_\xi \restriction B.$$

Note that (2) says that $B\backslash\{a \in A : \mathrm{Ch}_N(a) \leq f_\xi(a)\} \in J_{<\lambda}[A]$.

Now $\lambda \notin \mathrm{pcf}(A\backslash B)$ by Lemma 24b.25(ii), and hence $J_{<\lambda}[A\backslash B] = J_{\leq\lambda}[A\backslash B]$. So by Theorem 24b.8 we see that $\prod(A\backslash B)/J_{<\lambda}[A\backslash B]$ is λ^+-directed, so $\langle f_\xi \restriction (A\backslash B) : \xi < \lambda\rangle$ has an upper bound $h \in \prod(A\backslash B)$. We may assume that $h \in N$, by the usual argument. Hence

$$f_\xi \restriction (A\backslash B) <_{J_{<\lambda}[A\backslash B]} h < \mathrm{Ch}_N \restriction (A\backslash B);$$

hence $\{a \in A\backslash B : \mathrm{Ch}_N(a) \leq f_\xi(a)\} \in J_{<\lambda}[A]$, and together with (2) and using Lemma 24b.25(ix) this finishes the proof. $\square$

Now suppose that A is progressive, δ is a limit ordinal, $f = \langle f_\xi : \xi < \delta\rangle$ is a sequence of members of $\prod A$, $|A|^+ \leq \mathrm{cf}(\delta) < \min(A)$, and E is a club of δ of order type $\mathrm{cf}(\delta)$. Then we define

$$h_E = \sup\{f_\xi : \xi \in E\}.$$

We call h_E the *supremum along E of f*. Thus $h_E \in \prod A$, since $\mathrm{cf}(\delta) < \min(A)$. Note that if $E_1 \subseteq E_2$ then $h_{E_1} \leq h_{E_2}$.

Lemma 24c.14. *Let A, δ, f be as above. Then there is a unique function g in $\prod A$ such that the following two conditions hold.*

(i) There is a club C of δ of order type $\mathrm{cf}(\delta)$ *such that $g = h_C$.*

(ii) If E is any club of C of order type $\mathrm{cf}(\delta)$*, then $g \leq h_E$.*

Proof. Clearly such a function g is unique if it exists.

Now suppose that there is no such function g. Then for every club C of δ of order type $\mathrm{cf}(\delta)$ there is a club D of order type $\mathrm{cf}(\delta)$ such that $h_C \not\leq h_D$, hence $h_C \not\leq h_{C\cap D}$. Hence there is a decreasing sequence $\langle E_\alpha : \alpha < |A|^+\rangle$ of clubs of δ such that for every $\alpha < |A|^+$ we have $h_{E_\alpha} \not\leq h_{E_{\alpha+1}}$. Now note that

$$|A|^+ = \bigcup_{a\in A}\{\alpha < |A|^+ : h_{E_\alpha}(a) > h_{E_{\alpha+1}}(a)\}.$$

Hence there is an $a \in A$ such that $M \stackrel{\mathrm{def}}{=} \{\alpha < |A|^+ : h_{E_\alpha}(a) > h_{E_{\alpha+1}}(a)\}$ has size $|A|^+$. Now $h_{E_\alpha}(a) \geq h_{E_\beta}(a)$ whenever $\alpha < \beta < |A|^+$, so this gives an infinite decreasing sequence of ordinals, contradiction. $\square$

The function g of this lemma is called the *minimal club-obedient bound* of f.

Corollary 24c.15. *Suppose that A is progressive, δ is a limit ordinal, $f = \langle f_\xi : \xi < \delta\rangle$ is a sequence of members of $\prod A$, $|A|^+ \leq \mathrm{cf}(\delta) < \min(A)$, J is an ideal on A, and f is $<_J$-increasing. Let g be the minimal club-obedient bound of f. Then g is a $\leq_J$-bound for f.* $\square$

Now suppose that A is progressive, $\lambda \in \mathrm{pcf}(A)$, and κ is a regular cardinal such that $|A| < \kappa < \min(A)$. We say that $f = \langle f_\alpha : \alpha < \lambda\rangle$ is *κ-minimally obedient for λ* iff f is a universal sequence for λ and for every $\delta < \lambda$ of cofinality κ, f_δ is the minimal club-obedient bound of f.

A sequence f is *minimally obedient for* λ iff $|A|^+ < \min(A)$ and f is minimally obedient for every regular κ such that $|A| < \kappa < \min(A)$.

Lemma 24c.16. *Suppose that* $|A|^+ < \min(A)$ *and* $\lambda \in \mathrm{pcf}(A)$*. Then there is a minimally obedient sequence for* λ*.*

Proof. By Theorem 24b.18, let $\langle f^0_\xi : \xi < \lambda\rangle$ be a universal sequence for λ. Now by induction we define functions f_ξ for $\xi < \lambda$. Let $f_0 = f^0_0$, and choose $f_{\xi+1}$ so that $\max(f_\xi, f^0_\xi) < f_{\xi+1}$.

For limit $\delta < \lambda$ such that $|A| < \mathrm{cf}(\delta) < \min(A)$, let f_δ be the minimally club-obedient bound of $\langle f_\xi : \xi < \delta\rangle$.

For other limit $\delta < \lambda$, use the λ-directedness (Theorem 24b.8) to get f_δ as a $<_{J_{<\lambda}}$-bound of $\langle f_\xi : \xi < \delta\rangle$.

Thus we have assured the minimally obedient property, and it is clear that $\langle f_\xi : \xi < \lambda\rangle$ is universal. $\square$

Lemma 24c.17. *Suppose that* A *is progressive, and* κ *is a regular cardinal such that* $|A| < \kappa < \min(A)$*. Also assume the following:*

(i) $\lambda \in \mathrm{pcf}(A)$*.*

(ii) $f = \langle f_\xi : \xi < \lambda\rangle$ *is a* κ*-minimally obedient sequence for* λ*.*

(iii) N *is a* κ*-presentable elementary substructure of* H_Ψ*, with* Ψ *large, such that* $\lambda, f, A \in N$*.*

Then the following conditions hold:

(iv) For every $\gamma \in \overline{N} \cap \lambda \backslash N$ *we have:*

(a) $\mathrm{cf}(\gamma) = \kappa$*.*

(b) There is a club C *of* γ *of order type* κ *such that* $f_\gamma = \sup\{f_\xi : \xi \in C\}$ *and* $C \subseteq N$*.*

(c) $f_\gamma(a) \in \overline{N} \cap a$ *for every* $a \in A$*.*

(v) If $\gamma = \mathrm{Ch}_N(\lambda)$*, then:*

(a) $\gamma \in \overline{N} \cap \lambda \backslash N$*; hence we let* C *be as in (iv)(b), with* $f_\gamma = \sup\{f_\xi : \xi \in C\}$*.*

(b) $f_\xi \in N$ *for each* $\xi \in C$*.*

(c) $f_\gamma \le (\mathrm{Ch}_N \restriction A)$*.*

(vi) $\gamma = \mathrm{Ch}_N(\lambda)$ *and* C *is as in (iv)(b), with* $f_\gamma = \sup\{f_\xi : \xi \in C\}$*, and* B *is a* λ *generator, then for every* $h \in N \cap \prod A$ *there is a* $\xi \in C$ *such that* $(h \restriction B) <_{J_{<\lambda}} (f_\xi \restriction B)$*.*

Proof. Assume (i)–(iii). Note that $A \subseteq N$, by Lemma 24c.5(ix).

For (iv), suppose also that $\gamma \in \overline{N} \cap \lambda \backslash N$. Then by Lemma 24c.7 we have $\mathrm{cf}(\gamma) = \kappa$, and there is a club E in γ of order type κ such that $E \subseteq N$. By (ii), we have $f_\gamma = f_C$ for some club C of γ of order type κ. By the minimally obedient property we have $f_C = f_{C\cap E}$, and thus we may assume that $C \subseteq E$. For any $\xi \in C$ and $a \in A$ we have $f_\xi(a) \in N$ by Lemma 24c.5(viii). So (iv) holds.

For (v), suppose that $\gamma = \mathrm{Ch}_N(\lambda)$. Then $\gamma \in \overline{N} \cap \lambda \backslash N$ because $|N| = \kappa < \min(A) \le \lambda$. For each $\xi \in C$ we have $f_\xi \in N$ by Lemma 24c.5(viii). For (c), if $a \in A$, then $f_\gamma(a) = \sup_{\xi\in C} f_\xi(a) \le \mathrm{Ch}_N(a)$, since $f_\xi(a) \in N \cap a$ for all $\xi \in C$.

Next, assume the hypotheses of (vi). By Lemma 24c.11, f is persistently cofinal in λ, so by Lemma 24c.13, B' is a λ-generator. By Lemma 24b.25(v) there is a $\xi \in C$ such that $h \upharpoonright B' <_{J_{<\lambda}} f_\xi \upharpoonright B'$. Now $B =_{J_{<\lambda}[A]} B'$ by Lemma 24b.25(xi), so

$$\{a \in B : h(a) \geq f_\xi(b)\} \subseteq (B \backslash B') \cup \{a \in B' : h(a) \geq f_\xi(b)\} \in J_{<\lambda}[A]. \qquad \square$$

We now define some abbreviations.

$H_1(A, \kappa, N, \Psi)$ abbreviates

A is a progressive set of regular cardinals, κ is a regular cardinal such that $|A| < \kappa <$ $\min(A)$, and N is a κ-presentable elementary substructure of H_Ψ, with Ψ big and $A \in N$.

$H_2(A, \kappa, N, \Psi, \lambda, f, \gamma)$ abbreviates

$H_1(A, \kappa, N, \Psi)$, $\lambda \in \mathrm{pcf}(A)$, $f = \langle f_\xi : \xi < \lambda \rangle$ is a sequence of members of $\prod A$, $f \in N$, and $\gamma = \mathrm{Ch}_N(\lambda)$.

$P_1(A, \kappa, N, \Psi, \lambda, f, \gamma)$ abbreviates

$H_2(A, \kappa, N, \Psi, \lambda, f, \gamma)$ and $\{a \in A : \mathrm{Ch}_N(a) \leq f_\gamma(a)\}$ is a λ-generator.

$P_2(A, \kappa, N, \Psi, \lambda, f, \gamma)$ abbreviates

$H_2(A, \kappa, N, \Psi, \lambda, f, \gamma)$ and the following hold:
(i) $f_\gamma \leq (\mathrm{Ch}_N \upharpoonright A)$.
(ii) For every $h \in N \cap \prod A$ there is a $d \in N \cap \prod A$ such that for any λ-generator B,

$$(h \upharpoonright B) <_{J_{<\lambda}} (d \upharpoonright B) \quad \text{and} \quad d \leq f_\gamma.$$

Thus $H_1(A, \kappa, N, \Psi)$ is part of the hypothesis of Lemma 24c.17, and $H_2(A, \kappa, N, \Psi, \lambda, f, \gamma)$ is a part of the hypotheses of Lemma 24c.17(v).

Lemma 24c.18. *If $H_2(A, \kappa, N, \Psi, \lambda, f, \gamma)$ holds and f is persistently cofinal for λ, then $P_1(A, \kappa, N, \Psi, \lambda, f, \gamma)$ holds.*

Proof. This follows immediately from Lemma 24c.13. $\square$

Lemma 24c.19. *If $H_2(A, \kappa, N, \Psi, \lambda, f, \gamma)$ holds and f is κ-minimally obedient for λ, then both $P_1(A, \kappa, N, \Psi, \lambda, f, \gamma)$ and $P_2(A, \kappa, N, \Psi, \lambda, f, \gamma)$ hold.*

Proof. Since f is κ-minimally obedient for λ, it is a universal sequence for λ, by definition. Hence by Lemma 24c.11 f is persistently cofinal for λ, and so property P_1 follows from Lemma 24c.18.

For P_2, note that $\lambda, A \in N$ since $f \in N$, by Lemma 24c.5(vii),(ix). Hence the hypotheses of Lemma 24c.17(v) hold. So (i) in P_2 holds by Lemma 24c.17(v)(c). For condition (ii), suppose that $h \in N \cap \prod A$. Take B and C as in Lemma 24c.17(vi), and choose $\xi \in C$ such that $h \upharpoonright B <_{J_{<\lambda}} f_\xi \upharpoonright B$. Let $d = f_\xi$. Clearly this proves condition (ii). $\square$

The following obvious extension of Lemma 24c.19 will be useful below.

Lemma 24c.20. *Assume $H_1(A,\kappa,N,\Psi)$, and also assume that $\gamma = \mathrm{Ch}_N(\lambda)$ and*

(i) $f \stackrel{\text{def}}{=} \langle f^\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ is a sequence of sequences $\langle f^\lambda_\xi : \xi < \lambda\rangle$ each of which is a κ-minimally obedient for λ.

Then for each $\lambda \in N \cap \mathrm{pcf}(A)$, $P_1(A,\kappa,N,\Psi,\lambda,f^\lambda,\gamma)$ and $P_2(A,\kappa,N,\Psi,\lambda,f^\lambda,\gamma)$ hold.

□

Lemma 24c.21. *Suppose that $P_1(A,\kappa,N,\Psi,\lambda,f,\gamma)$ and $P_2(A,\kappa,N,\Psi,\lambda,f,\gamma)$ hold. Then*

(i) $\{a \in A : \mathrm{Ch}_N(a) = f_\gamma(a)\}$ is a λ-generator.

(ii) If $\lambda = \max(\mathrm{pcf}(A))$, then

$$< (f_\gamma, \mathrm{Ch}_N \restriction A) = \{a \in A : f_\gamma(a) < \mathrm{Ch}_N(a)\} \in J_{<\lambda}[A].$$

Proof. By (i) of $P_2(A,\kappa,N,\Psi,\lambda,f,\gamma)$ we have $f_\gamma \leq (\mathrm{Ch}_N \restriction A)$, so (i) holds by $P_1(A,\kappa,N,\Psi,\lambda,f,\gamma)$. (ii) follows from $P_1(A,\kappa,N,\Psi,\lambda,f,\gamma)$ and Lemma 24b.25(xii).

□

Lemma 24c.22. *Assume that $P_1(A,\kappa,N,\Psi,\lambda,f,\gamma)$ and $P_2(A,\kappa,N,\Psi,\lambda,f,\gamma)$ hold. Let*

$$b = \{a \in A : \mathrm{Ch}_N(a) = f_\gamma(a)\}.$$

Then

(i) b is a λ-generator.

(ii) There is a set $b' \subseteq b$ such that:

(a) $b' \in N$;

(b) $b \backslash b' \in J_{<\lambda}[A]$;

(c) b' is a λ-generator.

Proof. (i) holds by Lemma 24c.21(i). For (ii), by Lemma 24c.12 choose $\alpha < \kappa$ such that, for every $a \in A$,

(1) $$f_\gamma(a) < \mathrm{Ch}_N(a) \quad \text{iff} \quad f_\gamma(a) < \mathrm{Ch}_{N_\alpha}(a).$$

Now by (i) of $P_2(A,\kappa,N,\Psi,\lambda,f,\gamma)$ we have $f_\gamma \leq (\mathrm{Ch}_N \restriction A)$. Hence by (1) we see that for every $a \in A$,

(2) $$a \in b \quad \text{iff} \quad \mathrm{Ch}_{N_\alpha}(a) \leq f_\gamma(a).$$

Now by (ii) of $P_2(A,\kappa,N,\Psi,\lambda,f,\gamma)$ applied to $h = \mathrm{Ch}_{N_\alpha} \restriction A$, there is a $d \in N \cap \prod A$ such that the following conditions hold:

(3) $(\mathrm{Ch}_{N_\alpha} \restriction b) <_{J_{<\lambda}} (d \restriction b)$.

(4) $d \leq f_\gamma$.

Now we define

$$b' = \{a \in A : \mathrm{Ch}_{N_\alpha}(a) \leq d(a)\}.$$

Clearly $b' \in N$. Also, by (3),

$$b \backslash b' = \{a \in b : d(a) < \mathrm{Ch}_{N_\alpha}(a)\} \in J_{<\lambda},$$

and so (ii)(b) holds. Thus $b \subseteq_{J_{<\lambda}} b'$. If $a \in b'$, then $\mathrm{Ch}_{N_\alpha}(a) \leq d(a) \leq f_\gamma(a)$ by (4), so $a \in b$ by (2). Thus $b' \subseteq b$. Now (ii)(c) holds by Lemma 24b.25(ix). $\square$

Lemma 24c.23. *Assume $H_1(A, \kappa, N, \Psi)$ and $A \in N$. Suppose that $\langle f^\lambda : \lambda \in \mathrm{pcf}(A)\rangle \in N$ is an array of sequences $\langle f^\lambda_\xi : \xi < \lambda\rangle$ with each $f^\lambda_\xi \in \prod A$. Also assume that for every $\lambda \in N \cap \mathrm{pcf}(A)$, both $P_1(A, \kappa, N, \Psi, \lambda, f^\lambda, \gamma(\lambda))$ and $P_2(A, \kappa, N, \Psi, \lambda, f^\lambda, \gamma(\lambda))$ hold.*

Then there exist cardinals $\lambda_0 > \lambda_1 > \cdots > \lambda_n$ in $\mathrm{pcf}(A) \cap N$ such that

$$(\mathrm{Ch}_N \restriction A) = \sup\{f^{\lambda_0}_{\gamma(\lambda_0)}, \ldots, f^{\lambda_n}_{\gamma(\lambda_n)}\}.$$

Proof. We will define by induction a descending sequence of cardinals $\lambda_i \in \mathrm{pcf}(A) \cap N$ and sets $A_i \in \mathscr{P}(A) \cap N$ (strictly decreasing under inclusion as i grows) such that if $A_i \neq \emptyset$ then $\lambda_i = \max(\mathrm{pcf}(A_i))$ and

(1) $$(\mathrm{Ch}_N \restriction (A \backslash A_{i+1})) = \sup\{(f^{\lambda_0}_{\gamma(\lambda_0)} \restriction (A \backslash A_{i+1})), \ldots, (f^{\lambda_i}_{\gamma(\lambda_i)} \restriction (A \backslash A_{i+1}))\}.$$

Since the cardinals are decreasing, there is a first i such that $A_{i+1} = \emptyset$, and then the lemma is proved. To start, $A_0 = A$ and $\lambda_0 = \max(\mathrm{pcf}(A))$. Clearly $\lambda_0 \in N$. Now suppose that λ_i and A_i are defined, with $A_i \neq 0$. By Lemma 24c.22(i) and Lemma 24b.25(x), the set

$$\{a \in A \cap (\lambda_i + 1) : \mathrm{Ch}_N(a) = f^{\lambda_i}_{\gamma(\lambda_i)}(a)\}$$

is a λ_i-generator. Hence by Lemma 24c.22(ii) we get another λ_i-generator b'_{λ_i} such that

(2) $b'_{\lambda_i} \in N$.

(3) $b'_{\lambda_i} \subseteq \{a \in A \cap (\lambda_i + 1) : \mathrm{Ch}_N(a) = f^{\lambda_i}_{\gamma(\lambda_i)}(a)\}$.

Note that $b'_{\lambda_i} \neq \emptyset$. Let $A_{i+1} = A_i \backslash b'_{\lambda_i}$. Thus $A_{i+1} \in N$. Furthermore,

(4) $A \backslash A_{i+1} = (A \backslash A_i) \cup b'_{\lambda_1}$.

Now by Lemma 9.25(ii) and $\lambda_i = \max(\mathrm{pcf}(A_i))$ we have $\lambda_i \notin \mathrm{pcf}(A_{i+1})$. If $A_{i+1} \neq \emptyset$, we let $\lambda_{i+1} = \max(\mathrm{pcf}(A_{i+1}))$. Now by (i) of $P_2(A, \kappa, N, \Psi, \lambda, f^{\lambda_j}, \gamma(\lambda_j))$ we have

(5) $f^{\lambda_j}_{\gamma(\lambda_j)} \leq (\mathrm{Ch}_N \restriction A)$ for all $j \leq i$.

Now suppose that $a \in A \backslash A_{i+1}$. If $a \in A_i$, then by (4), $a \in b'_{\lambda_1}$, and so by (3), $\mathrm{Ch}_N(a) = f^{\lambda_i}_{\gamma(\lambda_1)}(a)$, and (1) holds for a. If $a \notin A_i$, then $A \neq A_i$, so $i \neq 0$. Hence by the inductive hypothesis for (1),

$$\mathrm{Ch}_N(a) = \sup\{f^{\lambda_0}_{\gamma(\lambda_0)}(a), \ldots, f^{\lambda_{i-1}}_{\gamma(\lambda_{i-1})}(a)\},$$

and (1) for a follows by (5). □

The cofinality of $([\mu]^\kappa, \subseteq)$

First we give some simple properties of the sets $[\mu]^\kappa$, not involving pcf theory.

Proposition 24c.24. *If $\kappa \le \mu$ are infinite cardinals, then*

$$(*) \qquad |[\mu]^\kappa| = \mathrm{cf}([\mu]^\kappa, \subseteq) \cdot 2^\kappa.$$

Proof. Let $\lambda = \mathrm{cf}([\mu]^\kappa, \subseteq)$, and let $\langle Y_i : i < \lambda\rangle$ be an enumeration of a cofinal subset of $\mathrm{cf}([\mu]^\kappa, \subseteq)$. For each $i < \lambda$ let f_i be a bijection from Y_i to κ. Now the inequality $\ge$ in $(*)$ is clear. For the other direction, we define an injection g of $[\mu]^\kappa$ into $\lambda \times \mathscr{P}(\kappa)$, as follows. Given $E \in [\mu]^\kappa$, let $i < \lambda$ be minimum such that $E \subseteq Y_i$, and define $g(E) = (i, f_i[E])$. Clearly g is one-one. □

Proposition 24c.25. *(i) If $\kappa_1 < \kappa_2 \le \mu$, then*

$$\mathrm{cf}([\mu]^{\kappa_1}, \subseteq) \le \mathrm{cf}([\mu]^{\kappa_2}, \subseteq) \cdot \mathrm{cf}([\kappa_2]^{\kappa_1}, \subseteq).$$

(ii) $\mathrm{cf}([\kappa^+]^\kappa, \subseteq) = \kappa^+$.
(iii) If $\kappa^+ \le \mu$, then $\mathrm{cf}([\mu]^\kappa, \subseteq) \le \mathrm{cf}([\mu]^{\kappa^+}, \subseteq) \cdot \kappa^+$.
(iv) If $\kappa \le \mu_1 < \mu_2$, then $\mathrm{cf}([\mu_1]^\kappa, \subseteq) \le \mathrm{cf}([\mu_2]^\kappa, \subseteq)$.
(v) If $\kappa \le \mu$, then $\mathrm{cf}([\mu^+]^\kappa, \subseteq) \le \mathrm{cf}([\mu]^\kappa, \subseteq) \cdot \mu^+$.
(vi) $\mathrm{cf}([\aleph_0]^{\aleph_0}, \subseteq) = 1$, *while for $m \in \omega\backslash 1$,* $\mathrm{cf}([\aleph_m]^{\aleph_0}) = \aleph_m$.
(vii) $\mathrm{cf}([\mu]^{\le\kappa}, \subseteq) = \mathrm{cf}([\mu]^\kappa, \subseteq)$.

Proof. (i): Let $M \subseteq [\mu]^{\kappa_2}$ be cofinal in $([\mu]^{\kappa_2}, \subseteq)$ of size $\mathrm{cf}([\mu]^{\kappa_2}, \subseteq)$, and let $N \subseteq ([\kappa_2]^{\kappa_1}, \subseteq)$ be cofinal in $([\kappa_2]^{\kappa_1}, \subseteq)$ of size $\mathrm{cf}([\kappa_2]^{\kappa_1}, \subseteq)$. For each $X \in M$ let $f_X : \kappa_2 \to X$ be a bijection. It suffices now to show that $\{f_X[Y] : X \in M, Y \in N\}$ is cofinal in $([\mu]^{\kappa_1}, \subseteq)$. Suppose that $W \in [\mu]^{\kappa_1}$. Choose $X \in M$ such that $W \subseteq X$. Then $f_X^{-1}[W] \in [\kappa_2]^{\kappa_1}$, so there is a $Y \in N$ such that $f_X^{-1}[W] \subseteq Y$. Then $W \subseteq f_X[Y]$, as desired.

(ii): The set $\{\gamma < \kappa^+ : |\gamma\backslash\kappa| = \kappa\}$ is clearly cofinal in $([\kappa^+]^\kappa$. If M is a nonempty subset of $[\kappa^+]^\kappa$ of size less than κ^+, then $|\bigcup M| = \kappa$, and $(\bigcup M) + 1$ is a member of $[\kappa^+]^\kappa$ not covered by any member of M. So (ii) holds.

(iii): Immediate from (i) and (ii).

(iv): Let $M \subseteq [\mu_2]^\kappa$ be cofinal of size $\mathrm{cf}([\mu_2]^\kappa, \subseteq)$. Let $N = \{X \cap \mu_1 : X \in M\}\backslash[\mu_1]^{<\kappa}$. It suffices to show that N is cofinal in $\mathrm{cf}([\mu_1]^\kappa, \subseteq)$. Suppose that $X \in [\mu_1]^\kappa$. Then also $X \in [\mu_2]^\kappa$, so we can choose $Y \in M$ such that $X \subseteq Y$. Clearly $X \subseteq Y \cap \mu_1 \in N$, as desired.

(v): For each $\gamma \in [\mu, \mu^+)$ let f_γ be a bijection from γ to μ. Let $E \subseteq [\mu]^\kappa$ be cofinal in $([\mu]^\kappa, \subseteq)$ and of size $\mathrm{cf}([\mu]^\kappa, \subseteq)$. It suffices to show that $\{f_\gamma^{-1}[X] : \gamma \in [\mu, \mu^+), X \in E\}$ is cofinal in $([\mu^+]^\kappa, \subseteq)$. So, take any $Y \in [\mu^+]^\kappa$. Choose $\gamma \in [\mu, \mu^+)$ such that $Y \subseteq \gamma$. Then $f_\gamma[Y] \in [\mu]^\kappa$, so we can choose $X \in E$ such that $f[Y] \subseteq X$. Then $Y \subseteq f_\gamma^{-1}[X]$, as desired.

(vi): Clearly $\mathrm{cf}([\aleph_0]^{\aleph_0}, \subseteq) = 1$. By induction it is clear from (v) that $\mathrm{cf}([\aleph_m]^{\aleph_0}) \le \aleph_m$. For $m > 0$ equality must hold, since if $X \subseteq [\aleph_m]^{\aleph_0}$ and $|X| < \aleph_m$, then $\bigcup X < \aleph_m$, and no denumerable subset of $\aleph_m\backslash\bigcup X$ is contained in a member of X.

(vii): Clear. □

The following elementary lemmas will also be needed.

Lemma 24c.26. *If $\alpha < \beta$ are limit ordinals, then*

$$|[\alpha,\beta]| = |\{\gamma : \alpha < \gamma < \beta,\ \gamma \text{ a successor ordinal}\}|.$$

Proof. For every $\delta \in [\alpha,\beta)$ let $f(\delta) = \delta + 1$. Then f is a one-one function from $[\alpha,\beta)$ onto $\{\gamma : \alpha < \gamma < \beta,\ \gamma$ a successor ordinal$\}$. □

Lemma 24c.27. *If $\alpha < \theta \leq \beta$ with θ limit, then*

$$|[\alpha,\beta]| = |\{\gamma : \alpha \leq \gamma \leq \beta,\ \gamma \text{ a successor ordinal}\}|.$$

Proof. Write $\beta = \delta + m$ with δ limit and $m \in \omega$. Then

$$[\alpha,\beta] = [\alpha,\alpha+\omega) \cup [\alpha+\omega,\delta] \cup (\delta,\beta],$$

and the desired conclusion follows easily from Lemma 24c.26. □

Theorem 24c.28. *Suppose that μ is singular and $\kappa < \mu$ is an uncountable regular cardinal such that $A \stackrel{\text{def}}{=} (\kappa,\mu)_{\text{reg}}$ has size $< \kappa$. Then*

$$\text{cf}([\mu]^\kappa, \subseteq) = \max(\text{pcf}(A)).$$

Proof. Note by the progressiveness of A that every limit cardinal in the interval (κ,μ) is singular, and hence every member of A is a successor cardinal.

First we prove $\geq$. Suppose to the contrary that $\text{cf}([\mu]^\kappa, \subseteq) < \max(\text{pcf}(A))$. For brevity write $\max(\text{pcf}(A)) = \lambda$. let $\{X_i : i \in I\} \subseteq [\mu]^\kappa$ be cofinal and of cardinality less than λ. Pick a universal sequence $\langle f_\xi : \xi < \lambda\rangle$ for λ by Theorem 24b.18. For every $\xi < \lambda$, $\text{rng}(f_\xi)$ is a subset of μ of size $\leq |A| \leq \kappa$, and hence $\text{rng}(f_\xi)$ is covered by some X_i. Thus $\lambda = \bigcup_{i\in I}\{\xi < \lambda : \text{rng}(f_\xi) \subseteq X_i\}$, so by $|I| < \lambda$ and the regularity of λ we get an $i \in I$ such that $|\{\xi < \lambda : \text{rng}(f_\xi) \subseteq X_i\}| = \lambda$. Now define for any $a \in A$,

$$h(a) = \sup(a \cap X_i).$$

Since $\kappa < a$ for each $a \in A$, we have $h \in \prod A$. Now the sequence $\langle f_\xi : \xi < \lambda\rangle$ is cofinal in $\prod A$ under $<_{J_{<\lambda}}$ by Lemma 24b.25(v),(iv). So there is a $\xi < \lambda$ such that $h <_{J_{<\lambda}} f_\xi$. Thus there is an $a \in A$ such that $h(a) < f_\xi(a) \in X_i$, contradicting the definition of h.

Second we prove $\leq$, by exhibiting a cofinal subset of $[\mu]^\kappa$ of size at most $\max(\text{pcf}(A))$. Take N and Ψ so that $H_1(A,\kappa,N,\Psi)$. Let $\mathscr{M}$ be the set of all κ-presented elementary substructures M of H_Ψ such that $A \subseteq M$, and let

$$F = \{M \cap \mu : M \in \mathscr{M}\}\backslash[\mu]^{<\kappa}.$$

Since $|M| = \kappa$, we have $|M \cap \mu| \le \kappa$, and so $\forall M \in F(|M \cap \mu| = \kappa)$.

(1) F is cofinal in $[\mu]^\kappa$.

In fact, for any $X \in [\mu]^\kappa$ we can find $M \in \mathscr{M}$ such that $X \subseteq M$, and (1) follows.

By (1) it suffices to prove that $|F| \le \max(\mathrm{pcf}(A))$.

Claim. If $M, N \in \mathscr{M}$ are such that $\mathrm{Ch}_M \restriction A = \mathrm{Ch}_N \restriction A$, then $M \cap \mu = N \cap \mu$.

For, if ν^+ is a successor cardinal $\le \mu$, then $\sup(M \cap \nu^+) = \mathrm{Ch}_M(\nu^+) = \mathrm{Ch}_N(\nu^+) = \sup(N \cap \nu^+)$. So the claim holds by Theorem 24c.9.

Now for each $M \in \mathscr{M}$, let $g(M)$ be the sequence $\langle(\lambda_0, \gamma_0), \ldots, (\lambda_n, \gamma_n)\rangle$ given by Lemma 24c.23. Clearly the range of g has size $\le \max(\mathrm{pcf}(A))$. Now for each $X \in F$, choose $M_X \in \mathscr{M}$ such that $X = M_X \cap \mu$. Then for $X, Y \in F$ and $X \neq Y$ we have $M_X \cap \mu \neq M_Y \cap \mu$, hence by the claim $\mathrm{Ch}_{M_X} \restriction A \neq \mathrm{Ch}_{M_Y} \restriction A$, and hence by Lemma 24c.23, $g(M_X) \neq g(M_Y)$. This proves that $|F| \le \max(\mathrm{pcf}(A))$. $\square$

Corollary 24c.29. *Let $A = \{\aleph_m : 0 < m < \omega\}$. Then for any $m \in \omega$ we have* $\mathrm{cf}([\aleph_\omega]^{\aleph_m}) = \max(\mathrm{pcf}(A))$. $\square$

Elevations and transitive generators

We start with some simple general notions about cardinals. If B is a set of cardinals, then a *walk* in B is a sequence $\lambda_0 > \lambda_1 > \cdots > \lambda_n$ of members of B. Such a walk is necessarily finite. Given cardinals $\lambda_0 > \lambda$ in B, a *walk from λ_0 to λ* is a walk as above with $\lambda_n = \lambda$. We denote by $F_{\lambda_0,\lambda}(B)$ the set of all walks from λ_0 to λ.

Now suppose that A is progressive and $\lambda_0 \in \mathrm{pcf}(A)$. A *special walk from λ_0 to λ_n in* $\mathrm{pcf}(A)$ is a walk $\lambda_0 > \cdots > \lambda_n$ in $\mathrm{pcf}(A)$ such that $\lambda_i \in A$ for all $i > 0$. We denote by $F'_{\lambda_0,\lambda}(A)$ the collection of all special walks from λ_0 to λ in $\mathrm{pcf}(A)$.

Next, suppose in addition that $f \stackrel{\text{def}}{=} \langle f^\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ is a sequence of sequences, where each f^λ is a sequence $\langle f^\lambda_\xi : \xi < \lambda\rangle$ of members of $\prod A$. If $\lambda_0 > \cdots > \lambda_n$ is a special walk in $\mathrm{pcf}(A)$, and $\gamma_0 \in \lambda_0$, then we define an associated sequence of ordinals by setting

$$\gamma_{i+1} = f^{\lambda_i}_{\gamma_i}(\lambda_{i+1})$$

for all $i < n$. Note that $\gamma_i < \lambda_i$ for all $i = 0, \ldots, n$. Then we define

$$\mathrm{El}_{\lambda_0,\ldots,\lambda_n}(\gamma_0) = \gamma_n.$$

Now we define the *elevation* of the sequence f, denoted by $f^e \stackrel{\text{def}}{=} \langle f^{\lambda,e} : \lambda \in \mathrm{pcf}(A)\rangle$, by setting, for any $\lambda_0 \in \mathrm{pcf}(A)$, any $\gamma_0 \in \lambda_0$, and any $\lambda \in A$,

$$f^{\lambda_0,e}_{\gamma_0}(\lambda) = \begin{cases} f^{\lambda_0}_{\gamma_0}(\lambda) & \text{if } \lambda_0 \le \lambda, \\ \max(\{\mathrm{El}_{\lambda_0,\ldots,\lambda_n}(\gamma_0) : (\lambda_0, \ldots, \lambda_n) \in F'_{\lambda_0,\lambda}\}) & \text{if } \lambda < \lambda_0, \\ & \quad\text{and this maximum exists,} \\ f^{\lambda_0}_{\gamma_0}(\lambda) & \text{if } \lambda < \lambda_0, \text{ otherwise.} \end{cases}$$

Note here that the superscript ^{e} is only notational, standing for "elevated".

Lemma 24c.30. *Assume the above notation. Then $f^{\lambda_0}_{\gamma_0} \leq f^{\lambda_0,e}_{\gamma_0}$ for all $\lambda_0 \in \mathrm{pcf}(A)$ and all $\gamma_0 \in \lambda_0$.*

Proof. Take any $\gamma_0 \in \lambda_0$ and any $\lambda \in A$. If $\lambda_0 \leq \lambda$, then $f^{\lambda_0,e}_{\gamma_0}(\lambda) = f^{\lambda_0}_{\gamma_0}(\lambda)$. Suppose that $\lambda < \lambda_0$. If the above maximum does not exist, then again $f^{\lambda_0,e}_{\gamma_0}(\lambda) = f^{\lambda_0}_{\gamma_0}(\lambda)$. Suppose the maximum exists. Now $(\lambda_0, \lambda) \in F'_{\lambda_0,\lambda}(A)$, so

$$f^{\lambda_0}_{\gamma_0}(\lambda) = \mathrm{El}_{\lambda_0,\lambda}(\gamma_0) \leq \max(\{\mathrm{El}_{\lambda_0,\ldots,\lambda_n}(\gamma_0) : (\lambda_0, \ldots, \lambda_n) \in F'_{\lambda_0,\lambda}\}) = f^{\lambda_0,e}_{\gamma_0}(\lambda). \qquad \square$$

Lemma 24c.31. *Suppose that A is progressive, κ is a regular cardinal such that $|A| < \kappa < \min(A)$, and $f \stackrel{\text{def}}{=} \langle f^\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ is a sequence of sequences f^λ such that f^λ is κ-minimally obedient for λ. Assume also $H_1(A, \kappa, N, \Psi)$ and $f \in N$.*

Then also $f^e \in N$.

Proof. The proof is a more complicated instance of our standard procedure for going from V to H_Ψ to N and then back. We sketch the details.

Assume the hypotheses. In particular, $A \in N$. Hence also $\mathrm{pcf}(A) \in N$. Also, $|A| < \kappa$, so $A \subseteq N$. Now clearly $F' \in N$. Also, $\mathrm{El} \in N$. (Note that El depends upon A.) Then by absoluteness,

$$H_\Psi \models \exists g \;\; g \text{ is a function, } \mathrm{dmn}(g) = \mathrm{pcf}(A) \wedge \forall \lambda_0 \in \mathrm{pcf}(A) \forall \gamma_0 \in \lambda_0 \forall \lambda \in A$$

$$g(\lambda) = \begin{cases} f^{\lambda_0}_{\gamma_0}(\lambda) & \text{if } \lambda_0 \leq \lambda, \\ \max(\{\mathrm{El}_{\lambda_0,\ldots,\lambda_n}(\gamma_0) : (\lambda_0, \ldots, \lambda_n) \in F'_{\lambda_0,\lambda}\}) & \text{if } \lambda < \lambda_0, \\ & \quad \text{and this maximum exists,} \\ f^{\lambda_0}_{\gamma_0}(\lambda) & \text{if } \lambda < \lambda_0, \text{ otherwise.} \end{cases}$$

Now the usual procedure can be applied. $\square$

Lemma 24c.32. *Suppose that A is progressive, κ is a regular cardinal such that $|A| < \kappa < \min(A)$, and $f \stackrel{\text{def}}{=} \langle f^\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ is a sequence of sequences f^λ such that f^λ is κ-minimally obedient for λ. Assume $H_1(A, \kappa, N, \Psi)$ and $f \in N$.*

Suppose that $\lambda_0 \in \mathrm{pcf}(A) \cap N$, and let $\gamma_0 = \mathrm{Ch}_N(\lambda_0)$.

(i) If $\lambda_0 > \cdots > \lambda_n$ is a special walk in $\mathrm{pcf}(A)$, and $\gamma_1, \ldots, \gamma_n$ are formed as above, then $\gamma_i \in \overline{N}$ for all $i = 0, \ldots, n$.

(ii) For every $\lambda \in A \cap \lambda_0$ we have $f^{\lambda_0,e}_{\gamma_0}(\lambda) \in \overline{N}$.

Proof. (i): By Lemma 24c.17(iv)(c), $f^{\lambda_0}_{\gamma_0}(\lambda) \in \overline{N}$, and (i) follows by induction using Lemma 24c.17(iv)(c).

(ii): immediate from (i). $\square$

Lemma 24c.33. *Assume the hypotheses of Lemma 24c.24c. Then* *(i) For any special walk $\lambda_0 > \cdots > \lambda_n = \lambda$ in $F'_{\lambda_0,\lambda}$, we have*

$$El_{\lambda_0,\ldots,\lambda_n}(\gamma_0) \le \mathrm{Ch}_N(\lambda).$$

(ii) $f^{\lambda_0,e}_{\gamma_0} \le \mathrm{Ch}_N \restriction A$ for every $\gamma_0 < \lambda_0$.
(iii) If there is a special walk $\lambda_0 > \cdots > \lambda_n = \lambda$ in $F'_{\lambda_0,\lambda}$ such that

$$El_{\lambda_0,\ldots,\lambda_n}(\gamma_0) = \mathrm{Ch}_N(\lambda),$$

then

$$\mathrm{Ch}_N(\lambda) = f^{\lambda_0,e}_{\gamma_0}(\lambda).$$

(iv) Suppose that $\mathrm{Ch}_N(\lambda) = f^{\lambda_0,e}_{\gamma_0}(\lambda) = \gamma$. If there is an $a \in A \cap \lambda$ such that $f^{\lambda,e}_{\gamma}(a) = \mathrm{Ch}_N(a)$, then also $f^{\lambda_0,e}_{\gamma_0}(a) = \mathrm{Ch}_N(a)$.

Proof. (i) is immediate from Lemma 24c.32(i) and Lemma 24c.8(iii). (ii) and (iii) follow from (i). For (iv), by Lemma 24c.32(i) and (i) there are special walks $\lambda_0 > \cdots > \lambda_n = \lambda$ and $\lambda = \lambda'_0 > \cdots > \lambda'_m = a$ such that

$$\begin{aligned} &f^{\lambda_0,e}_{\gamma_0}(\lambda) = \mathrm{Ch}_N(\lambda) = \mathrm{El}_{\lambda_0,\ldots,\lambda_n}(\gamma_0) \quad \text{and} \\ &f^{\lambda,e}_{\gamma}(a) = \mathrm{Ch}_N(a) = \mathrm{El}_{\lambda'_0,\ldots,\lambda'_m}(a). \end{aligned}$$

It follows that

$$\mathrm{El}_{\lambda_0,\ldots,\lambda_n,\lambda'_1,\ldots,a}(\gamma_0) = \mathrm{Ch}_N(a),$$

and (iii) then gives $f^{\lambda_0,e}_{\gamma_0}(a) = \mathrm{Ch}_N(a)$. □

Definition. Suppose that A is progressive and $A \subseteq P \subseteq \mathrm{pcf}(A)$. A system $\langle b_\lambda : \lambda \in P\rangle$ of subsets of A is *transitive* iff for all $\lambda \in P$ and all $\mu \in b_\lambda$ we have $b_\mu \subseteq b_\lambda$.

Theorem 24c.34. *Suppose that $H_1(A,\kappa,N,\Psi)$, and $f = \langle f^\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ is a system of functions, and each f^λ is κ-minimally obedient for λ. Let f^e be the derived elevated array. For every $\lambda_0 \in \mathrm{pcf}(A) \cap N$ put $\gamma_0 = \mathrm{Ch}_N(\lambda_0)$ and define*

$$b_{\lambda_0} = \{a \in A : \mathrm{Ch}_N(a) = f^{\lambda_0,e}_{\gamma_0}(a)\}.$$

Then the following hold for each $\lambda_0 \in \mathrm{pcf}(A) \cap N$:
(i) b_{λ_0} is a λ_0-generator.
(ii) There is a $b'_{\lambda_0} \subseteq b_{\lambda_0}$ such that
(a) $b_{\lambda_0} \backslash b'_{\lambda_0} \in J_{<\lambda_0}[A]$.
(b) $b'_{\lambda_0} \in N$ (each one individually, not the sequence).
(c) b'_{λ_0} is a λ_0-generator.
(iii) The system $\langle b_\lambda : \lambda \in \mathrm{pcf}(A) \cap N\rangle$ is transitive.

Proof. Note that $H_2(A,\kappa,N,\Psi,\lambda_0,f^{\lambda_0,e},\gamma_0)$ holds by Lemma 24c.24b. By definition, minimally obedient implies universal, so f^{λ_0} is persistently cofinal by Lemma 24c.11. Hence by Lemma 24c.24, $f^{\lambda_0,e}$ is persistently cofinal, and so $P_1(A,\kappa,N,\Psi,\lambda_0,f^{\lambda_0,e},\gamma_0)$ holds by

Lemma 24c.18. Also, by Lemma 24c.19 $P_2(A, \kappa, N, \Psi, \lambda_0, f^{\lambda_0}, \gamma_0)$ holds, so the condition $P_2(A, \kappa, N, \Psi, \lambda_0, f^{\lambda_0,e}, \gamma_0)$ holds by Lemmas 24c.30 and 24c.24a(ii). Now (i) and (ii) hold by Lemma 24c.29.

Now suppose that $\lambda_0 \in \mathrm{pcf}(A) \cap N$ and $\lambda \in b_{\lambda_0}$. Thus

$$\mathrm{Ch}_N(\lambda) = f^{\lambda_0,e}_{\gamma_0}(\lambda),$$

where $\gamma_0 = \mathrm{Ch}_N(\lambda_0)$. Write $\gamma = \mathrm{Ch}_N(\lambda)$. We want to show that $b_\lambda \subseteq b_{\lambda_0}$. Take any $a \in b_\lambda$. So $\mathrm{Ch}_N(a) = f^{\lambda,e}_\gamma(a)$. By Lemma 24c.24a(iv) we get $f^{\lambda_0,e}_{\gamma_0}(a) = \mathrm{Ch}_N(a)$, so $a \in b_{\lambda_0}$, as desired. $\square$

Localization

Theorem 24c.35. *Suppose that A is a progressive set. Then there is no subset $B \subseteq \mathrm{pcf}(A)$ such that $|B| = |A|^+$ and, for every $b \in B$, $b > \max(\mathrm{pcf}(B \cap b))$.*

Proof. Assume the contrary. We may assume that $|A|^+ < \min(A)$. In fact, if we know the result under this assumption, and now $|A|^+ = \min(A)$, suppose that $B \subseteq \mathrm{pcf}(A)$ with $|B| = |A|^+$ and $\forall b \in B[b > \max(\mathrm{pcf}(B \cap b))]$. Let $A' = A\backslash\{|A|^+\}$. Then let $B' = B\backslash\{|A|^+\}$. Hence we have $B' \subseteq \mathrm{pcf}(A')$. Clearly $|B'| = |A'|^+$ and $\forall b \in B'[b > \max(\mathrm{pcf}(B' \cap b))]$, contradiction.

Also, clearly we may assume that B has order type $|A|^+$.

Let $E = A \cup B$. Then $|E| < \min(E)$. Let $\kappa = |E|$. By Lemma 24c.16, we get an array $\langle f^\lambda : \lambda \in \mathrm{pcf}(E)\rangle$, with each f^λ κ-minimally obedient for λ. Choose N and Ψ so that $H_1(A, \kappa, N, \Psi)$, with N containing $A, B, E, \langle f^\lambda : \lambda \in \mathrm{pcf}(E)\rangle$. Now let $\langle b_\lambda : \lambda \in \mathrm{pcf}(E) \cap N\rangle$ be the set of transitive generators as guaranteed by Theorem 24c.24b. Let $b'_\lambda \in N$ be such that $b'_\lambda \subseteq b_\lambda$ and $b_\lambda\backslash b'_\lambda \in J_{<\lambda}$.

Now let F be the function with domain $\{a \in A : \exists\beta \in B(a \in b_\beta)\}$ such that for each such a, $F(a)$ is the least $\beta \in B$ such that $a \in b_\beta$. Define $B_0 = \{\gamma \in B : \exists a \in \mathrm{dmn}(F)(\gamma \leq F(a)\}$. Thus B_0 is an initial segment of B of size at most $|A|$. Clearly $B_0 \in N$. We let $\beta_0 = \min(B\backslash B_0)$; so $B_0 = B \cap \beta_0$.

Now we claim

(1) There exists a finite descending sequence $\lambda_0 > \cdots > \lambda_n$ of cardinals in $N \cap \mathrm{pcf}(B_0)$ such that $B_0 \subseteq b_{\lambda_0} \cup \ldots \cup b_{\lambda_n}$.

We prove more: we find a finite descending sequence $\lambda_0 > \cdots > \lambda_n$ of cardinals in $N \cap \mathrm{pcf}(B_0)$ such that $B_0 \subseteq b'_{\lambda_0} \cup \ldots \cup b'_{\lambda_n}$. Let $\lambda_0 = \max(\mathrm{pcf}(B_0))$. Since $B_0 \in N$, we clearly have $\lambda_0 \in N$ and hence $b'_{\lambda_0} \in N$. So $B_1 \stackrel{\text{def}}{=} B_0\backslash b'_{\lambda_0} \in N$. Now suppose that $B_k \subseteq B_0$ has been defined so that $B_k \in N$. If $B_k = \emptyset$, the construction stops. Suppose that $B_k \neq \emptyset$. Let $\lambda_k = \max(\mathrm{pcf}(B_k))$. Clearly $\lambda_k \in N$, so $b'_{\lambda_k} \in N$ and $B_{\kappa+1} \stackrel{\text{def}}{=} B_k\backslash b'_{\lambda_k} \in N$. Since $B_{\kappa+1} = B_k\backslash b'_{\lambda_k}$ and b'_{λ_k} is a λ_k-generator, from Lemma 9.25(xii) it follows that

$\lambda_0 > \lambda_1 > \cdots$. So the construction eventually stops; say that $B_{n+1} = \emptyset$. So $B_n \subseteq b'_{\lambda_n}$. So

$$\begin{aligned}
B_0 &\subseteq b'_{\lambda_0} \cup (B_0 \backslash b'_{\lambda_0}) \\
&= b'_{\lambda_0} \cup B_1 \\
&\subseteq b'_{\lambda_0} \cup b'_{\lambda_1} \cup B_2 \\
&\cdots\cdots\cdots \\
&\subseteq b'_{\lambda_0} \cup b'_{\lambda_1} \cup \ldots \cup B_n \\
&\subseteq b'_{\lambda_0} \cup b'_{\lambda_1} \cup \ldots \cup b'_{\lambda_n}.
\end{aligned}$$

This proves (1).

Note that $\beta_0 > \max(\text{pcf}(B \cap \beta_0) = \max(\text{pcf}(B_0)) \geq \lambda_0, \ldots, \lambda_n$ by the initial assumption of the proof. Next, we claim

(2) $b_{\beta_0} \subseteq b_{\lambda_0} \cup \ldots \cup b_{\lambda_n}$.

To prove this, first note that $b_{\beta_0} \subseteq A \cup B_0$. For, $b_{\beta_0} \subseteq E$ by definition, and $E = A \cup B$; $b_{\beta_0} \cap B = B_0$, so indeed $b_{\beta_0} \subseteq A \cup B_0$. Also, $B_0 \subseteq b_{\lambda_0} \cup \ldots \cup b_{\lambda_n}$. So it suffices to prove that $b_{\beta_0} \cap A \subseteq b_{\lambda_0} \cup \ldots \cup b_{\lambda_n}$.

Consider any cardinal $a \in b_{\beta_0} \cap A$. Since $\beta_0 \in B$, we have $a \in \text{dmn}(F)$, and since $\beta_0 \notin B_0$ we have $F(a) < \beta_0$. Let $\beta = F(a)$. So $a \in b_\beta$, and $\beta < \beta_0$, so by the minimality of β_0, $\beta \in B_0$. Since $B_0 \subseteq b_{\lambda_0} \cup \ldots \cup b_{\lambda_n}$, it follows that $\beta \in b_{\lambda_i}$ for some $i = 0, \ldots, n$. But transitivity implies that $b_\beta \subseteq b_{\lambda_i}$, and hence $a \in b_{\lambda_i}$, as desired. So (2) holds.

By (2) we have

$$\text{pcf}(b_{\beta_0}) \subseteq \text{pcf}(b_{\lambda_0}) \cup \ldots \cup \text{pcf}(b_{\lambda_n}),$$

and hence by Lemma 24b.25(vii) we get $\beta_0 = \max(\text{pcf}(b_{\beta_0})) \leq \max\{\lambda_i : i = 0, \ldots, n\} < \beta_0$, contradiction. □

Theorem 24c.36. (Localization) *Suppose that A is a progressive set of regular cardinals. Suppose that $B \subseteq \text{pcf}(A)$ is also progressive. Then for every $\lambda \in \text{pcf}(B)$ there is a $B_0 \subseteq B$ such that $|B_0| \leq |A|$ and $\lambda \in \text{pcf}(B_0)$.*

Proof. We prove by induction on λ that if A and B satisfy the hypotheses of the theorem, then the conclusion holds. Let C be a λ-generator over B. Thus $C \subseteq B$ and $\lambda = \max(\text{pcf}(C))$ by Lemma 24b.25(vii). Now $C \subseteq \text{pcf}(A)$ and C is progressive. It suffices to find $B_0 \subseteq C$ with $|B_0| \leq |A|$ and $\lambda \in \text{pcf}(B_0)$.

Let $C_0 = C$ and $\lambda_0 = \lambda$. Suppose that $C_0 \supseteq \cdots \supseteq C_i$ and $\lambda_0 > \cdots > \lambda_i$ have been constructed so that $\lambda = \max(\text{pcf}(C_i))$ and C_i is a λ-generator over B. If there is no maximal element of $\lambda \cap \text{pcf}(C_i)$ we stop the construction. Otherwise, let λ_{i+1} be that maximum element, let D_{i+1} be a λ_{i+1}-generator over B, and let $C_{i+1} = C_i \backslash D_{i+1}$. Now $D_{i+1} \in J_{\leq \lambda_{i+1}}[B] \subseteq J_{<\lambda}[B]$, so C_{i+1} is still a λ-generator of B by Lemma 9.25(ix), and $\lambda = \max(\text{pcf}(C_{i+1}))$ by Lemma 24b.25(vii). Note that $\lambda_{i+1} \notin \text{pcf}(C_{i+1})$, by Lemma 24b.25(ii).

This construction must eventually stop, when $\lambda \cap C_i$ does not have a maximal element; we fix the index i.

(1) There is an $E \subseteq \lambda \cap \mathrm{pcf}(C_i)$ such that $|E| \le |A|$ and $\lambda \in \mathrm{pcf}(E)$.

In fact, suppose that no such E exists. We now construct a strictly increasing sequence $\langle \gamma_j : j < |A|^+\rangle$ of elements of $\mathrm{pcf}(C_i)$ such that $\gamma_k > \max(\mathrm{pcf}(\{\gamma_j : j < k\}\rangle$ for all $k < |A|^+$. (This contradicts Theorem 24c.24c.) Suppose that $\{\gamma_j : j < k\} = E$ has been defined. Now $\lambda \notin \mathrm{pcf}(E)$ by the supposition after (1), and $\lambda < \max(\mathrm{pcf}(E))$ is impossible since $\mathrm{pcf}(E) \subseteq \mathrm{pcf}(C_i)$ and $\lambda = \max(\mathrm{pcf}(C_i))$. So $\lambda > \max(\mathrm{pcf}(E))$. Hence, because $\lambda \cap C_i$ does not have a maximal element, we can choose $\gamma_k \in \lambda \cap C_i$ such that $\gamma_k > \max(\mathrm{pcf}(E))$, as desired. Hence (1) holds.

We take E as in (1). Apply the inductive hypothesis to each $\gamma \in E$ and to A, E in place of A, B; we get a set $G_\gamma \subseteq E$ such that $|G_\gamma| \le |A|$ and $\gamma \in \mathrm{pcf}(G_\gamma)$. Let $H = \bigcup_{\gamma \in E} G_\gamma$. Note that $|H| \le |A|$. Thus $E \subseteq \mathrm{pcf}(H)$. Since $\mathrm{pcf}(E) \subseteq \mathrm{pcf}(H)$ by Theorem 9.15, we have $\lambda \in \mathrm{pcf}(H)$, completing the inductive proof.

The size of $\mathrm{pcf}(A)$

Theorem 24c.37. *If A is a progressive interval of regular cardinals, then* $|\mathrm{pcf}(A)| < |A|^{+4}$.

Proof. Assume that A is a progressive interval of regular cardinals but $|\mathrm{pcf}(A)| \ge |A|^{+4}$. Let $\rho = |A|$. We will define a set B of size ρ^+ consisting of cardinals in $\mathrm{pcf}(A)$ such that each cardinal in B is greater than $\max(\mathrm{pcf}(B \cap b))$. This will contradict Theorem 24c.24c.

Let $S = S^{\rho^{+3}}_{\rho^+}$; so S is a stationary subset of ρ^{+3}. By Theorem 24b.40 let $\langle C_k : k \in S\rangle$ be a club guessing sequence. Thus

(1) C_k is a club in k of order type ρ^+, for each $k \in S$.

(2) If D is a club in ρ^{+3}, then there is a $k \in D \cap S$ such that $C_k \subseteq D$.

Let σ be the ordinal such that $\aleph_\sigma = \sup(A)$. Now $\mathrm{pcf}(A)$ is an interval of regular cardinals by Theorem 24b.13. So $\mathrm{pcf}(A)$ contains all regular cardinals in the set $\{\aleph_{\sigma+\alpha} : \alpha < \rho^{+4}\}$.

Now we are going to define a strictly increasing continuous sequence $\langle \alpha_i : i < \rho^{+3}\rangle$ of ordinals less than ρ^{+4}.

1. Let $\alpha_0 = \rho^{+3}$.

9. For i limit let $\alpha_i = \bigcup_{j<i} \alpha_j$.

3. Now suppose that α_j has been defined for all $j \le i$; we define α_{i+1}. For each $k \in S$ let $e_k = \{\aleph_{\sigma+\alpha_j} : j \in C_k \cap (i+1)\}$. Thus $e_k^{(+)}$ is a subset of $\mathrm{pcf}(A)$. If $\max(\mathrm{pcf}(e_k^{(+)})) < \aleph_{\sigma+\rho^{+4}}$, let β_k be an ordinal such that $\max(\mathrm{pcf}(e_k^{(+)})) < \aleph_{\sigma+\beta_k}$ and $\beta_k < \rho^{+4}$; otherwise let $\beta_k = 0$. Let α_{i+1} be greater than α_i and all β_k for $k \in S$, with $\alpha_{i+1} < \rho^{+4}$. This is possible because $|S| = \rho^{+3}$. Thus

(3) For every $k \in S$, if $\max(\mathrm{pcf}(e_k^{(+)})) < \aleph_{\sigma+\rho^{+4}}$, then $\max(\mathrm{pcf}(e_k^{(+)})) < \aleph_{\sigma+\alpha_{i+1}}$.

This finishes the definition of the sequence $\langle \alpha_i : i < \rho^{+3}\rangle$. Let $D = \{\alpha_i : i < \rho^{+3}\}$, and let $\delta = \sup(D)$. Then D is club in δ. Let $\mu = \aleph_{\sigma+\delta}$. Thus μ has cofinality ρ^{+3}, and it is singular since $\delta > \alpha_0 = \rho^{+3}$. Now we apply Corollary 9.24c: there is a club C_0 in μ such that $\mu^+ = \max(\mathrm{pcf}(C_0^{(+)}))$. We may assume that $C_0 \subseteq [\aleph_\sigma, \mu)$. so we can write $C_0 = \{\aleph_{\sigma+i} : i \in D_0\}$ for some club D_0 in δ. Let $D_1 = D_0 \cap D$. So D_1 is a club of δ. Let

$E = \{i \in \rho^{+3} : \alpha_i \in D_1\}$. It is clear that E is a club in ρ^{+3}. So by (2) choose $k \in E \cap S$ such that $C_k \subseteq E$. Let $C'_k = \{\beta \in C_k :$ there is a largest $\gamma \in C_k$ such that $\gamma < \beta\}$. Set $B = \{\aleph^+_{\sigma+\alpha_i} : i \in C'_k\}$. We claim that B is as desired. Clearly $|B| = \rho^+$.

Take any $j \in C'_k$. We want to show that

$$(*) \qquad \aleph^+_{\sigma+\alpha_j} > \max(\mathrm{pcf}(B \cap \aleph^+_{\sigma+\alpha_j})).$$

Let $i \in C_k$ be largest such that $i < j$. So $i + 1 \leq j$. We consider the definition given above of α_{i+1}. We defined $e_k = \{\aleph_{\sigma+\alpha_l} : l \in C_k \cap (i + 1)\}$. Now

(4) $B \cap \aleph^+_{\sigma+\alpha_j} \subseteq e^{(+)}_k$.

For, if $b \in B \cap \aleph^+_{\sigma+\alpha_j}$, we can write $b = \aleph^+_{\sigma+\alpha_l}$ with $l \in C'_k$ and $l < j$. Hence $l \leq i$ and so $b = \aleph^+_{\sigma+\alpha_l} \in e^{(+)}_k$. So (4) holds.

Now if $l \in C_k \cap (i + 1)$, then $l \in E$, and so $\alpha_l \in D_1 \subseteq D_0$. Hence $\aleph_{\sigma+\alpha_l} \in C_0$. This shows that $e^{(+)}_k \subseteq C^{(+)}_0$. So $\max(\mathrm{pcf}(e^{(+)}_k)) \leq \max(\mathrm{pcf}(C^{(+)}_0)) = \mu^+ < \aleph_{\sigma+\rho^{+4}}$. Hence by (3) we get $\max(\mathrm{pcf}(e^{(+)}_k)) < \aleph_{\sigma+\alpha_{i+1}}$. So

$$\begin{aligned}
\max(\mathrm{pcf}(B \cap \aleph^+_{\sigma+\alpha_j})) &\leq \max(\mathrm{pcf}(e^{(+)}_k)) \quad \text{by (4)}\\
&< \aleph^+_{\sigma+\alpha_{i+1}}\\
&\leq \aleph^+_{\sigma+\alpha_j},
\end{aligned}$$

which proves $(*)$. □

Theorem 24c.38. *If $\aleph_\delta$ is a singular cardinal such that $\delta < \aleph_\delta$, then*

$$\mathrm{cf}([\aleph_\delta]^{|\delta|}, \subseteq) < \aleph_{|\delta|^{+4}}.$$

Proof. Let $\kappa = |\delta|^+$ and $A = (\kappa, \aleph_\delta)_{\mathrm{reg}}$. By Lemma 24c.25(iii) and Lemma 24c.28,

$$\begin{aligned}
\mathrm{cf}([\aleph_\delta]^{|\delta|}, \subseteq) &\leq \max(|\delta|^+, \mathrm{cf}([\aleph_\delta]^{|\delta|^+}, \subseteq))\\
&\leq \max(|\delta|^+, \max(\mathrm{pcf}(A))).
\end{aligned}$$

Hence it suffices to show that $\max(\mathrm{pcf}(A)) < \aleph_{|\delta|^{+4}}$.

By Theorem 24c.37, $|\mathrm{pcf}(A)| < |A|^{+4}$. Write $\max(\mathrm{pcf}(A)) = \aleph_\alpha$ and $\kappa = \aleph_\beta$. We want to show that $\alpha < |\delta|^{+4}$. Now $\mathrm{pcf}(A) = (\kappa, \max(\mathrm{pcf}(A))]_{\mathrm{reg}} = (\aleph_\beta, \aleph_\alpha]_{\mathrm{reg}}$. By Lemma 24c.27, $|(\beta, \alpha)| = |\mathrm{pcf}(A)| < |A|^{+4} \leq |\delta|^{+4}$. Also, $\beta \leq \aleph_\beta = \kappa = |\delta|^+ < |\delta|^{+4}$. So $|\alpha| < |\delta|^{+4}$, and hence $\alpha < |\delta|^{+4}$. □

Theorem 24c.39. *If δ is a limit ordinal, then*

$$\aleph^{\mathrm{cf}(\delta)}_\delta < \max\left(\left(|\delta|^{\mathrm{cf}(\delta)}\right)^+, \aleph_{|\delta|^{+4}}\right).$$

Proof. If $\delta = \aleph_\delta$, then $|\delta| = \aleph_\delta$ and the conclusion is obvious. So assume that $\delta < \aleph_\delta$. Now

(1) $\aleph_\delta^{\mathrm{cf}(\delta)} \leq |\delta|^{\mathrm{cf}(\delta)} \cdot \mathrm{cf}([\aleph_\delta]^{|\delta|}, \subseteq)$.

In fact, let $B \subseteq [\aleph_\delta]^{|\delta|}$ be cofinal and of size $\mathrm{cf}([\aleph_\delta]^{|\delta|}, \subseteq)$. Now $\mathrm{cf}(\delta) \leq |\delta|$, so

$$[\aleph_\delta]^{\mathrm{cf}(\delta)} = \bigcup_{Y \in B} [Y]^{\mathrm{cf}(\delta)},$$

and (1) follows. Hence the theorem follows by Theorem 24c.38. $\square$

Corollary 24c.40. $\aleph_\omega^{\aleph_0} < \max\left((2^{\aleph_0})^+, \aleph_{\omega_4}\right)$. $\square$

25. Descriptive set theory

We follow Marker.

Recall that a *metric* on a set X is a function $d : X \times X \to [0, \infty)$ such that:

(i) $d(x, y) = 0$ iff $x = y$.

(ii) $d(x, y) = d(y, x)$.

(iii) $d(x, y) + d(y, z) \geq d(x, z)$.

We define $B(x, r) = \{y : d(x, y) < r\}$ for $r \in (0, \infty)$. The set $\{B(x, r) : x \in X, r > 0\}$ is a base for a topology on X, the *metric topology determined by d*. A sequence $\langle x_n : n \in \omega\rangle$ *converges to y* iff $\forall\varepsilon > 0\exists N\forall n \geq N[d(x_n, y) < \varepsilon]$. A sequence is *Cauchy* iff $\forall\varepsilon > 0\exists N\forall n \geq N[d(x_N, x_n) < \varepsilon$. A metric is *complete* iff every Cauchy sequence converges. A *Polish space* is a separable space metrizable by a complete metric.

Proposition 25.1. m2 *Any countable discrete space is Polish.*

Proof. Let X be countable and discrete. So X is separable. Define $d : X\times X \to [0, \infty)$ by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

First, d is a metric on X. Only (iii) is non-trivial.

Case 1. $x = y = z$: clear.

Case 2. $x = y \neq z$: $0 + 1 = 1$.

Case 3. $x \neq y = z$: $1 + 0 = 1$.

Case 4. $x \neq y \neq z$. $1 + 1 \geq d(x, z)$.

d is complete: Suppose that $\langle x_n : n \in \omega\rangle$ is Cauchy. Choose N so that $\forall n \geq N[d(x_N, x_n) < 1]$. Then $\forall n \geq N[x_n = x_N]$, so the sequence converges.

Now take any $Y \subseteq X$; we want to show that Y is open in the metric topology. Given $y \in Y$, $B(y, \frac{1}{2}) = \{y\}$. Thus every point is open, hence Y is open. $\square$

Proposition 25.2. m2 *Let d be a metric on a set X. Define*

$$\hat{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Then:

(i) $\hat{d}$ is a metric on X.

(ii) d and $\hat{d}$ induce the same topology on X.

(iii) $\hat{d}(x, y) < 1$ for all x, y.

Proof. (i): Clearly $\hat{d}(x, y) = 0$ iff $x = y$, and $\hat{d}(x, y) = \hat{d}(y, x)$. Next,

$$\hat{d}(x, y) + \hat{d}(y, z) - \hat{d}(x, z) \geq 0 \quad \text{iff}$$
$$\frac{d(x, y)}{1 + d(x, y)} + \frac{d(y, z)}{1 + d(y, z)} - \frac{d(x, z)}{1 + d(x, z)} \geq 0 \quad \text{iff}$$

$$\begin{aligned}
&d(x,y)+d(x,y)d(y,z)+d(x,y)d(x,z)+d(x,y)d(y,z)d(x,z)\\
&+d(y,z)+d(y,z)d(x,y)+d(y,z)d(x,z)+d(y,z)d(x,y)d(x,z)\\
&-d(x,z)-d(x,z)d(x,y)-d(x,z)d(y,z)-d(x,z)d(x,y)d(y,z)\geq 0 \quad \text{iff}\\
&d(x,y)+d(x,y)d(y,z)+d(y,z)+d(y,z)d(x,y)+d(x,y)d(y,z)d(x,z)-d(x,z)\geq 0,
\end{aligned}$$

and the last statement is true.

(ii): $B_d(x,\varepsilon)$ is open in the topology determined by $\hat{d}$: First note that $B_d(x,\varepsilon)\subseteq B_{\hat{d}}(x,\varepsilon)$, since for all $y\in B_d(x,\varepsilon)$ we have

$$\hat{d}(x,y)=\frac{d(x,y)}{1+d(x,y)}\leq d(x,y)<\varepsilon.$$

It follows that $B_{\hat{d}}(x,\varepsilon)$ is open in the d-topology. Now suppose that $y\in B_d(x,\varepsilon)$. We want to find δ such that $B_{\hat{d}}(y,\delta)\subseteq B_d(x,\varepsilon)$. Let $\varepsilon'=\varepsilon-d(x,y)$ and $\delta=\frac{\varepsilon'}{1+\varepsilon'}$. So $\delta<1$. Hence $\delta+\delta\varepsilon'=\varepsilon'$, hence $\varepsilon'(1-\delta)=\delta$, hence $\varepsilon'=\frac{\delta}{1-\delta}$. Suppose that $z\in B_{\hat{d}}(y,\delta)$. Thus

$$\begin{aligned}
&\hat{d}(y,z)<\delta \text{ hence } \frac{d(y,z)}{1+d(y,z)}<\delta \text{ hence } d(y,z)<\delta+\delta d(y,z) \text{ hence}\\
&d(y,z)(1-\delta)<\delta \text{ hence } d(y,z)<\frac{\delta}{1-\delta} \text{ hence } d(y,z)<\varepsilon' \text{ hence}\\
&d(y,z)<\varepsilon-d(x,y) \text{ hence } d(x,z)\leq d(y,z)+d(x,y)<\varepsilon.
\end{aligned}$$

(iii): clear. □

Theorem 25.3. m2 *If $X_0,X_1,\ldots$ are Polish spaces, then $\prod_{n\in\omega}X_n$ is Polish.*

Proof. Suppose that d_n is a complete metric on X_n such that $\forall x,y\in X_n[d(x,y)<1]$. Define $\hat{d}$ on $\prod_{n\in\omega}X_n$ by

$$\hat{d}(x,y)=\sum_{n\in\omega}\frac{1}{2^{n+1}}d_n(x(n),y(n)).$$

Clearly $\hat{d}(x,y)=0$ iff $x=y$, and $\hat{d}(x,y)=\hat{d}(y,x)$. Next,

$$\begin{aligned}
\hat{d}(x,y)+\hat{d}(y,z)&=\sum_{n\in\omega}\frac{1}{2^{n+1}}d_n(x(n),y(n))+\sum_{n\in\omega}\frac{1}{2^{n+1}}d_n(y(n),z(n))\\
&=\sum_{n\in\omega}\frac{1}{2^{n+1}}(d_n(x(n),y(n))+d_n(y(n),z(n))\\
&\geq\sum_{n\in\omega}\frac{1}{2^{n+1}}d_n(x(n),z(n))=\hat{d}(x,z).
\end{aligned}$$

Next, suppose that $\langle x^n:n\in\omega\rangle$ is a Cauchy sequence. Take any $m\in\omega$. We claim that $\langle x^n(m):n\in\omega\rangle$ is a Cauchy sequence. For, take any $\varepsilon>0$, let $\varepsilon'=\frac{\varepsilon}{\frac{1}{2^{m+1}}}$ and choose N so that $\forall n\geq N[\hat{d}(x^N,x^n)<\varepsilon']$. Thus for all $n\geq N$,

$$\sum_{p\in\omega}\frac{1}{2^{p+1}}d_p(x^N(p),x^n(p))<\varepsilon'.$$

Then $\frac{1}{2^{m+1}}d_m(x^N(m), x^n(m)) < \varepsilon'$, and hence $d_m(x^N(m)x^n(m)) < \varepsilon$.

This proves the claim. For each $m \in \omega$ let $y(m) = \lim_{n\in\omega} x^n(m)$. Then $\lim_{n\in\omega} x^n = y$. For, let $\varepsilon > 0$. Choose M so that $\frac{1}{2^M} < \frac{\varepsilon}{2}$. Note that

$$\sum_{m\geq M} \frac{1}{2^{m+1}} = \frac{1}{2^M} < \frac{\varepsilon}{2}.$$

Choose $N \geq M$ so that for all $i < M$ and all $m \geq N$, $d_i(x^i(m), y(m)) < \frac{\varepsilon}{2M}$. Then for any $n \geq N$,

$$\begin{aligned}\hat{d}(x^n, y) &= \sum_{m\in\omega} \frac{1}{2^{m+1}} d_m(x^n(m), y(m)) \\ &\leq \sum_{i<M} d_i(x^i(m), y(m)) + \sum_{i\geq M} \frac{1}{2^{i+1}} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Thus $\hat{d}$ is a complete metric on $\prod_{n\in\omega} X_n$. Next, $\hat{d}$ determines the usual topology on $\prod_{n\in\omega} X_n$. For, let U be basic open in $\prod_{n\in\omega} X_n$. Then we can write $U = \prod_{n\in\omega} Y_n$, where each Y_n is open in X_n and there is a finite $F \subseteq \omega$ such that $Y_n = X_n$ for all $n \notin F$. Let $x \in U$. We want to find an $\varepsilon > 0$ such that $B_{\hat{d}}(x, \varepsilon) \subseteq U$. Choose $\varepsilon > 0$ so that for all $n \in F$, $B_{d_n}(x_n, \varepsilon) \subseteq Y_n$. Let

$$\varepsilon' = \frac{\varepsilon}{\prod_{n\in F} 2^{n+1}}.$$

Suppose that $y \in B_{\hat{d}}(x, \varepsilon')$. Thus $\hat{d}(x, y) < \varepsilon'$, i.e.,

$$\sum_{n\in\omega} \frac{1}{2^{n+1}} d_n(x(n), y(n)) < \varepsilon'.$$

If $n \in F$, then $\frac{1}{2^{n+1}}d_n(x(n), y(n)) < \varepsilon'$, and hence $d_n(x(n), y(n)) < \varepsilon$. It follows that $B_{\hat{d}}(x, \varepsilon) \subseteq U$.

Conversely, given $\varepsilon > 0$ and $x \in \prod_{n\in\omega} X_n$, we want to find a basic open subset V of $\prod_{n\in\omega} X_n$ such that $x \in V \subseteq B_{\hat{d}}(x, \varepsilon)$. Choose N so that

$$\sum_{n\geq N} \frac{1}{2^{n+1}} < \frac{\varepsilon}{2},$$

and then define

$$\varepsilon' = \frac{\varepsilon}{2N \prod_{n<N} 2^{n+1}}.$$

For each $n < N$ let $Y_n = B_{d_n}(x(n), \varepsilon')$ and $Y_n = X_n$ for all $n \geq N$. Then $x \in \prod_{n\in\omega} Y_n \subseteq B_{\hat{d}}(x, \varepsilon)$.

It remains only to show that $\prod_{n\in\omega} X_n$ is separable. For each $n \in \omega$ let D_n be a countable dense subset of X_n. For each $n \in \omega$ let $a_n \in X_n$. Let $E = \{x \in \prod_{n\in\omega} X_n :$ there is a finite $F \subseteq \omega$ such that $\forall n \in F[x_n \in D_n]$ and $\forall n \in \omega\backslash F[x_n = a_n]$. Thus E is countable. Let U be a basic open subset of $\prod_{n\in\omega} X_n$. Say $F \subseteq \omega$ is finite and $U = \prod_{n\in\omega} V_n$ with each V_n open in X_n and $V_n = X_n$ for all $n \notin F$. Clearly there is an $x \in E \cap U$. □

The *Hilbert cube* is the space $\mathbf{H} =^{\omega} [0, 1]$. It is Polish.

Theorem 25.4. m3 *Every Polish space is homeomorphic to a subspace of* $\mathbf{H}$.

Proof. Let X be Polish. Say the topology on X is produced by a complete metric d such that $\forall x, y \in X[d(x, y) < 1]$. Let $\{x_n : n \in \omega\}$ be a dense subset of X. Define $f : X \to \mathbf{H}$ by $f(x) = \langle d(x, x_0), d(x, x_1), \ldots\rangle$.

(1) The following set is a base for the topology on $\mathbf{H}$:

$$\left\{\prod_{n\in\omega} Y_n : \exists F \in [\omega]^{<\omega}[\forall n \in \omega\backslash F[Y_n = [0, 1]] \text{ and}\right.$$

$$\left.\forall n \in F\exists u_n, v_n[-1 < u_n < v_n < 2 \text{ and } Y_n = [0, 1] \cap (u_n, v_n)]\right\}.$$

In fact, let U be a basic open set in the product topology. Say $F \in [\omega]^{<\omega}$ and $U = \prod_{n\in\omega} Z_n$ such that $\forall n \in \omega\backslash F[Z_n = [0, 1]]$ and $\forall n \in F[Z_n$ is open in $[0, 1]]$. Then for each $n \in F$ there is a collection T_n of open intervals in $[0, 1]$ such that $Z_n = \bigcup T_n$. Clearly this gives (1).

Now to show that f is continuous, suppose that $\prod_{n\in\omega} Y_n$ is as in (1), and $x \in f^{-1}[\prod_{n\in\omega} Y_n]$. Thus $f(x) \in \prod_{n\in\omega} Y_n$, so for all $n \in F$, $u_n < d(x, x_n) < v_n$. Let $\varepsilon = \frac{1}{2}\min(\{v_n - d(x, x_n) : n \in F\} \cup \{d(x, x_n) - u_n\})$. Then $x \in \bigcap_{n\in F} B(x, \varepsilon) \subseteq \prod_{n\in\omega} Y_n$. In fact, if $y \in B(x, \varepsilon)$ and $n \in F$, then

$$d(y, x_n) \leq d(y, x) + d(x, x_n) < \varepsilon + d(x, x_n) \in (u_n, v_n),$$

and hence $f(y) \in \prod_{n\in\omega} Y_n$.

Next, f is one-one. For, suppose that $x, y \in X$ and $x \neq y$. Say $d(x, y) = \varepsilon$. Choose $x_n \in B(x, \frac{\varepsilon}{2})$. Then $d(x, x_n) < \frac{\varepsilon}{2}$ and $\varepsilon = d(x, y) \leq d(x, x_n) + d(y, x_n)$, so $d(y, x_n) > \frac{\varepsilon}{2}$. Hence $x \neq y$.

Finally, $f^{-1} : \operatorname{rng}(f) \to X$ is continuous. For, suppose that $f(x^m) \to f(x)$; we want to show that $x^m \to x$. So, let $\varepsilon > 0$. Choose n so that $d(x, x_n) < \frac{\varepsilon}{3}$. Then $\exists M\forall m \geq M[\hat{d}(f(x^m), f(x)) < \frac{1}{3\cdot 2^{n+1}}\varepsilon]$, i.e., for all $m \geq M$,

$$\sum_{n\in\omega} \frac{1}{2^{n+1}}|d(x^m, x_n) - d(x, x_n)| < \frac{1}{3 \cdot 2^{n+1}}\varepsilon.$$

Then $|d(x^m, x_n) - d(x, x_n)| < \frac{\varepsilon}{3}$, so $|d(x^m, x_n) - d(x, x_n)| < \frac{\varepsilon}{3}$. Hence $d(x^m, x_n) - d(x, x_n) < \frac{\varepsilon}{3}$. Hence $d(x^m, x) \leq d(x, x_n) + d(x^m, x_n) < 2d(x, x_n) + \frac{\varepsilon}{3} < \varepsilon$. □

If X is a metric space and $Y \subseteq X$, the $\operatorname{diam}(Y) = \sup\{d(x,y) : x, y \in Y\}$.

Lemma 25.5. m3 *Suppose that X is Polish and $X_0 \supseteq X_1 \supseteq \cdots$ are closed sets such that $\lim_{n\to\infty} \operatorname{diam}(X_n) = 0$. Then there is an $x \in X$ such that $\bigcap_{n\in\omega} X_n = \{x\}$.*

Proof. For each $n \in \omega$ choose $x_n \in X_n$.

(1) x is a Cauchy sequence.

In fact, let $\varepsilon > 0$. Choose N so that $\forall n \geq N[\operatorname{diam}(X_n) < \varepsilon]$. Then for all $n \geq N$, $d(x_n, x_N) < \varepsilon$ since $x_n, x_N \in X_n$. So (1) holds.

Let a be the limit of x. Then $a \in \bigcap_{n\in\omega} X_n$. If also $b \in \bigcap_{n\in\omega} X_n$, then $a = b$ since $d(a,b) = 0$. □

Lemma 25.6. m4 *If X is a Polish space, $U \subseteq X$ is open, and $\varepsilon > 0$, then there are open sets $U_0, U_1, \ldots$ such that $U = \bigcup_{n\in\omega} U_n = \bigcup_{n\in\omega} \overline{U}_n$, and $\forall n \in \omega[\operatorname{diam}(U_n) < \varepsilon]$.*

Proof. Let D be a countable dense subset of X.

(1) There is a $d \in D$ and an $n \in \omega\backslash\{0\}$ such that $\frac{1}{n} < \frac{\varepsilon}{2}$ and $\overline{B(d, \frac{1}{n})} \subseteq U$.

For, fix $d \in U \cap D$. Then there is a $\delta > 0$ such that $B(d,\delta) \subseteq U$. Choose $n \in \omega\backslash\{0\}$ such that $\frac{1}{n} < \frac{\varepsilon}{2}, \frac{\delta}{2}$. Then $\overline{B(d, \frac{1}{n})} \subseteq U$.

Now let $U_0, U_1, \ldots$ list all such sets $\overline{B(d, \frac{1}{n})} \subseteq U$. Suppose that $x \in U$. Choose $n \in \omega\backslash\{0\}$ so that $\frac{1}{n} < \varepsilon$ and $B(x, \frac{1}{n}) \subseteq U$. Choose $d \in D \cap B(x.\frac{1}{3n})$. Then $x \in B(d, \frac{1}{3n})$ and $\overline{B(d, \frac{1}{3n})} \subseteq U$. So there is an i such that $B(d, \frac{1}{3n}) = U_i$. Hence $x \in U_i$. □

The *Baire space* is ${}^{\omega}\omega$ with the product topology, ω having the discrete topology. This is a Polish space, with complete metric given by the above as

$$\hat{d}(x,y) = \sum_{n\in\omega} \frac{1}{2^{n+1}} d(x(n), y(n)),$$

where for $i, j \in \omega$,

$$d(i,j) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{otherwise.} \end{cases}$$

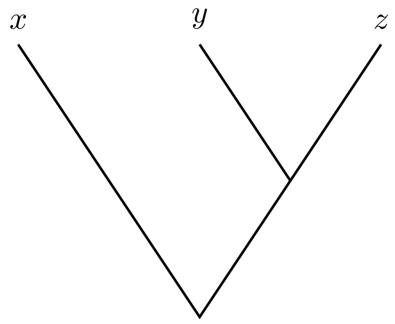
Proposition 25.7. m4 *Another complete metric on ${}^{\omega}\omega$ giving the product topology is*

$$\bar{d}(x,y) = \begin{cases} 0 & \textit{if } x = y, \\ \frac{1}{2^{n+1}} & \textit{if } x \neq y \textit{ and } n \textit{ is minimum such that } x(n) \neq y(n). \end{cases}$$

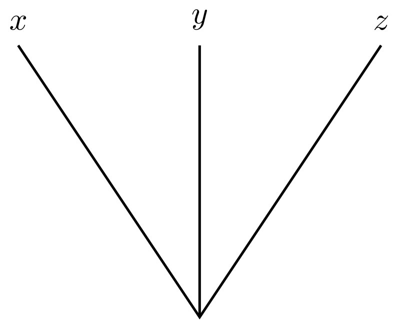
Proof. Clearly (i) and (ii) in the definition of metric hold. Now suppose that $x, y, z \in {}^{\omega}\omega$; we want to show that $\bar{d}(x,y) + \bar{d}(y,z) \geq \bar{d}(x,z)$. This is clear if the x, y, z are not distinct; so suppose that they are distinct. Let

$$m = \min\{i : x(i) \neq y(i)\};$$
$$n = \min\{i : y(i) \neq z(i)\};$$
$$p = \min\{i : x(i) \neq z(i)\}.$$

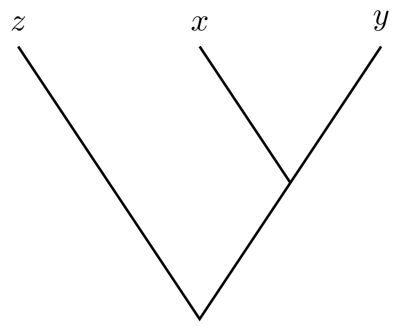
Case 1. $m < n$. Then $m = p$.



Case 2. $m = n$. Again $m = p$



Case 3. $n < m$. Then $p = n$.



For the complete metric condition, suppose that $\langle x^m : m \in \omega\rangle$ is a Cauchy sequence of members of ${}^\omega\omega$. We may assume that $\langle x^m : m \in \omega\rangle$ is not eventually constant. Now we define $y \in {}^\omega\omega$. For each $n \in \omega$ choose $M(n)$ minimum so that $\forall m \geq M(n)[\bar{d}(x^{M(n)}, x^m) < \frac{1}{2^{n+1}}$. Thus for all $m \geq M(n)$ the least p such that $x^{M(n)}(p) \neq x^m(p)$ is greater than n. Let $y(n) = x^{M(n)}(n)$. We claim that $\langle x^m : m \in \omega\rangle$ converges to y. For, let $\varepsilon > 0$ be given. Choose n so that $\frac{1}{2^{n+1}} < \varepsilon$. Take any $m \geq \max\{M(n') : n' \leq n\}$. We claim that $\bar{d}(x^m, y) < \varepsilon$. Suppose that $p \leq n$. Then $x^m(p) = x^{M(p)}(p) = y(p)$, as desired.

Now to show that the topology given by $\bar{d}$ is the product topology, first suppose that $B_{\hat{d}}(x, \varepsilon)$ is given; we show that it is open in the topology given by $\bar{d}$. So suppose that

$y \in B_{\hat{d}}(x, \varepsilon)$. Thus

$$\hat{d}(x, y) = \sum_{n \in \omega} \frac{1}{2^{n+1}} d(x(n), y(n)) < \varepsilon.$$

If n is minimum such that $x(n) \neq y(n)$, then

$$\bar{d}(x, y) = \frac{1}{2^{n+1}} \leq \hat{d}(x, y).$$

This shows that $B_{\bar{d}}(x, \varepsilon) \subseteq B_{\hat{d}}(x, \varepsilon)$.

Conversely, given $B_{\bar{d}}(x, \varepsilon)$, we show that it is open in the product topology. Suppose that $y \in B_{\bar{d}}(x, \varepsilon)$. Let n be minimum such that $x(n) \neq y(n)$. So $\bar{d}(x, y) = \frac{1}{2^{n+1}} < \varepsilon$. Let $z = y|(n+1)$ and $U_z = \{f \in {}^\omega\omega : z \subseteq f\}$. So U_z is open in the product topology and $y \in U_z$. If $f \in U_z$ then $\bar{d}(x, f) = \frac{1}{2^{n+1}} < \varepsilon$. $\square$

Let

$$C = \left\{ x \in [0, 1] : \exists t \in {}^{\omega \backslash 1}\{0, 2\} \left[x = \sum_{i=1}^{\infty} \frac{t_i}{3^i} \right] \right\}.$$

C is the *Cantor set*. For $a < b$ let

$$f([a, b]) = \left\{ \left[a, a + \frac{1}{3}(b - a) \right], \left[a + \frac{2}{3}(b - a), b \right] \right\}.$$

Define A with domain ω recursively by

$$A_0 = \{[0, 1]\};$$
$$A_{n+1} = \bigcup_{X \in A_n} f(X).$$

Lemma 25.8. *For every positive integer n and every set Y, $Y \in A_n$ iff there is a $t : (n + 1)\backslash 1 \to \{0, 2\}$ such that*

$$Y = \left[\sum_{i=1}^{n} \frac{t_i}{3^i}, \sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{1}{3^n} \right].$$

Proof. For $n = 1$ we have $A_1 = f([0, 1]) = \{[0, \frac{1}{3}], [\frac{2}{3}, 1]\}$. With $t_1 = 0$ we have $[\sum_{i=1}^{n} \frac{t_i}{3^i}, \sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{1}{3^n}] = [0, \frac{1}{3}]$, and with $t_1 = 2$ we have $[\sum_{i=1}^{n} \frac{t_i}{3^i}, \sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{1}{3^n}] = [\frac{2}{3}, 1]$, as desired.

Now assume the equality for $n \geq 1$. First suppose that $Y \in A_{n+1}$. Then there is an $X \in A_n$ such that $Y \in f(X)$. By the inductive hypothesis choose $t : (n + 1)\backslash 1 \to \{0, 2\}$ such that

$$X = \left[\sum_{i=1}^{n} \frac{t_i}{3^i}, \sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{1}{3^n} \right].$$

Note that X has size $\frac{1}{3^n}$.

Case 1.

$$Y = \left[\sum_{i=1}^{n} \frac{t_i}{3^i}, \sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{1}{3^{n+1}}\right].$$

Let $s|((n+1)\backslash 1) = t|((n+1)\backslash 1)$ and $s(n+1) = 0$. Then

$$(*) \qquad Y = \left[\sum_{i=1}^{n+1} \frac{s_i}{3^i}, \sum_{i=1}^{n+1} \frac{s_i}{3^i} + \frac{1}{3^{n+1}}\right].$$

Case 2.

$$Y = \left[\sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{2}{3^{n+1}}, \sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{1}{3^n}\right].$$

Let $s|((n+1)\backslash 1) = t|((n+1)\backslash 1)$ and $s(n+1) = 2$. Then $(*)$ holds.

Second, suppose that $(*)$ holds. Let $t = s|((n+1)\backslash 1)$. If $s(n+1) = 0$, then

$$Y = \left[\sum_{i=1}^{n} \frac{t_i}{3^i}, \sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{1}{3^{n+1}}\right];$$

If $s(n+1) = 2$, then

$$Y = \left[\sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{2}{3^{n+1}}, \sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{1}{3^n}\right].$$

Hence in either case,

$$Y \in f\left(\left[\sum_{i=1}^{n} \frac{t_i}{3^i}, \sum_{i=1}^{m} \frac{t_i}{3^i} + \frac{1}{3^n}\right]\right),$$

and

$$\left[\sum_{i=1}^{n} \frac{t_i}{3^i}, \sum_{i=1}^{m} \frac{t_i}{3^i} + \frac{1}{3^n}\right] \in A_n$$

by the inductive hypothesis. Hence $Y \in A_{n+1}$. □

Theorem 25.9. $C = \bigcap_{n\in\omega} \bigcup A_n$.

Proof. Suppose that $x \in C$ and $n \in \omega$. Choose $s \in {}^{\omega\backslash 1}\{0,2\}$ such that $x = \sum_{i=1}^{\infty} \frac{s_i}{3^i}$. Let $t = s|((n+1)\backslash 1)$. Then

$$x \in \left[\sum_{i=1}^{n} \frac{t_i}{3^i}, \sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{1}{3^n}\right]$$

since $\sum_{i=n+1}^{\infty} \frac{t_i}{3^i} \leq \frac{1}{3^n}$. Thus by Lemma 25.8, $x \in \bigcup A_n$.

Now suppose that $x \notin C$. If $x \notin [0,1]$, clearly $x \notin \bigcap_{n\in\omega}(\bigcup A_n)$. So suppose that $x \in [0,1]$, and choose $t \in {}^{\omega\backslash 1}3$ such that $x = \sum_{i=1}^{\infty}\frac{t_i}{3^i}$. Choose n minimal such that $t_n = 1$. Suppose $x \in \bigcup A_n$. By Lemma 25.8, choose $s \in {}^{(n+1)\backslash 1}\{0,2\}$ such that

$$\sum_{i=1}^{n}\frac{s_i}{3^i} \le x \le \sum_{i=1}^{n}\frac{s_i}{3^i} + \frac{1}{3^n}.$$

We claim that $t|n = s|n$. Otherwise there is a least $m < n$ such that $t_m \neq s_m$. If $t_m < s_m$, then $t_m = 0$ since $t_m \in \{0,2\}$ because $m < n$. Hence

$$x = \sum_{i=1}^{\infty}\frac{t_i}{3^i} = \sum_{i=1}^{m}\frac{t_i}{3^i} + \sum_{i=m+1}^{\infty}\frac{t_i}{3^i} \le \sum_{i=1}^{m}\frac{t_i}{3^i} + \frac{1}{3^m} < \sum_{i=1}^{m}\frac{s_i}{3^i} \le \sum_{i=1}^{n}\frac{s_i}{3^i} \le x,$$

contradiction. If $s_m < t_m$, then $s_m = 0$ and $t_m = 2$, and

$$x \le \sum_{i=1}^{n}\frac{s_i}{3^i} + \frac{1}{3^n} \le \sum_{i=1}^{m}\frac{s_i}{3^i} + \sum_{i=m+1}^{\infty}\frac{2}{3^i} = \sum_{i=1}^{m}\frac{s_i}{3^i} + \frac{1}{3^m} < \sum_{i=1}^{m}\frac{t_i}{3^i} \le x,$$

contradiction.

So $s|n = t|n$.

Case 1. $s_n = 2$. Then

$$x = \sum_{i=1}^{\infty}\frac{t_i}{3^i} = \sum_{i=1}^{n}\frac{t_i}{3^i} + \sum_{i=n+1}^{\infty}\frac{t_i}{3^i} \le \sum_{i=1}^{n}\frac{t_i}{3^i} + \frac{1}{3^n} = \sum_{i=1}^{n}\frac{s_i}{3^i} \le x$$

It follows that $t_i = 2$ for all $i \ge n+1$, and hence $x = (t|n)^\frown\langle 2,0,0,0,\ldots\rangle \in C$, contradiction.

Case 2. $s_n = 0$. Then

$$x = \sum_{i=1}^{\infty}\frac{t_i}{3^i} \le \sum_{i=1}^{n}\frac{s_i}{3^i} + \frac{1}{3^n} = \sum_{i=1}^{n}\frac{t_i}{3^i} \le x;$$

it follows that $t_i = 0$ for all $i > n$; hence $x = (t|n)^\frown\langle 0,2,2,2,\ldots\rangle \in C$, contradiction. □

Theorem 25.10. m4 *C is homeomorphic to ${}^{\omega}2$.*

Proof. For each $t \in {}^{\omega}2$ let

$$f(t) = \sum_{i=1}^{\infty}\frac{2t_{i-1}}{3^i}.$$

Clearly f maps onto C. It is one-one; for suppose that $s, t \in {}^{\omega}2$ with $s \neq t$. Let n be minimum such that $s_n \neq t_n$. Say $s_n = 0$ and $t_n = 1$. Then

$$\begin{aligned} f(s) &= \sum_{i=1}^{\infty}\frac{2s_{i-1}}{3^i} = \sum_{i=1}^{n}\frac{2s_{i-1}}{3^i} + \sum_{i=n+1}^{\infty}\frac{2s_{i-1}}{3^i} \\ &\le \sum_{i=1}^{n}\frac{2s_{i-1}}{3^i} + \sum_{i=n+1}^{\infty}\frac{2}{3^i} = \sum_{i=1}^{n}\frac{2s_{i-1}}{3^i} + \frac{1}{3^n} < \sum_{i=1}^{n}\frac{2s_{i-1}}{3^i} + \frac{2}{3^n} \le \sum_{i=1}^{\infty}\frac{2t_{i-1}}{3^i} = f(t). \end{aligned}$$

So f is one-one. Now f is continuous. For, suppose that U is open in $\mathbf{R}$ and $t \in f^{-1}[U]$. Say $(f(t)-\varepsilon, f(t)+\varepsilon) \subseteq U$. Choose n so that $3^{-n} < \varepsilon$. Let $V = \{s \in {}^{\omega}2 : s|n = t|n\}$. For any $s \in V$ we have

$$|f(s)-f(t)| = \left| \sum_{n+1 \le i}^{\infty} \frac{2(s(i-1)-t(i-1))}{3^i} \right| \le \sum_{n+1 \le i}^{\infty} \frac{2}{3^i} = \frac{1}{3^n} < \varepsilon.$$

Thus $V \subseteq f^{-1}[U]$. So f is continuous. By Engelking Theorem 3.1.13, f is a homeomorphism. □

Suppose that $S \subseteq {}^{<\omega}\omega$. Then we define

$$T_S = \{\sigma \in {}^{<\omega}\omega : \forall \tau \subseteq \sigma[\tau \notin S]\}.$$

Proposition 25.11. *T_S is a tree under $\subseteq$.*

Proof. Clearly for any $\sigma \in T_S$ the set $\{\tau \in {}^{<\omega}\omega : \tau \subseteq \sigma\} \subseteq T_S$ and is finite and simply ordered. □

An element $f \in {}^{\omega}\omega$ is a *path* through T_S iff $\forall m \in \omega[f|m \in T_S]$. For $\sigma \in {}^{<\omega}\omega$ we let $\omega_\sigma = \{f \in {}^{\omega}\omega : \sigma \subseteq f\}$. The set $\{\omega_\sigma : \sigma \in {}^{<\omega}\omega\}$ is a base for the usual topology on ${}^{\omega}\omega$. We let $[T_S] = \{f \in {}^{\omega}\omega : f$ is a path through T_S.

Proposition 25.12. *The following are equivalent:*
(i) $f \in [T_S]$.
(ii) $\forall \sigma \in S[\sigma \not\subseteq f]$.
(iii) $f \notin \bigcup_{\sigma \in S} \omega_\sigma$.

Proof. (i)⇒(ii): Assume that $f \in [T_S]$, $\sigma \in S$, and $\sigma \subseteq f$. Now $\sigma \in T_S$, so $\sigma \notin S$, contradiction.

(ii)⇒(iii): Assume (ii) and $f \bigcup_{\sigma \in S} \omega_\sigma$. Choose $\sigma \in S$ so that $\sigma \subseteq f$. This contradicts (ii).

(iii)⇒(i): Assume (iii), but suppose that $f \notin [T_S]$. Then there is an $m \in \omega$ such that $(f|m) \notin T_S$. Hence there is a $\tau \subseteq (f|m)$ such that $\tau \in S$. Then (iii) is contradicted. □

Proposition 25.13. m5 *$C \subseteq {}^{\omega}\omega$ is closed iff there is an $S \subseteq {}^{<\omega}\omega$ such that $C = [T_S]$.*

Proof. ⇒: Suppose that C is closed. Then there is an $S \subseteq {}^{<\omega}\omega$ such that $({}^{\omega}\omega \backslash C) = \bigcup_{\sigma \in S} \omega_\sigma$. By Proposition 25.12, $[T_S] = C$.

⇐: Assume that $S \subseteq {}^{<\omega}\omega$ and $C = [T_S]$. By Proposition 25.12, C is closed. □

A tree $T \subseteq {}^{<\omega}\omega$ is *pruned* iff $\forall \sigma \in T \exists \tau \in {}^{<\omega}\omega$ such that $\sigma \subset \tau$.

Proposition 25.14. m5 *Suppose that $T \subseteq {}^{<\omega}\omega$ is a tree of height ω. Let $T' = \{\sigma \in T$:there is a branch f of length ω in T such that $\sigma \subseteq f\}$. Then T' is pruned.* □

Proposition 25.15. m5 *If* $f : {}^{\omega}\omega \to {}^{\omega}\omega$*, then* f *is continuous iff*

$$\forall x \in {}^{\omega}\omega \forall \sigma \in {}^{<\omega}\omega[\sigma \subseteq f(x) \to \exists \tau \subseteq x \forall y \in {}^{\omega}\omega[\tau \subseteq y \to \sigma \subseteq f(y)]]$$

Proof. $\Rightarrow$: Assume that f is continuous, $x \in {}^{\omega}\omega$, $\sigma \in {}^{<\omega}\omega$ and $\sigma \subseteq f(x)$. Thus $x \in f^{-1}[\omega_\sigma]$, so there is a $\tau \in {}^{<\omega}\omega$ such that $x \in \omega_\tau \subseteq f^{-1}[\omega_\sigma]$. Thus $\tau \subseteq x$ and $\forall y \in {}^{\omega}\omega[\tau \subseteq y \to \sigma \subseteq f(y)]$.

$\Leftarrow$: similarly. $\square$

Proposition 25.16. m6 *If* $k, d \in \omega$ *with* $k \neq 0$*, then* ${}^{\omega}\omega$ *is homeomorphic to* ${}^{d}\omega \times {}^{k}({}^{\omega}\omega)$.

Proof. For any $(n, f) \in {}^{d}\omega \times {}^{k}({}^{\omega}\omega)$ let

$$\begin{aligned}\varphi(n, f) = &(n_0, \ldots, n_{d-1}, f_0(0), f_1(0), \ldots, f_{k-1}(0), f_0(1), f_1(1), \ldots, f_{k-1}(1),\\ &\ldots, f_0(n), f_1(n), \ldots, f_{k-1}(n), \ldots).\end{aligned}$$

Thus φ maps ${}^{d}\omega \times {}^{k}({}^{\omega}\omega)$ to ${}^{\omega}\omega$. Clearly φ is one-one and onto. To see that it is continuous, take a subbase U for ${}^{\omega}\omega$ and assume that $(n, f) \in f^{-1}[U]$.

Case 1. There exist an $i < d$ and $A \subseteq \omega$ such that $U = \{g \in {}^{\omega}\omega : g(i) \in A\}$. Let $V = \{(n, f) \in {}^{d}\omega \times {}^{k}({}^{\omega}\omega : n_i \in A\}$ Then $(n, f) \in V \subseteq f^{-1}[u]$.

The other case is similar, and proving that f^{-1} is continuous is similar. $\square$

Proposition 25.17. ${}^{\omega}({}^{\omega}\omega)$ *is homeomorphic to* ${}^{\omega}\omega$.

Proof. Define $k : \omega \to \omega \times \omega$ as follows: $k(0) = (0, 0)$. If $k(n) = (i, j)$, then

$$k(n+1) = \begin{cases} (0, i+1) & \text{if } j = 0; \\ (i+1, j-1) & \text{otherwise.} \end{cases}$$

Then k is a bijection. Now for any $f \in {}^{\omega}({}^{\omega}\omega)$ define $\varphi(f) \in {}^{\omega}\omega$ as follows. Take any $n \in \omega$, let $k(n) = (i, j)$, and define $(\varphi(f))(n) = f_i(j)$. Then φ is the desired homeomorphism. $\square$

Proposition 25.18. m6 *If* X *is a Polish space, then there is a continuous surjection* $\varphi : {}^{\omega}\omega \to X$.

Proof.

Claim. There is a function U with domain ${}^{<\omega}[\omega]$ satisfying the following conditions:

(i) $U_\emptyset = X$.

(ii) $\forall \sigma \in [\omega]^{<\omega}[U_\sigma$ is an open subset of $X]$.

(iii) $\forall \sigma \in [\omega]^{<\omega}[\mathrm{diam}(U_\sigma) < \frac{1}{\mathrm{dmn}(\sigma)}]$.

(iv) $\forall \sigma, \tau \in [\omega]^{<\omega}[\sigma \subset \tau \to \overline{U_\tau} \subseteq U_\sigma]$.

(v) $\forall \sigma \in [\omega]^{<\omega}[U_\sigma = \bigcup_{i \in \omega} U_{\sigma \frown \langle i \rangle}]$.

Proof of claim. Let $U_\emptyset = X$. If U_σ has been defined for $|\sigma| = m$, by Lemma 25.6 we can write $U_\sigma = \bigcup_{i\in\omega} U_{\sigma\frown\langle i\rangle}$, with $\mathrm{diam}(U_{\sigma\frown\langle i\rangle}) < \frac{1}{m+1}$ for each i, each $U_{\sigma\frown\langle i\rangle}$ open. So (i)–(v) hold. $\square$

(1) $\bigcap_{n\in\omega} U_{f|n} = \bigcap_{n\in\omega} \overline{U_{f|n}}$.

In fact,

$$\bigcap_{n\in\omega} \overline{U_{f|n}} = \bigcap_{n\in\omega} \overline{U_{(f|n)\frown\langle f(n)\rangle}} \subseteq \bigcap_{m\in\omega} U_{f|n} \subseteq \bigcap_{n\in\omega} \overline{U_{f|n}}$$

Now by Lemma 25.5 we can define $\varphi(f)$ to be the unique x such that $\bigcap_{n\in\omega} U_{f|n} = \{x\}$.

(2) φ is a surjection.

For, let $x \in X$. We define $f \in {}^\omega\omega$. Suppose that $f|n$ has been defined so that $x \in U_{f|n}$. By (v), let $f(n)$ be such that $x \in U_{f|(n+1)}$. Clearly $\varphi(f) = x$.

To show that φ is continuous, suppose that $f \in \varphi^{-1}[B_\varepsilon(x)]$. Thus $d(\varphi(f), x) < \varepsilon$. Choose n so that $\frac{1}{n} < \varepsilon$. Then $\varphi(f) \in U_{f|n}$. We claim that $\{g \in {}^\omega\omega : g|n = f|n\} \subseteq \varphi^{-1}[B_\varepsilon(x)]$. Suppose that $g \in {}^\omega\omega$ and $g|n = f|n$. Now $\varphi(g) \in U_{f|n}$, so by (iii), $d(\varphi(f), \varphi(g)) < \frac{1}{n} < \varepsilon$. $\square$

If X is a Polish space, then a subset $P \subseteq X$ is *perfect* iff P is closed and has no isolated points. Obviously $\emptyset$ is perfect.

Theorem 25.19. m8 *If X is a Polish space and $P \subseteq X$ is nonempty and perfect, then there is a perfect $F \subseteq P$ which is homeomorphic to ${}^\omega 2$. In particular, $|P| = 2^\omega$.*

Proof. We claim that there is a system $\langle U_\sigma : \sigma \in {}^{<\omega}2\rangle$ of nonempty open subsets of X such that the following conditions hold:

(1) $U_\emptyset = X$,
(2) If $\sigma \subset \tau$, then $\overline{U_\tau} \subseteq U_\sigma$.
(3) $U_{\sigma\frown\langle 0\rangle} \cap U_{\sigma\frown\langle 1\rangle} = \emptyset$.
(4) For $\sigma \neq \emptyset$, $\mathrm{diam}(U_\sigma) < \frac{1}{\mathrm{dmn}(\sigma)}$.
(5) $U_\sigma \cap P \neq \emptyset$.

In fact, we find U_σ by induction on $\mathrm{dmn}(\sigma)$. (1) forces the definition of $U_\emptyset$, and clearly (1)–(5) hold. Now suppose that U_σ has been defined so that (1)–(5) hold. Since P has no isolated points, there are distinct $x_0, x_1 \in U_\sigma \cap P$. Let $U_{\sigma\frown\langle 0\rangle}$ and $U_{\sigma\frown\langle 1\rangle}$ be disjoint open neighborhoods of x_0, x_1 respectively such that $\overline{U_{\sigma\frown\langle 0\rangle}}, \overline{U_{\sigma\frown\langle 1\rangle}}$ are subsets of U_σ with diameters less than $\frac{1}{\mathrm{dmn}(\sigma)+1}$. Clearly this extends (1)–(5) to σ of one greater length.

Now by Lemma 25.5, for each $x \in {}^\omega 2$ there is an $f(x) \in X$ such that $\{f(x)\} = \bigcap_{n\in\omega}(\overline{U_{x|n}} \cap P)$. Clearly f is one-one. f is continuous: for, suppose that $x \in f^{-1}[V_\varepsilon(f(x))]$. Choose a positive integer n such that $\frac{1}{n} < \varepsilon$. Then $x \in f^{-1}[U_{x|n}] \subseteq f^{-1}[V_\varepsilon(f(x))]$.

Since ${}^\omega 2$ is compact and $\mathrm{dmn}(f) = {}^\omega 2$, $\mathrm{rng}(f)$ is compact and hence closed. It has no isolated points; see Proposition 25.20.

Now $f : {}^\omega 2 \to \mathrm{rng}(f)$ is a homeomorphism. For, suppose V_τ is open in ${}^\omega 2$ and $x \in f[V_\tau]$; so f is a homeomorphism. $\square$

Proposition 25.20. m8 *If X is a polish space and $f : {}^{\omega}2 \to X$ is continuous and one-one, then* $\mathrm{rng}(f)$ *is perfect.*

Proof. Clearly $\mathrm{rng}(f)$ is closed. Suppose that $x \in {}^{\omega}2$ and $f(x)$ is isolated. For each $n \in \omega$ choose $y^n] \in {}^{\omega}2$ such that $y^n|n = x|n$ and $y^n \neq x$. Clearly $y^n \to x$, so $f(y_n) \to f(x)$, contradicting $f(x)$ being isolated. □

Corollary 25.21. m8 **Q** *is not a Polish space.*

Proof. By Proposition 25.19. □

Assume that $F \subseteq X$ is closed. Then we define

$$\begin{aligned}
\Gamma(F) &= \{x \in F : x \text{ is not an isolated point of } F;\\
\Gamma^0(F) &= F;\\
\Gamma^{\alpha+1}(F) &= \Gamma(\Gamma^\alpha(F))\\
\Gamma^\alpha(F) &= \bigcap_{\beta<\alpha} \Gamma^\beta(F) \quad \text{for } \alpha \text{ limit}
\end{aligned}$$

Proposition 25.22. m9 *For every ordinal α, $\Gamma^\alpha(F)$ is closed.*

Proof. If $x \in F \backslash \Gamma(F)$, then x is isolated in F, so there is an open subset U_x of X such that $U_x \cap F = \{x\}$. Thus

$$X\backslash\Gamma(F) = (X\backslash F) \cup (F\backslash\Gamma(F)) = (X\backslash F) \cup \bigcup_{x\in F\backslash\Gamma(F)} U_x,$$

so $X\backslash\Gamma(F)$ is open. □

Proposition 25.23. *If X is a Polish space and $F \subseteq X$ is closed, then $\{x \in F : x$ is isolated in $F\}$ is countable.*

Proof. By Engelking Theorem 4.1.15. □

Proposition 25.24. m9 $|F\backslash\Gamma(F)| \le \omega$.

Proof. Note that $F\backslash\Gamma(F) = \{x \in F : x \text{ is isolated in } F\}$. □

Proposition 25.25. m9 *If $\Gamma(F) = F$, then F is perfect, and $\Gamma^\alpha(F) = F$ for all α.* □

Proposition 25.26. m9 *There is an ordinal $\alpha < \omega_1$ such that $\Gamma^\alpha(F) = \Gamma^{\alpha+1}(F)$.*

Proof. Let $U_0, U_1, \ldots$ be a countable base for X. If $\Gamma^\alpha(F)\backslash\Gamma^{\alpha+1}(F) \neq \emptyset$, then there is an isolated point x_α of $\Gamma^\alpha(F)$ and hence there is an $n_\alpha \in \omega$ such that $U_{n_\alpha} \cap \Gamma^\alpha(F) = \{x_\alpha\}$. If $\beta < \alpha$, then

$$(U^\beta(F)\backslash U^{\beta+1}(F)) \cap (U^\alpha(F)\backslash U^{\alpha+1}(F)) = \emptyset,$$

and so $n_\alpha \neq n_\beta$. If $\forall \alpha < \omega_1[\Gamma^\alpha(F)\backslash\Gamma^{\alpha+1}(F) \neq \emptyset]$, then $\langle n_\alpha : \alpha < \omega_1\rangle$ is one-one, contradiction. $\square$

For every closed $F \subseteq X$, the *Cantor-Bendixson rank* of F is the least ordinal α such that $\Gamma^\alpha(F) = \Gamma^{\alpha_1}(F)$.

Proposition 25.27. m9 *If X is Polish and $F \subseteq X$ is closed, then there exist P and A such that $F = P \cup A$, where P is empty or perfect, A is countable, and $P \cap A = \emptyset$.*

Proof. Let α be the Cantor-Bendixson rank of F; let $P = \Gamma^\alpha(F)$, and let $A = \bigcup_{\beta<\alpha}(\Gamma^\alpha(F)\backslash\Gamma^{\alpha+1}(F)$. $\square$

Proposition 25.28. *If X is a Polish space and $F \subseteq X$ is uncountable and closed, then F contains a nonempty perfect set, and $|F| = 2^\omega$.*

Proof. F contains a nonempty perfect set by Proposition 25.27, and $|F| = 2^\omega$ by Proposition 25.19. $\square$

Proposition 25.29. *If X is Polish and $F \subseteq X$ is nonempty and closed, then F is Polish.*

Proof. By Theorem 4.1.15 of Engelking, F is separable. If $\langle x_n : n \in \omega\rangle$ is convergent and each $x_n \in F$, then the limit of $\langle x_n : n \in \omega\rangle$ is in F. $\square$

Proposition 25.30. $(0, 1)$ *is not Polish.*

Proof. $\langle \frac{1}{n} : n$ a positive integer$\rangle$ is Cauchy but does not converge. $\square$

Lemma 25.31. m9 *If X is Polish and $U \subseteq X$ is open, then U is Polish.*

Proof. Let d be a complete metric on X giving the topology. By Proposition 25.2 we may assume that $\forall x, y \in X[d(x, y) < 1]$. For $x, y \in U$ let

$$\hat{d}(x, y) = d(x, y) + \left|\frac{1}{d(x, X\backslash U)} - \frac{1}{d(y, X\backslash U)}\right|.$$

(1) $\hat{d}$ is a metric on U.

For, clearly $\hat{d}(x, y) = 0$ iff $x = y$; and $\hat{d}(x, y) = \hat{d}(y, x)$. Also.

$$\begin{aligned}\hat{d}(x, y) + \hat{d}(y, z) &= d(x, y) + \left|\frac{1}{d(x, X\backslash U)} - \frac{1}{d(y, X\backslash U)}\right| \\ &\quad + d(y, z) + \left|\frac{1}{d(y, X\backslash U)} - \frac{1}{d(z, X\backslash U)}\right| \\ &\geq d(x, z) + \left|\frac{1}{d(x, X\backslash U)} - \frac{1}{d(z, X\backslash U)}\right| \\ &= \hat{d}(x, z).\end{aligned}$$

Thus (1) holds.

(2) $\hat{d}$ determines the topology on U.

In fact, first suppose that W is open in U according to d. Take any $x \in W$ and choose $\varepsilon > 0$ such that $\{y : d(x.y) < \varepsilon\} \subseteq W$. Since $\hat{d}(x,y) > d(x,y)$, it follows that $\{y : \hat{d}(x.y) < \varepsilon\} \subseteq W$. So W is open according to $\hat{d}$.

Second, suppose that W is open in U according to $\hat{d}$. Take any $x \in W$ and choose $\varepsilon > 0$ such that $\{y : \hat{d}(x.y) < \varepsilon\} \subseteq W$. Let δ be such that

$$\begin{aligned}
&\delta < \frac{\varepsilon}{2} \text{ and}\\
&\frac{1}{r-\delta} - \frac{1}{r} < \frac{\varepsilon}{2} \text{ and}\\
&\frac{1}{r} - \frac{1}{\delta + r} < \frac{\varepsilon}{2}.
\end{aligned}$$

Suppose that $d(x,y) < \delta$. Let $d(x, X\backslash U) = r$.

If $z \in X\backslash W$ then $d(y, X\backslash U) \leq d(y,z) \leq d(x,y) + d(x,z)$, so $d(y, X\backslash W) \leq \delta + r$. Hence

$$\frac{1}{\delta + r} \leq \frac{1}{d(y, X\backslash U)}.$$

$r \leq d(x,z) \leq d(x,y) + d(y,z)$, so $r \leq d(x,y) + d(y, X\backslash U) \leq \delta + d(y, X\backslash U)$. Hence $r - \delta \leq d(y, X\backslash U)$. so

$$\frac{1}{d(y, X\backslash U)} \leq \frac{1}{r-\delta}.$$

Hence

$$\frac{1}{\delta + r} - \frac{1}{r} \leq \frac{1}{d(y, X\backslash U)} - \frac{1}{r}$$

and

$$\frac{1}{d(y, X\backslash U)} - \frac{1}{r} \leq \frac{1}{r-\delta} - \frac{1}{r}$$

Case 1. $1/d(y, X\backslash U) \geq \frac{1}{r}$. Then

$$\begin{aligned}
\left|\frac{1}{d(y, X\backslash U)} - \frac{1}{r}\right| &= \frac{1}{d(y, X\backslash U)} - \frac{1}{r} \leq \frac{1}{r-\delta} - \frac{1}{r}\\
&< \frac{\varepsilon}{2}.
\end{aligned}$$

Case 2. $1/d(y, X\backslash U) < \frac{1}{r}$. Then

$$\begin{aligned}
\left|\frac{1}{d(y, X\backslash U)} - \frac{1}{r}\right| &= \frac{1}{r} - \frac{1}{d(y, X\backslash U)}\\
&\leq \frac{1}{r} - \frac{1}{\delta + r} < \frac{\varepsilon}{2}.
\end{aligned}$$

It follows that U is open according to $\hat{d}$. Hence (2) holds.

Now suppose that $\langle x_n : n \in \omega \rangle$ is a Cauchy sequence under $\hat{d}$ with each $x_n \in U$. Thus $\forall \varepsilon > 0 \exists N \forall m, n > N[\hat{d}(x_m, x_n) < \varepsilon]$. Hence $\langle x_n : n \in \omega \rangle$ is a Cauchy sequence under d. Say $\langle x_n : n \in \omega \rangle$ converges to y; we want to show that $y \in U$.

$$\lim_{i,j\to\infty} \left| \frac{1}{d(x_i, X\backslash U)} - \frac{1}{d(x_j, X\backslash U)} \right| = 0. \tag{3}$$

This is true since $\lim_{i\to\infty} \frac{1}{d(x_i, X\backslash U)}$ exists. By (3),

$$\left\langle \frac{1}{d(x_i, X\backslash U)} : i \in \omega \right\rangle$$

is a Cauchy sequence under d, and hence it has a limit r. Hence

$$\exists M \forall i \geq M \left[\left| \frac{1}{d(x_i, X\backslash U)} - r \right| \leq 1 \right]$$

It follows that

$$\exists N \exists \varepsilon > 0 \forall i \geq N \left[\frac{1}{d(x_i, X\backslash U)} > \varepsilon \right]$$

Hence $d(y, X\backslash U) > 0$, and so $y \in U$. □

Proposition 25.32. m10 *If X is Polish, and $Y \subseteq X$ is G_δ, then Y is Polish.*

Proof. Let $Y = \bigcap_{n\in\omega} U_n$, with each U_n open. Let d_n be a complete metric on U_n compatible with the topology. We may assume that each $d_n < 1$. Define

$$\hat{d}(x, y) = \sum_{n\in\omega} \frac{1}{2^n} d_n(x, y).$$

Now assume that $\langle x_n : n \in \omega \rangle$ is $\hat{d}$-Cauchy.

(1) For each n, $\langle x_i : i \in \omega \rangle$ is d_n-Cauchy.

For, suppose that $\varepsilon > 0$. Choose M so that $\forall i, j > M[\hat{d}(x_i, x_j) < \frac{\varepsilon}{2^n}]$. Then $\frac{1}{2^n} d_n(x, y) < \frac{\varepsilon}{2^n}$. Hence $\forall i, j > M[d_n(x, y) < \varepsilon]$. Hence (1) holds.

For each $n \in \omega$ say $\lim_{i\in\omega} x_i = y_n$. Now there is a z such that each $y_n = z$. Thus $z \in Y$. □

Proposition 25.33. *If X is Polish and $Y \subseteq X$ is closed, then Y is a G_δ in X.*

Proof.

$$|d(x, Y) - d(y, Y)| \leq d(x, y). \tag{1}$$

In fact, if $z \in Y$ then $d(x, Y) \leq d(x, z) \leq d(x, y) + d(y, z)$, so $d(x, Y) - d(x, y) \leq d(y, z)$. This is true for all $z \in Y$, so $d(x, Y) - d(x, y) \leq d(y, Y)$. Thus $d(x, Y) - d(y, Y) \leq d(x, y)$. Similarly, $d(y, Y) - d(x, Y) \leq d(x, y)$, so (1) holds.

Let $B(Y,\varepsilon)=\{x:d(x,Y)<\varepsilon\}$. If $x\in B(Y,\varepsilon)$ and $d(x,y)<\varepsilon-d(x,Y)$, then by (1), $d(y,Y)<d(x,Y)+d(x,y)<\varepsilon$. Thus $B(Y,\varepsilon)$ is open.

Now we may assume that $Y\neq\emptyset$. We claim

$$Y=\bigcap_{n\in\omega}B\left(Y,\frac{1}{n+1}\right). \tag{$*$}$$

In fact, $\subseteq$ is clear. Now suppose that $x\notin Y$. Then $\inf\{d(x,y):y\in Y\}$ is positive since Y is closed. Say $\frac{1}{n+1}<\inf\{d(x,y):y\in Y\}$, so $x\notin B(Y,\frac{1}{n+1}$. Thus $\supseteq$ holds. □

We define

$$\begin{aligned} diam(B)&=\sup\{d(x,y):x,y\in B\}\\ osc_f(x)&=\inf\{diam(f[U\cap A]):U \text{ an open neighborhood of } x\},\end{aligned}$$

where $f:A\to Y$ with $A\subseteq X$ and X and Y are metric spaces and $x\in X$.

Proposition 25.34. *If X and Y are metric spaces, $A\subseteq X$, $f:A\to Y$, and $x\in A$, then the following are equivalent:*

(i) $\forall\varepsilon>0\exists\delta>0\forall y\in A[d(x,y)<\delta\to d(f(x),f(y))<\varepsilon]$.

(ii) $osc_f(x)=0$.

Proof. (i)⇒(ii): Assume (i), and suppose that $\varepsilon>0$. Choose $\delta>0$ such that $\forall y\in A[d(x,y)\le\delta\to d(f(x),f(y))<\varepsilon]$. Then $U\stackrel{\text{def}}{=}\{y:d(x,y)<\delta/2\}$ is an open neighborhood of x. If $u,v\in U\cap A$, then $d(x,u),d(x,v)<\delta/2$, so $d(u,v)<\delta$, hence $d(f(u),f(v))<\varepsilon$. Hence

$$diam(f[U\cap A])=\sup\{d(u,v):u,v\in f[U\cap A]|]\}=\sup\{d(f(y),f(z)):y,z\in U\cap A\}\le\varepsilon.$$

It follows that $osc_f(x)=0$.

Now assume (ii), and suppose that $\varepsilon>0$. Since $osc_f(x)=0$, let U be an open neighborhood of x such that $diam(f[U\cap A])<\varepsilon$. Choose $\delta>0$ such that $\{y\in A:d(x,y)<\delta\}\subseteq U$. So if $d(x,y)<\delta$, and $y\in A$ then $y\in U$, hence $d(f(x),f(y))\le diam(f[U\cap A])<\varepsilon$. □

Proposition 25.35. *If X and Y are metric spaces, and $f:X\to Y$, then $\{x:osc_f(x)=0\}$ is a G_δ.*

Proof. For any $\varepsilon>0$, the set $\{x:osc_f(x)<\varepsilon\}$ is open. In fact, suppose that $\varepsilon>0$, and $osc_f(x)<\varepsilon$. Thus $\inf\{diam(f[U]):U$ is an open neighborhood of $x\}<\varepsilon$. Let U be an open neighborhood of x such that $diam(f[U])<\varepsilon$. Suppose that $y\in U$. Then $d(f(x),f(y))<\varepsilon$, so $osc_f(y)\le diam(f[U])<\varepsilon$. This shows that $\{x:osc_f(x)<\varepsilon\}$ is open. The Proposition follows. □

Theorem 25.36. *Suppose that X and Y are completely metrizable, $A\subseteq X$, and $f:A\to Y$ is continuous. Then there is a G_δ set G such that $A\subseteq G\subseteq\overline{A}$, and there is a continuous extension $g:G\to Y$ of f.*

Proof. Let $G = \overline{A} \cap \{x : osc_f(x) = 0\}$. By Propositions 25.33 and 25.35, G is a G_δ. If $x \in A$, then by Proposition 25.34, $osc_f(x) = 0$. So $A \subseteq G \subseteq \overline{A}$. Now let $x \in G$. Since $x \in \overline{A}$, there is a sequence $y \in {}^\omega A$ which converges to x. Then

$$\lim_{n\in\omega} diam(f[\{y_{n+1}, y_{n+2}, \ldots\}]) = 0.$$

so $\langle f(y_n) : n \in \omega\rangle$ is a Cauchy sequence in Y.

(1) If $y, z \in {}^\omega A$ converge to x, then $\lim_{n\in\omega} f(y_n) = \lim_{n\in\omega} f(z_n)$.

In fact, define

$$w(n) = \begin{cases} y_n & \text{if } n \text{ is even,} \\ z_n & \text{otherwise.} \end{cases}$$

Then w converges to x and

$$\lim_{n\in\omega} diam(f[\{w_{n+1}, w_{n+2}, \ldots\}]) = 0.$$

Hence y, z, w are Cauchy sequences, and

$$\lim_{n\to\infty} w_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n.$$

Thus (1) holds.

For any $x \in G$ let $g(x) = \lim_{n\to\infty} f(y_n)$, where y is as above.

Clearly g extends f.

(2) If U is open in X, then $g[U] \subseteq \overline{f[U]}$.

In fact, suppose that U is open in X and $x \in G \cap U$. Let $y \in^\omega A$ converge to x. We may assume that $y \in {}^\omega U$. Then clearly $g(x) \in \overline{f[U]}$. It follows that $diam(g[U \cap G]) \leq diam(f[U \cap A])$, so $osc_g(x) \leq osc_f(x) = 0$. Thus g is continuous. □

Proposition 25.37. *If X and Y are Polish, with $Y \subseteq X$, then Y is a G_δ in X.*

Proof. We apply Proposition 25.36 to id_Y; we get a G_δ set G such that $Y \subseteq G \subseteq \overline{Y}$ and an extension $g : G \to Y$ of id_Y. Since G is dense in Y we have $g = id_G$. In fact, suppose that $x \in G$ and $x \neq g(x)$. Let U and V be disjoint open sets such that $x \in U$ and $g(x) \in V$. Then $x \in U \cap g^{-1}[V]$ so there is a $z \in Y \cap U \cap g^{-1}[V]$. Then $g(z) = z \in U$ and $g(z) \in V$, contradiction. □

Proposition 25.38. *Every separable metrizable metric space is homeomorphic to a subspace of ${}^\omega[0, 1]$.*

Proof. See theorem 4.7 of Kelley. □

Proposition 25.39. *Every Polish space is homeomorphic to a closed subspace of ${}^\omega\mathbf{R}$.*

Proof. Let X be a Polish space. By Propositions 25.37 and 25.38 we may assume that X is a G_δ subspace of ${}^\omega[0, 1]$. Let $\langle U_n : n \in \omega\rangle$ be a system of open subsets of ${}^\omega[0, 1]$

such that $\bigcap_{n\in\omega} U_n = X$. For each $n \in \omega$ let $F_n = {}^{\omega}\mathbf{R}\backslash U_n$, and let d be a metric on ${}^{\omega}[0,1]$ giving the topology. For each $x \in X$ define $f(x) \in {}^{\omega}\mathbf{R}$ by

$$\begin{aligned}(f(x))(2n+1) &= x_n;\\ (f(x))(2n) &= \frac{1}{d(x,F_n)}.\end{aligned}$$

(1) f is one-one.

In fact, suppose that $x, y \in X$ with $x \neq y$. Say $x_n \neq y_n$. Then $(f(x))(2n+1) = x_n \neq y_n = (f(y))(2n+1)$. So (1) holds

(2) f is continuous.

For, let W be a typical basic open set in ${}^{\omega}\mathbf{R}$. Say $W = \{z \in {}^{\omega}\mathbf{R} : \forall i \in F[z_i \in V_i]\}$, where F is a finite subset of ω and $\forall i \in F[V_i$ is an open subset of $\mathbf{R}]$. Suppose that $x \in f^{-1}[W]$. Thus $f(x) \in W$. So $\forall i \in F[((f(x)))(i) \in V_i]$. Hence

$$\forall i \in F \forall n \in \omega \left[[i = 2n+1 \to x_n \in V_i] \text{ and } \left[i = 2n \to \frac{1}{d(x,F_n)} \in V_i \right] \right]$$

For each $i \in F$ let V'_i be an open subset of $[0,1]$ such that $V'_i \subseteq V_i$ and

$$\forall i \in F \forall n \in \omega \left[[i = 2n+1 \to x_n \in V'_i] \text{ and } \left[i = 2n \to \frac{1}{d(x,F_n)} \in V'_i \right] \right]$$

Let

$$T = \left\{ t \in X : \forall i \in F \forall n \in \omega \left[[i = 2n+1 \to t_n \in V'_i] \text{ and } \left[i = 2n \to \frac{1}{d(t,F_n)} \in V'_i \right] \right] \right\}$$

Clearly T is an open subset of X and $x \in T$. Clearly also $T \subseteq f^{-1}[W]$. Hence (2) holds.

(3) f^{-1} is a continuous function from $\text{rng}(f)$ onto X.

In fact, suppose that U is basic open in X. Say $U = \{z \in {}^{\omega}[0,1] : \forall i \in F[z_i \in V_i]\}$, where F is a finite subset of ω and $\forall i \in F[V_i$ is an open subset of $[0,1]]$. Suppose that $y \in f[U]$; say $y = f(x)$ with $x \in U$. Thus

$$\forall i \in F \forall n \in \omega \left[[i = 2n+1 \to x_n \in V_i] \text{ and } \left[i = 2n \to \frac{1}{d(x,F_n)} \in V_i \right] \right]$$

Let

$$T = \left\{ t \in X : \forall i \in F \forall n \in \omega \left[[i = 2n+1 \to t_n \in V_i] \text{ and } \left[i = 2n \to \frac{1}{d(t,F_n)} \in V_i \right] \right] \right\}$$

Then $y \in T$ and $T \subseteq f[U]$. Thus (3) holds.

(4) $\text{rng}(f)$ is a closed subset of ${}^{\omega}\mathbf{R}$.

In fact, let $\langle f(x^n) : n \in \omega\rangle \to s$. Then

(5) $\langle x^n : n \in \omega\rangle$ is Cauchy.

This holds by Proposition 4.1.8 of Engelking.

Now if $\langle x^n : n \in \omega\rangle \to t$, then $\langle f(x^n)\rangle \to f(t)$, so $f(t) = s$. Thus (4) holds. □

Proposition 25.40. m10 *If X is a Polish space and $Y \subseteq X$ is an uncountable G_δ set, then Y contains a perfect set.*

Proof. By Propositions 25.39 and 25.28. □

Theorem 25.41. m11 *If X is a Polish space and $Y \subseteq X$, then Y is a Polish space iff Y is a G_δ set.*

Proof. By Propositions 25.32 and 25.37. □

For X any set, a *σ-algebra* on X is a collection of subsets of X closed under complement and countable union.

A *measure space* is a pair (X, Ω) such that Ω is a σ-algebra on X.

If (X, Ω_X) and (Y, Ω_Y) are measure spaces, then $f : X \to Y$ is a *measurable function* iff $\forall A \in \Omega_Y[f^{-1}[A] \in \Omega_X]$.

(X, Ω_X) and (Y, Ω_Y) are *isomorphic* iff there is a measurable bijection from (X, Ω_X) to (Y, Ω_Y) whose inverse is measurable.

If X is a topological space, then the class $\mathbb{B}(X)$ of *Borel sets* is the intersection of all σ-algebras on X which contain the class of open sets.

If X and Y are topological spaces then a function $f : X \to Y$ is *Borel measurable* iff it is a measurable map from $\mathbb{B}(X)$ to $\mathbb{B}(Y)$.

(X, Ω) is a *standard Borel space* iff there is a Polish space Y such that (X, Ω) is isomorphic to $(Y, \mathbb{B}Y))$.

Proposition 25.42. m14 *If X and Y are topological spaces and $f : X \to Y$, then f is Borel measurable iff $\forall A[A$ open in $Y \to f^{-1}[A] \in \mathbb{B}(X)$.*

Proof. $\Rightarrow$ is trivial. For $\Leftarrow$, let $\Xi = \{A \subseteq Y : f^{-1}[A] \in \mathbb{B}(X)\}$. Then every open set is in Ξ. If $\forall n \in \omega[A_n \in \Xi]$, then

$$f^{-1}\left[\bigcup_{n\in\omega} A_n\right] = \bigcup_{n\in\omega} f^{-1}[A_n] \in \Xi.$$

Thus Ξ is a σ-algebra of subsets of Y containing the collection of open subsets of Y, so $\mathbb{B}(Y) \subseteq \Xi$. □

Let X be a Polish space. $A \subseteq X$ is a *Borel* set in X iff it belongs to the smallest σ-field of subsets of X containing all closed subsets. Now we define

$$\begin{aligned}
\mathbf{\Sigma}^0_1(X) = {} & \text{ the collection of all open sets} \\
\mathbf{\Pi}^0_1(X) = {} & \text{ the collection of all closed sets} \\
\text{for } \alpha > 1: \quad \mathbf{\Sigma}^0_\alpha(X) = {} & \left\{ \bigcup_{n\in\omega} A_n : \forall n \in \omega \left[A_n \in \bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta(X) \right] \right\} \\
\text{for } \alpha > 1: \quad \mathbf{\Pi}^0_\alpha(X) = {} & \text{ the collection of all complements of sets in } \mathbf{\Sigma}^0_\alpha(X)
\end{aligned}$$

Proposition 25.43. *In any metric space, every open set is the union of countably many closed sets.*

Proof. Let U be open. For each positive integer n and each $x \notin U$ let $V_{x,n}$ be an open ball around x of radius $1/n$. Let $W_n = \bigcup_{x\notin U} V_{x,n}$. So $X\backslash U \subseteq W_n$ for each n. Let $F_n = X\backslash W_n$. So F_n is a closed set contained in U. We claim that $U = \bigcup_{n\in\omega\backslash 1} F_n$ (as desired). For, let $y \in U$. Choose a positive integer n and an open ball W about y of radius $1/n$ such that $W \subseteq U$. We claim that $y \in F_n$. For, suppose not. So $y \in W_n$, and so we can choose $x \in X\backslash U$ such that $y \in V_{x,n}$. Thus $d(x, y) < n$, so $x \in W \subseteq U$, contradiction. □

Proposition 25.44. m15 *For all α, β, if $1 \leq \alpha < \beta$ then*
(1) $\Sigma^0_\alpha(X) \subseteq \Sigma^0_\beta(X)$,
(2) $\Sigma^0_\alpha(X) \subseteq \Pi^0_\beta(X)$,
(3) $\Pi^0_\alpha(X) \subseteq \Sigma^0_\beta(X)$,
(4) $\Pi^0_\alpha(X) \subseteq \Pi^0_\beta(X)$,
(5) $\Sigma^0_\beta(X) = \{\bigcup_{n\in\omega} B_n : \forall n \in \omega[B_n \in \bigcup_{\alpha<\beta} \Pi^0_\alpha(X)]\}$.
(6) $\Pi^0_\beta(X) = \{\bigcap_{n\in\omega} B_n : \forall n \in \omega[B_n \in \bigcup_{\alpha<\beta} \Sigma^0_\alpha(X)]\}$.

Proof. For $\beta = 1$, (1)–(6) hold vacuously.

Now assume (1)–(6) hold for β; we prove them for $\beta + 1$. (5) holds by definition. For (6),

$$\begin{aligned}
Y \in \Pi^0_{\beta+1}(X) \quad & \text{iff} \quad (X\backslash Y) \in \Sigma^0_{\beta+1}(X) \\
& \text{iff} \quad \exists B \in {}^\omega\bigcup_{\alpha<\beta+1} \Pi^0_\alpha(X) \left[(X\backslash Y) = \bigcup_{n\in\omega} B_n \right] \\
& \text{iff} \quad \exists B \in {}^\omega\bigcup_{\alpha<\beta+1} \Pi^0_\alpha(X) \left[Y = \bigcap_{n\in\omega} (X\backslash B_n) \right] \\
& \text{iff} \quad \exists B \in {}^\omega\bigcup_{\alpha<\beta+1} \Sigma^0_\alpha(X) \left[Y = \bigcap_{n\in\omega} B_n \right]
\end{aligned}$$

This gives (6) for $\beta + 1$. (2) follows. For (1) we take two cases.

Case 1. $\alpha = 1$. Then $\Sigma^0_1(X) \subseteq \Sigma^0_{\beta+1}(X)$ by Proposition 25.43

Case 2. $\alpha > 1$. Suppose that $A \in \mathbf{\Sigma}^0_\alpha(X)$. Hence $A \in \mathbf{\Sigma}^0_{\beta+1}(X)$ by definition. So (1) holds.

(4) is clear by (1). (2) is clear by (6). For (3), if $A \in \mathbf{\Pi}^0_\alpha(X)$ then $A \in \Sigma^0_{\beta+1}(X)$ by definition.

Now assume inductively that β is limit. (5) and (3) are true for β by definition. For (2), if $A \in \Sigma^0_\alpha(X)$ with $\alpha < \beta$, then $X\backslash A \in \Pi^0_\alpha(X)$, hence $X\backslash A \in \Sigma^0_\beta(X)$ by definition, and so $A \in \Pi^0_\beta(X)$. For (1), suppose that $A \in \Sigma^0_\alpha(X)$ with $\alpha < \beta$. Then $A \in \Pi^0_{\alpha+1}(X)$ by (2), hence $A \in \Sigma^0_\beta(X)$ by definition. For (4), if $A \in \Pi^0_\alpha(X)$ with $\alpha < \beta$, then $X\backslash A \in \Sigma^0_\alpha(X)$, hence $X\backslash A \in \Sigma^0_\beta(X)$ by (1), so $A \in \Pi^0_\beta(X)$. For (6),

$$\begin{aligned}
Y \in \Pi^0_\beta(X) \quad &\text{iff} \quad (X\backslash Y) \in \Sigma^0_\beta(X) \\
&\text{iff} \quad \exists B \in {}^\omega \bigcup_{\alpha<\beta} \Pi^0_\alpha(X) \left[(X\backslash Y) = \bigcup_{n\in\omega} B_n\right] \\
&\text{iff} \quad \exists B \in {}^\omega \bigcup_{\alpha<\beta} \Pi^0_\alpha \left[Y = \bigcap_{n\in\omega} (X\backslash B_n)\right] \\
&\text{iff} \quad \exists B \in {}^\omega \bigcup_{\alpha<\beta} \Sigma^0_\alpha \left[Y = \bigcap_{n\in\omega} B_n\right]
\end{aligned}$$ □

Proposition 25.45. $\mathbb{B}(X) = \bigcup_{0<\alpha<\omega_1} \Sigma^0_\alpha(X)$. □

Proposition 25.46. *If X is infinite, then $|\mathbb{B}(X)| = 2^\omega$.*

Proof. Let $\langle U_n : n \in \omega\rangle$ be a base for the topology on X. For each $\Gamma \subseteq \omega$ let $U_\Gamma = \bigcup_{n\in\Gamma} U_n$. Then $\langle U_\Gamma : \Gamma \subseteq \omega\rangle$ maps $\mathscr{P}(\omega)$ onto the set of open subsets of X. Thus $|\Sigma^0_1(X)| \le 2^\omega$. By induction, $|\Sigma^0_\alpha| \le 2^\omega$ and $|\Pi^0_\alpha| \le 2^\omega$ for all $\alpha < \omega_1$. Hence $|\mathbb{B}(X)| \le 2^\omega$.

(1) Every countable subset of X is in $\Sigma^0_2(X)$.

In fact, let B be a countable subset of X. Then $\forall a \in B[\{a\}$ is closed], and hence $B = \bigcup_{a\in B}\{a\} \in \Sigma^0_2(X)$. □

Proposition 25.47.

(i) $\Sigma^0_\alpha(X)$ is closed under countable unions and finite intersections.

(ii) $\Pi^0_\alpha(X)$ is closed under countable intersections and finite unions.

Proof. We prove (i) and (ii) simultaneously by induction on α. Both are clear for $\alpha = 1$. Now assume that they hold for α.

Suppose that $A_i \in \Sigma^0_{\alpha+1}(X)$ for all $i \in \omega$. Say $\forall i \in \omega[A_i = \bigcup_{n\in\omega} B_{in}]$ such that $\forall n \in \omega[B_{in} \in \bigcup_{\beta<\alpha+1} \Pi^0_\beta]$. Then $\bigcup_{i\in\omega} A_i = \bigcup_{i\in\omega}\bigcup_{n\in\omega} B_{in}$ and so $\bigcup_{i\in\omega} A_i \in \Sigma^0_{\alpha+1}(X)$.

Suppose that $A_i \in \Pi^0_{\alpha+1}(X)$ for all $i \in \omega$. Then $X\backslash A_i \in \Sigma^0_{\alpha+1}(X)$ for all $i \in \omega$, hence $\bigcup_{i\in\omega}(X\backslash A_i) \in \Sigma^0_{\alpha+1}(X)$, and hence $\bigcap_{i\in\omega} A_i \in \Pi^0_{\alpha+1}$.

Suppose that $A_0, A_1 \in \Sigma^0_{\alpha+1}(X)$. Say $A_i = \bigcup_{n\in\omega} B_{in}$ such that

$$B_{in} \in \bigcup_{\beta<\alpha+1} \Pi^0_\beta(X) \text{ for } i \in 2$$

Then

$$A_0 \cap A_1 = \bigcup_{m,n\in\omega} (B_{0m} \cap B_{1n}).$$

and

$$B_{0m} \cap B_{1n} \in \bigcup_{\beta,\gamma<\alpha+1} (\Pi^0_\beta(X) \cap \Pi^0_\gamma(X)).$$

By Proposition 25.44 it follows that $A_0 \cap A_1 \in \Sigma^0_{\alpha+1}(X)$. If $A_0, A_1 \in \Pi^0_{\alpha+1}(X)$, then $X\backslash A_0, X\backslash A_1 \in \Sigma^0_{\alpha+1}(X)$, hence $(X\backslash A_0)\cap(X\backslash A_1) \in \Sigma^0_{\alpha+1}(X)$, hence $A_0 \cap A_1 \in \Pi^0_{\alpha+1}(X)$.

Thus (i) and (ii) hold for $\alpha + 1$.

The inductive step with α limit is treated similarly. □

Proposition 25.48. $\Delta^0_\alpha(X)$ *is closed under finite unions, finite intersections, and complements.* □

Proposition 25.49. *If* $f : X \to Y$ *is continuous and* $A \in \Sigma^0_\alpha(Y)$*, then* $f^{-1}[A] \in \Sigma^0_\alpha(X)$*. Similarly for* $\Pi^0_\alpha(Y)$ *and* $\Delta^0_\alpha(Y)$*.*

Proof. Clear for $\alpha = 1$. Now suppose that $\alpha > 1$ and $\forall n \in \omega[A_n \in \bigcup_{\beta<\alpha} \Pi^0_\beta(Y)$. Then

$$f^{-1}[A_n] \in f^{-1}\left[\bigcup_{\beta<\alpha} \Pi^0_\beta(Y)\right] = \bigcup_{\beta<\alpha} \Pi^0_\alpha(X);$$

so $f^{-1}[A] \in \Sigma^0_\alpha(X)$. $\Pi^0_\alpha(Y)$ and $\Delta^0_\alpha(Y)$ are treated similarly. □

Proposition 25.50. m16 *If* $A \subseteq X \times Y$ *is* $\Sigma^0_\alpha(X \times Y)$ *and* $a \in Y$*, then* $\{x \in X : (x, a) \in A\} \in \Sigma^0_\alpha(X)$ *and similarly for* Π^0_α *and* Π^0_α*.*

Proof. $x \mapsto (x, a)$ is continuous. □

Proposition 25.51. *Let* $A = \{x \in {}^\omega\omega : x$ *is eventually constant. Then* $A \in \Sigma^0_2({}^\omega\omega)$*.*

Proof. For each $n \in \omega$ let $B_n = \{x \in {}^\omega\omega : x(n) = x(n + 1)\}$. Then B_n is clopen and

$$A = \bigcup_{m\in\omega} \bigcap_{n>m} B_n.$$ □

Proposition 25.52. m16 *Let* $A = \{x \in {}^\omega\omega : x$ *is a bijection. Then* $A \in \Pi^0_2({}^\omega\omega)$*.*

Proof. Let

$$A_0 = \bigcap_{n\in\omega} \bigcap_{m\neq n} \{x \in {}^\omega\omega : x(n) \neq x(m)\}$$ □

Then A_0 consists of all one-one $x \in {}^\omega\omega$ and A_0 is closed. Let $A_1 = \{x \in {}^\omega\omega : \forall n \exists m[x(m) = n\}$. Then

$$A_n = \bigcap_{n\in\omega} \bigcup_{m\in\omega} \{x \in {}^\omega\omega : x(m) = n$$

is $\Pi^0_2({}^\omega\omega)$ and $A_0 \cap A_1$ is the collection of all bijections in ${}^\omega\omega$. $\square$

If $x \in {}^{({}^2\omega)}2$ then $R_x = \{(i,j) : x(i,j) = 2\}$.

Proposition 25.53. m17 *Let LO$= \{x : x \in {}^{({}^2\omega)}2$ and R_x is a linear order$\}$. Then LO is a closed subset of ${}^{({}^2\omega)}2$.*

Proof. For all $m, n \in \omega$ let

$$\begin{aligned}
M_{0mn} &= \{x \in {}^{({}^2\omega)}2 : x(m,n) = 0\};\\
M_{1mn} &= \{x \in {}^{({}^2\omega)}2 : x(m,n) = 1\};\\
M_{2mn} &= \{x \in {}^{({}^2\omega)}2 : m = n\}.
\end{aligned}$$

Clearly each of these set is clopen. Hence by Prposition 25.47, each of the following is closed:

$$\begin{aligned}
N_0 &\stackrel{\text{def}}{=} \bigcap_{m\in\omega} \bigcap_{n\in\omega} (M_{0mn} \cup M_{0nm});\\
N_1 &\stackrel{\text{def}}{=} \bigcap_{m\in\omega} \bigcap_{n\in\omega} (M_{2mn} \cup M_{1mn} \cup M_{1nm});\\
N_2 &\stackrel{\text{def}}{=} \bigcap_{m\in\omega} \bigcap_{n\in\omega} \bigcap_{k\in\omega} (-M_{1mn} \cup -M_{1nk} \cup M_{1nk})
\end{aligned}$$

Clearly LO$= N_0 \cap N_1 \cap N_2$. $\square$

Proposition 25.54. m17 *Let DLO$= \{x : x \in {}^{({}^2\omega)}2$ and R_x is a dense linear order$\}$. Then DLO$\in \Pi^0_2({}^{({}^2\omega)}2)$.*

Proof.

$$DLO = LO \cap \bigcap_{m,n\in\omega} \left[-M_{1nm} \cup \bigcup_{k\in\omega} [M_{1nk} \cap M_{1km}\right].$$

Now $\bigcup_{k\in\omega}[M_{1nk} \cap M_{1km}]$ is open, so DLO is $\Pi^0_2({}^{({}^2\omega)}2)$. $\square$

Proposition 25.55. m20 *Suppose that X and Y are disjoint Polish spaces. $X \oplus Y$ is $X \cup Y$ where $U \subseteq X \cup Y$ is open iff $U \cap X$ and $U \cap Y$ are both open. Then $X \oplus Y$ is a Polish space.*

Proof. Let d_X and d_Y be metrics on X and Y giving their topologies. By Proposition 25.2 we may assume that $d_X, d_Y < 1$. Define $\hat{d}$ on $X \oplus Y$ by

$$\hat{d}(x,y) = \begin{cases} d_X(x,y) & \text{if } x, y \in X,\\ d_Y(x,y) & \text{if } x, y \in Y,\\ 2 & \text{otherwise.}\end{cases}$$

(1) $\hat{d}$ is a metric on $X \oplus Y$.

Only the triangle inequality is questionable. Suppose that $x, y, z \in X \oplus Y$.

Case 1. $x, y, z \in X$ or $x, y, z \in Y$. The inequality is clear.

Case 2. $\neg$Case 1 and $x, z \in X$. Then $y \in Y$ and the inequality is clear.

Case 3. $\neg$Case 1 and $x, z \in Y$. Similar to Case 2.

Case 4. $\neg$Case 1 and $x \in X$ and $z \in Y$ and $y \in X$. The inequality is clear.

Other cases are similar. So (1) holds.

(2) $\hat{d}$ gives the topology on $X \oplus Y$.

In fact, first suppose that $U \subseteq X \oplus Y$ is open. Take any $x \in U$. Say $x \in X$. Choose $\varepsilon > 0$ so that $B_{d_X}(x, \varepsilon) \subseteq U$. Then $B_{\hat{d}}(x, \varepsilon) \subseteq U$. Hence U is open under $\hat{d}$.

Second, suppose that $U \subseteq X \oplus Y$ is open under $\hat{d}$. Suppose that $U \cap X \neq \emptyset$. Take $x \in U \cap X$. Choose $\varepsilon > 0$ such that $B_{\hat{d}}(x, \varepsilon) \subseteq U$. We may assume that $\varepsilon < 2$. Then $B_{d_X}(x, \varepsilon) \subseteq U \cap X$. Thus $U \cap X$ is open under d_X. Similarly $U \cap Y$ is open under d_Y. Thus (2) holds.

(3) $X \oplus Y$ is separable.

In fact, let D be countable and dense in X, and let E be countable and dense in Y. Then $D \cup E$ is countable and dense in $X \oplus Y$. So (3) holds.

(4) $\hat{d}$ is complete.

In fact, let $\langle x_n : n \in \omega \rangle$ be a Cauchy sequence in $X \oplus Y$. Let $\varepsilon > 0$. we may assume that $\varepsilon < 2$. Choose M such that $\forall m \geq M[\hat{d}(x_m, x_M) < \varepsilon$. Say $x_M \in X$. Then $\forall m \geq M[x_m \in X]$. It follows that $\forall m \geq M[d_X(x_m, x_M) < \varepsilon$. Thus x is a Cauchy sequence in X, and hence it converges to some $y \in X$, with respect to d_X. Clearly then it converges to some y with respect to $\hat{d}$. So (4) holds. □

Proposition 25.56. m20 *Let X be a Polish space with topology τ. Suppose that $F \subseteq X$ is closed. Then there is a Polish topology τ_1 on X such that τ_1 refines τ and F is clopen under τ_1 and τ and τ_1 have the same Borel sets.*

Proof. By Proposition 25.55 let τ_1 be the Polish topology on X determined by $\{U \cap F : U \in \tau\}$ and $\{U \backslash F : U \in \tau\}$. Clearly if $U \in \tau$, then $U \in \tau_1$, and F is clopen in τ_1. Clearly if U is open in τ_1 then it is Borel in τ. Hence τ and τ_1 have the same Borel sets. □

Proposition 25.57. *Let X be a Polish space with topology τ. Suppose that $A \subseteq X$ is Borel. Then there is a Polish topology τ^* on X such that A is clopen under τ^*, τ has the same Borel sets as τ^*, and $\tau \subseteq \tau^*$.*

Proof. Let $\Omega = \{B \in B(X)$: there is a Polish topology σ on X such that B is clopen, τ and σ have the same Borel sets, $\tau \subseteq \sigma\}$, By Proposition 25.56 every closed set $C \in B(X)$ is in Ω. Applying Proposition 25.56 to $X \backslash U$, also each open set $U \in B(X)$ is in Ω. Clearly also Ω is closed under complements.

Now suppose that $A_i \in \Omega$ for each $i \in \omega$. Let $B = \bigcap_{i\in\omega} A_i$. For each $i \in \omega$ let τ_i be a Polish topology on X such that A_i is clopen under τ_i and τ, τ_i have the same Borel sets, and $\tau \subseteq \tau_i$. By Theorem 25.3, $\prod_{i\in\omega}(X, \tau_i)$ is a Polish space; let σ be the topology on $\prod_{i\in\omega}(X, \tau_i)$. Define $j : X \to \prod_{i\in\omega}(X, \tau_i)$ by $j(x) = (x, x, \ldots)$.

(1) $\text{rng}(j)$ is a closed subset of $\prod_{i\in\omega}(X, \tau_i)$.

In fact, if $a \in \prod_{i\in\omega}(X, \tau_i)\backslash\text{rng}(j)$ then there exist distinct $i, j \in \omega$ such that $a_i \neq a_j$. Then

$$a \in \left\{ b \in \prod_{i\in\omega}(X, \tau_i) : b_i \neq b_j \right\} \subseteq \prod_{i\in\omega}(X, \tau_i)\backslash\text{rng}(j),$$

so $\prod_{i\in\omega}(X, \tau_i)\backslash\text{rng}(j)$ is open, and (1) follows.

Now let $\tau^* = \{j^{-1}[U] : U \text{ open in } \prod_{i\in\omega}(X, \tau_i)\}$. Clearly τ^* is a topology on X. Clearly

(2) j is a homeomorphism from (X, τ^*) onto $(\text{rng}(j), \sigma)$.

(3) τ^* is a Polish space on X.

This is true by (2), (1), and Proposition 25.29.

(4) $\forall i \in \omega[\tau_i \subseteq \tau^*]$.

In fact, if $U \in \tau_i$, then

$$X \times X \times U \times X \times X \times \cdots$$

is open in $\prod_{i\in\omega}(X, \tau_i)$, and

$$j^{-1}[X \times X \times U \times X \times X \times \cdots] = U.$$

Thus (4) follows.

If $i \in \omega$ and $\langle U_{ni} : n \in \omega \rangle$ is a base for τ_i, then

$$\{\{x \in X : x_i \in U_{ni}\} : i \in \omega\}$$

is a subbase for σ, and it follows that $B(X, \tau^*) = B(X, \tau)$.

Now clearly B is closed. Hence the conclusion of the proposition follows from Proposition 25.56. $\square$

Proposition 25.58. *If X is a Polish space and $B \subseteq X$ is an uncountable Borel set, then B contains a perfect set.*

Proof. Let τ be the topology on X. By Proposition 25.57 let τ^* be a Polish topology on X such that B is clopen under τ^*, τ has the same Borel sets as τ^*, and $\tau \subseteq \tau^*$. By Proposition 25.27 there is a $F \subseteq B$ which is perfect under τ^*. By Theorem 25.19 there is a perfect $P \subseteq F$ under τ^* such that (P, τ^*) is homeomorphic to ${}^\omega 2$, say under f. If $U \in \tau$ and $x \in f[U]$, then there is an open $V \subseteq {}^\omega 2$ such that $x \in V \subseteq f[U]$. Thus f^{-1} is a one-one continuous mapping of ${}^\omega 2$ onto (P, τ). By Theorem 3.1.13 of Engelking, f^{-1} is a homeomorphism of ${}^{\omega}2$ onto (P, τ). Since ${}^{\omega}2$ is compact, (P, τ) is closed. Since (P, τ^*) has no isolated points, also (P, τ) has no isolated points. Thus (P, τ) is perfect. $\square$

Proposition 25.59. *If X is a Polish space and $\emptyset \neq B \subseteq X$ is uncountable and Borel, then there is an $f : {}^{\omega}\omega \to X$ which is continuous and* $\text{rng}(f) = B$.

Proof. By Proposition 25.57 let τ^* be a Polish topology on X such that B is clopen under τ^*, τ has the same Borel sets as τ^*, and $\tau \subseteq \tau^*$. By Proposition 25.29 (B, τ^*) is Polish. By Proposition 25.18 there is a continuous surjection $f : {}^{\omega}\omega \to (B, \tau^*)$. Clearly f is continuous with respect to τ. $\square$

If $U \subseteq Y \times X$ and $a \in Y$, we let $U_a = \{b \in X : (a, b) \in U\}$. We say that U is *universal-*Σ^0_α iff

$U \in \Sigma^0_\alpha(Y \times X)$ and
$\forall A \in \Sigma^0_\alpha(X) \exists a \in Y [A = U_a]$.

Universal-Π^0_α is defined similarly.

Lemma 25.60. *Suppose that X is an uncountable Polish space and $\alpha \geq 1$. Then there exist $U, V \subseteq {}^{\omega}\omega \times X$ such that:*
(i) $U \in \Sigma^0_\alpha$, and $\forall A \subseteq X[A \in \Sigma^0_\alpha \leftrightarrow \exists x \in {}^{\omega}\omega[A = U_x]]$.
(ii) $V \in \Pi^0_\alpha$, and $\forall A \subseteq X[A \in \Pi^0_\alpha \leftrightarrow \exists x \in {}^{\omega}\omega[A = V_x]]$.

Proof. Induction on α. First we take $\alpha = 1$. Let $\langle W_n : n \in \omega \rangle$ enumerate a base for X. Then we set

$$U = \left\{ (x, y) \in {}^{\omega}\omega \times X : y \in \bigcup_{n \in \omega} W_{x(n)} \right\}.$$

Clearly U_x is open for any $x \in {}^{\omega}\omega$. If $A \subseteq X$ is open, write $A = \bigcup_{n \in \omega} W_{x(n)}$. So $A = U_x$. Next we show that U is open in ${}^{\omega}\omega \times X$. Take any $(x, y) \in U$. Choose $n \in \omega$ such that $y \in W_{x(n)}$. Then

$$(x, y) \in \{w \in {}^{\omega}\omega : w(n) = x(n)\} \times W_{x(n)} \subseteq U.$$

This shows that U is open.

Now let $V = ({}^{\omega}\omega \times X) \backslash U$. Then V is closed. If $A \subseteq X$ is closed, choose $x \in {}^{\omega}\omega$ such that $(X \backslash A) = U_x$. Then for any $y \in X$,

$$y \in V_x \quad \text{iff} \quad (x, y) \in V \quad \text{iff} \quad (x, y) \notin U \quad \text{iff} \quad y \notin U_x \quad \text{iff} \quad y \in A.$$

This takes care of $\alpha = 1$.

Now assume inductively that $\alpha > 1$.

Case 1. α is a limit ordinal less than ω_1. Let $\langle \beta_n < n \in \omega \rangle$ be a sequence of ordinals ≥ 1, each less than α, with supremum α. For each $n \in \omega$ let $V_n \subseteq {}^{\omega}\omega \times X$ be a $\Pi^0_{\beta_n}$ set universal for $\Pi^0_{\beta_n}$. For each $x \in {}^{\omega}\omega$ and $n \in \omega$ define $x^n \in {}^{\omega}\omega$ by

$$x^n(m) = x(2^n(2m + 1) - 1).$$

(1) For each $n \in \omega$ the function $f_n : {}^{\omega}\omega \to {}^{\omega}\omega$ defined by $f_n(x) = x^n$ is continuous.

In fact, suppose that $n \in \omega$, $s \in {}^q\omega$, and $x \in f_n^{-1}[\{y \in {}^\omega\omega : s \subseteq y\}]$. Thus $s \subseteq f_n(x) = x^n$, so $\forall m < q[s(m) = x^n(m) = x(2^n(2m+1)-1)]$. Let $t = x \restriction (2^n(2q+1)-1)$. Thus $x \in \{z \in {}^\omega\omega : t \subseteq z\}$. If $z \in {}^\omega\omega$ and $t \subseteq z$, then $s \subseteq f_n(z)$, since for any $m < q$ we have $s(m) = x^n(m) = x(2^n(2m+1)-1)$. Thus (1) holds.

Now define

$$U = \{(x, y) \in {}^\omega\omega \times X : \exists n[(x^n, y) \in V_n].$$

Now fix $n \in \omega$. Define $g : {}^\omega\omega \times X \to {}^\omega\omega \times X$ by $g(x, y) = (x^n, y) = (f_n(x), y)$. Clearly g is continuous. Now $U''_n \stackrel{\text{def}}{=} \{(x, y) \in {}^\omega\omega \times X : (x^n, y) \in V_n\}$ is in $\Pi^0_{\alpha_n}$, since $U''_n = g^{-1}[V_n]$. Hence $U = \bigcup_{n\in\omega} U''_n \in \Sigma^0_\alpha$.

Now to see that U is universal, suppose that $A \subseteq X$ is Σ^0_α. Then we can write $A = \bigcup_{n\in\omega} B_n$ where each B_n is $\Pi^0_{\beta_n}$. Choose $z_n \in {}^\omega\omega$ such that $B_n = (V_n)_{z_n}$. Define $x(2^n(2m+1)-1) = z_n(m)$. Then for any $y \in X$,

$$\begin{aligned} y \in A \quad &\text{iff} \quad \exists n \in \omega[y \in B_n] \quad \text{iff} \quad \exists n \in \omega[y \in (V_n)_{z_n}] \quad \text{iff} \quad \exists n \in \omega[(z_n, y) \in V_n] \\ &\text{iff} \quad [(x^n, y) \in V_n] \quad \text{iff} \quad [(x, y) \in U] \quad \text{iff} \quad [y \in U_x]. \end{aligned}$$

This takes care of Σ^0_α. Suppose that $B \in \Pi^0_\alpha$. Then we choose U for $({}^\omega\omega \times X)\backslash B$. Then $({}^\omega\omega \times X)\backslash U$ is as desired.

The non-limit case is similar. $\square$

Proposition 25.61. *If X is an uncountable Polish space and $\alpha < \omega_1$, then $\Sigma^0_\alpha(X) \neq \Pi^0_\alpha(X)$.*

Proof. Let $U \subseteq {}^\omega\omega \times X$ be such that $U \in \Sigma^0_\alpha$ and $\forall A \subseteq X[A \in \Sigma^0_\alpha \leftrightarrow \exists x \in {}^\omega\omega[A = U_x]]$. Let $Y = \{x \in X : (x, x) \notin U_\alpha\}$. Clearly $Y \in \Pi^0_\alpha$. If $Y \in \Sigma^0_\alpha$, then there is a $y \in X$ such that $\forall x \in X[x \in Y \leftrightarrow (y, x) \in U_\alpha]$. Then $y \in Y \leftrightarrow (y, y) \in U_\alpha \leftrightarrow y \notin Y$, contradiction. $\square$

26. The real line

We give several ways of thinking about the real numbers, and illustrate them with four invariants, concerning certain ideals.

Let $\mathrm{Fn}(I,J,\kappa)=\{f\in[I\times J]^{<\kappa}: f \text{ is a function}\}$. Let I be an ideal on a set A such that $[A]^{<\omega}\subseteq I$ and $A\notin I$.

$$\begin{aligned}
\mathrm{add}(I)&=\min\left\{\kappa:\exists E\in[I]^\kappa\left[\bigcup E\notin I\right]\right\};\\
\mathrm{cov}(I)&=\min\left\{\kappa:\exists E\in[I]^\kappa\left[A=\bigcup E\right]\right\};\\
\mathrm{non}(I)&=\min\{\kappa:\exists X\in[A]^\kappa[X\notin I]\}\\
\mathrm{cof}(I)&=\min\{\kappa:\exists X\in[I]^\kappa\forall C\in I\exists B\in X[C\subseteq B]\}.
\end{aligned}$$

Lemma 26.1. *For A infinite, the cardinals* $\mathrm{add}(I)$, $\mathrm{cov}(I)$, $\mathrm{non}(I)$, $\mathrm{cof}(I)$ *are well-defined and infinite.*

Proof.

$\mathrm{add}(I)$: Let $E=\{\{a\}:a\in A\}$. Then $\bigcup E=A\notin I$. If $F\subseteq I$ is finite, then $\bigcup F\in I$.

$\mathrm{cov}(I)$: Let $E=\{\{a\}:a\in A\}$. Then $\bigcup E=A$. If $F\subseteq I$ is finite, then $\bigcup F\in I$, hence $\bigcup F\neq A$.

$\mathrm{non}(I)$: $A\in[A]^{|A|}$ and $A\notin I$. $X\in I$ for all $X\in[A]^{<\omega}$.

$\mathrm{cof}(I)$: $I\in[I]^{|I|}$ and $\forall C\in I\exists B\in I[C\subseteq B]$. If $X\subseteq I$ is finite, then $\bigcup X\in I$, and if $a\in A\subseteq\bigcup X$ then there is no $B\in X$ such that $\{a\}\subseteq B$. ☐

Lemma 26.2. *For all $X\in[I]^\kappa$ with $\kappa<\mathrm{add}(I)$ we have $\bigcup X\in I$.* ☐

Lemma 26.3. *Let*

$$\mathrm{add}'(I)=\sup\left\{\kappa:\forall X\in[I]^\kappa\left[\bigcup X\in I\right]\right\}$$

Then $\mathrm{add}(I)=(\mathrm{add}'(I))^+$ *if* $\mathrm{add}'(I)$ *is a successor cardinal, or is a limit cardinal and the supremum is attained;* $\mathrm{add}(I)=\mathrm{add}'(I)$ *if* $\mathrm{add}'(I)$ *is a limit cardinal and the supremum is not attained.* ☐

Lemma 26.4. *For all $X\in[I]^\kappa$ with $\kappa<\mathrm{cov}(I)$ we have $\bigcup X\neq A$.* ☐

Lemma 26.5. *Let*

$$\mathrm{cov}'(I)=\sup\left\{\kappa:\forall X\in[I]^\kappa\left[\bigcup X\neq A\right]\right\}$$

Then $\mathrm{cov}(I)=(\mathrm{cov}'(I))^+$ *if* $\mathrm{cov}'(I)$ *is a successor cardinal, or is a limit cardinal and the supremum is attained;* $\mathrm{cov}(I)=\mathrm{cov}'(I)$ *if* $\mathrm{cov}'(I)$ *is a limit cardinal and the supremum is not attained.* ☐

Lemma 26.6. *For all $X\in[A]^\kappa$ with $\kappa<\mathrm{non}(I)$ we have $X\in I$.* ☐

Lemma 26.7. *Let*
$$\mathrm{non}'(I) = \sup\{\kappa : \forall X \in [A]^\kappa[X \in I]\}$$
Then $\mathrm{non}(I) = (\mathrm{non}'(I))^+$ *if* $\mathrm{non}'(I)$ *is a successor cardinal, or is a limit cardinal and the supremum is attained;* $\mathrm{non}(I) = \mathrm{non}'(I)$ *if* $\mathrm{non}'(I)$ *is a limit cardinal and the supremum is not attained.* $\square$

Lemma 26.8. (III.1.7.1) $\mathrm{add}(I) \leq \mathrm{cf}(\mathrm{non}(I)) \leq \mathrm{non}(I) \leq |A|$.

Proof. Let $\kappa = \mathrm{non}(I)$. Choose $X \in [A]^\kappa$ such that $X \notin I$. Let $X = \bigcup_{\alpha<\mathrm{cf}(\kappa)} Y_\alpha$ with each $|Y_\alpha| < \kappa$. Hence $Y \in {}^{\mathrm{cf}(\kappa)}I$. Then $X = \bigcup_{\alpha<\mathrm{cf}(\kappa)} Y_\alpha \notin I$, so $\mathrm{add}(I) \leq \mathrm{cf}(\kappa)$.

We have $A \notin I$, so $\mathrm{non}(I) \leq |A|$. $\square$

Lemma 26.9. (III.1.7.2) $\mathrm{add}(I) \leq \mathrm{cov}(I) \leq |A|$.

Proof. Let $\kappa = \mathrm{cov}(I)$, and choose $X \in [A]^\kappa$ such that $A = \bigcup X$. Then $\bigcup X \notin I$, so $\mathrm{add}(I) \leq \kappa$. Since $A \notin I$, we have $\mathrm{cov}(I) \leq |A|$. $\square$

Lemma 26.10. (III.1.7.3) $\mathrm{add}(I)$ *is regular.*

Proof. Suppose that $\lambda \stackrel{\mathrm{def}}{=} \mathrm{add}(I)$ is singular, and let $\kappa \in {}^{\mathrm{cf}(\lambda)}\lambda$ be such that $\sup_{\mu<\mathrm{cf}(\lambda)} \kappa_\mu = \lambda$. By the definition of add, let $X \in {}^\lambda I$ be such that $\bigcup_{\mu<\lambda} X_\mu \notin I$. For each $\mu < \mathrm{cf}(\lambda)$ we have $\bigcup_{\xi<\kappa_\mu} X_\xi \in I$. Since $\mathrm{cf}(\lambda) < \lambda$, it follows that

$$\bigcup_{\mu<\mathrm{cf}(\lambda)} \left(\bigcup_{\xi<\kappa_\mu} X_\xi \right) = \bigcup_{\xi<\lambda} X_\xi \in I,$$

contradiction. $\square$

Proposition 26.11. $\mathrm{add}(I) \leq \mathrm{cf}(\mathrm{cof}(I))$.

Proof. Let $\mathrm{cof}(I) = \kappa$, and let $X \in [I]^\kappa$ be such that $\forall C \in I \exists D \in X[C \subseteq D]$. Write $X = \bigcup_{\alpha<\mathrm{cf}(\kappa)} Y_\alpha$ with each $|Y_\alpha| < \kappa$. Then for each $\alpha < \mathrm{cf}(\kappa)$ there is a $C_\alpha \in I$ such that for all $D \in Y_\alpha[C_\alpha \not\subseteq D]$. Let $E = \bigcup_{\alpha<\mathrm{cf}(\kappa)} C_\alpha$. Then $E \notin I$. In fact, otherwise there is a $D \in X$ such that $E \subseteq D$. Say $D \in Y_\alpha$. Then $C_\alpha \subseteq D$, contradiction.

Thus $\mathrm{add}(I) \leq \mathrm{cf}(\mathrm{cof}(I))$. $\square$

Proposition 26.12. $\mathrm{cov}(I) \leq \mathrm{cof}(I)$.

Proof. Let $\kappa = \mathrm{cof}(I)$, and let $X \in [I]^\kappa$ be such that $\forall C \in I \exists B \in X[C \subseteq B]$. For each $a \in A$ choose $B_a \in X$ such that $\{a\} \subseteq B_a$. Then $A = \bigcup_{a\in A} B_a = \bigcup X$. So $\mathrm{cov}(I) \leq \mathrm{cof}(I)$. $\square$

Proposition 26.13. $\mathrm{non}(I) \leq \mathrm{cof}(I)$.

Proof. Let $\kappa = \mathrm{cof}(I)$, and let $X \in [I]^\kappa$ be such that $\forall C \in I \exists B \in X[C \subseteq B]$. For each $B \in X$ we have $B \neq A$; let $x_B \in A \backslash B$. Let $C = \{x_B : B \in X\}$. If $C \in I$, choose

$B \in X$ such that $C \subseteq B$. Then $x_B \in C$, so $x_B \in B$, contradiction. Thus $C \notin I$. So $\text{non}(I) \leq \text{cof}(I)$. □

Proposition 26.14. *Let* $A = \omega_1$*,* $I = [\omega_1]^{<\omega}$*. Then* $\text{add}(I) = \omega\chi$*,* $\text{cov}(I) = \omega_1$*,* $\text{non}(I) = \omega$*, and* $\text{cof}(I) = \omega_1$*.*

Proof. These statements are clear, except possibly for $\text{cof}(I) = \omega_1$. Clearly $\text{cof}(I) \leq \omega_1$. Suppose that $X \in [I]^{\leq\omega}$ and $\forall C \in I \exists B \in X[B \subseteq C]$. Choose $\alpha \in \omega_1 \backslash \bigcup X$. Choose $B \in X$ such that $\{\alpha\} \subseteq B$; this is impossible. □

Proposition 26.15. *Let* $A = \lambda$ *be singular,* $I = [\lambda]^{<\lambda}$*. Then* $\text{add}(I) = \text{cf}(\lambda)$*.* □

Proposition 26.16. *Let* $A = \lambda$ *be singular,* $I = [\lambda]^{<\lambda}$*. Then* $\text{cov}(I) = \text{cf}(\lambda)$*.* □

Proposition 26.17. *Let* $A = \lambda$ *be singular,* $I = [\lambda]^{<\lambda}$*. Then* $\text{non}(I) = \lambda$*.* □

Proposition 26.26. *Let* $A = \lambda$ *be singular,* $I = [\lambda]^{<\omega}$*. Then* $\text{add}(I) = \omega$*.* □

Proposition 26.19. *Let* $A = \lambda$ *be singular,* $I = [\lambda]^{<\omega}$*. Then* $\text{non}(I) = \omega$*.* □

Proposition 26.20. *Let* $A = \lambda$ *be singular,* $I = [\lambda]^{<\omega}$*. Then* $\text{cov}(I) = \lambda$*.* □

A *relational triple* is a triple $\mathbf{A} = (A_0, A_1, A)$ such that A_0 and A_1 are sets and $A \subseteq A_0 \times A_1$. The norm of a relational triple $\mathbf{A} = (A_0, A_1, A)$ is $\min\{|Y| : Y \subseteq A_1 \text{ and } \forall x \in A_0 \exists y \in Y[xAy]\}$; the norm is denoted by $||\mathbf{A}||$. The dual of a relational triple $\mathbf{A} = (A_0, A_1, A)$ is the relational triple $(A_1, A_0, \{(x, y) : (y, x) \notin A\})$; it is denoted $\mathbf{A}^\perp$. We also let $A_0^\perp = A_1$, $A_1^\perp = A_0$, and $A^\perp = \{(x, y) : (y, x) \notin A\}$, A *morphism* from a relational triple (A_0, A_1, A) to a relational triple $(B_0, B_1.B)$ is a pair $\varphi \overset{\text{def}}{=} (\varphi_0, \varphi_1)$ such that:

$\varphi_0 : B_0 \to A_0$;
$\varphi_1 : A_1 \to B_1$;
$\forall a \in A_1 \forall b \in B_0[\varphi_0(b)Aa \to bB\varphi_1(a)]$.

Given such a morphism φ, we define $\varphi^\perp = (\varphi_1, \varphi_0)$.

Proposition 26.21. *If* $\varphi : \mathbf{A} \to \mathbf{B}$*, then* $\varphi^\perp : \mathbf{B}^\perp \to \mathbf{A}^\perp$*.*

Proof. We have $\varphi_1 : A_1 \to B_1$, so $\varphi_1 : A_0^\perp \to B_0^\perp$. Similarly, $\varphi_0 : B_0 \to A_0$, so $\varphi_0 : B_1^\perp \to A_1^\perp$. Finally, if $b \in B_1^\perp$ and $a \in A_0^\perp$, then $b \in B_0$ and $a \in A_1$, so $\varphi_0(b)Aa \to bB\varphi_1(a)]$, hence $\text{not}(bB\varphi_1(a)) \to \text{not}(\varphi_0(b)Aa)$, hence $\varphi_1(a)B^\perp b \to aA^\perp\varphi_0(a)$. □

Proposition 26.22. *If there is a morphism* $\varphi : \mathbf{A} \to \mathbf{B}$*, then* $||\mathbf{B}|| \leq ||\mathbf{A}||$ *and* $||\mathbf{A}^\perp|| \leq ||\mathbf{B}^\perp||$*.*

Proof. Let $Y \subseteq A_1$ be such that $\forall x \in A_0 \exists y \in Y[xAy]\}$, and $|Y| = ||\mathbf{A}||$. Then $\varphi_1[Y] \subseteq B_1$ and for all $x \in B_0$ there is a $y \in Y$ such that $\varphi_0(x)Ay$, hence $xB\varphi_1(y)$. So $||\mathbf{B}|| \leq ||\varphi_1[Y]|| \leq ||Y|| = ||\mathbf{A}||$.

Applying this to $\varphi^\perp : \mathbf{B}^\perp \to \mathbf{A}^\perp$, we get $||\mathbf{A}^\perp|| \leq ||\mathbf{B}^\perp||$. □

For any ideal I on $\mathscr{P}({}^{\omega}2)$ let $\mathrm{Cov}(I) = ({}^{\omega}2, I. \in)$.

Proposition 26.23. $||\mathrm{Cov}(I)|| = \mathrm{cov}(I)$.

Proof. $||\mathrm{Cov}(I) = \min\{|Y| : Y \subseteq I \text{ and } \forall x \in {}^{\omega}2 \exists y \in Y[x \in y]\} = \mathrm{cov}(I)$. $\square$

Proposition 26.24. $||\mathrm{Cov}^{\perp}(I)|| = \mathrm{non}(I)$.

Proof. We have $\mathrm{Cov}^{\perp}(I) = (I, {}^{\omega}2, \{(a, f) : f \notin a\})$. Hence $||\mathrm{Cov}^{\perp}(I)|| = \min\{|X| : X \subseteq {}^{\omega}2 \text{ and } \forall a \in I \exists f \in X[f \notin a]\} = \mathrm{non}(I)$. $\square$

Proposition 26.25. *Suppose that f is a homeomorphism of a space X onto a space Y. Then*

(i) $\mathrm{add}(\mathrm{meager}_X) = \mathrm{add}(\mathrm{meager}_Y)$*;*
(ii) $\mathrm{cov}(\mathrm{meager}_X) = \mathrm{cov}(\mathrm{meager}_Y)$*;*
(iii) $\mathrm{non}(\mathrm{meager}_X) = \mathrm{non}(\mathrm{meager}_Y)$*;*
(iv) $\mathrm{cof}(\mathrm{meager}_X) = \mathrm{cof}(\mathrm{meager}_Y)$*.* $\square$

Now we show that add, non, cov, and cof have the same values for meager for each of the following notions of reals:

irrat	$\mathbb{R}$	${}^{\omega}2$	$\mathscr{P}(\omega)$	${}^{\omega}\omega$	C
$(0, 1)$	Ω	$[\omega]^{\omega}$	$[0, 1]$	Θ	

Here Ω, Θ, and C are defined below.

The irrationals and ${}^{\omega}\omega$

Theorem 26.26. *${}^{\omega}\omega$ under the product topology is homeomorphic to the irrationals.*

Proof. Let $a = \langle a_0, a_1, \ldots\rangle$ be an infinite sequence of integers such that $a_i > 0$ for all $i > 0$. We want to give a precise definition of the continued fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots}}}}$$

To start with, we assume that a is a sequence of positive real numbers with domain either ω or some positive integer. We define $[a_0, \ldots, a_l]$ for each $l < \mathrm{dmn}(a)$ by recursion:

$$[a_0] = a_0;$$
$$[a_0, \ldots, a_{k+1}] = a_0 + \frac{1}{[a_1, \ldots, a_{k+1}]}$$

We want to be very explicit as to how these approximations can be written as certain fractions. To this end we make the following recursive definitions:

$$p(a, 0) = a_0; \quad q(a, 0) = 1;$$
$$p(a, 1) = a_0 a_1 + 1; \quad q(a, 1) = a_1.$$

For $k \geq 2$:

$$p(a,k) = a_k p(a,k-1) + p(a,k-2);$$
$$(1) \qquad q(a,k) = a_k q(a,k-1) + q(a,k-2).$$

Note that $p(a,k) > 0$ and $q(a,k) > 0$ for all $k \geq 0$. Also, let $a' = \langle a_1, a_2, \ldots \rangle$. Now we claim that for all $i \in \omega$,

$$\begin{aligned} p(a,i+1) &= a_0 p(a',i) + q(a',i); \\ q(a,i+1) &= p(a',i). \end{aligned}$$

We prove these equations by induction on i. For $i = 0$ we have

$$\begin{aligned} p(a,1) &= a_0 a_1 + 1 = a_0 p(a',0) + q(a',0); \\ q(a,1) &= a_1 = p(a',0), \end{aligned}$$

as desired. For $i = 1$,

$$\begin{aligned} p(a,2) &= a_2 p(a,1) + p(a,0) \\ &= a_0 a_1 a_2 + a_2 + a_0 \\ &= a_0(a_1 a_2 + 1) + a_2 \\ &= a_0 p(a',1) + q(a',1); \\ q(a,2) &= a_2 q(a,1) + q(a,0) \\ &= a_1 a_2 + 1 \\ &= p(a',1), \end{aligned}$$

as desired. Now we do the inductive step for $i \geq 2$:

$$\begin{aligned} p(a,i+1) &= a_{i+1} p(a,i) + p(a,i-1) \\ &= a_{i+1}(a_0 p(a',i-1) + q(a',i-1)) + a_0 p(a',i-2) + q(a',i-2) \\ &= a_0(a_{i+1} p(a',i-1) + p(a',i-2)) + a_{i+1} q(a',i-1) + q(a',i-2) \\ &= a_0 p(a',i) + q(a',i); \\ q(a,i+1) &= a_{i+1} q(a,i) + q(a,i-1) \\ &= a_{i+1} p(a',i-1) + p(a',i-2) \\ &= p(a',i), \end{aligned}$$

as desired. So the above equations hold.

Note by an easy induction that $p(a,k), q(a,k) > 0$ for all k. Now we claim:

$$(2) \qquad [a_0, \ldots, a_k] = \frac{p(a,k)}{q(a,k)}$$

for every $k \in \omega$. We prove (2) by induction on k. For $k = 0$, we have

$$[a_0] = a_0 = \frac{p(a,0)}{q(a,0)},$$

as desired. For $k = 1$, we have

$$[a_0, a_1] = a_0 + \frac{1}{a_1} = \frac{a_0a_1 + 1}{a_1} = \frac{p(a, 1)}{q(a, 1)},$$

as desired. Inductively, for $k \geq 2$,

$$\begin{aligned}
[a_0, \ldots, a_k] &= a_0 + \frac{1}{[a_1, \ldots, a_k]} \\
&= a_0 + \frac{q(a', k-1)}{p(a', k-1)} \\
&= \frac{a_0 p(a', k-1) + q(a', k-1)}{p(a', k-1)} \\
&= \frac{p(a, k)}{q(a, k)},
\end{aligned}$$

as desired.

From now on we shall write p_k, q_k in place of $p(a, k), q(a, k)$ if a is understood. We also define $p_{-1} = 1$ and $q_{-1} = 0$. Then the equations (1) also hold for $k = 1$, since

$$\begin{aligned}
&a_1 p_0 + p_{-1} = a_0 a_1 + 1 = p_1 \quad \text{and} \\
&a_1 q_0 + q_{-1} = a_1 = q_1.
\end{aligned}$$

Next we claim that for $k \geq 1$,

$$q_k p_{k-1} - p_k q_{k-1} = -(q_{k-1} p_{k-2} - p_{k-1} q_{k-2}). \tag{3}$$

In fact, multiply the equations (1) by q_{k-1} and p_{k-1} respectively:

$$\begin{aligned}
p_k q_{k-1} &= a_k p_{k-1} q_{k-1} + p_{k-2} q_{k-1}; \\
q_k p_{k-1} &= a_k q_{k-1} p_{k-1} + q_{k-2} p_{k-1}.
\end{aligned}$$

Subtracting the first of these equations from the second gives (3).

Now $q_0 p_{-1} - p_0 q_{-1} = 1$, so by (3) and induction we get, for $k \geq 0$,

$$q_k p_{k-1} - p_k q_{k-1} = (-1)^k. \tag{4}$$

Hence for $k \geq 1$ we have

$$\frac{p_{k-1}}{q_{k-1}} - \frac{p_k}{q_k} = \frac{(-1)^k}{q_k q_{k-1}}\ . \tag{5}$$

Next, for any $k \geq 1$,

$$q_k p_{k-2} - p_k q_{k-2} = (-1)^{k-1} a_k\ . \tag{6}$$

To see this, multiply the equations (1) by q_{k-2} and p_{k-2} respectively:

$$p_k q_{k-2} = a_k p_{k-1} q_{k-2} + p_{k-2} q_{k-2};$$
$$q_k p_{k-2} = a_k q_{k-1} p_{k-2} + q_{k-2} p_{k-2}.$$

Now subtract the first from the second and use (4): (6) follows.

From (6):

$$\frac{p_{k-2}}{q_{k-2}} - \frac{p_k}{q_k} = \frac{(-1)^{k-1} a_k}{q_k q_{k-2}}\ . \tag{7}$$

Hence:

$$\left\langle \frac{p_{2k}}{q_{2k}} : k \in \omega \right\rangle \text{ is an increasing sequence;} \tag{8}$$

$$\left\langle \frac{p_{2k+1}}{q_{2k+1}} : k \in \omega \right\rangle \text{ is an decreasing sequence;} \tag{9}$$

Next we claim

$$\frac{p_{2k}}{q_{2k}} < \frac{p_{2l+1}}{q_{2l+1}} \text{ for all } k, l \in \omega \tag{10}$$

In fact, let $m = \max(k, l)$. Then

$$\begin{aligned} \frac{p_{2k}}{q_{2k}} &\leq \frac{p_{2m}}{q_{2m}} \quad \text{by (8)} \\ &< \frac{p_{2m+1}}{q_{2m+1}} \quad \text{by (5)} \\ &\leq \frac{p_{2l+1}}{q_{2l+1}} \quad \text{by (9)} \end{aligned}$$

So (10) holds. Next we claim:

$$p_k < p_{k+1} \text{ and } q_{k+1} < q_{k+2} \quad \text{for all } k \in \omega. \tag{11}$$

In fact, this is clear from the recursive definitions.

Now we assume that our sequence a is infinite, and all a_i are positive integers. It follows from (8), (9), (10), (11), and (5) that the approximations $\frac{p_k}{q_k}$ converge, and by definition the limit is the value of the infinite continued fraction described at the beginning. For a_0 a negative integer but all a_i positive integers for $i > 0$, we define $a' = \langle 1, a_1, a_2, \ldots \rangle$ and define the continued fraction to be

$$a_0 - 1 + \lim_{k\to\infty} \frac{p(a', k)}{q(a', k)}$$

Now we want to see how to represent any real number as a finite or infinite continued fraction. We make a recursive definition for any real number $\alpha > 1$. Let $r(\alpha, 0) = \alpha$.

Suppose that we have defined $r(\alpha,i)>1$. Write $r(\alpha,i)=a(\alpha,i)+s(\alpha,i+1)$ with $a(\alpha,i)$ a positive integer and $s(\alpha,i+1)$ a nonnegative real <1. If $s(\alpha,i+1)=0$, the construction stops. Otherwise we define $r(\alpha,i+1)=\frac{1}{s(\alpha,i+1)}$. This finishes the construction. Let $l(\alpha)$ be the index i such that $s(\alpha,i+1)=0$, or $l(\alpha)=\omega$ if there is no such index. We need the following technical fact.

(12) If $\alpha>1$ and $l(\alpha)>1$, then $l(r(\alpha,1))=l(\alpha)-1$, and for each $j\le l(\alpha)-1$ we have $r(r(\alpha,1),j)=r(\alpha,j+1)$ and $a(r(\alpha,1),j)=a(\alpha,j+1)$.

By induction on j we prove that $r(r(\alpha,1),j)$ is defined and equals $r(\alpha,j+1)$ for each $j\le l(\alpha)-1$. For $j=0$ we have $r(r(\alpha,1),0)$ defined and it equals $r(\alpha,1)$, as desired. Now assume our result for j, with $j+1\le l(\alpha)-1$. Then

$$r(r(\alpha,1),j)=r(\alpha,j+1)=a(\alpha,j+1)+s(\alpha,j+2).$$

Now $j+2\le l(\alpha)$, so $s(\alpha,j+2)>0$, and hence by definition, $r(\alpha,j+2)=\frac{1}{s(\alpha,j+2)}=r(r(\alpha,1),j+1)$. This completes the inductive proof.

Now if $j\le l(\alpha)-1$, then

$$\begin{aligned} r(r(\alpha,1),j)&=a(r(\alpha),1),j)+s(r(\alpha,1),j+1);\\ r(\alpha,j+1)&=a(\alpha,j+1)+s(\alpha,j+2); \end{aligned}$$

so $a(r(\alpha,1),j)=a(\alpha,j+1)$. Finally, if $j=l(\alpha)$, then $r(\alpha,j)=a(\alpha,j)$, and hence $r(r(\alpha,1),j-1)=r(\alpha,j)=a(\alpha,j)$ and so $l(r(\alpha,1))=j-1$, as desired in (12).

(13) If $\alpha>1$ and $n\le l(\alpha)$, then $\alpha=[a(\alpha,0),a(\alpha,1),\ldots,a(\alpha,n-1),r(\alpha,n)]$.

We prove this by induction on n. For $n=0$, $[r(\alpha,0)]=\alpha$. Assume that our condition is true for n, and $n+1\le l(\alpha)$. Then

$$\begin{aligned} [a(\alpha,0),&a(\alpha,1),\ldots,a(\alpha,n),r(\alpha,n+1)]\\ &=a(\alpha,0)+\frac{1}{[a(\alpha,1),a(\alpha,2),\ldots,a(\alpha,n),r(\alpha,n+1)]}\\ &=a(\alpha,0)+\frac{1}{[a(r(\alpha,1),0)a(r(\alpha,1),1),\ldots,a(r(\alpha,1),n-1),r(r(\alpha,1),n)]}\\ &=a(\alpha,0)+\frac{1}{r(\alpha,1)}\\ &=a(\alpha,0)+s(\alpha,1)\\ &=\alpha, \end{aligned}$$

completing the inductive proof.

(14) If $\alpha>1$ is rational, then the above definition of $r(\alpha,i)$'s terminates after finitely many steps.

In fact, it suffices to show that if $r(\alpha,i)=\frac{b}{c}$ with b,c positive integers and g.c.d$(b,c)=1$, and $r(\alpha,i+1)$ is defined, then $r(\alpha,i+1)$ has the form $\frac{d}{e}$, with d and e positive integers with $e < c$. To prove this, recall that $r(\alpha, i) = a(\alpha, i) + s(\alpha, i+1)$, with $s(\alpha, i+1)$ a nonnegative real < 1, and $r(\alpha, i+1) = \frac{1}{s(\alpha,i+1)}$. Thus

$$\frac{b}{c} = r(\alpha, i) = a(\alpha, i) + s(\alpha, i+1) \quad \text{and hence}$$

(15) $$b = ca(\alpha, i) + cs(\alpha, i+1);$$

Hence

$$\begin{aligned} r(\alpha, i+1) &= \frac{1}{s(\alpha, i+1)} \\ &= \frac{1}{r(\alpha, i) - a(\alpha, i)} \\ &= \frac{1}{\frac{b}{c} - a(\alpha, i)} \\ &= \frac{c}{b - ca(\alpha, i)} \\ &= \frac{c}{cs(\alpha, i+1)} \quad \text{by (15),} \end{aligned}$$

and $cs(\alpha, i+1)$ is a positive integer $< c$, as desired.

(16) If α is rational, then there exist integers $a_0, a_1, \ldots, a_n$ with $a_i > 0$ for all $i > 0$ such that $\alpha = [a_0, a_1, \ldots, a_n]$.

In fact, let m be an integer such that $\alpha + m > 1$; if $\alpha > 1$, let $m = 0$. By (14), $n \stackrel{\text{def}}{=} l(\alpha + m)$ is finite. We then have $r(\alpha + m, n) = a(\alpha + m, n)$. Hence by (13) we have $\alpha + m = [a(\alpha + m, 0), \ldots, a(\alpha + m, n)]$, and the desired conclusion follows.

(17) If $\langle a_0, a_1, \ldots \rangle$ is a sequence of rational numbers each greater than 0, then also $[a_0, a_1, \ldots, a_n]$ is rational for each n.

This is clear from the basic definition, by induction.

(26) Let $\alpha > 1$ be irrational. Then by (17), the sequence

$$b \stackrel{\text{def}}{=} \langle a(\alpha, 0), a(\alpha, 1), \ldots \rangle$$

never terminates. We claim that for each positive integer n,

$$\alpha = \frac{p(b, n-1)r(\alpha, n) + p(b, n-2)}{q(b, n-1)r(\alpha, n) + q(b, n-2)}.$$

We prove by induction that for every positive integer n, this holds for all irrationals $\alpha > 1$. First, the case $n = 1$:

$$\begin{aligned} \frac{p(b, 0)r(\alpha, 1) + p(b, -1)}{q(b, 0)r(\alpha, 1) + q(b, -1)} &= \frac{a(\alpha, 0)r(\alpha, 1) + 1}{r(\alpha, 1)} \\ &= a(\alpha, 0) + \frac{1}{r(\alpha, 1)} \\ &= a(\alpha, 0) + s(\alpha, 1) \\ &= r(\alpha, 0) \\ &= \alpha, \end{aligned}$$

as desired. Now we assume our statement for n. In fact, we apply it to $r(\alpha,1)$ rather than α. Note that $r(\alpha,1)>1$, and it is irrational by (17) and (13). Let

$$\begin{aligned}c&=\langle a(\alpha,1),a(\alpha,2),\ldots\rangle\\&=\langle a(r(\alpha,1),0),a(r(\alpha,1),1),\ldots\rangle,\end{aligned}$$

by (12). Hence, starting with the inductive hypothesis,

$$\begin{aligned}r(\alpha,1)&=\frac{p(c,n-1)r(r(\alpha,1),n)+p(c,n-2)}{q(c,n-1)r(r(\alpha,1),n)+q(c,n-2)}\\&=\frac{p(c,n-1)r(\alpha,n+1)+p(c,n-2)}{q(c,n-1)r(\alpha,n+1)+q(c,n-2)}.\end{aligned}$$

Hence, using the equations following (1),

$$\begin{aligned}\alpha&=r(\alpha,0)\\&=a(\alpha,0)+s(\alpha,1)\\&=a(\alpha,0)+\frac{1}{r(\alpha,1)}\\&=a(\alpha,0)+\frac{q(c,n-1)r(\alpha,n+1)+q(c,n-2)}{p(c,n-1)r(\alpha,n+1)+p(c,n-2)}\\&=\frac{a(\alpha,0)p(c,n-1)r(\alpha,n+1)+a(\alpha,0)p(c,n-2)+q(c,n-1)r(\alpha,n+1)+q(c,n-2)}{p(c,n-1)r(\alpha,n+1)+p(c,n-2)}\\&=\frac{p(b,n)r(\alpha,n+1)+p(b,n-1)}{q(b,n)r(\alpha,n+1)+q(b,n-1)},\end{aligned}$$

which finishes the inductive proof of (26).

We now omit the parameter b, as it is understood in what follows.

(19) Let $\alpha>1$ be irrational. Then for every positive integer n,

$$\alpha-\frac{p_n}{q_n}=\frac{(p_{n-1}q_{n-2}-q_{n-1}p_{n-2})(r_n-a_n)}{(q_{n-1}r_n+q_{n-2})(q_{n-1}a_n+q_{n-2})}.$$

To prove this, first note by (26) and (1) that

$$\alpha-\frac{p_n}{q_n}=\frac{p_{n-1}r_n-p_{n-2}}{q_{n-1}r_n+q_{n-2}}-\frac{p_{n-1}a_n+p_{n-2}}{q_{n-1}a_n+q_{n-2}}. \tag{20}$$

Now we have

$$\begin{aligned}&(p_{n-1}r_n-p_{n-2})(q_{n-1}a_n+q_{n-2})-(p_{n-1}a_n+p_{n-2})(q_{n-1}r_n+q_{n-2})\\&\quad=p_{n-1}q_{n-1}a_nr_n+p_{n-1}q_{n-2}r_n+p_{n-2}q_{n-1}a_n+p_{n-2}q_{n-2}\\&\qquad-p_{n-1}q_{n-1}a_nr_n-p_{n-1}q_{n-2}a_n-p_{n-2}q_{n-1}r_n-p_{n-2}q_{n-2}\\&\quad=(p_{n-1}q_{n-2}-q_{n-1}p_{n-2})(r_n-a_n).\end{aligned}$$

Hence from (20) we get (19).

(21) For irrational $\alpha > 1$ we have

$$\alpha = [a(\alpha, 0), a(\alpha, 1), \ldots].$$

In fact, note from (4) that $p_{n-1}q_{n-2} - q_{n-1}p_{n-2} = (-1)^{n-1}$, while by definition we have $r(\alpha, n) - a(\alpha, n) = s(\alpha, n+1) < 1$. Hence by (19),

$$\left|\alpha - \frac{p_n}{q_n}\right| \leq \frac{1}{(q_{n-1}r_n + q_{n-2})(q_{n-1}a_n + q_{n-2})} < \frac{1}{q_{n-2}^2},$$

and hence (21) follows from (11).

Now for any irrational $\alpha > 1$, define

$$f(\alpha) = \langle a(\alpha, 0), a(\alpha, 1), \ldots \rangle.$$

Then by the above results, f is a one-to-one function mapping the set $\mathcal{N}$ of irrationals $>$ 1 onto the set ${}^{\omega}(\omega \backslash 1)$. The latter set is clearly homeomorphic to ${}^{\omega}\omega$.

(22) The set of irrationals > 1 is homeomorphic to the entire set of irrationals.

To see this, define g by setting, for each irrational $x > 1$,

$$g(x) = \begin{cases} x + m & \text{if } 0 < m < x < m+1 \text{ with } m \in \omega, \\ x + 3m + 1 & \text{if } -m < x < -m+1 \text{ with } m \in \omega. \end{cases}$$

Then g maps $(m, m+1)_{\mathrm{irr}}$ one-one onto $(2m, 2m+1)_{\mathrm{irr}}$ for each positive integer m, and $(-m, -m+1)_{\mathrm{irr}}$ one-one onto $(2m+1, 2m+2)_{\mathrm{irr}}$ for each $m \in \omega$. Clearly g is the desired homeomorphism.

Thus to finish the proof of Theorem 26.26 it suffices to show that f, defined above, is a homeomorphism. To do this, we need the following fact.

(23) Suppose that $a_0, \ldots, a_n, b_0, \ldots, b_{n-1}$ are positive integers and r is a real number > 1. Assume that

$$\begin{aligned} &[a_0, \ldots, a_{n-1}] < [b_0, \ldots, b_{n-1}, r] < [a_0, \ldots, a_n] \quad \text{if } n \text{ is odd} \\ &[a_0, \ldots, a_{n-1}] > [b_0, \ldots, b_{n-1}, r] > [a_0, \ldots, a_n] \quad \text{if } n \text{ is even} \end{aligned}$$

Then $a_i = b_i$ for all $i < n$. Cf here (2), (8), (9), (10).

We prove (23) by induction on n. For $n = 1$ the assumption is that $a_0 < b_0 + \frac{1}{r} < a_0 + \frac{1}{a_1}$. So clearly $a_0 = b_0$. Now assume (23) for an odd n; we prove it for $n+1$ and $n+2$. So, first suppose that

$$[a_0, \ldots, a_n] > [b_0, \ldots, b_n, r] > [a_0, \ldots, a_{n+1}].$$

Thus

$$a_0 + \frac{1}{[a_1, \ldots, a_n]} > b_0 + \frac{1}{[b_1, \ldots, b_n, r]} > a_0 + \frac{1}{[a_1, \ldots, a_{n+1}]},$$

and it follows that $a_0 = b_0$ and

$$[a_1, \ldots, a_n] < [b_1, \ldots, b_n, r] < [a_1, \ldots, a_{n+1}];$$

then the inductive hypothesis yields $a_i = b_i$ for all $i = 1, \ldots, n$, which proves our statement for $n+1$.

The inductive step to $n+2$ is clearly similar. So (23) holds.

Now to show that f is continuous, suppose that $s \in {}^n(\omega \backslash 1)$; we want to show that $f^{-1}[O(s)]$ is open. We may assume that $n = 2m+1$ for some natural number m. Let $\alpha \in f^{-1}[O_s]$. Define $a_i = a(\alpha, i)$ for all i. Thus $a_0 = s_0, \ldots, a_{2m} = s_{2m}$. By (2) and (8)–(10) we have $[a_0, \ldots, a_{2m}] < \alpha < [a_0, \ldots, a_{2m+1}]$. Choose ε so that $[a_0, \ldots, a_{2m}] + \varepsilon < \alpha < \alpha + \varepsilon < [a_0, \ldots, a_{2m+1}]$. We claim:

(24) For every irrational $\beta > 1$, if $|\alpha - \beta| < \varepsilon$, then $\beta \in f^{-1}[O(s)]$.

This will prove continuity of f. To prove (24), assume its hypothesis, and let $b_i = b(\beta, i)$ for all i.

Case 1. $\beta < \alpha$. Thus $\alpha - \beta < \varepsilon$. Hence $[a_0, \ldots, a_{2m}] < [a_0, \ldots, a_{2m}] + \varepsilon < \alpha < \beta + \varepsilon$, so $[a_0, \ldots, a_{2m}] < \beta$. If $[a_0, \ldots, a_{2m+1}] \le \beta$, then by (8)–(10), $\alpha < \beta$, contradiction. So $\beta < [a_0, \ldots, a_{2m+1}]$. Now $\beta = [b_0, \ldots, b_{2m}, r_{2m+1}]$ by (13), so by (23), $a_i = b_i$ for all $i \le 2m$, as desired.

Case 2. $\alpha < \beta$. Thus $\beta - \alpha < \varepsilon$, so $\beta < \alpha + \varepsilon$. Hence

$$[a_0, \ldots, a_{2m}] < \alpha < \beta < \alpha + \varepsilon < [a_0, \ldots, a_{2m+1}],$$

and the argument is finished as in Case 1.

So (24) holds, and f is continuous.

(25) f is an open mapping.

For, suppose that $\alpha > 1$ is irrational, and ε is a positive real number; we want to show that $f[S_\varepsilon(\alpha)]$ is open. Let $b \in f[S_\varepsilon(\alpha)]$; we want to find a finite sequence s such that $b \in O(s) \subseteq f[S_\varepsilon(\alpha)]$. Say $b = f(\beta)$ with $\beta \in S_\varepsilon(\alpha)$. So $|\alpha - \beta| < \varepsilon$. Choose m such that

$$\frac{1}{q(b, 2m)q(b, 2m+1)} < \varepsilon - |\alpha - \beta|.$$

This is possible by (11). Let $s = \langle b_0, \ldots, b_{2m+1} \rangle$. So $b \in O(s)$. Now suppose that $c \in O(s)$. Then

$$[b_0, \ldots, b_{2m}] = [c_0, \ldots, c_{2m}] < [c] < [c_0, \ldots, c_{2m+1}] = [b_0, \ldots, b_{2m+1}]$$

by (8)–(10). Also,

$$[b_0, \ldots, b_{2m}] = [c_0, \ldots, c_{2m}] < \beta < [c_0, \ldots, c_{2m+1}] = [b_0, \ldots, b_{2m+1}]$$

by (8)–(10). Now

$$\begin{aligned} [b_0, \ldots, b_{2m+1}] - [b_0, \ldots, b_{2m}] &= \frac{p(b, 2m+1)}{q(b, 2m+1)} - \frac{p(b, 2m)}{q(b, 2m)} \quad \text{by (2)} \\ &= \frac{1}{q(b, 2m)q(b, 2m+1)} \\ &< \varepsilon - |\alpha - \beta|. \end{aligned}$$

Hence

$$|[c]-\alpha| \le |[c]-\beta| + |\beta-\alpha| < \varepsilon,$$

and so $c = f([c]) \in f[S_\varepsilon(\alpha)]$, as desired. $\square$

Meager for $\mathbb{R}$ and $(0,1)$.

Proposition 26.27. *$\mathbb{R}$ is homeomorphic to $(0,1)$.*

Proof. For each $x \in (0,1)$ let $f(x) = -\frac{1}{x} + \frac{1}{1-x}$. Then if $x < y$ we have

$$1 < \frac{y}{x}; \quad \frac{1}{y} < \frac{1}{x} \quad -\frac{1}{x} < -\frac{1}{y};$$
$$-y < -x; \quad 1-y < 1-x; \quad 1 < \frac{1-x}{1-y}; \quad \frac{1}{1-x} < \frac{1}{1-y};$$
$$f(x) < f(y).$$

In particular, f is one-one. Also, $\lim_{x\to 0} f(x) = -\infty$ and $\lim_{x\to 1} f(x) = \infty$. So the proposition follows. $\square$

Proposition 26.28. *If $A \subseteq (0,1)$ is nowhere dense in $(0,1)$, then A is nowhere dense in $[0,1]$.*

Proof. Suppose that $A \subseteq (0,1)$ is nowhere dense in $(0,1)$. Take any $a < b$ with $(a,b) \cap [0,1] \neq \emptyset$; we want to show that $(a,b) \cap [0,1]\backslash A \neq \emptyset$. Clearly $(a,b) \cap (0,1) \neq \emptyset$, so $(a,b) \cap (0,1)\backslash A \neq \emptyset$. So $(a,b) \cap [0,1]\backslash A \neq \emptyset$. $\square$

Corollary 26.29. *If $A \subseteq (0,1)$ is meager in $(0,1)$, then A is meager in $[0,1]$.* $\square$

Proposition 26.30. *If $A \subseteq [0,1]$ is nowhere dense in $[0,1]$, then $A \cap (0,1)$ is nowhere dense in $(0,1)$.*

Proof. Suppose that $A \subseteq [0,1]$ is nowhere dense in $[0,1]$. Take any $a < b$ with $(a,b)\cap(0,1) \neq \emptyset$; we want to show that $(a,b)\cap(0,1)\backslash(A\cap(0,1)) \neq \emptyset$. Now $(\max(a,0),\min(b,1)) = (a,b)\cap(0,1) \neq \emptyset$ and $(\max(a,0),\min(b,1) \subseteq [0,1]$, so $(\max(a,0),\min(b,1))\backslash A \neq \emptyset$. Clearly $(\max(a,0),\min(b,1))\backslash A \subseteq (a,b) \cap (0,1)\backslash(A \cap (0,1))$.qed

Corollary 26.31. *If $A \subseteq [0,1]$ is meager in $[0,1]$, then $A\cap(0,1)$ is meager in $(0,1)$.* $\square$

Proposition 26.32. *(i)* $\mathrm{add}(\mathrm{meager}_{[0,1]}) = \mathrm{add}(\mathrm{meager}_{(0,1)})$*;*
(ii) $\mathrm{cov}(\mathrm{meager}_{[0,1]}) = \mathrm{cov}(\mathrm{meager}_{(0,1)})$*;*
(iii) $\mathrm{non}(\mathrm{meager}_{[0,1]}) = \mathrm{non}(\mathrm{meager}_{(0,1)})$*;*
(iv) $\mathrm{cof}(\mathrm{meager}_{[0,1]}) = \mathrm{cof}(\mathrm{meager}_{(0,1)})$.

Proof. (i): First let $\kappa = \mathrm{add}(\mathrm{meager}_{[0,1]})$ and suppose that $E \in [\mathrm{add}(\mathrm{meager}_{[0,1]})]^\kappa$ with $\bigcup E \notin \mathrm{add}(\mathrm{meager}_{[0,1]})$. Then by Corollary 26.31, $E' = \{A \cap (0,1) : A \in E\} \subseteq \mathscr{P}(\mathrm{meager}_{(0,1)})$. If $\bigcup E' \in \mathrm{add}(\mathrm{meager}_{(0,1)})$, then clearly

$$\bigcup E \subseteq \bigcup E' \cup \{0,1\} \in \mathrm{add}(\mathrm{meager}_{[0,1]}),$$

contradiction.

Second let $\kappa = \text{add}(\text{meager}_{(0,1)})$ and suppose that $E \in [\text{add}(\text{meager}_{(0,1)})]^\kappa$ with $\bigcup E \notin \text{add}(\text{meager}_{(0,1)})$. Then by Corollary 26.29, $E \subseteq \mathscr{P}(\text{meager}_{[0,1]})$. If $\bigcup E \in \text{add}(\text{meager}_{[0,1]})$, then by Corollary 26.31, $\bigcup E = (\bigcup E) \cap (0,1) \in \text{add}(\text{meager}_{(0,1)})$, contradiction.

(ii): First let $\kappa = \text{cov}(\text{meager}_{[0,1]})$ and suppose that $E \in [\text{add}(\text{meager}_{[0,1]})]^\kappa$ with $[0,1] = \bigcup E$. Then by Corollary 26.31, $E' = \{A \cap (0,1) : A \in E\} \subseteq \mathscr{P}(\text{meager}_{(0,1)})$. Hence $(0,1) = \bigcup E'$.

Second let $\kappa = \text{cov}(\text{meager}_{(0,1)})$ and suppose that $E \in [\text{add}(\text{meager}_{(0,1)})]^\kappa$ with $(0,1) = \bigcup E$. Then by Corollary 26.29, $E \subseteq \mathscr{P}(\text{meager}_{[0,1]})$. Now $[0,1] = \bigcup E \cup \{0,1\}$.

(iii): First let $\kappa = \text{non}(\text{meager}_{[0,1]})$ and $X \in [[0,1]]^\kappa$ with $X \notin \text{non}(\text{meager}_{[0,1]})$. Then $X \backslash \{0,1\} \in [(0,1)]^\kappa$ and $X \backslash \{0,1\} \notin \text{non}(\text{meager}_{(0,1)})$ by Corollary 26.29.

Second let $\kappa = \text{non}(\text{meager}_{(0,1)})$ and $X \in [(0,1)]^\kappa$ with $X \notin \text{non}(\text{meager}_{(0,1)})$. Then $X \in [(0,1)]^\kappa$ with $X \notin \text{non}(\text{meager}_{[0,1]})$ by Corollary 26.31.

(iv): First let $\kappa = \text{cof}(\text{meager}_{[0,1]})$ and $X \in [\text{cof}(\text{meager}_{[0,1]})]^\kappa$ such that $\forall A \in \text{cof}(\text{meager}_{[0,1]}) \exists B \in X[A \subseteq B]$. Let $X' = \{B \cap (0,1) : B \in X\}$. Then

$$X' \in \mathscr{P}(\text{cof}(\text{meager}_{(0,1)}))$$

by Corollary 26.31. Suppose that $A \in \text{cof}(\text{meager}_{(0,1)})$. Then by Corollary 26.29,

$$A \in \text{cof}(\text{meager}_{[0,1]}).$$

So there is a $B \in X$ such that $A \subseteq B$. Hence $A \subseteq B \cap (0,1)$.

Second let $\kappa = \text{non}(\text{meager}_{(0,1)})$ and $X \in [\text{cof}(\text{meager}_{(0,1)})]^\kappa$ such that

$$\forall A \in \text{cof}(\text{meager}_{(0,1)}) \exists B \in X[A \subseteq B].$$

Let $X' = \{B \cup \{0,1\} : B \in X\}$. Clearly $X' \in [\text{cof}(\text{meager}_{[0,1]})]^\kappa$. Suppose that $A \in \text{cof}(\text{meager}_{[0,1]})$. Then $A \cap (0,1) \in \text{cof}(\text{meager}_{(0,1)})$ by Corollary 26.31. Choose $B \in X$ such that $A \cap (0,1) \subseteq B$. Then $A \subseteq B \cup \{0,1\}$. □

Meager for irrat and $\mathbb{R}$

Lemma 26.33. *If $A \subseteq \mathbb{R}$ is nowhere dense in $\mathbb{R}$, then $A \cap \text{irrat}$ is nowhere dense in* irrat.

Proof. Assume that $A \subseteq \mathbb{R}$ is nowhere dense in $\mathbb{R}$. Then $\forall a, b \in \mathbb{R}[a < b$ implies that $(a,b) \backslash \overline{A} \neq \emptyset]$, so $\forall a, b \in \mathbb{R}[a < b$ implies that $\exists c, d \in \mathbb{R}[c < d$ and $(c,d) \subseteq (a,b) \backslash \overline{A}]]$. Now the closure of $A \cap \text{irrat}$ in irrat is $\overline{A} \cap \text{irrat}$. Given $a, b \in \mathbb{R}$ with $a < b$, choose $c, d \in \mathbb{R}$ such that $c < d$ and $(c,d) \subseteq (a,b) \backslash \overline{A}$. Then

$$(c,d) \cap \text{irrat} \subseteq ((a,b) \cap \text{irrat}) \backslash (\overline{A} \cap \text{irrat}).$$

Thus $A \cap \text{irrat}$ is nowhere dense in irrat. □

Lemma 26.34. *If A is* $\text{meager}_\mathbb{R}$*, then $A \cap \text{irrat}$ is* $\text{meager}_{\text{irrat}}$. □

Lemma 26.35. *If* $A \subseteq \text{irrat}$ *is nowhere dense in* irrat*, then it is nowhere dense in* $\mathbb{R}$.

Proof. Assume that $A \subseteq \text{irrat}$ is nowhere dense in irrat. Then $\forall a, b \in \mathbb{R}[a < b$ implies that $(a, b) \cap \text{irrat} \backslash (\overline{A} \cap \text{irrat}) \neq \emptyset$. Given $a, b \in \mathbb{R}$ with $a < b$, choose $c, d \in \mathbb{R}$ with $c < d$ and $(c, d) \cap \text{irrat} \subseteq (a, b) \cap \text{irrat} \backslash (\overline{A} \cap \text{irrat})$. We claim that $(c, d) \subseteq (a, b) \backslash \overline{A}$. Now $(c, d) \cap (a, b) = (\max(c, a), \min(d, b))$. Suppose that $(\max(c, a), \min(d, b)) \cap \overline{A} \neq \emptyset$. Then there is an $x \in (\max(c, a), \min(d, b)) \cap A$. So x is irrational, contradiction. $\square$

Lemma 26.36. *If* A *is* $\text{meager}_{\text{irrat}}$*, then* A *is* $\text{meager}_{\mathbb{R}}$. $\square$

Lemma 26.37. $\text{add}(\text{meager}_{\mathbb{R}}) = \text{add}(\text{meager}_{\text{irrat}})$.

Proof. First let $\kappa = \text{add}(\text{meager}_{\text{irrat}})$, and let $E \subseteq \text{meager}_{\text{irrat}}$ be such that $|E| = \kappa$ and $\bigcup E \notin \text{meager}_{\text{irrat}}$. By Lemma 26.35, $E \subseteq \text{meager}_{\mathbb{R}}$. We have $\bigcup E \notin \text{meager}_{\mathbb{R}}$ by Lemma 26.34.

Second, let $\kappa = \text{add}(\text{meager}_{\mathbb{R}})$, and let $E \subseteq \text{meager}_{\mathbb{R}}$ be such that $|E| = \kappa$ and $\bigcup E \notin \text{meager}_{\mathbb{R}}$. Then $E' = \{X \cap \text{irrat} : X \in E\} \subseteq \text{meager}_{\text{irrat}}$ by Lemma 26.34. Hence $E' \cup \{\mathbb{Q}\} \subseteq \text{meager}_{\text{irrat}}$. If $\bigcup(E' \cup \{\mathbb{Q}\}) \in \text{meager}_{\text{irrat}}$, then $\bigcup E \subseteq \bigcup(E' \cup \{\mathbb{Q}\}) \in \text{meager}_{\text{irrat}} \subseteq \text{meager}_{\mathbb{R}}$ by Lemma 26.36, contradiction. $\square$

Lemma 26.38. $\text{cov}(\text{meager}_{\mathbb{R}}) = \text{cov}(\text{meager}_{\text{irrat}})$.

Proof. First let $\kappa = \text{cov}(\text{meager}_{\mathbb{R}})$, and let $E \in [\text{meager}_{\mathbb{R}}]^\kappa$ be such that $\mathbb{R} = \bigcup E$. For each $A \in E$ we have $A \cap \text{irrat} \in \text{meager}_{\text{irrat}}$ by Corollary 26.34. Moreover, $\bigcup_{A \in E}(A \cap \text{irrat}) = \text{irrat}$. So $\text{cov}(\text{meager}_{\text{irrat}}) \leq \kappa$.

Second let $\kappa = \text{cov}(\text{meager}_{\text{irrat}})$, and let $E \in [\text{meager}_{\text{irrat}}]^\kappa$ be such that $\text{irrat} = \bigcup E$. Let $E' = E \cup \{\{q\} : q \in \mathbb{Q}\}$. Then by Lemma 26.36, $E' \in \mathscr{P}(\text{meager}_{\mathbb{R}})$, and $\bigcup E' = \mathbb{R}$. So $\text{cov}(\text{meager}_{\mathbb{R}}) \leq \kappa$. $\square$

Lemma 26.39. $\text{non}(\text{meager}_{\mathbb{R}}) = \text{non}(\text{meager}_{\text{irrat}})$.

Proof. First let $\kappa = \text{non}(\text{meager}_{\mathbb{R}})$, and let $X \in [\mathbb{R}]^\kappa$ such that $X \notin \text{meager}_{\mathbb{R}}$. Then $X \cap \text{irrat} \notin \text{meager}_{\text{irrat}}$, as otherwise, with $Y = (X \cap \text{irrat}) \cup \{\{q\} : q \in \mathbb{Q}\}$ we would have $X \subseteq Y \in \text{meager}_{\mathbb{R}}$, using Lemma 26.36.

Second let $\kappa = \text{non}(\text{meager}_{\text{irrat}})$, and let $X \in [\text{irrat}]^\kappa$ such that $X \notin \text{meager}_{\text{irrat}}$. By Lemma 26.34, $X \notin \text{meager}_{\mathbb{R}}$. Hence $\text{non}(\text{meager}_{\mathbb{R}}) \leq \kappa$. $\square$

Lemma 26.40. $\text{cof}(\text{meager}_{\mathbb{R}}) = \text{cof}(\text{meager}_{\text{irrat}})$.

Proof. First let $\kappa = \text{cof}(\text{meager}_{\mathbb{R}})$, and let $X \in [\text{meager}_{\mathbb{R}}]^\kappa$ be such that $\forall A \in \text{meager}_{\mathbb{R}} \exists B \in X[A \subseteq B]$. Let $Y = \{A \cap \text{irrat} : A \in X\}$. Then by Lemma 26.34, $Y \in \mathscr{P}(\text{meager}_{\text{irrat}})$, and $\forall A \in \text{meager}_{\text{irrat}} \exists B \in Y[A \subseteq B]$, using also Lemma 26.36. Hence $\text{cof}(\text{meager}_{\text{irrat}}) \leq \kappa$.

Second let $\kappa = \text{cof}(\text{meager}_{\text{irrat}})$, with $X \in [\text{meager}_{\text{irrat}}]^\kappa$ so that $\forall A \in \text{meager}_{\text{irrat}} \exists B \in X[A \subseteq B]$. Let $Y = \{A \cup \mathbb{Q} : A \in X$. Then by Lemma 26.36, $Y \in \mathscr{P}(\text{meager}_{\mathbb{R}})$. If $A \in \text{meager}_{\mathbb{R}}$, then $A \cap \text{irrat} \in \text{meager}_{\text{irrat}}$ by Lemma 26.34, and so there is a $B \in X$ such that $A \cap \text{irrat} \subseteq B$. Then $A \subseteq B \cup \mathbb{Q} \in Y$. Hence $\text{cof}(\text{meager}_{\mathbb{R}}) \leq \kappa$. $\square$

The Cantor set and ${}^\omega 2$.

Let

$$C = \left\{ x \in [0,1] : \exists t \in {}^{\omega\backslash 1}\{0,2\} \left[x = \sum_{i=1}^{\infty} \frac{t_i}{3^i} \right] \right\}.$$

C is the *Cantor set*. For $a < b$ let

$$f([a,b]) = \left\{ \left[a, a + \frac{1}{3}(b-a) \right], \left[a + \frac{2}{3}(b-a), b \right] \right\}.$$

Define A with domain ω recursively by

$$A_0 = \{[0,1]\};$$
$$A_{n+1} = \bigcup_{X \in A_n} f(X).$$

Lemma 26.41. *For every positive integer n and every set Y, $Y \in A_n$ iff there is a $t : (n+1)\backslash 1 \to \{0,2\}$ such that*

$$Y = \left[\sum_{i=1}^{n} \frac{t_i}{3^i}, \sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{1}{3^n} \right].$$

Proof. For $n = 1$ we have $A_1 = f([0,1]) = \{[0, \frac{1}{3}], [\frac{2}{3}, 1]\}$. With $t_1 = 0$ we have $[\sum_{i=1}^{n} \frac{t_i}{3^i}, \sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{1}{3^n}] = [0, \frac{1}{3}]$, and with $t_1 = 2$ we have $[\sum_{i=1}^{n} \frac{t_i}{3^i}, \sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{1}{3^n}] = [\frac{2}{3}, 1]$, as desired.

Now assume the equality for $n \geq 1$. First suppose that $Y \in A_{n+1}$. Then there is an $X \in A_n$ such that $Y \in f(X)$. By the inductive hypothesis choose $t : (n+1)\backslash 1 \to \{0,2\}$ such that

$$X = \left[\sum_{i=1}^{n} \frac{t_i}{3^i}, \sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{1}{3^n} \right].$$

Note that X has size $\frac{1}{3^n}$.

Case 1.

$$Y = \left[\sum_{i=1}^{n} \frac{t_i}{3^i}, \sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{1}{3^{n+1}} \right].$$

Let $s \restriction ((n+1)\backslash 1) = t \restriction ((n+1)\backslash 1)$ and $s(n+1) = 0$. Then

$$(*) \qquad Y = \left[\sum_{i=1}^{n+1} \frac{s_i}{3^i}, \sum_{i=1}^{n+1} \frac{s_i}{3^i} + \frac{1}{3^{n+1}} \right].$$

Case 2.

$$Y = \left[\sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{2}{3^{n+1}}, \sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{1}{3^n} \right].$$

Let $s \restriction ((n+1)\backslash 1) = t \restriction ((n+1)\backslash 1)$ and $s(n+1) = 2$. Then $(*)$ holds.

Second, suppose that $(*)$ holds. Let $t = s \restriction ((n+1)\backslash 1)$. If $s(n+1) = 0$, then

$$Y = \left[\sum_{i=1}^{n} \frac{t_i}{3^i}, \sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{1}{3^{n+1}}\right];$$

If $s(n+1) = 2$, then

$$Y = \left[\sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{2}{3^{n+1}}, \sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{1}{3^n}\right].$$

Hence in either case,

$$Y \in f\left(\left[\sum_{i=1}^{n} \frac{t_i}{3^i}, \sum_{i=1}^{m} \frac{t_i}{3^i} + \frac{1}{3^n}\right]\right),$$

and

$$\left[\sum_{i=1}^{n} \frac{t_i}{3^i}, \sum_{i=1}^{m} \frac{t_i}{3^i} + \frac{1}{3^n}\right] \in A_n$$

by the inductive hypothesis. Hence $Y \in A_{n+1}$. $\square$

Theorem 26.42. $C = \bigcap_{n\in\omega} \bigcup A_n$.

Proof. Suppose that $x \in C$ and $n \in \omega$. Choose $s \in {}^{\omega\backslash 1}\{0,2\}$ such that $x = \sum_{i=1}^{\infty} \frac{s_i}{3^i}$. Let $t = s \restriction ((n+1)\backslash 1)$. Then

$$x \in \left[\sum_{i=1}^{n} \frac{t_i}{3^i}, \sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{1}{3^n}\right]$$

since $\sum_{i=n+1}^{\infty} \frac{t_i}{3^i} \le \frac{1}{3^n}$. Thus by Lemma 26.41, $x \in \bigcup A_n$.

Now suppose that $x \notin C$. If $x \notin [0,1]$, clearly $x \notin \bigcap_{n\in\omega} (\bigcup A_n)$. So suppose that $x \in [0,1]$, and choose $t \in {}^{\omega\backslash 1}3$ such that $x = \sum_{i=1}^{\infty} \frac{t_i}{3^i}$. Choose n minimal such that $t_n = 1$. Suppose $x \in \bigcup A_n$. By Lemma 26.41, choose $s \in {}^{(n+1)\backslash 1}\{0,2\}$ such that

$$\sum_{i=1}^{n} \frac{s_i}{3^i} \le x \le \sum_{i=1}^{n} \frac{s_i}{3^i} + \frac{1}{3^n}.$$

We claim that $t \restriction n = s \restriction n$. Otherwise there is a least $m < n$ such that $t_m \ne s_m$. If $t_m < s_m$, then $t_m = 0$ since $t_m \in \{0,2\}$ because $m < n$. Hence

$$x = \sum_{i=1}^{\infty} \frac{t_i}{3^i} = \sum_{i=1}^{m} \frac{t_i}{3^i} + \sum_{i=m+1}^{\infty} \frac{t_i}{3^i} \le \sum_{i=1}^{m} \frac{t_i}{3^i} + \frac{1}{3^m} < \sum_{i=1}^{m} \frac{s_i}{3^i} \le \sum_{i=1}^{n} \frac{s_i}{3^i} \le x,$$

contradiction. If $s_m < t_m$, then $s_m = 0$ and $t_m = 2$, and

$$x \le \sum_{i=1}^{n} \frac{s_i}{3^i} + \frac{1}{3^n} \le \sum_{i=1}^{m} \frac{s_i}{3^i} + \sum_{i=m+1}^{\infty} \frac{2}{3^i} = \sum_{i=1}^{m} \frac{s_i}{3^i} + \frac{1}{3^m} < \sum_{i=1}^{m} \frac{t_i}{3^i} \le x,$$

contradiction.

So $s \restriction n = t \restriction n$.

Case 1. $s_n = 2$. Then

$$x = \sum_{i=1}^{\infty} \frac{t_i}{3^i} = \sum_{i=1}^{n} \frac{t_i}{3^i} + \sum_{i=n+1}^{\infty} \frac{t_i}{3^i} \le \sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{1}{3^n} = \sum_{i=1}^{n} \frac{s_i}{3^i} \le x$$

It follows that $t_i = 2$ for all $i \ge n+1$, and hence $x = (t \restriction n)^\frown\langle 2, 0, 0, 0, \ldots\rangle \in C$, contradiction.

Case 2. $s_n = 0$. Then

$$x = \sum_{i=1}^{\infty} \frac{t_i}{3^i} \le \sum_{i=1}^{n} \frac{s_i}{3^i} + \frac{1}{3^n} = \sum_{i=1}^{n} \frac{t_i}{3^i} \le x;$$

it follows that $t_i = 0$ for all $i > n$; hence $x = (t \restriction n)^\frown\langle 0, 2, 2, 2, \ldots\rangle \in C$, contradiction. $\square$

Theorem 26.43. *C is homeomorphic to ${}^\omega 2$.*

Proof. For each $t \in {}^\omega 2$ let

$$f(t) = \sum_{i=1}^{\infty} \frac{2t_{i-1}}{3^i}.$$

Clearly f maps onto C. It is one-one; for suppose that $s, t \in {}^\omega 2$ with $s \ne t$. Let n be minimum such that $s_n \ne t_n$. Say $s_n = 0$ and $t_n = 1$. Then

$$\begin{aligned} f(s) &= \sum_{i=1}^{\infty} \frac{2s_{i-1}}{3^i} = \sum_{i=1}^{n} \frac{2s_{i-1}}{3^i} + \sum_{i=n+1}^{\infty} \frac{2s_{i-1}}{3^i} \\ &\le \sum_{i=1}^{n} \frac{2s_{i-1}}{3^i} + \sum_{i=n+1}^{\infty} \frac{2}{3^i} = \sum_{i=1}^{n} \frac{2s_{i-1}}{3^i} + \frac{1}{3^n} < \sum_{i=1}^{n} \frac{2s_{i-1}}{3^i} + \frac{2}{3^n} \le \sum_{i=1}^{\infty} \frac{2t_{i-1}}{3^i} = f(t). \end{aligned}$$

So f is one-one. Now f is continuous. For, suppose that U is open in $\mathbb{R}$ and $t \in f^{-1}[U]$. Say $(f(t) - \varepsilon, f(t) + \varepsilon) \subseteq U$. Choose n so that $3^{-n} < \varepsilon$. Let $V = \{s \in {}^\omega 2 : s \restriction n = t \restriction n\}$. For any $s \in V$ we have

$$|f(s) - f(t)| = \left| \sum_{n+1 \le i}^{\infty} \frac{2(s(i-1) - t(i-1))}{3^i} \right| \le \sum_{n+1 \le i}^{\infty} \frac{2}{3^i} = \frac{1}{3^n} < \varepsilon.$$

Thus $V \subseteq f^{-1}[U]$. So f is continuous. By Engelking Theorem 3.1.13, f is a homeomorphism. $\square$

Ω and ${}^\omega 2$.

If $z \in {}^{\omega}2$ let $z^{\circ} = \sum_{i=1}^{\infty}(z_{i-1}2^{-i})$. Note that clearly $0 \leq z^{\circ}$. Also, if $z \neq \langle 1, 1, \ldots \rangle$ then $z^{\circ} < 1$. In fact, choose j such that $z_j = 0$. then

$$\begin{aligned} z^{\circ} &= \sum_{i=1}^{\infty}(z_{i-1}2^{-i}) = \sum_{i=1}^{j}(z_{i-1}2^{-i}) + \sum_{i=j+2}^{\infty}(z_{i-1}2^{-i}) \\ &\leq \sum_{i=1}^{j}(z_{i-1}2^{-i}) + \sum_{i=j+2}^{\infty} 2^{-i} = \sum_{i=1}^{j}(z_{i-1}2^{-i}) + 2^{-j-1} < 1. \end{aligned}$$

For $z \in [0, 1)$ let $z' \in {}^{\omega}2$ be such that $z = \sum_{i=1}^{\infty}(z'_{i-1}2^{-i})$, with z' not eventually 1. Let $\Omega = \{x \in {}^{\omega}2 : x \text{ is not eventually 1 and } x \neq \langle 0, 0, 0, \ldots \rangle\}$. For each $m \in \omega$ and each $f \in {}^{m}2$ let $W_f = \{x \in {}^{\omega}2 : f \subseteq x\}$ and $W'_f = \{x \in \Omega : f \subseteq x\}$. Let $M = \{x \in {}^{\omega}2 : x \text{ is eventually 1 or } x = \langle 0, 0, 0, \ldots \rangle\}$.

Lemma 26.44. *If $x, y \in {}^{\omega}2$ and neither x nor y is eventually 1, and if $x \neq y$, then $\sum_{i=1}^{\infty}(x_{i-1}2^{-i}) \neq \sum_{i=1}^{\infty}(y_{i-1}2^{-i})$.*

Proof. Suppose that $x, y \in {}^{\omega}2$ and neither is eventually 1, and $x \neq y$. Let j be minimum such that $x_j \neq y_j$. Wlog $x_j = 0$ and $y_j = 1$. Choose $k > j$ such that $x_k = 0$. Then

$$\begin{aligned} x &= \sum_{i=1}^{k}(x_{i-1}2^{-i}) + \sum_{i=k+2}^{\infty} x_{i-1}2^{-i} \leq \sum_{i=1}^{k}(x_{i-1}2^{-i}) + \sum_{i=k+2}^{\infty} 2^{-i} \\ &= \sum_{i=1}^{k}(x_{i-1}2^{-i}) + 2^{-k-1} < \sum_{i=1}^{k}(x_{i-1}2^{-i}) + \sum_{i=k+1}^{\infty} 2^{-i} \\ &= \sum_{i=1}^{j}(x_{i-1}2^{-i}) + 2^{-k} \leq \sum_{i=1}^{\infty}(y_i 2^{-i}), \end{aligned}$$

so $\sum_{i=1}^{\infty}(x_{i-1}2^{-i}) \neq \sum_{i=1}^{\infty}(y_{i-1}2^{-i})$. ☐

Lemma 26.45. *(i) If $x \in {}^{\omega}2$ is not eventually 1, then $x^{\circ\prime} = x$.*

(ii) If $x \in [0, 1)$, then $x'^{\circ} = x$.

Proof. (i): Suppose that $x \in {}^{\omega}2$ is not eventually 1. Now $x^{\circ\prime} = z'$, where $x^{\circ} = \sum_{i=1}^{\infty}(z'_{i-1}2^{-i})$ with z' not eventualy 1; but also $x^{\circ} = \sum_{i=1}^{\infty}(x_{i-1}2^{-i})$. So $x = z'$ by Lemma 26.44.

(ii): obvious. ☐

Lemma 26.46. *Ω is dense in ${}^{\omega}2$.*

Proof. Given $m \in \omega$ and $f \in {}^{m}2$, let $x \in W'_f$ be such that x is not eventually 1. ☐

Lemma 26.47. *If $X \subseteq {}^{\omega}2$ is nowhere dense in ${}^{\omega}2$, then $X \cap \Omega$ is nowhere dense in Ω.*

Proof. Suppose that $X \subseteq {}^\omega 2$ is nowhere dense in ${}^\omega 2$. Suppose that W'_f is given. Now ${}^\omega 2 \backslash \overline{X}$ is dense in ${}^\omega 2$, so $W_f \backslash \overline{X} \neq \emptyset$. Choose g with $W_g \subseteq W_f \backslash \overline{X}$. Take $x \in W_G \cap \Omega$. Then $x \in W'_f \backslash (\overline{X} \cap \Omega)$. This shows that $X \cap \Omega$ is nowhere dense in Ω. □

Corollary 26.48. *If $X \subseteq {}^\omega 2$ is meager in ${}^\omega 2$, then $X \cap \Omega$ is meager in Ω.* □

Lemma 26.49. *If $X \subseteq \Omega$ is nowhere dense in Ω, then X is nowhere dense in ${}^\omega 2$.*

Proof. Suppose that $X \subseteq \Omega$ is nowhere dense in Ω, We want to show that for any $f \in {}^{<\omega}2$, $W_f \backslash \overline{X} \neq \emptyset$. We have $W'_f \backslash (\overline{X} \cap \Omega) \neq \emptyset$, so $W_f \backslash \overline{X} \neq \emptyset$. □

Corollary 26.50. *If $X \subseteq \Omega$ is meager in Ω, then X is meager in ${}^\omega 2$.* □

Lemma 26.51. $\mathrm{add}(\mathrm{meager}_{{}^\omega 2}) = \mathrm{add}(\mathrm{meager}_\Omega)$.

Proof. First let $\kappa = \mathrm{add}(\mathrm{meager}_{{}^\omega 2})$ and suppose that $E \in [\mathrm{meager}_{{}^\omega 2}]^\kappa$ with $\bigcup E \notin \mathrm{meager}_{{}^\omega 2}$. Let $E' = \{A \cap \Omega : A \in E\}$. Then by Corollary 26.48, $E' \subseteq \mathrm{meager}_\Omega$, and $|E'| \leq \kappa$. Suppose that $\bigcup E' \in \mathrm{meager}_\Omega$. Then by Corollary 26.50, $\bigcup E' \in \mathrm{meager}_{{}^\omega 2}$. Now M is countable, and $\bigcup E \subseteq \bigcup E' \cup M$, so $\bigcup E \in \mathrm{meager}_{{}^\omega 2}$, contradiction. Thus $\mathrm{add}(\mathrm{meager}_\Omega) \leq \mathrm{add}(\mathrm{meager}_{{}^\omega 2})$.

Second let $\kappa = \mathrm{add}(\mathrm{meager}_\Omega)$ and suppose that $E \in [\mathrm{meager}_\Omega]^\kappa$ with $\bigcup E \notin \mathrm{meager}_\Omega$. By Corollary 26.50, $E \in [\mathrm{meager}_{{}^\omega 2}]^\kappa$. Suppose that $\bigcup E \in \mathrm{meager}_{{}^\omega 2}$. By Corollary 26.48, $\bigcup E = (\bigcup E) \cap \Omega \in \mathrm{meager}_\Omega$, contradiction. □

Lemma 26.52. $\mathrm{cov}(\mathrm{meager}_{{}^\omega 2}) = \mathrm{cov}(\mathrm{meager}_\Omega)$.

Proof. First let $\kappa = \mathrm{cov}(\mathrm{meager}_{{}^\omega 2})$ and suppose that $E \in [\mathrm{meager}_{{}^\omega 2}]^\kappa$ with $\bigcup E = {}^\omega 2$. Let $E' = \{A \cap \Omega : A \in E\}$. Then by Corollary 26.48, $E' \subseteq \mathscr{P}(\mathrm{meager}_\Omega)$, and $|E'| \leq \kappa$. Clearly $\bigcup E' = \Omega$. So $\mathrm{cov}(\mathrm{meager}_\Omega) \leq \kappa$.

Second let $\kappa = \mathrm{cov}(\mathrm{meager}_\Omega)$ and suppose that $E \in [\mathrm{meager}_\Omega]^\kappa$ with $\bigcup E = \Omega$. By Corollary 26.50, $E \in [\mathrm{meager}_{{}^\omega 2}]^\kappa$. Then $M \cup \bigcup E = {}^\omega 2$. Hence $\mathrm{cov}(\mathrm{meager}_{{}^\omega 2}) = \mathrm{cov}(\mathrm{meager}_\Omega)$. □

Lemma 26.53. $\mathrm{non}(\mathrm{meager}_{{}^\omega 2}) = \mathrm{non}(\mathrm{meager}_\Omega)$.

Proof. First let $\kappa = \mathrm{non}(\mathrm{meager}_{{}^\omega 2})$ and suppose that $X \in [{}^\omega 2]^\kappa$ with $X \notin \mathrm{meager}_{{}^\omega 2}$. Then $|X \cap \Omega| \leq \kappa$. Suppose that $X \cap \Omega \in \mathrm{meager}_\Omega$. Then by Corollary 26.50, $X \cap \Omega \in \mathrm{meager}_{{}^\omega 2}$) so $X \subseteq (X \cap \Omega) \cup M \in \mathrm{meager}_{{}^\omega 2}$, contradiction.

Second let $\kappa = \mathrm{non}(\mathrm{meager}_\Omega)$ and suppose that $X \in [\Omega]^\kappa$ with $X \notin \mathrm{meager}_\Omega$. Then $X \in [{}^\omega 2]^\kappa$ and by Corollary 26.48 $X \notin \mathrm{meager}_{{}^\omega 2}$. □

Lemma 26.54. $\mathrm{cof}(\mathrm{meager}_{{}^\omega 2}) = \mathrm{cof}(\mathrm{meager}_\Omega)$.

Proof. First let $\kappa = \mathrm{cof}(\mathrm{meager}_{{}^\omega 2})$ and suppose that $X \in [\mathrm{meager}_{{}^\omega 2}]^\kappa$ is such that $\forall A \in \mathrm{meager}_{{}^\omega 2} \exists B \in X[A \subseteq B]$. Let $X' = \{A \cap \Omega : A \in X\}$. So $|X| \leq \kappa$, and by Corollary 26.48, $X' \subseteq \mathscr{P}(\mathrm{meager}_\Omega)$. Suppose that $A \in \mathrm{meager}_\Omega$. Then by Corollary 26.50, $A \in \mathrm{meager}_{{}^\omega 2}$, so there is a $B \in X$ such that $A \subseteq B$. Then $A \subseteq B \cap \Omega \in X'$. Hence $\mathrm{cof}(\mathrm{meager}_\Omega) \leq \kappa$.

Second let $\kappa = \mathrm{cof}(\mathrm{meager}_\Omega)$ and suppose that $X \in [\mathrm{meager}_\Omega]^\kappa$ is such that $\forall A \in \mathrm{meager}_\Omega \exists B \in X[A \subseteq B]$. Let $X' = \{A \cup M : A \in X\}$. So $|X'| \le \kappa$. Suppose that $A \in \mathrm{meager}_{^\omega 2}$. Then by Corollary 26.48, $A \cap \Omega \in \mathrm{meager}_\Omega$, so there is a $B \in X$ such that $A \cap \Omega \subseteq B$. Then $A \subseteq B \cup M \in X'$. Hence $\mathrm{cof}(\mathrm{meager}_{^\omega 2}) = \mathrm{cof}(\mathrm{meager}_\Omega)$. □

$(0,1)$ and Ω

Lemma 26.55. *If $h \in {}^\omega 2$ is not eventually 1 and $h_m = 0$, then*

$$\sum_{i=1}^{\infty}(h_{i-1}2^{-i}) < \sum_{i=1}^{m}(h_{i-1}2^{-i}) + 2^{-m-1}.$$

Proof. Assume that $h \in {}^\omega 2$ is not eventually 1 and $h_m = 0$. Choose $n > m$ so that $h_n = 0$. Then

$$\begin{aligned}
\sum_{i=1}^{\infty}(h_{i-1}2^{-i}) &\le \sum_{i=1}^{n}(h_{i-1}2^{-i}) + \sum_{i=n+2}^{\infty} 2^{-i} \\
&= \sum_{i=1}^{n}(h_{i-1}2^{-i}) + 2^{-n-1} \\
&< \sum_{i=1}^{n}(h_{i-1}2^{-i}) + 2^{-n-1} + 2^{-n-2} \\
&\le \sum_{i=1}^{m}(h_{i-1}2^{-i}) + \sum_{i=m+2}^{\infty} 2^{-i} \\
&= \sum_{i=1}^{m}(h_{i-1}2^{-i}) + 2^{-m-1}.
\end{aligned}$$

□

If $A \subseteq {}^\omega 2$, we let $A^\circ = \{x^\circ : x \in A\}$; and for $X \subseteq [0,1)$ we let $X^p = \{x' : x \in X\}$.

Lemma 26.56. *If $0 < a < b < 1$, then there is an f such that $(W'_f)^\circ \subseteq (a,b)$.*

Proof. Assume that $0 < a < b < 1$. Let m be minimum such that $a'_m \ne b'_m$. If $a'_m = 1$ and $b'_m = 0$, then

$$\begin{aligned}
b = \sum_{i=1}^{\infty}(b'_{i-1}2^{-i}) &\le \sum_{i=1}^{m}(b'_{i-1}2^{-i}) + \sum_{i=m+2}^{\infty} 2^{-i} \\
&= \sum_{i=1}^{m}(b'_{i-1}2^{-i}) + 2^{-m-1} = \sum_{i=1}^{m+1}(a'_{i-1}2^{-i}) \le a,
\end{aligned}$$

contradiction. Hence $a'_m = 0$ and $b'_m = 1$.

Choose $p > n > m$ such that $a'_n = a'_p = 0$. Let $f = \langle a'_i : i < n\rangle^\frown\langle 1\rangle^\frown\langle a'_i : n+1 \le i \le p\rangle$. We claim that $(W'_f)^\circ \subseteq (a,b)$. Take any $g \in W'_f$; we want to show that $a < g^\circ < b$, i.e.,

$$\sum_{i=1}^{\infty}(a'_{i-1}2^{-i}) < \sum_{i=1}^{\infty}(g_{i-1}2^{-i}) < \sum_{i=1}^{\infty}(b'_{i-1}2^{-i}) \tag{1}$$

We have

$$\begin{aligned}
\sum_{i=1}^{\infty}(a'_{i-1}2^{-i}) &< \sum_{i=1}^{n}(a'_{i-1}2^{-i}) + 2^{-n-1} \quad \text{by Lemma 26.55}\\
&= \sum_{i=1}^{n+1}(f_{i-1}2^{-i}) = \sum_{i=1}^{n+1}(g_{i-1}2^{-i}) \le \sum_{i=1}^{\infty}(g_{i-1}2^{-i})\\
&\le \sum_{i=1}^{p+1}(g_{i-1}2^{-i}) + \sum_{i=p+2}^{\infty}2^{-i} = \sum_{i=1}^{p+1}(g_{i-1}2^{-i}) + 2^{-p-1}\\
&< \sum_{i=1}^{n+1}(g_{i-1}2^{-i}) + \sum_{i=n+2}^{\infty}2^{-i}\\
&= \sum_{i=1}^{n+1}(g_{i-1}2^{-i}) + 2^{-n-1} = \sum_{i=1}^{n}(a'_{i-1}2^{-i}) + 2^{-n} + 2^{-n-1}\\
&\le \sum_{i=1}^{m}(b'_{i-1}2^{-i}) + \sum_{i=m+1}^{\infty}2^{-i} = \sum_{i=1}^{m+1}(b'_{i-1}2^{-i}) \le \sum_{i=1}^{\infty}(b'_{i-1}2^{-i}).
\end{aligned}$$
$\square$

Lemma 26.57. *For any $f \in {}^{<\omega}2$ there exist a, b with $0 < a < b < 1$ and $(a,b) \subseteq (W'_f)^\circ$.*

Proof. Say $f \in {}^m2$. Let $a = f^\frown\langle 1,0,0,0,\ldots\rangle$ and $b = f^\frown\langle 1,1,0,0,\ldots\rangle$. Clearly $0 < a^\circ < b^\circ < 1$. We claim that $(a^\circ, b^\circ) \subseteq {}^\circ W'_f$. Suppose that $a^\circ < z < b^\circ$. In particular, $z \in (0,1)$ and $z' \in \Omega$. If $a = z'$, then $a^\circ = z'^\circ = z$, contradiction. So $a \ne z'$. Similarly $b \ne z'$. Let n be minimum such that $a_n \ne z'_n$. Suppose that $n < m$.

Subcase 1. $a_n = 0$, $z'_n = 1$. Then

$$\begin{aligned}
b^\circ &= \sum_{i=1}^{\infty}(b_{i-1}2^{-i}) = \sum_{i=1}^{n+1}(a_{i-1}2^{-i}) + \sum_{i=n+2}^{\infty}(b_{i-1}2^{-i})\\
&\le \sum_{i=1}^{n}(a_{i-1}2^{-i}) + \sum_{i=n+2}^{\infty}2^{-i} = \sum_{i=1}^{n}(a_{i-1}2^{-i}) + 2^{-n-1}\\
&= \sum_{i=1}^{n}(z'_{i-1}2^{-i}) + 2^{-n-1} \le \sum_{i=1}^{\infty}(z'_{i-1}2^{-i}) = z,
\end{aligned}$$

contradiction.

Subcase 2. $a_n = 1$, $z'_n = 0$. Then

$$\begin{aligned} z = \sum_{i=1}^{\infty}(z'_{i-1}2^{-i}) &\le \sum_{i=1}^{n}(z'_{i-1}2^{-i}) + \sum_{i=n+2}^{\infty} 2^{-i} \\ &= \sum_{i=1}^{n}(a_{i-1}2^{-i}) + 2^{-n-1} \le \sum_{i=1}^{\infty}(a_{i-1}2^{-i}) = a^\circ, \end{aligned}$$

contradiction.

It follows that $m \le n$, hence $f \subseteq z'$ and so $z' \in W'_f$. Thus $z = z'^\circ \in (W'_f)^\circ$. $\square$

Lemma 26.58. *If $A \subseteq \Omega$ is nowhere dense, then A° is nowhere dense in $(0,1)$.*

Proof. Suppose that $A \subseteq \Omega$ is nowhere dense, U is open, $U \cap (0,1) \neq \emptyset$. We want to show that $U \backslash \overline{A^\circ} \neq \emptyset$. Wlog $U = (a,b)$ with $0 \le a < b \le 1$. We want to find $0 \le c < d \le 1$ such that $(c,d) \subseteq (a,b)\backslash \overline{A^\circ}$. By Lemma 26.56 choose f such that $(W'_f)^\circ \subseteq (a,b)$. Now $W'_f \backslash \overline{A} \neq \emptyset$, so there is a g such that $W'_g \subseteq W'_f \backslash \overline{A}$. By Lemma 26.57 choose $0 < c < d < 1$ so that $(c,d) \subseteq (W'_g)^\circ$. Thus $(c,d) \subseteq (W'_g)^\circ \subseteq (W'_f)^\circ \subseteq (a,b)$. Suppose that $x \in (c,d) \cap \overline{A^\circ}$; we want to get a contradiction. Say $y \in (c,d) \cap A^\circ$. Say $y = z^\circ$ with $z \in A$. But also $y \in (c,d) \subseteq (W'_g)^\circ$, so there is a $w \in W'_g$ such that $y = w^\circ$. Since $W'_g \subseteq W'_f \backslash \overline{A}$, we have $w \notin A$. But $z, w \in \Omega$, so $z = z^{\circ\prime} = y' = w^{\circ\prime} = w$, contradiction. $\square$

Corollary 26.59. *If $A \subseteq \Omega$ is meager, then A° is meager in $(0,1)$.* $\square$

Lemma 26.60. *If $A \subseteq (0,1)$ is nowhere dense in $(0,1)$, then A^p is nowhere dense in Ω.*

Proof. Assume that $A \subseteq (0,1)$ is nowhere dense in $(0,1)$ Let W_f be given; we want to show that $W_f \cap \Omega \backslash \overline{A^p} \neq \emptyset$. By Lemma 26.57 choose $0 < a < b < 1$ so that $(a,b) \subseteq (W'_f)^\circ$. Now $(a,b)\backslash \overline{A}$ is dense, hence nonempty and open. Choose $0 < c < d < 1$ with $(c,d) \subseteq (a,b)\backslash \overline{A}$. By Lemma 26.56 choose W'_g such that $(W'_g)^\circ \subseteq (c,d)$. So $(W'_g)^\circ \subseteq (W'_f)^\circ$, and hence $W'_g \subseteq W'_f$. Suppose that $W'_g \subseteq \overline{A^p}$; we want to get a contradiction. Then $W'_g \cap A^p \neq \emptyset$. Say $x \in W'_g \cap A_p$. Choose $y \in A$ with $x = y'$. Then $y = x^\circ \in (W'_g)^\circ \subseteq (c,d) \subseteq (a,b)\backslash \overline{A}$, contradiction. $\square$

Corollary 26.61. *If $A \subseteq (0,1)$ is meager in $(0,1)$. then A^p is meager in Ω.*

Lemma 26.62. $\operatorname{add}(\text{meager}_\Omega) = \operatorname{add}(\text{meager}_{(0,1)})$.

Proof. First let $\kappa = \operatorname{add}(\text{meager}_\Omega)$ and suppose that $E \in [\text{meager}_\Omega]^\kappa$ with $\bigcup E \notin \text{meager}_\Omega$. Let $E' = \{A^\circ : A \in E\}$. So by Corollary 26.59, $E' \in \mathscr{P}(\text{meager}_{(0,1)})$. Clearly $|E'| \le \kappa$. Suppose that $\bigcup E' \in \text{meager}_{(0,1)}$. By Corollary 26.61, $(\bigcup E')^p \in \text{meager}_\Omega$. Take any $A \in E$. Then $A^\circ \in E'$, so $A^\circ \subseteq \bigcup E'$. Hence $A = A^{\circ p} \subseteq (\bigcup E')^p$. Thus $\bigcup E \subseteq (\bigcup E')^p \in \text{meager}_\Omega$, so $\bigcup E \in \text{meager}_\Omega$, contradiction.

Second let $\kappa = \operatorname{add}(\text{meager}_{(0,1)})$ and suppose that $E \in [\text{meager}_{(0,1)}]^\kappa$ with $\bigcup E \notin \text{meager}_{(0,1)}$. Let $E' = \{A^p : A \in E\}$. So by Corollary 26.61, $E' \subseteq \mathscr{P}(\text{meager}_\Omega)$. Suppose that $\bigcup E' \in \text{meager}_\Omega$. By Corollary 26.59, $(\bigcup E')^\circ \in \text{meager}_{(0,1)}$. If $A \in E$, then $A^p \in E'$, so $A^p \subseteq \bigcup E'$, hence $A = A^{p\circ} \subseteq (\bigcup E')^\circ$, so $\bigcup E \subseteq (\bigcup E')^\circ$, contradiction. $\square$

Lemma 26.63. $\text{cov}(\text{meager}_\Omega) = \text{cov}(\text{meager}_{(0,1)})$.

Proof. First let $\kappa = \text{cov}(\text{meager}_\Omega)$ and suppose that $E \in [\text{meager}_\Omega]^\kappa$ with $\Omega = \bigcup E$. Let $E' = \{A^\circ : A \in E\}$. Then by Corollary 26.59, $E' \subseteq \mathscr{P}(\text{meager}_{(0,1)})$. If $a \in (0,1)$, then $a' \in \Omega$, hence there is an $A \in E$ such that $a' \in A$. So $a = a^{p\circ} \in {}^\circ[A] \in E'$. Thus $(0,1) \subseteq \bigcup E'$.

Seecond let $\kappa = \text{cov}(\text{meager}_{(0,1)})$ and suppose that $E \in [\text{meager}_{(0,1)}]^\kappa$ with $(0,1) = \bigcup E$. Let $E' = \{A^p : A \in E\}$. Then $E' \subseteq \mathscr{P}(\text{meager}_\Omega)$ by Corollary 26.61. Suppose that $x \in \Omega$. Then $x^\circ \in (0,1)$, so there is an $A \in E$ such that $x^\circ \in A$. Hence $x = x^{\circ p} \in A^p \in E'$. This shows that $\bigcup E' = \Omega$. $\square$

Lemma 26.64. $\text{non}(\text{meager}_\Omega) = \text{non}(\text{meager}_{(0,1)})$.

Proof. First let $\kappa = \text{non}(\text{meager}_\Omega)$ and suppose that $X \in [\Omega]^\kappa$ such that $X \notin \text{meager}_\Omega$. Suppose that X° is meager in $(0,1)$. Then $X = X^{\circ p}$ is meager in Ω by Corollary 26.61, contradiction. It follows that $\text{non}(\text{meager}_{(0,1)}) \leq \kappa$.

Second let $\kappa = \text{non}(\text{meager}_{(0,1)})$ and suppose that $X \in [(0,1)]^\kappa$ such that $X \notin \text{meager}_{(0,1)}$. Suppose that $X^p \in \text{meager}_\Omega$. Then $X = X^{p\circ} \in \text{meager}_{(0,1)}$ by Lemma 26.59, contradiction. Hence $\text{non}(\text{meager}_\Omega) = \text{non}(\text{meager}_{(0,1)})$. $\square$

Lemma 26.65. $\text{cof}(\text{meager}_\Omega) = \text{cof}(\text{meager}_{(0,1)})$.

Proof. First let $\kappa = \text{cof}(\text{meager}_\Omega)$ and suppose that $X \in [\text{meager}_\Omega]^\kappa$ is such that $\forall A \in \text{meager}_\Omega \exists B \in X[A \subseteq B]$. Let $X' = \{A^\circ : A \in X\}$. Then by Corollary 26.59, $X' \subseteq \mathscr{P}(\text{meager}_{(0,1)})$, and clearly $|X'| \leq \kappa$. Suppose that $A \in \text{meager}_{(0,1)}$. Then by Corollary 26.61, $A^p \in \text{meager}_\Omega$. Hence there is a $B \in X$ such that $A^p \subseteq B$. Then $A = A^p\circ \subseteq {}^\circ[B] \in X'$. It follows that $\text{cof}(\text{meager}_{(0,1)}) \leq \kappa$.

Second let $\kappa = \text{cof}(\text{meager}_{(0,1)})$ and suppose that $X \in [\text{meager}_{(0,1)}]^\kappa$ is such that $\forall A \in \text{meager}_{(0,1)} \exists B \in X[A \subseteq B]$. Let $X' = \{A^p : A \in X\}$. Then by Corollary 26.61, $X' \subseteq \mathscr{P}(\text{meager}_\Omega)$, and clearly $|X'| \leq \kappa$. Suppose that $A \in \text{meager}_\Omega$. Then $A^\circ \in \text{meager}_{(0,1)}$ by Corollary 26.59, so there is a $B \in X$ such that $A^\circ \subseteq B$. Then $A = A^{\circ p} \subseteq {}'[B] \in X'$. It follows that $\text{cof}(\text{meager}_\Omega) = \text{cof}(\text{meager}_{(0,1)})$. $\square$

For finite disjoint $F, G \subseteq \omega$ we define $U_{FG} = \{X \subseteq \omega : F \subseteq X \text{ and } X \cap G = \emptyset\}$.

Lemma 26.66. *$\{U_{FG} : F, G \in [\omega]^{<\omega}$ and $F \cap G = \emptyset\}$ forms a basis for a topology on $\mathscr{P}(\omega)$, and χ is a homeomorphism from ${}^\omega 2$ onto $\mathscr{P}(\omega)$.* $\square$

Let $\Theta = \{x \in {}^\omega 2 : \{i \in \omega : x(i) = 1\}$ is infinite$\}$.

Lemma 26.67. *Θ is dense in ${}^\omega 2$, and if $X \subseteq {}^\omega 2$ is nowhere dense, then $X \cap \Theta$ is nowhere dense in Θ.*

Proof. Clearly Θ is dense in ${}^\omega 2$. Now ${}^\omega 2 \backslash \overline{X}$ is nonempty and open, so there is a $W_f \subseteq {}^\omega 2 \backslash \overline{X}$. Then $W_f \cap \Theta \subseteq \Theta \backslash (\overline{X} \cap \Theta)$. $\square$

Lemma 26.68. *If $X \subseteq {}^\omega 2$ is meager$_{^\omega 2}$, then $X \cap \Theta$ is meager in Θ.* $\square$

Lemma 26.69. *If $X \subseteq \Theta$ is nowhere dense in Θ, then X is nowhere dense in ${}^\omega 2$.*

Proof. Let U be a nonempty open set in ${}^\omega 2$. We want to show that $U \cap ({}^\omega 2 \backslash \overline{X}) \neq \emptyset$. Now $U \cap \Theta \neq \emptyset$, and $\overline{X} \cap \Theta$ is the closure of X in Θ. Hence

$$\emptyset \neq U \cap \Theta \cap (\Theta \backslash (\overline{X} \cap \Theta)) = U \cap \Theta \backslash \overline{X} \subseteq U \cap ({}^\omega 2 \backslash \overline{X}),$$

as desired. $\square$

Lemma 26.70. *If $X \subseteq \Theta$ is meager in Θ, then X is meager in ${}^\omega 2$.* $\square$

Lemma 26.71. $\mathrm{add}(\mathrm{meager}_{^\omega 2}) = \mathrm{add}(\mathrm{meager}_\Theta)$.

Proof. Suppose that $E \in [\mathrm{meager}_{^\omega 2}]^\kappa$ and $\bigcup E \notin \mathrm{meager}_{^\omega 2}$, with $\kappa = \mathrm{add}(\mathrm{meager}_{^\omega 2})$. Let $E' = \{X \cap \Theta : X \in E\}$. Then $E' \subseteq \mathrm{meager}_\Theta$ by Lemma 26.68. Suppose that $\bigcup E' \in \mathrm{meager}_\Theta$. By Lemma 26.70, $\bigcup E' \in \mathrm{meager}_{^\omega 2}$. Let $M = \{x \in {}^\omega 2 : \{i \in I : x(i) = 1\}$ is finite$\}$. Then M is countable, and $\bigcup E \subseteq \bigcup E' \cup M$, so $\bigcup E$ is meager, contradiction.

Second suppose that $E \in [\mathrm{meager}_\Theta]^\kappa$ and $\bigcup E \notin \mathrm{meager}_\Theta$, with $\kappa = \mathrm{add}(\mathrm{meager}_\Theta)$. By Lemma 26.70, $E \subseteq \mathrm{meager}_{^\omega 2}$. Suppose that $\bigcup E \in \mathrm{meager}_{^\omega 2}$. By Lemma 26.68, $\bigcup E \in \mathrm{meager}_\Theta$, contradiction. $\square$

Lemma 26.72. $\mathrm{cov}(\mathrm{meager}_{^\omega 2}) = \mathrm{cov}(\mathrm{meager}_\Theta)$.

Proof. First suppose that $E \in [\mathrm{meager}_{^\omega 2}]^\kappa$ and ${}^\omega 2 = \bigcup E$, with $\kappa = \mathrm{cov}(\mathrm{meager}_{^\omega 2})$. Let $E' = \{X \cap \Theta : X \in E\}$. Then $E' \subseteq \mathrm{meager}_\Theta$ by Lemma 26.68. We have $\bigcup E' = (\bigcup E) \cap \Theta = {}^\omega 2 \cap \Theta = \Theta$.

Second suppose that $E \in [\mathrm{meager}_\Theta]^\kappa$ and $\bigcup E = \Theta$, $\kappa = \mathrm{cov}(\mathrm{meager}_\Theta)$. Let $M = \{x \in {}^\omega 2 : \{i \in I : x(i) = 1\}$ is finite$\}$. Then M is countable, hence meager in ${}^\omega 2$, and $\bigcup E \cup M = {}^\omega 2$. $\square$

Lemma 26.73. $\mathrm{non}(\mathrm{meager}_{^\omega 2}) = \mathrm{non}(\mathrm{meager}_\Theta)$.

Proof. First suppose that $X \in [{}^\omega 2]^\kappa$ with $X \notin \mathrm{meager}_{^\omega 2}$ and $|X| = \mathrm{non}(\mathrm{meager}_{^\omega 2})$. If $X \cap \Theta \in \mathrm{meager}_\Theta$, then $X \subseteq (X \cap \Theta) \cup M \in \mathrm{meager}_{^\omega 2}$ by Lemma 26.70, with M as above, contradiction.

Second suppose that $X \in [\Theta]^\kappa$ with $X \notin \mathrm{meager}_\Theta$ and $|X| = \mathrm{non}(\mathrm{meager}_\Theta)$. Then by Lemma 26.68, X is not meager in ${}^\omega 2$. $\square$

Lemma 26.74. $\mathrm{cof}(\mathrm{meager}_{^\omega 2}) = \mathrm{cof}(\mathrm{meager}_\Theta)$.

Proof. First suppose that $X \in [\mathrm{meager}_{^\omega 2}]^\kappa$ such that $\forall A \in \mathrm{meager}_{^\omega 2} \exists B \in X[A \subseteq B]$, with $\kappa = \mathrm{cof}(\mathrm{meager}_{^\omega 2})$. Let $Y = \{B \cap \Theta : B \in X\}$. Then $Y \subseteq \mathrm{meager}_\Theta$ by Lemma 26.68. Suppose that $A \in \mathrm{meager}_\Theta$. Then $A \in \mathrm{meager}_{^\omega 2}$ by Lemma 26.70. Hence there is a $B \in X$ such that $A \subseteq B$. So $B \cap \Theta \in Y$ and $A \subseteq B \cap \Theta$.

Second suppose that $X \in [\mathrm{meager}_\Theta]^\kappa$ such that $\forall A \in \mathrm{meager}_\Theta \exists B \in X[A \subseteq B]$, with $\kappa = \mathrm{cof}(\mathrm{meager}_\Theta)$. Let M be as above. Let $Y = \{B \cup M : B \in X\}$. Then $Y \subseteq \mathrm{meager}_{^\omega 2}$

by Lemma 26.70. Suppose that $A \in \text{meager}_{^\omega 2}$. Then $A \cap \Theta \in \text{meager}_\Theta$ by Lemma 26.68. Choose $B \in X$ such that $A \cap \Theta \subseteq B$. Then $A \subseteq B \cup M \in Y$. $\square$

Lemma 26.75. *With the relative topology on $[\omega]^\omega$, Θ is homeomorphic to $[\omega]^\omega$.* $\square$

measures

We give background on measures, and prove that add, non, cov, and cof are the same applied to null sets in the sense of $[0,1]$, ${}^\omega 2$, Θ, or $[\omega]^\omega$.

If A is a σ-algebra of subsets of X, then a *measure* on A is a function $\mu : A \to [0,\infty]$ such that $\mu(\emptyset) = 0$ and $\mu(\bigcup_{i\in\omega} a_i) = \sum_{i\in\omega} \mu(a_i)$ if $a \in {}^\omega A$ and $a_i \cap a_j = \emptyset$ for all $i \neq j$. Note that $a_i = \emptyset$ is possible for some $i \in \omega$.

We give some important properties of measures:

Proposition 26.76. *Suppose that μ is a measure on a σ-algebra A of subsets of X. Then:*

(i) If $Y, Z \in A$ and $Y \subseteq Z$, then $\mu(Y) \leq \mu(Z)$.

(ii) If $Y \in {}^\omega A$, then $\mu(\bigcup_{n\in\omega} Y_n) \leq \sum_{n\in\omega} \mu(Y_n)$.

(iii) If $Y \in {}^\omega A$ and $Y_n \subseteq Y_{n+1}$ for all $n \in \omega$, then $\mu(\bigcup_{n\in\omega} Y_n) = \sup_{n\in\omega} \mu(Y_n)$.

(iv) If $Y \in {}^\omega A$ and $\mu(Y_0) < \infty$ and $Y_n \supseteq Y_{n+1}$ for all $n \in \omega$, then $\mu(\bigcap_{n\in\omega} Y_n) = \inf_{n\in\omega} \mu(Y_n)$.

Proof. (i): We have $\mu(Z) = \mu(Y) + \mu(Z\backslash Y) \geq \mu(Y)$.

(ii): Let $Z_n = Y_n \backslash \bigcup_{m<n} Y_m$. By induction, $\bigcup_{m\leq n} Z_m = \bigcup_{m\leq n} Y_m$, and hence $\bigcup_{m\in\omega} Z_m = \bigcup_{m\in\omega} Y_m$. Now

$$\mu\left(\bigcup_{m\in\omega} Y_m\right) = \mu\left(\bigcup_{m\in\omega} Z_m\right) = \sum_{m\in\omega} \mu(Z_m) \leq \sum_{m\in\omega} \mu(Y_m).$$

(iii): Again let $Z_n = Y_n \backslash \bigcup_{m<n} Y_m$. By induction, $Y_n = \bigcup_{m\leq n} Z_m$. Hence

$$\begin{aligned}
\mu\left(\bigcup_{n\in\omega} Y_n\right) &= \mu\left(\bigcup_{n\in\omega} Z_n\right) \\
&= \sum_{n\in\omega} \mu(Z_n) \\
&= \lim_{n\to\infty} \sum_{m\leq n} \mu(Z_m) \\
&= \lim_{n\to\infty} \mu\left(\bigcup_{m\leq n} Z_m\right) \\
&= \lim_{n\to\infty} \mu(Y_n) \\
&= \sup_{n\in\omega} \mu(Y_n).
\end{aligned}$$

(iv): $Y_0\backslash Y_n \subseteq Y_0\backslash Y_{n+1}$ for all n, so, by (iii),

$$\mu\left(\bigcup_{n\in\omega}(Y_0\backslash Y_n)\right) = \sup_{n\in\omega}\mu(Y_0\backslash Y_n).$$

Hence

$$\mu(Y_0) = \mu\left(Y_0\backslash\bigcap_{n\in\omega}Y_n\right) + \mu\left(\bigcap_{n\in\omega}Y_n\right),$$

so

$$\mu\left(\bigcap_{n\in\omega}Y_n\right) = \mu(Y_0) - \mu\left(Y_0\backslash\bigcap_{n\in\omega}Y_n\right) = \mu(Y_0) - \sup_{n\in\omega}\mu(Y_0\backslash Y_n).$$

Now for any $n \in \omega$, $\mu(Y_0) = \mu(Y_0\backslash Y_n) + \mu(Y_n)$, and hence

$$\mu(Y_0) - \sup_{n\in\omega}\mu(Y_0\backslash Y_n) \leq \mu(Y_0) - \mu(Y_0\backslash Y_n) = \mu(Y_n).$$

Also, if $x \leq \mu(Y_n)$ for all n, then $x \leq \mu(Y_0) - \mu(Y_0\backslash Y_n)$, hence $\mu(Y_0\backslash Y_n) \leq \mu(Y_0) - x$ for all n, so $\sup_{n\in\omega}\mu(Y_0\backslash Y_n) \leq \mu(Y_0) - x$, and so $x \leq \mu(Y_0) - \sup_{n\in\omega}\mu(Y_0\backslash Y_n)$. This proves (iv). □

measure spaces and outer measures

A *measure space* is a triple (X, Σ, μ) such that:

(1) X is a set

(2) Σ is a σ-algebra of subsets of X.

(3) μ is a measure on Σ.

Given a measure space as above, a subset A of X is a μ-*null* set iff there is an $E \in \Sigma$ such that $A \subseteq E$ and $\mu(E) = 0$.

Theorem 26.77. *If (X, Σ, μ) is a measure space, then the collection of μ-null sets is a σ-ideal of subsets of X.*

Proof. Let I be the collection of all μ-null sets. Clearly $\emptyset \in I$, and $B \subseteq A \in I$ implies that $B \in I$. Now suppose that $\langle A_i : i \in \omega\rangle$ is a system of members of I. For each $i \in \omega$ choose $E_i \in \Sigma$ such that $A_i \subseteq E_i$ and $\mu(E_i) = 0$. Then $\bigcup_{i\in I} A_i \subseteq \bigcup_{i\in I} E_i$, and

$$\mu\left(\bigcup_{i\in\omega}E_i\right) \leq \sum_{i\in\omega}\mu(E_i) = 0.$$ □

An *outer measure* on a set X is a function $\mu : \mathscr{P}(X) \to [0, \infty]$ satisfying the following conditions:

(1) $\mu(\emptyset) = 0$.

(2) If $A \subseteq B \subseteq X$, then $\mu(A) \leq \mu(B)$.

(3) For every $A \in {}^{\omega}\mathscr{P}(X)$, $\mu(\bigcup_{n\in\omega} A_n) \leq \sum_{n\in\omega} \mu(A_n)$.

If θ is an outer measure on a set X, then a subset E of X is *θ-measurable* iff for every $A \subseteq X$,

$$\theta(A) = \theta(A \cap E) + \theta(A \backslash E).$$

Note that every subset $E \subseteq X$ such that $\theta(E) = 0$ is automatically θ-measurable.

Theorem 26.78. *Let θ be an outer measure on a set X. Let Σ be the collection of all θ-measurable subsets of X. Then $(X, \Sigma, \theta \restriction \Sigma)$ is a measure space. Moreover, if $E \subseteq X$ and $\theta(E) = 0$, then $E \in \Sigma$.*

Proof. Note that Σ is obviously closed under complementation. Obviously

(1) If $A, E \subseteq X$, then $\theta(A) \leq \theta(A \cap E) + \theta(A \backslash E)$.

Clearly $\emptyset \in \Sigma$ and Σ is closed under complements. Next we show that Σ is closed under $\cup$. Suppose that $E, F \in \Sigma$ and $A \subseteq X$. Then

$$\begin{aligned}
\theta(A \cap (E \cup F)) + \theta(A \backslash (E \cup F)) &\leq \theta((A \cap (E \cup F) \cap E)) + \theta(A \cap (E \cup F) \backslash E))) \\
&\qquad + \theta(A \backslash (E \cup F)) \\
&= \theta(A \cap E) + \theta((A \backslash E) \cap F) + \theta((A \backslash E) \backslash F) \\
&= \theta(A \cap E) + \theta(A \backslash E) \\
&= \theta(A) \\
&\leq \theta(A \cap (E \cup F)) + \theta(A \backslash (E \cup F)) \quad \text{by (1).}
\end{aligned}$$

This proves that $E \cup F \in \Sigma$. Thus we have shown that Σ is a field of subsets of X.

Next we show that Σ is closed under countable unions. So, suppose that $E \in {}^{\omega}\Sigma$, and let $K = \bigcup_{n\in\omega} E_n$. For every $m \in \omega$ let

$$G_m = \bigcup_{n \leq m} E_n.$$

Then clearly each G_m is in Σ. Now we define $F_0 = G_0$, and for $m > 0$, $F_m = G_m \backslash G_{m-1}$. Then also each F_m is in Σ. By induction, $\bigcup_{n\leq m} F_n = G_m$. Hence $\bigcup_{n\in\omega} F_n = \bigcup_{n\in\omega} E_n$. Now temporarily fix a positive integer n and an $A \subseteq X$. Then

$$\theta(A \cap G_n) = \theta(A \cap G_n \cap G_{n-1}) + \theta(A \cap G_n \backslash G_{n-1}) = \theta(A \cap G_{n-1}) + \theta(A \cap F_n);$$

hence by induction $\theta(A \cap G_n) = \sum_{m\leq n} \theta(A \cap F_m)$.

Now we unfix n. Now $A \cap K = \bigcup_{n\in\omega}(A \cap F_n)$, so

$$\theta(A \cap K) \leq \sum_{n\in\omega} \theta(A \cap F_n) = \lim_{n\to\infty} \sum_{m\leq n} \theta(A \cap F_m) = \lim_{n\to\infty} \theta(A \cap G_m).$$

Also, note that if $m < n$ then $G_m \subseteq G_n$, hence $X \backslash G_n \subseteq X \backslash G_m$, and so

$$\theta(A \backslash K) = \theta\left(A \backslash \bigcup_{n\in\omega} G_n\right) = \theta\left(\bigcap_{n\in\omega}(A\backslash G_n)\right) \leq \inf_{n\in\omega}\theta(A\backslash G_n) = \lim_{n\to\infty}\theta(A\backslash G_n).$$

Hence

$$\begin{aligned}\theta(A\cap K)+\theta(A\backslash K) &\leq \lim_{n\to\infty}\theta(A\cap G_n)+\lim_{n\to\infty}\theta(A\backslash G_n)\\ &= \lim_{n\to\infty}(\theta(A\cap G_n)+\theta(A\backslash G_n))\\ &= \theta(A)\\ &\leq \theta(A\cap K)+\theta(A\backslash K).\end{aligned}$$

This proves that $K \in \Sigma$, so that Σ is closed under countable unions.

Finally, suppose that $\langle E_n : n \in \omega\rangle$ is a system of pairwise disjoint members of Σ. Let $K = \bigcup_{n\in\omega} E_n$. Hence $\theta(K) \leq \sum_{n\in\omega}\theta(E_n)$. Conversely, for each $n \in \omega$ let $G_n = \bigcup_{m\leq n} E_m$. Then

$$\theta(G_{n+1}) = \theta(G_{n+1}\cap E_{n+1}) + \theta(G_{n+1}\backslash E_{n+1}) = \theta(E_{n+1}) + \theta(G_n).$$

Hence by induction, $\theta(G_n) = \sum_{m\leq n}\theta(E_m)$ for every n, and hence

$$\theta(K) \geq \theta(G_n) = \sum_{m\leq n}\theta(E_m),$$

and so $\theta(K) \geq \sum_{n\in\omega}\theta(E_n)$.

For the "moreover" statement, suppose that $E \subseteq X$ and $\theta(E) = 0$, Then for any $A \subseteq X$, $\theta(A) \leq \theta(A\cap E) + \theta(A\backslash E) = \theta(A\backslash E) \leq \theta(A)$. ☐

measure on ${}^\kappa 2$

Let κ be an infinite cardinal. For each $f \in \mathrm{Fn}(\kappa, 2, \omega)$ let $U_f = \{g \in {}^\kappa 2 : f \subseteq g\}$. Hence $U_\emptyset = {}^\kappa 2$. Note that the function taking f to U_f is one-one. For each $f \in \mathrm{Fn}(\kappa, 2, \omega)$ let $\theta_0(U_f) = 1/2^{|\mathrm{dmn}(f)|}$. Thus $\theta_0(U_\emptyset) = 1$. Let $\mathcal{C} = \{U_f : f \in \mathrm{Fn}(\kappa, 2, \omega)\}$. Note that ${}^\kappa 2 \in \mathcal{C}$. For any $A \subseteq {}^\kappa 2$ let

$$\theta(A) = \inf\left\{\sum_{n\in\omega}\theta_0(C_n) : C \in {}^\omega\mathcal{C} \text{ and } A \subseteq \bigcup_{n\in\omega} C_n\right\}.$$

Proposition 26.79. *θ is an outer measure on ${}^\kappa 2$.*

Proof. For (1), for any $m \in \omega$ let $f \in \mathrm{Fn}(\kappa, 2, \omega)$ have domain of size m. Then $\emptyset \subseteq U_f$ and $\theta_0(U_f) = \frac{1}{2^m}$. Hence $\theta(\emptyset) = 0$.

For (2), if $A \subseteq B \subseteq {}^{\kappa}2$, then

$$\left\{C \in {}^{\omega}\mathcal{C} : B \subseteq \bigcup_{n\in\omega} C_n\right\} \subseteq \left\{C \in {}^{\omega}\mathcal{C} : A \subseteq \bigcup_{n\in\omega} C_n\right\},$$

and hence $\mu(A) \leq \mu(B)$.

For (3), assume that $A \in {}^{\omega}\mathscr{P}({}^{\kappa}2)$. We may assume that $\sum_{n\in\omega} \theta(A_n) < \infty$. Let $\varepsilon > 0$; we show that $\theta(\bigcup_{n\in\omega} A_n) \leq \sum_{n\in\omega} \theta(A_n) + \varepsilon$, and the arbitrariness of ε then gives the desired result. For each $n \in \omega$ choose $C^n \in {}^{\omega}\mathcal{C}$ such that $A_n \subseteq \bigcup_{m\in\omega} C^n_m$ and $\sum_{m\in\omega} \theta_0(C^n_m) \leq \theta(A_n) + \frac{\varepsilon}{2^n}$. Then $\bigcup_{n\in\omega} A_n \subseteq \bigcup_{n\in\omega} \bigcup_{m\in\omega} C^n_m$ and

$$\theta\left(\bigcup_{n\in\omega} A_n\right) \leq \sum_{n\in\omega}\sum_{n\in\omega} \theta_0(C^n_m) \leq \sum_{n\in\omega} \theta(A_n) + \varepsilon,$$

as desired. □

Let Σ_0 be the set of all θ-measurable subsets of ${}^{\omega}2$.

Proposition 26.80. *If $\varepsilon \in 2$ and $\alpha < \kappa$, then $\{f \in {}^{\kappa}2 : f(\alpha) = \varepsilon\} \in \Sigma_0$.*

Proof. Let $E = \{f \in {}^{\kappa}2 : f(\alpha) = \varepsilon\}$, and let $X \subseteq {}^{\kappa}2$; we want to show that $\theta(X) = \theta(X \cap E) + \theta(X \backslash E)$. $\leq$ holds by the definition of outer measure. Now suppose that $\delta > 0$. Choose $C \in {}^{\omega}\mathcal{C}$ such that $X \subseteq \bigcup_{n\in\omega} C_n$ and $\sum_{n\in\omega} \theta_0(C_n) < \theta(X) + \delta$. For each $n \in \omega$ let $C_n = U_{f_n}$ with $f_n \in \mathrm{Fn}(\kappa, 2, \omega)$. For each $n \in \omega$, if $\alpha \notin \mathrm{dmn}(f_n)$, replace C_n by U_g and U_h, where $g = f_n \cup \{(\alpha, 0)\}$ and $h = f_n \cup \{(\alpha, 1)\}$; let the new sequence be $C' \in {}^{\omega}\mathcal{C}$. Note that

$$\theta_0(C_n) = \theta_0(U_{f_n}) = \frac{1}{2^{|\mathrm{dmn}(f_n)|}} = \theta_0(U_g) + \theta_0(U_h).$$

Then $\sum_{n\in\omega} \theta(C_n) = \sum_{n\in\omega} \theta(C'_n)$ and $X \subseteq \bigcup_{n\in\omega} C'_n$. Say $C'_n = U_{g_n}$ for each $n \in \omega$. Note that $\alpha \in \mathrm{dmn}(g_n)$ for each $n \in \omega$. Let $M = \{n \in \omega : g_n(\alpha) = \varepsilon\}$ and $N = \{n \in \omega : g_n(\alpha) = 1-\varepsilon\}$. Then M, N is a partition of ω such that $X \cap E \subseteq \bigcup_{n\in M} C'_n$ and $X\backslash E \subseteq \bigcup_{n\in N} C'_n$. Hence

$$\theta(X \cap E) + \theta(X\backslash E) \leq \sum_{n\in M} \theta(C'_n) + \sum_{n\in N} \theta(C'_n) = \sum_{n\in\omega} \theta(C'_n) < \theta(X) + \delta.$$

Since δ is arbitrary, it follows that $\theta(X) = \theta(X \cap E) + \theta(X\backslash E)$. □

For $f : 2 \to \mathbb{R}$ we define $\int f = \frac{1}{2}f(0) + \frac{1}{2}f(1)$.

Proposition 26.81. *If $f_n : 2 \to [0, \infty)$ for each $n \in \omega$ and $\forall t < 2[\sum_{n\in\omega} f_n(t) < \infty]$, then $\sum_{n\in\omega} \int f_n < \infty$, and $\sum_{n\in\omega} \int f_n = \int \sum_{n\in\omega} f_n$.*

Proof.

$$\int \sum_{n\in\omega} f_n = \frac{1}{2}\sum_{n\in\omega} f_n(0) + \frac{1}{2}\sum_{n\in\omega} f_n(1) = \sum_{n\in\omega}\left(\frac{1}{2}f_n(0) + \frac{1}{2}f_n(1)\right) = \sum_{n\in\omega}\int f_n.$$

Proposition 26.82. $\theta({}^{\kappa}2) = 1$.

Proof. It is obvious that ${}^{\kappa}2 \in \Sigma_0$, and that $\theta({}^{\kappa}2) \leq \theta_0({}^{\kappa}2) = 1$. Suppose that $\theta({}^{\kappa}2) < 1$. Choose $C \in {}^{\omega}\mathcal{C}$ such that $2^{\kappa} = \bigcup_{n\in\omega} C_n$ and $\sum_{n\in\omega} \theta_0(C_n) < 1$, with C one-one. For each $n \in \omega$ let $C_n = U_{f_n}$, where $f_n \in \mathrm{Fn}(\kappa, 2, \omega)$.

(1) $\forall g \in \mathrm{Fn}(\kappa, 2, \omega)\exists n \in \omega[f_n \subseteq g \text{ or } g \subseteq f_n]$.

In fact, let $g \in \mathrm{Fn}(\kappa, 2, \omega)$. Let $h \in {}^{\kappa}2$ with $g \subseteq h$. Choose n such that $h \in C_n$. Then $f_n \subseteq h$. So $f_n \subseteq g$ or $g \subseteq f_n$.

(2) Let $M = \{n \in \omega : \forall m \neq n[f_m \not\subseteq f_n]\}$. Then ${}^{\kappa}2 \subseteq \bigcup_{n\in M} U_{f_n}$.

For, given $g \in {}^{\kappa}2$ choose $m \in \omega$ such that $g \in C_m$. Thus $f_m \subseteq g$. Let $n \in \omega$ with $f_n \subseteq f_m$ and $|\mathrm{dmn}(f_n)|$ minimum. Then $f_n \subseteq g$ and $n \in M$, as desired.

(3) $|M| \geq 2$.

In fact, obviously $M \neq \emptyset$. Suppose that $M = \{n\}$. Since $\sum_{n\in M} \theta_0(C_n) < 1$, we have $f_n \neq \emptyset$. Then ${}^{\kappa}2 \subseteq U_{f_n}$, contradiction.

(4) M is infinite.

In fact, suppose that M is finite, and let $m = \sup\{|\mathrm{dmn}(f_n)| : n \in M\}$. Let $g \in \mathrm{Fn}(\kappa, 2, \omega)$ be such that $|\mathrm{dmn}(g)| = m+1$. Then by (1), $f_n \subseteq g$ for all $n \in M$. By (3), this contradicts the definition of M.

Let $J = \bigcup_{n\in M} \mathrm{dmn}(f_n)$.

(5) J is infinite.

For, suppose that J is finite. Now $M = \bigcup_{G\subseteq J}\{n \in M : \mathrm{dmn}(f_n) = G\}$, so there is a $G \subseteq J$ such that $\{n \in M : \mathrm{dmn}(f_n) = G\}$ is infinite. But clearly $|\{n \in M : \mathrm{dmn}(f_n) = G\}| \leq 2^{|G|}$, contradiction.

Let $i : \omega \to J$ be a bijection. For $n, k \in \omega$ let f'_{nk} be the restriction of f_n to the domain $\{\alpha \in \mathrm{dmn}(f_n) : \forall j < k[\alpha \neq i_j]\}$, and let

$$\alpha_{nk} = \frac{1}{2^{|\mathrm{dmn}(f'_{nk})|}}.$$

Now for $n, k \in \omega$ and $t < 2$ we define

$$\varepsilon_{nk}(t) = \begin{cases} \alpha_{n,k+1} & \text{if } i_k \notin \mathrm{dmn}(f_n), \\ \alpha_{n,k+1} & \text{if } i_k \in \mathrm{dmn}(f_n) \text{ and } f_n(i_k) = t, \\ 0 & \text{otherwise.} \end{cases}$$

(6) $\int \varepsilon_{nk} = \alpha_{nk}$ for all $n, k \in \omega$.

In fact,

$$\begin{aligned} \int \varepsilon_{nk} &= \frac{1}{2}\varepsilon_{nk}(0) + \frac{1}{2}\varepsilon_{nk}(1) \\ &= \begin{cases} \alpha_{n,k+1} & \text{if } i_k \notin \mathrm{dmn}(f_n), \\ \frac{1}{2}\alpha_{n,k+1} & \text{if } i_k \in \mathrm{dmn}(f_n) \end{cases} \\ &= \alpha_{nk}. \end{aligned}$$

Now we define by induction elements $t_k \in 2$ and subsets M_k of M. Let $M_0 = M$. Note that

$$\alpha_{n0} = \frac{1}{2^{|\mathrm{dmn}(f_n)|}}; \quad \sum_{n\in M} \alpha_{n0} = \sum_{n\in M} \frac{1}{2^{|\mathrm{dmn}(f_n)|}} = \sum_{n\in M} \theta_0(C_n) < 1.$$

Now suppose that M_k and t_i have been defined for all $i < k$, so that $\sum_{n\in M_k} \alpha_{nk} < 1$. Note that this holds for $k = 0$. Now

$$\begin{aligned} 1 > \sum_{n\in M_k} \alpha_{nk} &= \sum_{n\in M_k} \int \varepsilon_{nk} \quad \text{by (6)} \\ &= \int \sum_{n\in M_k} \varepsilon_{nk} \quad \text{by Proposition 26.81.} \end{aligned}$$

It follows that there is a $t_k < 2$ such that $\left(\sum_{n\in M_k} \varepsilon_{nk}\right)(t_k) < 1$. Let

$$M_{k+1} = \{n \in M : \forall j < k+1[i_j \notin \mathrm{dmn}(f_n), \text{ or } i_j \in \mathrm{dmn}(f_n) \text{ and } f_n(i_j) = t_j]\}.$$

If $n \in M_{k+1}$, then $\varepsilon_{nk}(t_k) = \alpha_{n,k+1}$. Hence

$$\sum_{n\in M_{k+1}} \alpha_{n,k+1} = \sum_{n\in M_{k+1}} \varepsilon_{nk}(t_k) \le \left(\sum_{n\in M_k} \varepsilon_{nk}\right)(t_k) < 1.$$

Also, $M_{k+1} \neq \emptyset$. For, let $g \in {}^\kappa 2$ such that $g(i_j) = t_j$ for all $j \le k$. Say $g \in C_n$ with $n \in M$. Then $f_n \subseteq g$. Hence $i_j \notin \mathrm{dmn}(f_n)$, or $i_j \in \mathrm{dmn}(f_n)$ and $f_n(i_j) = t_j$. Thus $n \in M_{k+1}$.

This finishes the construction. Now let $g \in {}^\kappa 2$ be such that $g(i_j) = t_j$ for all $j \in \omega$. Say $g \in C_n$ with $n \in M$. Then $f_n \subseteq g$. The domain of f_n is a finite subset of J. Choose $k \in \omega$ so that $\mathrm{dmn}(f_n) \subseteq \{i_j : j < k\}$. Then $n \in M_k$. Hence $f'_{nk} = \emptyset$ and so $\alpha_{nk} = 1$. This contradicts $\sum_{m\in M_k} \alpha_{mk} < 1$. $\square$

Let ν be the tiny function with domain 2 which interchanges 0 and 1. For any $f \in {}^\kappa 2$ let $F(f) = \nu \circ f$.

Proposition 26.83.
(i) F is a permutation of ${}^\kappa 2$.
(ii) For any $f \in \mathrm{Fn}(\kappa, 2, \omega)$ we have $F[U_f] = U_{\nu\circ f}$.
(iii) For any $X \subseteq {}^\kappa 2$ we have $\theta(X) = \theta(F[X])$.
(iv) $\forall E \in \Sigma_0[F[E] \in \Sigma_0]$.

Proof. (i): Clearly F is one-one, and $F(F(f)) = f$ for any $f \in {}^\kappa 2$. So (i) holds.
(ii): For any $g \in {}^\kappa 2$,

$$\begin{aligned} g \in F[U_f] \quad &\text{iff} \quad \exists h \in U_f[g = F(h)] \\ &\text{iff} \quad \exists h \in {}^\kappa 2[f \subseteq h \text{ and } g = \nu \circ h] \\ &\text{iff} \quad \exists h \in {}^\kappa 2[\nu \circ f \subseteq \nu \circ h \text{ and } g = \nu \circ h] \\ &\text{iff} \quad \nu \circ f \subseteq g \\ &\text{iff} \quad g \in U_{\nu\circ f} \end{aligned}$$

(iii): Clearly $\theta_0(U_f) = \theta_0(F[U_f])$ for any $f \in \mathrm{Fn}(\kappa, 2, \omega)$. Also, $A \subseteq \bigcup_{n\in\omega} C_n$ iff $F[A] \subseteq \bigcup_{n\in\omega} F[C_n]$. So (iii) holds.

(iv): Suppose that $E \in \Sigma_0$. Let $X \subseteq {}^\kappa 2$. Then

$$\begin{aligned}\theta(X \cap F[E]) + \theta(X \backslash F[E]) &= \theta(F[F[X]] \cap F[E]) + \theta(F[F[X]] \backslash F[E]) \\ &= \theta(F[F[X] \cap E]) + \theta(F[F[X] \backslash E]) \\ &= \theta(F[X] \cap E) + \theta(F[X] \backslash E) \\ &= \theta(E) = \theta(F[E]).\end{aligned}$$

□

Proposition 26.84. *If $\alpha < \kappa$ and $\varepsilon < 2$, then $\theta(U_{\{(\alpha,\varepsilon)\}}) = \frac{1}{2}$.*

Proof. By Proposition 26.83 we have $\theta(U_{\{(\alpha,\varepsilon)\}}) = \theta(U_{\{(\alpha,1-\varepsilon)\}})$, so the result follows from Proposition 26.82. □

Proposition 26.85. *For each $f \in \mathrm{Fn}(\kappa, 2, \omega)$ we have $U_f \in \Sigma_0$ and $\theta(U_f) = \frac{1}{2^{|\mathrm{dmn}(f)|}}$.*

Proof. We have $U_f = \bigcap_{\alpha\in\mathrm{dmn}(f)} U_{\{(\alpha,f(\alpha))\}}$. Note that if $\alpha \in \mathrm{dmn}(f)$, then $U_{\{(\alpha,f(\alpha))\}} = \{g \in {}^\kappa 2 : g(\alpha) = f(\alpha)\}$; hence $U_{\{(\alpha,f(\alpha))\}} \in \Sigma_0$ by Proposition 26.80, and so $U_f \in \Sigma_0$. We prove that $\theta(U_f) = \frac{1}{2^{|\mathrm{dmn}(f)|}}$ by induction on $|\mathrm{dmn}(f)|$. For $|\mathrm{dmn}(f)| = 1$, this holds by Proposition 26.84. Now assume that it holds for $|\mathrm{dmn}(f)| = m$. For any f with $|\mathrm{dmn}(f)| = m$ and $\alpha \notin \mathrm{dmn}(f)$ we have $2^{-|\mathrm{dmn}(f)|} = \theta(U_f) = \theta(U_{f\cup\{(\alpha,0)\}}) + \theta(U_{f\cup\{(\alpha,1)\}})$. Since $\theta(U_{f\cup\{(\alpha,\varepsilon)\}}) \le \theta_0(U_{f\cup\{(\alpha,\varepsilon)\}}) = 2^{-|\mathrm{dmn}(f)|-1}$ for each $\varepsilon \in 2$, it follows that $\theta(U_{f\cup\{(\alpha,\varepsilon)\}}) = 2^{-|\mathrm{dmn}(f)|-1}$ for each $\varepsilon \in 2$. □

Proposition 26.86. *If F is a finite subset of ${}^\kappa 2$, then $F \in \Sigma_0$ and $\theta(F) = 0$.*

Proof. This is obvious if $F = \emptyset$. For $F = \{f\}$ we have $F \subseteq U_{f\restriction n}$ for each $n \in \omega$, and so $\theta(F) = 0$. Then it is clear that $F \in \Sigma_0$. Now the general case follows easily. □

Proposition 26.87. *If $X \subseteq {}^\kappa 2$ is measurable, then $\theta(X) = \inf\{\varphi(U) : X \subseteq U$ and U is open$\}$.*

Proof. By Proposition 26.85, $\theta(U_f) = \theta_0(U_f)$ for each $f \in \mathrm{Fn}(\kappa, 2, \omega)$. Hence by the definition preceding Proposition 26.79,

$$\begin{aligned}\theta(X) &\le \inf\left\{\theta\left(\bigcup_{n\in\omega} U_{f_n}\right) : f \in {}^\omega\mathrm{Fn}(\kappa, 2, \omega),\ X \subseteq \bigcup_{n\in\omega} U_{f_n}\right\} \\ &\le \inf\left\{\sum\{\theta(U_{f_n}) : f \in {}^\omega\mathrm{Fn}(\kappa, 2, \omega),\ X \subseteq \bigcup_{n\in\omega} U_{f_n}\right\} \\ &= \inf\left\{\sum\{\theta_0(U_{f_n}) : f \in {}^\omega\mathrm{Fn}(\kappa, 2, \omega),\ X \subseteq \bigcup_{n\in\omega} U_{f_n}\right\} \\ &= \theta(X).\end{aligned}$$

□

Proposition 26.88. *If $X \subseteq {}^{\kappa}2$ is measurable, then there is a system $\langle f^n_m : n, m \in \omega\rangle$ with each $f^n_m \in \mathrm{Fn}(\kappa, 2, \omega)$ such that $X \subseteq \bigcap_{n\in\omega} \bigcup_{m\in\omega} U_{f^n_m}$ and $\theta((\bigcap_{n\in\omega} \bigcup_{m\in\omega} U_{f^n_m})\backslash X) = 0$.*

Proof. By the proof of Proposition 26.87, for each $n \in \omega$ let $\langle f^n_m : m \in \omega\rangle$ be such that each $f^n_m \in \mathrm{Fn}(\kappa, 2, \omega)$, $X \subseteq \bigcup_{m\in\omega} U_{f^n_m}$, and $\theta(\bigcup_{m\in\omega} U_{f^n_m}) - \theta(X) \le \frac{1}{n+1}$. Then

$$\forall n \in \omega \left[X \subseteq \bigcap_{p\in\omega} \bigcup_{m\in\omega} U_{f^p_m} \subseteq \bigcup_{m\in\omega} U_{f^n_m}\right];$$

$$\forall n \in \omega \left[\theta\left(\bigcap_{n\in\omega} \bigcup_{m\in\omega} U_{f^n_m}\right) - \theta(X) \le \theta\left(\bigcup_{m\in\omega} U_{f^n_m}\right) - \theta(X) \le \frac{1}{n+1}\right];$$

$$\theta\left(\bigcap_{n\in\omega} \bigcup_{m\in\omega} U_{f^n_m}\right) - \theta(X) = 0;$$

$$\theta\left(\bigcap_{n\in\omega} \bigcup_{m\in\omega} U_{f^n_m}\right) = \theta\left(\left(\bigcap_{n\in\omega} \bigcup_{m\in\omega} U_{f^n_m}\right)\backslash X\right) + \theta(X);$$

$$\theta\left(\left(\bigcap_{n\in\omega} \bigcup_{m\in\omega} U_{f^n_m}\right)\backslash X\right) = 0. \qquad \square$$

measure on $\mathbb{R}$

For any $a, b \in \mathbb{R}$ let $[a, b) = \{x \in \mathbb{R} : a \le x < b\}$. Note that if $a \ge b$, then $[a, b) = \emptyset$. Note that if $[a, b) = [c, d)$, $a < b$, and $c < d$, then $a = c$ and $b = d$. For any $a, b \in \mathbb{R}$ we define

$$\lambda([a, b)) = \begin{cases} 0 & \text{if } a \ge b, \\ b - a & \text{if } a < b. \end{cases}$$

A set of the form $[a, b)$ is called a *half-open interval.*

Lemma 26.89. *Suppose that I is a half-open interval, $\langle J_i : i \in \omega\rangle$ is a system of half-open intervals, and $I \subseteq \bigcup_{i\in\omega} J_i$. Then*

$$\lambda(I) \le \sum_{j\in\omega} \lambda(J_i).$$

Proof. If $I = \emptyset$ this is obvious. So suppose that $I \ne \emptyset$. Then there exist real numbers $a < b$ such that $I = [a, b)$. Let

$$A = \left\{x \in [a, b] : x - a \le \sum_{j\in\omega} \lambda(J_j \cap (-\infty, x))\right\}.$$

Obviously $a \in A$, and A is bounded above by b, so $c \overset{\text{def}}{=} \sup(A)$ exists. Now

$$\begin{aligned} c - a &= \sup_{x\in A}(x - a) \\ &\le \sup_{x\in A} \sum_{j\in\omega} \lambda(J_j \cap (-\infty, x)) \\ &\le \sum_{j\in\omega} \lambda(J_j \cap (-\infty, c)). \end{aligned}$$

Hence $c \in A$. Now suppose that $c < b$. Thus $c \in [a, b)$, so there is a $k \in \omega$ such that $c \in J_k$. Say $J_k = [u, v)$. Then $x \stackrel{\text{def}}{=} \min(v, b) > c$. Then $\lambda(J_j \cap (-\infty, c)) \leq \lambda(J_j \cap (-\infty, x))$ for each j, and $\lambda(J_k \cap (-\infty, x)) = \lambda(J_k \cap (-\infty, c)) + x - c$. Hence

$$\begin{aligned}\sum_{j\in\omega} \lambda(J_j \cap (-\infty, x)) &\geq \sum_{j\in\omega} \lambda(J_j \cap (-\infty, c)) + x - c \\ &\geq c - a + x - c = x - a.\end{aligned}$$

Here we used the above inequality on $c - a$. Thus we have shown that $x \in A$. But $x > c = \sup(A)$, contradiction.

Hence $c = b$, so $b \in A$. □

Now for any $A \subseteq \mathbb{R}$ let

$$\theta'(A) = \inf \left\{ \sum_{j\in\omega} \lambda(I_j) : \langle I_j : j \in \omega\rangle \text{ is a sequence of half-open intervals} \right.$$
$$\left. \text{such that } A \subseteq \bigcup_{j\in\omega} I_j \right\}.$$

Lemma 26.90. *(i) θ' is an outer measure on $\mathbb{R}$.*

(ii) $\theta'(I) = \lambda(I)$ for every half-open interval I.

Proof. (i): Clearly (1) and (2) hold. Now for (3), suppose that $\langle A_i : i \in \omega\rangle$ is a sequence of subsets of X. Let $B = \bigcup_{i\in\omega} A_i$. For each $i \in \omega$ let $\langle I_{ij} : j \in \omega\rangle$ be a sequence of half-open intervals such that $A_i \subseteq \bigcup_{j\in\omega} I_{ij}$ and

$$\sum_{j\in\omega} \lambda(I_{ij}) \leq \theta'(A_i) + \frac{\varepsilon}{2^i}.$$

Note that this holds even if $\theta'(A_i) = \infty$. Let $p : \omega \to \omega \times \omega$ be a bijection.

$$B \subseteq \bigcup_{m\in\omega} I_{1^{st}(p(m)),2^{nd}(p(m))}. \tag{1}$$

In fact, if $b \in B$, choose $i \in I$ such that $b \in A_i$, and then choose $j \in \omega$ such that $b \in I_{ij}$. Let $m = p^{-1}(i, j)$. Then

$$b \in I_{1^{st}(p(m)),2^{nd}(p(m))},$$

as desired in (1).

$$\sum_{m\in\omega} \lambda(I_{1^{st}(p(m)),2^{nd}(p(m))}) \leq \sum_{i\in\omega}\sum_{j\in\omega} \lambda(I_{ij}). \tag{2}$$

In fact, let $m \in \omega$, and set

$$n = \max(\{1^{st}(p(i)) : i \leq m\} \cup \{2^{nd}(p(i)) : i \leq m\}).$$

Then
$$\sum_{i=0}^{m} \lambda(I_{1^{st}(p(m)),2^{nd}(p(m))}) \le \sum_{i=0}^{n}\sum_{j=0}^{n} \lambda(I_{ij}) \le \sum_{i\in\omega}\sum_{j\in\omega} \lambda(I_{ij}),$$
and (2) follows.

Hence using (1) we have
$$\begin{aligned}
\theta'\left(\bigcup_{i\in\omega} A_i\right) &= \theta'(B) \\
&\le \sum_{m\in\omega} \lambda(I_{1^{st}(p(m)),2^{nd}(p(m))}) \\
&\le \sum_{i\in\omega}\sum_{j\in\omega} \lambda(I_{ij}) \\
&\le \sum_{i\in\omega}\left(\theta'(A_i) + \frac{\varepsilon}{2^i}\right) \\
&= \sum_{i\in\omega}\theta'(A_i) + \sum_{i\in\omega}\frac{\varepsilon}{2^i} \\
&= \sum_{i\in\omega}\theta'(A_i) + 2\varepsilon.
\end{aligned}$$
Hence (3) in the definition of outer measure holds.

Clearly $\theta'(I) \le \lambda(I)$. The other inequality follows from Lemma 26.89. □

Corollary 26.91. *For θ' the explicit outer measure defined above on $\mathbb{R}$, and with*
$$\begin{aligned}\Sigma_1 = \{E \subseteq \mathbb{R} : \textit{for every } A \subseteq X,& \\ \theta'(A) = \theta'(A\cap E) + \theta'(A\backslash E)\},&\end{aligned}$$
the system $(\mathbb{R}, \Sigma_1, \theta' \restriction \Sigma_1)$ is a measure space. □

Lemma 26.92. $(-\infty, x)$ *is measurable for every $x \in \mathbb{R}$.*

Proof. First we show

(1) $\lambda(I) = \lambda(I\cap(-\infty,x)) + \lambda(I\backslash(-\infty,x))$ for every half-open interval I.

This is obvious if $I \subseteq (-\infty, x)$ or $I \subseteq [x,\infty)$. So assume that neither of these cases hold. Then with $I = [a,b)$ we must have $a < x < b$. Then
$$\begin{aligned}
\lambda(I\cap(-\infty,x)) + \lambda(I\backslash(-\infty,x)) &= \lambda([a,x)) + \lambda([x,b)) \\
&= \lambda([a,x)) + \lambda([x,b)) \\
&= x - a + b - x \\
&= b - a \\
&= \lambda([a,b)) \\
&= \lambda(I).
\end{aligned}$$

So (1) holds.

Now for the proof of the lemma, let $A \subseteq \mathbb{R}$ and let $\varepsilon > 0$. We show that $\theta'(A \cap (-\infty, x)) + \theta'(A \backslash (-\infty, x)) \leq \theta'(A) + \varepsilon$, which will prove the lemma. By the definition of θ', there is a sequence $\langle I_j : j \in \omega \rangle$ of half-open intervals such that $A \subseteq \bigcup_{j\in\omega} I_j$ and $\sum_{j\in\omega} \lambda(I_j) \leq \theta'(A) + \varepsilon$. Now $\langle I_j \cap (-\infty, x) : j \in \omega \rangle$ and $\langle I_j \backslash (-\infty, x) : j \in \omega \rangle$ are sequences of half-open intervals, $A \cap (-\infty, x) \subseteq \bigcup_{j\in\omega}(I_j \cap (-\infty, x))$, and $A \backslash (-\infty, x) \subseteq \bigcup_{j\in\omega}(I_j \backslash (-\infty, x))$, so

$$\begin{aligned}\theta'(A \cap (-\infty, x)) + \theta'(A \backslash (-\infty, x)) &\leq \sum_{j=0}^{\infty} \lambda(I_j \cap (-\infty, x)) + \sum_{j=0}^{\infty} \lambda(I_j \backslash (-\infty, x)) \\ &= \sum_{j=0}^{\infty} \lambda(I_j) \leq \theta'(A) + \varepsilon. \end{aligned}$$ □

Theorem 26.93. *Every Borel subset of* $\mathbb{R}$ *is Lebesgue measurable.*

Proof. It suffices to show that every open set is Lebesgue measurable. It then suffices to prove the following:

(1) If U is a nonempty open subset of $\mathbb{R}$, then there is a family $\mathscr{A}$ of half-open intervals with rational coefficients such that $U = \bigcup \mathscr{A}$.

To prove (1), let $\mathscr{A}$ be the set of all half-open intervals contained in U. Now take any $x \in U$. Since U is open, there are real numbers $y < z$ such that $x \in (y, z) \subseteq U$. Choose rational numbers r, s such that $y < r < x < s < z$. Then $x \in [r, s) \subseteq U$, as desired. □

Corollary 26.94. *Every Lebesgue null set is Lebesgue measurable. Every singleton is a null set, and every countable set is a null set.* □

Lemma 26.95. *Suppose that* μ *is a measure and* E, F, G *are* μ*-measurable. Then*

$$\mu(E \triangle F) \leq \mu(E \triangle G) + \mu(G \triangle F).$$

Proof.

$$\begin{aligned}\mu(E \triangle F) &= \mu(E \backslash F) + \mu(F \backslash E) \\ &= \mu((E \backslash F) \cap G) + \mu((E \backslash F) \backslash G) + \mu(F \backslash E) \cap G) + \mu((F \backslash E) \backslash G) \\ &\leq \mu(G \backslash F) + \mu(E \backslash G) + \mu(G \backslash E) + \mu(F \backslash G) \\ &= \mu(E \triangle G) + \mu(G \triangle F).\end{aligned}$$ □

Lemma 26.96. *If* E *is Lebesgue measurable with finite measure, then for any* $\varepsilon > 0$ *there is an open set* $U \supseteq E$ *such that* $\theta'(E) \leq \theta'(U) \leq \theta'(E) + \varepsilon$*. Moreover, there is a system* $\langle K_j : j < \omega \rangle$ *of open intervals such that* $U = \bigcup_{j<\omega} K_j$ *and* $\theta'(U) \leq \sum_{j<\omega} \theta'(K_j) \leq \theta'(E) + \varepsilon$*.*

Proof. By the basic definition of Lebesgue measure,

$$0 = \theta'(E) = \inf \left\{ \sum_{j\in\omega} \theta'(I_j) : \langle I_j : j \in \omega\rangle \text{ is a sequence of half-open intervals} \right.$$
$$\left. \text{such that } A \subseteq \bigcup_{j\in\omega} I_j \right\}.$$

Hence we can choose a sequence $\langle I_j : j \in \omega\rangle$ of half-open intervals such that $E \subseteq \bigcup_{j\in\omega} I_j$ and

$$\theta'\left(\bigcup_{j\in\omega} I_j\right) \leq \sum_{j\in\omega} \theta'(I_j) \leq \theta'(E) + \frac{\varepsilon}{2}.$$

Write $I_j = [a_j, b_j)$ with $a_j < b_j$. Define

$$\begin{aligned}
K_j &= \left(a_j - \frac{\varepsilon}{2^{j+2}}, b_j\right); \quad \text{then} \\
E &\subseteq \bigcup_{j\in\omega} K_j \quad \text{and} \\
\theta'\left(\bigcup_{j\in\omega} K_j\right) &\leq \sum_{j\in\omega} \theta'(K_j) \\
&= \sum_{j\in\omega} \left(\frac{\varepsilon}{2^{j+2}} + \theta'(I_j)\right) \\
&= \sum_{j\in\omega} \frac{\varepsilon}{2^{j+2}} + \sum_{j\in\omega} \theta'(I_j) \\
&\leq \frac{\varepsilon}{2} + \theta'(E) + \frac{\varepsilon}{2} = \theta'(E) + \varepsilon.
\end{aligned}$$

□

Corollary 26.97. *(i) If A is Lebesgue measurable and $\theta'(A)$ is finite, then $\theta'(A) = \inf\{\theta'(U) : U$ open, $A \subseteq U\}$.*

(ii) If A is Lebesgue measurable with finite measure, then $\theta'(A) = \sup\{\theta'(C) : C$ closed, $C \subseteq A\}$.

(iii) If A is measurable and $\theta'(A) = \infty$, then $\sup\{\theta'(C) : C$ closed, $C \subseteq A\} = \infty$.

Proof. Only (iii) needs a proof. Let $\varepsilon > 0$. For each $n \in \omega$ let

$$\begin{aligned}
a_{2n} &= n; \\
b_{2n} &= n + 1; \\
a_{2n+1} &= -n - 1; \\
b_{2n+1} &= -n.
\end{aligned}$$

For each $n \in \omega$ let C_n be a closed subset of $[a_n, b_n) \cap A$ such that

$$\theta'([a_n, b_n) \cap A \backslash C_n) < \frac{\varepsilon}{2^n}.$$

Then

$$\begin{aligned}
\theta'(A) &= \sum_{n\in\omega} \theta'([a_n, b_n) \cap A) \\
&= \lim_{n=0}^{\infty} \theta'([[a_0, b_0) \cap A] \cup \ldots \cup [[a_n, b_n) \cap A]) \\
&= \lim_{n=0}^{\infty} \theta'([[a_0, b_0) \cap A\backslash C_0] \cup \ldots \cup [[a_n, b_n) \cap A\backslash C_n]) \\
&\qquad + \theta'(C_0 \cup \ldots \cup C_n) \\
&= \lim_{n=0}^{\infty} \theta'([[a_0, b_0) \cap A\backslash C_0] \cup \ldots \cup [[a_n, b_n) \cap A\backslash C_n]) \\
&\qquad + \lim_{n\to\infty} \theta'(C_0 \cup \ldots \cup C_n) \\
&= \varepsilon + \lim_{n\to\infty} \theta'(C_0 \cup \ldots \cup C_n),
\end{aligned}$$

as desired.

The following is an elementary lemma concerning the topology of the reals.

Lemma 26.98. *Suppose that U is a bounded open set.*

(i) There is a collection $\mathscr{A}$ of pairwise disjoint open intervals such that $U = \bigcup \mathscr{A}$.

(ii) There exist a countable subset C of $\mathbb{R}$ and a collection $\mathscr{B}$ of pairwise disjoint open intervals with rational endpoints such that $U = C \cup \bigcup \mathscr{B}$ and $C \cap \bigcup \mathscr{B} = \emptyset$.

Proof. (i): For $x, y \in \mathbb{R}$, define $x \equiv y$ iff one of the following conditions holds: (1) $x = y$; (2) $x < y$ and $[x, y] \subseteq U$; (3) $y < x$ and $[y, x] \subseteq U$. Clearly $\equiv$ is an equivalence relation on $\mathbb{R}$. If $x < z < y$ and $x \equiv y$, then obviously $x \equiv z$. Thus each equivalence class is convex. If C is an equivalence class with more than one element, then it must be an open interval (a, b), since if for example the left endpoint a is in C then some real to the left of a must be in C, contradiction. It follows now that the collection $\mathscr{A}$ of all equivalence classes with more than one element is as desired in (i).

(ii): First note that the set $\mathscr{A}$ of (i) must be countable. Now take any $(a, b) \in \mathscr{A}$, $a < b$. Let $c_0 < c_1 < \cdots < c_m < \cdots$ be rational numbers in (a, b) which converge to b, and $c_0 = d_0 > d_1 > \cdots > d_m > \cdots$ rational numbers which converge to a. Then let $L^{ab}_{2i} = (c_i, c_{i+1})$ and $L^{ab}_{2i+1} = (d_{i+1}, d_i)$ for all $i \in \omega$. Let $D^{ab} = \{c_i : i < \omega\} \cup \{d_i : i < \omega\}$. Define $\mathscr{B} = \{L^{ab}_i : (a, b) \in \mathscr{A}, i < \omega\}$ and $C = \bigcup_{(a,b)\in\mathscr{A}} D^{ab}$. Clearly this works for (ii). $\square$

Lemma 26.99. *If E is Lebesgue measurable and $\varepsilon > 0$, then there is an $m \in \omega$ and a sequence $\langle I_i : i < m\rangle$ of open intervals with rational endpoints such that $\theta'\left(E \triangle \bigcup_{i<m} I_i\right) \leq \varepsilon$.*

Proof. By Corollary 26.97, let $U \supseteq E$ be open such that $\theta'(E) \leq \theta'(U) \leq \theta'(E) + \frac{\varepsilon}{2}$. Then choose C and $\mathscr{B}$ as above. Let $W = \bigcup \mathscr{B}$. So $\theta'(W) = \sum_{I\in\mathscr{B}} \theta'(I)$. Then choose $m \in \omega$ and $\langle I_i : i < m\rangle$ elements of $\mathscr{B}$ such that $\sum_{I\in\mathscr{B}} \theta'(I) - \sum_{i<m} \theta'(I_i) \leq \frac{\varepsilon}{2}$. Now $\theta'(W) = \sum_{I\in\mathscr{B}} \theta'(I)$ and $\theta'(\bigcup_{i<m} I_i) = \sum_{i<m} \theta'(I_i)$. Let $V = \bigcup_{i<m} I_i$. Thus $\theta'(W) - \theta'(V) \leq \frac{\varepsilon}{2}$. Hence $V \subseteq W \subseteq U$, and

$$\begin{aligned}\theta'(E\triangle V) &\leq \theta'(E\triangle U) + \theta'(U\triangle W) + \theta'(W\triangle V)\\ &= \theta'(U\backslash E) + \theta'(C) + \theta'(W\backslash V)\\ &= \theta'(U) - \theta'(E) + \theta'(W) - \theta'(V)\\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$ □

Lemma 26.100. *(i)* $\theta'([a,b)) = b - a$ *if* $a < b$.

(ii) $\theta'([a,b]) = b - a$ *if* $a < b$.

Proof. (i) holds by Lemma 26.90(ii). Then for $a < b$, $\theta'([a,b]) = \theta'([a,b)) + \theta'(\{b\}) = \varphi'([a,b)) = b - 1$ using Corollary 26.94. □

Connections between different measures

Lemma 26.101. *If* (X, Σ, μ) *is a measure space and* $Y \subseteq X$*, then*

$$(Y, \{A \cap Y : A \in \Sigma\}, \mu \restriction \{A \cap Y : A \in \Sigma\})$$

is a measure space. □

At this point we have two important measure spaces: $({}^{\omega}2, \Sigma_0, \theta)$ and $(\mathbb{R}, \Sigma_1, \theta')$. We now define $\Sigma_2 = \{A \cap \Omega : A \in \Sigma_0\}$ and $\theta_2 = \theta \restriction \{A \cap \Omega : A \in \Sigma_0\}$. Thus

Corollary 26.102. $(\Omega, \Sigma_2, \theta_2)$ *is a measure space.* □

Let $\Sigma_3 = \{A \cap [0,1] : A \in \Sigma_1\}$ and $\theta_3 = \theta' \restriction \{A \cap [0,1] : A \in \Sigma_1\}$.

Corollary 26.103. $([0,1], \Sigma_3, \theta_3)$ *is a measure space.* □

If (X, Σ, μ) is a measure space, then $A \in \Sigma$ is an *atom* iff $\mu(A) > 0$. and for all $B \in \Sigma$ with $B \subseteq A$, either B or $A\backslash B$ has measure 0.

Let λ be the usual measure on ${}^{\omega}2$ and μ Lebesgue measure on $[0,1]$. Consider the measure spaces $({}^{\omega}2, \Sigma_0, \lambda)$ and $([0,1], \Sigma_1, \mu)$. For each $x \in {}^{\omega}2$ let $\varphi(x) = \sum_{i=0}^{\infty}(2^{-i-1}x_i)$.

Theorem 26.104. *There is a bijection* $\tilde{\varphi} : {}^{\omega}2 \to [0,1]$ *which is equal to* φ *except at countably many points, and any such bijection is an isomorphism from* $({}^{\omega}2, \Sigma_0, \lambda)$ *to* $([0,1], \Sigma_1, \mu)$*. That is:*

(a) $\forall X \subseteq {}^{\omega}2[X \in \Sigma_0$ *iff* $\tilde{\varphi}[X] \in \Sigma_1]$*;*

(b) $\forall X \subseteq [0,1][X \in \Sigma_1$ *iff* $\tilde{\varphi}^{-1}[X] \in \Sigma_0]$*;*

(c) $\forall X \in \Sigma_0[\lambda(X) = \mu(\tilde{\varphi}[X])]$*;*

(d) $\forall X \in \Sigma_1[\mu(X) = \lambda(\tilde{\varphi}^{-1}[X])]$.

Proof. Let $H = \{x \in {}^\omega 2 : \exists m \in \omega \forall i \geq m[x_i = x_m]\}$ and $H' = \{2^{-n}k : n \in \omega, k \leq 2^n\}$. Then H and H' are countable.

(1) $\varphi \restriction ({}^\omega 2 \backslash H)$ is a bijection from ${}^\omega 2 \backslash H$ onto $[0,1] \backslash H'$.

For, first we show that $\varphi \restriction ({}^\omega 2 \backslash H)$ maps into $[0,1] \backslash H'$. Let $x \in ({}^\omega 2 \backslash H)$. Thus

(2) $\forall m \in \omega \exists i > m[x_i \neq x_m]$.

It follows that $\varphi(x) \neq 1$, for by (2) there is a j such that $x_j = 0$, and then

$$\varphi(x) = \sum_{i=0}^{\infty}(2^{-i-1}x_i) \leq \sum_{i=0}^{j-1}(2^{-i-1}x_i) + \sum_{i=j+1}^{\infty} 2^{-i-1} = \sum_{i=0}^{j-1}(2^{-i-1}x_i) + 2^{-j-1} < 1.$$

Suppose that $\varphi(x) \in H'$. Thus there exist $n \in \omega$ and $k < 2^n$ such that

$$\sum_{i=0}^{\infty}(2^{-i-1}x_i) = 2^{-n}k. \tag{3}$$

Since $\varphi(x) \neq 1$, we can write $2^{-n}k = \sum_{i=0}^{n-1}(2^{-i-1}y_i)$ with each $y_i \in 2$. Thus by (3) we have

$$\sum_{i=0}^{\infty}(2^{-i-1}x_i) = \sum_{i=0}^{n-1}(2^{-i-1}y_i). \tag{4}$$

Now we claim that $y \subseteq x$. For, suppose not, and let $j < n$ be minimum such that $x_j \neq y_j$. Hence by (4) we have

$$\sum_{i=j}^{\infty}(2^{-i-1}x_i) = \sum_{i=j}^{n-1}(2^{-i-1}y_i).$$

Case 1. $x_j = 0$ and $y_j = 1$. By (2) choose $k > j$ so that $x_k = 1$ and choose $l > k$ so that $x_l = 0$. Then

$$\begin{aligned}\sum_{i=j}^{\infty}(2^{-i-1}x_i) &\leq \sum_{i=j}^{l-1}(2^{-i-1}x_i) + \sum_{i=l+1}^{\infty} 2^{-i-1} = \sum_{i=j}^{l-1}(2^{-i-1}x_i) + 2^{-l-1}\\ &< \sum_{i=0}^{j-1}(2^{-i-1}x_i) + \sum_{i=j+1}^{\infty} 2^{-i-1} \leq \sum_{i=j}^{n-1}(2^{-i-1}y_i),\end{aligned}$$

contradiction.

Case 2. $x_j = 1$ and $y_j = 0$. Then

$$\sum_{i=j}^{n-1}(2^{-i-1}y_i) \leq \sum_{i=j+1}^{n} 2^{-i-1} < \sum_{i=j+1}^{\infty} 2^{-i-1} = 2^{-j-1} \leq \sum_{i=j}^{\infty}(2^{-i-1}x_i),$$

contradiction.

Thus $y \subseteq x$. Now by (2) there is a $j \geq n$ such that $x_j = 1$. Hence

$$\sum_{i=j}^{\infty}(2^{-i-1}x_i) > \sum_{i=j}^{n-1}(2^{-i-1}y_i),$$

contradiction.

Thus $\varphi(x) \notin H'$.

To show that $\varphi \restriction ({}^{\omega}2 \backslash H)$ maps onto $[0,1]\backslash H'$, let $t \in [0,1]\backslash H'$. Since $1 \in H'$, we can write $t = \sum_{i=0}^{\infty}(2^{-i-1}x_i)$ with x not eventually 1. We claim that $x \notin H$. For, suppose that $x \in H$. Say $m \in \omega$ and $\forall i > m[x_i = x_m]$. Since x is not eventually 1, we have $x_m = 0$. Since $t \notin H'$, we have $t \neq 0$, so x is not the all 0 sequence. Choose n maximum such that $x_n \neq 0$. Thus $t = \sum_{i=0}^{n}(2^{-i-1}x_i)$. Hence

$$\begin{aligned} 2^{n+1}t &= 2^{n+1}2^{-1}x_0 + 2^{n+1}2^{-2}x_1 + \cdots + x_0 \\ &= 2^n x_0 + 2^{n-1}x_1 + \cdots + x_0. \end{aligned}$$

Hence with $k = 2^n x_0 + 2^{n-1}x_1 + \cdots + x_0$ we have $k \leq 2^{n+1}$ and $t = 2^{-n-1}k \in H'$, contradiction. So $x \notin H$. Clearly $\varphi(x) = t$.

For $\varphi \restriction ({}^{\omega}2 \backslash H)$ one-one, suppose that $x, y \in ({}^{\omega}2 \backslash H)$ and $x \neq y$. Let m be minimum such that $x_m \neq y_m$. By symmetry, say $x_m = 0$ and $y_m = 1$. Choose $n > m$ so that $x_n = 0$; this is possible since $x \notin H$. Then

$$\sum_{i=0}^{\infty}(2^{-i-1}x_i) \leq \sum_{i=0}^{n-1}(2^{-i-1}x_i) + \sum_{i=n+1}^{\infty} 2^{-i-1} = \sum_{i=0}^{n-1}(2^{-i-1}x_i) + 2^{-n} < \sum_{i=0}^{n-1}(2^{-i-1}y_i).$$

This finishes the proof of (1).

Now H and H' are countable and infinite. Hence there is an extension of $\varphi \restriction ({}^{\omega}2 \backslash H)$ to a bijection of ${}^{\omega}2$ onto $[0,1]$. Let $\tilde{\varphi}$ be any bijection of ${}^{\omega}2$ onto $[0,1]$ which is equal to φ except for countably many points. Let M be the countable set $\{x \in {}^{\omega}2 : \varphi(x) \neq \tilde{\varphi}(x)\}$ and let N be the countable set $\varphi[M] \cup \tilde{\varphi}[M]$.

(5) $\forall A \subseteq {}^{\omega}2[\varphi[A] \triangle \tilde{\varphi}[A] \subseteq N]$.

In fact, if $b \in \varphi[A]\backslash\tilde{\varphi}[A]$, then there is an $x \in A$ such that $b = \varphi(x)$. Since $b \notin \tilde{\varphi}[A]$, we have $\tilde{\varphi}(x) \neq b$. Hence $x \in M$, so $b \in \varphi[M] \subseteq N$. Now suppose that $b \in \tilde{\varphi}[A]\backslash\varphi[A]$. Say $b = \tilde{\varphi}(x)$ with $x \in A$. Since $b \notin \varphi[A]$, we have $\varphi(x) \neq b$. So $x \in M$ and $b \in \tilde{\varphi}[M] \subseteq N$. So (5) holds.

(6) If $t \in [0,1]$, then $\lambda(\tilde{\varphi}^{-1}[\{t\}]) = 0$ and hence $\lambda(\tilde{\varphi}^{-1}[\{t\}]) = \mu(\{t\})$.

We have $\tilde{\varphi}^{-1}[\{t\}] = \{\tilde{\varphi}^{-1}(t)\}$, so $\lambda(\tilde{\varphi}^{-1}[\{t\}]) = 0$ by Proposition 26.86. $\mu(\{t\}) = 0$ by Corollary 26.94.

(7) If $n \in \omega$, $k < 2^n$, and $E = [2^{-n}k, 2^{-n}(k+1)]$, then $\tilde{\varphi}^{-1}[E] \in \Sigma_0$ and $\lambda(\tilde{\varphi}^{-1}[E]) = \mu(E) = 2^{-n}$.

$\mu(E) = 2^{-n}$ by Lemma 26.100. Further,

$$\varphi^{-1}[E] = \left\{ x \in {}^{\omega}2 : 2^{-n}k \leq \sum_{i=0}^{\infty}(2^{-i-1}x_i) \leq 2^{-n}(k+1) \right\}.$$

Let $k = 2^{n-1}y_0 + 2^{n-2}y_1 + \cdots + y_{n-1}$ with each $y_i \in 2$. Then $2^{-n}k = 2^{-1}y_0 + 2^{-2}y_1 + \cdots + 2^{-n}y_{n-1} = \sum_{i=0}^{n-1}(2^{-i-1}y_i)$.

Case 1. $y_{n-1} = 0$. Then $k+1 = \sum_{i=0}^{n-2}(2^{n-i-1}y_i) + 1$ and so $2^{-n}(k+1) = \sum_{i=0}^{n-2}(2^{-i-1}y_i) + 2^{-n}$.

(8) If $x \in \varphi^{-1}[E]$ and x is not eventually 1, then $\forall i < n[x_i = y_i]$.

For, suppose that $j < n$ is minimum such that $x_i \neq y_i$. Choose $l > k > j$ with $x_l = x_k = 0$.

Subcase 1.1. $x_j = 0$, $y_j = 1$. Then

$$\begin{aligned}\sum_{i=0}^{\infty}(2^{-i-1}x_i) &\leq \sum_{i=0}^{l-1}(2^{-i-1}x_i) + \sum_{i=l+1}^{\infty}2^{-i-1} = \sum_{i=0}^{l-1}(2^{-i-1}x_i) + 2^{-l-1}\\ &< \sum_{i=0}^{j-1}(2^{-i-1}x_i) + 2^{-j-1} \leq \sum_{i=0}^{n-1}(2^{-i-1}y_i) = 2^{-n}k.\end{aligned}$$

This contradicts $x \in \varphi^{-1}[E]$.

Subcase 1.2. $x_j = 1$, $y_j = 0$. Then

$$2^{-n}(k+1) = \sum_{i=0}^{n-2}(2^{-i-1}y_i) + 2^{-n} < \sum_{i=0}^{\infty}(2^{-i-1}x_i),$$

contradiction.

Thus (8) holds.

(9) If $x \in {}^{\omega}2$ and $\forall i < n[x_i = y_i]$, then $x \in \varphi^{-1}[E]$.

In fact, assume that $x \in {}^{\omega}2$ and $\forall i < n[x_i = y_i]$. Then

$$\begin{aligned}2^{-n}k &= \sum_{i=0}^{n-1}(2^{-i-1}y_i) \leq \sum_{i=0}^{\infty}(2^{-i-1}x_i)\\ &\leq \sum_{i=0}^{n-1}(2^{-i-1}x_i) + \sum_{i=n+1}^{\infty}2^{-i-1} = \sum_{i=0}^{n-2}(2^{-i-1}x_i) + 2^{-n} = 2^{-n}(k+1).\end{aligned}$$

This proves (9).

Case 2. $y_{n-1} = 1$ and there is a $j < n-1$ such that $y_j = 0$. Take the greatest such j. Then $k+1 = 2^{n-1}y_0 + 2^{n-2}y_1 + \cdots + 2^{n-j}y_{j-1} + 2^{n-j-1}$, and hence $2^{-n}(k+1) = \sum_{i=0}^{j-1}(2^{-i-1}y_i) + 2^{-j-1}$. Now suppose that $x \in \varphi^{-1}[E]$. Again we claim that (8) and (9) hold. For (8), suppose that $x \in \varphi^{-1}[E]$, x is not eventually 1, and $l < n$ is minimum such that $x_l \neq y_l$. Choose $t > s > l$ with $x_t = x_s = 0$.

Subcase 2.1. $x_l = 0$, $y_l = 1$. Then

$$\begin{aligned}\sum_{i=0}^{\infty}(2^{-i-1}x_i) &\leq \sum_{i=0}^{t-1}(2^{-i-1}x_i) + \sum_{i=t+1}^{\infty} 2^{-i-1} = \sum_{i=0}^{t-1}(2^{-i-1}x_i) + 2^{-t-1} \\ &< \sum_{i=0}^{l-1}(2^{-i-1}x_i) + 2^{-l-1} \leq \sum_{i=0}^{n-1}(2^{-i-1}y_i) = 2^{-n}k.\end{aligned}$$

This contradicts $x \in \varphi^{-1}[E]$.

Subcase 2.2. $x_l = 1$, $y_l = 0$. Then $l \leq j$, and

$$\begin{aligned}2^{-n}(k+1) &= \sum_{i=0}^{j-1}(2^{-i-1}y_i) + 2^{-j-1} = \sum_{i=0}^{l-1}(2^{-i-1}y_i) + \sum_{i=l}^{j-1}(2^{-i-1}y_i) + 2^{-j-1} \\ &< \sum_{i=0}^{l-1}(2^{-i-1}y_i) + \sum_{i=l+1}^{\infty} 2^{-i-1} = \sum_{i=0}^{l-1}(2^{-i-1}x_i) + 2^{-i-1} \leq \sum_{i=0}^{\infty}(2^{-i-1}x_i,\end{aligned}$$

contradiction.

Hence (8) holds.

Now for (9), assume that $x \in {}^{\omega}2$ and $\forall i < n[x_i = y_i]$. Then

$$\begin{aligned}2^{-n}k &= \sum_{i=0}^{n-1}(2^{-i-1}y_i) \leq \sum_{i=0}^{\infty}(2^{-i-1}x_i) \\ &\leq \sum_{i=0}^{j-1}(2^{-i-1}x_i) + \sum_{i=j+1}^{\infty} 2^{-i-1} = \sum_{i=0}^{j-1}(2^{-i-1}y_i) + 2^{-j-1} = 2^{-n}(k+1).\end{aligned}$$

So (9) holds. This finishes Case 2.

Case 3. $\forall i < n[y_i = 1]$. Then $k + 1 = 2^n$ and $2^{-n}k = 1$. To check (8), suppose that $x \in \varphi^{-1}[E]$, x is not eventually 1, and j is minimum such that $x_j \neq y_j$. Take $s > t > j$ with $x_s = x_t = 0$. Then $x_j = 0$, and

$$\sum_{i=0}^{\infty}(2^{-i-1}x_i) \leq \sum_{i=0}^{s-1}(2^{-i-1}x_i) + \sum_{i=s+1}^{\infty} 2^{-i-1} = \sum_{i=0}^{s-1}(2^{-i-1}x_i) + 2^{-s-1} < \sum_{i=0}^{n-1}(2^{-i-1}y_i),$$

contradiction. Thus (8) holds.

For (9), assume that $x \in {}^{\omega}2$ and $\forall i < n[x_i = y_i]$. Clearly $x \in \varphi^{-1}[E]$.

So (8) and (9) hold in all cases.

Now let $S = \{x \in {}^{\omega}2 : x \text{ is eventually } 1\}$. So S is countable. By (8) we have $\varphi^{-1}[E]\backslash S \subseteq \{x \in {}^{\omega}2 : x \restriction n = y \restriction n\}$, and by (9) we have $\{x \in {}^{\omega}2 : x \restriction n = y \restriction n\} \subseteq \varphi^{-1}[E]$. Let $T = \varphi^{-1}[E]\backslash\{x \in {}^{\omega}2 : x \restriction n = y \restriction n\}$. Now $\{x \in {}^{\omega}2 : x \restriction n = y \restriction n\} \in \Sigma_0$

and $\lambda(\{x \in {}^\omega 2 : x \restriction n = y \restriction n\}) = 2^{-n}$ by Proposition 26.85. Note that $T \subseteq S$, so T is countable. Since $\varphi^{-1}[E] = \{x \in {}^\omega 2 : x \restriction n = y \restriction n\} \cup T$, it follows that $\varphi^{-1}[E] \in \Sigma_0$ and $\lambda(\varphi^{-1}[E]) = 2^{-n}$. Since $\varphi^{-1}[E] = (\varphi^{-1}[E] \cap M) \cup (\varphi^{-1}[E] \backslash M)$ and M is countable, it follows that $(\varphi^{-1}[E] \backslash M) \in \Sigma_0$ and $\lambda(\varphi^{-1}[E] \backslash M) = 2^{-n}$. Clearly $\tilde{\varphi}^{-1}[E] \backslash M = \varphi^{-1}[E] \backslash M$, so $(\tilde{\varphi}^{-1}[E] \backslash M) \in \Sigma_0$ and $\lambda(\tilde{\varphi}^{-1}[E] \backslash M) = 2^{-n}$. Hence $\tilde{\varphi}^{-1}[E] \in \Sigma_0$ and $\lambda(\tilde{\varphi}^{-1}[E]) = 2^{-n}$. This proves (7).

(10) If $n \in \omega$ and $k < l \leq 2^n$, and $E = [2^{-n}k, 2^{-n}l]$, then $E \in \Sigma_0$, and $\lambda(\tilde{\varphi}^{-1}[E]) = 2^{-n}(l-k) = \mu(E)$.

This is true by (7) since $E = \bigcup_{k \leq i < l} [2^{-n}i, 2^{-n}(i+1)] \backslash \{2^{-n}i : 0 < i < l\}$.

(11) Suppose that $0 \leq t < u \leq 1$ and $E = [t, u)$. Then $\tilde{\varphi}^{-1}[E]) \in \Sigma_0$, and $\lambda(\tilde{\varphi}^{-1}[E]) = u - t = \mu(E)$.

In fact, for each $n \in \omega$ let $k_n = \lfloor 2^n t \rfloor$ and $l_n = \lfloor 2^n u \rfloor$. Then $k_n \leq 2^n t < k_n + 1$ and $l_n \leq 2^n u < l_n + 1$; hence $2^{-n}k_n \leq t < 2^{-n}k_n + 2^{-n}$ and $2^{-n}l_n \leq u < 2^{-n}l_n + 2^{-n}$. It follows that $\bigcap_{n\in\omega} [2^{-n}k_n, 2^{-n}(l_n+1)] = [t, u]$. Hence by (10), $\tilde{\varphi}^{-1}[E] = \tilde{\varphi}^{-1}[[t, u]] = \bigcap_{n\in\omega} \tilde{\varphi}^{-1}[[2^{-n}k_n, 2^{-n}(l_n+1)]] \in \Sigma_0$. Also, if $m < n$ then $[2^{-n}k_n, 2^{-n}(l_n+1)] \subseteq [2^{-m}k_m, 2^{-m}(l_m+1)]]$, hence $\tilde{\varphi}^{-1}[[2^{-n}k_n, 2^{-n}(l_n+1)]] \subseteq \tilde{\varphi}^{-1}[[2^{-m}k_m, 2^{-m}(l_m+1)]]]$. Hence by Proposition 26.76(iv) we have $\lambda(\tilde{\varphi}^{-1}[E]) = u - t$. Clearly $\mu(E) = u - t$.

Now for each $X \subseteq {}^\omega 2$ define

$$\lambda^*(X) = \inf\{\lambda(E) : X \subseteq E \in \Sigma_0\}.$$

(12) For every $X \subseteq {}^\omega 2$ there is an $E \in \Sigma_0$ such that $X \subseteq E$ and $\lambda^*(X) = \lambda(E)$.

In fact, suppose that $X \subseteq {}^\omega 2$. For each $n \in \omega$ choose $E_n \in \Sigma_0$ such that $X \subseteq E_n$ and $\lambda(E_n) \leq \lambda^*(X) + \frac{1}{2^n}$. Then $E \stackrel{\text{def}}{=} \bigcap_{n\in\omega} E_n \in \Sigma_0$, $X \subseteq E$, and

$$\lambda^*(X) \leq \lambda(E) \leq \inf_{n\in\omega} \lambda(E_n) \leq \lambda^*(X),$$

proving (12).

(13) If $E \in \Sigma_1$, then $\lambda^*(\tilde{\varphi}^{-1}[E]) \leq \mu(E)$ and there is a $V \in \Sigma_0$ such that $\tilde{\varphi}^{-1}[E] \subseteq V$ and $\lambda(V) \leq \mu(E)$.

To prove (13), assume that $E \in \Sigma_1$. By the basic definition of Lebesgue measure,

$$\mu(E\backslash\{1\}) = \inf\left\{ \sum_{n\in\omega} \mu(I_n) : \langle I_n : n \in \omega\rangle \text{ is a sequence of half-open} \right.$$
$$\left. \text{subintervals of } [0,1] \text{ such that } E \subseteq \bigcup_{n\in\omega} I_n \right\}$$

Hence for every $\varepsilon > 0$ there is a system $\langle I_n : n \in \omega\rangle$ of half-open subintervals of $[0, 1]$ such that $E \subseteq \bigcup_{n\in\omega} I_n$ and $\sum_{n\in\omega} \mu(I_n) \leq \mu(E\backslash\{1\}) + \varepsilon$. Hence

$$\tilde{\varphi}^{-1}[E] \subseteq \{\tilde{\varphi}^{-1}[\{1\}] \cup \bigcup_{n\in\omega} \tilde{\varphi}^{-1}[I_n],$$

and hence

$$\lambda^*(\tilde{\varphi}^{-1}[E]) \leq \lambda\left(\bigcup_{n\in\omega} \tilde{\varphi}^{-1}[I_n]\right) \leq \sum_{n\in\omega} \lambda(\tilde{\varphi}^{-1}[I_n]) = \sum_{n\in\omega} \mu(I_n) \leq \mu(E) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $\lambda^*(\tilde{\varphi}^{-1}[E]) \leq \mu(E)$. By (12) there is a $V \in \Sigma_0$ such that $\tilde{\varphi}^{-1}[E] \subseteq V$ and $\lambda^*(\tilde{\varphi}^{-1}[E]) = \lambda(V)$. So (13) holds.

(14) If $E \in \Sigma_1$, then $\tilde{\varphi}^{-1}[E] \in \Sigma_0$ and $\lambda(\tilde{\varphi}^{-1}[E]) = \mu(E)$.

For, by symmetry with (13) there is a $V' \in \Sigma_0$ such that $\tilde{\varphi}^{-1}[[0,1]\backslash E] \subseteq V'$ and $\lambda(V') \leq \mu([0,1]\backslash E)$. Then ${}^\omega 2\backslash\tilde{\varphi}^{-1}[E] = \tilde{\varphi}^{-1}[[0,1]\backslash E] \subseteq V'$ and $\tilde{\varphi}^{-1}[E] \subseteq V$, so $V \cup V' = {}^\omega 2$. Now

$$\lambda(V) + \lambda(V') \leq \mu(E) + \mu([0,1]\backslash E) = 1 \leq \lambda(V \cup V') \leq \lambda(V) + \lambda(V').$$

So $\lambda(V) + \lambda(V') = \lambda(V \cup V')$. Hence

$$\begin{aligned}\lambda(V) + \lambda(V') &= \lambda(V\backslash V') + \lambda(V \cap V') + \lambda(V'\backslash V) + \lambda(V \cap V') \\ &= \lambda(V \cup V') + \lambda(V \cap V') = \lambda(V) + \lambda(V') + \lambda(V \cap V').\end{aligned}$$

It follows that $\lambda(V \cap V') = 0$. In particular, $V \cap V' \cap \tilde{\varphi}^{-1}[E] \in \Sigma_0$. Now $\tilde{\varphi}^{-1}[[0,1]\backslash E] = \tilde{\varphi}^{-1}[[0,1]]\backslash\tilde{\varphi}^{-1}[E] = ({}^\omega 2\backslash\tilde{\varphi}^{-1}[E]) \subseteq V'$, so $({}^\omega 2\backslash V') \subseteq \tilde{\varphi}^{-1}[E]$. Also, $\tilde{\varphi}^{-1}[E] \subseteq V$. Hence $\tilde{\varphi}^{-1}[E] = ({}^\omega 2\backslash V') \cup (V' \cap \tilde{\varphi}^{-1}[E] = ({}^\omega 2)\backslash V') \cup (V' \cap V \cap \tilde{\varphi}^{-1}[E]) \in \Sigma_0$.

Now

$$\lambda(\tilde{\varphi}^{-1}[E]) \leq \lambda(V) \leq \mu(E) \quad \text{and} \quad 1 - \lambda(\tilde{\varphi}^{-1}[E]) \leq \lambda(V') \leq 1 - \mu(E),$$

so $\lambda(\tilde{\varphi}^{-1}[E]) = \mu(E)$. Thus (14) holds.

(15) If $n \in \omega$, $\varepsilon \in {}^{n+1}2$, $t = \sum_{i=0}^n (2^{-i-1}\varepsilon_i)$, and $C = \{x \in {}^\omega 2 : x \restriction (n+1) = \varepsilon\}$, then $\varphi[C] = [t, t + 2^{-n-1}]$.

For, first let $x \in C$. Then

$$t = \sum_{i=0}^n (2^{-i-1}\varepsilon_i) \leq \sum_{i=0}^\infty (2^{-i-1}x_i) \leq \sum_{i=0}^n (2^{-i-1}\varepsilon_i) + \sum_{i=n+1}^\infty 2^{-i-1} = t + 2^{-n-1}.$$

Thus $\varphi(x) \in [t, t + 2^{-n-1}]$.

Second, suppose that $u \in [t, t + 2^{-n-1}]$.

Case 1. $\varepsilon_n = 0$. Let $u = \sum_{i=0}^\infty (2^{-i-1}x_i)$, with x not eventually 1.

(16) $x \restriction (n+1) = \varepsilon$.

For, suppose that j is minimum such that $x_j \neq \varepsilon_j$. Choose $s > t > j$ such that $x_s = x_t = 0$.

Subcase 2.1. $x_j = 0$ and $\varepsilon_j = 1$. Then

$$u = \sum_{i=0}^\infty (2^{-i-1}x_i) \leq \sum_{i=0}^{s-1} (2^{-i-1}x_i) + \sum_{i=s+1}^\infty 2^{-i-1} < \sum_{i=0}^n (2^{-i-1}\varepsilon_i) = t,$$

contradiction.

Subcase 2.2. $x_j = 1$ and $\varepsilon_j = 0$. Then

$$t + 2^{n-1} < \sum_{i=0}^{\infty}(2^{-i-1}x_i) = u,$$

contradiction.

Thus (16) holds, as desired in Case 1.

Case 2. ε is the all 1 sequence, and $u = t + 2^{-n-1}$. Then $u = 1$. Let $x = \langle 1 : i \in \omega\rangle$. Then $\sum_{i=0}^{\infty}(2^{-i-1}x_i) = 1$. Again (16) holds.

Case 3. ε is the all 1 sequence, and $u < t + 2^{-n-1}$. Let $u = \sum_{i=0}^{\infty}(2^{-i-1}x_i)$, with x not eventually 1. We claim that (16) holds again. Otherwise there is a $j \leq n$ such that $x_j = 0$. Then $u = \sum_{i=0}^{\infty}(2^{-i-1}x_i) < \sum_{i=0}^{n} 2^{-i-1} = t$, contradiction.

Case 4. $\varepsilon_n = 1$, ε not the all 1 sequence, $u < t + 2^{-n-1}$. Let $u = \sum_{i=0}^{\infty}(2^{-i-1}x_i)$, with x not eventually 1. We claim that (16) holds. Otherwise let j be minimum such that $x_j \neq \varepsilon_j$.

Subcase 4.1. $x_j = 0$ and $\varepsilon_j = 1$. Then

$$u = \sum_{i=0}^{\infty}(2^{-i-1}x_i) < \sum_{i=0}^{n}(2^{-i-1}\varepsilon_i) = t,$$

contradiction.

Subcase 4.2. $x_j = 1$ and $\varepsilon_j = 0$. Then

$$\begin{aligned}
t + 2^{-n-1} &= \sum_{i=0}^{n}(2^{-i-1}\varepsilon_i) + 2^{-n-1} \\
&= \sum_{i=0}^{j-1}(2^{-i-1}\varepsilon_i) + \sum_{i=j+1}^{n}(2^{-i-1}\varepsilon_i) + 2^{n-1} \\
&\leq \sum_{i=0}^{j-1}(2^{-i-1}\varepsilon_i) + \sum_{i=j+1}^{n} 2^{-i-1} + 2^{n-1} \\
&= \sum_{i=0}^{j-1}(2^{-i-1}\varepsilon_i) + 2^{j-1} \\
&\leq \sum_{i=0}^{\infty}(2^{-i-1}x_i) = u,
\end{aligned}$$

contradiction.

Case 5. $\varepsilon_n = 1$, ε not the all 1 sequence, $u = t + 2^{-n-1}$. Let $x = \varepsilon^\frown\langle 1 : i \in \omega\rangle$. Then with j maximum such that $\varepsilon_j = 0$ we have

$$u = t + 2^{-n-1} = \sum_{i=0}^{j-1}(2^{-i-1}\varepsilon_i) + 2^{-j-1} = \sum_{i=0}^{\infty}(2^{-i-1}x_i).$$

This finishes the proof of (15).

(17) If $n \in \omega$, $\varepsilon \in {}^{n+1}2$, $t = \sum_{i=0}^{n}(2^{-i-1}\varepsilon_i)$, and $C = \{x \in {}^{\omega}2 : x \restriction (n+1) = \varepsilon\}$, then $\mu(\varphi[C]) = \lambda(C) = 2^{-n-1}$.

This is clear from (15).

(26) If $n \in \omega$, $\varepsilon \in {}^{n+1}2$, $t = \sum_{i=0}^{n}(2^{-i-1}\varepsilon_i)$, and $C = \{x \in {}^{\omega}2 : x \restriction (n+1) = \varepsilon\}$, then $\mu(\tilde{\varphi}[C]) = \lambda(C) = 2^{-n-1}$.

Recall that M is countable, and $N = \varphi[M] \cup \tilde{\varphi}[M]$ is countable. Clearly $\varphi[C] \backslash N = \tilde{\varphi}[C] \backslash N$. Hence

$$\begin{aligned}\lambda(C) = \mu(\varphi[C]) &= \mu(\varphi[C] \cap N) + \mu(\varphi[C] \backslash N)\\ &= \mu(\varphi[C] \backslash N) = \mu(\tilde{\varphi}[C] \backslash N)\\ &= \mu(\tilde{\varphi}[C] \backslash N) + \mu(\tilde{\varphi}[C] \cap N) = \mu(\tilde{\varphi}[C]).\end{aligned}$$

(19) If $F \in [\omega]^{<\omega}$, $h \in {}^{F}2$, and $C = \{x \in {}^{\omega}2 : h \subseteq x\}$, then $\mu(\tilde{\varphi}[C]) = \lambda(C)$.

In fact, choose $m \in \omega$ such that $F \subseteq m$. Then

$$C = \bigcup\{\{x \in {}^{\omega}2 : k \subseteq x\} : k \in {}^{m}2 \text{ and } h \subseteq k\}.$$

For each $k \in {}^{m}2$ such that $h \subseteq k$ let $D_k = \{x \in {}^{\omega}2 : k \subseteq x\}$. Note that $D_k \cap D_l = \emptyset$ when $k \neq l$. Let $I = \{k \in {}^{m}2 : h \subseteq k\}$ Note that $|\{k \in {}^{m}2 : h \subseteq k\}| = 2^{m-|F|}$. Now $\lambda(C) = 2^{-|F|}$ by Proposition 26.85 and by (26),

$$\mu(\tilde{\varphi}[C]) = \mu\left(\bigcup_{k \in I} \tilde{\varphi}[D_k]\right) = \sum_{k \in I} 2^{-m} = 2^{-m}2^{m-|F|} = 2^{-|F|}.$$

So (19) holds.

Now for each $X \subseteq [0,1]$ define

$$\mu^*(X) = \inf\{\mu(E) : X \subseteq E \in \Sigma_1\}.$$

(20) For every $X \subseteq [0,1]$ there is an $E \in \Sigma_1$ such that $X \subseteq E$ and $\mu^*(X) = \mu(E)$.

In fact, suppose that $X \subseteq [0,1]$. For each $n \in \omega$ choose $E_n \in \Sigma_1$ such that $X \subseteq E_n$ and $\mu(E) \leq \mu^*(X) + \frac{1}{2^n}$. Then $E \overset{\text{def}}{=} \bigcap_{n \in \omega} E_n \in \Sigma_1$, $X \subseteq E$, and

$$\mu^*(X) \leq \lambda(E) \leq \inf_{n \in \omega} \mu(E_n) \leq \mu^*(X),$$

proving (20).

(21) If $E \in \Sigma_0$, then $\mu^*(\tilde{\varphi}[E]) \leq \lambda(E)$ and there is a $V \in \Sigma_1$ such that $\tilde{\varphi}[E] \subseteq V$ and $\mu(V) \leq \lambda(E)$.

To prove (21), assume that $E \in \Sigma_0$. By the basic definition of measure on ${}^\omega 2$,

$$\lambda(E) = \inf\left\{\sum_{n\in\omega} \theta_0(U_{f_n}) : E \subseteq \bigcup_{n\in\omega} U_{f_n}\right\}.$$

(For θ_0 see before Proposition 26.29.) Hence for every $\varepsilon > 0$ there is a system $\langle f_n : n \in \omega\rangle$ such that $E \subseteq \bigcup_{n\in\omega} U_{f_n}$ and $\sum_{n\in\omega} \lambda(U_{f_n}) \le \lambda(E) + \varepsilon$. Hence

$$\tilde{\varphi}[E] \subseteq \bigcup_{n\in\omega} \tilde{\varphi}[U_{f_n}],$$

and hence, using (19),

$$\mu^*(\tilde{\varphi}[E]) \le \mu\left(\bigcup_{n\in\omega} \tilde{\varphi}[U_{f_n}]\right) \le \sum_{n\in\omega} \mu(\tilde{\varphi}[U_{f_n}]) = \sum_{n\in\omega} \lambda(U_{f_n}) \le \lambda(E) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $\mu^*(\tilde{\varphi}[E]) \le \lambda(E)$. By (20) there is a $V \in \Sigma_1$ such that $\tilde{\varphi}[E] \subseteq V$ and $\mu^*(\tilde{\varphi}[E]) = \mu(V)$. So (21) holds.

(22) If $E \in \Sigma_0$, then $\tilde{\varphi}[E] \in \Sigma_1$ and $\mu(\tilde{\varphi}[E]) = \lambda(E)$.

For, by symmetry with (21) there is a $V' \in \Sigma_1$ such that $\tilde{\varphi}[{}^\omega 2 \backslash E] \subseteq V'$ and $\mu(V') \le \lambda({}^\omega 2 \backslash E)$. Then $V \cup V' = [0,1]$ and

$$\mu(V) + \mu(V') \le \lambda(E) + \lambda({}^\omega 2\backslash E) = 1 \le \mu(V \cup V') \le \mu(V) + \mu(V').$$

So $\mu(V) + \mu(V') = \mu(V \cup V')$. Hence

$$\begin{aligned}\mu(V) + \mu(V') &= \mu(V\backslash V') + \mu(V \cap V') + \mu(V'\backslash V) + \mu(V \cap V')\\ &= \mu(V \cup V') + \mu(V \cap V') = \mu(V) + \mu(V') + \mu(V \cap V').\end{aligned}$$

It follows that $\mu(V \cap V') = 0$. In particular, $V \cap V' \cap \tilde{\varphi}[E] \in \Sigma_1$. Now $\tilde{\varphi}[{}^\omega 2\backslash E] = \tilde{\varphi}[{}^\omega 2]\backslash\tilde{\varphi}[E] = [0,1]\backslash\tilde{\varphi}[E] \subseteq V'$, so $[0,1]\backslash V' \subseteq \tilde{\varphi}[E]$. Hence $\tilde{\varphi}[E] = ([0,1]\backslash V') \cup (V' \cap \tilde{\varphi}[E]) = [0,1]\backslash V') \cup (V' \cap V \cap \tilde{\varphi}[E]) \in \Sigma_1$.

Now

$$\mu(\tilde{\varphi}[E]) \le \mu(V) \le \mu(E) \quad \text{and} \quad 1 - \mu(\tilde{\varphi}[E]) \le \mu(V') \le 1 - \mu(E),$$

so $\mu(\tilde{\varphi}[E]) = \mu(E)$. Thus (22) holds.

Now (14) gives (d) of Theorem 26.104 and $\Rightarrow$ of (b). (22) gives (c) and $\Rightarrow$ in (a). For $\Leftarrow$ of (a), suppose that $X \subseteq {}^\omega 2$ and $\tilde{\varphi}[X] \in \Sigma_1$. By $\Rightarrow$ of (b), $X = \tilde{\varphi}^{-1}[\tilde{\varphi}[X] \in \Sigma_0$. For $\Leftarrow$ of (b), suppose that $X \subseteq [0,1]$ and $\tilde{\varphi}^{-1}[X] \in \Sigma_0$. Then by (a), $X = \tilde{\varphi}[\tilde{\varphi}^{-1}[X]] \in \Sigma_0 \in \Sigma_1$. □

Lemma 26.105. *If $E \subseteq \mathscr{P}(\Sigma_0)$, then $\tilde{\varphi}\left[\bigcup E\right] = \bigcup_{A\in E} \tilde{\varphi}[A]$.* □

Proposition 26.106. $\mathrm{add}(\mathrm{null}_{^\omega 2}) = \mathrm{add}(\mathrm{null}_{[0,1]})$.

Proof. First let $\kappa = \mathrm{add}(\mathrm{null}_{^\omega 2})$, and let $E \in [\mathrm{null}_{^\omega 2}]^\kappa$ with $\bigcup E \notin \mathrm{null}_{^\omega 2}$. For each $A \in E$ let $A' = \tilde{\varphi}[A]$, and let $E' = \{A' : A \in E\}$. Then by Theorem 26.104(c), $E' \subseteq \mathscr{P}(\mathrm{null}_{[0,1]})$. Suppose that $\bigcup E' \in \mathrm{null}_{[0,1]}$. By Theorem 26.104(d),

$$\bigcup E = \bigcup_{A \in E} \tilde{\varphi}^{-1}[\tilde{\varphi}[A]] = \bigcup_{B \in E'} \tilde{\varphi}^{-1}[B] = \tilde{\varphi}^{-1}\left[\bigcup E'\right] \in \mathrm{null}_{^\omega 2},$$

contradiction.

Second let $\kappa = \mathrm{add}(\mathrm{null}_{[0,1]})$, and let $E \in [\mathrm{null}_{[0,1]}]^\kappa$ with $\bigcup E \notin \mathrm{null}_{[0,1]}$. For each $A \in E$ let $A' = \tilde{\varphi}^{-1}[A]$. Thus $A' \in \mathrm{null}_{^\omega 2}$ by Theorem 26.104(d). Continue as in the first case. $\square$

Proposition 26.107. $\mathrm{cov}(\mathrm{null}_{^\omega 2}) = \mathrm{cov}(\mathrm{null}_{[0,1]})$.

Proof. First let $\kappa = \mathrm{cov}(\mathrm{null}_{^\omega 2})$, and let $E \in [\mathrm{null}_{^\omega 2}]^\kappa$ with ${}^\omega 2 = \bigcup E$.

$$[0,1] = \tilde{\varphi}[{}^\omega 2] = \tilde{\varphi}\left[\bigcup E\right] = \bigcup_{A \in E} \tilde{\varphi}[A],$$

and each $\tilde{\varphi}[A] \in \mathrm{null}_{[0,1]}$.

The other direction is similar. $\square$

Proposition 26.108. $\mathrm{non}(\mathrm{null}_{^\omega 2}) = \mathrm{non}(\mathrm{null}_{[0,1]})$.

Proof. First let $\kappa = \mathrm{non}(\mathrm{null}_{^\omega 2})$, and let $X \in [{}^\omega 2]^\kappa$ such that $X \notin \mathrm{null}_{^\omega 2}$. If $\tilde{\varphi}[X] \in \mathrm{null}_{[0,1]}$, this is a contradiction.

The other direction is similar. $\square$

Proposition 26.109. $\mathrm{cof}(\mathrm{null}_{^\omega 2}) = \mathrm{cof}(\mathrm{null}_{[0,1]})$.

Proof. First let $\kappa = \mathrm{non}(\mathrm{null}_{^\omega 2})$, and let $X \in [{}^\omega 2]^\kappa$ such that $\forall A \in \mathrm{null}_{^\omega 2} \exists B \in X[A \subseteq B]$. Let $X' = \{\tilde{\varphi}[C] : C \in X\}$. Take any $A \in \mathrm{null}_{[0,1]}$. Then $\tilde{\varphi}^{-1}[A] \in \mathrm{null}_{^\omega 2}$, so there is a $B \in X$ such that $\tilde{\varphi}^{-1}[A] \subseteq B$. Then $\tilde{\varphi}[\tilde{\varphi}^{-1}[A]] = A \subseteq \tilde{\varphi}[B]$.

The other direction is similar. $\square$

Let $\mathscr{A} = \{X \cap \Theta : X \subseteq {}^\omega 2 \text{ is measurable}\}$. Then $\mathscr{A}$ is a σ-field of subsets of Θ. For any X measurable in ${}^\omega 2$ the set $X \cap \Theta$ is also measurable.

Proposition 26.110. $\mathrm{add}(\mathrm{null}_{^\omega 2}) = \mathrm{add}(\mathrm{null}_\Theta)$.

Proof. First suppose that $E \in [\mathrm{null}_{^\omega 2}]^\kappa$, $\bigcup E \notin \mathrm{null}_{^\omega 2}$, and $|E| = \mathrm{add}(\mathrm{null}_{^\omega 2})$. Let $E' = \{X \cap \Theta : X \in E\}$. Then $E' \subseteq \mathrm{null}_\Theta$. If $\bigcup E' \in \mathrm{null}_\Theta$, then $\bigcup E \subseteq \bigcup E' \cup N \in \mathrm{null}_{^\omega 2}$, contradiction, where $N = \{x \in {}^\omega 2 : \{i \in \omega : x(i) = 1\} \text{ is finite}\}$.

Second suppose that $E \in [\mathrm{null}_\Theta]^\kappa$, $\bigcup E \notin \mathrm{null}_\Theta$, and $|E| = \mathrm{add}(\mathrm{null}_\Theta)$. Then $E \subseteq \mathrm{null}_{^\omega 2}$ and $\bigcup E \notin \mathrm{null}_{^\omega 2}$. $\square$

Proposition 26.111. $\text{cov}(\text{null}_{^\omega 2}) = \text{cov}(\text{null}_\Theta)$.

Proof. First suppose that $E \in [\text{null}_{^\omega 2}]^\kappa$, ${}^\omega 2 = \bigcup E$, and $|E| = \text{cov}(\text{null}_{^\omega 2})$. Let $E' = \{X \cap \Theta : X \in E\}$. Then $\bigcup E' = \Theta$ and $E' \subseteq \text{null}_\Theta$.

Second suppose that $E \in [\text{null}_\Theta]^\kappa$, $\Theta = \bigcup E$, and $|E| = \text{cov}(\text{null}_\Theta)$. Then $E \subseteq \text{null}_{^\omega 2}$ and ${}^\omega 2 = \bigcup E \cup N$, with N as above. $\square$

Proposition 26.112. $\text{non}(\text{null}_{^\omega 2}) = \text{non}(\text{null}_\Theta)$.

Proof. First let $X \in [{}^\omega 2]^\kappa$ such that $X \notin \text{null}_{^\omega 2}$ and $\kappa = \text{non}(\text{null}_{^\omega 2})$. Then $X \cap \Theta \subseteq \Theta$ and $X \cap \Theta \notin \text{null}_\Theta$, as otherwise $X \subseteq (X \cap \Theta) \cup N \in \text{null}_{^\omega 2}$, with N as above.

Second let $X \in [\Theta]^\kappa$ such that $X \notin \text{null}_\Theta$ and $\kappa = \text{non}(\text{null}_\Theta)$. Then $X \notin \text{null}_{^\omega 2}$. $\square$

Proposition 26.113. $\text{cof}(\text{null}_{^\omega 2}) = \text{cof}(\text{null}_\Theta)$.

Proof. First suppose that $X \in [\text{null}_{^\omega 2}]^\kappa$, $\forall A \in \text{null}_{^\omega 2} \exists B \in X[A \subseteq B]$, and $\kappa = \text{cof}(\text{null}_{^\omega 2})$. Let $Y = \{B \cap \Theta : B \in X\}$. Thus $Y \subseteq \text{null}_\Theta$. Suppose that $A \in \text{null}_\Theta$. Then $A \in \text{null}_{^\omega 2}$, so there is a $B \in X$ such that $A \subseteq B$. Hence $A \subseteq B \cap \Theta \in Y$.

Second suppose that $X \in [\text{null}_\Theta]^\kappa$, $\forall A \in \text{null}_\Theta \exists B \in X[A \subseteq B]$, and $\kappa = \text{cof}(\text{null}_\Theta)$. Let $Y = \{B \cup N : B \in X\}$, with N as above. So $Y \subseteq \text{null}_{^\omega 2}$. Suppose that $A \in \text{null}_{^\omega 2}$. Then $A \cap \Theta \in \text{null}_\Theta$, so there is a $B \in X$ such that $A \cap \Theta \subseteq B$, so $A \subseteq B \cup N \in Y$. $\square$

There is a bijection f from Θ onto $[\omega]^\omega$. So the measure on Θ can be carried over to a measure on $[\omega]^\omega$.

29. More combinatorial set theory

• Suppose that ρ is a nonzero cardinal number, $\langle \lambda_\alpha : \alpha < \rho \rangle$ is a sequence of cardinals, and σ, κ are cardinals. We also assume that $1 \le \sigma \le \lambda_\alpha \le \kappa$ for all $\alpha < \rho$. Then we write

$$\kappa \to (\langle \lambda_\alpha : \alpha < \rho \rangle)^\sigma$$

provided that the following holds:

For every $f : [\kappa]^\sigma \to \rho$ *there exist* $\alpha < \rho$ *and* $\Gamma \in [\kappa]^{\lambda_\alpha}$ *such that* $f[[\Gamma]^\sigma]] \subseteq \{\alpha\}$.

In this case we say that Γ is *homogeneous for* f. The following colorful terminology is standard. We imagine that α is a color for each $\alpha < \rho$, and we color all of the σ-element subsets of κ. To say that Γ is homogeneous for f is to say that all of the σ-element subsets of Γ get the same color. Usually we will take σ and ρ to be a positive integers. If $\rho = 2$, we have only two colors, which are conventionally taken to be red (for 0) and blue (for 1). If $\sigma = 2$ we are dealing with ordinary graphs.

Note that if $\rho = 1$ then we are using only one color, and so the arrow relation obviously holds by taking $\Gamma = \kappa$. If κ is infinite and $\sigma = 1$ and ρ is a positive integer, then the relation holds no matter what σ is, since

$$\kappa = \bigcup_{i<\rho} \{\alpha < \kappa : f(\{\alpha\}) = i\},$$

and so there is some $i < \rho$ such that $|\{\alpha < \kappa : f(\{\alpha\}) = i\}| = \kappa \ge \lambda_i$, as desired.

The general infinite Ramsey theorem is as follows.

Theorem 29.1. (Ramsey) *If n and r are positive integers, then*

$$\omega \to (\underbrace{\omega, \ldots, \omega}_{r \text{ times}})^n.$$

Proof. We proceed by induction on n. The case $n = 1$ is trivial, as observed above. So assume that the theorem holds for $n \geq 1$, and now suppose that $f : [\omega]^{n+1} \to r$. For each $m \in \omega$ define $g_m : [\omega \backslash \{m\}]^n \to r$ by:

$$g_m(X) = f(X \cup \{m\}).$$

Then by the inductive hypothesis, for each $m \in \omega$ and each infinite $S \subseteq \omega$ there is an infinite $H_m^S \subseteq S \backslash \{m\}$ such that g_m is constant on $[H_m^S]^n$. We now construct by recursion two sequences $\langle S_i : i \in \omega \rangle$ and $\langle m_i : i \in \omega \rangle$. Each m_i will be in ω, and we will have $S_0 \supseteq S_1 \supseteq \cdots$. Let $S_0 = \omega$ and $m_0 = 0$. Suppose that S_i and m_i have been defined, with S_i an infinite subset of ω. We define

$$\begin{aligned} S_{i+1} &= H_{m_i}^{S_i} \quad \text{and} \\ m_{i+1} &= \text{the least element of } S_{i+1} \text{ greater than } m_i. \end{aligned}$$

Clearly $S_0 \supseteq S_1 \supseteq \cdots$ and $m_0 < m_1 < \cdots$. Moreover, $m_i \in S_i$ for all $i \in \omega$.

(1) For each $i \in \omega$, the function g_{m_i} is constant on $[\{m_j : j > i\}]^n$.

In fact, $\{m_j : j > i\} \subseteq S_{i+1}$ by the above, and so (1) is clear by the definition.

Let $p_i < r$ be the constant value of $g_{m_i} \restriction [\{m_j : j > i\}]^n$, for each $i \in \omega$. Hence

$$\omega = \bigcup_{j<r} \{i \in \omega : p_i = j\};$$

so there is a $j < r$ such that $K \stackrel{\text{def}}{=} \{i \in \omega : p_i = j\}$ is infinite. Let $L = \{m_i : i \in K\}$. We claim that $f[[L]^{n+1}] \subseteq \{j\}$, completing the inductive proof. For, take any $X \in [L]^{n+1}$; say $X = \{m_{i_0}, \ldots, m_{i_n}\}$ with $i_0 < \cdots < i_n$. Then

$$f(X) = g_{m_{i_0}}(\{m_{i_1}, \ldots, m_{i_n}\}) = p_{i_0} = j. \qquad \square$$

The finite version of Ramsey's theorem is as follows.

Theorem 29.2. (Ramsey) *Suppose that $n, r, l_0, \ldots, l_{r-1}$ are positive integers, with $n \leq l_i$ for each $i < r$. Then there is a $k \geq l_i$ for each $i < r$ and $k \geq n$ such that*

$$k \to (l_0, \ldots, l_{r-1})^n.$$

Proof. Assume the hypothesis, but suppose that the conclusion fails. Thus for every k such that $k \geq l_i$ for each $i < r$ with $k \geq n$ also, we have $k \not\to (l_0, \ldots, l_{r-1})^n$, which means that there is a function $f_k : [k]^n \to r$ such that for each $i < r$, there is no set $S \in [k]^{l_i}$ such that $f_k[[S]^n] \subseteq \{i\}$. We use these functions to define a certain $g : [\omega]^n \to r$ which will contradict the infinite version of Ramsey's theorem. Let $M = \{k \in \omega : k \geq l_i$ for each $i < r$ and $k \geq n\}$.

To define g, we define functions $h_i : [i]^n \to r$ by recursion. h_0 has to be the empty function. Now suppose that we have defined h_i so that $S_i \stackrel{\text{def}}{=} \{s \in M : f_s \restriction [i]^n = h_i\}$ is infinite. This is obviously true for $i = 0$. Then

$$S_i = \bigcup_{s:[i+1]^n \to r} \{k \in S_i : f_k \restriction [i+1]^n = s\},$$

and so there is a $h_{i+1} : [i+1]^n \to r$ such that $S_{i+1} \stackrel{\text{def}}{=} \{k \in S_i : f_k \restriction [i+1]^n = h_{i+1}\}$ is infinite, finishing the construction.

Clearly $h_i \subseteq h_{i+1}$ for all $i \in \omega$. Hence $g = \bigcup_{i\in\omega} h_i$ is a function mapping $[\omega]^n$ into r. By the infinite version of Ramsey's theorem choose $v < r$ and $Y \in [\omega]^\omega$ such that $g[[Y]^n] \subseteq \{v\}$. Take any $Z \in [Y]^{l_v}$. Choose i so that $Z \subseteq i$, and choose $k \in S_i$. Then for any $X \in [Z]^n$ we have

$$f_k(X) = h_i(X) = g(X) = v,$$

so Z is homogeneous for f_k, contradiction. $\square$

Hindman's Theorem

We prove Hindman's theorem, following Graham, Rothschild, Spencer. A *semigroup* is an algebraic structure $(A, \cdot)$ where $\cdot$ is associative. A *topological semigroup* is a semigroup $(A, \cdot)$ together with a Hausdorff topology on A under which $\cdot$ is continuous.

Theorem 29.3. *Let E be a semigroup with a topology which is compact. Define $\mathscr{R}_b(a) = a \cdot b$ for all $a, b \in E$. Assume that $\forall b \in E[\mathscr{R}_b$ is continuous$]$. Then there is an $e \in E$ such that $e^2 = e$.*

Proof. Let $\mathscr{A}$ be the set of all subsemigroups of E which are compact under the relativized topology. If $\mathscr{C} \subseteq \mathscr{A}$ is a chain, then $\bigcap \mathscr{C} \in \mathscr{A}$. Note that compact subspaces are closed, and hence $\bigcap \mathscr{C} \neq \emptyset$. By Zorn's lemma there is a minimal element A of $\mathscr{A}$. Fix $e \in A$. Then Ae is a subsemigroup, since $a_1ea_1e = (a_1ea_2)e \in Ae$ for $a_1, a_2 \in A$. Now $\mathscr{R}_e$ is a continuous mapping of A onto Ae, so Ae is compact. Now $Ae \subseteq A$, so $Ae = A$ by minimality. Let $B = \{f \in A : fe = e\}$. Since $Ae = A$, $B \neq \emptyset$. B is a subsemigroup, since if $f_1, f_2 \in B$ then $f_1f_2e = f_1e = e$. Now B is closed in A, and hence is compact. For, suppose that $f \in A\backslash B$. Then $f \in \mathscr{R}_f^{-1}[A\backslash\{e\}] \subseteq A\backslash B$. Hence $A = B$ by minimality. Since $e \in B$ it follows that $e^2 = e$. $\square$

For brevity let $\omega' = \omega\backslash\{0\}$. If $S \subseteq \omega'$, let $\Sigma(S)$ be the set of all finite sums of members of S.

Theorem 29.4. (Hindman) *If $m \in \omega$ and $f : \omega' \to m$, then there exist an $i < m$ and an infinite $S \subseteq \omega'$ such that $f(x) = i$ for all $x \in \Sigma(S)$.*

Proof. ${}^{\omega'}2$ is a compact Hausdorff space under the product topology. This transfers to a compact Hausdorff topology on $\mathscr{P}(\omega')$. Namely, let χ_A be the characteristic function of $A \subseteq \omega'$. Then χ is a bijection from $\mathscr{P}(\omega')$ onto ${}^{\omega'}2$, and we call $U \subseteq \mathscr{P}(\omega')$ open iff $\chi[U]$ is open. A basic open set in $\mathscr{P}(\omega')$ has the form U_{FG}, where F and G are finite disjoint subsets of $\mathscr{P}(\omega')$ and $U_{FG} = \{X \subseteq \mathscr{P}(\omega') : F \subseteq X \text{ and } X \cap G = \emptyset\}$.

Let $\mathscr{U}$ be the set of all ultrafilters on ω'.

(1) $\mathscr{U}$ is a closed subset of $\mathscr{P}(\omega')$.

In fact, suppose that $X \subseteq \mathscr{P}(\omega')$ and $X \notin \mathscr{U}$.

Case 1. $\omega' \notin X$. Let $G = \{\omega'\}$. Then $X \in U_{\emptyset G}$ and $U_{\emptyset G} \cap \mathscr{U} = \emptyset$.

Case 9. $a \in X$, $a \subseteq b$, $b \notin X$. Then $X \in U_{\{a\}\{b\}}$ and $U_{\{a\}\{b\}} \cap \mathscr{U} = \emptyset$.

Case 3. $a, b \in X$ but $a \cap b \notin X$. Then $X \in U_{\{a,b\}\{a\cap b\}}$ and $U_{\{a,b\}\{a\cap b\}} \cap \mathscr{U} = \emptyset$.

Case 4. $a, \omega' \backslash a \notin X$. Then $X \in U_{\emptyset\{a,\omega'\backslash a\}}$ and $U_{\emptyset\{a,\omega'\backslash a\}} \cap \mathscr{U} = \emptyset$.

Thus (1) holds.

Now for $F, G \in \mathscr{U}$ we define

$$F + G = \{A \subseteq \omega' : \{n : \{m : m + n \in A\} \in G\} \in F\}.$$

(2) $F + G$ is an ultrafilter.

In fact, for any n, $\{m : m + n \in \omega'\} = \omega' \in G$, and hence $\omega' \in F + G$. Now for any n, $\{m : m + n \in \emptyset\} = \emptyset \notin G$, so $\{n : \{m : m + n \in \emptyset\} \in G\} = \emptyset \notin F$. So $\emptyset \notin F + G$. Suppose that $A \in F + G$ and $A \subseteq B$. Then $H = \{n : \{m : m+n \in A\} \in G\} \in F$, and for $n \in H$ we have $\{m : m+n \in A\} \in G$, hence $\{m : m+n \in B\} \in G$, so $H \subseteq \{n : \{m : m+n \in B\} \in G\}$ and so $B \in F + G$. Now suppose that $A, A' \in F + G$. Then

$$\begin{aligned} F \ni &\{n : \{m : m + n \in A\} \in G\} \cap \{n : \{m : m + n \in A'\} \in G\} = \\ &\{n : \{m : m + n \in A \cap A'\} \in G\}, \end{aligned}$$

so $A \cap A' \in F + G$. Now suppose that $A \subseteq \omega'$ and $A \notin F + G$. Then $\{n : \{m : m + n \in A\} \in G\} \notin F\}$, so $\omega' \backslash \{n : \{m : m + n \in A\} \in G\} \in F$. Now $\omega' \backslash \{n : \{m : m + n \in A\} = \{n : \{m : m + n \in \omega' \backslash A\}\}$. Hence $\omega' \backslash A \in F + G$.

(3) $+$ is associative.

For, if $A \subseteq \omega'$ let $A_n = \{m : m + n \in A\}$. Then

$$\begin{aligned} F + (G + H) &= \{A \subseteq \omega' : \{n : \{m : m + n \in A\} \in G + H\} \in F\} \\ &= \{A \subseteq \omega' : \{n : A_n \in G + H\} \in F\} \\ &= \{A \subseteq \omega' : \{n : \{p : \{m : m + p \in A_n\} \in H\} \in G\} \in F\} \\ &= \{A \subseteq \omega' : \{n : \{p : \{m : m + p + n \in A\} \in H\} \in G\} \in F\} \end{aligned}$$

Also, let $B = \{n : \{m : m + n \in A\} \in H\}$. Then

$$\begin{aligned} (F + G) + H &= \{A \subseteq \omega' : \{n : \{m : m + n \in A\} \in H\} \in F + G\} \\ &= \{A \subseteq \omega' : B \in F + G\} \\ &= \{A \subseteq \omega' : \{n : \{p : p + n \in B\} \in G\} \in F\} \\ &= \{A \subseteq \omega' : \{n : \{p : \{m + p + q \in A\} \in H\} \in G\} \in F\}, \end{aligned}$$

which is the same as the above. So (3) holds.

(4) For each ultrafilter G define $\mathscr{R}_G : \mathscr{U} \to \mathscr{U}$ by $\mathscr{R}_G(F) = F+G$. Then $\mathscr{R}_G$ is continuous.

Let $\mathscr{S} = \{H \in \mathscr{U} : A \in H, A \subseteq \omega'\} \cup \{H \in \mathscr{U} : A \notin H, A \subseteq \omega'\}$. $\mathscr{S}$ is a subbase for the topology on $\mathscr{U}$, and it suffices to show that if $B \in \mathscr{S}$ and $F \in \mathscr{R}_G^{-1}[B]$ then there is an open set V such that $F \in V \subseteq \mathscr{R}_G^{-1}[B]$.

Case 1. $B = \{H \in \mathscr{U} : A \in H\}$. Thus $A \in F+G$, so $\{n : \{m : m+n \in A\} \in G\} \in F\}$. Then $F \in \{H \in \mathscr{U} : \{n : \{m : m+n \in A\} \in G\} \in H\}$, and if $K \in \{n : \{m : m+n \in A\} \in G\} \in H$ then $K + G \in B$, as desired.

Case 9. $B = \{H \in \mathscr{U} : A \notin H\}$. Thus $A \notin F+G$, so $\{n : \{m : m+n \in A\} \in G\} \notin F\}$. Then $F \in \{H \in \mathscr{U} : \{n : \{m : m+n \in A\} \in G\} \notin H\}$, and if $K \in \{n : \{m : m+n \in A\} \in G\} \notin H$ then $K + G \in B$, as desired.

So (4) holds.

Now by Theorem 29.3 there is an ultrafilter F such that $F + F = F$.

(5) For each $i \in \omega'$ let $K_i = \{A \subseteq \omega' : i \in A\}$. Then K_i is a principal ultrafilter, and $K_i + K_i = K_{2i} \neq K_i$.

In fact,

$$\begin{aligned} K_i + K_i &= \{A \subseteq \omega' : \{n : \{m : m + n \in A\} \in K_i\} \in K_i\} \\ &= \{A \subseteq \omega' : \{n : i + n \in A\} \in K_i\} \\ &= \{A \subseteq \omega' : 2i \in A\} = K_{2i}. \end{aligned}$$

So (5) holds.

Hence F is nonprincipal.

Now suppose that $f : \omega' \to m$. Then $\omega' = \bigcup_{i<m} f^{-1}[\{i\}]$, so there is an $i < m$ such that $A_0 \overset{\text{def}}{=} f^{-1}[\{i\}] \in F$. For each $B \subseteq \omega'$ and $n \in \omega'$ let $B - n = \{m : n + m \in B\}$ and $B^* = \{n : B - n \in F\}$.

(6) If $B \in F$, then $B^* \in F$ and so $B \cap B^* \in F$.

In fact, if $B \in F$, then $B \in F + F$, so $\{n : \{m : m + n \in B\} \in F\} \in F\}$, hence $\{n : B - n \in F\} \in F$, hence $B^* \in F$.

Now if $A_n \in F$ has been defined, pick $a_{n+1} \in A_n \cap A_n^*$. Since $a_{n+1} \in A_n^*$, we have $A_n - a_{n+1} \in F$ Let $A_{n+1} = (A_n \cap (A_n - a_{n+1}))\backslash\{a_{n+1}\}$. So $A_{n+1} \in F$.

(7) $a_{n+1} + A_{n+1} \subseteq A_n$.

In fact, if $m \in A_{n+1}$, then $m \in A_n - a_{n+1}$, so $a_{n+1} + m \in A_n$.

Now let $S = \{a_n : n \in \omega'\}$. We claim that $x \in A_0$, hence $f(x) = i$, for all $x \in \Sigma(S)$. Take any $x \in \Sigma(S)$. Say $x = a_{i_0} + \cdots + a_{i_m}$ with $0 < i_0 < \ldots < i_m$. We prove that $x \in A_0$ by induction on m. It is clear for $m = 0$, since each $a_i \in A_0$ because $A_0 \supseteq A_1 \supseteq \cdots$. Assume it for m and suppose that $x = a_{i_0} + \cdots + a_{i_{m+1}}$. Then $a_{i_0} + \cdots + a_{i_m} \in A_0$ and $a_{i_{m+1}} \in A_1$, so $x \in A_0$ by (7). $\square$

van der Waerden's Theorem

We prove van der Waerden's theorem, following Mauro de Nasso. A set $A \subseteq \omega'$ is *thick* iff $\forall m \in \omega \exists a \in \omega'[[a, a+m] \subseteq A]$. Recall that for any $A \subseteq \omega'$ and $n \in \omega$, $A - n = \{m : m + n \in A\} = \{p - n : p \in A, p > n\}$.

Proposition 29.5. *A is thick iff for every finite set $F \subseteq \omega'$ there is an $x \in \omega'$ such that $F + x \subseteq A$.*

Proof. $\Rightarrow$: Assume that A is thick and $F \in [\omega']^{<\omega}$. Say $F = \{b_0, \ldots, b_{m-1}\}$ with $b_0 < \cdots < b_{m-1}$. Choose $a \in \omega'$ such that $[a, a + b_{m-1}] \subseteq A$. Then $F + a = \{b_0 + a, \ldots, b_{m-1} + a\} \subseteq [a, a + b_{m-1}] \subseteq A$.

$\Leftarrow$: Assume the indicated condition, and suppose that $m \in \omega$. Now $[1, m]$ is a finite subset of ω', so there is an $x \in \omega'$ such that $[1, m] + x \subseteq A$. Thus $[x+1, x+m] \subseteq A$. $\square$

Proposition 29.6. *A is thick iff for every finite set $F \subseteq \omega'$ there is an $x \in A$ such that $F + x \subseteq A$.*

Proof. $\Rightarrow$: Assume that A is thick, and $F = \{b_0, \ldots, b_{m-1}\} \subseteq \omega'$ with $b_0 < \cdots < b_{m-1}$. Apply the condition in Proposition 29.5 to $[1, b_{m-1} + 1]$; this gives $x \in \omega'$ such that $1 + x, \ldots, b_{m-1} + 1 + x] \subseteq A$. Then $1 + x \in A$ and $\{b_0 + 1 + x, \ldots, b_{m-1} + 1 + x\} \subseteq A$.

$\Leftarrow$: see the proof of Proposition 29.5. $\square$

Proposition 29.7. *A is thick iff $\forall n_0, \ldots, n_{k-1} \in \omega[\bigcap_{i<k}(A - n_i) \neq \emptyset]$.*

Proof. $\Rightarrow$: Assume that A is thick, and $n_0, \ldots, n_{k-1} \in \omega$. Say $n_0 < \cdots < n_{k-1}$. Note that $A - 0 = A$. Hence we may assume that $0 < n_0$. Choose m so that $\{n_0 + m, \ldots, n_{k-1} + m\} \subseteq A$. Then $m \in \bigcap_{i<k}(A - n_i)$.

$\Leftarrow$: clear by reversing the above argument. $\square$

A is *syndetic* iff $\exists k \in \omega' \forall l[l, l + k - 1] \cap A \neq \emptyset]$.

Corollary 29.8. *A is thick, then so is $A - n$.* $\square$

Proposition 29.9. *A is syndetic iff there are $n_0, \ldots, n_{k-1} \in \omega$ such that $\omega' = \bigcup_{i<k}(A - n_i)$.*

Proof. $\Rightarrow$: Assume that A is syndetic. Choose $k \in \omega'$ such that $\forall l[l, l+k-1] \cap A \neq \emptyset]$. Let $n_0 = 0, n_1 = 1, \ldots, n_{k-1} = k - 1$. Suppose that $s \in \omega'$. Now $[s, s + k - 1] \cap A \neq \emptyset$. Choose $i < k$ with $s + i \in A$. Then $s \in (A - n_i)$.

$\Leftarrow$: Assume the indicated condition. Assume that $n_0 < \cdots < n_{k-1}$. Given l, choose $i < k$ such that $l \in (A - n_i)$. Then $n_i + l \in A$. So $[l, l + n_{k-1} + 1 - 1] \cap A \neq \emptyset$. $\square$

Proposition 29.10. *A is syndetic iff $A \cap B \neq \emptyset$ for every thick B.*

Proof. $\Rightarrow$: Assume that A is syndetic and B is thick. By Proposition 29.9 let $n_0, \ldots, n_{k-1} \in \omega$ be such that $\omega' = \bigcup_{i<k}(A - n_i)$. By Proposition 29.7 choose $b \in \bigcap_{i<k}(B - n_i)$. Choose $i < k$ such that $b + n_i \in A$. Also $b + n_i \in B$.

$\Leftarrow$: Suppose that A is not syndetic. By Proposition 29.9, for every finite subset F of ω' we have $\omega' \neq \bigcup_{x \in F}(A - x)$. Then

(1) For every finite subset F of ω' we have $\bigcap_{x\in F}((\omega'\backslash A)-x)\neq\emptyset$.

In fact, for F a finite subset of ω' choose $y\notin\bigcup_{x\in F}(A-x)$. Thus $\forall x\in F[x+y\in(\omega'\backslash A)]$, and hence $y\in((\omega'\backslash A)-x)$. So (1) holds.

By (1) and Proposition 29.7, $\omega'\backslash A$ is thick. This proves $\Leftarrow$. $\square$

Proposition 29.11. *A is syndetic iff $\omega'\backslash A$ is not thick.*

Proof. $\Rightarrow$: by Proposition 29.10. $\Leftarrow$: see the proof of Proposition 29.10, second part. $\square$

A is *piecewise syndetic* iff there exist a thick B and a syndetic C such that $A=B\cap C$.

Proposition 29.12. *The following are equivalent:*

(i) A is piecewise syndetic.

(ii) There is a finite $F\subseteq\omega'$ such that for every finite $G\subseteq\omega'$ there is an $s\in\omega'$ such that for every $t\in G$ there is an $x\in F$ such that $s+t+x\in A$.

(iii) There is a finite $F\subseteq\omega$ such that $\bigcup_{x\in F}(A-x)$ is thick.

Proof. (i)$\Rightarrow$(ii): Choose $k\in\omega'$ such that $\forall l[[l,l+k-1]\cap C\neq\emptyset]$, and let $F=[0,k-1]$. Suppose that $G\subseteq\omega'$ is finite. Let $H=\{t+i:t\in G,i<k\}$. So H is finite. By Proposition 29.5 choose $s\in\omega'$ such that $H+s\subseteq B$. Suppose that $t\in G$. Then $t\in G$, so $t+s\in B$. Choose $x\in F$ such that $t+s+x\in C$. Now $t+x\in H$, so $t+x+s\in B$. So $s+t+x\in A$.

(ii)$\Rightarrow$(iii): Assume (ii), and choose F as indicated. We claim that $\bigcup_{x\in F}(A-x)$ is thick. To prove this we use Proposition 29.5. Suppose that G is a finite subset of ω'. Choose s as in the indicated condition. Then we claim that $G+s\subseteq\bigcup_{x\in F}(A-x)$. For, take any $t\in G$. By the indicated condition there is an $x\in F$ such that $s+t+x\in A$, as desired.

(iii)$\Rightarrow$(i): Assume (iii). Let $F'=F\cup\{0\}$. Then $\bigcup_{x\in F'}(A-x)$ is thick, and $A\subseteq\bigcup_{x\in F'}(A-x)$. We claim that $A\cup(\omega'\backslash B)$ is syndetic; its intersection with B is A, as desired. Suppose that $A\cup(\omega'\backslash B)$ is not syndetic. Say $F=\{0,\ldots,k\}$. There is an l such that $[l,l+k+1]\cap(A\cup(\omega'\backslash B))=\emptyset$. So $[l,l+k]\cap A=\emptyset$ and $[l,l+k]\subseteq B$. Since $l\notin A$, there is an $m\in F'$ with $m\neq 0$ such that $l\in A-m$. So $l+m\in A$. But $l+m\leq l+k$, contradiction. $\square$

Corollary 29.13. *If A is piecewise syndetic, then so is $A-n$.* $\square$

Lemma 29.14. *Suppose that $\mathscr{S}\subseteq\mathscr{P}(\omega')$ is closed upwards. Let $\mathscr{T}=\{T\subseteq\omega':\forall S\in\mathscr{S}[T\cap S\neq\emptyset]\}$. Let $\mathscr{A}=\{S\cap T:S\in\mathscr{S},T\in\mathscr{T}\}$.*

Suppose that $A\in\mathscr{A}$ and $A=B\cup C$ with $B\cap C=\emptyset$. Then $B\in\mathscr{A}$ or $C\in\mathscr{A}$.

Proof. Say $A=S\cap T$ with $S\in\mathscr{S}$ and $T\in\mathscr{T}$. Let $\tilde{S}=B\cup(S\backslash A)$.

(1) $B=\tilde{S}\cap T$.

In fact, $\tilde{S}\cap T=(B\cap T)\cup((S\cap T)\backslash A)=B\cap T=B$ since $B\subseteq A\subseteq T$.

So if $\tilde{S}\in\mathscr{S}$ we have $B\in\mathscr{A}$, as desired.

Suppose that $\tilde{S}\notin\mathscr{S}$. Let $\tilde{T}=\omega'\backslash\tilde{S}$.

(2) $\tilde{T} \in \mathscr{T}$.

In fact, if $U \in \mathscr{S}$ and $U \cap \tilde{T} = \emptyset$, then $U \subseteq \tilde{S}$ and so $\tilde{S} \in \mathscr{S}$, contradiction.

(3) $C = \tilde{T} \cap S$.

For, $\tilde{T} \cap S = S \backslash \tilde{S} = S \backslash (B \cup (S \backslash A)) = (S \backslash B) \cap (S \cap A) = A \subseteq B = C$. ☐

Theorem 29.15. *If A is piecewise syndetic and $A = B \cup C$ with $B \cap C = \emptyset$, then B is piecewise syndetic or C is piecewise syndetic.*

Proof. Let $\mathscr{S}$ be the collection of all thick subsets of $\mathscr{P}(\omega')$ and $\mathscr{T}$ be the collection of all syndetic subsets of $\mathscr{P}(\omega')$. By Proposition 29.7, the hypotheses of Lemma 29.14 hold. ☐

A set $\mathscr{C} \subseteq \mathscr{P}(\omega')$ is *translation invariant* iff $\forall A \in \mathscr{C}[(A-1) \in \mathscr{C}]$.

Proposition 29.16. *If $\mathscr{C}$ is a collection of translation invariant set algebras $(A, \cup, \omega' \backslash)$ on ω', then $\bigcap \mathscr{C}$ is a translation invariant set algebra on ω'.* ☐

A filter F on a translation invariant set algebra A on ω' is *translation invariant* iff $\forall A \in F[(A-1) \in F]$. TIF abbreviates translation invariant filter.

Proposition 29.17. *A is thick iff $A \in F$ for some TIF F.*

Proof. $\Rightarrow$: Assume that A is thick. By Proposition 29.7, $\mathscr{A} \overset{\text{def}}{=} \{A - n : n \in \omega\}$ has fip. Let F be the filter generated by $\mathscr{A}$. So

$$F = \left\{ B : \exists G \in [\omega']^{<\omega} \left[\bigcap_{n \in G} (A - n) \subseteq B \right] \right\}.$$

Suppose that $p \in \omega$ and $B \in F$. Say $G \in [\omega']^{<\omega}$ and $\bigcap_{n \in G}(A-n) \subseteq B$. Let $H = G + p$. Suppose that $q \in \bigcap_{n \in H}(A-n)$. Thus $\forall n \in H[n+q \in A]$, so for all $n \in G[n+p+q \in A]$. Hence $\forall n \in G[p+q \in (A-n)]$, hence $p+q \in \bigcap_{n \in G}(A-n)$, hence $p+q \in B$, hence $q \in (B-p)$. So we have shown that $\bigcap_{n \in H}(A-n) \subseteq (B-p)$, and so $(B-p) \in F$. Thus $A \in F$ and F is a TIF.

$\Leftarrow$: Suppose that $A \in F$ with F a TIF. If $G \subseteq \omega$ is finite, then for all $n \in G$, $A-n \in F$. Hence $\bigcap_{n \in G}(A-n) \in F$, and so it is nonempty. By Proposition 29.7, A is thick. ☐

Proposition 29.26. *Every TIF is a subset of a maximal TIF.*

Proof. Zorn's lemma. ☐

Proposition 29.19. *If $F \subseteq \mathscr{P}(\omega')$ and $n \in \omega$, then $(\bigcap F) - n = \bigcap_{A \in F}(A - n)$.*

Proof.

$$\forall m\in\omega'\Big[m\in\Big(\Big(\bigcap F\Big)-n\Big) \quad \text{iff} \quad n+m\in\bigcap F$$
$$\text{iff} \quad \forall A\in F[n+m\in A]$$
$$\text{iff} \quad \forall A\in F[m\in A-n]$$
$$\text{iff} \quad m\in\bigcap_{A\in F}(A-n)\Big].$$ ☐

Proposition 29.20. *Suppose that $F\subseteq\mathscr{P}(\omega')$. Let $\mathscr{F}=\{(Y,n):Y\in F,\ n\in\omega\}$. Then the TIF generated by F is*

$$\left\{X:\exists G\in[\mathscr{F}]^{<\omega}\left[\bigcap_{(Y,n)\in G}(Y-n)\subseteq X\right]\right\}.$$

Proof. Let K be the indicated set. Note that $F\subseteq K$, since if $Y\in F$ then we can take $G=\{(y,0)\}$. Clearly K is closed upwards and is also closed under $\cap$. Now suppose that G is as indicated, and $m\in\omega$. Then by Proposition 29.19,

$$\left(\bigcap_{(Y,n)\in G}(Y-n)\right)-m=\bigcap_{(Y,n)\in G}((Y-n)-m)=\bigcap_{(Y,n)\in G}(Y-(m+n))$$
$$=\bigcap_{(Y,n)\in G'}(Y-n),$$

where $G'=\{(Y,n+m):(Y,n)\in G\}$. It follows that K is closed under $-$. ☐

Proposition 29.21. *Let B be a translation invariant field of subsets of ω', let M be a maximal TIF, and let U be an ultrafilter extending M. Then every $B\in U$ is piecewise syndetic.*

Proof. First we claim

(1) $\forall B\in U\exists F\in[\omega']^{<\omega}[\bigcup_{x\in F}(B-x)\in M]$.

In fact, let $\Lambda=\{(\omega'\backslash B)-n:n\in\omega\}$. Then $M\cup\Lambda$ does not have fip. For, suppose that it has fip. Then the translation invariant filter M' generated by it is proper. Otherwise, by Proposition 29.20 we get a finite subset F of ω and a $Y\in M$ such that $Y\cap\bigcap_{n\in F}((\omega'\backslash B)-n)=\emptyset$, contradiction. But $\omega'\backslash B\in M'$ while $\omega'\backslash B\notin M$, as otherwise $\omega'\backslash B\in U$, contradiction. So $M\subset M'$, contradicion. So, this proves that $M\cup\Lambda$ does not have fip. Hence there exist $Y\in M$ and a finite $F\subseteq\omega'$ such that $Y\cap\bigcap_{n\in F}((\omega'\backslash B)-n))=\emptyset$. Thus $Y\subseteq\bigcup_{n\in F}(\omega'\backslash((\omega'\backslash B)-n))$. Now $\omega'\backslash((\omega'\backslash B)-n)=\{m:m+n\notin(\omega'\backslash B)=\{m:m+n\in B\}=\{m:m\in(B-n)\}$, so $\bigcup_{n\in F}(B-n)\in M$. So (1) holds.

Now for any $B \in U$ we take F as in (1). By Proposition 29.17, $\bigcup_{x\in F}(B-x)$ is thick. By Proposition 29.12, B is piecewise syndetic. $\square$

Proposition 29.22. *Let B be a translation invariant field of subsets of ω', let M be a maximal TIF, and let U be an ultrafilter extending M. Then for every $B \in U$, the set $B_U = \{n \in \omega : (B-n) \in U\}$ is syndetic.*

Proof. By (1) in the proof of Proposition 29.21, there is an $F \in [\omega']^{<\omega}$ such that $\bigcup_{x\in F}(B-x) \in M$. Now the proposition follows by Proposition 29.9 from

(1) $\omega' = \bigcup_{x\in F}(B_U - x)$

For, by translation invariance of M, for every $m \in \omega$ we have $(\bigcap_{x\in F}(B-x)) - m = \bigcap_{x\in F}(B-x-m) \in M \in U$, so there is an $x \in F$ such that $(B-x-m) \in U$, so that $m \in (B_U - x)$. This proves (1). $\square$

Lemma 29.23. *Let A be a translation invariant field of subset of ω', let M be a maximal TIF contained in A, and let U be an ultrafilter on A extending M. Suppose that $B \subseteq \omega'$, $l \in \omega$ and $B - l \in U$. Then for every $k \in \omega'$, $B_U - l$ contains an arithmetic progression of length k.*

Proof. Induction on k. For $k = 1$, we just need to show that $B_U - l$ is nonempty. Now $\forall n \in \omega[n \in (B_U - l)$ iff $n + l \in B_U$ iff $B - l - n \in U$ iff $n \in (B-l)_U$. Since $(B-l)_U$ is syndetic by Proposition 29.22, it follows that $B_U - l$ is syndetic, and hence is nonempty.

Now we assume that $B_U - l$ contains an arithmetic progression of length k; we want to show that it contains one of length $k+1$. Let $l_0 = l$. Since $B_U - l_0$ is syndetic, by Proposition 29.8 there is a finite $F \subseteq \omega$ such that $\omega' = \bigcup_{x\in F}(B_U - l_0 - x)$. We may assume that $0 \in F$. Thus

(1) $\forall n \in \omega' \exists x \in F[l_0 + x + n \in B_U]$.

Now by the inductive hypothesis choose $l_1 \in \omega$ and $y_1 \in \omega'$ such that

(2) $l_1 + iy_1 \in (B_U - l)$ for $i = 1, \ldots, k$.

(If $k = 1$ take $l_1 = 0$ and y_1 any member of $B_U - l$.) Let $x_0 = 0$. Thus

(3) For all $i = 1, \ldots, k$, $l_0 + l_1 + x_0 + iy_1 \in B_U$.

By (1) choose $x_1 \in F$ so that $l_0 + l_1 + x_1 \in B_U$. Thus

(4) $B - (l_0 + l_1 + x_0 + iy_1) \in U$ for all $i = 1, \ldots, k$ and

(5) $B - (l_0 + l_1 + x_1) \in U$.

Case 1. $x_1 = 0$. Then

(6) $l_1 + iy_1 \in (B_U - l)$ for $i = 0, \ldots, k$ is an arithmetic progression of length $k+1$.

Case 9. $x_1 \neq 0$.

We now define sequences $B_1, B_2, \ldots$ and $x_1, x_2, \ldots \in F$ and $l_0, l_1, \ldots$ so that for each $s = 1, 2, \ldots$ we have

$$B_s = (B - x_s) \cap \bigcap_{i=1}^{k} (B - x_{s-1} - iy_s) \cap \bigcap_{i=1}^{k} (B - x_{s-2} - i(y_{s-1} + y_s))$$
$$\cap \ldots \cap \bigcap_{i=1}^{k} (B - x_0 - i(y_1 + y_2 + \cdots + y_s)) \qquad (*)$$

and

$$(**) \qquad (B_s - (l_0 + \cdots + l_s)) \in U.$$

Let

$$B_1 = (B - x_1) \cap \bigcap_{i=1}^{k} (B - x_0 - iy_1).$$

So $(*)$ and $(**)$ hold for $s = 1$. Assume that they hold for s. Then

$$B_s - (l_0 + \cdots + l_s) = (B - (l_0 + \cdots + l_s + x_s) \cap$$
$$\bigcap_{i=1}^{k} (B - (l_0 + \cdots + l_s + x_{s-1} + iy_s)$$
$$\cap \ldots \cap \bigcap_{i=1}^{k} (B - (x_0 + i(y_1 + \cdots + y_s) + l_0 + \cdots + l_s)$$

By the inductive hypothesis there are $l_{s+1} \in \omega$ and $y_{s+1} \in \omega'$ such that

(7) $l_{s+1} + iy_{s+1} \in (B_s)_U - (l_0 + \cdots + l_s)$ for $i = 1, \ldots, k$. Thus

(8) $(l_0 + \cdots + l_{s+1} + iy_{s+1}) \in (B_s)_U$ for $i = 1, \ldots, k$.

Hence

(9) $l_0 + \cdots + l_{s+1} + x_{s-t} + i(y_{s-t+1} + \cdots + y_{s+1}) \in B_U$ for $i = 1, \ldots, k$ and $0 \leq t \leq s$.

By (1) choose $x_{s+1} \in F$ so that $l_0 + \cdots + l_{s+1} + x_{s+1} \in B_U$.

Subcase 9.1. $x_{s+1} = x_0$. Then $x_0 + l_0 + \cdots + l_{s+1} + i(y_1 + \cdots + y_s)$ is a $(k+1)$-ary arithmetic progression in $B_U - l$ for $i = 0, \ldots, k$.

Subcase 9.9. $x_{s+1} = x_j$ for some $j = 1, \ldots, s$. Then we have $l_0 + l_1 + \cdots + l_{s+1} + x_j + i(y_j + \cdots + y_{s+1}) \in B_U$ for $i = 0, \ldots, k$.

it Subcase 9.3. $x_{s+1} \neq x_j$ for all $j \leq s$. Then the construction continues.

Since F is finite, the construction eventually stops. $\square$

Theorem 29.24. *Suppose that A is a piecewise syndetic set. Then for every $k \in \omega'$, $\{x \in A : \exists y \in \omega' \forall i = 1, \ldots, k[x + iy \in A]\}$ is piecewise syndetic.*

Proof. Let B be the translation invariant field of subsets of ω' generated by $\{A-n : n \in \omega\}$. By Proposition 29.12 there is a finite subset F of ω such that $T \stackrel{\text{def}}{=} \bigcup_{n\in F}(A-n)$ is thick. By Proposition 29.7, $\mathscr{G} \stackrel{\text{def}}{=} \{T-m : m \in \omega\}$ has fip and so it is contained in a maximal TIF M on B. Let U be an ultrafilter on B with $M \subseteq U$. Now $T \in \mathscr{G} \subseteq M \subseteq U$, so there is an $n \in F$ such that $(A-n) \in U$. By Lemma 29.23, $A_U - n$ has an arithmetic progression of every length $k \in \omega$. Now for each $k \in \omega$ choose x and y so that $x+iy \in (A_U - n)$ for all $i = 1, \ldots, k$. Thus $C \stackrel{\text{def}}{=} \bigcap_{i=1}^{k}(A-(n+x+iy)) \in U$.

(1) $C \subseteq \{z : \exists y \in \omega' \forall i = 1, \ldots, k[z+iy \in A]\} - n - x$.

In fact,

$$\begin{aligned}
&\{z : \exists y \in \omega' \forall i = 1, \ldots, k[z+iy \in A]\} - n - x \\
&= \{m : m+n+x \in \{z : \exists y \in \omega' \forall i = 1, \ldots, k[z+iy \in A]\}\} \\
&= \{m : \exists y \in \omega' \forall i = 1, \ldots, k[m+n+x+iy \in A]\}\}
\end{aligned}$$

and

$$C = \{m : \forall i = 1, \ldots, k[m+n+x+iy \in A\},$$

so (1) holds.

Now by Proposition 29.9 and Corollary 29.13 the theorem follows. $\square$

Theorem 29.23. *If $\omega' = C_1 \cup \ldots \cup C_n$ is a partition of ω', then there is an i such that for every $k \in \omega'$, $\{x \in C_i : \exists y \in \omega' : \forall i = 1, \ldots, k[x+iy \in C_i]\}$ is piecewise syndetic.*

Proof. Clearly, for example by Proposition 29.10, ω' is piecewise syndetic. By Theorem 29.15 there is an i such that C_i is piecewise syndetic. Now apply Theorem 29.29. $\square$

Theorem 29.24. (van der Waerden) If $\omega' = C_1 \cup \ldots \cup C_n$ is a partition of ω', then there is an i such that for every $k \in \omega'$, C_i has an arithmetic progression of length k. $\square$

The Hales-Jewitt Theorem

We prove the Hales-Jewitt theorem, following the proof in the book of Stasys Jukna, which is based on a sketch of Alon Nilli, which in turn is a simplified version of Shelah's proof.

For t a positive integer let $[t] = \{1, \ldots, t\}$. With t, n, r positive integers, we are going to consider colorings of ${}^n[t]$ with r colors, i.e., functions $f : {}^n[t] \to r$. A (t,n)-*root* is a member of ${}^n([t] \cup \{*\})$ taking the value $*$ at least once. For τ a (t,n)-root and $i < t$, we denote by $\tau(i)$ the result of replacing all $*$ in τ by i. A subset $L \subseteq {}^n[t]$ is a *line* iff there is a (t,n)-root τ such that $L = \{\tau(1), \ldots, \tau(t)\}$. For example, the following is a line in ${}^3[4]$, given by the root $(*, 1, *)$:

$(1,1,1)$
$(2,1,2)$
$(3,1,3)$
$(4,1,4)$
$(5,1,5)$

Theorem 29.25. (Hales-Jewitt) *For all positive integers t and r there is a positive integer $n \stackrel{\text{def}}{=} HJ(r,t)$ such every coloring of ${}^n[t]$ with r colors has a monochromatic line.*

Proof. We go by induction on t. For $t = 1$ the only line is $\{(1, 1, \ldots, 1)\}$ and the desired conclusion is obvious: $HJ(r, 1) = 1$.

Now suppose the result is true for $t - 1 \geq 1$. Let

$$\begin{aligned} n &= HJ(r, t-1); \\ N_i &= r^{t^{n+\sum_{j=1}^{i-1} N_j}} \quad \text{for } i = 1, \ldots, n; \\ N &= N_1 + \cdots + N_n. \end{aligned}$$

We suppose that $\chi : {}^N[t] \to r$. If $\tau = \tau_1 \ldots \tau_n$ is a sequence of n roots, with each τ_i of length N_i, and $a \in {}^n[t]$, we define

$$\tau(a) = \tau_1(a_1) \ldots \tau_n(a_n).$$

Thus $\tau(a) \in {}^N[t]$.

Two members $a, b \in {}^n[t]$ are *neighbors* iff they differ at exactly one place, where one of them has value 1 and the other has value 9.

(1) There is a sequence $\tau = \tau_1 \ldots \tau_n$ of roots, with each τ_i of length N_i, such that $\chi(\tau(a)) = \chi(\tau(b))$ for any two neighbors a, b.

We define τ_i by downward induction on i. First we take the case $i = n$. Let $L_{n-1} = N_1 + \cdots + N_{n-1}$. For $k = 0, \ldots, N_n$ let W_k be the following member of ${}^{N_n}[t]$:

$$W_k = \underbrace{1 \ldots 1}_{k} \underbrace{2 \ldots 2}_{N_n - k}$$

For each $k = 0, \ldots, N_n$ we define a coloring $\chi_k : {}^{L_{n-1}}[t] \to r$ by

$$\chi_k(x_1, \ldots, x_{L_{n-1}}) = \chi(x_1, \ldots, x_{L_{n-1}} W_k).$$

Now the number of colorings of ${}^{L_{n-1}}[t]$ is

$$r^{t^{L_{n-1}}} < N_n,$$

and we have $N_n + 1$ colorings. So there exist $s < k \leq N_n$ such that $\chi_s = \chi_k$. We then define

$$\tau_n = \underbrace{1 \ldots 1}_{s} \underbrace{* \ldots *}_{k-s} \underbrace{2 \ldots 2}_{N_n - k}$$

The inductive step from τ_{i+1} to τ_i is similar. Let $L_{i-1} = N_1 + \cdots + N_{i-1}$. For $k = 0, \ldots, N_i$ let W_k be the following member of ${}^{N_i}[t]$:

$$W_k = \underbrace{1 \ldots 1}_{k} \underbrace{2 \ldots 2}_{N_i - k}$$

For each $k = 0, \ldots, N_i$ we define a coloring $\chi_k : {}^{L_{i-1}+n-i}[t] \to r$ by

$$\chi_k(x_1, \ldots, x_{L_{i-1}}, y_{i+1}, \ldots, y_n) = \chi(x_1, \ldots, x_{L_{i-1}} W_k, \tau_{i+1}(y_{i+1}), \ldots, \tau_n(y_n)).$$

Now the number of colorings of ${}^{L_{i-1}+n-i}[t]$ is

$$r^{t^{L_{n-1}+n-i}} \leq N_i,$$

and we have $N_i + 1$ colorings. So there exist $s < k \leq N_n$ such that $\chi_s = \chi_k$. We then define

$$\tau_i = \underbrace{1 \ldots 1}_{s} \underbrace{* \ldots *}_{k-s} \underbrace{2 \ldots 2}_{N_i - k}$$

Now to check that this works, suppose that a and b are neighbors in the i-th place. Say $a_i = 1$ and $b_i = 2$.

Case 1. $i = n$. Thus

$$\begin{aligned} a &= a_0, \ldots, a_{n-2}, 1; \\ b &= a_0, \ldots, a_{n-2}, 2 \end{aligned}$$

Then

$$\begin{aligned} \tau(a) &= \tau_1(a_0) \ldots \tau_{n-1}(a_{n-2}) \tau_n(1); \\ \tau(b) &= \tau_1(a_0) \ldots \tau_{n-1}(a_{n-2}) \tau_n(2). \end{aligned}$$

Now

$$\tau_n(1) = \underbrace{1 \ldots 1}_{k} \underbrace{2 \ldots 2}_{N_n - k} = W_k \quad \text{and} \quad \tau_n(2) = \underbrace{1 \ldots 1}_{s} \underbrace{2 \ldots 2}_{N_n - s} = W_s.$$

Hence

$$\begin{aligned} \chi(\tau(a)) &= \chi(\tau_1(a_0) \ldots \tau_{n-1}(a_{n-2}) \tau_n(1)) = \chi(\tau_1(a_0) \ldots \tau_{n-1}(a_{n-2}) W_k) \\ &= \chi_k(\tau_1(a_0) \ldots \tau_{n-1}(a_{n-2})) = \chi_s(\tau_1(a_0) \ldots \tau_{n-1}(a_{n-2})) \\ &= \chi(\tau_1(a_0) \ldots \tau_{n-1}(a_{n-2}) W_s) = \chi(\tau(b)). \end{aligned}$$

Case 9. $i < n$. Similar to Case 1.

Now to prove the theorem, let τ be as in (1). Define $\chi' : {}^n\{2, \ldots, t\} \to r$ by defining $\chi'(a) = \chi(\tau(a))$ for any $a \in {}^n\{2, \ldots, t\}$. By the inductive hypothesis there is a root $\nu \in {}^n(\{2, \ldots, t\} \cup \{*\})$ such that $\{\nu(2), \ldots, \nu(t)\}$ is monochromatic under χ'. Now the string $\rho \stackrel{\text{def}}{=} \tau_1(\nu_0) \cdots \tau_n(\nu_{n-1})$ has length N and it is a root since ν is. We claim that

$$M \stackrel{\text{def}}{=} \{\rho(1), \ldots, \rho(t)\}$$

is a monochromatic line under χ. Note that for any $i = 1, \ldots, t$, $\rho(i) = \tau(\nu(i))$. Now $\chi'(\nu(2)) = \cdots = \chi'(\nu(t))$; hence $\chi(\tau(\nu(2))) = \cdots = \chi(\tau(\nu(t)))$. We claim that also

$\chi(\tau(\nu(1))) = \chi(\tau(\nu(2)))$. Clearly there are members $\sigma^1, \ldots, \sigma^s$ of ${}^n[t]$ such that $\nu(1) = \sigma^1$, $\nu(2) = \sigma^s$, and successive members of $\sigma^1, \ldots, \sigma^s$ are neighbors. Hence by (1), $\chi(\tau(\nu(1))) = \chi(\tau(\nu(2)))$. $\square$

The Halpern-Läuchli Theorem

We expand the origonal proof. We deal with trees of height ω, finitely branching, with a unique root, and with no maximal nodes. A set S of nodes is (h,k)-*dense* iff there is a node x of height h such that S dominates the nodes of height $h+k$ which are above x. k-*dense* means $(0,k)$-dense, and ∞-*dense* means k-dense for all k.

Proposition 1. *S is k-dense iff S dominates the nodes of height k.* $\square$

Proposition 2. *S is ∞-dense iff S dominates all nodes of T.* $\square$

We define $T \uparrow t = \{s : t \leq s\}$. For each $n \in \omega$, $n(T) = \{T \uparrow x : |x| = n\}$. For $B \subseteq T$, $n(T,B) = \{(T \uparrow t) \cap B : |t| = n\}$. If $\mathbf{T} = (T_1, \ldots, T_d)$ is a system of trees, then an (h,k)-*matrix* for $\mathbf{T}$ is a product $\prod_{i=1}^d A_i$ with each A_i (h,k)-dense in T_i. A k-*matrix* is a $(0,k)$-matrix.

Theorem 29.28. (Halpern-Läuchli) *Let $\mathbf{T} = (T_1, \ldots, T_d)$ be a system of trees, each finitely branching, with a single root, and of height ω. Suppose that $Q \subseteq \prod_{i=1}^d T_i$. Then one of the following conditions holds:*

(i) For all $k \in \omega$ there is a k-matrix contained in Q.

(ii) There is an $h \in \omega$ such that for each k there is an (h,k)-matrix contained in $(\prod_{i=1}^d T_i)\backslash Q$.

Proof. We first introduce a certain algebra of symbols. *Atomic symbols* are

$$\exists A_i, \forall x_i, \forall a_i, \exists x_i \quad \text{for each positive integer } i.$$

For each positive integer d we define

$L_d = \{\sigma : \sigma$ is a function with domain $\{1, \ldots, 2d\}$, and for each $i \in \{1, \ldots, d\}$ exactly one of the following holds:

(i) Each of $\exists A_i$ and $\forall x_i$ occurs exactly once in σ, with $\exists A_i$ before $\forall x_i$.

(ii) Each of $\forall a_i$ and $\exists x_i$ occurs exactly once in σ, with $\forall a_i$ before $\exists x_i$.

Examples:

$L_1 = \{\langle \exists A_1, \forall x_1\rangle, \langle \forall a_1 \exists x_1\rangle\}$.
$L_2 = \{\langle \exists A_1, \forall x_1, \exists A_2, \forall x_2\rangle, \langle \exists A_1, \exists A_2, \forall x_1, \forall x_2\rangle, \ldots\}$.

Now we define a relation $\vdash_d$ on L_d. α, β stand for A_i, a_i, x_i and U and V are strings of atomic symbols each of length $d-1$.

Rules 1.

$U\exists\alpha\exists\beta V \vdash_d U\exists\beta\exists\alpha V$, if $U\exists\alpha\exists\beta V, U\exists\beta\exists\alpha V \in L_d$.

$U\forall\alpha\forall\beta V \vdash_d U\forall\beta\forall\alpha V$, if $U\forall\alpha\forall\beta V, U\forall\beta\forall\alpha V \in L_d$.
$U\exists\alpha\forall\beta V \vdash_d U\forall\beta\exists\alpha V$, if $U\exists\alpha\forall\beta V, U\forall\beta\exists\alpha V \in L_d$,

Rules 2.
$U\forall a_i\exists x_i V \vdash_d U\exists A_i\forall x_i V$ for all $i = 1, \ldots, d$ such that $U\forall a_i\exists x_i V, U\exists A_i\forall x_i V \in L_d$.
$U\exists A_i\forall x_i V \vdash U\forall a_i\exists x_i V$ for all $i = 1, \ldots, d$ such that $U\forall a_i\exists x_i V, U\exists A_i\forall x_i V \in L_d$.

To state rules 3, we first define, if $\langle V_i : r \le i \le k\rangle$ is a sequence of strings of atomis symbols, then $(V_i)_r^k$ is the concatenation $V_r \cdots V_k$.

Rules 3.
If σ is a permutation of $\{1, \ldots, d\}$, then
$(\forall a_{\sigma(i)})_1^r(\exists A_{\sigma(i)})_{r+1}^d V \vdash_d (\exists A_{\sigma(i)})_{r+1}^d(\forall a_{\sigma(i)})_1^r V$ for $r = 1 \ldots d-1$.

Example

$$d = 4, \quad r = 2, \quad V = \exists x_3\forall x_1\forall x_4\exists x_2, \quad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$$

gives

$$\forall a_2\forall a_3\exists A_1\exists A_4\exists x_3\forall x_1\forall x_4\exists x_2 \vdash_d \exists A_1\exists A_4\forall a_2\forall a_3\exists x_3\forall x_1\forall x_4\exists x_9.$$

$\models_d$ is the transitive closure of $\vdash_d$.

(1) $\forall a_d(\exists A_i)_1^{d-1}(\forall x_i)_1^{d-1}\exists x_d \models_d \exists A_d(\forall a_i)_1^{d-1}(\exists x_i)_1^{d-1}\forall x_d$.

Proof of (1): Let $\sigma(1) = d$, $\sigma(i+1) = i$ for $i = 2, \ldots, d-1$, $r = 1$. Then an instance of rules 3 is

$$(\forall a_{\sigma(i)})_1^1(\exists A_{\sigma(i)})_2^d(\forall x_i)_1^{d-1}\exists x_d \models_d (\exists A_{\alpha(i)})_2^d(\forall a_{\sigma(i)})_1^1(\forall x_i)_1^{d-1}\exists x_d,$$

or

$$\forall a_d(\exists A_i)_1^{d-1}(\forall x_i)_1^{d-1}\exists x_d \models_d (\exists A_i)_1^{d-1}\forall a_d(\forall x_i)_1^{d-1}\exists x_d \tag{1a}$$

By Rules 1,

$$(\exists A_i)_1^{d-1}\forall a_d(\forall x_i)_1^{d-1}\exists x_d \models_d (\exists A_i)_1^{d-1}(\forall x_i)_1^{d-1}\forall a_d\exists x_d \tag{1b}$$

Again using rules 1,

$$\begin{aligned}(\exists A_i)_1^{d-1}(\forall x_i)_1^{d-1}\forall a_d\exists x_d &\models_d \exists A_1\forall x_1(\exists A_i)_2^{d-1}(\forall x_i)_2^{d-1}\forall a_d\exists x_d \\ &\cdots \\ &\models_d (\exists A_i\forall x_i)_1^{d-1}\forall a_d\exists x_d \qquad (1c)\end{aligned}$$

By rules 2,

$$(\exists A_i\forall x_i)_1^{d-1}\forall a_d\exists x_d \models_d (\forall a_i\exists x_i)_1^{d-1}\exists A_d\forall x_d \tag{1d}$$

By Rules 1,

$$\begin{aligned}
(\forall a_i \exists x_i)_1^{d-1} \exists A_d \forall x_d &\models_d (\forall a_i \exists x_i)_1^{d-2} \forall a_{d-1} \exists x_{d-1} \exists A_d \forall x_d \\
&\models_d (\forall a_i \exists x_i)_1^{d-3} \forall a_{d-2} \exists x_{d-2} \forall a_{d-1} \exists x_{d-1} \exists A_d \forall x_d \\
&\models_d (\forall a_i \exists x_i)_1^{d-3} \forall a_{d-2} \exists x_{d-2} \forall a_{d-1} \exists A_d \exists x_{d-1} \forall x_d \\
&\models_d (\forall a_i \exists x_i)_1^{d-3} \forall a_{d-2} \forall a_{d-1} \exists x_{d-2} \exists A_d \exists x_{d-1} \forall x_d \\
&\models_d (\forall a_i \exists x_i)_1^{d-3} \forall a_{d-2} \forall a_{d-1} \exists A_d \exists x_{d-2} \exists x_{d-1} \forall x_d \\
&\cdots \\
&\models_d (\forall a_i)_1^{d-1} \exists A_d (\exists x_i)_1^{d-1} \forall x_d \qquad (1e)
\end{aligned}$$

Now with σ the identity and $r = d-1$, rules 3 give

$$(1f) \qquad (\forall a_i)_1^{d-1} \exists A_d (\exists x_i)_1^{d-1} \forall x_d \models_d \exists A_d (\forall a_i)_1^{d-1} (\exists x_i)_1^{d-1} \forall x_d$$

Now (1a)–(1f) give (1).

(2) Suppose that $UV \in L_d$, U has length d, no atoms of the forms $\forall x_i, \exists x_i$ occur in U, and $\bar{U}$ is any rearrangement of U. Then $UV \models_d \bar{U}V$.

In fact, assume the hypotheses. So only $\exists A_i$ and $\forall a_i$ occur in U. If no $\forall a_i$ occurs, or no $\exists A_i$ occurs, the conclusion is clear by rules 1. So suppose some $\forall a_i$ occurs and some $\exists A_i$ occurs. By rules 1 there is a permutation σ of $\{1, \ldots, d\}$ such that

$$UV \models_d (\forall a_{\sigma(i)})_1^r (\exists A_{\sigma(i)})_{r+1}^d V$$

By rules 3,

$$(\forall a_{\sigma(i)})_1^r (\exists A_{\sigma(i)})_{r+1}^d V \models_d (\exists A_{\sigma(i)})_{r+1}^d (\forall a_{\sigma(i)})_1^r V$$

Then by rules 1,

$$(\forall a_{\sigma(i)})_1^r (\exists A_{\sigma(i)})_{r+1}^d V \models_d \bar{U}V.$$

Thus (2) holds.

(3) If $W \models_{d-1} \bar{W}$, then $\forall a_d W \exists x_d \models_d \forall a_d \bar{W} \exists x_d$.

For, assume that $W \models_{d-1} \bar{W}$. Say

$$W = S_0 \vdash_{d-1} S_1 \vdash_{d-1} S_2 \cdots \vdash_{d-1} S_n = \bar{W}.$$

We claim that

$$\forall a_d W \exists x_d = \forall a_d S_0 \exists x_d \vdash_d \forall a_d S_1 \exists x_d \cdots \vdash_d \forall a_d S_n \exists x_d = \forall a_d \bar{W} \exists x_d.$$

Consider the step from S_i to S_{i+1}. If rules (1) or rules (2) are used in going from S_i to S_{i+1}, clearly the same rules go from $\forall a_d S_i \exists x_d$ to $\forall a_d S_{i+1} \exists x_d$. Suppose that rules (3) are used. Say S_i is $(a_{\sigma(i)})_1^r (\exists A_{\sigma(i)})_{r+1}^{d-1} V$ and S_{i+1} is $(\exists A_{\sigma(i)})_{r+1}^{d-1} (\forall a_{\sigma(i)})_1^r V$. Then $\forall a_d S_i \exists x_d$

is $\forall a_d (a_{\sigma(i)})_1^r (\exists A_{\sigma(i)})_{r+1}^{d-1} V \exists x_d$ and $\forall a_d S_{i+1} \exists x_d$ is $\forall a_d (\exists A_{\sigma(i)})_{r+1}^{d-1} (\forall a_{\sigma(i)})_1^r V \exists x_d$. Hence $\forall a_d S_i \exists x_d \models_d \forall a_d S_{i+1} \exists x_d$ by (2). Hence (3) holds.

(4) $(\forall a_i)_1^d (\exists x_i)_1^d \models_d (\exists A_i)_1^d (\forall x_i)_1^d$.

In fact, we prove this by induction on d. For $d = 1$ the assertion is that $\forall a_1 \exists x_1 \models_d \exists A_1 \forall x_1$, which is an instance of rules 2. Now assume (4) for $d - 1 \geq 1$. Then rules 1 give

$$(4a) \qquad (\forall a_i)_1^d (\exists x_i)_1^d \models_d \forall a_d (\forall a_i)_1^{d-1} (\exists x_i)_1^{d-1} \exists x_d$$

By the inductive hypothesis and (3) we have

$$(4b) \qquad \forall a_d (\forall a_i)_1^{d-1} (\exists x_i)_1^{d-1} \exists x_d \models_d \forall a_d (\exists A_i)_1^{d-1} \forall x_i)_1^{d-1} \exists x_d$$

By (1) we have

$$(4c) \qquad \forall a_d (\exists A_i)_1^{d-1} (\forall x_i)_1^{d-1} \exists x_d \models_d \exists A_d (\forall a_i)_1^{d-1} (\exists x_i)_1^{d-1} \forall x_d$$

By the inductive hypothesis and (3) we have

$$(4d) \qquad \exists A_d (\forall a_i)_1^{d-1} (\exists x_i)_1^{d-1} \forall x_d \models_d \exists A_d (\exists A_i)_1^{d-1} (\forall x_i)_1^{d-1} \forall x_d$$

Now by rules (1) we get

$$(4e) \qquad \exists A_d (\exists A_i)_1^{d-1} (\forall x_i)_1^{d-1} \forall x_d \models_d (\exists A_i)_1^d (\forall x_i)_1^d$$

Now (4a)–(4e) give (4).

Now suppose that $\mathbf{T} = \langle T_i : 1 \leq i \leq d \rangle$ is a vector tree and $Q \subseteq \prod_{i=1}^d T_i$. We define a $(d+1)$-sorted language $\mathscr{L}$. The sorts are $S_1, \ldots, S_{d+1}$. Additional constants are as follows.

A d-ary function symbol Seq acting on d-tuples from $S_1 \times \cdots \times S_d$ with values in S_{d+1}.
For each $i = 1, \ldots, d$, a binary relation symbol $<_i$ acting on S_i.
$x_1, \ldots, x_d$ are variables ranging over $S_1, \ldots, S_d$ respectively.
$B_1, \ldots, B_d$ are constants for subsets of $S_1, \ldots, S_d$ respectively.
$A_1, \ldots, A_d$ are variables ranging over subsets of $S_1, \ldots, S_d$ respectively.
v_{ik} for $i = 1, \ldots, d$ and $k \in \omega$ are variables ranging over S_i,
$a_1, \ldots, a_d$ are variables ranging over subsets of $S_1, \ldots, S_d$ respectively.
Q, a constant for a subset of S_{d+1}

A structure for this language assigns T_i to S_i for $i = 1, \ldots, d$, the product $\prod_{i=1}^d T_i$ to S_{d+1}, and subsets B_i of T_1 for $i = 1, \ldots, d$, with Q assigned to Q.

Now with each sequence $\mathbf{n} = (n_1, \ldots, n_d)$ of positive integers and each sequence W of atomic symbols we associate a formula $\varphi = \varphi_{W\mathbf{n}}$. This is done by induction on the length of W

If W is empty, we let $\varphi_{W\mathbf{n}}$ be the formula $Seq(x_1, \ldots, x_d) \in Q$.

If $W = \exists A_i W'$, then we let $\varphi_{W\mathbf{n}}$ be the formula $\exists A_i[A_i \subseteq B_i \wedge A_i$ is n_i-dense in $S_i \wedge \varphi_{W'\mathbf{n}}]$. Here "$A_i$ is n_i-dense in S_i" is the formula

$$\forall t \in S_i[|t| = n_i \to \exists s \in A_i[t \leq_i s]].$$

We use the variables v_{ik} to express this.

If $W = \forall x_i W'$, then we let $\varphi_{W\mathbf{n}}$ be the formula $\forall x_i[x_i \in A_i \to \varphi_{W'\mathbf{n}}]$.

If $W = \forall a_i W'$, then we let $\varphi_{W\mathbf{n}}$ be the formula $\forall a_i[a_i \in n_i(S_i, B_i) \to \varphi_{W'\mathbf{n}}]$ Here $a_i \in n_i(S_i, B_i)$ is the formula

$$\exists t \in S_i[|t| = n_i \wedge \forall s[s \in a_i \leftrightarrow [t \leq s \wedge s \in B_i]]].$$

If $W = \exists x_i W'$, then we let $\varphi_{W\mathbf{n}}$ be the formula $\exists x_i[x_i \in a_i \wedge \varphi_{W'\mathbf{n}}]$.

Now we let $\psi(W, n, p)$ be the statement "$\forall B_1 \subseteq S_1 \cdots \forall B_d \subseteq S_d[[\forall i = 1, \dots, d[B_i$ is p-dense in $S_i] \to \varphi_{W\mathbf{n}}]]$"

(5) Suppose that W, W', ρ are sequences of atomic symbols. Suppose that under every assignment of values to the variables, φ_{Wn} implies $\varphi_{W'n}$. Then $\varphi_{\rho Wn}$ under any assignment implies $\varphi_{\rho W'n}$ under that assignment.

We prove this by induction on ρ. If ρ is empty, it is obvious. The induction step is clear upon looking at what $\varphi_{\rho Wn}$ is:

Case 1. $\rho = \exists A_i \rho'$. Then $\varphi_{\rho Wn}$ is

$$\exists A_i[A_i \subseteq B_i \wedge A_i \text{ is } n_i\text{-dense in } S_i \wedge \varphi_{\rho' Wn}]$$

Case 2. $\rho = \forall x_i \rho'$. Then $\varphi_{\rho Wn}$ is

$$\forall x_i[x_i \in A_i \to \varphi_{\rho' Wn}].$$

Case 3. $\rho = \forall a_i \rho'$. Then $\varphi_{\rho Wn}$ is

$$\forall a_i[a_i \in n_i(S_i, B_i) \to \varphi_{\rho' Wn}]$$

Case 4. $\rho = \exists x_i \rho'$. Then $\varphi_{\rho Wn}$ is

$$\exists x_i[x_i \in a_i \wedge \varphi_{\rho' Wn}]$$

This proves (5).

(6) If under some assignment of values to the variables, φ_{Wn} implies that $\varphi_{W'n}$, then under that assignment, $\varphi_{\exists A_i \exists A_j Wn}$ implies that $\varphi_{\exists A_j \exists A_i W'n}$.

In fact, $\varphi_{\exists A_i \exists A_j Wn}$ is

$$\exists A_i[A_i \subseteq B_i \wedge A_i \text{ is } n_i\text{-dense in } S_i \wedge \varphi_{A_j W\mathbf{n}}];$$

expanding we get

$$\exists A_i[A_i \subseteq B_i \wedge A_i \text{ is } n_i\text{-dense in } S_i \wedge \exists A_j[A_j \subseteq B_j \wedge A_j \text{ is } n_j\text{-dense in } S_j \wedge \varphi_{W\mathbf{n}}]].$$

This is logically equivalent to

$$\exists A_i \exists A_j[A_i \subseteq B_i \wedge A_i \text{ is } n_i\text{-dense in } S_i \wedge A_j \subseteq B_j \wedge A_j \text{ is } n_j\text{-dense in } S_j \wedge \varphi_{W\mathbf{n}}].$$

Hence (6) holds.

(7) If under some assignment of values to the variables, φ_{Wn} implies that $\varphi_{W'n}$, then under that assignment, $\varphi_{\exists A_i \exists x_j Wn}$ implies that $\varphi_{\exists x_j \exists A_i W'n}$.

In fact, $\varphi_{\exists A_i \exists x_j Wn}$ is

$$\exists A_i[A_i \subseteq B_i \wedge A_i \text{ is } n_i\text{-dense in } S_i \wedge \varphi_{x_j W\mathbf{n}}];$$

expanding we get

$$\exists A_i[A_i \subseteq B_i \wedge A_i \text{ is } n_i\text{-dense in } S_i \wedge \exists x_j[x_j \in a_j \wedge \varphi_{W\mathbf{n}}]].$$

This is logically equivalent to

$$\exists A_i \exists x_j[A_i \subseteq B_i \wedge A_i \text{ is } n_i\text{-dense in } S_i \wedge x_j \in a_j \wedge \varphi_{W\mathbf{n}}].$$

Hence (7) holds.

(8) If under some assignment of values to the variables, φ_{Wn} implies that $\varphi_{W'n}$, then under that assignment, $\varphi_{\exists x_i \exists A_j Wn}$ implies that $\varphi_{\exists A_j \exists x_i W'n}$.

This is proved as for (7).

(9) If under some assignment of values to the variables, φ_{Wn} implies that $\varphi_{W'n}$, then under that assignment, $\varphi_{\exists x_i \exists x_j Wn}$ implies that $\varphi_{\exists x_j \exists x_i W'n}$.

In fact, $\varphi_{\exists x_i \exists x_j Wn}$ is

$$\exists x_i[x_i \in a_i \wedge \varphi_{x_j W\mathbf{n}}];$$

expanding we get

$$\exists x_i[x_i \in a_i \wedge \exists x_j[x_j \in a_j \wedge \varphi_{W\mathbf{n}}]].$$

This is logically equivalent to

$$\exists x_i \exists x_j[x_i \in a_i \wedge x_j \in a_j \wedge \varphi_{W\mathbf{n}}].$$

Hence (9) holds.

(10) If under some assignment of values to the variables, φ_{Wn} implies that $\varphi_{W'n}$, then under that assignment, $\varphi_{\forall x_i \forall x_j Wn}$ implies that $\varphi_{\forall x_j \forall x_i Wn}$.

In fact, $\varphi_{\forall x_i \forall x_j Wn}$ is

$$\forall x_i[x_i \in A_i \to \varphi_{\forall x_j Wn}];$$

expanding we get

$$\forall x_i[x_i \in A_i \to \forall x_j[x_j \in A_j \to \varphi_{Wn}]]$$

This is logically equivalent to

$$\forall x_i \forall x_j[x_i \in A_i \to [x_j \in A_j \to \varphi_{Wn}]]$$

Hence (10) follows.

(11) If under some assignment of values to the variables, φ_{Wn} implies that $\varphi_{W'n}$, then under that assignment, $\varphi_{\forall a_i \forall x_j Wn}$ implies that $\varphi_{\forall x_j \forall a_i Wn}$.

In fact, $\varphi_{\forall a_i \forall x_j Wn}$ is

$$\forall a_i[a_i \in n_i(S_i, B_i) \to \varphi_{\forall x_j Wn}];$$

expanding we get

$$\forall a_i[a_i \in n_i(S_i, B_i) \to \forall x_j[x_j \in A_j \to \varphi_{Wn}]]$$

This is logically equivalent to

$$\forall a_i \forall x_j[x_i \in n_i(S_i, B_i) \to [x_j \in A_j \to \varphi_{Wn}]]$$

Hence (11) follows.

(12) If under some assignment of values to the variables, φ_{Wn} implies that $\varphi_{W'n}$, then under that assignment, $\varphi_{\forall x_i \forall a_j Wn}$ implies that $\varphi_{\forall a_j \forall x_i Wn}$.

This is similar to (11).

(13) If under some assignment of values to the variables, φ_{Wn} implies that $\varphi_{W'n}$, then under that assignment, $\varphi_{\forall a_i \forall a_j Wn}$ implies that $\varphi_{\forall a_j \forall a_i Wn}$.

This is similar to (11).

(14) If under some assignment of values to the variables, φ_{Wn} implies that $\varphi_{W'n}$, then under that assignment, $\varphi_{\exists A_i \forall a_j Wn}$ implies that $\varphi_{\forall a_j \exists A_i Wn}$.

In fact, $\varphi_{\exists A_i \forall a_j Wn}$ is

$$\exists A_i[A_i \subseteq B_i \wedge A_i \text{ is } n_i\text{-dense in } S_i \wedge \varphi_{\forall a_j Wn}]$$

Expanding, we get

$$\exists A_i[A_i \subseteq B_i \wedge A_i \text{ is } n_i\text{-dense in } S_i \wedge \forall a_j[a_j \in n_j(S_j, B_j) \to \varphi_{Wn}]].$$

This is logically equivalent to

$$\exists A_i \forall a_j[A_i \subseteq B_i \wedge A_i \text{ is } n_i\text{-dense in } S_i \wedge [a_j \in n_j(S_j, B_j) \to \varphi_{Wn}]].$$

This implies

$$\forall a_j \exists A_i [A_i \subseteq B_i \wedge A_i \text{ is } n_i\text{-dense in } S_i \wedge [a_j \in n_j(S_j, B_j) \to \varphi_{Wn}]].$$

Hence (14) holds.

One similarly treats other sequences of the form $\exists\alpha\forall\beta$.

(15) $A_i \subseteq B_i$ is n_i-dense in T_i iff $A_i \subseteq B_i$ and $\forall a_i \in n_i(T_i, B_i)[a_i \cap A_i \neq \emptyset]$.

In fact, for $\Rightarrow$, suppose that $A_i \subseteq B_i$ is n_i-dense in T_i and $a_i \in n_i(T_i, B_i$. Say $a_i = (T_i \uparrow t) \cap B_i$ with $|t| = n_i$. There is an $s \in A_i$ such that $t \leq s$. Thus $s \in a_i \cap A_i$.

For $\Leftarrow$, suppose that $A_i \subseteq B_i$ and $\forall a_i \in n_i(T_i, B_i)[a_i \cap A_i \neq \emptyset]$ and $|t| = n_i$. Set $a_i = (T_i \uparrow t) \cap B_i$. Choose $u \in a_i \cap A_i$. Then $t \leq u$, as desired.

(16) If under some assignment of values to the variables, φ_{Wn} implies that $\varphi_{W'n}$, then under that assignment, $\varphi_{\forall a_i \exists x_i Wn}$ implies that $\varphi_{\exists A_i \forall x_i Wn}$.

In fact, $\varphi_{\forall a_i \exists x_i Wn}$ is

$$\forall a_i [a_i \in n_i(S_i, B_i) \to \varphi_{\exists x_i Wn}];$$

expanding, we get

$$\forall a_i [a_i \in n_i(S_i, B_i) \to \exists x_i [x_i \in a_i \wedge \varphi_{Wn}]].$$

Now assume $\varphi_{\forall a_i \exists x_i Wn}$. For each $a_i \in n_i(T_i, B_i)$ choose $x_i(a_i) \in a_i$ such that φ_{Wn}. Let $A_i = \{x_i(a_i) : a_i \in n_i(T_i, B_i)\}$. Note that $\forall a_i \in n_i(T_i, B_i)[a_i \subseteq B_i]$. Hence $A_i \subseteq B_i$. Hence by (15), A_i is n_i-dense in T_i. Now $\forall x_i \in A_i \varphi_{Wn}$. So (16) holds.

(17) If $W \vdash W'$ using rules 1 or 2, then $\forall n \exists p \psi(W.n.p)$ implies that $\forall n \exists p \psi(W'.n.p)$.

In fact, $\forall n \exists p \psi(W.n.p)$ is

$$\forall n \exists p \forall B_1 \subseteq S_1 \cdots \forall B_d \subseteq S_d [[\forall i = 1, \ldots, d[B_i \text{ is } p\text{-dense in } S_i] \to \varphi_{W\mathbf{n}}]],$$

and similarly for $\forall n \exists p \psi(W'.n.p)$. Hence (17) follows from (5)–(16).

Now suppose that σ is a permutation of $\{1, \ldots, d\}$. Let $W = (\forall a_{\sigma(i)})_1^r (\exists A_{\sigma(i)})_{r+1}^d V$ and $\overline{W} = (\exists A_{\sigma(i)})_{r+1}^d (\forall a_{\sigma(i)})_1^r V$ with $r \in \{1 \ldots d-1\}$. Now for simplicity we assume that σ is the identity. Note that V is a string of length d whose entries are $\forall x_i$ for $r+1 \leq i \leq d$ and $\exists x_j$ for $1 \leq j \leq r$; moreover, only A_i for $i = r+1, \ldots, d$ and a_i for $i = 1 \ldots, r$ are free. If V is such a string, $\mathbf{a}$ is an assignment of values to the a_i's, $A_{r+1}, \ldots, A_d$ an assignment of values to the A_i's, then the assertion $\varphi_{Vn}[\mathbf{a}, A_{r+1}, \ldots, A_d]$ has the natural meaning.

(26) If V is such a string, $\mathbf{a}$ assigns values to the a_i for $i = 1, \ldots, r$, $A_{r+1}, \ldots, A_d$ an assignment of values to the A_i's, $A'_{r+1} \subseteq A_{r+1}, \ldots A'_d \subseteq A_d$, and $\varphi_{Vn}[\mathbf{a}, A_{r+1}, \ldots, A_d]$, then $\varphi_{Vn}[\mathbf{a}, A'_{r+1}, \ldots, A'_d]$.

We prove (26) by induction on the length of V. It is trivial for the empty string. Now suppose that the string is $\forall x_i V'$. Then $\varphi_{\forall x_i V'n}[\mathbf{a}, A_{r+1}, \ldots, A_d]$ is

$$\forall x_i [x_i \in A_i \to \varphi_{V'n}[\mathbf{a}, A_{r+1}, \ldots, A_d]],$$

so

$$\forall x_i[x_i \in A'_i \to \varphi_{V'n}[\mathbf{a}, A'_{r+1}, \ldots, A'_d]],$$

If the string is $\exists x_i V'$, then $\varphi_{\exists x_i V'n}[\mathbf{a}, A_{r+1}, \ldots, A_d]$ is

$$\exists x_i[x_i \in a_i \wedge \varphi_{V'n}[\mathbf{a}, A_{r+1}, \ldots, A_d]],$$

and the conclusion is obvious. So (26) holds.

To prove the implication in (17) for rules 3, suppose that $\forall \mathbf{n} \exists p \psi(W, \mathbf{n}, p)$. Let F be such that $\forall \mathbf{n} \psi(W, \mathbf{n}, F(\mathbf{n}))$. Thus

(19) $\qquad \forall \mathbf{n}[\forall B_1 \subseteq T_1 \cdots \forall B_d \subseteq T_d \forall i = 1, \ldots, d[B_i \text{ is } F(\mathbf{n})\text{-dense in } T_i] \to \varphi_{W\mathbf{n}}]]].$

Since p'-density implies p-density for $p < p'$, we may assume that for all $\mathbf{n}$ and all $i = 1, \ldots, d$, $F(\mathbf{n}) > n_i$.

Now fix a sequence $\mathbf{n} = (n_1, \ldots, n_d)$ of positive integers. We want to find p such that $\psi(\overline{W}, \mathbf{n}, p)$. Define G by induction, as follows.

$$G(0) = \max\{n_i : r < i \leq d\};$$
$$G(j+1) = F(\mathbf{k}), \text{ where } k_i = \begin{cases} n_i & \text{if } 1 \leq i \leq r, \\ G(j) & \text{if } r < i \leq d. \end{cases}$$

Now for each $i = 1, \ldots, r$ let z_i be the number of elements of T_i of height n_i, and let $m = \prod_{i=1}^r z_i$. For each $j \leq m$ let $p_j = G(m-j)$.

(20) If $j < m$, then $p_{j+1} \leq p_j$.

For,

$$p_j = G(m-j) = G(m-j-1+1) = F(\mathbf{k}^j), \text{ where } k_i^j = \begin{cases} n_i & \text{if } 1 \leq i \leq r, \\ G(m-j-1) & \text{if } r < i \leq d \end{cases}$$
$$= \begin{cases} n_i & \text{if } 1 \leq i \leq r, \\ p_{j+1} & \text{if } r < i \leq d \end{cases}$$

Since p_{j+1} is an entry of $\mathbf{k}^j$, (20) holds.

It follows that

(21) If a set is p_j-dense in T_i, then it is also p_{j+1}-dense in T_i.

We claim that $\psi(\overline{W}, \mathbf{n}, p_0)$. Now $\psi(\overline{W}, \mathbf{n}, p_0)$ is

$$\forall B_1 \subseteq T_1 \cdots \forall B_d \subseteq T_d[\forall i = 1, \ldots, d[B_i \text{ is } p_0\text{-dense in } T_i \to \varphi_{\overline{W}\mathbf{n}}]].$$

So, assume that $B_1 \subseteq T_1 \cdots \forall B_d \subseteq T_d$ and $\forall i = 1, \ldots, d[B_i$ is p_0-dense in $T_i]$.

(22) If $a_1 \in n_1(T_1, B_1) \wedge \ldots \wedge a_r \in n_r(T_r, B_r)$, $0 \leq j < m$, $A_{r+1} \subseteq B_{r+1}, \ldots, A_d \subseteq B_d$, and $A_{r+1}, \ldots, A_d$ are p_j-dense in $T_{r+1}, \ldots, T_d$ respectively, then there exist $A'_{r+1} \subseteq A_{r+1}, \ldots, A'_d \subseteq A_d$ which are p_{j+1}-dense such that $\varphi_{V\mathbf{k}^j}[\vec{a}, A'_{r+1}, \ldots, A'_d]$.

By (19), $\varphi_{W\mathbf{k}^j}[A_{r+1},\ldots,A_d]$, and hence by the form of W, there are $A'_{r+1}\subseteq A_{r+1}$,. . .,$A'_d\subseteq A_d$ such that $A'_{r+1},\ldots,A'_d$ are k^j_{r+1}-,. . . , k^j_d-dense and $\varphi_{V\mathbf{k}^j}[\vec{a},A'_{r+1},\ldots,A'_d]$. Now $k^j_{r+1}=\cdots=k^j_d=p_{j+1}$, as desired.

Now clearly $|\prod_{i=1}^r n_i(T_i,B_i)|\le|\prod_{i=1}^r z_i|=m$.

(23) For any $J\subseteq\prod_{i=1}^r n_i(T_i,B_i)$ with $|J|=j\le m$, there are $A_{r+1}\subseteq B_{r+1}$, ..., $A_d\subseteq B_d$ such that each A_i is p_j-dense in T_i, and for every $\mathbf{a}\in J$, $\varphi_{V\mathbf{n}}[\mathbf{a},A_{r+1},\ldots,A_d]$.

We prove this by induction on j. It is obvious for $j=0$. Now assume that $\mathbf{b}\notin J$ and $J\cup\{\mathbf{b}\}\subseteq\prod_{i=1}^r n_i(T_i,B_i)$ and the assertion is true for J. So $j<m$ and there are $A_{r+1}\subseteq B_{r+1}$, ..., $A_d\subseteq B_d$ such that each A_i is p_j-dense in T_i, and for every $\mathbf{a}\in J$, $\varphi_{V\mathbf{n}}[\mathbf{a},A_{r+1},\ldots,A_d]$. Now by (22) there exist $A'_{r+1}\subseteq A_{r+1}$, ..., $A'_d\subseteq A_d$ such that $A'_{r+1},\ldots,A'_d$ are p_{j+1}-dense in $T_{r+1},\ldots,T_d$ respectively, and $\varphi_{V\mathbf{k}^j}[\mathbf{b},A'_{r+1},\ldots,A'_d]$. By (26), $\varphi_{V\mathbf{n}}[\mathbf{c},A'_{r+1},\ldots,A'_d]$ for all $\mathbf{c}\in J\cup\{\mathbf{b}\}$. This proves (23).

This completes the proof of (17) for rules 3.

Now the proof of the theorem goes as follows. Let $W_0=(\forall a_i)_1^d(\exists x_i)_1^d$, $W_1=(\exists A_i)_1^d(\forall x_i)_1^d$. By (4), $W_0\models_d W_1$. By (17) as extended, $\forall n\exists p\psi(W_0,n,p)$ implies $\forall n\exists p\psi(W_1,n,p)$.

Case 1. $\forall n\exists p\psi(W_0,n,p)$. Hence $\forall n\exists p\psi(W_1,n,p)$. For any $k\in\omega$ let $\mathbf{n}$ be constantly k. Then choose p so that $\psi(W_1,\mathbf{n},p)$. Then there exist A_i for $i=1,\ldots,d$ such that $A_i\subseteq B_I$ for all $i=1,\ldots,d$, A_i is n_i-dense in T_i for all $i=1,\ldots,d$, and for all $i=1,\ldots,d$, $\forall x_i\in A_i$, $\varphi_{V\mathbf{n}}[A_1,\ldots,A_d,x_1,\ldots,x_d]$. Then $\forall i=1,\ldots,d[A_i$ is k-dense in T_i and $(x_1,\ldots,x_d)\in Q$. Thus (i) in the theorem holds.

Case 2. There is an n such that for all p, $\neg\psi(W_0,n,p)$. Thus for every p there are $B_1,\ldots,B_d$ which are p-dense in their respective trees, and $a_i\in n_i(T_i,B_i)$ for $i=1,\ldots d$ such that $\prod_{i=1}^d a_i\subseteq\prod_{i=1}^d T_i\backslash Q$. Let $h=\max\{n_i:1\le i\le d\}$. For any k, take $p=h+k$. Then $a_1,\ldots,a_d$ is an (m,k)-matrix contained in $\prod_{i=1}^d T_i\backslash Q$. □

Gaps in $[\omega]^\omega$

We now treat gaps in $[\omega]^\omega$. For $a,b\in[\omega]^\omega$ we define $a\subseteq^* b$ iff $a\backslash b$ is finite. It is convenient to use Boolean algebra terminology. The set $[\omega]^{<\omega}$ is an ideal in $\mathscr{P}(\omega)$, and the quotient $\mathscr{P}(\omega)/[\omega]^{<\omega}$ is a BA which we use. The equivalence class of $X\subseteq\omega$ under $[\omega]^{<\omega}$ is denoted by $[X]$. Note that $[a]\le[b]$ iff $a\subseteq^* b$.

Proposition 29.29. *If $\mathscr{A},\mathscr{B}$ are nonempty countable subsets of $[\omega]^\omega$ and $a\subseteq^* b$ whenever $a\in\mathscr{A}$ and $b\in\mathscr{B}$, then there is a $c\in[\omega]^\omega$ such that $a\subseteq^* c\subseteq^* b$ whenever $a\in\mathscr{A}$ and $b\in\mathscr{B}$.*

Proof. Write $\mathscr{A}=\{a_n:n\in\omega\}$ and $\mathscr{B}=\{b_n:n\in\omega\}$. Let

$$c=\bigcup_{n\in\omega}\left[\left(\bigcup_{m\le n}a_m\right)\cap\bigcap_{m\le n}b_m\right].$$

Now suppose that $p \in \omega$. Then

$$\begin{aligned}
a_p \backslash c &= \bigcap_{n \in \omega} \left[a_p \cap \left(\bigcap_{m \leq n} (\omega \backslash a_m) \cup \bigcup_{m \leq n} (\omega \backslash b_m) \right) \right] \\
&= \bigcap_{n < p} \left[a_p \cap \left(\bigcap_{m \leq n} (\omega \backslash a_m) \cup \bigcup_{m \leq n} (\omega \backslash b_m) \right) \right] \\
&\quad \cap \bigcap_{n \geq p} \left[a_p \cap \left(\bigcap_{m \leq n} (\omega \backslash a_m) \cup \bigcup_{m \leq n} (\omega \backslash b_m) \right) \right] \\
&\subseteq \bigcap_{n < p} \left[a_p \cap \left(\bigcap_{m \leq n} (\omega \backslash a_m) \cup \bigcup_{m \leq n} (\omega \backslash b_m) \right) \right] \\
&\quad \cap \bigcap_{n \geq p} \left[a_p \cap \bigcup_{m \leq n} (\omega \backslash b_m) \right] \\
&\subseteq a_p \cap \bigcup_{m \leq p} (\omega \backslash b_m),
\end{aligned}$$

and this last set is finite.

Furthermore,

$$\begin{aligned}
c \backslash b_p &= \bigcup_{n < p} \left[\left(\bigcup_{m \leq n} a_m \right) \cap \bigcap_{m \leq n} b_m \cap (\omega \backslash b_p) \right] \\
&\subseteq \left(\bigcup_{m < p} a_m \right) \backslash b_p,
\end{aligned}$$

and this last set is finite.

The set c is infinite, as otherwise $a_0 = (a_0 \cap c) \cup (a_0 \backslash c)$ would be finite. $\square$

Proposition 29.30. *If $\mathscr{A}$ is an infinite countable collection of almost disjoint members of $[\omega]^\omega$, then $\mathscr{A}$, then $\mathscr{A}$ is not maximal.*

Proof. Say $\mathscr{A} = \{a_n : n \in \omega\}$. Note that $a_n \cap \bigcup_{m<n} a_m$ is finite, for any $n \in \omega$. Let $x_n \in (a_n \backslash \bigcup_{m<n} a_m) \backslash \{x_m : m < n\}$. Then $\{x_n : n \in \omega\} \in [\omega]^\omega$ and $a_n \cap \{x_n : n \in \omega\}$ is finite, for each $n \in \omega$. $\square$

Proposition 29.31. *Suppose that $\mathscr{A}$ is a nonempty countable family of members of $[\omega]^\omega$, and $\forall a, b \in \mathscr{A}[a \subseteq^* b$ or $b \subseteq^* a]$. Also suppose that $\forall a \in \mathscr{A}[a \subset^* d]$, where $d \in [\omega]^\omega$. Then there is a $c \in [\omega]^\omega$ such that $\forall a \in \mathscr{A}[a \subseteq^* c \subset^* d]$.*

Proof. If $\exists a \in \mathscr{A} \forall b \in \mathscr{A}[b \subseteq^* a]$, then the conclusion is obvious. So suppose that no such a exists. Then there is a sequence $\langle a_n : n \in \omega\rangle$ of elements of $\mathscr{A}$ such that $a_n \subset^* a_m$ for $n < m$, and the sequence is cofinal in $\mathscr{A}$ in the $\subseteq^*$-sense. In fact, let $\langle a'_n : n \in \omega\rangle$ be a list of all of the elements of $\mathscr{A}$. Let $a_0 = a'_0$. If a_n has been defined, then by hypothesis $a_n \subseteq^* a'_n$ or $a'_n \subseteq^* a_n$; choose $a_{n+1} \in \mathscr{A}$ such that $a_n, a'_n \subset^* a_{n+1}$. Let $\mathscr{C} = \{a_0\} \cup \{a_{m+1} \backslash a_m : m \in \omega\} \cup \{\omega\backslash d\}$. Then $\mathscr{C}$ is an almost disjoint family, except that possibly $\omega\backslash d$ is finite. By Proposition 29.30, let $e \subseteq \omega$ be infinite and almost disjoint from each member of $\mathscr{C}$. Let $c = d\backslash e$. Then for any $n \in \omega$,

$$\begin{aligned} a_{n+1}\backslash c &= (a_{n+1}\backslash d) \cup (a_{n+1} \cap e) \\ &\subseteq (a_{n+1}\backslash d) \cup \left[\bigcup_{i \le n}(a_{i+1}\backslash a_i) \cup a_0\right] \cap e, \end{aligned}$$

and the last set is finite. Thus $a_{n+1} \subseteq^* c$, hence $b \subseteq^* c$ for all $b \in \mathscr{A}$.

Since $c \subseteq d$, we have $c \subseteq^* d$. Also, $d\backslash c = d \cap e$, and this is infinite since $e\backslash d$ is finite. Thus $c \subset^* d$

Note that c is infinite, since $a \subseteq^* c$ for all $a \in \mathscr{A}$.

Proposition 29.32. *If $a, b \in [\omega]^\omega$ and $a \subset^* b$, then there is a $c \in [\omega]^\omega$ such that $a \subset^* c \subset^* b$.*

Proof. Write $b\backslash a = d \cup e$ with d, e infinite and disjoint. Let $c = a \cup d$. $\square$

Proposition 29.33. *Suppose that $\mathscr{A}$ and $\mathscr{B}$ are nonempty countable subsets of $[\omega]^\omega$, $\forall x, y \in \mathscr{A}[x \subseteq^* y$ or $y \subseteq^* x]$, $\forall x, y \in \mathscr{B}[x \subseteq^* y$ or $y \subseteq^* x]$, and $\forall x \in \mathscr{A}\forall y \in \mathscr{B}[a \subset^* b]$. Then there is a $c \in [\omega]^\omega$ such that $a \subset^* c \subset^* b$ for all $a \in \mathscr{A}$ and $b \in \mathscr{B}$.*

Proof. By Proposition 29.29 choose $d \subseteq \omega$ such that $\forall a \in \mathscr{A}\forall b \in \mathscr{B}[a \subseteq^* d \subseteq^* b]$. Thus either $\forall a \in \mathscr{A}[a \subset^* d]$ or $\forall b \in \mathscr{B}[d \subset^* b]$.

Case 1. $\forall a \in \mathscr{A}[a \subset^* d]$. By Proposition 29.31 choose $e \subseteq \omega$ such that $\forall a \in \mathscr{A}[a \subseteq^* e \subset^* d]$. By Proposition 29.32 choose $c \in [\omega]^\omega$ such that $e \subset^* c \subset^* d$.

Case 9. $\forall b \in \mathscr{B}[d \subset^* b]$. Then $\forall b \in \mathscr{B}[(\omega\backslash b) \subset^* (\omega\backslash d)]$. By Proposition 29.31 choose $e \subseteq \omega$ such that $\forall b \in \mathscr{B}[(\omega\backslash b) \subseteq^* e \subset^* (\omega\backslash d)]$. By Proposition 29.32 choose $c \subseteq \omega$ such that $e \subset^* c \subset^* (\omega\backslash d)$. Then $\forall a \in \mathscr{A}\forall b \in \mathscr{B}[a \subset^* (\omega\backslash c) \subset^* b]$. $\square$

Now we need some more terminology. Let $\mathscr{A} \subseteq [\omega]^\omega$, $b \in [\omega]^\omega$, and $\forall a \in \mathscr{A}[a \subset^* b]$. We say that b is *near* to $\mathscr{A}$ iff for all $m \in \omega$ the set $\{a \in \mathscr{A} : a\backslash b \subseteq m\}$ is finite.

Proposition 29.34. *Suppose that $a_m \in [\omega]^\omega$ for all $m \in \omega$, $a_m \subset^* a_n$ whenever $m < n \in \omega$, $b \in [\omega]^\omega$, and $a_m \subset^* b$ for all $m \in \omega$. Then there is a $c \in [\omega]^\omega$ such that $\forall m \in \omega[a_m \subset^* c \subset^* b]$ and c is near to $\{a_n : n \in \omega\}$.*

Proof. By Proposition 29.24a choose $d \subseteq \omega$ such that $\forall m \in \omega[a_n \subset^* d \subset^* b]$. Now for each $m \in \omega$, $\bigcup_{i \le m}(a_i\backslash a_m)$ is finite, and $a_{m+1}\backslash \bigcup_{i \le m} a_i = (a_{m+1}\backslash a_m)\backslash \bigcup_{i \le m}(a_i\backslash a_m)$, so $a_{m+1}\backslash \bigcup_{i \le m} a_i$ is infinite. Choose $e_m \subseteq a_{m+1}\backslash \bigcup_{i \le m} a_i$ such that $|e_m| = m$. Let $c = d\backslash \bigcup_{m \in \omega} e_m$. Thus $c \subseteq^* d \subset^* b$.

If $n \in \omega$, then

$$a_n \backslash c = (a_n \backslash d) \cup \bigcup_{m \in \omega} (a_n \cap e_m) = (a_n \backslash d) \cup \bigcup_{m < n} (a_n \cap e_m),$$

and this last set is finite. Hence $a_n \subseteq^* c$. Since n is arbitrary, it follows that $a_n \subset^* c$ for all $n \in \omega$.

Also for any $m \in \omega$ we have $a_{m+1} \backslash c \supseteq a_{m+1} \cap e_m = e_m$, and so $|a_{m+1} \backslash c| \geq m$. So if $p \in \omega$ then $|a_{p+1} \backslash c| \geq p$ and so $\{a_m : a_m \backslash c \subseteq n\} \subseteq \{a_0, \ldots, a_n\}$. So c is near to $\{a_m : m \in \omega\}$. □

Proposition 29.35. *Suppose that $\mathscr{A} \subseteq [\omega]^\omega$, $\forall x, y \in \mathscr{A}[x \subset^* y$ or $y \subset^* x]$, $b \in [\omega]^\omega$, $\forall x \in \mathscr{A}[x \subset^* b]$, and $\forall a \in \mathscr{A}[b$ is near to $\{d \in \mathscr{A} : d \subset^* a\}]$.*

Then there is a $c \in [\omega]^\omega$ such that $\forall a \in \mathscr{A}[a \subset^* c \subset^* b]$ and c is near to $\mathscr{A}$.

Proof. We consider several cases.

Case 1. $\exists a \in \mathscr{A} \forall d \in \mathscr{A}[d \subseteq^* a]$. By Proposition 29.32, choose c such that $a \subset^* c \subset^* b$. Choose $n \in \omega$ such that $c \backslash b \subseteq n$. Then for any $m \in \omega$ and any $d \in \mathscr{A}$, if $d \backslash c \subseteq m$ then $d \backslash b \subseteq (d \backslash c) \cup (c \backslash b) \subseteq \max(m, n)$. Hence

$$\{d \in \mathscr{A} : d \backslash c \subseteq m\} \subseteq \{a\} \cup \{d \in \mathscr{A} : d \subset^* a \text{ and } d \backslash b \subseteq \max(m, n)\},$$

and the later set is finite, since b is near to $\{d \in \mathscr{A} : d \subset^* a\}$. Thus c is as desired.

Case 9. $\forall a \in \mathscr{A} \exists d \in \mathscr{A}[a \subset^* d]$ and b is near to $\mathscr{A}$. By Proposition 29.24a choose c so that $\forall a \in \mathscr{A}[a \subset^* c \subset^* b]$. Choose $n \in \omega$ such that $c \backslash b \subseteq n$. Then for any $m \in \omega$ and any $d \in \mathscr{A}$, if $d \backslash c \subseteq m$ then $d \backslash b \subseteq (d \backslash c) \cup (c \backslash b) \subseteq \max(m, n)$. Hence

$$\{d \in \mathscr{A} : d \backslash c \subseteq m\} \subseteq \{a\} \cup \{d \in \mathscr{A} : d \backslash b \subseteq \max(m, n)\},$$

and the later set is finite, since b is near to $\mathscr{A}$. Thus c is as desired.

Case 3. $\forall a \in \mathscr{A} \exists d \in \mathscr{A}[a \subset^* d]$ and b is not near to $\mathscr{A}$. For each $m \in \omega$ let $\mathscr{B}_m = \{a \in \mathscr{A} : a \backslash b \subseteq m\}$. Since b is not near to $\mathscr{A}$, choose m so that $\mathscr{B}_m$ is infinite. Note that $p < q \to \mathscr{B}_p \subseteq \mathscr{B}_q$. Hence $\mathscr{B}_n$ is infinite for every $n \geq m$. Now we claim

$$\forall n \geq m \forall a \in \mathscr{A} \exists d \in \mathscr{B}_n[a \subseteq^* d]. \tag{1}$$

In fact, otherwise we get $n \geq m$ and $a \in \mathscr{A}$ such that $\forall d \in \mathscr{B}_n[d \subset^* a]$. Now b is near to $\{d \in \mathscr{A} : d \subset^* a\}$ by a hypothesis of the lemma, so $\{d \in \mathscr{A} : d \subset^* a$ and $d \backslash b \subseteq n\}$ is finite. But $\mathscr{B}_n \subseteq \{d \in \mathscr{A} : d \subset^* a$ and $d \backslash b \subseteq n\}$, contradiction. So (1) holds.

Next we claim

$$\forall n \geq m \forall d \in \mathscr{B}_n[\{e \in \mathscr{B}_n : e \subset^* d\} \text{ is finite}]. \tag{2}$$

In fact, suppose that $n \geq m$, $d \in \mathscr{B}_n$ and $\{e \in \mathscr{B}_n : e \subset^* d\}$ is infinite. Since b is near to $\{a \in \mathscr{A} : a \subset^* d\}$, the set $\{a \in \mathscr{A} : a \subset^* d$ and $a \backslash b \subseteq n\}$ is finite. But $\{e \in \mathscr{B}_n : e \subset^* d\} \subseteq \{a \in \mathscr{A} : a \subset^* d$ and $a \backslash b \subseteq n\}$, contradiction. So (2) holds.

From (2) it follows that $\mathscr{B}_n$ has order type ω under $\subset^*$, for each $n \geq m$. Now clearly $\mathscr{A} = \bigcup_{p\in\omega} \mathscr{B}_p$, so $\mathscr{A}$ is countable.

Now by Proposition 29.24b, choose c_m such that $\forall d \in \mathscr{B}_m[d \subset^* c_m \subset^* b]$ and c_m is near to $\mathscr{B}_m$. By (1), $a \subset^* c_m$ for each $a \in \mathscr{A}$. Now suppose that $n \geq m$ and c_n has been defined so that $a \subset^* c_n$ for each $a \in \mathscr{A}$. Again by Proposition 29.24b choose c_{n+1} such that $\forall d \in \mathscr{B}_{n+1}[d \subset^* c_{n+1} \subset^* c_n]$ and c_{n+1} is near to $\mathscr{B}_{n+1}$. Thus we have

$$\forall a \in \mathscr{A}[a \subset^* \cdots \subset^* c_{n+1} \subset^* c_n \subset^* \cdots \subset^* c_m \subset^* b].$$

By Proposition 29.33, choose d so that $\forall a \in \mathscr{A} \forall n \geq m[a \subset^* d \subset^* c_n]$. We claim that d is near to $\mathscr{A}$, completing the proof. For, let $n \in \omega$. Let $p = \max(m, n)$, and choose $q \geq p$ such that $d \backslash c_p \subseteq q$. Then

$$\begin{aligned}\{a \in \mathscr{A} : a\backslash d \subseteq n\} &\subseteq \{a \in \mathscr{A} : a\backslash d \subseteq p\}\\ &= \{a \in \mathscr{B}_p : a\backslash d \subseteq p\}\\ &\subseteq \{a \in \mathscr{B}_p : a\backslash c_p \subseteq q\},\end{aligned}$$

where the last inclusion holds since $a\backslash c_p = (a\backslash d) \cup (d\backslash c_p)$. The last set is finite since c_p is near to $\mathscr{B}_p$, as desired. $\square$

Proposition 29.36. *(The Hausdorff gap) There exist sequences $\langle a_\alpha : \alpha < \omega_1\rangle$ and $\langle b_\alpha : \alpha < \omega_1\rangle$ of members of $[\omega]^\omega$ such that $\forall \alpha, \beta < \omega_1[\alpha < \beta \to a_\alpha \subset^* a_\beta$ and $b_\beta \subset^* b_\alpha]$, $\forall \alpha, \beta < \omega_1[a_\alpha \subset^* b_\beta]$, and there does not exist a $c \subseteq \omega$ such that $\forall \alpha < \omega_1[a_\alpha \subset^* c$ and $c \subset^* b_\alpha]$.*

Proof. We construct by recursion $a_\alpha, b_\alpha \subseteq \omega$ for $\alpha < \omega_1$ so that $a_\alpha \subset^* b_\alpha$, $\alpha < \beta \to a_\alpha \subset^* a_\beta$ and $b_\beta \subset^* b_\alpha$, and for all $\alpha < \omega_1$, b_β is near to $\{a_\alpha : \alpha < \beta\}$.

Let $a_0 = \emptyset$, $b_0 = \omega$. Suppose that a_α and b_α have been constructed for all $\alpha < \beta$ so that $a_\alpha \subset^* b_\alpha$, $\alpha < \gamma < \beta \to a_\alpha \subset^* a_\gamma$ and $b_\gamma \subset^* b_\beta$, and $\alpha < \beta \to b_\alpha$ is near to $\{a_\gamma : \gamma < \alpha\}$. By Proposition 29.24a choose c such that $\forall \alpha < \beta[a_\alpha \subset^* c \subset^* b_\alpha]$. Suppose that $\alpha < \beta$. We claim that c is near to $\{a_\gamma : \gamma < \alpha\}$. In fact, suppose that $m \in \omega$. Choose $n \geq m$ such that $c\backslash b_\alpha \subseteq n$. Now for any $\gamma < \alpha$ we have $a_\gamma\backslash b_\alpha \subseteq (a_\gamma\backslash c) \cup (c\backslash b_\alpha)$, so

$$\{a_\gamma : \gamma < \alpha \text{ and } a_\gamma\backslash c \subseteq m\} \subseteq \{a_\gamma : \gamma < \alpha \text{ and } a_\gamma\backslash b_\alpha \subseteq n\},$$

and the latter set is finite since b_α is near to $\{a_\gamma : \gamma < \alpha\}$. Thus indeed c is near to $\{a_\gamma : \gamma < \alpha\}$. Now by Proposition 29.24c there is a b_β such that $\forall \alpha < \beta[a_\alpha \subset^* b_\beta \subset^* c]$ and b_β is near to $\{a_a : \alpha < \beta\}$. By Proposition 29.24a choose a_β so that $\forall \alpha < \beta[a_\alpha \subset^* a_\beta \subset^* b_\beta]$. This finishes the construction.

Now suppose that $d \subseteq \omega$ and $\forall \alpha < \omega_1[a_\alpha \subset^* d \subset^* b_\alpha]$. Now $\omega_1 = \bigcup_{m\in\omega}\{\alpha < \omega_1 : a_\alpha\backslash d \subseteq m\}$, so we can choose $m \in \omega$ such that $|\{\alpha < \omega_1 : a_\alpha\backslash d \subseteq m\}| = \omega_1$. Hence there is an $\alpha < \omega_1$ such that $\{\beta < \alpha : a_\beta\backslash d \subseteq m\}$ is infinite. Choose $p \geq m$ such that $d\backslash b_\alpha \subseteq p$. Now $a_\beta\backslash b_\alpha \subseteq (a_\beta\backslash d) \cup (d\backslash b_\alpha)$, so $\{\beta < \alpha : a_\beta\backslash d \subseteq m\} \subseteq \{\beta < \alpha : a_\beta\backslash b_\alpha \subseteq p\}$, contradicting b_α near to $\{a_\beta : \beta < \alpha\}$. $\square$

Now we discuss the open coloring axiom. Suppose that $X \subseteq \mathbb{R}$, and let $K \subseteq [X]^2$. We say that K is *open* iff $\{(x, y) : \{x, y\} \in K\}$ is open in $X \times X$.

OCA: $\forall X \subseteq \mathbb{R} \forall$ partition $[X]^2 = K_0 \cup K_1 [K_0$ open$\rightarrow$ one of the following holds:

(i) there is an uncountable $Y \subseteq X$ such that $[Y]^2 \subseteq K_0$.

(ii) there exist H_n for $n \in \omega$ such that $X = \bigcup_{n\in\omega} H_n$ and $\forall n \in \omega[[H_n]^2 \subseteq K_1]]]$.

Theorem 29.37. *Assuming OCA, every uncountable subset of $\mathscr{P}(\omega)$ contains an uncountable chain or an uncountable antichain.*

Proof. Assume OCA. By full Theorem 26.43, ${}^\omega 2$ is homeomorphic to a subset of $\mathbb{R}$, and by full Lemma 26.66, ${}^\omega 2$ is homeomorphic to $\mathscr{P}(\omega)$. So we can apply OCA to $\mathscr{P}(\omega)$.

Suppose that $X \subseteq \mathscr{P}(\omega)$ is uncountable. Let $K_0 = \{\{a, b\} : a, b \in X$ and $a \not\subseteq b$ and $b \not\subseteq a\}$.

(1) $\{(a, b) : \{a, b\} \in K_0\}$ is an open subset of $X \times X$.

In fact, suppose that (a, b) is such that $\{a, b\} \in K_0$. Choose $x \in a\backslash b$ and $y \in b\backslash a$. Then $(a, b) \in U_{\{x\}\{y\}} \times U_{\{y\}\{x\}} \subseteq \{(u, v) : \{u, v\} \in K_0\}$, as desired.

Now the desired conclusion is clear. □

Proposition 29.38. *Suppose that $X \subseteq {}^\omega\omega$ and $|X| = \omega_1$. Then there is an $X' \subseteq {}^\omega\omega$ such that X' is well-ordered by $<^*$ with type at most ω_1, each member of X' is stricly increasing, and $\forall f \in X \exists g \in X'[f <^* g]$.*

Proof. Let $\langle h_\alpha : \alpha < \omega_1\rangle$ enumerate X'. Define $h'_0(m) = h_0(m) + 1$ for all $m \in \omega$, and for $0 < \alpha < \omega$ and each $m \in \omega$ define

$$h'_\alpha(m) = \sup\{h_\beta(m) + 1, h'_\beta(m) + 1 : \beta \le \alpha\}.$$

Then for $\beta < \alpha < \omega$ we have $\forall m[h'_\beta(m) < h'_\alpha(m)]$; so $h'_\beta <^* h'_\alpha$.

Now suppose that $\omega \le \alpha < \omega_1$.

Case 1. $\alpha = \beta + 1$ for some β. For each $m \in \omega$ let $h'_\alpha(m) = \max\{h(\alpha), h_\beta(m) + 1\}$. Thus $h'_\beta <^* h'_\alpha$.

Case 9. α is a limit ordinal. Let $\langle\beta_m : m \in \omega\rangle$ be strictly increasing with supremum α. For each $m \in \omega$ let $h'_\alpha(m) = \max\{h_\alpha(m), h'_{\beta_n}(m) + 1 : n \le m\}$. Clearly $h'_{\beta_n} <^* h'_\alpha$ for all $n \in \omega$. Hence $h'_\gamma <^* h'_\alpha$ for all $\gamma < \alpha$. □

Lemma 29.39. *Assume OCA. Then every subset of ${}^\omega\omega$ of size $\aleph_1$ is bounded.*

Proof. Suppose that $X \in [{}^\omega\omega]^{\aleph_1}$. Choose X' as in Proposition 9. Let $K_0 = \{\{h'_\alpha, h'_\beta\} : \alpha < \beta$ and $\exists k[h'_\alpha(k) > h'_\beta(k)]\}$ and $K_1 = X'\backslash K_0$,

(1) K_0 is open.

For, suppose that $(h'_\alpha, h'_\beta) \in K_0 \times K_0$ with $\alpha < \beta$. Say $h'_\alpha(k) \ge h'_\beta(k)$. Then

$$(h'_\alpha, h'_\beta) \in U_{\{(k,h'_\alpha(k))\},\emptyset} \times U_{\{(k,h'_\beta(k))\},\emptyset} \subseteq K_0,$$

showing that K_0 is open.

Note that ${}^\omega\omega$ is isomorphic to the irrationals, a subset of $\mathbb{R}$. Thus we can apply OCA to get the following two cases:

Case 1. $X' = \bigcup_{n\in\omega} H_n$ and $\forall n \in \omega[[H_n]^2 \subseteq K_1]$. Choose $n \in \omega$ such that H_n is uncountable. Then if $\alpha < \beta$ and $h'_\alpha, h'_\beta \in H_n$, then $\forall k[h'_\alpha \le h'_\beta]$. Also, since $h'_\alpha <^* h'_\beta$, there is a k such that $h'_\alpha(k) < h'_\beta(k)$. Say $\langle h'_{\beta(\alpha} : \alpha < \omega_1\rangle$ enumerates H_n. For each $\alpha < \omega_1$ let $S_\alpha = \{(m,k) : m \le f'_{\beta(\alpha)}(k)\}$. Then

(2) If $\alpha < \gamma < \omega_1$, then $S_\alpha \subset S_\gamma$

In fact, suppose that $\alpha < \gamma < \omega_1$ and $(m,k) \in S_\alpha$. Then $m \le f'_{\beta(\gamma)}(k)$. Also, there is a k such that $h'_{\beta(\alpha)}(k) < h'_{\beta(\gamma)}(k)$, so $(f'_{\beta(\alpha)}(k)+1,k) \in S_\gamma \backslash S_\alpha$. So (2) holds.

But $S_\alpha \subseteq \omega \times \omega$, contradiction.

Case 9. There is an uncountable $Y \subseteq X'$ such that $[Y]^2 \subseteq K_0$. Let g, with domain ω_1 enumerate Y such that $\alpha < \beta < \omega_1$ implies $g_\alpha <^* g_\beta$. Let $A = \{t \in {}^{<\omega}\omega : \exists g \in Y[t \subseteq g]\}$. For each $t \in A$ choose $\alpha_t \in \omega_1$ such that $t \subseteq g_{\alpha_t}$. Let $\gamma = (\sup_{t\in A} \alpha_t) + 1$. Thus $\gamma < \omega_1$.

(3) $\exists k_0[\{\beta \in (\gamma, \omega_1) : \forall k \ge k_0[g_\gamma(k) < g_\beta(k)]\}$ is uncountable].

In fact,

$$(\gamma, \omega_1) = \bigcup_{k_0\in\omega} \{\beta \in (\gamma, \omega_1) : \forall k \ge k_0[g_\gamma(k) < g_\beta(k)], \}$$

and (3) follows.

(4) There is an uncountable $Z \subseteq \omega_1 \backslash \gamma$ such that $\forall \beta \in Z \forall k \ge k_0[g_\gamma(k) < g_\beta(k)]$ and $\forall \beta_1, \beta_2 \in Z[g_{\beta_1} \restriction k_0 = g_{\beta_2} \restriction k_0]$.

This is clear from (3), since ${}^{k_0}\omega$ is countable.

Now since Y is uncountable, it is cofinal in ω_1. Hence if Y is bounded, then so is X'. Assume that Y is unbounded. Then so is Z.

(5) There is an $m \ge k_0$ such that $\{g_\beta(m) : \beta \in Z\}$ is infinite.

In fact, suppose not. Define $f \in {}^\omega\omega$ by $f(m) = 0$ for $m < k_0$ and $f(m) = \sup\{g_\beta(m) : \beta \in Z\}$ for $m \ge k_0$. Then $g_\beta \le^* f$ for all $\beta \in Z$, contradicting Z being unbounded.

Let m be minimum such that $\{g_\beta(m) : \beta \in Z\}$ is infinite.

(6) There exist $t \in {}^m\omega$ and $W \subseteq Z$ such that $\forall \beta \in W[g_\beta \restriction m = t]$ and $\{g_\beta(m) : \beta \in W\}$ is infinite.

In fact, let $h(k) = \{g_\beta(k) : \beta \in Z\}$ for each $k < m$. So $h(k)$ is finite for each $k < m$. Let $B = \{t : \mathrm{dmn}(t) = m \text{ and } \forall k < m[t(k) \in h(k)]\}$. So B is finite. For all $\beta \in Z$ we have $g_\beta \restriction m \in B$. Now by (5) let T be an infinite subset of Z such that $\langle g_\beta(m) : \beta \in T\rangle$ is an injection of T into $\{g_\beta(m) : \beta \in Z\}$. Define $\beta \equiv \gamma$ iff $\beta, \gamma \in T$ and $g_\beta \restriction m = g_\gamma \restriction m$. Clearly there are finitely many $\equiv$-classes. So some $\equiv$-class is infinite, and this gives (6).

Now let $\alpha = \alpha_t$. Thus $\alpha < \gamma$, so $g_\alpha <^* g_\gamma$. Hence there is a $k_1 \ge m$ such that $\forall k \ge k_1[g_\alpha(k) < g_\gamma(k)]$. Choose $\beta \in W$ such that $g_\beta(m) \ge g_\alpha(k_1)$.

(7) $\forall k \le k_1[g_\alpha(k) \le g_\beta(k)]$

In fact, $g_\beta \restriction m = t = g_\alpha \restriction m$. From the above, $g_\alpha(m) \le g_\alpha(k_1) \le g_\beta(m)$. If $m < k \le k_1$, then $g_\alpha(k) \le g_\alpha(k_1) \le g_\beta(m) \le g_\beta(k)$. So (7) holds.

(8) $\forall k[g_\alpha(k) \le g_\beta(k)$.

For, suppose that $k_1 < k$. Now $k_0 \leq m \leq k_1 < k$, so $g_\alpha(k) \leq g_\gamma(k) < g_\beta(k)$. Then (8) follows from (7).

Now (8) contradicts $\{g_\alpha, g_\beta\} \in K_0$. It follows that Y is bounded; hence X' and X are also bounded. $\square$

We now consider gaps in ${}^\omega\omega$, which is different from gaps in $[\omega]^\omega$. We write $f <^* g$ iff $f, g \in {}^\omega\omega$ and $\exists m \forall n \geq m[f(n) < g(n)]$. For infinite cardinals κ, λ, a (κ, λ)-gap in ${}^\omega\omega$ is a pair (f, g) such that

(1) $f \in {}^\kappa({}^\omega\omega)$ and $g \in {}^\lambda({}^\omega\omega)$.
(2) $\forall \alpha, \beta < \kappa[\alpha < \beta \to f_\alpha <^* f_\beta]$.
(3) $\forall \alpha, \beta < \kappa[\alpha < \beta \to g_\beta <^* g_\alpha]$.
(4) $\forall \alpha < \kappa \forall \beta < \lambda[f_\alpha <^* g_\beta]$.
(5) There is no $h \in {}^\omega\omega$ such that $\forall \alpha < \kappa \forall \beta < \lambda[f_\alpha <^* h <^* g_\beta]$.

We define $f \prec g$ iff $f, g \in {}^\omega\omega$ and $\lim_{n\to\infty}(g(n) - f(n)) = \infty$.

Proposition 29.40. *If $f \prec g$ then $f <^* g$.* $\square$

Proposition 29.41. *If $f_0 <^* f_1 <^* \cdots$ and $\forall n[f_n <^* g]$, then $f_0 \prec g$.*

Proof. Assume that $f_0 <^* f_1 <^* \cdot$, $\forall n[f_n <^* g]$, and $M > 0$. For each $i \leq M$ choose m_i so that $\forall n \geq m_i[f_i(n) < f_{i+1}(n)$. Also, let m' be such that $\forall n \geq m'[f_M(n) < g(n)]$. Then for all $n \geq \max_{i \leq M} m_i$ and $n \geq m'$ we have

$$f_0(n) < f_1(n) < \cdots < f_M(n) < g(n),$$

and so $g(n) - f_0(n) \geq M$. $\square$

For distinct $f, g \in {}^\omega\omega$, $G(f, g)$ is the least $n \in \omega$ such that $f(n) \neq g(n)$.

Lemma 29.42. *If κ and λ are uncountable and regular, then there is a (κ, λ)-gap iff there is a (λ, κ)-gap.*

Proof. Assume that κ and λ are uncountable and regular and (f, g) is a (κ, λ)-gap. For each $\alpha < \kappa$ and $k \in \omega$ let

$$h_\alpha(k) = \begin{cases} g_0(k) - f_\alpha(k) & \text{if } f_\alpha(k) \leq g_0(k), \\ 0 & \text{otherwise;} \end{cases}$$

$$m_\beta(k) = \begin{cases} g_0(k) - g_\beta(k) & \text{if } g_\beta(k) \leq g_0(k), \\ 0 & \text{otherwise.} \end{cases}$$

(1) $\forall \beta < \lambda \forall \alpha < \kappa[m_\beta <^* h_\alpha]$,

For, suppose that $\beta < \lambda$ and $\alpha < \kappa$. Choose n_0 so that $\forall k > n_0[g_0(k) > f_\alpha(k)]$. Take $k > n_0$. Then $h_\alpha(k) = g_0(k) - f_\alpha(k)$. Let n_1 be such that $\forall k \geq n_1[g_\beta(k) < g_0(k)]$. Then for $k \geq n_0, n_1$,

$$h_\alpha(k) - m_\beta(k) = g_0(k) - f_\alpha(k) - g_0(k) + g_\beta(k) = g_\beta(k) - f_\alpha(k) > 0.$$

(2) If $\nu < \beta$, then $m_\nu <^* m_\beta$

For, suppose that $\nu < \beta$. Choose n_2 so that $\forall k \geq n_2[g_\beta(k) < g_\nu(k)]$. Also, choose n_3 so that $\forall k \geq n_3[g_\nu(k) \leq g_0(k)$ and $g_\beta(k) \leq g_0(k)$. Then

$$\forall k \geq n_2, n_3[m_\nu(k) = g_0(k) - g_\nu(k) < g_0(k) - g_\beta(k) = m_\beta(k).$$

(3) If $\alpha < \beta$, then $h_\beta <^* h_\alpha$.

Suppose that $\alpha < \beta$. Choose n_4 so that $\forall k \geq n_4[f_\alpha(k) < f_\beta(k)]$. Choose n_5 so that $\forall k \geq n_5[g_\alpha(k) \leq g_0(k)$ and $g_\beta(k) \leq g_0(k)]$. Then for any $k \geq n_4, n_5$.

$$h_\beta(k) = l(k) - f_\beta(k) < l(k) - f_\alpha(k) = h_\alpha(k).$$

(4) (m, h) is a (λ, κ) gap.

In fact, suppose that $l \in {}^\omega\omega$ and $\forall \alpha < \lambda \forall \beta < \kappa[m_\alpha <^* l <^* h_\beta]$. Define $s \in {}^\omega\omega$ by

$$s(\kappa) = \begin{cases} g_0(k) - l(k) & \text{if } l(k) \leq g_0(k), \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $\alpha < \lambda$; we want to show that $s <^* g_\alpha$. Choose n_6 so that $\forall k \geq n_6[m_\alpha(k) < l(k)]$. Choose n_7 such that $\forall k \geq n_7[g_\alpha(k) \leq g_0(k)]$; then $m_\alpha(k) = g_0(k) - g_\alpha(k)$. Hence for $k \geq n_6, n_7$ we have $g_0(k) - g_\alpha(k) < l(k)$, hence $g_0(k) - l(k) < g_\alpha(k)$. Choose n_8 so that $\forall k \geq n_8[l(k) < h_0(k) \leq g_0(k)]$. Hence for $k \geq n_6, n_7, n_8$ we have $s(k) < g_\alpha(k)$.

Finally, suppose that $\beta < \kappa$; we want to show that $f_\beta <^* s$. Choose n_9 so that $\forall k \geq n_9[l(k) < h_\beta(k)]$. Choose n_{10} so that $\forall k \geq n_{10}[f_\beta(k) < g_0(k)]$. Hence $\forall k \geq n_9, n_{10}[l(k) < g_0(k) - f_\beta(k)]$, hence $f_\beta(k) < g_0(k) - l(k) = s(k)$. □

Theorem 29.43. (OCA) *There is no (κ, λ)-gap in ${}^\omega\omega$ such that κ and λ are regular uncountable and $\kappa > \omega_1$.*

Proof. Suppose that (f, g) is a (κ, λ)-gap in ${}^\omega\omega$ with κ and λ regular uncountable and $\kappa > \omega_1$. By Lemma 6 we may assume that $\kappa \geq \lambda$.

(1) $\forall \alpha < \kappa \exists m_\alpha[\forall k \geq m_\alpha\{\beta < \lambda : f_\alpha(k) < g_\beta(k)\}$ has size $\lambda]$.

In fact, let $\alpha < \kappa$. Then

$$\lambda = \bigcup_{m \in \omega} \{\beta < \lambda : \forall k \geq m[f_\alpha(k) < g_\beta(k)]\},$$

and (1) follows.

Now there exist $K \in [\kappa]^\kappa$ and n such that $\forall \alpha \in K[m_\alpha = n]$. Thus

(2) $\forall \alpha \in K[\{\beta < \lambda : \forall k \geq n[f_\alpha(k) < g_\beta(k)]\}$ has size $\lambda]$.

(3) There is a (κ, λ)-gap (f', g') such that $\forall \alpha < \kappa[\{\beta < \lambda : \forall k[f'_\alpha(k) < g'_\beta(k)]\}$ has size $\lambda]$.

For, let $\langle \alpha_\xi : \xi < \kappa \rangle$ be a bijection from κ onto K. For each $\xi < \kappa$ and $k \in \omega$ let $f'_\xi(k) = f_{\alpha_\xi}(k + n)$, and for each $\eta < \lambda$ and $k \in \omega$ let $g'_\eta(k) = g_\eta(k + n)$.

If $\xi < \eta < \kappa$, choose $n \in \omega$ such that $\forall k \geq n[f_{\alpha_\xi}(k) < f_{\alpha_\eta}(k)]$ Then $\forall k \geq n[f'_\xi(k) = f_{\alpha_\xi}(k+n) < f_{\alpha_\eta}(k+n) = f'_\eta(k)]$. Thus if $\xi < \eta$ then $f'_\xi <^* f_\eta$.

If $\xi < \eta < \lambda$, choose $p \in \omega$ such that $\forall k \geq p[g_\eta(k) < g_\xi(k)]$. Then $\forall k \geq p[g'_\eta(k) = g_\eta(k+n) < g_\xi(k+n) = g'_\xi(k)]$.

Now suppose that $\xi < \kappa$ and $\eta < \lambda$. Choose m so that $\forall k \geq m[f_{\alpha_\xi}(k) < g_\eta(k)]$. Then $\forall k \geq m[f'_\xi(k) = f_{\alpha_\xi}(k+n) < g_\eta(k+n) = g'_\eta(k)$.

Suppose that $h \in {}^\omega\omega$ and $\forall \xi < \kappa \forall \eta < \lambda[f'_\xi <^* h <^* g'_\eta]$. Define $s \in {}^\omega\omega$ by

$$s(k) = \begin{cases} 0 & \text{if } k < n, \\ h(k-n) & \text{if } n \leq k. \end{cases}$$

Suppose that $\xi < \kappa$. Choose m so that $\forall k \geq m[f'_\xi(k) < h(k)]$. Then $\forall k \geq m+n[f_{\alpha_\xi}(k) = f_{\alpha_\xi}(k-n+n) = f'_\xi(k-n) < h(k-n) = s(k)]$. Hence $f_{\alpha_\xi} <^* s$.

Now suppose that $\xi < \lambda$. Choose m so that $\forall k \geq m[h(k) < g'_\xi(k)]$. Then $\forall k \geq m+n[g_{\alpha_\xi}(k) = g_{\alpha_\xi}(k-n+n) = g'_\xi(k-n) > h(k-n) = s(k)]$. Hence $s <^* g_{\alpha_\xi}$.

Clearly this contradicts (f,g) being a (κ,λ)-gap. Hence (f',g') is a (κ,λ)-gap.

(4) $\forall \xi < \kappa[S_\xi \stackrel{\text{def}}{=} \{\eta < \lambda : \forall k[f'_\xi(k) < g'_\eta(k)]\}$ has size $\lambda]$.

For, take any $\xi < \kappa$. Then

$$\{\eta < \lambda : \forall k[f'_\xi(k) < g'_\eta(k)]\} = \{\eta < \lambda : \forall k[f_{\alpha_\xi}(k+n) < g_\eta(k+n)]\}$$

This set has size λ by (2).

Now let

$$\begin{aligned} X &= \{(f'_\alpha, g'_\beta) : \alpha < \kappa, \beta \in S_\alpha\}, \\ K_0 &= \{(f'_\alpha, g'_\beta), (f'_\gamma, g'_\delta)\} : (f'_\alpha, g'_\beta), (f'_\gamma, g'_\delta) \in X \text{ and} \\ &\quad \exists k[(f'_\alpha(k) \geq g'_\delta(k)) \text{ or } (f'_\gamma(k) \geq g'_\beta(k))]\} \\ K_1 &= [X]^2 \backslash K_0. \end{aligned}$$

Note that $K_0 \subseteq ({}^\omega\omega \times {}^\omega\omega$; we claim that it open. In fact, suppose that $(f'_\alpha, g'_\beta), (f'_\gamma, g'_\delta) \in K_0$. Choose k accordingly.

Case 1. There is a $k \in \omega$ such that $(f'_\alpha(k) > g'_\delta(k))$. Then

$$(f'_\alpha, g'_\delta) \in U_{(k, f'_\alpha(k)} \times U_{(k, g'_\delta(k)} \subseteq K_0.$$

Case 2, Similarly.

Thus K_0 is open. So by OCA we have two cases.

Case 1. There exist H_n for $n \in \omega$ such that $X = \bigcup_{n\in\omega} H_n$ and $\forall n[[H_n]^2 \subseteq K_1]$.

(5) $\exists A \subseteq \kappa[|A| = \kappa \wedge \forall \alpha < \kappa \exists T_\alpha \subseteq S_\alpha[|T_\alpha| = \lambda \wedge \exists n \in \omega \forall \alpha \in A \forall \beta \in T_\alpha[(f'_\alpha, g'_\beta) \in H_n]]]$.

In fact, for each $\alpha \in \kappa$ we have

$$\lambda = \bigcup_{n\in\omega} \{\beta < \lambda : (f'_\alpha, g'_\beta) \in H_n\}$$

so there is an n_α such that $\{\beta < \lambda : (f'_\alpha, g'_\beta) \in H_{n_\alpha}\}$ has size λ. Then there is an n such that $\{\alpha < \kappa : n_\alpha = n\}$ has size κ. This gives (5).

Let A, $\langle T_\alpha : \alpha < \kappa\rangle$, and $\langle n_\alpha : \alpha < \kappa\rangle$ be as in (5).

(6) $\forall \alpha, \gamma \in A \forall \delta \in T_\gamma \forall k \in \omega[f'_\alpha(k) < g'_\delta(k)]$.

In fact, let $\alpha, \gamma \in A$, $\delta \in T_\gamma$, and $k \in \omega$. Choose $\beta \in T_\gamma$ with $\beta \neq \delta$. Then by (5), $(f'_\alpha, g'_\beta), f'_\alpha, g'_\delta) \in H_n$. By the case assumption it follows that $f'_\alpha(k) < g'_\beta(k)$ and $f'_\alpha(k) < g'_\delta(k)$. Thus (6) holds.

Now fix $\gamma \in A$ and let $B = T_\gamma$. Now A is cofinal in κ, B is cofinal in λ, and $\forall \alpha \in A$ and $\beta \in B$, if $k \in \omega$ then $f'_\alpha(k) < g'_\beta(k)$ by (6). Define $h \in {}^\omega\omega$ by

$$h(k) = \min_{\beta \in B} g'_\beta(k).$$

Then $\forall \alpha < \kappa \forall \beta \in \lambda[f'_\alpha <^* h <^* g'_\beta$. This contradicts (κ, λ) beirg a gap.

Case 9. There is an uncountable $Y \subseteq X$ such that $[Y]^2 \subseteq K_0$.

(7) Y is a one-one function.

For, first suppose that $(f'_\alpha, g'_\beta).(f'_\alpha.g'_\gamma) \in Y$ with $\beta \neq \gamma$. Then by the definition of K_0, there is a $k \in \omega$ such that $f'_\alpha(k) \geq g'_\beta(k)$ or $f'_\alpha(k) \geq g'_\gamma(k)$. Both of these contradict the definition of S_β. Thus Y is a function.

If $(f'_\alpha.g'_\beta), (f'_\gamma.g'_\beta) \in Y$, a similar contradiction follows. So Y is one-one. Now define

α_0 minimum such that $f'_{\alpha_0} \in \mathrm{dmn}(Y)$;
α_ν minimum such that $\forall \beta < \nu[f'_{\alpha_\beta} < f'_{\alpha_\nu} \in \mathrm{dmn}(Y)]$ for $\nu < \omega_1$;
β_ν such that $(f'_{\alpha_\nu}, g'_{\beta_\nu}) \in Y$ for $\nu < \omega_1$.

Thus α and β are increasing ω_1-sequences such that $\{(f'_{\alpha_\nu}, g'_{\beta_\nu}) : \nu < \omega_1\} \subseteq Y$. Since κ is regular and greater than ω_1, there is an f'_δ with $\forall \nu < \omega_1[\alpha_\nu < \delta]$. Then $\forall \nu < \omega_1[f'_{\alpha_\nu} <^* f'_\delta <^* g'_{\beta_\nu}$.

(8) $\exists Z \in [\omega_1]^{\omega_1} \exists m \in \omega[\forall \nu, \eta \in Z \forall k \geq m[f'_{\alpha_\nu}(k) < f'_\delta(k) < g'_{\beta_\eta}(k)]$ and $f'_{\alpha_\nu} \restriction m = f'_{\alpha_\eta} \restriction m$ and $g'_{\beta_\nu} \restriction m = g'_{\beta_\eta} \restriction m]$.

For, for each $\nu < \omega_1$ let q_ν be such that $\forall k \geq q_\nu[f'_{\alpha\nu}(k) < f'_\delta(k) < g'_{\beta_\eta}(k)]$ Then there exist a $P \in [\omega_1]^{\omega_1}$ and an m such that $q_\nu = m$ for all $\nu \in P$. Now

$$P = \bigcup\{\nu \in P : f'_{\alpha_\nu} \restriction m = s \text{ and } g'_{\beta_\nu} \restriction m = t : s \in {}^m\omega \text{ and } t \in {}^m\omega\}.$$

Hence (8) follows.

Now suppose that $\nu, \eta \in P$ with $\nu \neq \eta$. If $k \geq m$, then $f'_{\alpha_\nu}(k) < g'_{\beta_\eta}(k)$ by (8). If $k < m$, then $f'_{\alpha\nu}(k) = f'_{\alpha_\eta}(k) < g'_{\beta_\eta}(k)$. Thus $\forall k \in \omega[f'_{\alpha_\nu}(k) < g'_{\beta_\eta}(k)]$. Similarly, $\forall k \in \omega[f'_{\alpha_\eta}(k) < g'_{\beta_\nu}(k)]$. But $\{(f'_{\alpha_\nu}, g'_{\beta_\nu}), (f'_{\alpha_\eta}, g'_{\beta_\eta})\} \in K_0$, contradiction. □

Lemma 29.44. *If $\mathfrak{b} > \omega_2$, then there is a regular uncountable cardinal λ such that there is an (ω_2, λ)-gap.*

Proof. Suppose that $\mathfrak{b} > \omega_2$.

(1) There is a $f \in {}^{\omega_2}({}^\omega\omega)$ such that each f_α is increasing and $f_\alpha <^* f_\beta$ for $\alpha < \beta < \omega_2$.

For, if f_α has been defined for $\alpha < \beta$, where $\beta < \omega_2$, let $g \in {}^\omega\omega$ be such that $f_\alpha <^* g$ for all $\alpha < \beta$. Now define f_β as follows.

$$f_\beta(n) = \max(g(n), \max\{f_\beta(m) + 1 : m < n\}).$$

This proves (1).

Since $\omega_2 < \mathfrak{b}$, there is a $g_0 \in {}^\omega\omega$ such that $\forall\alpha < \omega_2[f_\alpha <^* g_0]$. Let θ be the largest ordinal such that there is a decreasing under $<^*$ sequence $\langle g_\alpha : \beta < \theta\rangle$ of members of ${}^\omega\omega$ all greater under $<^*$ than each f_α. Then θ is a limit ordinal. For, suppose that $\theta = \beta + 1$. Define $h \in {}^\omega\omega$ by

$$h(k) = \begin{cases} 0 & \text{if } g_\beta(k) = 0, \\ g_\beta(k) - 1 & \text{otherwise.} \end{cases}$$

If $\alpha < \omega_2$, choose $m \in \omega$ such that $\forall k > m[f_\alpha(k) < f_{\alpha+1}(k) < g_\beta(k)]$. Then $\forall k > m[f_\alpha(k) < h(k)]$. Thus $f_\alpha <^* h$. Now choose p so that $\forall k > p[f_0(k) < g_\beta(k)]$. Then $\forall k > p[h(k) < g_\beta(k)]$. So $h <^* g_\beta$. This contradicts the maximality of $\langle g_\beta : \beta < \theta\rangle$. So θ is a limit ordinal.

Now it suffices to show that $\mathrm{cf}(\theta) > \omega$. Suppose not, and let $\langle \beta_n : n \in \omega\rangle$ be strictly increasing with sup θ. Now define $g \in {}^\omega\omega$ by $g(n) = min(\{g_{\beta_k}(n) : k \le n\}$.

(2) $\forall\alpha < \omega_2[f'_\alpha <^* g]$

In fact, suppose that $\alpha < \omega_2$. Choose p such that $\forall k \ge p \forall s \le n[f'_\alpha(k) < g_{\beta_s}(k)]$. Hence $f'_\alpha <^* g$.

(3) $\forall n \in \omega[g <^* g_{\beta_n}]$.

For, take any $n \in \omega$. If $k \ge n$, then $g(k) \le g_{\beta_n}(k)$.

So (3) holds, contradiction. □

Corollary 29.45. (OCA) $\mathfrak{b} = \omega_2$.

Proof. Assume OCA. By Lemma 29.39, $\mathfrak{b} \ge \omega_2$. If $\mathfrak{b} > \omega_2$, then by Theorem 29.44 there is an $(\omega_9.\lambda)$-gap. This contradicts Theorem 29.43. □

30. Complete Boolean algebras

Let B be a complete BA. A *measure* on B is a function $\mu : B \to \mathbb{R}$ such that

(i) $\mu(0) = 0$.

(ii) $\forall a \in B[\mu(a) \geq 0]$.

(iii) For all pairwise disjoint a_n, $n \in \omega$,

$$\mu\left(\sum_{n\in\omega} a_n\right) = \sum_{n\in\omega} \mu(a_n).$$

A measure μ is *strictly positive* iff $\forall a \in B^+[\mu(a) > 0]$. A measure μ is *probabilistic* iff $\mu(1) = 1$. A function $\mu : B \to \mathbb{R}$ is a *signed measure* iff (i) and (iii) hold.

Lemma 30.1. *Let B be a ccc complete BA. If ν is a signed measure on B, then there is an $a \in B$ such that $\forall x \leq a[\nu(x) \geq 0]$ and $\forall x \leq -a[\nu(x) \leq 0]$.*

Proof.

(1) If $\nu(a) > 0$ then there is a $b \leq a$ such that $\nu(b) > 0$ and $\forall x \leq b[\nu(x) \geq 0]$.

In fact, if (1) fails, then for every $b \leq a$, $\nu(b) \leq 0$ or $\exists x \leq b[\nu(x) < 0]$. So

(2) for every nonzero $b \leq a$ there is a nonzero $x \leq b$ such that $\nu(x) \leq 0$.

In fact, either $\nu(b) \leq 0$ or there is such an x.

Let W be a maximal antichain of nonzero $b \leq a$ with $\nu(b) \leq 0$. Then $\sum W = a$, as otherwise we could apply (2) to $a \cdot -\sum W$ and contradict the maximality of W. Now it follows that $\nu(a) \leq 0$, contradiction. So (1) holds.

If $\forall a \in B[\nu(a) \leq 0]$, then the lemma holds trivially. Otherwise let Z be a maximal antichain of elements b such that $\nu(b) > 0$ and $\forall x \leq b[\nu(x) \geq 0]$; by (1), Z is nonempty. Then let $a = \sum Z$. Then $\nu(a) > 0$, and $\forall x \leq a[\nu(x) \geq 0]$. If $x \leq -a$, then $x \cdot \sum Z = 0$. If $\nu(x) > 0$, then by (1) there is a $b \leq x$ such that $\nu(b) > 0$ and $\forall x \leq b[\nu(x) \leq 0]$, contradicting the maximality of Z. □

Lemma 30.2. *Let B be a complete BA and μ, ν measures on B. Suppose that $a \in B$ with $\nu(a) > 0$. Then there exist $b \leq a$ with $b \neq 0$ and $\varepsilon > 0$ such that $\forall x \leq b[\varepsilon \cdot \mu(x) \leq \nu(x)]$.*

Proof. Choose $\varepsilon > 0$ so that $\nu(a) > \varepsilon \cdot \mu(a)$. For all $x \leq a$ let $\rho(x) = \nu(x) - \varepsilon \cdot \mu(x)$. Clearly

(1) ρ is a signed measure on $B \upharpoonright a$.

By Lemma 30.1 there is a $b \leq a$ such that $\forall x \leq b[\rho(x) \geq 0]$ and $\forall x \leq a \cdot -b[\rho(x) \leq 0]$. Hence $\forall x \leq b[\nu(x) \geq \varepsilon \cdot \mu(x)]$. Also, $\nu(b) - \varepsilon \cdot \mu(b) = \rho(b) \geq \rho(a) > 0$, so $b \neq 0$. □

A *measure algebra* is a pair (B, m) such that B is a complete BA, $m : B \to \mathbb{R}$, and

(i) $m(0) = 0$; $\forall a \in B^+[m(a) > 0]$; $m(1) = 1$.

(ii) $\forall a, b \in B[a \leq b \to m(a) \leq m(b)]$.

(iii) For all pairwise disjoint $a \in {}^{\omega}B$,

$$m\left(\sum_{n\in\omega} a_n\right) = \sum_{n\in\omega} m(a_n).$$

Lemma 30.3. *Every measure algebra is ccc.*

Proof. Suppose that (B, m) is a measure algebra and $a \in {}^{\omega_1}B^+$ is pairwise disjoint. Then $\omega_1 = \bigcup_{n\in\omega}\{\alpha < \omega_1 : m(a_\alpha) \geq \frac{1}{n+1}\}$. So there is an $n \in \omega$ such that $|\{\alpha < \omega_1 : m(a_\alpha) \geq \frac{1}{n+1}\}| = \omega_1$, contradiction. □

Lemma 30.4. *If μ and ν are measures on a complete BA B and $\forall b \in B[\mu(b) \leq \nu(b)]$, then $\nu - \mu$ is a measure on B, where $(\nu - \mu)(b) = \nu(b) - \mu(b)$ for all $b \in B$.* □

Lemma 30.5. *If (B, m) is a measure algebra, $Y \in {}^{\omega}B$, and $\forall n \in \omega[Y_n \leq Y_{n+1}]$, then $m(\sum_{n\in\omega} Y_n) = \sup_{n\in\omega} m(Y_n)$.*

Let $Z_n = Y_n \cdot -\sum_{m<n} Y_m$. By induction, $\forall n \in \omega[Y_n = \sum_{m\leq n} Z_m$, and hence $\sum_{n\in\omega} Y_n = \sum_{n\in\omega} Z_n$. Hence

$$\begin{aligned}
m\left(\sum_{n\in\omega} Y_n\right) &= m\left(\sum_{n\in\omega} Z_n\right)\\
&= \sum_{n\in\omega} m(Z_n)\\
&= \lim_{n\to\infty} \sum_{m\leq n} m(Z_m)\\
&= \lim_{n\to\infty} m\left(\sum_{m\leq n} Z_m\right)\\
&= \lim_{n\to\infty} m(Y_n)\\
&= \sup_{n\in\omega} m(Y_n).
\end{aligned}$$

□

Theorem 30.6. *Let (A, m) be a measure algebra and μ a strictly positive measure on A. Suppose that B is a complete subalgebra of A and ν is a measure on B such that $\forall b \in B[\nu(b) \leq \mu(b)]$. Assume that*

$$(*) \qquad \forall a \in A^+[A \restriction a \neq \{a \cdot b : b \in B\}].$$

Then there is an $a \in A$ such that

$$\forall b \in B[\nu(b) = \mu(a \cdot b)].$$

Proof. For each $a \in A$ define $\nu_a : B \to \mathbb{R}$ by setting, for any $b \in B$, $\nu_a(b) = \mu(a \cdot b)$.

(1) ν_a is a measure on B.

For, $\nu_a(0) = \mu(a \cdot 0) = \mu(0) = 0$; for any $b \in B$, $\nu_a(b) = \mu(a \cdot b) \geq 0$; if $b \in {}^\omega B$ is pairwise disjoint, then

$$\nu_a\left(\sum_{n\in\omega} b_n\right) = \mu\left(\sum_{n\in\omega}(a \cdot b_n)\right) = \sum_{n\in\omega}\mu(a \cdot b_n) = \sum_{n\in\omega}\nu_a(b_n).$$

(2) $\forall a \in A^+ \exists c \in (A \restriction a)^+ \forall b \in B[\nu_c(b) \leq \frac{1}{2}\nu_a(b)]$.

In fact, suppose that $a \in A^+$. By $(*)$ there is a $d < a$ such that $\forall b \in B[d \neq a \cdot b]$. Define $\nu' : B \to \mathbb{R}$ by setting $\nu'(b) = \frac{1}{2}\nu_a(b) - \nu_d(b)$.

(3) ν' is a signed measure on B.

In fact, $\nu'(0) = \frac{1}{2}\nu_a(0) - \nu_d(0) = 0$; if $b \in {}^\omega B$ is pairwise disjoint, then

$$\nu'\left(\sum_{n\in\omega} b_n\right) = \frac{1}{2}\nu_a\left(\sum_{n\in\omega} b_n\right) - \nu_d\left(\sum_{n\in\omega} b_n\right) = \frac{1}{2}\sum_{n\in\omega}\nu_a(b_n) - \sum_{n\in\omega}\nu_d(b_n) = \sum_{n\in\omega}\nu'(b_n).$$

Now by Lemma 30.1 there is a $b \in B$ such that $\forall x \leq b[\nu'(x) \geq 0]$ and $\forall x \leq -b[\nu'(x) \leq 0]$. Thus $\forall x \leq b[\frac{1}{2}\nu_a(x) \geq \nu_d(x)]$ and $\forall x \leq -b[\frac{1}{2}\nu_a(x) \leq \nu_d(x)]$.

Case 1. $b \cdot d > 0$. Let $c = b \cdot d$. Then for all $b' \in B$,

$$\nu_c(b') = \nu_{b\cdot d}(b') = \mu(b \cdot d \cdot b') = \nu_d(b \cdot b') \leq \frac{1}{2}\nu_a(b \cdot b') \leq \frac{1}{2}\nu_a(b').$$

Case 9. $b \cdot d = 0$. Then $d \leq a \cdot -b$. Let $c = (a \cdot -b) \cdot (a \cdot -d)$. If $c = 0$, then $a \cdot -b \leq d$ so $a \cdot -b = d$, contradiction. So $c \neq 0$. Now take any $x \in B$. Then

$$\mu(a \cdot -b \cdot -d \cdot x) + \mu(a \cdot -b \cdot d \cdot x) = \mu(a \cdot -b \cdot x).$$

Hence

$$\begin{aligned}
\nu_c(x) &= \mu(a \cdot -b \cdot -d \cdot x) = \mu(a \cdot -b \cdot x) - \mu(a \cdot -b \cdot d \cdot x) \\
&= \mu(a \cdot -b \cdot x) - \mu(-b \cdot d \cdot x) \\
&= \nu_a(x \cdot -b) - \nu_d(x \cdot -b) \leq \frac{1}{2}\nu_a(x \cdot -b) \leq \frac{1}{2}\nu_a(x).
\end{aligned}$$

This proves (2).

Repeating (2) we obtain

(3) $\forall a \in A^+ \forall \varepsilon > 0 \exists c \in (A \restriction a)^+ \forall b \in B[\nu_c(b) \leq \varepsilon\nu_a(b)]$.

(4) There is a maximal $a \in A$ such that $\forall b \in B[\nu_a(b) \leq \nu(b)]$.

In fact, first note that $\forall b \in B[\nu_0(b) = \mu(0) \leq \nu(b)]$; so there is an a of the sort given in (4). Now suppose that C is a chain of such elements. By ccc we may assume that $C = \{a_n : n \in \omega\}$ with $a_n \leq a_{n+1}$ for all $n \in \omega$. Then for any $b \in B$,

$$\nu_{\sup(\{a_n:n\in\omega\})}(b) = \mu\left(\sum_{n\in\omega}(a_n \cdot b)\right) = \sup\{\mu(a_n \cdot b) : n \in \omega\} = \sup\{\nu_{a_n}(b) : n \in \omega\} \leq \nu(b).$$

Hence (4) follows by Zorn's lemma.

We take a as in (4).

(5) $\forall b \in B[\nu_a(b) = \nu(b)]$.

Note that (5) gives the result desired in the theorem. Suppose that (5) fails. Say $b_1 \in B$ and $\nu_a(b_1) < \nu(b_1)$. By Lemma 30.2 choose $b_2 \leq b_1$ and $\varepsilon > 0$ so that $b_2 \neq 0$ and $\forall x \leq b_2[(\nu - \nu_a)(x) \geq \varepsilon\mu(x)]$.

(6) $b_2 \not\leq a$

In fact, otherwise $\nu_a(b_2) = \mu(b_2) \geq \nu(b_2)$, so that $\mu(b_2) = 0$ and hence $b_2 = 0$, contradiction.

Applying (3) to $b_2 \cdot -a$ we get $c \in (A \restriction (b_2 \cdot -a))^+$ such that $\forall b \in B[\nu_c(b) \leq \varepsilon\nu_{b_2\cdot -a}(b)]$. Now $\varepsilon\nu_{b_2\cdot -a}(b) = \varepsilon\mu(b_2 \cdot -a \cdot b) \leq \varepsilon\mu(b) \leq (\nu - \nu_a)(b)$. So $\forall b \in B[\nu_c(b) + \nu_a(b) \leq \nu(b)]$. Since $a \cdot c = 0$ we have $a < a + c$ and $\forall b \in B[\nu_{a+c}(b) \leq \nu(b)]$. This contradicts the maximality of a. $\square$

If G is a subset of a measure algebra B, we say that G σ-generates B iff B is the smallest σ-algebra containing G. The *weight* of B is the least size of a G σ-generating B. B is weight-*homogeneous* iff each $B \restriction a$ with $a \in B^+$ has the same weight.

Lemma 30.7. *Let A be a σ-algebra and $X \subseteq A$. Define for $\alpha < \omega_1$*

$$\begin{aligned} Y_0 &= X; \\ Y_{\alpha+1} &= Y_\alpha \cup \left\{\sum Z : Z \in [Y_\alpha]^\omega\right\} \cup \{-Z : Z \in Y_\alpha\}; \\ Y_\gamma &= \bigcup_{\alpha<\gamma} Y_\alpha \quad \textit{for } \gamma \textit{ limit less than } \omega_1. \end{aligned}$$

Then $\bigcup_{\alpha<\omega_1} Y_\alpha$ is the σ-subalgebra of A generated by X. $\square$

Theorem 30.8. *Every measure algebra is the direct product of countably many weight-homogeneous measure algebras.*

Proof. Let B be a measure algebra.

(1) If $0 \neq a \leq b$, then weight$(B \restriction a) \leq$ weight$(B \restriction b)$.

For, let G σ-generate $B \restriction b$ with $|G| =$ weight$(B \restriction b)$. Let $G' = \{x \cdot a : x \in G\}$ we claim that G' σ-generates $B \restriction a$. For, let $\langle Y_\alpha : \alpha < \omega_1\rangle$ be defined as in the proof of Lemma

30.7 with G in place of X, and with $-$ in the sense of $B \upharpoonright b$. Let $\langle Y'_\alpha : \alpha < \omega_1 \rangle$ be defined similarly with G' in place of X and with $-$ in the sense of $B \upharpoonright a$.

(2) $\forall \alpha < \omega_1 [Y'_\alpha = \{x \cdot a : x \in Y_\alpha\}]$

We prove this by induction on α. It is clear for $\alpha = 0$. Now assume it for α. Let $x \in Y'_{\alpha+1}$.

Case 1. There is a $Z \in [Y'_\alpha]^\omega$ such that $x = \sum Z$. For each $c \in Z$ there is an $x_c \in Y_\alpha$ such that $c = x_c \cdot a$. Then $\sum_{c \in Z} x_c \in Y_{\alpha+1}$ and $\sum Z = a \cdot \sum_{c \in Z} x_c$.

Case 9. There is a $z \in Y'_\alpha$ such that $x = -^a z$. Say $z = w \cdot a$ with $w \in Y_\alpha$. Then $x = -^b w \cdot a$ and $-^b w \in Y_{\alpha+1}$.

This proves $\subseteq$ in (2) for $\alpha + 1$. Now suppose that $x \in Y_\alpha + 1$; we want to show that $x \cdot a \in Y'_{\alpha+1}$.

Case 1. There is a $Z \in [Y_\alpha]^\omega$ such that $x = \sum Z$. By the inductive hypothesis, $\forall z \in Z[z \cdot a \in Y'_\alpha]$. Hence $x \cdot a = a \cdot \sum Z = \sum\{z \cdot a : z \in Z\}$ and so $x \cdot a \in Y'_{\alpha+1}$.

Case 9. There is a $w \in Y_\alpha$ such that $x = -^b w$. Then $x \cdot a = -^a(w \cdot a)$ and $x \cdot a \in Y'_{\alpha+1}$.

Thus (2) holds for $\alpha + 1$. The limit case is clear, so (2) holds.

Now take any $x \in B \upharpoonright a$. Since $x \in B \upharpoonright b$, there is an $\alpha < \omega_1$ such that $x \in Y_\alpha$. By (2), $x \in Y'_\alpha$. So G' generates $A \upharpoonright a$, and so (1) holds.

Now let $X \subseteq B$ be maximal with respect to being pairwise disjoint with each $x \in X$ such that $B \upharpoonright x$ is weight-homogeneous. By (1), $\sum X = 1$, as desired in the theorem. $\square$

If μ and ν are probabilistic measures on complete BAs A, B respectively, then an isomorphism f of A onto B is *measure preserving* iff $\forall a \in A[\nu(f(a)) = \mu(a)]$.

Lemma 30.9. *Suppose that A is a complete subalgebra of a complete BA B and $b \in B \backslash A$. Then the complete subalgebra C of B generated by $A \cup \{b\}$ is $\{x \cdot b + y \cdot -b : x, y \in A\}$.*

Proof. Clearly $\{x \cdot b + y \cdot -b : x, y \in A\} \subseteq C$. It suffices to show that this set is closed under $-$ and $\sum$. Clearly $-(x \cdot b + y \cdot -b) = -x \cdot b + -y \cdot -b$. Also,

$$\sum_{i \in I}(x_i \cdot b + y_i \cdot -b) = \left(\sum_{i \in I} x_i\right) \cdot b + \left(\sum_{i \in I} y_i\right) \cdot -b. \qquad \square$$

Lemma 30.10. *Let A_1 and A_2 be weight homogeneous complete BAs, each with weight κ, and let μ_1 and μ_2 be probabilistic measures on A_1 and A_2. Suppose that B_1 and B_2 are complete subalgebras of A_1 and A_2 respectively, and suppose that f is a measure preserving isomorphism of B_1 onto B_2. Also, suppose that B_1 is σ-generated by a set with less than κ elements.*

Then for every $a_1 \in A_1$ there exist an $a_2 \in A_2$ and a measure preserving isomorphism $g \supseteq f$ of $\langle B_1 \cup \{a_1\} \rangle^{\mathrm{cm}}$ onto $\langle B_2 \cup \{a_2\} \rangle^{\mathrm{cm}}$.

Proof. First note:

(1) $\forall a \in A_1^+[A_1 \upharpoonright a \neq \{a \cdot b : b \in B_1\}]$.

For, suppose that $a \in A_1^+$. Now $\{a \cdot b : b \in B_1\}$ is σ-generated by a set with less than κ elements, while $A_1 \upharpoonright a$ is not. So (1) holds.

Similarly,

(2) $\forall a \in A_2^+[A_2 \restriction a \neq \{a \cdot b : b \in B_2\}]$.

Now for each $a_1 \in A_1$ and $b \in B_1$ define $\nu_{a_1}(f(b)) = \mu_1(a_1 \cdot b)$. Clearly ν_{a_1} is a measure on B_2. Also, for any $b \in B_1$, $\mu_2(f(b)) = \mu_1(b) \geq \mu_1(a_1 \cdot b) = \nu_{a_1}(f(b))$; so $\nu_{a_1} \leq \mu_2$. Applying Theorem 30.6 to B_2, ν_{a_1}, and μ_2 we get $a_2 \in A_2$ such that for all $b \in B_1$, $\nu_{a_1}(f(b)) = \mu_2(a_2 \cdot f(b))$.

(3) There is a homomorphism g of $\langle B_1 \cup \{a_1\}\rangle$ onto $\langle B_2 \cup \{a_2\}\rangle$ extending f, with $g(a_1) = a_2$.

To prove this we apply Corollary 5.8 of the BA handbook. So, suppose that $b, b' \in B_1$ and $b \leq a_1 \leq b'$; we want to show that $f(b) \leq a_2 \leq f(b')$. We have $\mu_2(f(b)) = \mu_1(b) = \mu_1(a_1 \cdot b) = \nu_{a_1}(f(b)) = \mu_2(a_2 \cdot f(b))$. It follows that $\mu_2(f(b) \cdot -a_2) = 0$, so $f(b) \cdot -a_2 = 0$, i.e., $f(b) \leq a_2$. Also, $\mu_2(a_2 \cdot f(b')) = \nu_{a_1}(f(b')) = \mu_1(a_1 \cdot b') = \mu_1(a_1) = \nu_{a_1}(1) = \mu_2(a_2)$. It follows that $a_2 \cdot -f(b') = 0$, i.e. $a_2 \leq f(b')$. This proves (3).

By symmetry the function g given in (3) is an isomorphism of $\langle B_1 \cup \{a_1\}\rangle^{\text{cmp}}$ onto $\langle B_2 \cup \{a_2\}\rangle^{\text{cmp}}$. It is measure preserving, since for $b, b' \in B_1$,

$$\mu_2(g((b \cdot a_1 + b' \dot{-} a_1))) = \mu_2(f(b) \cdot a_2 + f(b') \cdot -a_2) = \mu_2(f(b) \cdot a_2) + \mu_2(f(b') \cdot -a_2).$$

Now $\mu_2(f(b) \cdot a_2) = \nu_{a_1}(f(b)) = \mu_1(a_1 \cdot b)$. Also $\mu_2(f(b') \cdot -a_2) + \mu_2(f(b') \cdot a_2) = \mu_2(f(b')) = \mu_1(b')$, so $\mu_2(f(b') \cdot -a_2) = \mu_1(b') - \mu_2(f(b') \cdot a_2) = \mu_1(b') - \mu_1(a_1 \cdot b') = \mu_1(b' \cdot -a_1)$. Thus $\mu_2(g((b \cdot a_1 + b' \dot{-} a_1))) = \mu_1(b \cdot a_1 + b' \cdot -a_1)$. □

Lemma 30.11. *Suppose that A and B are BAs with completions $\overline{A}, \overline{B}$ and f is an isomorphism of A onto B. Then there is an isomorphism g of $\overline{A}$ onto $\overline{B}$ which extends f.*

Proof. By Sikorski's extension theorem let $g : \overline{A} \to \overline{B}$ extend f and $h : \overline{B} \to \overline{A}$ extend f^{-1}. Then $g \circ h : \overline{A} \to \overline{A}$ is one-one. In fact, suppose that $a \in \overline{A}^+$ and $g(h(a)) = 0$. Choose $x \in A^+$ such that $x \leq a$. Then $x = g(h(x)) \leq g(h(a))$, contradiction. Similarly, $h \circ g$ is one-one. We claim that $\text{rng}(h \circ g) = \overline{A}$. For, suppose that $a \in \overline{A}$. Write $a = \sum X$ with $X \subseteq A$. Clearly $\sum^{\overline{A}} X \leq \sum^{\text{rng}(h\circ g)} X$. Suppose that $\sum^{\text{rng}(h\circ g)} X \cdot -\sum^{\overline{A}} X \neq 0$. Choose $a \in A^+$ such that $a \leq \sum^{\text{rng}(h\circ g)} X \cdot -\sum^{\overline{A}} X$. Since $a \leq -\sum^{\overline{A}} X$, we have $\forall x \in X[a \cdot x = 0]$. This contradicts $a \leq \sum^{\text{rng}(h\circ g)} X$. □

Theorem 30.12. *If A and B are infinite weight homogeneous measure algebras with the same weight, and μ and ν are strictly positive probabilistic measures on A and B, then there is an isomorphism f from A onto B which is measure preserving.*

Proof. Assume the hypotheses. Say A and B have weight κ. Let $\langle a_\alpha : \alpha < \kappa\rangle$ and $\langle b_\alpha : \alpha < \kappa\rangle$ be σ-generating sequences for A and B. We construct $A_0 \subseteq \cdots \subseteq A_\alpha \subseteq \cdots$, $B_0 \subseteq \cdots \subseteq B_\alpha \subseteq \cdots$, $f_0, \ldots, f_\alpha \ldots$ so that for every $\alpha < \kappa$, A_α is a complete subalgebra of A of weight less than κ, $a_\alpha \in A_{\alpha+1}$, similarly for B_α, B and b_α, f_α is a measure preserving isomorphism from A_α onto B_α, $\bigcup_{\alpha<\kappa} A_\alpha = A$, and $\bigcup_{\alpha<\kappa} B_\alpha = B$. We start with $A_0 = B_0 = 2$ and f_0 the identity. For the successor step we apply Lemma 30.10 to A_α and then to B_α. Now suppose that α is limit. Let $A'_\alpha = \bigcup_{\beta<\alpha} A_\beta$ and $f'_\alpha = \bigcup_{\beta<\alpha} f_\beta$. Let A_α be the completion of A'_α.

(1) $A_\alpha = \{\sum X : X \subseteq A'_\alpha\}$.

In fact, let $C = \{\sum X : X \subseteq A'_\alpha\}$. Clearly C is closed under $\sum$. Given $X \subseteq A'_\alpha$, let $Y = \{a \in A'_\alpha : a \cdot \sum X = 0\}$. Clearly $\sum Y \cdot \sum X = 0$. Suppose that $a \stackrel{\text{def}}{=} -(\sum X + \sum Y) \neq 0$. Choose $b \leq a$ with $b \in A'_\alpha$ and $b \neq 0$. Then $b \cdot \sum X = 0$, so $b \in Y$. But $b \cdot \sum Y = 0$, contradiction. It follows that $\sum Y = -\sum X$, and (1) holds.

We similarly define B'_α and B_α and have

(2) $B_\alpha = \{\sum X : X \subseteq B'_\alpha\}$.

By Lemma 30.11, let f_α be an extension of f'_α to an isomorphism of A_α onto B_α. Now f_α is measure preserving. In fact, if $a \in A_\alpha$, write $a = \sum X$ with $X \subseteq A'_\alpha$. Write $\sum X = \sum X'$ with X' pairwise disjoint. Then

$$\begin{aligned}\nu(f_\xi(a)) &= \nu\left(\sum\{f_\xi(x) : x \in X'\}\right)\\ &= \sum\{\nu(f_\xi(x)) : x \in X'\} = \sum\{\mu(x) : x \in X'\} = \mu(a).\end{aligned}$$

This completes the construction, and $\bigcup_{\alpha<\kappa} f_\alpha$ is as desired. □

For each infinite cardinal κ, let $P_\kappa = \{f : f$ is a finite function with $\mathrm{dmn}(f) \subseteq \kappa$ and $\mathrm{rng}(f) \subseteq \{0,1\}\}$, with the order $\supseteq$. Let $C_\kappa = \mathrm{RO}(P_\kappa)$. A BA B is a *Cohen algebra* iff $\overline{B}^{\mathrm{cmp}} \cong C_\kappa$ for some infinite cardinal κ.

If A is a subalgebra of B and $b \in B$, then $A \restriction b = \{a \in A : a \leq b\}$.

A subalgebra A of a BA B is a *regular* subalgebra, $A \leq_{reg} B$, iff for any $X \subseteq A$, if $\sum^A X$ exists, then $\sum^B X$ exists and $\sum^A X = \sum^B X$.

If I is an ideal in A, then $I^d = \{b \in A : \forall a \in I[a \cdot b = 0]\}$. An ideal I is *regular* iff $I = I^{dd}$.

Lemma 30.13. *The following are equivalent:*

(i) $A \leq_{reg} B$.

(ii) Every maximal antichain in A is a maximal antichain in B.

(iii) $\forall b \in B^+ \exists a \in A^+[A \restriction (a \cdot -b) = \{0\}]$.

(iv) There is a dense subset M of B such that $\forall b \in M \exists a \in A^+[A \restriction (a \cdot -b) = \{0\}]$.

(v) For every regular ideal I of B, the ideal $I \cap A$ of A is regular.

(vi) $\forall b \in B[A \restriction b$ *is regular*$]$.

Proof. (i)⇒(ii): Assume (i), and suppose that X is a maximal antichain in A. Then $\sum^A X = 1$, so $\sum^B X = 1$ and hence X is a maximal antichain in B.

(ii)⇒(iii): Assume (ii), and suppose that $b \in B^+$. Choose maximal disjoint subsets M_0 and M_1 in $A \restriction b$ and $A \restriction -b$.

Case 1. There is an $a \in A^+$ disjoint from each member of $M_0 \cup M_1$. Then $a \cdot -b \neq 0$. We claim that $A \restriction (a \cdot -b) = \{0\}$. Suppose that $x \in A^+$ and $x \leq a \cdot -b$. Then $x \in A \restriction -b$, so there is a $y \in M_1$ such that $x \cdot y \neq 0$. But $a \cdot y = 0$ and $x \leq a$, so $x \cdot y = 0$, contradiction.

Case 9. $M_0 \cup M_1$ is maximal disjoint in A. Hence it is also maximal disjoint in B. Hence $b \cdot a \neq 0$ for some $a \in M_0 \cup M_1$. Now b is disjoint from all elements of M_1, so $a \in M_0$. Then $A \restriction (a \cdot -b) = \{0\}$.

(iii)⇒(iv): Assume (iii). Since $B^+ = \{b \in B : \exists a \in A^+ \forall a' \in A^+[a' \leq a \rightarrow a \cdot b \neq 0]\}$, we can take $M = B^+$.

(iv)⇒(v): Assume (iv), and suppose that I is a regular ideal in B. Clearly $I \cap A \subseteq (I \cap A)^{dd}$. Suppose that

(1) $c \in (I \cap A)^{dd} \backslash I$.

Since $I = I^{dd}$, there is a $b \in I^d$ such that $c \cdot b \neq 0$. By (iv), choose $m \in M$ such that $m \leq c \cdot b$. By the definition of M, choose $a \in A^+$ such that $A \restriction (a \cdot -m) = \{0\}$. We claim that

(2) $a \in (I \cap A)^d$.

For, take any $d \in I \cap A$. Since $b \in I^d$, we have $0 = b \cdot d \geq m \cdot d$. Now $a \geq a \cdot d$, and $a \cdot d \cdot b = 0$, so $a \cdot d \in A \restriction (a \cdot -m) = \{0\}$, proving (2)

Now $c \in (I \cap A)^{dd}$, so by (2), $a \cdot c = 0$. But $c \cdot a \geq m \cdot a$ and $m \cdot a > 0$ as otherwise $a \leq -m$, contradicting $A \restriction (a \cdot -m) = \{0\}$. This is a contradiction.

(v)⇒(vi): The ideal $B \restriction b$ in B is regular, since $(B \restriction b)^d = B \restriction (-b)$ and hence $(B \restriction b)^{dd} = B \restriction b$. Now (vi) follows from (v).

(vi)⇒(i): Assume (vi), and suppose that $X \subseteq A$ and $a \stackrel{\text{def}}{=} \sum^A X$ exists. Suppose that $b \in B$ is an upper bound for X; we want to show that $a \leq b$. Suppose that $x \in (A \restriction b)^d$. Then $x \in A$ and $x \cdot b = 0$. Hence $x \cdot y = 0$ for all $y \in X$, so $x \cdot a = 0$. Since x is arbitrary, $a \in (A \restriction b)^{dd} = A \restriction b$. □

The *density* of B is the least size of a dense subset of B B *has uniform density* iff $\forall b, c \in B^+[B \restriction b$ and $B \restriction c$ have the same density].

Lemma 30.14. $-e(p) = \{q : p \perp q\}$.

Proof. $-e(p) = \text{int}(P \backslash e(p)) = \{q : e(q) \cap e(p) = \emptyset\} = \{q : p \perp q\}$. □

If $i : P \rightarrow Q$, then i is a *complete embedding* iff

(i) $i(\mathbb{1}_P) = \mathbb{1}_Q$.

(ii) $\forall p_1, p_2 \in P[p_1 \leq p_2 \rightarrow i(p_1) \leq i(p_2)$.

(iii) $\forall p_1, p_2 \in P[p_1 \perp p_2 \leftrightarrow i(p_1) \perp i(p_2)$.

(iv) $\forall q \in Q \exists p \in P \forall p' \in P[p' \leq p \rightarrow i(p')$ and q are compatible].

Theorem 30.15. *If $i : P \rightarrow Q$ is a complete embedding, then there is a unique complete embedding f of* RO(P) *into* RO(Q) *such that* $\forall p \in P[f(e(p)) = e(i(p))]$.

Proof. We first prove that the following two statements are equivalent for any $p_0, \ldots, p_{m-1}, \ldots, p_n \in P$:

$$e(p_0) \cap \ldots \cap e(p_{m-1}) \cap -e(p_m) \cap \ldots \cap -e(p_n) = \emptyset; \tag{1}$$

$$e(i(p_0)) \cap \ldots \cap e(i(p_{m-1})) \cap -e(i(p_m)) \cap \ldots \cap -e(i(p_n)) = \emptyset. \tag{2}$$

First suppose that $e(r) \subseteq e(p_0) \cap \ldots \cap e(p_{m-1}) \cap -e(p_m) \cap \ldots \cap -e(p_n)$. Then by Theorem 14.6(vi), $\{s : s \leq p_0, \ldots, p_{m-1}\}$ is dense below r, and $r \perp p_m, \ldots, p_n$. By (iii), $i(r) \perp i(q_m), \ldots, i(p_n)$. Now we claim that

(3) $\{t : t \leq i(p_0), \ldots, i(p_{m-1})\}$ is dense below $i(r)$.

For, suppose that $q \leq i(r)$. By (iv), choose $s \in P$ such that for all $s' \in P$, if $s' \leq s$ then $i(s')$ and q are compatible. Then $i(s)$ and q are compatible, so $i(s)$ and $i(r)$ are compatible. Hence by (iii), s and r are compatible. Say $t \leq s, r$. Choose $u \leq t, p_0, \ldots, p_{m-1}$. Then $i(u) \leq i(p_0), \ldots, i(p_{m-1})$. Since $u \leq s$ it follows that $i(u)$ and q are compatible. Say $v \leq i(u), q$. Also $v \leq i(p_0), \ldots, i(p_{m-1})$. This proves (3).

It follows that $e(i(r)) \subseteq (2)$, and this proves that (2) implies (1).

Now suppose that $q \in Q$ and $e(q) \subseteq (2)$. Then by Theorem 14.6(vi), $\{s : s \leq i(p_0), \ldots i(p_{m-1})\}$ is dense below q, and $q \perp i(p_m), \ldots, i(p_n)$. Choose $p \in P$ so that $\forall p' \in P[p' \leq p \rightarrow i(p')$ and q are compatible]. Now suppose that $m \leq k \leq n$ and p and p_k are compatible. Say $p' \leq p, p_k$. Then $i(p')$ and q are compatible. Say $s \leq i(p'), q$. Since $q \perp i(p_k)$ it follows that $s \perp i(p_k)$. But $i(p') \leq i(p_k)$, so $s \leq i(p_k)$, contradiction. Thus $\forall k = m, \ldots, n[p \perp p_k]$.

Next we claim that $\{s : s \leq p_0, \ldots, p_{m-1}\}$ is dense below p. For, suppose that $p' \leq p$. Then $i(p')$ and q are compatible. Say $r \leq i(p'), q$. Choose $t \leq r, i(p_0), \ldots, i(p_{m-1})$. Then $i(p')$ is compatible with $i(p_0), \ldots, i(p_{m-1}$. By (iii) say $s_0 \leq p', p_0$. Then $i(s_0)$ is compatible with $i(p_1)$; say $s_1 \leq s_0, p_1$. Continuing, we obtain $t \leq p', p_0, \ldots, p_{m-1}$.

It follows that $e(p) \subseteq (1)$, and this proves that (1) implies (2).

Therefore there is an isomorphism f of $\mathrm{RO}(P)$ onto a subalgebra A of $\mathrm{RO}(Q)$ such that $f(e(p)) = e(i(p))$ for all $p \in P$.

Now suppose that $X \subseteq \mathrm{RO}(P)$. We claim that $f(\sum X) = \sum f[X]$ in $\mathrm{RO}(Q)$. If $x \in X$, then $x \leq \sum X$ and hence $f(x) \leq f(\sum X)$. Hence $\sum f[X] \leq f(\sum X)$. Suppose that $f(\sum X) \cdot -\sum f[X] \neq 0$. Choose $q \in Q$ such that $e(q) \subseteq f(\sum X) \cdot -\sum f[X]$. For each $x \in X$ let $Y_x \subseteq P$ be such that $x = \sum_{p \in Y_x} e(p)$. Then $f(x) = \sum_{p \in Y_x} e(i(p))$. It follows that $q \perp i(p)$ for each $p \in Y_x$. Let $Z = \bigcup_{x \in X} Y_x$. Then $\sum X = \sum_{p \in Z} e(p)$, and $f(\sum X) = \sum_{p \in Z} e(i(p))$. Hence $e(q) \cap f(\sum X) = 0$, contradiction. This proves the existence of the complete embedding.

For the uniqueness. suppose that also f' is a complete embedding of $\mathrm{RO}(P)$ into $\mathrm{RO}(Q)$ such that $\forall p \in P[f'(e(p)) = e(i(p))]$. Thus f and f' agree on $e[P]$. Since $e[P]$ is dense in $\mathrm{RO}(P)$, it follows that $f = f'$. $\square$

Theorem 30.16. *Suppose that κ is an infinite cardinal and $S \subseteq \kappa$. Let $P_S = \{f : f$ is a finite function $\subseteq S \times \{0, 1\}\}$, with the ordering $\supseteq$. Then the identity map is a complete embedding of P_S into P_κ.*

Proof. Clearly (i) and (ii) hold. If $p_1, p_2 \in P_S$ and they are compatible in P_S, then they are compatible in P_κ. Now suppose that they are compatible in P_κ. Say $p_3 \leq p_1, p_2$. Then $p_3 \restriction S \leq p_1, p_2$. So (iii) holds. For (iv), suppose that $q \in P_\kappa$. Let $p = q \restriction S$. Suppose that $p' \leq p$ with $p' \in P_S$. Then p' and q are compatible. Namely, let $s = p' \cup (q \restriction (\kappa \backslash S)$. Clearly $p' \subseteq s$. If $\alpha \in S$ then $q(\alpha) = p(\alpha) = p'(\alpha) = s(\alpha)$ and if $\alpha \in \kappa \backslash S$ then $q(\alpha) = s(\alpha)$. So $s \leq q$. $\square$

Lemma 30.17. *Suppose that A, B, C are BAs such that $A \subseteq_{reg} B$. Then*

(i) If C is a subalgebra of B, and $C \cap A$ is dense in A, then $C \cap A \subseteq_{reg} B$.

(ii) If C is a dense subalgebra of B, then $C \cap A \subseteq_{reg} B$.

Proof. (i): Let X be a maximal antichain in $C \cap A$. Then X is a maximal antichain in A, so X is a maximal antichain in B.

(ii): by (i). □

In the following results, recall the notion of a club $\subseteq [B]^\omega$ from page 100 of Jech.

Theorem 30.26. *Suppose that B is an infinite BA and $\{A \in [B]^\omega : A \leq_{reg} B\}$ is stationary. Then B has ccc.*

Proof. Let W be a partition of unity in B.

(1) There is an elementary submodel $(A, \leq, W')$ of $(B, \leq, W)$ such that $A \leq_{reg} B$.

To prove this, let $C = \{A \in [B]^\omega : (A, \leq, W') \preceq (B, \leq, W)\}$. Clearly C is closed. For unboundedness, suppose that $X \in [B]^\omega$. By the downward Löwenheim-Skolem-Tarski theorem we get a member of C with $X \subseteq A$. Now by the hypothesis of the theorem, let $A \in C$ with $A \leq_{reg} B$. Now $\forall b \in B[b \neq 0 \to \exists w \in W[b \cdot w \neq 0]]$, so $\forall b \in A[b \neq 0 \to \exists w \in W \cap A[b \cdot w \neq 0]]$. Thus $W \cap A$ is a maximal antichain in A, and hence it is a maximal antichain in B. So $W = W \cap A$ is countable. □

Lemma 30.19. *Suppose that B is an infinite BA of uniform density κ. Suppose that C is a club of countable regular subalgebras of B such that $\forall A_1, A_2 \in C[\langle A_1 \cup A_2 \rangle \in C]$. Let*

$$\mathcal{S} = \left\{\left\langle \bigcup X \right\rangle : X \subseteq C\right\}.$$

Then $\forall A \in \mathcal{S}[A \leq_{reg} B]$.

Proof. Let $X \subseteq C$, and let M be a partition of unity in $\langle \bigcup X \rangle$; we want to show that M is a partition of unity in B. By Lemma 30.26, B has ccc; hence M is countable. Hence $M \subseteq \langle \bigcup Y \rangle$ for some countable subset Y of X. Say $Y = \{D_n : n \in \omega\}$ with each $D_n \in C$. Define $E_0 = D_0$ and if $E_n \in C$ has been defined, let $E_{n+1} = \langle E_n \cup D_{n+1} \rangle$. So $E_{n+1} \in C$ by a hypothesis of the lemma. Now $F = \bigcup_{n \in \omega} E_n \in C$ since C is a club. Clearly $M \subseteq F$. It is a disjoint set in F, and in fact is a partition of unity in F. In fact, if $a \in F^+$ and $\forall x \in M[a \cdot x = 0]$, then $a \in \langle X \rangle$, contradicting the maximality of M. Now $F \leq_{reg} B$, so M is a partition of unity in B. □

If A is a subalgebra of B and $b \in B$, then b is *independent over A* iff $\forall a \in A^+[a \cdot b \neq 0 \neq a \cdot -b]$.

Lemma 30.20. *Suppose that D is a complete BA of uniform density and A is a complete subalgebra with smaller density. Then there is an element $u \in D$ independent over A.*

Proof. Let $\kappa = \pi(D)$. Let $X = \{x \in D^+ : A \restriction x = \{0\}\}$.

(1) X is dense in D.

For, suppose that $b \in D^+$. Let $Y \subseteq A$ be dense with $|Y| < \kappa$. Now Y is not dense in $B \upharpoonright b$, so let $c \neq 0$ with $c \leq b$ and with no $y \in Y$ with $y \leq c$. Suppose that $A \upharpoonright c \neq \{0\}$. Choose $a \in A^+$ with $a \leq c$. Then choose $y \in Y^+$ with $y \leq c$; this is a contradiction. So $c \in X$, as desired.

For any $x \in D$ let $prr^A(x) = \prod\{y \in A : x \leq y\}$. Let $Y = \{prr^A(x) : x \in X\}$.

(2) Y is dense in A.

In fact, suppose that $a \in A^+$. By (1), choose $x \in X$ such that $x \leq a$. Let $y = prr^A(x)$. Thus $y \leq a$ and $y \in Y$.

Let $W \subseteq Y$ be a maximal antichain. Then there is a $Z \subseteq X$ such that $W = \{prr^A(z) : z \in Z\}$, with $prr^A(z) \neq prr^A(z')$ if $z \neq z'$. Let $u = \sum Z$. Now suppose that $a \in A^+$. Choose $w \in W$ such that $a \cdot w \neq 0$. Say $w = prr^A(z)$ with $z \in Z$. If $a \cdot z = 0$ then $z \leq -a$, so $w \leq -a$ and $a \cdot w = 0$, contradiction. Hence $a \cdot z \neq 0$, hence $a \cdot u \neq 0$.

Now $a \cdot w \in A^+$ and $z \in Z \subseteq X$, so $a \cdot w \cdot -z \neq 0$. Now $w \cdot -z \cdot u = \sum_{v \in Z}(w \cdot -z \cdot v)$. If $v \in Z$ and $v = z$, then $w \cdot -z \cdot v = 0$. If $v \in Z$ and $v \neq z$, then $w \cdot -z \cdot v \leq prr^A(z) \cdot -z \cdot prr^A(v) = 0$. Thus $w \cdot -z \cdot u = 0$; hence $w \cdot -z \leq -u$ and so $a \cdot -u \neq 0$. □

Lemma 30.21. *Suppose that D is a complete BA of uniform density and A is a complete subalgebra with smaller density.*

Then $\forall v \in D \backslash A \exists u \in D[u$ is independent over A and $v \in A(u)$.

Proof. Let prr^A be as in the proof of Lemma 30.20. Also, for each $x \in D$ let $prr_A(x) = \sum\{y \in A : y \leq x\}$. Define $z = prr_A(v) + -prr^A(v)$.

Case 1. $z = 0$. Let $u = v$. If $a \in A^+$ then $a \not\leq v$, i.e., $a \cdot -v \neq 0$. Also $prr^A(v) = 1$, so $v \not\leq -a$, i.e., $a \cdot v \neq 0$.

Case 9. $z \neq 0$. By Lemma 30.20 applied to $D \upharpoonright z$ and $A \upharpoonright z$ there is a $w \in D \upharpoonright z$ such that w is independent over $A \upharpoonright z$. Let $u = w + v \cdot -z$.

(1) $v \in A(u)$.

In fact, $v \cdot -z = v \cdot -prr_A(v)$, and

$$w \cdot prr^A(v) \leq z \cdot prr^A(v) = prr_A(v) \cdot prr^A(v) = prr_A(v),$$

and so

$$\begin{aligned} prr_A(v) + u \cdot prr^A(v) &= prr_A(v) + (w + v \cdot -prr_A(v)) \cdot prr^A(v) \\ &= prr_A(v) + w \cdot prr^A(v) + v \cdot -prr_A(v) \\ &= prr_A(v) + v \cdot -prr_A(v) = v. \end{aligned}$$

(2) u is independent of A.

For, let $a \in A^+$.

Case 1. $a \cdot z \neq 0$. Then $a \cdot z \cdot w \neq 0$, hence $a \cdot z \cdot u \neq 0$. Also, $-u = -w \cdot (-v + z)$ so $a \cdot -u \geq a \cdot -w \cdot z \neq 0$.

Case 9. $a \cdot z = 0$. Now $a \cdot u = a \cdot (w + v \cdot -z) = a \cdot v$. If $a \cdot v = 0$ then $v \leq -a$, so $prr^A(v) \leq -a$. Then $a \leq -z = -prr_A(v) \cdot prr^A(v) \leq -a$ so $a = 0$, contradiction. Thus $a \cdot u = a \cdot v \neq 0$.

Also, $a \cdot -u = a \cdot -w \cdot (-v + z) = a \cdot -w \cdot -v$. Now $w \le z$, so $a \le -z \le -w$. Thus $a \cdot -u = a \cdot -v$. If $a \cdot -v = 0$, then $a \le v$, so $a \le prr_A(v)$. But $a \le -z = -prr_A(v) \cdot prr^A(v)$, so $a = 0$, contradiction. Hence $a \cdot -u = a \cdot -v \ne 0$. □

Lemma 30.22. *Suppose that D is a complete BA of uniform density and A is a complete subalgebra with smaller density.*

Then $\forall X \in [D]^{\le\omega} \exists U \in [D]^{\le\omega}[U$ is independent over A and $X \subseteq A(U)$.

Proof. A simple induction using Lemma 30.21. □

Theorem 30.23. *Let B be an infinite BA of uniform density, and assume that B is a Cohen algebra. Then the set $\{A \in [B]^\omega : A \le_{reg} B\}$ contains a club C with the property that*

$$(*) \qquad \forall A_1, A_2 \in C[\langle A_1 \cup A_2 \rangle \in C].$$

Proof. Suppose that $\overline{B}^{cmp} = C_\kappa$ Note that if $|B| = \omega$ then $\{B\}$ is a club as desired. So we assume that B is uncountable. For each $S \subseteq \kappa$ let $P_S = \{f : f$ is a finite function with $\mathrm{dmn}(f) \subseteq S$ and $\mathrm{rng}(f) \subseteq \{0,1\}\}$ ordered by $\supseteq$. Thus P_S is a subset of P_κ. By Theorems 30.15 and 30.16 there is a complete embedding f_S of $\mathrm{RO}(P_S)$ into $\mathrm{RO}(P_\kappa)$. Now let C be the set of all countable subalgebras A of B such that there is a countable $S \subseteq \kappa$ such that

(1) $\qquad A$ is dense in $B \cap \mathrm{rng}(f_S)$ and $B \cap \mathrm{rng}(f_S)$ is dense in $\mathrm{rng}(f_S)$.

Now we claim

(2) C is club in $[B]^\omega$, and $\forall A \in C[A \subseteq_{reg} B]$.

First we show that every $A \in C$ is a regular subalgebra of B. Since $B \cap \mathrm{rng}(f_S)$ is dense in $\mathrm{rng}(f_S)$ and $\mathrm{rng}(f_S)$ is a regular subalgebra of $\mathrm{RO}(P_\kappa)$, it follows from Lemma 30.17(i) with A, B, C replaced by $\mathrm{rng}(f_S)$, $\mathrm{RO}(P_\kappa)$ and B that $B \cap \mathrm{rng}(f_S)$ is a regular subalgebra of $\mathrm{RO}(P_\kappa)$. Now if X is a maximal antichain in $B \cap \mathrm{rng}(f_S)$ then it is a maximal antichain in $\mathrm{RO}(P_\kappa)$, and since $B \subseteq \mathrm{RO}(P_\kappa)$, it is a maximal antichain in B. So $B \cap \mathrm{rng}(f_S) \subseteq_{reg} B$. Now A is dense in $B \cap \mathrm{rng}(f_S)$, so $A \subseteq_{reg} B$ by Lemma 30.17(ii), with A, B, C replaced by $B \cap \mathrm{rng}(f_S)$, B, and A, noting that $A \cap B \cap \mathrm{rng}(f_S) = A$.

To show that C is closed, let $\langle A_n : n \in \omega \rangle$ be an increasing chain of members of C; say S_n shows that $A_n \in C$, for every $n \in \omega$.

(3) $S_n \subseteq S_{n+1}$.

For, let $\alpha \in S_n$. Let $p_\alpha = \{(\alpha, 0)\}$. Then $e_\kappa(p_\alpha) \in \mathrm{rng}(f_{S_n})$, so there is a $b_\alpha \in A_n$ such that $b_\alpha \le e_\kappa(p_\alpha)$. Now $b_\alpha \in A_{n+1}$, so $b_\alpha \in \mathrm{rng}(f_{S_{n+1}})$. Say $f_{S_{n+1}}(c_\alpha) = b_\alpha$ with $c_\alpha \in \mathrm{RO}(P_{S_{n+1}})$. Say $q_\alpha \in P_{S_{n+1}}$ and $e_{S_{n+1}}(q_\alpha) \le c_\alpha$. Then

$$e_\kappa(q_\alpha) = f_{S_{n+1}}(e_{S_{n+1}}(q_\alpha)) \le f_{S_{n+1}}(c_\alpha) = b_\alpha \le e_\kappa(p_\alpha).$$

Hence by Theorem 14.6(v), $\{r : r \le p_\alpha, q_\alpha\}$ is dense below q_α. If $\alpha \notin S_{n+1}$, then there is an extension t of q_α such that $\alpha \in \mathrm{dmn}(t)$ and $t(\alpha) = 1$. But then $p_\alpha \perp t$, contradiction. Hence $\alpha \in S_{n+1}$. This proves (3).

Let $A = \bigcup_{n\in\omega} A_n$ and $S = \bigcup_{n\in\omega} S_n$. Thus A is a countable subalgebra of B.

(4) $\mathrm{rng}(f_{S_n}) \subseteq \mathrm{rng}(f_S)$.

In fact, by Theorems 30.15 and 30.16 there is a complete embedding g of $\mathrm{RO}(P_{S_n})$ into $\mathrm{RO}(P_S)$ such that $g(e_{S_n}(p)) = e_S(p)$ for all $p \in P_{S_n}$. Then $f_S(g(e_{S_n})) = f_S(e_S(p)) = e_\kappa(p) = f_{S_n}(p)$. By the uniqueness statement in Theorem 30.15 (4) follows.

(5) A is dense in $B \cap \mathrm{rng}(f_S)$.

In fact, let $b \in B \cap \mathrm{rng}(f_S)$, $b \ne 0$. Say $b = f_S(c)$ with $c \in \mathrm{RO}(P_S)$. Say $e_S(p) \subseteq c$. Then $f_S(e_S(p)) \subseteq b$. Now $f_S(e_S(p)) = e_\kappa(p)$. Say $p \in P_{S_n}$. Then $f_{S_n}(e_{S_n}(p)) = e_\kappa(p)$. Choose $a \in A_n^+$ such that $a \le f_{S_n}(e_{S_n}(p))$. Now $f_{S_n}(e_{S_n}(p)) = e_\kappa(p) = f_S(e_S(p)) \subseteq b$. This proves (5).

(6) $B \cap \mathrm{rng}(f_S)$ is dense in $\mathrm{rng}(f_S)$.

For, let $a \in \mathrm{rng}(f_S)$. Say $b \in \mathrm{RO}(P_S)$ and $a = f_S(b)$. Choose $p \in P_S$ such that $e_S(p) \subseteq b$. Then $e_\kappa(p) = f_S(e_S(p)) \subseteq f_S(b) = a$. Say $p \in P_{S_n}$. Then $e_{S_n}(p) \in \mathrm{RO}(P_{S_n})$. Hence $e_\kappa(p) = f_{S_n}(e_{S_n}(p)) \in \mathrm{rng}(f_{S_n})$. Choose $c \in B \cap \mathrm{rng}(f_{S_n})$ such that $c \ne 0$ and $c \le e_\kappa(p)$. Then $c \le a$. By (4), $\mathrm{rng}(f_{S_n}) \subseteq \mathrm{rng}(f_S)$, so $c \in B \cap \mathrm{rng}(f_S)$. This proves (6).

Thus C is closed.

To see that C is unbounded, suppose that X is a countable subset of B. For each $b \in X$ let Y_b be a countable subset of P_κ such that $b = \bigcup_{p\in Y_b} e_\kappa(p)$. For each $p \in Y_b$ say $p \in P_{S_p}$, with S_p a finite subset of κ. Let $S = \bigcup\{S_p : b \in X, p \in Y_b\}$. Thus S is a countable subset of κ.

(7) $X \subseteq B \cap \mathrm{rng}(f_S)$.

In fact, if $b \in X$ then $b = f_S(\bigcup_{p\in Y_b} e_S(p))$.

Now we can find $S_1 \supseteq S$ with S_1 countable such that $B \cap \mathrm{rng}(f_{S_1})$ is dense in $\mathrm{rng}(f_S)$. Continuing by induction we get $S_n \supseteq \cdots \supseteq S$ with $B \cap \mathrm{rng}(f_{S_{n+1}})$ dense in $\mathrm{rng}(f_{S_n})$. Let $S' = \bigcup_{n\in\omega} S_n$. Then $X \subseteq B \cap \mathrm{rng}(f_{S'})$ and $B \cap \mathrm{rng}(f_{S'})$ is dense in $\mathrm{rng}(f_{S'})$. This shows that C is unbounded, and proves (2).

It remains to check $(*)$. Suppose that $A_1, A_2 \in C$. Say

$$A_1 \text{ is dense in } B \cap \mathrm{rng}(f_{S_1}) \text{ and } B \cap \mathrm{rng}(f_{S_1}) \text{ is dense in } \mathrm{rng}(f_{S_1}) \text{ and}$$
$$A_2 \text{ is dense in } B \cap \mathrm{rng}(f_{S_2}) \text{ and } B \cap \mathrm{rng}(f_{S_2}) \text{ is dense in } \mathrm{rng}(f_{S_2}).$$

Let $S = S_1 \cup S_2$ and $A = \langle A_1 \cup A_2\rangle$. Suppose that $0 \ne a \in B \cap \mathrm{rng}(f_S)$. Say $a = f_S(b_a)$ with $b_a \in \mathrm{RO}(P_S)$. Choose $p_a \in P_S$ so that $e_S(p_a) \le b_a$. Let $p'_a = p_a \restriction S_1$ and $p''_a = p_a \restriction S_2$. Choose $c_a \in B \cap \mathrm{rng}(f_{S_1})$ with $c_a \le f_{S_1}(e_{S_1}(p'_a))$ and $d_a \in B \cap \mathrm{rng}(f_{S_2})$ with $d_a \le f_{S_2}(e_{S_2}(p''_a))$. Then choose $a^3 \in A_1^+$ such that $a^3 \le c_a$ and choose $a^4 \in A_2^+$ such that $a^4 \le d_a$. Then

$$a^3 \cdot a^4 \le c_a \cdot d_a \le f_{S_1}(e_{S_1}(p'_a)) \cap f_{S_2}(e_{S_2}(p''_a)) = e_\kappa(p'_a) \cap e_\kappa(p''_a).$$

Now by Proposition 14.7(ii),

$$\begin{aligned}e_\kappa(p'_a) \cap e_\kappa(p''_a) &= \{q : \forall r \le q(r \text{ and } p'_a \text{ are compatible, and } r \text{ and } p''_a \text{ are compatible})\}\\ &= \{q : \forall r \le q \exists s \le r, p'_a, p''_a\} = \{q : \forall r \le q[r \text{ and } p_a \text{ are compatible}]\}\\ &= e_\kappa(p_a) = f_S(e_S(p_a)) \le f_S(b_a) = a\end{aligned}$$

This shows that $\langle A_1 \cup A_2\rangle$ is dense in $B \cap \mathrm{rng}(f_S)$.

Now suppose that $0 \ne a \in \mathrm{rng}(f_S)$. Say $a = f_S(b_a)$ with $b_a \in \mathrm{RO}(P_S)$. Choose $p_a \in P_S$ so that $e_S(p_a) \le b_a$. Let $p'_a = p_a \restriction S_1$ and $p''_a = p_a \restriction S_2$. Choose $c_a \in B \cap \mathrm{rng}(f_{S_1})$ with $c_a \le f_{S_1}(e_{S_1}(p'_a))$ and $d_a \in B \cap \mathrm{rng}(f_{S_2})$ with $d_a \le f_{S_2}(e_{S_2}(p''_a))$. Then $c_a \cdot d_a \in B$. Say $c_a = f_{S_1}(c'_a)$ with $c'_a \in \mathrm{RO}(P_{S_1})$ and $d_a = f_{S_1}(d'_a)$ with $d'_a \in \mathrm{RO}(P_{S_2})$. Say $c'_a = \sum_{q \in Y_a} e_{S_1}(q)$ with Y_a a countable subset of $\mathrm{RO}(P_{S_1})$, and say $d'_a = \sum_{q \in Z_a} e_{S_2}(q)$ with Z_a a countable subset of $\mathrm{RO}(P_{S_2})$. Then $c_a = \sum_{q \in Y_a} e_\kappa(q)$ and $d_a = \sum_{q \in Z_2} e_\kappa(q)$ and hence $c_a \cdot d_a$ has the form $\sum_{s \in W} e_\kappa(s)$ with W a countable subset of $\mathrm{rng}(f_S)$. So $c_a \cdot d_a \in \mathrm{rng}(f_S)$. Hence we have shown that $B \cap \mathrm{rng}(f_S)$ is dense in $\mathrm{rng}(f_S)$. □

The notion of Cohen algebra above is due to Balcar, Jech, and Zapletal. Similar but different notions are due to Koppelberg. We describe these now, using somewhat different terminology.

A *standard Cohen algebra of* π*-weight* κ is the completion of the free BA on κ generators. A *general Cohen algebra* is an algebra A whose completion is isomorphic to a product of standard Cohen algebras. We begin with an extension of Lemma 30.13.

Lemma 30.24. *Assume that* $A \le B$. *The following are equivalent:*

(i) $A \le_{reg} B$.

(ii) For every nonempty open $V \subseteq \mathrm{Ult}(B)$ *there is a nonempty open* $U \subseteq \mathrm{Ult}(A)$ *such that* $U \subseteq f[V]$, *where* $f : \mathrm{Ult}(B) \to \mathrm{Ult}(A)$ *is defined by* $f(q) = q \cap A$ *for all* $q \in \mathrm{Ult}(B)$.

Proof. (i)⇒(ii): First note that (ii) is equivalent to

(1) For every $b \in B^+$ there is a nonempty open $U \subseteq \mathrm{Ult}(A)$ such that $U \subseteq f[\mathcal{S}(b)]$.

In fact clearly (ii) implies (1). Now assume that (1) holds. Suppose that V is a nonempty open subset of $\mathrm{Ult}(B)$. For each $b \in B^+$ with $\mathcal{S}(b) \subseteq V$ let U_b be a nonempty open subset of A such that $U_b \subseteq f[\mathcal{S}(b)]$. Let $U' = \bigcup\{U_b : b \in B^+,\ \mathcal{S}(b) \subseteq V\}$. Suppose that $x \in U'$. Choose $b \in B^+$ such that $\mathcal{S}(b) \subseteq V$ and $x \in U_b$. Then $x \in f[\mathcal{S}(b)] \subseteq f[V]$. So $U' \subseteq f[V]$.

Now suppose that (1) fails. Thus there is a $b \in B^+$ such that for every nonempty open $U \subseteq \mathrm{Ult}(A)$ we have $U \not\subseteq f[\mathcal{S}(b)]$.

(2) $\forall a \in A^+ \exists p \in \mathrm{Ult}(A)[a \in p \text{ and } \exists x \in p[x \cdot b = 0]]$.

In fact, if $a \in A^+$ then $\mathcal{S}(a) \not\subseteq f[\mathcal{S}(b)]$, so there is a $p \in \mathcal{S}(a)$ such that $p \notin f[\mathcal{S}(b)]$. Thus $a \in p$ and there is no $q \in \mathrm{Ult}(B)$ with $b \in q$ and $p = q \cap A$. So $p \cup \{b\}$ does not have fip. Hence there is an $x \in p$ such that $x \cdot b = 0$. So (2) holds.

Now let $M = \{a \in A^+ : a \cdot b = 0\}$.

(3) M is dense in A.

For, suppose that $a \in A^+$ while $\forall x \le a[x \ne 0 \to x \cdot b \ne 0]$. By (2) choose $p \in \mathrm{Ult}(A)$ with $a \in p$ and with $x \in p$ such that $x \cdot b = 0$. Then $a \cdot x \in p$, hence $a \cdot x \ne 0$, and $a \cdot x \cdot b = 0$, contradiction. So (3) holds.

Now $A \restriction (-b) = \{a \in A : a \le -b\} = M \cup \{0\}$. Hence $(A \restriction (-b))^d = \{x \in A : \forall y \in (A \restriction (-b))[x \cdot y = 0]\} = \{x \in A : \forall y \in M \cup \{0\}[x \cdot y = 0]\} = \{0\}$. Hence $(A \restriction (-b))^{dd} = A$. By Lemma 30.13(vi), $A \restriction (-b)$ is regular, so $A \restriction (-b) = A$. Hence $-b = 1$ and so $b = 0$, contradiction. This proves (ii).

(ii)⇒(i): Suppose that (i) fails. Say $X \subseteq A$ and $\sum^A X$ exists but there is an upper bound $c \in B$ for X with $c < \sum^A X$. Let $b = \sum^A X \cdot -c$. By the equivalent (1) above, let U be a nonempty open set in $\mathrm{Ult}(A)$ such that $U \subseteq f[\mathcal{S}(b)]$. Choose $a \in A^+$ such that $\mathcal{S}(a) \subseteq U$. Take any $p \in \mathcal{S}(a)$. Then $p \in f[\mathcal{S}(b)]$, so p extends to $q \in \mathrm{Ult}(B)$ with $b \in q$. Then $a \cdot b \ne 0$, so $a \cdot \sum^A X \ne 0$. Hence there is an $x \in X$ such that $a \cdot x \ne 0$. Let p' be an ultrafilter on A such that $a \cdot x \in p'$. Thus $p' \in \mathcal{S}(a)$, so $p' \in f[\mathcal{S}(b)]$ and hence there is an ultrafilter q' on B such that $p' \subseteq q'$ and $b \in q'$. Also $x \in q'$. Since $x \cdot -c = 0$, this is a contradiction. □

If A is a BA and $A \le B$, then B is a *simple extension* of A provided that there is a $b \in B$ such that $B = \langle A \cup \{b\}\rangle$. Then we write $B = A(b)$.

Proposition 30.25. *If I and J are regular ideals in A, then $I \cap J$ is regular.*

Proof.

$$\begin{aligned} x \in (I \cap J)^d \quad &\text{iff} \quad \forall a \in (I \cap J[a \cdot x = 0] \\ &\text{iff} \quad \forall a \in I[a \cdot x = 0] \text{ and } \forall a \in J[a \cdot x = 0] \\ &\text{iff} \quad x \in I^d \text{ and } x \in J^d \\ &\text{iff} \quad x \in I^d \cap J^d; \end{aligned}$$

similarly, $x \in (I \cap J)^{dd}$ iff $x \in I^{dd} \cap J^{dd}$. So the proposition follows. □

Proposition 30.26. *I and J are regular ideals in A and $I \cap J = \{0\}$, then $\langle I \cup J\rangle^{id}$ is a regular ideal.*

Proof. Note that $\langle I \cup J\rangle^{id} = \{x : \exists y \in I \exists z \in J[x = y + z]\}$. Hence

$$\begin{aligned} (\langle I \cup J\rangle^{id})^d &= \{w : \forall y \in I \forall z \in J[w \cdot (y + z) = 0]\} \\ &= \{w : \forall y \in I[w \cdot y = 0]\} \cap \{w : \forall z \in J[w \cdot z = 0]\} \\ &= I^d \cap J^d. \end{aligned}$$

As in the proof of Proposition 30.25, $(I^d \cap J^d)^d = I^{dd} \cap J^{dd}$. Hence the proposition follows. □

Proposition 30.27. *For $B = A(b)$, $A \le_{reg} B$ iff $A \restriction b$ and $A \restriction -b$ are regular ideals of A.*

Proof. ⇒: Suppose that $A \leq_{reg} B$. By Lemma 30.13(vi), $A \restriction b$ and $A \restriction (-b)$ are regular.

⇐: Assume that $A \restriction b$ and $A \restriction -b$ are regular ideals of A. We apply Theorem 30.13(vi) by showing that for each $c \in B$ the ideal $A \restriction c$ is regular. Now $B = \{u \cdot b + v \cdot -b + w : u, v, w \in A$ are pairwise disjoint$\}$. So write $c = u \cdot b + v \cdot -b + w$ with $u, v, w \in A$ pairwise disjoint. Now $A \restriction (u \cdot b) = (A \restriction u) \cap (A \restriction b)$. We are assuming that $A \restriction b)$ is a regular ideal, and obviously $A \restriction u$ is a regular ideal. Hence by Proposition 30.20, $A\langle(u \cdot b)$ is a regular ideal. Similarly, $A \restriction (v \cdot -b)$ is a regular ideal. Obviously $A \restriction w$ is a regular ideal. So by Proposition 30.21, $A \restriction c$ is a regular ideal. □

Proposition 30.28. *If $A \leq_{reg} B \leq_{reg} C$ then $A \leq_{reg} C$.* □

Proposition 30.29. *If $A \leq B \leq C$ and $A \leq_{reg} C$, then $A \leq_{reg} B$.*

Proof. Suppose that $X \subseteq A$ and $\sum^A X$ exists. Then $\sum^C X$ exists and equals $\sum^A X$. Now $\sum^A X \in B$. If b is an upper bound for X in B then $\sum^A X = \sum^C X \leq b$. So $\sum^B X$ exists and equals $\sum^A X$. □

Proposition 30.30. *If $\langle A_\alpha : \alpha < \sigma\rangle$ is an increasing chain of BAs and $A_\alpha \leq_{reg} A_\beta$ for all $\alpha < \beta < \sigma$, then $A_\alpha \leq_{reg} \bigcup_{\beta<\sigma} A_\beta$ for all $\alpha < \sigma$.*

Proof. This is obvious if σ is a successor ordinal. Suppose that it is a limit ordinal. Suppose that $X \subseteq A_\alpha$ and $\sum^{A_\alpha} X$ exists. Suppose that $b \in \bigcup_{\beta<\sigma} A_\beta$ is an upper bound for X. Say $b \in A_\beta$ with $\alpha < \beta$. Then $\sum^{A_\alpha} X = \sum^{A_\beta} X \leq b$. □

Proposition 30.31. *If $A \leq B \leq C$, $A \leq_{reg} C$, and B is finitely generated over A, then $B \leq_{reg} C$.*

Proof. Clearly it suffices to consider the case where $B = A(u)$. Suppose that $B \not\leq_{reg} C$. Then as in the proof of Lemma 30.24 we get $c \in C^+$ such that

(1) $M \stackrel{\text{def}}{=} \{b \in B^+ : b \cdot c = 0\}$ is dense in B.

Case 1. $c \cdot u > 0$. Now since $A \leq_{reg} C$, by Lemma 30.13(iii) there is an $a^+ \in A$ such that $A \restriction (a^* \cdot -(c \cdot u)) = \{0\}$.

(2) $a^* \cdot u > 0$.

For, otherwise $a^* \cdot u \cdot c = 0$, hence $a^* = a^* \cdot -(c \cdot u)$ and so $0 \neq a^* \in A \restriction (a^* \cdot -(c \cdot u))$, contradiction.

Now since M is dense in B, there is a nonzero $b \in M$ with $b \leq a^* \cdot u$. Since $b \leq u$, we have $b = a_0 \cdot u$ for some $a_0 \in A$. Now $0 < b \leq b \cdot a^* \leq a_0 \cdot a^*$ and $a_0 \cdot a^* \cdot c \cdot u \leq b \cdot c = 0$. So $0 < a_0 \cdot a^* \leq a^* \cdot -(c \cdot u)$. Thus $0 < a_0 \cdot a^* \in A \restriction (a^* \cdot -(c \cdot u))$, contradiction.

Case 9. $c \cdot -u > 0$. Symmetric to Case 1. □

Lemma 30.32. *If A is dense in B, then $A \leq_{reg} B$.*

Proof. Suppose that $X \subseteq A$ and $\sum^A X$ exists, but $b \in B$ is an upper bound for X with $b < \sum^A X$. Choose $a \in A^+$ with $a \leq \sum^A X \cdot -b$. Then $a \cdot b = 0$, so $\forall x \in X[a \cdot x = 0]$. Hence $a \cdot \sum^A X = 0$, contradiction. $\square$

Lemma 30.33. *If $A \leq B \leq C$, $A \leq_{reg} C$, and A is dense in B, then $B \leq_{reg} C$.*

Proof. Let X be a maximal antichain in B. Let $Y \subseteq A$ be maximal disjoint such that $\forall y \in Y \exists x \in X[y \leq x]$. Then Y is a maximal antichain in A. In fact, suppose that $a \in A^+$ and $\forall y \in Y[a \cdot y = 0]$. Choose $x \in X$ such that $a \cdot x \neq 0$ and then choose $y \in A^+$ such that $y \leq a \cdot x$. Then $y \notin Y$ and $Y \cup \{y\}$ is disjoint, contradiction. Since Y is a maximal antichain in A, it is also a maximal antichain in C. Suppose that X is not a maximal antichain in C. Choose $c \in C^+$ such that $\forall x \in X[c \cdot x = 0]$. If $y \in Y$, choose $x \in X$ with $y \leq x$. Then $c \cdot y = 0$, contradiction. $\square$

Proposition 30.34. *Suppose that $A \leq B$ and B is complete. Then*

(i) There is an isomorphic embedding of $\overline{A}^{cmp}$ into B which is the identity on A.

Moreover, the following conditions are equivalent:

(ii) $A \leq_{reg} B$.

(iii) There is a unique embedding e of $\overline{A}^{cmp}$ into B which is the identity on A. Moreover, $\mathrm{rng}(e) \leq_{reg} B$. *Here $\overline{A}^{cmp}$ is the completion of A.*

(iv) There is an isomorphism of $\langle A\rangle^{cmp}$ onto $\overline{A}^{cmp}$ which is the identity on A. Here $\langle A\rangle^{cmp}$ is the complete subalgebra of B generated by A.

(v) A is dense in $\langle A\rangle^{cmp}$.

Proof. (i) holds by Sikorski's extension theorem.

(ii)⇒(iii): Assume (ii). e exists by Sikorski's extension theorem. Suppose that e' also is an embedding of $\overline{A}^{cmp}$ into B which is the identity on A. Now A is dense in $\overline{A}^{cmp}$, so clearly A is dense in $rng(e)$.

(1) Every element a of $\mathrm{rng}(e)$ has the form $\sum^{\mathrm{rng}(e)} X$ for some $X \subseteq A$.

In fact, let $X = \{x \in A : x \leq a\}$. Then clearly $\sum^{\mathrm{rng}(e)} X \leq a$. Equality holds since A is dense in $\mathrm{rng}(e)$.

Now A is dense in $\mathrm{rng}(e)$ and $A \leq_{reg} B$, so by Lemma 30.28, $\mathrm{rng}(e) \leq_{reg} B$. Now for any $a \in A^{cmp}$, by (1) choose $X \subseteq A$ such that $e(a) = \sum^{\mathrm{rng}(e)} X$, so $e(a) = \sum^B X$. By symmetry, $e'(a) = \sum^B X$.

(iii)⇒(iv): Assume (iii). In particular, $\mathrm{rng}(e) \leq_{reg} B$. Now $\mathrm{rng}(e) = \{\sum^{\mathrm{rng}(e)} X : X \subseteq A\} = \{\sum^B X : X \subseteq A\}$, and (iv) follows.

(iv)⇒(v): This is true since A is dense in $\overline{A}^{cmp}$.

(v)⇒(ii): By Lemma 30.27, $A \leq_{reg} \langle A\rangle^{cmp} \leq_{reg} B$ and hence $A \leq_{reg} B$ by Proposition 30.23. $\square$

$A \leq_d B$ means that A is a dense subalgebra of B. If $A \leq B$ and $U \subseteq B$, then $A(U)$ is the subalgebra of B generated by $A \cup U$.

Lemma 30.35. *Suppose that $C \le D \le E$ and $C \le_d D \le_{reg} E$. Suppose that $U \subseteq E$. Then*

(i) $C \le_{reg} C(U)$,

(ii) $D \le_{reg} D(U)$.

(iii) $C(U) \le_d D(U)$.

Proof. (i): By Propositions 30.28 and 30.32, $C \le_{reg} E$. Hence $C \le_{reg} C(U)$ by Proposition 30.29.

(ii): Since $D \le D(U) \le E$, this follows from Proposition 30.29.

(iii): Clearly $C(U) = C(\langle U\rangle)$ and $D(U) = D(\langle U\rangle)$. Now suppose that $a \in D(\langle U\rangle)$, with $a \ne 0$. Clearly every element of $D(\langle U\rangle)$ is a finite sum of elements $d \cdot u$ with $d \in D$ and $u \in \langle U\rangle$. So we may assume that $a = d \cdot u$ with $d \in D$ and $u \in \langle U\rangle$. Since $C \le_d D$ there is an $X \subseteq C$ such that $d = \sum^D X$. By (ii) we have $d = \sum^{D(U)} X$. Since $u \in D(U)$ we have $d \cdot u = \sum^{D(U)}\{x \cdot u : x \in X\}$. Since $0 \ne a = c\dot{d}$, there is an $x \in X$ such that $x \cdot u \ne 0$. Then $x \cdot u \in C(U)$ and $x \cdot u \le d \cdot U$, as desired. □

Lemma 30.36. *If $C \le_d D \le_{reg} E$ and $C' \le_d D' \le_{reg} E$, then $C(C') \le_d D(D')$.*

Proof. By Lemma 30.24c, $C(C') \le_d D(C')$ and $C'(D) \le_d D'(D)$. Hence

$$C(C') \le_d D(C') = C'(D) \le D'(D) = C(D'). \qquad \square$$

A is *relatively complete* in B, $A \le_{rc} B$, iff for every $b \in B$ there is a greatest $a \in A$ such that $a \le b$.

Proposition 30.37. *If $A \le B$ then the following are equivalent:*

(i) $A \le_{rc} B$.

(ii) For all $b \in B$ there is a least $a \in A$ such that $b \le a$.

Proof. Assume that $A \le_{rc} B$, and let $b \in B$. Let $a \in A$ be greatest such that $a \le -b$. Then $-a$ is least such that $b \le -a$. By symmetry the proposition follows. □

If $A \le_{rc} B$, then

$$\begin{aligned} \mathrm{pr}_A(b) &= \text{smallest } a \in A \text{ such that } b \le a; \\ \mathrm{pr}^A(b) &= \text{greatest } a \in A \text{ such that } a \le b. \end{aligned}$$

Proposition 30.38. *If $A \le_{rc} B$ then $A \le_{reg} B$.*

Proof. Assume $A \le_{rc} B$ and suppose that $X \subseteq A$ and $\sum^A X$ exists. Suppose that $b \in B$ is an upper bound for X. Let $a \in A$ be greatest such that $a \le b$. Now $\forall x \in X[x \le b]$, so $\forall x \in X[x \le a]$. Hence $\sum^A X \le a \le b$. □

Proposition 30.39. *If $A \le_{rc} B \le_{rc} C$, then $A \le_{rc} C$.*

Proof. Assume that $A \le_{rc} B \le_{rc} C$, and suppose that $c \in C$. Let $b \in B$ be greatest (in B) such that $b \le c$. Let $a \in A$ be greatest (in A) such that $a \le b$. Suppose that $d \in A$ and $d \le c$. Then $d \le b$, so $d \le a$. □

If $A \leq B$, then we say that B is *projective* over A, $A \leq_{proj} B$ iff there is a continuous chain $\langle D_\alpha : \alpha < \rho \rangle$ of subalgebras of B such that $B = \bigcup_{\alpha<\rho} D_\alpha$, $D_0 = A$, and for $\alpha + 1 < \rho$, $D_\alpha \leq_{rc} D_{\alpha+1}$ and $D_{\alpha+1}$ is a simple extension of D_α.

Proposition 30.40. *If $A \leq_{proj} B$ then $A \leq_{rc} B$.*

Proof. Assume that $A \leq_{proj} B$ with notation as above. Suppose that $b \in B$. Say $b \in D_\alpha$ with $\alpha < \rho$. By Proposition 30.39, $A \leq_{rc} D_\alpha$ for all $\alpha < \rho$. Hence choose $a \in A$ greatest $\leq b$. □

Proposition 30.41. *Suppose that $A(u)$ is a simple extension of A. Define*

$$I = A \restriction u; \quad J = A \restriction (-u); \quad K = \{a \in A : \forall x \in I \cup J[a \cdot x = 0]\}.$$

Let $\langle a_s : s \in S \rangle$ be a system of pairwise disjoint elements of K, and let

$$v_s = a_s \cdot u \text{ for all } s \in S; \quad A' = A(\{v_s : s \in S\}).$$

Then $A \leq_{proj} A'$.

Proof. Let $<$ be a well-order of S, and for each $s \in S$ let $A_s = A(\{v_t : t < s\})$. Then $\langle A_s : s \in S \rangle$ is a continuous chain with union A' if S has no greatest element, and $A' = (\bigcup_{s \in S} A_s)(v_s^*)$ if S has a greatest element s^*. So it suffices to prove that $A(\{v_t : t < s\}) \leq_{rc} A(\{v_t : t \leq s\})$ for all $s \in S$. Suppose that $x \in A(\{v_t : t \leq s\})$. So we can write $x = b \cdot v_s + c \cdot -v_s$ with $b, c \in A(\{v_t : t < s\})$. Then

$$c \cdot -a_s \cdot x = c \cdot -a_s \cdot b \cdot a_s \cdot u + c \cdot -a_s \cdot c \cdot (-a_s + -u) = c \cdot -a_s.$$

So $c \cdot -a_s \leq x$. Also, $b \cdot c = b \cdot c \cdot v_s + b \cdot c \cdot -v_s \leq x$. So $b \cdot c + c \cdot -a_s \leq x$.

Suppose that $y \in A(\{v_t : t < s\})$ and $y \leq x$. Thus

$$0 = y \cdot -x = y \cdot (-b + -v_s) \cdot (-c + v_s) = y \cdot -b \cdot -c + y \cdot -b \cdot v_s + y \cdot -v_s \cdot -c.$$

Hence

$$0 = y \cdot -b \cdot a_s \cdot u = y \cdot -c \cdot (-a_s + -u) = y \cdot -c \cdot -a_s + y \cdot -c \cdot -u.$$

so

$$0 = y \cdot -b \cdot a_s = y \cdot -c,$$

and it follows that $y \leq c$ and $y \leq b + -a_s$, so $y \leq b \cdot c + c \cdot -a_s$. □

We now describe an alternative construction of the completion of a Boolean algebra. An ideal I in A is *complete* iff for all $X \subseteq I$, if $\sum X$ exists then $\sum X \in I$. We will see later that an ideal is complete iff it is regular.

Proposition 30.42. *If X is a collection of complete ideals of A, then $\bigcap X$ is a complete ideal.* □

Proposition 30.43. *If I is a complete ideal of A, then I^d is a complete ideal.*

Proof. Clearly I^d is an ideal. Suppose that $X \subseteq I^d$ and $\sum X$ exists. Now $\forall y \in I \forall x \in X[x \cdot y = 0]$, so $\forall y \in I[y \cdot \sum X = 0]$. Thus $\sum X \in I^d$. □

Theorem 30.44. *The collection B of all complete ideals of A forms a Boolean algebra under the following operations:*
(i) $I + J = \bigcap\{K : K$ *is a complete ideal with* $I \cup J \subseteq K\}$.
(ii) $I \cdot J = I \cap J$.
(iii) $-I = I^d$.
(iv) $0 = \{0\}$.
(v) $1 = A$.

Proof. The commutative laws are clear, as is the associative law for $\cdot$. The associative law for $+$:

$$\begin{aligned} I + (J + K) &= \bigcap\{L : L \text{ is a complete ideal with } I \cup (J + K) \subseteq L\} \\ &= \bigcap\{L : L \text{ is a complete ideal with} \\ &\qquad I \cup \bigcap\{M : M \text{ is a complete ideal with } J \cup K \subseteq M\} \subseteq L\} \qquad (*) \\ &= \bigcap\{L : L \text{ is a complete ideal with } I \cup J \cup K \subseteq L\}. \qquad (**) \end{aligned}$$

To see this last equation, note that $(**)$ is a complete ideal containing $J \cup K$, so $\bigcap\{M : M$ is a complete ideal with $J \cup K \subseteq M\} \subseteq (**)$. Similarly $I \subseteq (**)$. Hence $(*) \subseteq (**)$. On the other hand, $(*)$ is a complete ideal containing $I \cup J \cup K$, so $(**) \subseteq (*)$.

Similarly, $(I + J) + K = \bigcap\{L : L$ is a complete ideal with $I \cup J \cup K \subseteq L\}$, so the associative law holds.

First distributive law:

$$\begin{aligned} [I \cap (J + K) &= I \cap \bigcap\{L : L \text{ is a complete ideal with } J \cup K \subseteq L\} \qquad (*) \\ &= \bigcap\{L : L \text{ is a complete ideal with } (I \cap J) \cup (I \cap K) \subseteq L\} \qquad (**) \\ &= I \cdot J + I \cdot K. \end{aligned}$$

To see that $(*) = (**)$, first suppose that L is a complete ideal with $J \cup K \subseteq L$. Then $I \cap L$ is a complete ideal with $(I \cap J) \cup (I \cap K) \subseteq I \cap L$. Then $(**) \subseteq (*)$ follows. Second, clearly $I \cap J \subseteq (*)$ and $I \cap K \subseteq (*)$, so $(I \cap J) \cup (I \cap K) \subseteq (*)$. Hence $(**) \subseteq (*)$.

First absorption law:

$$I \cdot (I + J) = I \cap \bigcap\{K : K \text{ is a complete ideal with } I \cup J \subseteq K\} = I.$$

Second absorption law:

$$I + (I \cap J) = \bigcap\{K : K \text{ is a complete ideal with } I \cup (I \cap J) \subseteq K\} = I.$$

The second distributive law follows from the first:

$$\begin{aligned}(I+J)\cdot(I+K) &= ((I+J)\cdot I)+((I+J)\cdot K)\\ &= (I\cdot(I+J))+(K\cdot(I+J))\\ &= I+((K\cdot I)+(K\cdot J))\\ &= I+((I\cdot K)+(J\cdot K))\\ &= (I+(I\cdot K))+(J\cdot K)\\ &= I+(J\cdot K).\end{aligned}$$ $\square$

First complementation law:

$$I+-I=I+I^d=\bigcup\{K : K \text{ is a complete ideal with } I\cup I^d\subseteq K\}.$$

Suppose that K is a complete ideal with $I\cup I^d\subseteq K$. We claim that $\sum(I\cup I^d)$ exists and equals 1 (Hence $K=A$.) For, suppose that a is an upper bound of $I\cup I^d$. Then $\forall x\in I[x\leq a]$, so $\forall x\in I[x\cdot -a=0]$, hence $-a\in I^d$, hence $-a\leq a$, so $a=1$.

Second complementation law: $I\cdot -I=I\cap I^d=\{0\}$. $\square$

Theorem 30.45. *The collection B of all complete ideals of A is a complete BA under the operations given in Theorem 30.44. Moreover, the mapping $a\mapsto A\restriction a$ is an isomorphism of A onto a dense subalgebra of B.*

Proof. Clearly $A\restriction a$ is a complete ideal, for all $a\in A$. Let f be the indicated mapping. Preservation of operations:

$$\begin{aligned}f(a\cdot b)&=A\restriction(a\cdot b)=(A\restriction a)\cap(A\restriction b)=f(a)\cdot f(b);\\ f(-a)&=A\restriction(-a)=(A\restriction a)^d=-f(a).\end{aligned}$$

f is one-one, since $a\neq 0$ implies that $f(a)=(A\restriction a)\neq\{0\}$.

$\mathrm{rng}(f)$ is dense in B: Suppose that $I\neq\{0\}$ is a complete ideal. Choose $a\in I\backslash\{0\}$. Then $f(a)=(A\restriction a)\subseteq I$. $\square$

Proposition 30.46. *I is complete iff I is regular.*

Proof. First suppose that I is regular, and suppose that $X\subseteq I$ and $\sum X$ exists. Then $I^d\subseteq X^d$ and hence $X^{dd}\subseteq I^{dd}=I$. Now suppose that $y\in X^d$. Thus $\forall x\in X[x\cdot y=0]$, so $y\cdot\sum X=0$. This is true for all $y\in X^d$, so $\sum X\in X^{dd}\subseteq I$.

Second suppose that I is complete. Then $I^{dd}=--I=I$. $\square$

Proposition 30.47. *If I and J are regular ideals, then $I+J=(I\cup J)^{dd}$.*

Proof. By definition, $I+J=\bigcap\{K : K \text{ is a regular ideal and } I\cup J\subseteq K\}$. Now $(I\cup J)^{dd}$ is clearly a regular ideal. If K is a regular ideal with $I\cup J\subseteq K$, then $(I\cup J)^{dd}\subseteq K^{dd}=K$. $\square$

Lemma 30.48. *Let $E \subseteq A$ and define $E\downarrow = \{x : x \le e \text{ for some } e \in E\}$. Then*

$$\bigcap\{K : K \text{ is a regular ideal with } E \subseteq K\} = \left\{\sum X : X \subseteq E\downarrow\right\}.$$

Proof. Let $M = \bigcap\{K : K$ is a regular ideal with $E \subseteq K\}$ and $N = \{\sum X : X \subseteq E\downarrow\}$.

(1) $E \subseteq N$.

For, if $p \in E$ then $\{p\} \subseteq E\downarrow$ and $\sum\{p\} = p$. So (1) holds.

(2) N is an ideal of A.

In fact, $0 = \sum\emptyset$, so $0 \in N$. If $x \le y \in N$, say $y = \sum X$ with $X \subseteq E\downarrow$. Then $x = \sum\{x \cdot z : z \in X\}$, and $\{x \cdot z : z \in X\} \subseteq E\downarrow$. If $x, y \in N$, say $x = \sum X$ and $y = \sum Y$ with $X, Y \subseteq E\downarrow$. Then $x + y = \sum(X \cup Y)$. Thus (2) holds.

(3) N is a complete ideal.

For, suppose that $X \subseteq N$ and $\sum X$ exists. For each $x \in X$ write $x = \sum Y_x$ with $Y_x \subseteq E\downarrow$. Then $\sum X = \sum Z$, where $Z = \sum_{x\in X} Y_x$, and $Z \subseteq E\downarrow$.

From (1)–(3) it follows that $M \subseteq N$.

Now suppose that $x \in N$ and K is a regular ideal with $E \subseteq K$. Say $x = \sum X$ with $X \subseteq E\downarrow$. Then $X \subseteq K$, so $x = \sum X \in K$. □

Proposition 30.49. *Assume the hypotheses of Proposition 30.41, and assume in addition that $A \le_{reg} A(u)$ and $\{a_s : s \in S\}$ is maximal disjoint in K.*

Then $A' \le_d A(u)$.

Proof. First note that I and J are regular ideals, by Lemma 30.13(vi). Also K is regular, since clearly $I \cup J \subseteq K^d$ and so $K^{dd} \subseteq (I \cup J)^d = K$. Now in the BA of regular ideals over A we have $K = -(I + J)$, so $I + J + K = A$. Then for any $a \in A$, by Lemma 30.48 we can write $a = \sum X$, where $X \subseteq (I \cup J \cup K)\downarrow = I \cup J \cup K$. This gives three cases:

(1) There is a $b \in I$ with $b \ne 0$ and $b \in X$, hence $b \le a$.

(2) There is a $b \in K$ with $b \ne 0$ and $b \in X$, hence $b \le a$.

(3) $X \subseteq J$ and so $a \in J$.

Now suppose that $0 \ne b \in A(u)$. Say $b = c \cdot u + d \cdot -u$ with $c, d \in A$. By symmetry we may assume that $c \cdot u \ne 0$. Thus $c \notin J$. By the above we then have two cases.

Case 1. There is a $b \in I$ with $b \ne 0$ and $b \le c$. Then $b \le c \cdot u$, as desired.

Case 9. There is a $b \in K$ with $b \ne 0$ and $b \le c$. Then by the maximality of $\{a_s : s \in S\}$ there is an $s \in S$ such that $c \cdot a_s \ne 0$. Then $c \cdot v_s \in A'$ and $c \cdot v_s = c \cdot a_s \cdot u \le c \cdot u$. Finally, $c \cdot v_s \ne 0$, as otherwise $c \cdot a_s \cdot u = 0$, hence $c \cdot a_s \in J$; but also $c \cdot a_s \in K$ since $a_s \in K$; so $c \cdot a_s = 0$ since $J \cap K = \{0\}$; contradiction. □

If $C \le D$, then the *weight of D over C, $wt(D/C)$*, is the least cardinal κ such that there is a subset X of D of size κ such that $D = \langle C \cup X\rangle$.

Proposition 30.50. *If in addition to the hypotheses of Proposition 30.49 we assume that A has ccc, then $wt(A'/A) \leq \omega$.* $\square$

Proposition 30.51. *Assume that $\langle A_\alpha : \alpha < \rho\rangle$ is a continuous chain of BAs such that for all α with $\alpha + 1 < \rho$ the following conditions hold:*

(i) $A_\alpha \leq_{reg} A_{\alpha+1}$.

(ii) Either $A_\alpha \leq_d A_{\alpha+1}$ or $A_{\alpha+1}$ is a simple extension of A_α.

Let $A = \bigcup_{\alpha<\rho} A_\alpha$. Then there is a subalgebra C of A such that $A_0 \leq_{proj} C \leq_d A$.

Proof. We construct $\langle C_\alpha : \alpha < \rho\rangle$ by induction. Let $C_0 = A_0$. Suppose that C_α has been constructed, where $\alpha + 1 < \rho$, such that $C_\alpha \leq_d A_\alpha$.

Case 1. $A_\alpha \leq_d A_{\alpha+1}$. Then we let $C_{\alpha+1} = C_\alpha$. Note that $C_{\alpha+1} = C_\alpha \leq_d A_\alpha \leq_d A_{\alpha+1}$, so $C_{\alpha+1} \leq_d A_{\alpha+1}$.

Case 9. $A_{\alpha+1} = A_\alpha(u)$. Now $C_\alpha \leq_d A_\alpha \leq_{reg} A_{\alpha+1} = A_\alpha(u)$. Hence by Lemma 30.24c, $C_\alpha \leq_{reg} C_\alpha(u) \leq_d A_\alpha(u) = A_{\alpha+1}$. Hence by Propositions 30.41 and 30.49 we get $C_{\alpha+1}$ such that $C_\alpha \leq_{proj} C_{\alpha+1} \leq_d C_\alpha(u)$.

Now in either case, $C_\alpha = \bigcup_{\beta<\alpha} C_\beta$ for α limit. Clearly $C_\alpha \leq_d A_\alpha$. This finishes the construction of the C_α's.

Let $C = \bigcup_{\alpha<\rho} C_\alpha$. Clearly $A_0 \leq_{proj} C \leq_d A$. $\square$

Theorem 30.52. *Assume that $A \leq_{reg} B$ and $\pi(B/A) \leq \omega$. Then there is a C such that $A \leq_{proj} C \leq_d B$.*

Proof. Say $B = A(\{u_n : n \in \omega\})$. For each $n \in \omega$ let $B'_n = A(\{u_i : i < n\})$. By Proposition 30.31, $B'_n \leq_{reg} B$. Hence clearly $B'_n \leq_{reg} B'_{n+1}$ for all $n \in \omega$. Hence the desired conclusion follows by Proposition 30.5. $\square$

If $A \leq B$, then $\pi(B/A)$ is the least cardinal κ such that there is a set $X \subseteq B$ of size κ such that $A \cup X$ generates a dense subalgebra of B.

$\mathbb{S}$ is a *Cohen skeleton* for A iff the following hold:

(i) The elements of $\mathbb{S}$ are regular subalgebras of A.

(ii) There is an $S \in \mathbb{S}$ such that $\pi(S) \leq \omega$.

(iii) If $S \in \mathbb{S}$ and X is a countable subset of A, then there is an $S' \in \mathbb{S}$ such that $S \cup X \subseteq S'$ and $\pi(S'/S) \leq \omega$.

(iv) For every nonempty chain $\mathbb{C}$ in $\mathbb{S}$ there is an $S \in \mathbb{S}$ such that $\bigcup \mathbb{C}$ is dense in S.

A is a *projective Boolean algebra* iff A is the union of a continuous chain $\langle A_\alpha : \alpha < \rho\rangle$ of subalgebras such that $A_0 = 2$, for each α with $\alpha+1 < \rho$, $A_\alpha \leq_{rc} A_{\alpha+1}$ and $wt(A_{\alpha+1}/A_\alpha) \leq \omega$.

If $C \leq B$ and B is complete, we define

$$C^*_B = \{b \in B : b \text{ and } -b \text{ are (in } B\text{) sums of elements of } C.$$

Proposition 30.53. *Let $C \leq B$ with B complete.*

(i) C^*_B is a subalgebra of B.

(ii) $C \leq_d C^*_B$.
(iii) $C^{**}_B = C^*_B$.

Proof. (i): Clearly C^*_B is closed under $-$. For $\cdot$, suppose that $b, c \in C^*_B$. Say $b = \sum X$, $-b = \sum Y$, $c = \sum Z$, and $-c = \sum W$, with $X, Y, Z, W \subseteq C$. Then $b \cdot c = \sum\{x \cdot y : x \in X, y \in Z\}$ and $-(b \cdot c) = -b + -c = \sum(Y \cup W)$.

(ii): Obvious.

(iii): $C^*_B \subseteq C^{**}_B$ by (ii). Now suppose that $x \in C^{**}_B$. Say $x = \sum X$ and $-x = \sum Y$ where $X, Y \subseteq C^*_B$. For each $y \in X$ write $y = \sum Z_y$ with $Z_y \subseteq C$; and for each $y \in Y$ write $y = \sum V_y$ with $V_y \subseteq C$. Then $x = \sum \bigcup_{y \in X} Z_y$ and $\bigcup_{y\in X} Z_y \subseteq C$, and $-x = \sum \bigcup_{y \in Y} V_y$ and $\bigcup_{y \in Y} V_y \subseteq C$. Thus $x \in C^*_B$. $\square$

Lemma 30.54. *Suppose that $S \leq_{rc} A$ and $\forall a \in A^+[|S| < |A \upharpoonright a|$. Let $x \in A$.*
Then there are $P \subseteq S$ and $u \in \overline{A}^{cmp}$ such that
(i) P is a partition of unity in S, hence in A and $\overline{A}^{cmp}$.
(ii) u is independent from S, i.e., $\forall a \in S^+[a \cdot u \neq 0 \neq a \cdot -u]$.
(iii) $\forall p \in P[p \cdot u \in A]$ and hence $\forall p \in P[S(u) \upharpoonright p \leq_{rc} A \upharpoonright p]$.
(iv) $x \in S(u)^*_{\overline{A}^{cmp}}$.
(v) $u \in S(x)^*_{\overline{A}^{cmp}}$.

Proof. Recall the definition of pr^C from just before Proposition 30.38. Now we define for any $a \in A$, $\mathrm{indp}^S(a) = -(\mathrm{pr}^C(a) + \mathrm{pr}^C(-a))$.

(1) $D \overset{\text{def}}{=} \{\mathrm{indp}^S(a) : a \in A\}\backslash\{0\}$ is dense in S.

For, take any $s \in S^+$. Since $|S| < |A \upharpoonright s|$, choose $a \in A\backslash S$ with $0 < a \leq s$. Let $d = \mathrm{indp}^S(a)$. Then $d \neq 0$, as otherwise we have $\mathrm{pr}^S(a) \leq a$, $\mathrm{pr}^S(a) \cdot \mathrm{pr}^S(-a) = 0$, $\mathrm{pr}^S(a) + \mathrm{pr}^S(-a) = 1$, hence $\mathrm{pr}^S(a) = a \in S$, contradiction. So $d \in D$. Now $-s \leq -a$, so $-s \leq \mathrm{pr}^S(-a)$ and hence $d \leq -\mathrm{pr}^S(-a) \leq s$. So (1) holds.

Now $\mathrm{pr}^S(x) + \mathrm{pr}^S(-x) + \mathrm{indp}^S(x) = 1$, so $D' \overset{\text{def}}{=} \{y \in D : y \leq \mathrm{pr}^S(x)$ or $y \leq \mathrm{pr}^S(-x)$ or $y \leq \mathrm{indp}^S(x)\}$ is dense in S. Hence there is a partition of unity P in S with $P \subseteq D'$. By Lemma 30.13(iii), P is a partition of unity in A. Since A is dense in $\overline{A}^{cmp}$, it is also a partition of unity in $\overline{A}^{cmp}$.

(2) If $p \in P$ and $p \leq \mathrm{indp}^S(x)$, then $\mathrm{indp}^S(p \cdot x) = p$.

For, suppose that $p \in P$ and $p \leq \mathrm{indp}^S(x)$. Then $p \cdot (\mathrm{pr}^S(x) + \mathrm{pr}^S(-x)) = 0$. Then $p \cdot \mathrm{pr}^S(p \cdot x) \leq p \cdot \mathrm{pr}^S(x) = 0$. Also, if $u \in S$ is greatest such that $u \leq -(p \cdot x)$, then it is greatest such that $u \cdot p \cdot x = 0$; in particular, $u \cdot p \leq \mathrm{pr}^S(-x)$, so $u \cdot p = 0$. Thus $p \cdot \mathrm{pr}^S(-(p \cdot x)) = 0$. So $p \leq \mathrm{indp}^S(p \cdot x)$. Also,

$$\mathrm{indp}^S(p \cdot x) \cdot -p = -\mathrm{pr}^S(p \cdot x) \cdot -\mathrm{pr}^S(-(p \cdot x)) \cdot -p \leq -\mathrm{pr}^S(-(p \cdot x)) \cdot -p$$

and $-p \leq \mathrm{pr}^S(-p + -x)$, so $-\mathrm{pr}^S(-(p \cdot x)) \leq p$ and (2) follows.

For each $p \in P$, since $p \in D$ we can fix $a_p \in A$ such that $p = \mathrm{indp}^S(a_p)$; if $p \leq \mathrm{indp}^S(x)$ we let $a_p = p \cdot x$; see (2). Now let

$$u = \sum_{p \in P}^{\overline{A}^{cmp}} (p \cdot a_p).$$

To check (ii), suppose that $a \in S^+$ and $a \cdot u = 0$. Choose $p \in P$ such that $a \cdot p \neq 0$. Then $a \cdot p \cdot a_p = 0$, so $a \cdot p \leq \mathrm{pr}^S(-a_p)$, hence $a \cdot p = a \cdot p \cdot \mathrm{indp}^S(-a_p) = 0$, contradiction. Next, suppose that $a \in S^+$ and $a \cdot -u = 0$. Choose $p \in P$ such that $a \cdot p \neq 0$. Then $a \cdot p \cdot -a_p = 0$, so $a \cdot p \leq \mathrm{pr}^S(a_p)$, hence $a \cdot p = a \cdot p \cdot \mathrm{indp}^S(a_p) = 0$, contradiction. So (ii) holds.

For (iii), if $p \in P$ then $p \cdot u = p \cdot a_p \in A$. Now

(3) $S(a_p) \leq_{rc} A$.

In fact, let $a \in A^+$ and let $b = \mathrm{pr}^S(-a_p + a) \cdot a_p + \mathrm{pr}^S(a_p + a) \cdot -a_p$ Thus $b \in S(a_p)$ and $b \leq a$. Suppose that $c \in S(a_p)$ and $c \leq a$. Write $c = d \cdot a_p + e \cdot -a_p$ with $d, e \in S$. Then $c \cdot a_p = d \cdot a_p \leq a$, hence $d \leq -a_p + a$, and so $d \leq \mathrm{pr}^S(-a_p + a)$. Hence $d \cdot a_p \leq \mathrm{pr}^S(-a_p + a) \cdot a_p$. Similarly, $e \cdot -a_p \leq \mathrm{pr}^S(a_p + a) \cdot -a_p$. Hence $c \leq b$, proving (3).

Now by (3), $S(u) \upharpoonright p = S(a_p) \upharpoonright p \leq_{rc} A \upharpoonright p$.

Hence (iii) holds.

For (iv), we have $x = \sum^{\overline{A}^{cmp}}_{p \in P}(x \cdot p)$ and $-x = \sum^{\overline{A}^{cmp}}_{p \in P}(-x \cdot p)$. Now we claim that $\forall p \in P[p \cdot x, p \cdot -x \in S(u)]$. For, take any $p \in P$.

Case 1. $p \leq \mathrm{pr}^S(x)$. Then $p \leq x$, so $p \cdot x = p \in S \subseteq S(u)$.

Case 9. $p \leq \mathrm{pr}^S(-x)$. Then $p \leq -x$, so $p \cdot x = 0 \in S(u)$.

Case 3. $p \leq \mathrm{indp}^S(x)$. Then $p \cdot x = a_p = u \cdot p \in S(u)$.

Similarly, $p \cdot -x \in S(u)$.

For (v), clearly $p \cdot a_p = p \cdot x \in S(x)$; so u is a sum of elements of $S(x)$. Now $-u = \sum_{p \in P}(p \cdot -a_p) = \sum_{p \in P}(p \cdot -(p \cdot x)) = \sum_{p \in P}(p \cdot -x)$, and hence $-u$ is a sum of elements of $S(x)$. So (v) holds.

Lemma 30.55. *Assume that $S \leq_{rc} A$ and $\forall a \in A^+[|S| < |A \upharpoonright a|]$. Let $\{x_n : n \in \omega\} \subseteq A$. Then there exists $\{u_n : n \in \omega \backslash \{0\}\} \subseteq \overline{A}^{cmp}$ such that $\forall n \in \omega[u_{n+1}$ is independent from $S(\{u_i : 1 \leq i \leq n\})$, $x_n \in S(\{\{u_i : 1 \leq i \leq n\})^*_{\overline{A}^{cmp}}$, and $u_{n+1} \in S(\{x_i : i \leq n\})^*_{\overline{A}^{cmp}}$.*

Proof. We define by induction $P_0, P_1, \ldots$ and $u_1, \ldots, u_n$ such that, with $C_n = S(\{u_1, \ldots, u_n\})$,

(i) $\forall n \in \omega[P_n \subseteq C_n \cap A$ and P_n is a partition of unity in $A]$.

(ii) $\forall n \in \omega[P_{n+1}$ refines $P_n]$.

(iii) $\forall n \in \omega[u_{n+1}$ is independent from $C_n]$.

(iv) $\forall n \in \omega \forall p \in P_n[\{p \cdot u_1, \ldots, p \cdot u_{n+1}\} \subseteq A$, $C_{n+1} \upharpoonright p \leq_{rc} A \upharpoonright p$, and $x_n \in C^*_{(n+1)\overline{A}^{cmp}}]$.

(v) $\forall n \in \omega[\{u_1, \ldots, u_{n+1}\} \subseteq (S(\{x_0, \ldots, x_n\}))^*_{\overline{A}^{cmp}}$.

We get P_0 and u_1 by Lemma 30.54. Given $n \geq 1$ and $P_0, \ldots, P_{n-1}$, $u_1, \ldots u_n$ so that
(1) $\forall i < n[P_i \subseteq C_i \cap A$ and P_i is a partition of unity in $A]$.
(2) $\forall i < n$ with $i + 1 < n[P_{i+1}$ refines $P_i]$.
(3) $\forall i < n[u_{i+1}$ is independent from $C_i]$.
(4) $\forall i < n \forall p \in P_i[\{p \cdot u_1, \ldots, p \cdot u_{i+1}\} \subseteq A$, $C_{i+1} \upharpoonright p \leq_{rc} A \upharpoonright p$, and $x_i \in C^*_{(i+1)\overline{A}^{cmp}}]$.
(5) $u_{n+1} \in (S(\{x_0, \ldots, x_n\}))^* \overline{A}^{cmp}$.

Note that $\forall p \in P_{n-1} \forall a \in A^+[C_n \upharpoonright p \leq_{rc} A \upharpoonright p$ and $|C_n| < |A \upharpoonright a|]$. Hence we can apply Lemma 30.54 to get a refinement P_n of P_{n-1} so that (1)-(5) hold with n replaced by $n + 1$. ∎

A Boolean algebra A is *card-homogeneous* iff $\forall a \in A^+[|A \restriction a| = |A|]$.

Theorem 30.56. *The collection $D \stackrel{\text{def}}{=} \{a \in A^+ : A \restriction a$ is card-homogeneous$\}$ is dense in A. Let P be a partition of unity with $P \subseteq D$. Then $\overline{A}^{cmp} \cong \prod_{a\in P}(\overline{A}^{cmp} \restriction a)$.*

Proof. Clearly D is dense in A. Now for each $b \in \overline{A}^{cmp}$ and $a \in P$ let $(f(b))_a = b \cdot a$. Then

$$(f(b+c))_a = (b+c)\cdot a = b\cdot a + c\cdot a = (f(b))_a + (f(c))_a = (f(b)+f(c))_a,$$

so $f(b+c) = f(b) + f(c)$. Also, $(f(-b))_a = (-b)\cdot a = -^{\overline{A}^{cmp}\restriction a} b = -(f(b)_a$, so $f(-b) = -f(b)$. Clearly f is one-one. Given $x \in \prod_{a\in P}(\overline{A}^{cmp} \restriction a)$, let $b = \sum_{a\in P} x(a)$. Then $f(b) = x$. □

Proposition 30.57. *Suppose that $A \leq_{rc} A(u)$, F is a free BA, and $e : A \to F$ and $f : F \to A$ are homomorphisms such that $f \circ e$ is the identity. Let $F(x)$ be the free extension of F by one new free generator x. Then there are homomorphisms $e' : A(u) \to F(x)$ and $f' : F(x) \to A(u)$ such that $e \subseteq e'$, $e'(u) = e(-(\mathrm{pr}^A(u) + \mathrm{pr}^A(-u)))\cdot x + e(\mathrm{pr}^A(u))$, $f \subseteq f'$, and $f' \circ e'$ is the identity.*

Proof. By Sikorski's extension criterion we need to prove that the following conditions are equivalent:

$$\prod_{i<m}(a_i\cdot u + b_i\cdot -u)\cdot\prod_{i<n}(-c_i\cdot u + -d_i\cdot -u) = 0; \tag{1}$$

$$\prod_{i<m}(e(a_i)\cdot e'(u)\cdot x + e(b_i)\cdot -e'(u))\cdot\prod_{i<n}(-e(c_i)\cdot e'(u) + -e(d_i)\cdot -e'(u)) = 0, \tag{2}$$

where each $a_i, b_i, c_i, d_i \in A$. Now (1) is equivalent to

$$\left(\prod_{i<m} a_i\cdot\prod_{j<n} -c_j\right)\cdot u + \left(\prod_{i<m} b_i\cdot\prod_{j<n} -d_j\right)\cdot -u = 0$$

and (2) is equivalent to

$$\left(\prod_{i<m} e(a_i)\cdot\prod_{j<n} -e(c_j)\right)\cdot e'(u) + \left(\prod_{i<m} e(b_i)\cdot\prod_{j<n} -e(d_j)\right)\cdot -e'(u) = 0.$$

Now let $v = \prod_{i<m} a_i\cdot\prod_{j<n} -c_j$ and $w = \prod_{i<m} b_i\cdot\prod_{j<n} -d_j$. Then (1) is equivalent to

$$v\cdot u + w\cdot -u = 0$$

and (2) is equivalent to

$$e(v) \cdot e'(u) + e(w) \cdot -e'(u) = 0.$$

Note that

$$\begin{aligned} -e'(u) &= -e(\mathrm{pr}^A(u)) \cdot -(e(-(\mathrm{pr}^A(u) + \mathrm{pr}^A(-u))) \cdot x) \\ &= -e(\mathrm{pr}^A(u)) \cdot (e(\mathrm{pr}^A(u) + \mathrm{pr}^A(-u)) + -x) \\ &= -e(\mathrm{pr}^A(u)) \cdot (e(\mathrm{pr}^A(-u)) + -x). \end{aligned}$$

Now $v \cdot u = 0$ iff $v \leq \mathrm{pr}^A(-u)$ iff $v \cdot -\mathrm{pr}^A(-u) = 0$, and

$$e(v) \cdot e'(u) = 0 \quad \text{iff} \quad e(v) \cdot -e(\mathrm{pr}^A(u)) \cdot -e(\mathrm{pr}^A(-u)) = 0 \text{ and } e(v) \cdot e(\mathrm{pr}^A(u)) = 0.$$

Now if $v \cdot u = 0$ then $v \cdot \mathrm{pr}^A(u) = 0$ and $e(v) \cdot -e(\mathrm{pr}^A(u)) \cdot -e(\mathrm{pr}^A(-u)) = 0$ and $e(v) \cdot e(\mathrm{pr}^A(u)) = 0$. Conversely, if $e(v) \cdot e'(u) = 0$ then $e(v) \cdot -e(\mathrm{pr}^A(-u)) = 0$, hence $v \cdot -\mathrm{pr}^A(-u) = 0$, and so $v \cdot u = 0$.

Next, if $w \cdot -u = 0$, then $w \leq u$, hence $w \in \mathrm{pr}^A(u)$, hence $w \cdot -\mathrm{pr}^A(u) = 0$. Then

$$\begin{aligned} e(w) \cdot -e'(u) &= e(w) \cdot -e(\mathrm{pr}^A(u)) \cdot (e(\mathrm{pr}^A(-u)) + -x) \\ &= e(w) \cdot -e(\mathrm{pr}^A(u)) \cdot e(\mathrm{pr}^A(-u)) + e(w) \cdot -e(\mathrm{pr}^A(u) \cdot -x \end{aligned}$$

Now $w \cdot \mathrm{pr}^A(-u) = 0$, so $e(w) \cdot e(\mathrm{pr}^A(-u)) = 0$ Also, $e(w) \cdot -e(\mathrm{pr}^A(u) = 0$; so $e(w) \cdot -e'(u) = 0$.

Conversely, suppose that $e(w) \cdot -e'(u) = 0$. Thus $e(w) \cdot -e(\mathrm{pr}^A(u)) \cdot e(\mathrm{pr}^A(-u)) = 0$ and $e(w) \cdot -e(\mathrm{pr}^A(u) = 0$. We have $\mathrm{pr}^A(u) \leq u$, so $-u \leq -e(\mathrm{pr}^A(u))$, so $w \cdot -u = 0$.

Thus e extends to an isomorphism e' into $F(x)$ as indicated.

Now clearly there is a homomorphism $f' : F(x) \to A(u)$ extending f such that $f'(x) = u$. To show that $f' \circ e'$ is the identity on $A(u)$ it suffices to show that $f'(e'(u)) = u$:

$$\begin{aligned} f'(e'(u)) &= f'(e(-(\mathrm{pr}^A(u) + \mathrm{pr}^A(-u))) \cdot x + e(\mathrm{pr}^A(u))) \\ &= f((e(-(\mathrm{pr}^A(u) + \mathrm{pr}^A(-u))))) \cdot u + f(e(\mathrm{pr}^A(u)) \\ &= -(\mathrm{pr}^A(u) + \mathrm{pr}^A(-u)) \cdot u + \mathrm{pr}^A(u) \\ &= -\mathrm{pr}^A(u) \cdot u + \mathrm{pr}^A(u) = u. \end{aligned}$$

□

Lemma 30.58. *Assume that $C \leq_{rc} A$ and $x \in A$. Then $C(x) \leq_{rc} A$.*

Proof. Given $a \in A$, let $b = \mathrm{pr}^C(-x+a) \cdot x + \mathrm{pr}^C(x+a) \cdot -x$. Clearly $b \leq a$. Suppose that $c \in C(x)$ and $c \leq a$. Say $c = d \cdot x + e \cdot -x$ with $d, e \in A$. Then $d \cdot x \leq a$, so $d \leq -x + a$, so $d \leq \mathrm{pr}^C(-x+a)$ and hence $d \cdot -\mathrm{pr}^C(-x+a) = 0$. Also $e \cdot -x \leq a$, so $e \leq x + a$ and hence $e \cdot -\mathrm{pr}^C(x+a) = 0$. Hence

$$c \cdot -b = (d \cdot -\mathrm{pr}^C(-x+a) \cdot x + e \cdot -\mathrm{pr}^C(x+a) \cdot -x = 0.$$

□

Lemma 30.59. *If A is a projective BA, then there is an increasing continuous sequence $\langle B_\alpha : \alpha < \rho$ with union A such that $B_0 = 2$, $B_\alpha \leq_{rc} B_{\alpha+1}$ and $B_{\alpha+1}$ is a simple extension of B_α for all α with $\alpha + 1 < \rho$.*

Proof. By definition there is an increasing continuous sequence $\langle B_\alpha : \alpha < \rho$ with union A such that $B_0 = 2$, $B_\alpha \leq_{rc} B_{\alpha+1}$ and $wt(B_{\alpha+1}/B_\alpha) \leq \omega$ for all α with $\alpha+1 < \rho$. For each α with $\alpha + 1 < \rho$ let $\{x_{\alpha 0}, x_{\alpha 1}, \ldots\}$ such that $B_{\alpha+1} = \langle B_\alpha \cup \{x_{\alpha 0}, x_{\alpha 1}, \ldots\}\rangle$. By Lemma 30.53 we get $B_\alpha(\{x_{\alpha 0}, \ldots, x_{\alpha n}\}) \leq_{rc} B_{\alpha+1}$ for all $n \in \omega$. Clearly then $B_\alpha(\{x_{\alpha 0}, \ldots, x_{\alpha n}\}) \leq_{rc} B_\alpha(\{x_{\alpha 0}, \ldots, x_{\alpha(n+1)}\})$ for all $n \in \omega$. □

Lemma 30.60. *If A is projective and card-homogeneous and $|A| = \kappa$, then there exist homomorphisms $e : A \to \mathrm{Fr}(\kappa)$ and $f : \mathrm{Fr}(\kappa) \to A$ such that $f \circ e$ is the identity.*

Proof. This is a matter of a transfinite construction using Lemma 30.59 and Proposition 30.57. □

A *projective skeleton* for A is a collection $\mathbb{S}$ such that the following conditions hold:

(i) $\forall S \in \mathbb{S}[S \leq_{rc} A]$.

(ii) $2 \in \mathbb{S}$.

(iii) $\forall S \in \mathbb{S} \forall X \in [A]^{\leq\omega} \exists S' \in \mathbb{S}[S \cup X \subseteq S'$ and $wt(S'/S) \leq \omega]$.

(iv) $\mathbb{S}$ is closed under the union of nonempty chains.

Proposition 30.61. *Suppose that $e : A \to \mathrm{Fr}(\kappa)$ and $f : \mathrm{Fr}(\kappa) \to A$ are homomorphisms such that $f \circ e$ is the identity. Then there is a projective skeleton $\mathbb{S}$ for A.*

Proof. Let U be a set of free generators for $\mathrm{Fr}(\kappa)$. For each $V \subseteq U$ let $F_V = \langle V \rangle$ and $A_V = f[F_V]$. We say that V is *closed* iff $e[A_V] \subseteq F_V$. Let

$$\mathbb{S} = \{A_V : V \subseteq U \text{ is closed}\}.$$

(ii): $F_\emptyset = 2$, $A_\emptyset = 2$, and $e[A_\emptyset] = 2 \subseteq F_\emptyset$. So $\emptyset$ is closed and $A_\emptyset = 2 \in \mathbb{S}$.

(i): suppose that V is closed and $a \in A^+$; we want to show that there is a largest $b \in A_V$ such that $b \leq a$. Now $F \cong F_V \oplus F_{U \setminus V}$, so F_V is relatively complete in F, by Proposition 11.8 of the handbook. Let $b = (f \restriction F_V)(\mathrm{pr}^F_{F_V}(e(a)))$. Then $\mathrm{pr}^F_{F_V}(e(a)) \leq e(a)$, so $b \leq f(e(a)) = a$. Suppose that $c \in A_V$ and $c \leq a$. Then $e(c) \leq e(a)$. Hence $e(c) \leq \mathrm{pr}^F_{F_V}(e(a)$, so $c = f(e(c)) \leq f(\mathrm{pr}^F_{F_V}(e(a))) = b$.

(iv): Suppose that $\mathscr{A} \subseteq \mathbb{S}$ is a nonempty chain. For each $B \in \mathscr{A}$ let $B = A_{V_B}$ where $V_B \subseteq U$ is closed. Let

$$W = \bigcup_{B \in \mathscr{A}} V_B \quad \text{and} \quad C = \bigcup \mathscr{A}.$$

(1) $C = \bigcup_{B \in \mathscr{A}} f[F_{V_B}]$.

In fact,

$$\begin{aligned} x \in C \quad \text{iff} \quad & \exists B \in \mathscr{A}[x \in B] \quad \text{iff} \quad \exists B \in \mathscr{A}[x \in A_{V_B}] \\ \text{iff} \quad & \exists B \in \mathscr{A}[x \in f[F_{V_B}]] \quad \text{iff} \quad x \in \bigcup_{B \in \mathscr{A}} f[F_{V_B}]. \end{aligned}$$

(2) $C = f[F_W]$.

For, if $x \in C$, choose $B \in \mathscr{A}$ such that $x \in f[F_{V_B}]$. Now $V_B \subseteq W$, so $F_{V_B} \subseteq F_W$ and so $f[V_B] \subseteq f[F_W]$ and hence $x \in f[F_W]$. So $C \subseteq f[F_W]$. To prove the converse, it suffices to show that $f[W] \subseteq C$, since f is a homomorphism. Now

$$f[W] = \bigcup_{B\in\mathscr{A}} f[V_B] \subseteq \bigcup_{B\in\mathscr{A}} f[F_{V_B}] = C.$$

So (2) holds.

(3) $\forall B \in \mathscr{A}[B \subseteq C]$.

For, if $B \in \mathscr{A}$, then $B = A_{V_B} = f[F_{V_B}] \subseteq C$.

W is closed:

$$\begin{aligned} e[A_W] &= e[f[F_W]] = e[C] \quad \text{by (2)} \\ &= \bigcup_{B\in\mathscr{A}} e[f[F_{V_B}]] = \bigcup_{B\in\mathscr{A}} e[A_{V_B}] \\ &\subseteq \bigcup_{B\in\mathscr{A}} F_{V_B} \subseteq F_W. \end{aligned}$$

Thus (iv) holds.

(iii): Suppose that $S \in \mathbb{S}$ and $X \subseteq A$ is countable. Say $S = A_V$ with $V \subseteq U$ closed. Define $X_0 = X$. If $X_0, \ldots, X_n$ and $V_0, \ldots, V_{n-1}$ have been defined, let $V_n \subseteq U$ be countable such that $e[X_n] \subseteq F_{V_n}$ and $V_0 \cup \ldots \cup V_{n-1} \subseteq V_n$. Then let $X_{n+1} = f[V_n] \cup X_0 \cup \ldots \cup X_n$. Let $W = V \cup \bigcup_{n\in\omega} V_n$ and let $S' = A_W$. Now $V \subseteq W$, so $F_V \leq F_W$ and $S = A_V = f[F_V] \leq f[F_W] = A_W = S'$. Also, $e[X] = e[X_0] \leq F_{V_0} \leq F_W$, so $X = f[e[X]] \leq f[F_W] = S'$. Now F_W is countably generated over F_V, so $S' = A_W = f[F_W]$ is countably generated over $f[F_V] = S$. Finally, W is closed: we want to show that $e[A_W] \subseteq F_W$. We have

$$\begin{aligned} e[A_W] = e[f[F_W]] &= e\left[f\left[\left\langle V \cup \bigcup_{n\in\omega} V_n \right\rangle\right]\right] \\ &= \left\langle e[f[V]] \cup \bigcup_{n\in\omega} e[f[V_n]] \right\rangle \\ &= \left\langle \langle e[f[V]]\rangle \cup \bigcup_{n\in\omega} e[f[V_n]] \right\rangle \\ &= \left\langle e[f[\langle V\rangle]]\rangle \cup \bigcup_{n\in\omega} e[f[V_n]] \right\rangle \\ &= \left\langle e[A_V] \cup \bigcup_{n\in\omega} e[f[V_n]] \right\rangle \\ &\subseteq \left\langle F_V \cup \bigcup_{n\in\omega} e[X_{n+1}] \right\rangle \\ &\subseteq \left\langle F_V \cup \bigcup_{n\in\omega} F_{V_n} \right\rangle = F_W \end{aligned}$$

□

Theorem 30.62. *If A is projective and card-homogeneous and $|A| = \kappa$, then $\overline{A}^{cmp} \cong \overline{\mathrm{Fr}(\kappa)}^{cmp}$.*

Proof. Assume that A is projective and card-homogeneous and $|A| = \kappa$.

Case 1. $\kappa = \omega$. Since A is card-homogeneous, it is atomless, and so is isomorphic to $\mathrm{Fr}(\omega)$.

Case 9. $\kappa > \omega$. Let $\{y_\alpha : \alpha < \kappa\}$ be an enumeration of A. By Lemma 30.59 and Proposition 30.60 let $\mathbb{S}$ be a projective skeleton for A. We now construct $\langle S_\alpha : \alpha < \kappa\rangle$, $\langle X_\alpha : \alpha < \kappa\rangle$, $\langle F_\alpha : \alpha < \kappa\rangle$, $\langle U_\alpha : \alpha < \kappa\rangle$ by recursion so that the following conditions hold for every $\alpha < \kappa$:

(1) $S_\alpha \in \mathbb{S}$.

(2) $S_{\alpha+1} = S_\alpha(X_\alpha)$, $X_\alpha \subseteq A$ is countable, and $y_\alpha \in X_\alpha$.

(3) $F_\alpha \leq \overline{A}^{cmp}$.

(4) F_α is free, $F_{\alpha+1} = F_\alpha(U_\alpha)$, U_α is countable, and U_α is independent of F_α.

(5) $F_\alpha \leq S^*_{\alpha\overline{A}^{cmp}}$ and $S_\alpha \leq F^*_{\alpha\overline{A}^{cmp}}$.

We define $S_0 = F_0 = 2$. Clearly (1)–(5) hold. Now suppose that S_α, F_α and X_β, U_β for all $\beta < \alpha$ have been defined. By (iii) in the projective skeleton, there is a countable $X_\alpha \subseteq A$ with $y_\alpha \in X_\alpha$, and $S_\alpha(X_\alpha) \in \mathbb{S}$. By Lemma 30.55 we get $U_\alpha \subseteq \overline{A}^{cmp}$ countable and independent over S_α, and with $X_\alpha \subseteq S(U_\alpha)^*_{\alpha\overline{A}^{cmp}}$ and $U_\alpha \subseteq S(X_\alpha)^*_{\overline{A}^{cmp}}$. We put $S_{\alpha+1} = S_\alpha(X_\alpha)$.

(6) U_α is independent over F_α.

For, let p be an elementary product from U_α, and let $f \in F_\alpha^+$. Now $F_\alpha \leq S^*_{\alpha\overline{A}^{cmp}}$, so there is an $a \in S_\alpha$ such that $0 < a \leq f$. Since U_α is independent over S_α, we get $p \cdot a \neq 0$. Hence $p \cdot f \neq 0$. This proves (6).

Let $F_{\alpha+1} = F_\alpha(U_\alpha)$.

(7) $F_{\alpha+1} \leq S^*_{\alpha+1,\overline{A}^{cmp}}$.

For, $F_\alpha \leq S^*_{\alpha\overline{A}^{cmp}} \leq S^*_{\alpha+1,\overline{A}^{cmp}}$ and $U_\alpha \subseteq S_\alpha(X_\alpha)^*_{\overline{A}^{cmp}} = S^*_{\alpha+1,\overline{A}^{cmp}}$. So (7) holds.

(8) $S_{\alpha+1} \leq F^*_{\alpha+1,\overline{A}^{cmp}}$.

For, $S_\alpha \leq F^*_{\alpha\overline{A}^{cmp}} \leq F^*_{\alpha+1,\overline{A}^{cmp}}$ and

$$X_\alpha \subseteq (S_\alpha(U_\alpha))^*_{\overline{A}^{cmp}} \leq (F^*_{\overline{A}^{cmp}}(U_\alpha))^*_{\overline{A}^{cmp}} \leq F^{**}_{\alpha+1,\overline{A}^{cmp}} = F^*_{\alpha+1,\overline{A}^{cmp}}.$$

Thus (1)–(5) hold. Now let $F = \bigcup_{\alpha<\kappa} F\alpha$. So $F \cong \mathrm{Fr}(\kappa)$. F is dense in $\overline{A}^{cmp}$, for suppose that $b \in \overline{A}^{cmp}$ and $b \neq 0$. Choose $a \in A$ such that $0 < a \leq b$. Say $a \in S_\alpha$. Since $S_\alpha \leq F^*_{\alpha\overline{A}^{cmp}}$ and $F_\alpha \leq_d F^*_{\alpha\overline{A}^{cmp}}$, there is an $a' \in F_\alpha$ such that $0 < a' \leq a \leq b$. □

Lemma 30.63. *Suppose that B is an infinite BA of uniform density κ. Suppose that C is a club of countable regular subalgebras of B such that $\forall A_1, A_2 \in C[\langle A_1 \cup A_2\rangle \in C]$. Let*

$$\mathcal{S} = \left\{\left\langle \bigcup X \right\rangle : X \subseteq C\right\}.$$

Then $\mathcal{S}$ is a Cohen skeleton for A.

Proof. By Lemma 30.19, the members of $\mathcal{S}$ are regular subalgebras of B. Since $C \subseteq \mathcal{S}$, there is an $A \in \mathcal{S}$ with $\pi(A) \leq \omega$. Now suppose that $S \in \mathcal{S}$ and X is a countable subset of A. Say $S = \langle \bigcup Y \rangle$ with $Y \subseteq C$. There is a $D \in C$ such that $X \subseteq D$. Then for each $E \in Y$ we have $\langle D \cup E \rangle \in C$. Let $Z = \langle \bigcup \{\langle D \cup E \rangle : E \in Y\}$. Then $Z \in \mathcal{S}$, $S \cup X \subseteq Z$, and $\pi(Z \backslash S) \leq \omega$. Clearly $\mathcal{S}$ is closed under the union of chains. □

Lemma 30.64. *If A has a Cohen skeleton, then A is the union of a continuous chain $\langle A_\alpha : \alpha < \rho \rangle$ where $\pi(A_0) \leq \omega$, for all α with $\alpha + 1 < \rho$, $A_\alpha \leq_{reg} A_{\alpha+1}$, and $\pi(A_{\alpha+1}/A_\alpha) \leq \omega$.*

Proof. Assume that A has a Cohen skeleton, with notation as in the definition. By (ii) in the definition, let $A_0 \in \mathbb{S}$ be such that $\pi(A_0) \leq \omega$. Let $\langle a_\alpha : \alpha < \rho \rangle$ enumerate A. Having defined $A_\alpha \in \mathbb{S}$, by (iii) in the definition let $A_{\alpha+1} \in \mathbb{S}$ be such that $A_\alpha \cup \{a_\alpha\} \subseteq A_{\alpha+1}$ and $\pi(A_{\alpha+1}/A_\alpha) \leq \omega$. Then $A_\alpha \leq_{reg} A_{\alpha+1}$ by Proposition 30.24. For α limit let $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$. Clearly $A = \bigcup_{\alpha<\rho} A_\alpha$, as desired. □

Lemma 30.65. *The following conditions are equivalent:*

(i) A is the union of a continuous chain $\langle A_\alpha : \alpha < \rho \rangle$ where $\pi(A_0) \leq \omega$, for all α with $\alpha + 1 < \rho$, $A_\alpha \leq_{reg} A_{\alpha+1}$, and $\pi(A_{\alpha+1}/A_\alpha) \leq \omega$.

(ii) As in (i), but in addition $A_0 = 2$ and $A_{\alpha+1}$ is a simple extension of A_α for all α with $\alpha + 1 < \rho$.

(iii) As in (ii), but in addition A_α is dense or relatively complete in $A_{\alpha+1}$ for all α with $\alpha + 1 < \rho$.

Proof. (i)⇒(iii): Assume (i). Let $\{a_n : n \in \omega\}$ be dense in A_0. Then define $B_0 = 2$ and $B_{n+1} = \langle B_n \cup \{a_n\} \rangle$ for all $n \in \omega$. Obviously $B_n \leq_{rc} B_{n+1}$ for all n, so by Proposition 30.24, $B_n \leq_{reg} B_{n+1}$ for all n. Let $B' = \bigcup_{n\in\omega} B_n$. Then B' is dense in A_0. This takes care of (iii) up to A_0.

Now by Proposition 30.51 there is a subalgebra C of A such that $A_0 \leq_{proj} C \leq_d A$. By the definition of $\leq_{proj}$ we get (iii).

(iii)⇒(ii)⇒(i): clear. □

Lemma 30.66. *Assume that Lemma 30.65(i) holds. Then A has a dense projective subalgebra.*

Proof. By Proposition 30.51. □

Theorem 30.67. *Let B be an infinite BA of uniform density. Then B is isomorphic to a standard Cohen algebra iff the set $\{A \in [B]^\omega : A \leq_{reg} B\}$ contains a club C with the property that*

$$(*) \qquad \forall A_1, A_2 \in C[\langle A_1 \cup A_2 \rangle \in C].$$

Proof. ⇒: by Theorem 30.23. ⇐: Assume the indicated condition. Then by Lemmas 30.63–30.66, B has a dense projective subalgebra A. Let $A \cong \prod_{i\in I} C_i$ with each C_i card-homogeneous. By Theorem 30.26, A and B have ccc, so I is countable. Say $C_i = A \restriction a_i$.

for each $i \in I$. By Theorem 30.62, $\overline{C}_i^{cmp} \cong \overline{\mathrm{Fr}(\kappa_i)}^{cmp}$, where $\kappa_i = |C_i|$. Also clearly $\pi(C_i) = \pi(\overline{\mathrm{Fr}(\kappa_i)}^{cmp}) = \kappa_i$. Now $A \upharpoonright a_i$ is dense in $B \upharpoonright a_i$, so $\kappa_i = \pi(A \upharpoonright a_i) = \pi(B \upharpoonright a_i)$. Since B has uniform density, all the κ_i's are equal, say to λ. Hence $\overline{B}^{cmp} \cong \overline{A}^{cmp} \cong \prod_{i\in I} \overline{C}_i^{cmp} \cong \overline{\mathrm{Fr}(\lambda)}^{cmp}$. □

If B is a complete BA, then $\mathbb{S}$ is a *complete skeleton* for B iff the following conditions hold:

(i) $\forall S \in \mathbb{S}[S$ is complete and $S \leq_{reg} B]$.

(ii) $\forall X \in [B]^{\leq\omega} \exists S \in \mathbb{S}[X \subseteq S$ and $\pi(S) \leq \omega]$.

(iii) $\forall S, S' \in \mathbb{S}[S \vee S' \stackrel{\text{def}}{=} \langle S \cup S'\rangle^{cmp} \in \mathbb{S}$ and $\langle S \cup S'\rangle \leq_d S \vee S']$.

(iv) $\forall S, S' \in \mathbb{S} \forall X, Y[X$ is dense in S and Y is dense in $S' \Rightarrow [\{x \cdot y : x \in X,\ y \in Y\}$ is dense in $\langle S \cup S'\rangle]]$.

(v) For every chain $\mathbb{C}$ in $\mathbb{S}$ there is an $S \in \mathbb{S}$ such that $\bigcup \mathbb{C} \leq_d S$.

Lemma 30.68. $\{2\}$ *is a complete skeleton for 9.* □

Lemma 30.69. *Let κ be an infinite cardinal, and let U be a set of independent generators of* $\mathrm{Fr}(\kappa)$. *Then* $\mathbb{S} \stackrel{\text{def}}{=} \{\langle M\rangle^{cmp} : M \subseteq U\}$ *is a complete skeleton for* $\overline{\mathrm{Fr}(\kappa)}^{cmp}$.

Proof. (i) is obvious. For (ii), suppose that $X \in [\overline{\mathrm{Fr}(\kappa)}^{cmp}]^{\leq\omega}$. For each $x \in X$ there is a countable $Y_x \subseteq \mathrm{Fr}(\kappa)$ such that $x = \sum Y_x$. Then there is a countable $M \subseteq U$ such that $\bigcup_{x\in X} Y_x \subseteq \langle M\rangle$. Clearly $S \stackrel{\text{def}}{=} \langle M\rangle^{cmp} \in \mathbb{S}$, $X \subseteq \langle M\rangle^{cmp}$, and $\pi(S) \leq \omega$. For (iii), suppose that $S, S' \in \mathbb{S}$. Say $S = \langle M\rangle^{cmp}$ and $S' = \langle M'\rangle^{cmp}$. Clearly $S \vee S' = \langle M \cup M'\rangle^{cmp} \in \mathbb{S}$ and $\langle S \cup S'\rangle \leq_d S \vee S'$. For (iv), suppose that $a \in S^+$ and $b \in S'^+$. Then $a = \sum\{x : x \in X$ and $x \leq a\}$ and $b = \sum\{y : y \in Y$ and $y \leq b\}$, so $a \cdot b = \sum\{x \cdot y : x \in X, x \leq a, y \in Y, y \leq b\}$, and (iv) follows. (v) is clear. □

Lemma 30.70. *If I is countable and for each $i \in I$, $\mathbb{S}_i$ is a complete skeleton for B_i, then*

$$\mathbb{S} \stackrel{\text{def}}{=} \left\{\prod_{i\in I} S_i : \forall i \in I[S_i \in \mathbb{S}_i]\right\}$$

is a complete skeleton for $\prod_{i\in I} B_i$.

Proof. (i): clear. (ii): Suppose that $X \in [\prod_{i\in I} B_i]^{\leq\omega}$. Then $\forall i \in I[\{x_i : x \in X\} \in [B_i]^{\leq\omega}]$, and so $\forall i \in I \exists S_i \in \mathbb{S}_i[\{x_i : x \in X\} \subseteq S_i$ and $\pi(S_i) \leq \omega]$. Hence $X \subseteq \prod_{i\in I} S_i$ and $\pi(\prod_{i\in I} S_i) \leq \omega$. In fact, for each $i \in I$ let $Y_i \subseteq S_i$ be dense with $|Y_i| \leq \omega$. Let

$$Z = \left\{s \in \prod_{i\in I} S_i : \exists j \in I[s_j \in Y_j \text{ and } \forall i \in I\backslash\{j\}[s_i = 0]]\right\}.$$

Then Z is dense in $\prod_{i\in I} S_i$ and $|Z| \leq \omega$.

For (iii), suppose that $\forall i \in I[S_i \in \mathbb{S}_i]$ and $\forall i \in I[S'_i \in \mathbb{S}_i]$. Then for each $i \in I$, $S_i \vee S'_i \in \mathbb{S}$ and $\langle S_i \cup S'_i\rangle \leq_d S_i \vee S'_i$. So $\prod_{i\in I}(S_i \vee S'_i) \in \mathbb{S}$. Clearly

$$\left(\prod_{i\in I} S_i\right) \vee \left(\prod_{i\in I} S'_i\right) = \prod_{i\in I}(S_i \vee S'_i)$$

and

$$\left\langle \left(\prod_{i\in I} S_i\right) \cup \left(\prod_{i\in I} S'_i\right) \right\rangle \leq_d \prod_{i\in I}(S_i \vee S'_i)$$

For (iv), suppose that $\forall i \in I[S_i \in \mathbb{S}_i]$ and $\forall i \in I[S'_i \in \mathbb{S}_i]$, X is dense in $\prod_{i\in I} S_i$, and Y is dense in $\prod_{i\in I} S'_i$. Then for each $i \in I$, $\{x_i : x \in X\}$ is dense in S_i and $\{y_i : y \in Y\}$ is dense in S'_i, so $\{x_i \cdot y_i : x \in X, y \in Y\}$ is dense in $\langle S_i \cup S'_i\rangle$. Hence $\{x \cdot y : x \in X, y \in Y\}$ is dense in $\langle \prod_{i\in I} S_i \cup \prod_{i\in I} S'_i\rangle$.

Clearly (v) holds. □

Theorem 30.71. *Any product of countably many standard Cohen algebras has a complete skeleton.*

Proof. By Lemmas 30.68–30.70. □

Lemma 30.72. *Suppose that B is a complete BA, $A \leq_d B$, and $\mathbb{S}$ is a complete skeleton for B. For each $S \in \mathbb{S}$ let $A_S = S \cap A$, and call S* **closed** *iff $A_S \leq_d S$. Then*

(i) If S is closed, then $S = (A_S)^{cmp}$.

(ii) If S and S' are closed, then $S \leq S'$ iff $A_S \leq A_{S'}$.

(iii) If S and S' are closed, then so is $S \vee S'$.

(iv) If $\langle S_i : i \in I\rangle$ is a chain under inclusion, each S_i is closed, $S \in \mathbb{S}$, and $\bigcup_{i\in I} S_i \leq_d S$, then S is closed.

(v) If S and S_0 are closed, then $\pi(A_{S\vee S_0}/A_S) \leq \pi(S_0)$.

(vi) For every $X \in [A]^{\leq\omega}$ there is a closed $S \in \mathbb{S}$ such that $X \subseteq A_S$ and $\pi(S) \leq \omega$.

Proof. (i): Since S is complete and A_S is dense in S, clearly $S = (A_S)^{cmp}$.

(ii):

$$S \leq S' \;\Rightarrow\; A_S \leq A_{S'} \;\Rightarrow\; (A_S)^{cmp} \leq (A_{S'})^{cmp} \;\Rightarrow\; S \leq S'.$$

(iii): Let $S'' = S \vee S'$. By Lemma 30.24 and (iii) in the definition of complete skeleton, $\langle A_S \cup A_{S'}\rangle \leq_d \langle S \cup S'\rangle \leq_d S''$. Also $\langle A_S \cup A_{S'}\rangle \leq A_{S''} \leq S''$, so $A_{S''} \leq_d S''$.

(iv): We have

$$\bigcup_{i\in I} A_{S_i} = A \cap \bigcup_{i\in I} S_i \leq A \cap S = A_S \leq S$$

and

$$\bigcup_{i\in I} A_{S_i} \leq_d \bigcup_{i\in I} S_i \leq_d S.$$

Hence $A_S \leq_d S$.

(v): Since $A_{S_0} \leq_d S_0$, there is a $D \leq_d A_{S_0}$ such that $|D| \leq \pi(S_0)$. Then

$$D \leq_d A_{S_0} \leq_d S_0 \leq_{reg} B \quad \text{and} \quad A_S \leq_d S \leq_{reg} B.$$

Hence by Lemma 30.36 we get $A_S(D) \leq_d \langle S_0 \cup S\rangle \leq_d S_0 \vee S$. Also $A_S(D) \leq A_{S\vee S_0} \leq S \vee S_0$. Hence $A_S(D) \leq_d A_{S\vee S_0}$.

(vi): We construct increasing chains $\langle S_n : n \in \omega\rangle$, each $S_i \in \mathbb{S}$ and $\langle A_n : n \in \omega\rangle$, each A_n countable and $\leq A$, such that $\forall s \in S_n^+ \exists a \in A_n^+[a \leq s]$.

By (ii) in the definition let $S_0 \in \mathbb{S}$ with $X \subseteq S_0$ and $\pi(S_0) \leq \omega$. Say Y is a countable subset of S_0 dense in S_0. Now $A \leq_d B$ and $\pi(S_0) \leq \omega$, so choose A_0 countable so that $A_0 \leq A$ and $\forall y \in Y^+ \exists a \in A_0^+$ such that $a \leq y$. If A_n and S_n have been defined with A_n countable and dense in S_n with $\pi(S_n) \leq \omega$, by (ii) in the definition let $S'_{n+1} \in \mathbb{S}$ be such that $A_n \leq S'_{n+1}$ and $\pi(S'_{n+1}) \leq \omega$, and then by (iii) in the definition get $S_{n+1} \in \mathbb{S}$ such that $S'_{n+1}, S_n \leq S_{n+1}$. By (iv) in the definition, $\pi(S_{n+1}) \leq \omega$. Let A'_{n+1} be countable and dense in S_{n+1}, and let $A_{n+1} = A_n \cup A'_{n+1}$.

By (v) in the definition, let $S \in \mathbb{S}$ be such that $\bigcup_{n\in\omega} S_n \leq_d S$. Then $\pi(S) \leq \omega$ since $\bigcup_{n\in\omega} A_n$ is dense in $\bigcup_{n\in\omega} S_n$, hence in S. Now we show that S is closed. Obviously $A_S \leq S$. For denseness, suppose that $b \in S^+$. Then there is an $n \in \omega$ and a $c \in S_n^+$ such that $c \leq b$. Then choose $a \in A_n^+$ such that $a \leq c$. Then $a \in A_n \leq A \cap S_{n+1} \leq A \cap S = A_S$. □

Theorem 30.73. *For any ccc BA A the following are equivalent:*

(A) A is a general Cohen algebra.

(B) A has a Cohen skeleton.

(C) A is the union of a continuous chain $\langle A_\alpha : \alpha < \rho\rangle$ where $\pi(A_0) \leq \omega$, for all α with $\alpha + 1 < \rho$, $A_\alpha \leq_{reg} A_{\alpha+1}$, and $\pi(A_{\alpha+1}/A_\alpha) \leq \omega$.

(D) As in (C), but in addition $A_0 = 2$ and $A_{\alpha+1}$ is a simple extension of A_α for all α with $\alpha + 1 < \rho$.

(E) As in (D), but in addition A_α is dense or relatively complete in $A_{\alpha+1}$ for all α with $\alpha + 1 < \rho$.

(F) A has a dense projective subalgebra.

Proof. (A)⇒(B): Let A be a general Cohen algebra satisfying ccc. By Theorem 30.71, A^{cmp} has a complete skeleton. As in Lemma 30.72 for each $S \in \mathbb{S}$ let $A_S = S \cap A$, and call S **closed** iff $A_S \leq_d S$. Let

$$\mathbb{T} = \{A_S : S \in \mathbb{S},\ S \text{ closed}\}.$$

We claim that $\mathbb{T}$ is a Cohen skeleton for A. For (i): $A_S \leq_d S \leq_{reg} A^{cmp}$. Since clearly $\leq_d$ implies $\leq_{reg}$, by Proposition 30.28 we get $A_S \leq_{reg} A^{cmp}$. Now $A_S \leq A \leq_{reg} A^{cmp}$, so $A_S \leq_{reg} A$. For (ii), by Lemma 30.72 choose $S \in \mathbb{A}$ closed such that $\pi(S) \leq \omega$. Now $A_S \leq_d S$, so $\pi(A_S) \leq \omega$. For (iii), suppose that $T \in \mathbb{T}$ and X is a countable subset of A. Say $T = A_S$ with $S \in \mathbb{S}$ closed. By Lemma 30.72(vi) there is a closed $S_0 \in \mathbb{S}$ such that $X \subseteq A_{S_0}$ and $\pi(S_0) \leq \omega$. Now $S' \stackrel{\text{def}}{=} S \vee S_0$ is closed by Lemma 30.72(iii), so $T' \stackrel{\text{def}}{=} A_{S\vee S_0} \in \mathbb{T}$. Now $T \cup X \subseteq A_S \cup A_{S_0} \subseteq A_{S'} = T'$. Now $\pi(T'/T) = \pi(A_{S\vee S_0}/A_S) \leq \pi(S_0) \leq \omega$. For (iv), suppose that $\mathbb{C} \subseteq \mathbb{T}$ is a chain. Say $\mathbb{C} = \{A_{S_i} : i \in I\}$ with each S_i closed. By Lemma 30.72(ii), $\{S_i : i \in I\}$ is a chain. Hence by (iv) in the definition, there is an $S \in \mathbb{S}$ such that $\bigcup_{i\in I} S_i \subseteq_d S$. By Lemma 30.72(iv), S is closed. Hence $A_S \in \mathbb{T}$. Finally,

$$\bigcup_{i\in I} A_{S_i} \leq_d \bigcup_{i\in I} S_i \leq_d S \quad \text{and} \quad \bigcup_{i\in I} A_{S_i} \leq A_S \leq S,$$

so $\bigcup_{i\in I} A_{S_i} \leq_d A_S$.

(B)$\Rightarrow$(C): See Lemma 30.64.

(C)$\Rightarrow$(D)$\Rightarrow$(E)$\Rightarrow$(C): Lemma 30.65.

(E)$\Rightarrow$(F): see Proposition 30.51.

(F)$\Rightarrow$(A): It suffices to show that if A is projective and ccc, then $\overline{A}^{cmp}$ is the product of countably many standard Cohen algebras. This follows from Theorems 30.56 and 30.69. $\square$

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