NOTES ON SET THEORY

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PREFACE

The edition of September 2022 has more material on linear orders.

The edition of January 2021 has added further infinite combinatorics to Chapter 24.

The edition of March 2020, differs from previous editions mainly in new material on cardinal arithmetic: theorem of Galvin and Hajnal.

Note that a large portion of the notes consists in giving details about Kunen 2011; there are references to the appropriate pages of that book.

Optional material in each section is indicated starting with

. . . .

LOGIC

1. Sentential logic

We go into the mathematical theory of the simplest logical notions: the meaning of "and", "or", "implies", "if and only if" and related notions. The basic idea here is to describe a formal language for these notions, and say precisely what it means for statements in this language to be true. The first step is to describe the language, without saying anything mathematical about meanings. We need very little background to carry out this development. ω is the set of all natural numbers $0, 1, 2, \ldots$. Let ω_+ be the set of all positive integers. For each positive integer m let $m' = \{0, \ldots, m-1\}$. A *finite sequence* is a function whose domain is m' for some positive integer m; the values of the function can be arbitary.

To keep the treatment strictly mathematical, we will define the basic "symbols" of the language to just be certain positive integers, as follows:

Negation symbol: the integer 1.
Implication symbol: the integer 2.
Sentential variables: all integers ≥ 3.

Let Expr be the collection of all finite sequences of positive integers; we think of these sequences as *expressions*. Thus an expression is a function mapping m' into ω_+, for some positive integer m. Such sequences are frequently indicated by $\langle \varphi_0, \ldots, \varphi_{m-1} \rangle$. The case $m = 1$ is important; here the notation is $\langle \varphi \rangle$.

The one-place function $\neg$ mapping Expr into Expr is defined by $\neg\varphi = \langle 1 \rangle ^\frown \varphi$ for any expression φ. Here in general $\varphi ^\frown \psi$ is the sequence φ followed by the sequence ψ.

The two-place function $\rightarrow$ mapping Expr$\times$Expr into Expr is defined by $\varphi \rightarrow \psi = \langle 2 \rangle ^\frown \varphi ^\frown \psi$ for any expressions φ, ψ. (For any sets A, B, $A \times B$ is the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$. So Expr $\times$ Expr is the set of all ordered pairs (φ, ψ) with φ, ψ expressions.)

For any natural number n, let $S_n = \langle n + 3 \rangle$.

Now we define the notion of a sentential formula—an expression which, suitably interpreted, makes sense. We do this definition by defining a *sentential formula construction*, which by definition is a sequence $\langle \varphi_0, \ldots, \varphi_{m-1} \rangle$ with the following property: for each $i < m$, one of the following holds:

$\varphi_i = S_j$ for some natural number j.

There is a $k < i$ such that $\varphi_i = \neg\varphi_k$.

There exist $k, l < i$ such that $\varphi_i = (\varphi_k \rightarrow \varphi_l)$.

Then a *sentential formula* is an expression which appears in some sentential formula construction.

The following proposition formulates the principle of *induction on sentential formulas.*

Proposition 1.1. *Suppose that M is a collection of sentential formulas, satisfying the following conditions.*

(i) S_i is in M, for every natural number i.

(ii) If φ is in M, then so is $\neg\varphi$.

(iii) If φ and ψ are in M, then so is $\varphi \to \psi$.

Then M consists of all sentential formulas.

Proof. Suppose that θ is a sentential formula; we want to show that $\theta \in M$. Let $\langle \tau_0, \ldots, \tau_m \rangle$ be a sentential formula construction with $\tau_t = \theta$, where $0 \leq t \leq m$. We prove by complete induction on i that for every $i \leq m$, $\tau_i \in M$. Hence by applying this to $i = t$ we get $\theta \in M$.

So assume that for every $j < i$, the sentential formula τ_j is in M.

Case 1. τ_i is S_s for some s. By (i), $\tau_i \in M$.

Case 2. τ_i is $\neg\tau_j$ for some $j < i$. By the inductive hypothesis, $\tau_j \in M$, so $\tau_i \in M$ by (ii).

Case 3. τ_i is $\tau_j \to \tau_k$ for some $j, k < i$. By the inductive hypothesis, $\tau_j \in M$ and $\tau_k \in M$, so $\tau_i \in M$ by (iii). $\square$

Proposition 1.2. *(i) Any sentential formula is a nonempty sequence.*

(ii) For any sentential formula φ, exactly one of the following conditions holds:

(a) φ is S_i for some $i \in \omega$.

(b) φ begins with 1, and there is a sentential formula ψ such that $\varphi = \neg\psi$.

(c) φ begins with 2, and there are sentential formulas ψ, χ such that $\varphi = \psi \to \chi$.

(iii) No proper initial segment of a sentential formula is a sentential formula.

(iv) If φ and ψ are sentential formulas and $\neg\varphi = \neg\psi$, then $\varphi = \psi$.

(v) If $\varphi, \psi, \varphi', \psi'$ are sentential formulas and $\varphi \to \psi = \varphi' \to \psi'$, then $\varphi = \varphi'$ and $\psi = \psi'$.

Proof. (i): Clearly every entry in a sentential formula construction is nonempty, so (i) holds.

(ii): First we prove by induction that one of (a)–(c) holds. This is true of sentential variables—in this case, (a) holds. If it is true of a sentential formula φ, it is obviously true of $\neg\varphi$; so (b) holds. Similarly for $\to$, where (c) holds.

Second, the first entry of a formula differs in cases (a),(b),(c), so exactly one of them holds.

(iii): We prove this by complete induction on the length of the formula. So, suppose that φ is a sentential formula and we know for any formula ψ shorter than φ that no proper initial segment of ψ is a formula. We consider cases according to (ii).

Case 1. φ is S_i for some i. Only the empty sequence is a proper initial segment of φ in this case, and the empty sequence is not a sentential formula, by (i).

Case 2. φ is $\neg\psi$ for some formula ψ. If χ is a proper initial segment of φ and it is a formula, then χ begins with 1 and so by (ii), χ has the form $\neg\theta$ for some formula θ. But then θ is a proper initial segment of ψ and ψ is shorter than φ, so the inductive hypothesis is contradicted.

Case 3. φ is $\psi \to \chi$ for some formulas ψ and χ. That is, φ is $\langle 2\rangle^\frown\psi^\frown\chi$. If θ is a proper initial segment of φ which is a formula, then by (ii), θ has the form $\langle 2\rangle^\frown\xi^\frown\eta$ for some formulas ξ, η. Now $\psi^\frown\chi = \xi^\frown\eta$, so ψ is an initial segment of ξ or ξ is an initial segment of ψ. Since ψ and ξ are both shorter than φ, it follows from the inductive hypothesis that $\psi = \xi$. Hence $\chi = \eta$, and $\varphi = \theta$, contradiction.

(iv) is rather obvious; if $\neg\varphi = \neg\psi$, then φ and ψ are both the sequence obtained by deleting the first entry.

(v): Assume the hypothesis. Then $\varphi \to \psi$ is the sequence $\langle 2\rangle^\frown\varphi^\frown\psi$, and $\varphi' \to \psi'$ is the sequence $\langle 2\rangle^\frown\varphi'^\frown\psi'$. Since these are equal, φ and φ' start at the same place in the sequence. By (iii) it follows that $\varphi = \varphi'$. Deleting the initial segment $\langle 2\rangle^\frown\varphi$ from the sequence, we then get $\psi = \psi'$. □

Parts (iv) and (v) of this proposition enable us to define values of sentential formulas, which supplies a mathematical meaning for the truth of formulas. A *sentential assignment* is a function mapping the set $\{0, 1, \ldots\}$ of natural numbers into the set $\{0, 1\}$. Intuitively we think of 0 as "false" and 1 as "true". The definition of values of sentential formulas is a special case of definition by recursion:

Proposition 1.3. *For any sentential assignment f there is a function F mapping the set of sentential formulas into $\{0, 1\}$ such that the following conditions hold:*

(i) $F(S_n) = f(n)$ for every natural number n.

(ii) $F(\neg\varphi) = 1 - F(\varphi)$ for any sentential formula φ.

(iii) $F(\varphi \to \psi) = 0$ iff $F(\varphi) = 1$ and $F(\psi) = 0$.

Proof. An *f-sequence* is a finite sequence $\langle(\varphi_0, \varepsilon_0), \ldots, (\varphi_{m-1}, \varepsilon_{m-1})\rangle$ such that each ε_i is 0 or 1, and such that for each $i < m$ one of the following holds:

(1) φ_i is S_n for some $n \in \omega$, and $\varepsilon_i = f(n)$.

(2) There is a $k < i$ such that $\varphi_i = \neg\varphi_k$ and $\varepsilon_i = 1 - \varepsilon_k$.

(3) There are $k, l < i$ such that $\varphi_i = \varphi_k \to \varphi_l$, and $\varepsilon_i = 0$ iff $\varepsilon_k = 1$ and $\varepsilon_l = 0$.

Now we claim:

(4) For any sentential formula ψ and any f-sequences $\langle(\varphi_0.\varepsilon_0), \ldots, (\varphi_{m-1}, \varepsilon_{m-1})\rangle$ and $\langle(\varphi'_0.\varepsilon'_0), \ldots, (\varphi'_{n-1}, \varepsilon'_{n-1})\rangle$ such that $\varphi_{m-1} = \varphi'_{n-1} = \psi$ we have $\varepsilon_{m-1} = \varepsilon'_{n-1}$.

We prove (4) by induction on ψ, thus using Proposition 1.1. If $\psi = S_n$, then $\varepsilon_{m-1} = f(n) = \varepsilon'_{n-1}$. Assume that the condition holds for ψ, and consider $\neg\psi$. There is a $k < m - 1$ such that $\neg\psi = \varphi_{m-1} = \neg\varphi_k$. By Proposition 1.2(iv) we have $\varphi_k = \psi$. Similarly, there is an $l < n - 1$ such that $\neg\psi = \varphi'_{n-1} = \neg\varphi'_l$ and so $\varphi'_l = \psi$. Applying the inductive hypothesis to ψ and the sequences $\langle\varphi_0, \ldots, \varphi_k\rangle$ and $\langle\varphi'_0, \ldots, \varphi'_l\rangle$ we get $\varepsilon_k = \varepsilon'_l$. Hence $\varepsilon_{m-1} = 1 - \varepsilon_k = 1 - \varepsilon'_l = \varepsilon'_{n-1}$.

Now suppose that the condition holds for ψ and χ, and consider $\psi \to \chi$. There are $k, l < m - 1$ such that $(\psi \to \chi) = (\varphi_k \to \varphi_l)$. By Proposition 1.2(v) we have $\varphi_k = \psi$ and $\varphi_l = \chi$. Similarly there are $s, t < n - 1$ such that $(\psi \to \chi) = (\varphi'_s \to \varphi'_t)$. By Proposition 1.2(v) we have $\varphi'_s = \psi$ and $\varphi'_t = \chi$. Applying the inductive hypotheis to ψ

and the sequences $\langle\varphi_0,\ldots,\varphi_k\rangle$ and $\langle\varphi'_0,\ldots,\varphi'_s\rangle$ we get $\varepsilon_k=\varepsilon'_s$. Similarly, we get $\varepsilon_l=\varepsilon'_t$. Hence

$$\begin{aligned}\varepsilon_{m-1}=0 \quad &\text{iff} \quad \varepsilon_k=1 \text{ and } \varepsilon_l=0\\ &\text{iff} \quad \varepsilon'_s=1 \text{ and } \varepsilon'_t=0\\ &\text{iff} \quad \varepsilon'_{n-1}=0.\end{aligned}$$

This finishes the proof of (4).

(5) If $\langle\varphi_0,\ldots,\varphi_{m-1}\rangle$ is a sentential formula construction, then there is an f-sequence of the form $(\varphi_0,\varepsilon_0),\ldots,(\varphi_{m-1},\varepsilon_{m-1})\rangle$.

We prove this by induction on m. First suppose that $m=1$. Then φ_0 must equal S_n for some n, and $\langle(\varphi_0,f(n))\rangle$ is as desired. Now suppose that $m>1$ and the statement is true for $m-1$. So let $\theta\overset{\text{def}}{=}\langle(\varphi_0,\varepsilon_0),\ldots.(\varphi_{m-2},\varepsilon_{m-2})\rangle$ be an f-sequence.

Case 1. $\varphi_{m-1}=S_p$. Then $\theta{}^\frown\langle(\varphi_{m-1},f(p))\rangle$ is as desired.

Case 2. There is a $k<m$ such that $\varphi_{m-1}=\neg\varphi_k$. Then $\theta{}^\frown\langle(\varphi_{m-1},1-\varepsilon_k)\rangle$ is as desired.

Case 3. There are $k,l<m$ such that $\varphi_{m-1}=\varphi_k\to\varphi_l$. Then $\theta{}^\frown\langle(\varphi_{m-1},\delta)\rangle$ is as desired, where $\delta=0$ iff $\varepsilon_k=1$ and $\varepsilon_l=0$.

Thus (5) holds. Now we can define the function F required in the Proposition. Let ψ be any sentential formula. Let $\langle\varphi_0,\ldots,\varphi_{m-1}\rangle$ be a sentential formula construction such that $\varphi_{m-1}=\psi$. By (5), let $\langle(\varphi_0,\varepsilon_0),\ldots,(\varphi_{m-1},\varepsilon_{m-1})\rangle$ be an f-sequence. We define $F(\psi)=\varepsilon_{m-1}$. This is unambiguous by (4).

Case 1. $\psi=S_n$ for some n. Then by the definition of f-sequence we have $F(\psi)=f(n)$.

Case 2. There is a $k<m$ such that $\psi=\varphi_{m-1}=\neg\varphi_k$. Then $\langle(\varphi_0,\varepsilon_0),\ldots,(\varphi_k,\varepsilon_k)\rangle$ is an f-sequence, so $F(\varphi_k)=\varepsilon_k$. So

$$F(\psi)=F(\varphi_{m-1})=\varepsilon_{m-1}=1-\varepsilon_k=1-F(\varphi_k).$$

Case 3. There are $k,l<m$ such that $\psi=\varphi_{m-1}=\varphi_k\to\varphi_l$. Then $\langle(\varphi_0,\varepsilon_0),\ldots,(\varphi_k,\varepsilon_k)\rangle$ is an f-sequence, so $F(\varphi_k)=\varepsilon_k$; and $\langle(\varphi_0,\varepsilon_0),\ldots,(\varphi_l,\varepsilon_k)\rangle$ is an f-sequence, so $F(\varphi_l)=\varepsilon_l$. So

$$\begin{aligned}F(\psi)=0 \quad \text{iff} \quad F(\varphi_{m-1})=0 \quad \text{iff} \quad e_{m-1}=0 \quad \text{iff}\\ \varepsilon_k=1 \text{ and } \varepsilon_l=0 \quad \text{iff} \quad F(\varphi_k)=1 \text{ and } F(\varphi_l)=0.\end{aligned}$$

□

With f a sentential assignment, and with F as in this proposition, for any sentential formula φ we let $\varphi[f]=F(\varphi)$. Thus:

$$\begin{aligned}S_i[f]&=f(i);\\ (\neg\varphi)[f]&=1-\varphi[f];\\ (\varphi\to\psi)[f]&=\begin{cases}0 & \text{if } \varphi[f]=1 \text{ and } \psi[f]=0,\\ 1 & \text{otherwise.}\end{cases}\end{aligned}$$

The definition can be recalled by using *truth tables*:

φ	$\neg\varphi$
1	0
0	1

φ	ψ	$\varphi \to \psi$
1	1	1
1	0	0
0	1	1
0	0	1

Other logical notions can be defined in terms of $\neg$ and $\to$. We define

$\varphi \wedge \psi = \neg(\varphi \to \neg\psi)$.
$\varphi \vee \psi = \neg\varphi \to \psi$.
$\varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi)$.

Working out the truth tables for these new notions shows that they mean approximately what you would expect:

φ	ψ	$\neg\psi$	$\varphi \to \neg\psi$	$\varphi \wedge \psi$	$\neg\varphi$	$\varphi \vee \psi$	$\varphi \to \psi$	$\psi \to \varphi$	$\varphi \leftrightarrow \psi$
1	1	0	0	1	0	1	1	1	1
1	0	1	1	0	0	1	0	1	0
0	1	0	1	0	1	1	1	0	0
0	0	1	1	0	1	0	1	1	1

(Note that $\vee$ corresponds to *non-exclusive or*: φ or ψ, or both.)

The following simple proposition is frequently useful.

Proposition 1.4. *If f and g map $\{0, 1, \ldots\}$ into $\{0, 1\}$ and $f(m) = g(m)$ for every m such that S_m occurs in φ, then $\varphi[f] = \varphi[g]$.*

Proof. Induction on φ. If φ is S_i for some i, then the hypothesis says that $f(i) = g(i)$; hence $S_i[f] = f(i) = g(i) = S_i[g]$. Assume that it is true for φ. Now S_m occurs in φ iff it occurs in $\neg\varphi$. Hence if we assume that $f(m) = g(m)$ for every m such that S_m occurs in $\neg\varphi$, then also $f(m) = g(m)$ for every m such that S_m occurs in φ, so $(\neg\varphi)[f] = 1 - \varphi[f] = 1 - \varphi[g] = (\neg\varphi)[g]$. Assume that it is true for both φ and ψ, and $f(m) = g(m)$ for every m such that S_m occurs in $\varphi \to \psi$. Now if S_m occurs in φ, then it also occurs in $\varphi \to \psi$, and hence $f(m) = g(m)$. Similarly for ψ. It follows that

$$(\varphi \to \psi)[f] = 0 \text{ iff } (\varphi[f] = 1 \text{ and } \psi[f] = 0) \text{ iff } (\varphi[g] = 1 \text{ and } \psi[g] = 0) \text{ iff } (\varphi \to \psi)[g] = 0.$$

□

This proposition justifies writing $\varphi[f]$ for a finite sequence f, provided that f is long enough so that m is in its domain for every m for which S_m occurs in φ.

A sentential formula φ is a *tautology* iff it is true under every assignment, i.e., $\varphi[f] = 1$ for every assignment f.

Here is a list of common tautologies:

(T1) $\varphi \to \varphi$.
(T2) $\varphi \leftrightarrow \neg\neg\varphi$.
(T3) $(\varphi \to \neg\varphi) \to \neg\varphi$.
(T4) $(\varphi \to \neg\psi) \to (\psi \to \neg\varphi)$.
(T5) $\varphi \to (\neg\varphi \to \psi)$.
(T6) $(\varphi \to \psi) \to [(\psi \to \chi) \to (\varphi \to \chi)]$.
(T7) $[\varphi \to (\psi \to \chi)] \to [(\varphi \to \psi) \to (\varphi \to \chi)]$.
(T8) $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$.
(T9) $(\varphi \wedge \psi) \to \varphi$.
(T10) $(\varphi \wedge \psi) \to \psi$.
(T11) $\varphi \to [\psi \to (\varphi \wedge \psi)]$.
(T12) $\varphi \to (\varphi \vee \psi)$.
(T13) $\psi \to (\varphi \vee \psi)$.
(T14) $(\varphi \to \chi) \to [(\psi \to \chi) \to ((\varphi \vee \psi) \to \chi)]$.
(T15) $\neg(\varphi \wedge \psi) \leftrightarrow (\neg\varphi \vee \neg\psi)$.
(T16) $\neg(\varphi \vee \psi) \leftrightarrow (\neg\varphi \wedge \neg\psi)$.
(T17) $[\varphi \vee (\psi \vee \chi)] \leftrightarrow [(\varphi \vee \psi) \vee \chi]$.
(T18) $[\varphi \wedge (\psi \wedge \chi)] \leftrightarrow [(\varphi \wedge \psi) \wedge \chi]$.
(T19) $[\varphi \wedge (\psi \vee \chi)] \leftrightarrow [(\varphi \wedge \psi) \vee (\varphi \wedge \chi)]$.
(T20) $[\varphi \vee (\psi \wedge \chi)] \leftrightarrow [(\varphi \vee \psi) \wedge (\varphi \vee \chi)]$.
(T21) $(\varphi \to \psi) \leftrightarrow (\neg\varphi \vee \psi)$.
(T22) $\varphi \wedge \psi \leftrightarrow \neg(\neg\varphi \vee \neg\psi)$.
(T23) $\varphi \vee \psi \leftrightarrow \neg(\neg\varphi \wedge \neg\psi)$.

Now we describe a proof system for sentential logic. Formulas of the following form are *sentential axioms*; φ, ψ, χ are arbitrary sentential formulas.

(1) $\varphi \to (\psi \to \varphi)$.

(2) $[\varphi \to (\psi \to \chi)] \to [(\varphi \to \psi) \to (\varphi \to \chi)]$.

(3) $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$.

Proposition 1.5. *Every sentential axiom is a tautology.*

Proof. For (1):

φ	ψ	$\psi \to \varphi$	$\varphi \to (\psi \to \varphi)$
1	1	1	1
1	0	1	1
0	1	0	1
0	0	1	1

For (2): Let ρ denote this formula:

φ	ψ	χ	$\psi \to \chi$	$\varphi \to (\psi \to \chi)$	$\varphi \to \psi$	$\varphi \to \chi$	$(\varphi \to \psi) \to (\varphi \to \chi)$	ρ
1	1	1	1	1	1	1	1	1
1	1	0	0	0	1	0	0	1
1	0	1	1	1	0	1	1	1
1	0	0	1	1	0	0	1	1
0	1	1	1	1	1	1	1	1
0	1	0	0	1	1	1	1	1
0	0	1	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1

For (3):

φ	ψ	$\neg\varphi$	$\neg\psi$	$\neg\varphi \to \neg\psi$	$\psi \to \varphi$	(3)
1	1	0	0	1	1	1
1	0	0	1	1	1	1
0	1	1	0	0	0	1
0	0	1	1	1	1	1

□

If Γ is a collection of sentential formulas, then a Γ-*proof* is a finite sequence $\langle \psi_0, \ldots, \psi_m \rangle$ such that for each $i \leq m$ one of the following conditions holds:

(a) ψ_i is a sentential axiom.

(b) $\psi_i \in \Gamma$.

(c) There exist $j, k < i$ such that ψ_k is $\psi_j \to \psi_i$. (Rule of *modus ponens*, abbreviated MP).

We write $\Gamma \vdash \varphi$ if there is a Γ-proof with last entry φ. We also write $\vdash \varphi$ in place of $\emptyset \vdash \varphi$.

Proposition 1.6. *(i) If $\Gamma \vdash \varphi$, f is a sentential assignment, and $\psi[f] = 1$ for all $\psi \in \Gamma$, then $\varphi[f] = 1$.*

(ii) If $\vdash \varphi$, then φ is a tautology.

Proof. For (i), let $\langle \psi_0, \ldots, \psi_m \rangle$ be a Γ-proof. Suppose that f is a sentential assignment and $\chi[f] = 1$ for all $\chi \in \Gamma$. We show by complete induction that $\psi_i[f] = 1$ for all $i \leq m$. Suppose that this is true for all $j < i$.

Case 1. ψ_i is a sentential axiom. Then $\psi_i[f] = 1$ by Proposition 1.5.

Case 2. $\psi_i \in \Gamma$. Then $\psi_i[f] = 1$ by assumption.

Case 3. There exist $j, k < i$ such that ψ_k is $\psi_j \to \psi_i$. By the inductive assumption, $\psi_k[f] = \psi_j[f] = 1$. Hence $\psi_i[f] = 1$.

(ii) clearly follows from (i), □

Now we are going to show that, conversely, if φ is a tautology then $\vdash \varphi$. This is a kind of completeness theorem, and the proof is a highly simplified version of the proof of the completeness theorem for first-order logic which will be given later.

Lemma 1.7. $\vdash \varphi \to \varphi$.

Proof.

(a)	$[\varphi \to [(\varphi \to \varphi) \to \varphi]] \to [[\varphi \to (\varphi \to \varphi)] \to (\varphi \to \varphi)]$	(2)
(b)	$\varphi \to [(\varphi \to \varphi) \to \varphi]$	(1)
(c)	$[\varphi \to (\varphi \to \varphi)] \to (\varphi \to \varphi)$	(a), (b), MP
(d)	$\varphi \to (\varphi \to \varphi)$	(1)
(e)	$\varphi \to \varphi$	(c), (d), MP

□

Theorem 1.8. (The deduction theorem) *If* $\Gamma \cup \{\varphi\} \vdash \psi$, *then* $\Gamma \vdash \varphi \to \psi$.

Proof. Let $\langle \chi_0, \ldots, \chi_m \rangle$ be a $(\Gamma \cup \{\varphi\})$-proof with last entry ψ. We replace each χ_i by several formulas so that the result is a Γ-proof with last entry $\varphi \to \psi$.

If χ_i is a logical axiom or a member of Γ, we replace it by the two formulas $\chi_i \to (\varphi \to \chi_i)$, $\varphi \to \chi_i$.

If χ_i is φ, we replace it by the five formulas in the proof of Lemma 1.7; the last entry is $\varphi \to \varphi$.

If χ_i is obtained from χ_j and χ_k by modus ponens, so that χ_k is $\chi_j \to \chi_i$, we replace χ_i by the formulas

$$[\varphi \to (\chi_j \to \chi_i)] \to [(\varphi \to \chi_j) \to (\varphi \to \chi_i)]$$
$$(\varphi \to \chi_j) \to (\varphi \to \chi_i)$$
$$\varphi \to \chi_i$$

Clearly this is as desired. □

Lemma 1.9. $\vdash \psi \to (\neg\psi \to \varphi)$.

Proof. By axiom (1) we have $\{\psi, \neg\psi\} \vdash \neg\varphi \to \neg\psi$. Hence axiom (3) gives $\{\psi, \neg\psi\} \vdash \psi \to \varphi$, and hence $\{\psi, \neg\psi\} \vdash \varphi$. Now two applications of Theorem 1.8 give the desired result. □

Lemma 1.10. $\vdash (\varphi \to \psi) \to [(\psi \to \chi) \to (\varphi \to \chi)]$.

Proof. Clearly $\{\varphi \to \psi, \psi \to \chi, \varphi\} \vdash \chi$, so three applications of Theorem 1.8 give the desired result. □

Lemma 1.11. $\vdash (\neg\varphi \to \varphi) \to \varphi$.

Proof. Clearly $\{\neg\varphi \to \varphi, \neg\varphi\} \vdash \varphi$ and also $\{\neg\varphi \to \varphi, \neg\varphi\} \vdash \neg\varphi$, so by Lemma 1.9, $\{(\neg\varphi \to \varphi, \neg\varphi\} \vdash \neg(\varphi \to \varphi)$. Then Theorem 1.8 gives $\{\neg\varphi \to \varphi\} \vdash \neg\varphi \to \neg(\varphi \to \varphi)$, and so using axiom (3), $\{\neg\varphi \to \varphi\} \vdash (\varphi \to \varphi) \to \varphi$. Hence by Lemma 1.7, $\{\neg\varphi \to \varphi\} \vdash \varphi$, and so Theorem 1.8 gives the desired result. $\square$

Lemma 1.12. $\vdash (\varphi \to \psi) \to [(\neg\varphi \to \psi) \to \psi]$.

Proof.

$$
\begin{array}{l}
\{\varphi \to \psi, \neg\varphi \to \psi, \neg\psi\} \vdash \neg\varphi \to \neg\psi \quad \text{using axiom (1)} \\
\{\varphi \to \psi, \neg\varphi \to \psi, \neg\psi\} \vdash \psi \to \varphi \quad \text{using axiom (3)} \\
\{\varphi \to \psi, \neg\varphi \to \psi, \neg\psi\} \vdash \neg\varphi \to \varphi \quad \text{using Lemma 1.10} \\
\{\varphi \to \psi, \neg\varphi \to \psi, \neg\psi\} \vdash \varphi \quad \text{by Lemma 1.11} \\
\{\varphi \to \psi, \neg\varphi \to \psi, \neg\psi\} \vdash \psi \\
\qquad \{\varphi \to \psi, \neg\varphi \to \psi\} \vdash \neg\psi \to \psi \quad \text{by Theorem 1.8} \\
\qquad \{\varphi \to \psi, \neg\varphi \to \psi\} \vdash \psi \quad \text{by Lemma 1.11}
\end{array}
$$

Now two applications of Theorem 1.8 give the desired result. $\square$

Theorem 1.13. *There is a sequence $\langle \varphi_0, \varphi_1, \ldots \rangle$ containing all sentential formulas.*

Proof. One can obtain such a sequence by the following procedure.

(1) Start with S_0.

(2) List all sentential formulas of length at most two which involve only S_0 or S_1; they are S_0, S_1, $\neg S_0$, and $\neg S_1$.

(3) List all sentential formulas of length at most three which involve only S_0, S_1, or S_2; they are S_0, S_1, S_2, $\neg S_0$, $\neg S_1$, $\neg S_2$, $\neg\neg S_0$, $\neg\neg S_1$, $\neg\neg S_2$, $S_0 \to S_0$, $S_0 \to S_1$, $S_0 \to S_2$, $S_1 \to S_0$, $S_1 \to S_1$, $S_1 \to S_2$, $S_2 \to S_0$, $S_2 \to S_1$, $S_2 \to S_2$.

(4) Etc. $\square$

Theorem 1.14. *If not($\Gamma \vdash \varphi$), then there is a sentential assignment f such that $\psi[f] = 1$ for all $\psi \in \Gamma$, while $\varphi[f] = 0$.*

Proof. Let $\langle \chi_0, \chi_1, \ldots \rangle$ list all the sentential formulas. We now define $\Delta_0, \Delta_1, \ldots$ by recursion. Let $\Delta_0 = \Gamma$. Suppose that Δ_i has been defined. If not($\Delta_i \cup \{\chi_i\} \vdash \varphi$) then we set $\Delta_{i+1} = \Delta_i \cup \{\chi_i\}$. Otherwise we set $\Delta_{i+1} = \Delta_i$.

Here is a detailed proof that Δ exists. Let $M = \{\Omega : \Omega$ is a function with domain m' for some positive integer m, $\Omega_1 = \Gamma$, and for every positive integer i with $i + 1 \leq m$ we have

$$\Omega_{i+1} = \begin{cases} \Omega_i \cup \{\chi_i\} & \text{if not}(\Omega_i \cup \{\chi_i\} \vdash \varphi), \\ \Omega_i & \text{otherwise.}\} \end{cases}$$

(1) If $\Omega, \Omega' \in M$ with domains m', n' respectively, with $m \leq n$, then $\forall i \leq m[\Omega_i = \Omega'_i]$.

This is easily proved by induction on i.

(2) For every positive integer m there is a $\Omega \in M$ with domain m'.

Again this is easily proved by induction on m.

Now we define $\Delta_i = \Omega_i$, where $\Omega \in M$ and $i < \mathrm{dmn}(\Omega)$. This is justified by (1) and (2).

Now it is easily verified that the defining conditions for Δ hold.

Let $\Theta = \bigcup_{i \in \omega} \Delta_i$. By induction we have not$(\Delta_i \vdash \varphi)$ for each $i \in \omega$. In fact, we have $\Delta_0 = \Gamma$, so not$(\Delta_0 \vdash \varphi)$ by assumption. If not$(\Delta_i \vdash \varphi)$, then not$(\Delta_{i+1} \vdash \varphi)$ by construction.

Hence also not$(\Theta \vdash \varphi)$, since $\Theta \vdash \varphi$ means that there is a Θ-proof with last entry φ, and any Θ-proof involves only finitely many formulas χ_i, and they all appear in some Δ_j, giving $\Delta_j \vdash \varphi$, contradiction.

$(*)$ For any formula χ_i, either $\chi_i \in \Theta$ or $\neg\chi_i \in \Theta$.

In fact, suppose that $\chi_i \notin \Theta$ and $\neg\chi_i \notin \Theta$. Say $\neg\chi_i = \chi_j$. Then by construction, $\Delta_i \cup \{\chi_i\} \vdash \varphi$ and $\Delta_j \cup \{\neg\chi_i\} \vdash \varphi$. So $\Theta \cup \{\chi_i\} \vdash \varphi$ and $\Theta \cup \{\neg\chi_i\} \vdash \varphi$. Hence by Theorem 1.8, $\Theta \vdash \chi_i \to \varphi$ and $\Theta \vdash \neg\chi_i \to \varphi$. So by Lemma 1.12 we get $\Theta \vdash \varphi$, contradiction.

$(**)$ If $\Theta \vdash \psi$, then $\psi \in \Theta$.

In fact, clearly not$(\Theta \cup \{\psi\} \vdash \varphi)$ by Theorem 1.8, so $(**)$ follows.

Now let f be the sentential assignment such that $f(i) = 1$ iff $S_i \in \Theta$. Now we claim

$(***)$ For every sentential formula ψ, $\psi[f] = 1$ iff $\psi \in \Theta$.

We prove this by induction on ψ. It is true for $\psi = S_i$ by definition. Now suppose that it holds for ψ. Suppose that $(\neg\psi)[f] = 1$. Thus $\psi[f] = 0$, so by the inductive assumption, $\psi \notin \Theta$, and hence by $(*)$, $\neg\psi \in \Theta$. Conversely, suppose that $\neg\psi \in \Theta$. If $(\neg\psi)[f] = 0$, then $\psi[f] = 1$, hence $\psi \in \Theta$ by the inductive hypothesis. Hence by Lemma 1.9, $\Theta \vdash \varphi$, contradiction. So $(\neg\psi)[f] = 1$.

Next suppose that $(***)$ holds for ψ and χ; we show that it holds for $\psi \to \chi$. Suppose that $(\psi \to \chi)[f] = 1$. If $\chi[f] = 1$, then $\chi \in \Theta$ by the inductive hypothesis. By axiom (1), $\Theta \vdash \psi \to \chi$. Hence by $(**)$, $(\psi \to \chi) \in \Theta$. Suppose that $\chi[f] = 0$. Then $\psi[f] = 0$ also, since $(\psi \to \chi)[f] = 1$. By the inductive hypothesis and $(*)$ we have $\neg\psi \in \Theta$. Hence $\Theta \vdash \neg\chi \to \neg\psi$ by axiom (1), so $\Theta \vdash \psi \to \chi$ by axiom (3). So $(\psi \to \chi) \in \Theta$ by $(**)$.

Conversely, suppose that $(\psi \to \chi) \in \Theta$. Working for a contradiction, suppose that $(\psi \to \chi)[f] = 0$. Thus $\psi[f] = 1$ and $\chi[f] = 0$. So $\psi \in \Theta$ and $\neg\chi \in \Theta$ by the inductive hypothesis and $(*)$. Since $(\psi \to \chi) \in \Theta$ and $\psi \in \Theta$, we get $\Theta \vdash \chi$. Since also $\neg\chi \in \Theta$, we get $\Theta \vdash \varphi$ by Lemma 1.9, contradiction.

This finishes the proof of $(***)$.

Since $\Gamma \subseteq \Theta$, $(***)$ implies that $\psi[f] = 1$ for all $\psi \in \Gamma$. Also $\varphi[f] = 0$ since $\varphi \notin \Theta$. □

Corollary 1.15. *If $\varphi[f] = 1$ whenever $\psi[f] = 1$ for all $\psi \in \Gamma$, then $\Gamma \vdash \varphi$.*

Proof. This is the contrapositive of Theorem 1.14. □

Theorem 1.16. $\vdash \varphi$ *iff* φ *is a tautology.*

Proof. $\Rightarrow$ is given by Proposition 1.6(ii). $\Leftarrow$ follows from Corollary 1.15 by taking $\Gamma = \emptyset$. □

. . . .

Proposition 1.17.

$$S_0 \to \neg S_1 = \langle 2, 3, 1, 4\rangle$$

and

$$(S_0 \to S_1) \to (\neg S_1 \to \neg S_0) = \langle 2, 2, 3, 4, 2, 1, 4, 1, 3\rangle.$$

Proof.

$$\begin{aligned} S_0 \to \neg S_1 &= \langle 2\rangle^\frown S_0^\frown \neg S_1 \\ &= \langle 2\rangle^\frown \langle 3\rangle^\frown \langle 1\rangle^\frown S_1 \\ &= \langle 2, 3, 1, 4\rangle; \end{aligned}$$

$$\begin{aligned} (S_0 \to S_1) \to (\neg S_1 \to \neg S_0) &= \langle 2\rangle^\frown (S_0 \to S_1)^\frown (\neg S_1 \to \neg S_0) \\ &= \langle 2\rangle^\frown \langle 2\rangle^\frown S_0^\frown S_1^\frown \langle 2\rangle^\frown \neg S_1^\frown \neg S_0 \\ &= \langle 2, 2, 3, 4, 2\rangle^\frown \langle 1\rangle^\frown S_1^\frown \langle 1\rangle^\frown S_0 \\ &= \langle 2, 2, 3, 4, 2, 1, 4, 1, 3\rangle. \end{aligned}$$ □

Proposition 1.18. *There is a sentential formula of each positive integer length.*

Proof. If m is a positive integer, then

$$\langle \overbrace{1, 1, \ldots, 1}^{m-1 \text{ times}}, S_0\rangle$$

is a formula of length m, it is

$$\overbrace{\neg\neg \cdots \neg}^{m-1 \text{ times}} S_0.$$ □

Proposition 1.19. m *is the length of a sentential formula not involving* $\neg$ *iff* m *is odd.*

Proof. $\Rightarrow$: We prove by induction on φ that if φ is a sentential formula not involving $\neg$, then the length of φ is odd. This is true of sentential variables, which have length 1. Suppose that it is true of φ and ψ, which have length $2m+1$ and $2n+1$ respectively. Then $\varphi \to \psi$, which is $\langle 1\rangle^\frown \varphi^\frown \psi$, has length $1 + 2m + 1 + 2n + 1 = 2(m + n + 1) + 1$, which is again odd. This finishes the inductive proof.

$\Leftarrow$. We construct formulas without $\neg$ with length any odd integer by induction. $\langle S_0\rangle$ is a formula of length 1. If φ has been constructed of length $2m+1$, then $S_0 \to \varphi$, which is $\langle 1, S_0\rangle^\frown\varphi$, has length $2m+3$. This finishes the inductive construction. $\square$

Proposition 1.20. *The truth table for a sentential formula involving n basic formulas has 2^n rows.*

Proof. We prove this by induction on n. For $n = 1$, there are two rows. Assume that for n basic formulas there are 2^n rows. Given $n+1$ basic formulas, let φ be one of them. For the others, by the inductive hypothesis there are 2^n rows. For each such row there are two possibilities, 0 or 1, for φ. So for the $n+1$ basic formulas there are $2^n \cdot 2 = 2^{n+1}$ rows. $\square$

Proposition 1.21. *The formula*

$$(\varphi \to \psi) \leftrightarrow (\neg\varphi \vee \psi)$$

is a tautology.

Proof.

φ	ψ	$\varphi \to \psi$	$\neg\varphi$	$\neg\varphi \vee \psi$	$(\varphi \to \psi) \leftrightarrow (\neg\varphi \vee \psi)$
1	1	1	0	1	1
1	0	0	0	0	1
0	1	1	1	1	1
0	0	1	1	1	1

$\square$

Proposition 1.22. *The formula*

$$[\varphi \vee (\psi \wedge \chi)] \leftrightarrow [(\varphi \vee \psi) \wedge (\varphi \vee \chi)]$$

is a tautology.

Proof. Let θ be the indicated formula.

φ	ψ	χ	$\varphi \vee \psi$	$\varphi \vee \chi$	$(\varphi \vee \psi) \wedge (\varphi \vee \chi)$	$\psi \wedge \chi$	$\varphi \vee (\psi \wedge \chi)$	θ
1	1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1	1
1	0	1	1	1	1	0	1	1
1	0	0	1	1	1	0	1	1
0	1	1	1	1	1	1	1	1
0	1	0	1	0	0	0	0	1
0	0	1	0	1	0	0	0	1
0	0	0	0	0	0	0	0	1

□

Proposition 1.23. *The formula*

$$(\varphi \to \psi) \to (\varphi \to \neg\psi)$$

is not a tautology.

Proof.

φ	ψ	$\varphi \to \psi$	$\neg\psi$	$\varphi \to \neg\psi$	$(\varphi \to \psi) \to (\varphi \to \neg\psi)$
1	1	1	0	0	0

□

Proposition 1.24. *The following is a tautology:*

$$S_0 \to (S_1 \to (S_2 \to (S_3 \to S_1))).$$

Proof. Suppose that f is an assignment making the indicated formula false; we work towards a contradiction. Thus

(1) $S_0[f] = 1$ and

(2) $(S_1 \to (S_2 \to (S_3 \to S_1)))[f] = 0$.

From (2) we get

(3) $S_1[f] = 1$ and

(4) $(S_2 \to (S_3 \to S_1))[f] = 0$.

From (4) we get

(5) $S_2[f] = 1$ and

(6) $(S_3 \to S_1)[f] = 0$.

From (6) we get $S_1[f] = 0$, contradicting (3). □

Proposition 1.25. *The following is a tautology.*

$$(\{[(\varphi \to \psi) \to (\neg\chi \to \neg\theta)] \to \chi\} \to \tau) \to [(\tau \to \varphi) \to (\theta \to \varphi)].$$

Proof. Suppose that f is an assignment which makes the given formula false; we want to get a contradiction. Thus we have

(1) $(\{[(\varphi \to \psi) \to (\neg\chi \to \neg\theta)] \to \chi\} \to \tau)[f] = 1$ and

(2) $[(\tau \to \varphi) \to (\theta \to \varphi)][f] = 0$.

By (2) we have

(3) $(\tau \to \varphi)[f] = 1$ and

(4) $(\theta \to \varphi)[f] = 0$.

By (4) we have

(5) $\theta[f] = 1$ and

(6) $\varphi[f] = 0$.

By (3) and (6) we get

(7) $\tau[f] = 0$.

By (1) and (7) we get

(8) $\{[(\varphi \to \psi) \to (\neg\chi \to \neg\theta)] \to \chi\}[f] = 0$.

It follows that

(9) $[(\varphi \to \psi) \to (\neg\chi \to \neg\theta)][f] = 1$ and

(10) $\chi[f] = 0$.

Now by (6) we have

(11) $(\varphi \to \psi)[f] = 1$,

and hence by (9),

(12) $(\neg\chi \to \neg\theta)[f] = 1$.

By (5) we have

(13) $(\neg\theta)[f] = 0$,

and hence by (12),

(14) $(\neg\chi)[f] = 0$.

This contradicts (10). ☐

Proposition 1.26. *The following statements are logically consistent: If the contract is valid, then Horatio is liable. If Horation is liable, he will go bankrupt. Either Horatio will go bankrupt or the bank will lend him money. However, the bank will definitely not lend him money.*

Proof. Let S_0 correspond to "the contract is valid", S_1 to "Horatio is liable", S_2 to "Horatio will go bankrupt", and S_3 to "the bank will lend him money". Then we want to see if there is an assignment of values which makes the following sentence true:

$$(S_0 \to S_1) \wedge (S_1 \to S_2) \wedge (S_2 \vee S_3) \wedge \neg S_3.$$

We can let $f(0) = f(1) = f(2) = 1$ and $f(3) = 0$, and this gives the sentence the value 1. ☐

Proposition 1.27. $\{\psi\} \vdash \neg\psi \to \varphi$.

Proof. Following the proof of Lemma 1.9, the following is a $\{\psi, \neg\psi\}$-proof:

(a)	$\neg\psi$	
(b)	$\neg\psi \to (\neg\varphi \to \neg\psi)$	(1)
(c)	$\neg\varphi \to \neg\psi$	(a), (b), MP
(d)	$(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$	(3)
(e)	$\psi \to \varphi$	(c), (d), MP
(f)	ψ	
(g)	φ	(e), (f), MP

Now applying the proof of the deduction theorem, the following is a $\{\psi\}$-proof:

(a)	$[\neg\psi \to [(\neg\psi \to \neg\psi) \to \neg\psi]] \to [[\neg\psi \to (\neg\psi \to \neg\psi)]$ $\to (\neg\psi \to \neg\psi)]$	(2)
(b)	$\neg\psi \to [(\neg\psi \to \neg\psi) \to \neg\psi]$	(1)
(c)	$[\neg\psi \to (\neg\psi \to \neg\psi)] \to (\neg\psi \to \neg\psi)$	(a), (b), MP
(d)	$\neg\psi \to (\neg\psi \to \neg\psi)$	(1)
(e)	$\neg\psi \to \neg\psi$	(c), (d), MP
(f)	$[\neg\psi \to (\neg\varphi \to \neg\psi)] \to [\neg\psi \to [\neg\psi \to (\neg\varphi \to \neg\psi)]]$	(1)
(g)	$\neg\psi \to (\neg\varphi \to \neg\psi)$	(1)
(h)	$\neg\psi \to [\neg\psi \to (\neg\varphi \to \neg\psi)]]$	(f), (g), MP
(i)	$[(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)] \to [\neg\psi \to [(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)]]$	(1)
(j)	$(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$	(3)
(k)	$\neg\psi \to [(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)]$	(i), (j), MP
(l)	$[\neg\psi \to [(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)]] \to [[\neg\psi \to (\neg\varphi \to \neg\psi)]$ $\to [\neg\psi \to (\psi \to \varphi)]]$	(2)
(m)	$[\neg\psi \to (\neg\varphi \to \neg\psi)] \to [\neg\psi \to (\psi \to \varphi)]$	(k), (l), MP
(n)	$\neg\psi \to (\psi \to \varphi)$	(g), (m), MP
(o)	$\psi \to (\neg\psi \to \psi)$	(1)

(p)	ψ	
(q)	$\neg\psi \to \psi$	(o), (p), MP
(r)	$[\neg\psi \to (\psi \to \varphi)] \to [(\neg\psi \to \psi) \to (\neg\psi \to \varphi)]$	(2)
(s)	$(\neg\psi \to \psi) \to (\neg\psi \to \varphi)$	(n), (r), MP
(t)	$\neg\psi \to \varphi$	(q), (s), MP

$\square$

2. First-order logic

Although set theory can be considered within a single first-order language, with only non-logical constant $\in$, it is convenient to have more complicated languages, corresponding to the many definitions introduced in mathematics.

All first-order languages have the following *symbols* in common. Again, as for sentential logic, we take these to be certain natural numbers.

1 (negation)
2 (implication)
3 (the equality symbol)
4 (the universal quantifier)
$5m$ for each positive integer m (variables ranging over elements, but not subsets, of a given structure) We denote $5m$ by v_{m-1}. Thus v_0 is 5, v_1 is 10, and in general v_i is $5(i+1)$.

Special first-order languages have additional symbols for the functions and relations and special elements involved. These will always be taken to be some positive integers not among the above; thus they are positive integers greater than 4 but not divisible by 5. So we have in addition to the above logical symbols some non-logical symbols:

Relation symbols, each of a certain positive rank.
Function symbols, also each having a specified positive rank.
Individual constants.

Formally, a first-order language is a quadruple (Rel, Fcn, Cn, rnk) such that Rel, Fcn, Cn are pairwise disjoint subsets of M (the sets of relation symbols, function symbols, and individual constants), and rnk is a function mapping $Rel \cup Fcn$ into the positive integers; $rnk(\mathbf{S})$ gives the *rank* of the symbol $\mathbf{S}$.

Now we will define the notions of terms and formulas, which give a precise formulation of meaningful expressions. Terms are certain finite sequences of symbols. A *term construction sequence* is a sequence $\langle \tau_0, \ldots, \tau_{m-1} \rangle$, $m > 0$, with the following properties: for each $i < m$ one of the following holds:

τ_i is $\langle v_j \rangle$ for some natural number j.

τ_i is $\langle \mathbf{c} \rangle$ for some individual constant $\mathbf{c}$.

τ_i is $\langle \mathbf{F} \rangle^\frown \sigma_0^\frown \sigma_1^\frown \cdots ^\frown \sigma_{n-1}$ for some n-place function symbol $\mathbf{F}$, with each σ_j equal to τ_k for some $k < j$, depending upon j.

A *term* is a sequence appearing in some term construction sequence. Note the similarity of this definition with that of sentential formula given in Chapter 1.

Frequently we will slightly simplify the notation for terms. Thus we might write simply v_j, or $\mathbf{c}$, or $\mathbf{F}\sigma_0 \ldots \sigma_{n-1}$ for the above.

The following two propositons are very similar, in statement and proof, to Propositions 1.1 and 1.2. The first one is the principle of *induction on terms*.

Proposition 2.1. *Let T be a collection of terms satisfying the following conditions:*

(i) Each variable is in T.

(ii) Each individual constant is in T.

(iii) If $\mathbf{F}$ is a function symbol of rank m and $\tau_0, \ldots, \tau_{m-1} \in T$, then also $\mathbf{F}\tau_0 \ldots \tau_{m-1} \in T$.

Then T consists of all terms.

Proof. Let τ be a term. Say that $\langle \sigma_0, \ldots, \sigma_{m-1} \rangle$ is a term construction sequence and $\sigma_i = \tau$. We prove by complete induction on j that $\sigma_j \in T$ for all $j < m$; hence $\tau \in T$. Suppose that $j < m$ and $\sigma_k \in T$ for all $k < j$. If $\sigma_j = \langle v_s \rangle$ for some s, then $\sigma_j \in T$. If $\sigma_j = \langle \mathbf{c} \rangle$ for some individual constant $\mathbf{c}$, then $s_j \in T$. Finally, suppose that σ_j is $\mathbf{F}\sigma_{k_0} \ldots \sigma_{k_{n-1}}$ with each $k_t < j$. Then $\sigma_{k_t} \in T$ for each $t < n$ by the inductive hypothesis, and it follows that $\sigma_j \in T$. This completes the inductive proof. □

Proposition 2.2. *(i) Every term is a nonempty sequence.*

(ii) If τ is a term, then exactly one of the following conditions holds:

(a) τ is an individual constant.

(b) τ is a variable.

(c) There exist a function symbol $\mathbf{F}$, say of rank m, and terms $\sigma_0, \ldots, \sigma_{m-1}$ such that τ is $\mathbf{F}\sigma_0 \ldots \sigma_{m-1}$.

(iii) No proper initial segment of a term is a term.

(iv) If $\mathbf{F}$ and $\mathbf{G}$ are function symbols, say of ranks m and n respectively, and if $\sigma_0, \ldots, \sigma_{m-1}, \tau_0, \ldots, \tau_{n-1}$ are terms, and if $\mathbf{F}\sigma_0 \ldots \sigma_{m-1}$ is equal to $\mathbf{G}\tau_0, \ldots \tau_{n-1}$, then $\mathbf{F} = \mathbf{G}$, $m = n$, and $\sigma_i = \tau_i$ for all $i < m$.

Proof. (i): This is clear since any entry in a term construction sequence is nonempty.

(ii): Also clear.

(iii): We prove this by complete induction on the length of a term. So suppose that τ is a term, and for any term σ shorter than τ, no proper initial segment of σ is a term. We consider cases according to (ii).

Case 1. τ is an individual constant. Then τ has length 1, and any proper initial segment of τ is empty; by (i) the empty sequence is not a term.

Case 2. τ is a variable. Similarly.

Case 3. There exist an m-ary function symbol $\mathbf{F}$ and terms $\sigma_0, \ldots, \sigma_{m-1}$ such that τ is $\mathbf{F}\sigma_0 \ldots \sigma_{m-1}$. Suppose that ρ is a term which is a proper initial segment of τ. By (i), ρ is nonempty, and the first entry of ρ is $\mathbf{F}$. By (ii), ρ has the form $\mathbf{F}\xi_0 \ldots \xi_{m-1}$ for certain terms $\xi_0, \ldots, \xi_{m-1}$. Since both σ_0 and ξ_0 are shorter terms than τ, and one of them is an initial segment of the other, the induction hypothesis gives $\sigma_0 = \xi_0$. Let $i < m$ be maximum such that $\sigma_i = \xi_i$. Since ρ is a proper initial segment of τ, we must have $i < m - 1$. But σ_{i+1} and ξ_{i+1} are shorter terms than τ and one is a segment of the other, so by the inductive hypthesis $\sigma_{i+1} = \xi_{i+1}$, contradicting the choice of i.

(iv): $\mathbf{F}$ is the first entry of $\mathbf{F}\sigma_0 \ldots \sigma_{m-1}$ and $\mathbf{G}$ is the first entry of $\mathbf{G}\tau_0, \ldots \tau_{n-1}$, so $\mathbf{F} = \mathbf{G}$. Then by (ii) we get $m = n$. By induction using (iii), each $\sigma_i = \tau_i$. □

We now give the general notion of a structure. This will be modified and extended for set theory later. For a given first-order language $\mathscr{L} = (Rel, Fcn, Cn, rnk)$, an *$\mathscr{L}$-structure* is a quadruple $\overline{A} = (A, Rel', Fcn', Cn')$ such that A is a nonempty set (the *universe* of the structure), Rel' is a function assigning to each relation symbol $\mathbf{R}$ a $rnk(\mathbf{R})$-ary relation on A, i.e., a collection of $rnk(\mathbf{R})$-tuples of elements of A, Fcn' is a function assigning to each function symbol $\mathbf{F}$ a $rnk(\mathbf{F})$-ary opeation on A, i.e., a function assigning a value in A to each $rnk(\mathbf{F})$-tuple of elements of A, and Cn' is a function assigning to each individual constant $\mathbf{c}$ an element of A. Usually instead of $Rel'(\mathbf{R})$, $Fcn'(\mathbf{F})$ and $Cn'(\mathbf{c})$ we write $\mathbf{R}^{\overline{A}}$, $\mathbf{F}^{\overline{A}}$, and $\mathbf{c}^{\overline{A}}$.

Now we define the "meaning" of terms. This is a recursive definition, similar to the definition of the values of sentential formulas under assignments:

Proposition 2.3. *Let $\overline{A}$ be a structure, and a a function mapping ω into A. (A is the universe of $\overline{A}$.) Then there is a function F mapping the set of terms into A with the following properties:*

(i) $F(v_i) = a_i$ for each $i \in \omega$.

(ii) $F(\mathbf{c}) = \mathbf{c}^{\overline{A}}$ for each individual constant $\mathbf{c}$.

(iii) $F(\mathbf{F}\sigma_0 \ldots \sigma_{m-1}) = \mathbf{F}^{\overline{A}}(F(\sigma_0), \ldots, F(\sigma_{m-1}))$ for every m-ary function symbol $\mathbf{F}$ and all terms $\sigma_0, \ldots, \sigma_{m-1}$

Proof. An $(\overline{A}, a)$-*term sequence* is a sequence $\langle(\tau_0, b_0), \ldots, (\tau_{m-1}, b_{m-1})\rangle$ such that each $b_i \in A$ and for each $i < m$ one of the following conditions holds:

(1) τ_i is $\langle v_j\rangle$ and $b_i = a_j$.

(2) τ_i is $\langle \mathbf{c}\rangle$ for some individual constant $\mathbf{c}$, and $b_i = \mathbf{c}^{\overline{A}}$.

(3) $\tau_i = \langle\mathbf{F}\rangle^\frown\tau_{k(0)}^\frown\cdots^\frown\tau_{k(n-1)}$ and $b_i = \mathbf{F}^{\overline{A}}(b_{k(0)}, \ldots, b_{k(n-1)})$ for some n-ary function symbol $\mathbf{F}$ and some $k(0), \ldots, k(n-1) < i$.

Now we claim

(4) For any term σ and any $(\overline{A}, a)$-term sequences

$$\langle(\tau_0, b_0), \ldots, (\tau_{m-1}, b_{m-1})\rangle \quad \text{and} \quad \langle(\tau'_0, b'_0), \ldots, (\tau'_{n-1}, b'_{n-1})\rangle$$

such that $\tau_{m-1} = \tau'_{n-1} = \sigma$ we have $b_{m-1} = b'_{n-1}$.

We prove (4) by induction on σ, thus using Proposition 2.1. If $\sigma = v_i$, then $b_{m-1} = a_i = b_{n-1}$. If σ is an individual constant $\mathbf{c}$, then $b_{m-1} = \mathbf{c}^{\overline{A}} = b'_{n-1}$. Finally, if $\sigma = \langle\mathbf{F}\rangle^\frown\rho_0\cdots^\frown\rho_{p-1}$, then we have:

$$\tau_{m-1} = \langle\mathbf{F}\rangle^\frown\tau_{k(0)}^\frown\cdots^\frown\tau_{k(p-1)} \quad \text{and} \quad b_{m-1} = \mathbf{F}^{\overline{A}}(b_{k(0)}, \ldots, b_{k(p-1)});$$
$$\tau'_{n-1} = \langle\mathbf{F}\rangle^\frown\tau'^\frown_{l(0)}\cdots^\frown\tau'_{l(p-1)} \quad \text{and} \quad b'_{m-1} = \mathbf{F}^{\overline{A}}(b'_{l(0)}, \ldots, b'_{l(p-1)})$$

with each $k(s)$ and $l(t)$ less than i. By Proposition 2.2(iv) we have $\tau_{k(s)} = \tau'_{l(s)}$ for every $s < p$. Now for every $s < p$ we can apply the inductive hypothesis to $\tau_{k(s)}$ and the sequences

$$\langle(\tau_0, b_0), \ldots, (\tau_{k(s)}, b_{k(s)}) \quad \text{and} \quad \langle(\tau'_0, b'_0), \ldots, (\tau'_{l(s)}, b'_{l(s)})$$

to obtain $b_{k(s)} = b'_{l(s)}$. Hence

$$b_{m-1} = \mathbf{F}^{\overline{A}}(b_{k(0)}, \ldots, b_{k(p-1)}) = \mathbf{F}^{\overline{A}}(b'_{l(0)}, \ldots, b'_{l(p-1)}) = b'_{n-1},$$

completing the inductive proof of (4).

(5) If $\langle \tau_0, \ldots, \tau_{m-1} \rangle$ is a term construction sequence, then there is an $(\overline{A}, a)$-term sequence of the form $\langle (\tau_0, b_0), \ldots, (\tau_{m-1}, b_{m-1} \rangle$.

We prove this by induction on m. For $m = 1$ we have two possibilities.

Case 1. τ_0 is v_j for some $j \in \omega$. Then $\langle (\tau_0, b_j) \rangle$ is as desired.

Case 2. τ_0 is $\mathbf{c}$, an individual constant. Then $\langle (\tau_0, \mathbf{c}^{\overline{A}}) \rangle$ is as desired.

Now assume the statement for $m - 1 \geq 1$. By the induction hypothesis there is an $(\overline{A}, a)$-term sequence of the form $\sigma \stackrel{\text{def}}{=} \langle (\tau_0, b_0), \ldots, (\tau_{m-2}, b_{m-2}) \rangle$. Then we have three possibilities:

Case 1. τ_{m-1} is v_j for some $j \in \omega$. Then $\sigma^\frown \langle (\tau_{m-1}, b_j) \rangle$ is as desired.

Case 2. τ_{m-1} is $\mathbf{c}$, an individual constant. Then $\sigma^\frown \langle (\tau_{m-1}, \mathbf{c}^{\overline{A}}) \rangle$ is as desired.

Case 3. τ_{m-1} is $\langle \mathbf{F} \rangle^\frown \tau_{k(0)}^\frown \cdots {}^\frown \tau_{k(p-1)}$ for some p-ary function symbol $\mathbf{F}$ with each $k(s) < i$. Then $\sigma^\frown \langle (\tau_{m-1}, \mathbf{F}^{\overline{A}}(b_{k(0)}, \ldots, b_{k(p-1)})) \rangle$ is as desired.

So (5) holds.

Now we can define F as needed in the Proposition. Let σ be a term. Let $\langle \tau_0, \ldots, \tau_{m-1} \rangle$ be a term construction sequence with $\tau_{m-1} = \sigma$. By (5), let $\langle (\tau_0, b_0), \ldots, (\tau_{m-1} b_{m-1}) \rangle$ be an $(\overline{A}, a)$-term sequence. Then we define $F(\sigma) = b_{m-1}$. This definition is unambiguous by (4). Now we check the conditions of the Proposition. Let σ be a term, and let $\langle (\tau_0, b_0), \ldots, (\tau_{m-1}, b_{m-1}) \rangle$ be an $(\overline{A}, a)$-term sequence with $\tau_{m-1} = \sigma$.

Case 1. $\sigma = v_j$ for some $j \in \omega$. Then $F(\sigma) = b_{m-1} = a_j$.

Case 2. $\sigma = \mathbf{c}$ for some individual constant $\mathbf{c}$. Then $F(\sigma) = b_{m-1} = \mathbf{c}^{\overline{A}}$.

Case 3. $\sigma = \langle \mathbf{F} \rangle^\frown \rho_0^\frown \cdots {}^\frown \rho_{p-1}$ with $\mathbf{F}$ a p-ary function symbol and each ρ_s a term. Then there exist $c(0), \ldots, c(p-1) < m - 1$ such that $\rho_s = \tau_{c(s)}$ for every $s < p$. Then $F(\tau_{c(s)}) = b_{c(s)} = \tau_{c(s)}^{\overline{A}}$ for each $s < p$, and hence

$$F(\sigma) = b_{m-1} = \mathbf{F}^{\overline{A}}(b_{s(0)}, \ldots, b_{s(p-1)}) = \mathbf{F}^{\overline{A}}(\tau_{s(0)}^{\overline{A}}, \ldots, \tau_{s(p-1)}^{\overline{A}}) = \mathbf{F}^{\overline{A}}(\rho_0^{\overline{A}}, \ldots, \rho_{p-1}^{\overline{A}}). \quad \square$$

With F as in Proposition 2.3, we denote $F(\sigma)$ by $\sigma^{\overline{A}}(a)$. Thus

$$\begin{aligned} v_i^{\overline{A}}(a) &= a_i; \\ \mathbf{c}^{\overline{A}}(a) &= \mathbf{c}^{\overline{A}}; \\ (\mathbf{F}\tau_0 \ldots \tau_{m-1})^{\overline{A}}(a) &= \mathbf{F}^{\overline{A}}(\tau_0^{\overline{A}}(a), \ldots, \tau_{m-1}^{\overline{A}}(a)). \end{aligned}$$

Here v_i is any variable, $\mathbf{c}$ any individual constant, and $\mathbf{F}$ any function symbol (of some rank, say m).

What $\sigma^{\overline{A}}(a)$ means intuitively is: replace the individual constants and function symbols by the actual members of A and functions on A given by the structure $\overline{A}$, and replace the variables v_i by coresponding elements a_i of A; calculate the result, giving an element of A.

Proposition 2.4. *Suppose that τ is a term, $\overline{A}$ is a structure, a, b assignments, and $a(i) = b(i)$ for all i such that v_i occurs in τ. Then $\tau^{\overline{A}}(a) = \tau^{\overline{A}}(b)$.*

Proof. By induction on τ:

$$\begin{aligned}
\mathbf{c}^{\overline{A}}(a) &= \mathbf{c}^{\overline{A}} = \mathbf{c}^{\overline{A}}(b);\\
v_i^{\overline{A}}(a) &= a(i) = b(i) = v_i^{\overline{A}}(b);\\
(\mathbf{F}\sigma_0 \ldots \sigma_{m-1})^{\overline{A}}(a) &= \mathbf{F}^{\overline{A}}(\sigma_0^{\overline{A}}(a), \ldots, \sigma_{m-1}^{\overline{A}}(a))\\
&= \mathbf{F}^{\overline{A}}(\sigma_0^{\overline{A}}(b), \ldots, \sigma_{m-1}^{\overline{A}}(b))\\
&= (\mathbf{F}\sigma_0 \ldots \sigma_{m-1})^{\overline{A}}(b).
\end{aligned}$$

The last step here is the induction step (many of them, one for each function symbol and associated terms). The inductive assumption is that $a(i) = b(i)$ for all i for which v_i occurs in $\mathbf{F}\sigma_0 \ldots \sigma_{m-1}$; hence also for each $j < m$, $a(i) = b(i)$ for all i for which v_i occurs in σ_j, so that the inductive hypothesis can be applied. $\square$

This proposition enables us to simplify our notation a little bit. If n is such that each variable occurring in τ has index less than n, then in the notation $\varphi^{\overline{A}}(a)$ we can just use the first n entries of a rather than the entire infinite sequence.

We turn to the definition of formulas. For any terms σ, τ we define $\sigma = \tau$ to be the sequence $\langle 3\rangle^\frown\sigma^\frown\tau$. Such a sequence is called an *atomic equality formula.* An *atomic non-equality formula* is a sequence of the form $\langle \mathbf{R}\rangle^\frown\sigma_0^\frown \cdots^\frown \sigma_{m-1}$ where $\mathbf{R}$ is an m-ary relation symbol and $\sigma_0, \ldots \sigma_{m-1}$ are terms. An *atomic formula* is either an atomic equality formula or an atomic non-equality formula.

We define $\neg$, a function assigning to each sequence φ of symbols of a first-order language the sequence $\neg\varphi \overset{\text{def}}{=} \langle 1\rangle^\frown\varphi$. $\rightarrow$ is the function assigning to each pair (φ, ψ) of sequences of symbols the sequence $\varphi \rightarrow \psi \overset{\text{def}}{=} \langle 2\rangle^\frown\varphi^\frown\psi$. $\forall$ is the function assigning to each pair (i, φ) with $i \in \omega$ and φ a sequence of symbols the sequence $\forall v_i\varphi \overset{\text{def}}{=} \langle 4, 5i+5\rangle^\frown\varphi$.

A *formula construction sequence* is a sequence $\langle\varphi_0, \ldots, \varphi_{m-1}\rangle$ such that for each $i < m$ one of the following holds:

(1) φ_i is an atomic formula.

(2) There is a $j < i$ such that φ_i is $\neg\varphi_j$

(3) There are $j, k < i$ such that φ_i is $\varphi_j \rightarrow \varphi_k$.

(4) There exist $j < i$ and $k \in \omega$ such that φ_i is $\forall v_k\varphi_j$.

A *formula* is an expression which appears as an entry in some formula construction sequence.

The following is the principle of *induction on formulas.*

Proposition 2.5. *Suppose that Γ is a set of formulas satisfying the following conditions:*

(i) Every atomic formula is in Γ.

(ii) If $\varphi \in \Gamma$, then $\neg\varphi \in \Gamma$.

(iii) If $\varphi, \psi \in \Gamma$, then $(\varphi \to \psi) \in \Gamma$.

(iv) If $\varphi \in \Gamma$ and $i \in \omega$, then $\forall v_i \varphi \in \Gamma$.

Then Γ is the set of all formulas.

Proof. It suffices to take any formula construction sequence $\langle \varphi_0, \ldots, \varphi_{m-1} \rangle$ and show by complete induction on i that $\varphi_i \in \Gamma$ for all $i \in \omega$. So, suppose that $i < m$ and $\varphi_j \in \Gamma$ for all $j < i$. By the definition of formula construction sequence, we have the following cases.

Case 1. φ_i is an atomic formula. Then $\varphi_i \in \Gamma$ by (i).

Case 2. There is a $j < i$ such that φ_i is $\neg\varphi_j$. By the inductive hypothesis, $\varphi_j \in \Gamma$. Hence by (ii), $\varphi_i \in \Gamma$.

Case 2. There are $j, k < i$ such that φ_i is $\varphi_j \to \varphi_k$. By the inductive hypothesis, $\varphi_j \in \Gamma$ and $\varphi_k \in \Gamma$. Hence by (iii), $\varphi_i \in \Gamma$.

Case 4. There exist $j < i$ and $k \in \omega$ such that φ_i is $\forall v_k \varphi_j$. By the inductive hypothesis, $\varphi_j \in \Gamma$. Hence by (iv), $\varphi_i \in \Gamma$.

This completes the inductive proof. $\square$

Proposition 2.6. *(i) Every formula is a nonempty sequence.*

(ii) If φ is a formula, then exactly one of the following conditions holds:

(a) φ is an atomic equality formula, and there are terms σ, τ such that φ is $\sigma = \tau$.

(b) φ is an atomic non-equality formula, and there exist a positive integer m, a relation symbol $\mathbf{R}$ of rank m, and terms $\sigma_0, \ldots, \sigma_{m-1}$, such that φ is $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$.

(c) There is a formula ψ such that φ is $\neg\psi$.

(d) There are formulas ψ, χ such that φ is $\psi \to \chi$.

(e) There exist a formula ψ and a natural number i such that φ is $\forall v_i \psi$.

(iii) No proper initial segment of a formula is a formula.

(iv) (a) If φ is an atomic equality formula, then there are unique terms σ, τ such that φ is $\sigma = \tau$.

(b) If φ is an atomic non-equality formula, then there exist a unique positive integer m, a unique relation symbol $\mathbf{R}$ of rank m, and unique terms $\sigma_0, \ldots, \sigma_{m-1}$, such that φ is $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$.

(c) If φ is a formula and the first symbol of φ is 1, then there is a unique formula ψ such that φ is $\neg\psi$.

(d) If φ is a formula and the first symbol of φ is 2, then there are unique formulas ψ, χ such that φ is $\psi \to \chi$.

(e) If φ is a formula and the first symbol of φ is 4, then there exist a unique natural number i and a unique formula ψ such that φ is $\forall v_i \psi$.

Proof. (i): First note that this is true of atomic formulas, since an atomic formula must have at least a first symbol 3 or some relation symbol. Knowing this about atomic formulas, any entry in a formula construction sequence is nonempty, since the entry is either an atomic formula or else begins with 1,2, or 4.

(ii): This is true on looking at any entry in a formula construction sequence: either the entry begins with 3 or a relation symbol and hence (a) or (b) holds, or it begins with 1, 2, or 4, giving (c), (d) or (e). Only one of (a)–(e) holds because of the first symbol in the entry.

(iii): We prove this by complete induction on the length of the formula. Thus suppose that φ is a formula of length m, and for any formula ψ of length less than m, no proper initial segment of ψ is a formula. Suppose that χ is a proper initial segment of φ and χ is a formula; we want to get a contradiction. By (ii) we have several cases.

Case 1. φ is an atomic equality formula $\sigma = \tau$ for certain terms σ, τ. Thus φ is $\langle 3\rangle^\frown\sigma^\frown\tau$. Since χ is a formula which begins with 3 (since χ is an initial segment of φ and is nonempty by (i)), (ii) yields that χ is $\langle 3\rangle^\frown\rho^\frown\xi$ for some terms ρ, ξ. Hence $\sigma^\frown\psi = \rho^\frown\xi$. Thus σ is an initial segment of ρ or ρ is an initial segment of σ. By Proposition 2.2(iii) it follows that $\sigma = \rho$. Then also $\tau = \xi$, so $\varphi = \chi$, contradiction.

Case 2. φ is an atomic non-equality formula $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$ for some m-ary relation symbol $\mathbf{R}$ and some terms $\sigma_0, \ldots, \sigma_{m-1}$. Then χ is a formula which begins with $\mathbf{R}$, and so there exist terms $\tau_0, \ldots, \tau_{m-1}$ such that χ is $\mathbf{R}\tau_0 \ldots \tau_{m-1}$. By induction using Proposition 2.2(iii), $\sigma_i = \tau_i$ for all $i < m$, so $\varphi = \chi$, contradiction.

Case 3. φ is $\neg\psi$ for some formula ψ. Then 1 is the first entry of χ, so by (ii) χ has the form $\neg\rho$ for some formula ρ. Thus ρ is a proper initial segment of ψ, contradicting the inductive hypothesis, since ψ is shorter than φ.

Case 4. φ is $\psi \to \theta$ for some formulas ψ, θ, i.e., it is $\langle 2\rangle^\frown\psi^\frown\theta$. Then χ starts with 2, so by (ii) χ has the form $\langle 2\rangle^\frown\sigma^\frown\tau$ for some formulas σ, τ. Now both ψ and σ are shorter than φ, and one is an initial segment of the other. So $\psi = \sigma$ by the inductive assumption. Then τ is a proper initial segment of θ, contradicting the inductive assumption.

Case 5. φ is $\langle 4, 5(i+1)\rangle^\frown\psi$ for some $i \in \omega$ and some formula ψ. Then by (ii), χ is $\langle 4, 5(i+1)\rangle^\frown\theta$ for some formula θ. So θ is a proper initial segment of ψ, contradiction.

(iv): These conditions follow from Proposition 2.2(iii) and (iii). $\square$

Now we come to a fundamental definition connecting language with structures. Again this is a definition by recursion; it is given in the following proposition. First a bit of notation. If $a : \omega \to A$, $i \in \omega$, and $s \in A$, then by a^i_s we mean the sequence which is just like a except that $a^i_s(i) = s$.

Proposition 2.7. *Suppose that $\overline{A}$ is an $\mathscr{L}$-structure. Then there is a function G assigning to each formula φ and each sequence $a : \omega \to A$ a value $G(\varphi, a) \in \{0, 1\}$, such that*

(i) For any terms σ, τ, $G(\sigma = \tau, a) = 1$ iff $\sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a)$.

(ii) For each m-ary relation symbol $\mathbf{R}$ and terms $\sigma_0, \ldots, \sigma_{m-1}$, $G(\mathbf{R}\sigma_0 \ldots \sigma_{m-1}, a) = 1$ iff $\langle \sigma_0^{\overline{A}}(a), \ldots, \sigma_{m-1}^{\overline{A}}(a)\rangle \in \mathbf{R}^{\overline{A}}$.

(iii) For every formula φ, $G(\neg\varphi, a) = 1 - G(\varphi, a)$.

(iv) For all formulas φ, ψ, $G(\varphi \to \psi, a) = 0$ iff $G(\varphi, a) = 1$ and $G(\psi, a) = 0$.

(v) For all formulas φ and any $i \in \omega$, $G(\forall v_i\varphi, a) = 1$ iff for every $s \in A$, $G(\varphi, a^i_s) = 1$.

Proof. An $(\overline{A}, a)$-formula sequence is a sequence $\langle(\varphi_0, b_0), \ldots, (\varphi_{m-1}, b_{m-1})\rangle$ such that each b_s is a function mapping $M \stackrel{\text{def}}{=} \{a : a : \omega \to A\}$ into $\{0, 1\}$ and for each $i < m$ one of the following holds:

(1) φ_i is an atomic equality formula $\sigma = \tau$, and $\forall a \in M[b_i(a) = 1 \text{ iff } \sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a)]$.

(2) φ_i is an atomic nonequality formula $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$, and

$$\forall a \in M[b_i(a) = 1 \quad \text{iff} \quad \langle \sigma_0^{\overline{A}}(a), \ldots, \sigma_{m-1}^{\overline{A}}(a)\rangle \in \mathbf{R}^{\overline{A}}].$$

(3) There is a $j < i$ such that $\varphi_i = \neg\varphi_j$, and $\forall a \in M[b_i(a) = 1 - b_j(a)]$.

(4) There are $j, k < i$ such that $\varphi_i = \varphi_j \to \varphi_k$, and $\forall a \in M[b_i(a) = 0 \text{ iff } b_j(a) = 1 \text{ and } b_k(a) = 0]$.

(5) There are $j < i$ and $k \in \omega$ such that $\varphi_i = \forall v_k \varphi_j$, and $\forall a \in M[b_i(a) = 1 \text{ iff } \forall u \in A[b_j(a^k_u) = 1]]$.

Now we claim

(6) For any formula ψ and any $(\overline{A}, a)$-formula sequences

$$\langle(\varphi_0, b_0), \ldots, (\varphi_{m-1}, b_{m-1})\rangle \quad \text{and} \quad \langle(\varphi'_0, b'_0), \ldots, (\varphi'_{n-1}, b'_{n-1})\rangle$$

such that $\varphi_{m-1} = \varphi'_{n-1} = \psi$ we have $b_{m-1} = b'_{n-1}$.

We prove (6) by induction on ψ, thus using Proposition 2.5. First suppose that ψ is an atomic equality formula $\sigma = \tau$. Then the desired conclusion is clear. Similarly for atomic nonequality formulas. Now suppose that ψ is $\neg\chi$. Then by Proposition 2.6(c) there are $j < m$ and $k < n$ such that $\chi = \varphi_j = \varphi'_k$. By the inductive hypothesis we have $b_j = b'_k$, and hence $\forall a \in M[b_{m-1}(a) = 1 - b_j(a) = 1 - b'_k(a) = b_{n-1}(a)]$, so that $b_{m-1} = b'_{n-1}$. Next suppose that ψ is $\chi \to \theta$. Then by Proposition 2.6(d) there are $j, k < m - 1$ such that $\chi = \varphi_j$ and $\theta = \varphi_k$, and there are $s, t < n - 1$ such that $\chi = \varphi'_s$ and $\theta = \varphi'_t$. Then $b_j = b'_s$ and $b_k = b'_t$ by the inductive hypothesis. Hence for any $a \in M$,

$$b_{m-1}(a) = 0 \quad \text{iff} \quad b_j(a) = 1 \text{ and } b_k = 0 \quad \text{iff} \quad b'_s = 1 \text{ and } b'_t = 0 \quad \text{iff} \quad b'_{n-1}(a) = 0.$$

Thus $b_{m-1} = b'_{n-1}$. Finally, suppose that ψ is $\forall v_k\theta$. Then by Proposition 2.6(e) there are $j, s < i$ such that $\varphi_j = \theta$ and $\varphi'_s = \theta$. So by the inductive hypothesis $b_j = b'_s$. Hence for any $a \in M$ we have

$$\begin{aligned} b_{m-1}(a) = 1 \quad &\text{iff} \quad \text{for every } u \in A[b_j(a^k_u) = 1] \\ &\text{iff} \quad \text{for every } u \in A[b'_s(a^k_u) = 1] \\ &\text{iff} \quad b'_{n-1}(a) = 1. \end{aligned}$$

Thus $b_{m-1} = b'_{n-1}$, finishing the proof of (6).

(7) If $\langle\varphi_0, \ldots, \varphi_{m-1}\rangle$ is a formula construction sequence, then there is an $(\overline{A}, a)$-formula sequence of the form $\langle(\varphi_0, b_0), \ldots, (\varphi_{m-1}, b_{m-1})\rangle$.

We prove (7) by induction on m. For $m = 1$ we have two possibilities.

Case 1. φ_0 is an atomic equality formula $\sigma = \tau$. Let $b_0(a) = 1$ iff $\sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a)$.

Case 2. φ_0 is an atomic nonequality formula $\mathbf{R}\sigma_0 \ldots, x_{m-1}$. Let $b_0(a) = 1$ iff $\langle \sigma_0^{\overline{A}}(a), \ldots, \sigma_{m-1}^{\overline{A}} \rangle \in \mathbf{R}^{\overline{A}}$.

Now assume the statement in (7) for $m - 1 \geq 1$. By the inductive hypothesis there is an $(\overline{A}, a)$-formula sequence of the form $\psi \stackrel{\text{def}}{=} \langle (\varphi_0, b_0), \ldots, (\varphi_{m-2}, b_{m-2}) \rangle$. Then we have these possibilities for φ_{m-1}.

Case 1. φ_{m-1} is $\sigma = \tau$ for some terms σ, τ. Define $b_{m-1}(a) = 1$ iff $\sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a)$. Then $\psi ^\frown \langle (\varphi_{m-1}, b_{m-1}) \rangle$ is as desired.

Case 2. φ_{m-1} is $\mathbf{R}\sigma_0 \ldots \sigma_{p-1}$ for some terms $\sigma_0, \ldots, \sigma_{p-1}$. Define $b_{m-1}(a) = 1$ iff $\langle \sigma_0^{\overline{A}}(a), \ldots, \sigma_{p-1}^{\overline{A}}(a) \rangle \in \mathbf{R}^{\overline{A}}$. Then $\psi ^\frown \langle (\varphi_{m-1}, b_{m-1}) \rangle$ is as desired.

Case 3. φ_{m-1} is $\neg\varphi_i$ with $i < m - 1$. Define $b_{m-1}(a) = 1 - b_i(a)$ for any a. Then $\psi ^\frown \langle (\varphi_{m-1}, b_{m-1}) \rangle$ is as desired.

Case 4. φ_{m-1} is $\varphi_i \to \varphi_j$ with $i, j < m - 1$. Define $b_{m-1}(a) = 0$ iff $b_i(a) = 1$ and $b_j(a) = 0$. Then $\psi ^\frown \langle (\varphi_{m-1}, b_{m-1}) \rangle$ is as desired.

Case 5. φ_{m-1} is $\forall v_k \varphi_i$ with $i < m - 1$. Define $b_{m-1}(a)$ iff for all $u \in A$, $b_i(a_u^k) = 1$. Then $\psi ^\frown \langle (\varphi_{m-1}, b_{m-1}) \rangle$ is as desired.

This completes the proof of (7).

Now we can define the function G needed in the Proposition. Let ψ be a formula and $a : \omega \to A$. Let $\langle \varphi_0, \ldots, \varphi_{m-1} \rangle$ be a formula construction sequence with $\varphi_{m-1} = \psi$. By (7) let $\langle (\varphi_0, b_0), \ldots, (\varphi_{m-1}, b_{m-1}) \rangle$ be an $(\overline{A}, a)$-formula sequence. Then we define $G(\psi, a) = b_{m-1}(a)$. The conditions in the Proposition are clear. □

With G as in Proposition 2.7, we write $\overline{A} \models \varphi[a]$ iff $G(\varphi, a) = 1$. $\overline{A} \models \varphi[a]$ is read: "$\overline{A}$ is a model of φ under a" or "$\overline{A}$ models φ under a" or "φ is satisfied by a in $\overline{A}$" or "φ holds in $\overline{A}$ under the assignment a". In summary:

$\overline{A} \models (\sigma = \tau)[a]$ iff $\sigma^{\overline{A}}(a) = \tau^{\overline{A}}(b)$. Here σ and τ are terms.

$\overline{A} \models (\mathbf{R}\sigma_0 \ldots \sigma_{m-1})[a]$ iff the m-tuple $\langle \sigma_0^{\overline{A}}, \ldots, \sigma_{m-1}^{\overline{A}} \rangle$ is in the relation $R^{\overline{A}}$. Here $\mathbf{R}$ is an m-ary relation symbol, and $\sigma_0, \ldots, \sigma_{m-1}$ are terms.

$\overline{A} \models (\neg\varphi)[a]$ iff it is not the case that $\overline{A} \models \varphi[a]$.

$\overline{A} \models (\varphi \to \psi)[a]$ iff either it is not true that $\overline{A} \models \varphi[a]$, or it is true that $\overline{A} \models \psi[a]$. (Equivalently, iff ($\overline{A} \models \varphi[a]$ implies that $\overline{A} \models \psi[a]$).

$\overline{A} \models (\forall v_i \varphi)[a]$ iff $\overline{A} \models \varphi[a_s^i]$ for every $s \in A$.

We define some additional logical notions:

$\varphi \vee \psi$ is the formula $\neg\varphi \to \psi$; $\varphi \vee \psi$ is called the *disjunction* of φ and ψ.

$\varphi \wedge \psi$ is the formula $\neg(\varphi \to \neg\psi)$; $\varphi \wedge \psi$ is called the *conjunction* of φ and ψ.

$\varphi \leftrightarrow \psi$ is the formula $(\varphi \to \psi) \wedge (\psi \to \varphi)$; $\varphi \leftrightarrow \psi$ is called the *equivalence* between φ and ψ.

$\exists v_i\varphi$ is the formula $\neg\forall v_i\neg\varphi$; $\exists$ is the *existential quantifier.*

These notions mean the following.

Proposition 2.8. *Let $\overline{A}$ be a structure and $a:\omega\to A$.*
(i) $\overline{A}\models(\varphi\vee\psi)[a]$ iff $\overline{A}\models\varphi[a]$ or $\overline{A}\models\psi[a]$ (or both).
(ii) $\overline{A}\models(\varphi\wedge\psi)[a]$ iff $\overline{A}\models\varphi[a]$ and $\overline{A}\models\psi[a]$.
(iii) $\overline{A}\models(\varphi\leftrightarrow\psi)[a]$ iff ($\overline{A}\models\varphi[a]$ iff $\overline{A}\models\psi[a]$).
(iv) $\overline{A}\models\exists v_i\varphi[a]$ iff there is a $b\in A$ such that $\overline{A}\models\varphi[a^i_b]$.

Proof. The proof consists in reducing the statements to ordinary mathematical usage. (i):

$$\begin{aligned}
\overline{A}\models(\varphi\vee\psi)[a] \quad &\text{iff}\quad \overline{A}\models(\neg\varphi\to\psi)[a]\\
&\text{iff}\quad \text{either it is not true that } \overline{A}\models(\neg\varphi)[a] \text{ or it is true that } \overline{A}\models\psi[a]\\
&\text{iff}\quad \text{not}(\text{not}(\overline{A}\models\varphi[a])) \text{ or } \overline{A}\models\psi[a]\\
&\text{iff}\quad \overline{A}\models\varphi[a] \text{ or } \overline{A}\models\psi[a].
\end{aligned}$$

(ii):

$$\begin{aligned}
\overline{A}\models(\varphi\wedge\psi)[a] \quad &\text{iff}\quad \text{not}(\overline{A}\models(\varphi\to\neg\psi)[a])\\
&\text{iff}\quad \text{not}(\text{not}(\overline{A}\models\varphi[a]) \text{ or } \overline{A}\models\neg\psi[a])\\
&\text{iff}\quad \text{not}(\text{not}(\overline{A}\models\varphi[a]) \text{ or not}(\overline{A}\models\psi[a]))\\
&\text{iff}\quad \overline{A}\models\varphi[a] \text{ and } \overline{A}\models\psi[a].
\end{aligned}$$

(iii):

$$\begin{aligned}
\overline{A}\models(\varphi\leftrightarrow\psi)[a] \quad &\text{iff}\quad \overline{A}\models((\varphi\to\psi)\wedge(\psi\to\psi))[a]\\
&\text{iff}\quad \overline{A}\models((\varphi\to\psi)[a] \text{ and } \overline{A}\models(\psi\to\psi))[a]\\
&\text{iff}\quad (\overline{A}\models\varphi[a] \text{ implies that } \overline{A}\models\psi[a]) \text{ and}\\
&\qquad\ (\overline{A}\models\psi[a] \text{ implies that } \overline{A}\models\varphi[a])\\
&\text{iff}\quad (\overline{A}\models\varphi[a] \text{ iff } \overline{A}\models\psi[a]).
\end{aligned}$$

(iv):

$$\begin{aligned}
\overline{A}\models\exists v_i\varphi[a] \quad &\text{iff}\quad \overline{A}\models\neg\forall v_i\neg\varphi[a]\\
&\text{iff}\quad \text{not}(\text{for all } b\in A(\overline{A}\models\neg\varphi[a^i_b]))\\
&\text{iff}\quad \text{not}(\text{for all } b\in A(\text{not}(\overline{A}\models\varphi[a^i_b]))\\
&\text{iff}\quad \text{there is a } b\in A \text{ such that } \overline{A}\models\varphi[a^i_b].
\end{aligned}$$
$\square$

We say that $\overline{A}$ is a *model* of φ iff $\overline{A}\models\varphi[a]$ for every $a:\omega\to A$. If Γ is a set of formulas, we write $\Gamma\models\varphi$ iff every structure which models each member of Γ also models φ. $\models\varphi$ means that every structure models φ. φ is then called *universally valid.*

Now we want to apply the material of Chapter 1 concerning sentential logic. By definition, a tautology in a first-order language is a formula ψ such that there exist formulas $\varphi_0, \varphi_1, \ldots$ and a sentential tautology χ such that ψ is obtained from χ by replacing each symbol S_i occurring in χ by φ_i, for each $i < \omega$.

Theorem 2.9. *If ψ is a tautology in a first-order language, then ψ holds in every structure for that language.*

Proof. Let $\overline{A}$ be any structure, and $b : \omega \to A$ any assignment. We want to show that $\overline{A} \models \psi[b]$. Let formulas $\varphi_0, \varphi_1, \ldots, \chi$ be given as in the above definition. For each sentential formula θ, let θ' be the first-order formula obtained from θ by replacing each sentential variable S_i by φ_i. Thus χ' is ψ. We define a sentential assignment f by setting, for each $i \in \omega$,

$$f(i) = \begin{cases} 1 & \text{if } \overline{A} \models \varphi_i[b], \\ 0 & \text{otherwise.} \end{cases}$$

Then we claim:

(*) For any sentential formula θ, $\overline{A} \models \theta'[b]$ iff $\theta[f] = 1$.

We prove this by induction on θ:

If θ is S_i, then θ' is φ_i, and our condition holds by definition. If inductively θ is $\neg\tau$, then θ' is $\neg\tau'$, and

$$\begin{aligned} \overline{A} \models \theta'[b] \quad & \text{iff} \quad \text{not}(\overline{A} \models \tau'[b]) \\ & \text{iff} \quad \text{not}(\tau[f] = 1) \\ & \text{iff} \quad \tau[f] = 0 \\ & \text{iff} \quad \theta[f] = 1. \end{aligned}$$

Finally if inductively θ is $\tau \to \xi$, then θ' is $\tau' \to \xi'$, and

$$\begin{aligned} \overline{A} \models \theta'[b] \quad & \text{iff} \quad (\overline{A} \models \tau'[b] \text{ implies that } \overline{A} \models \xi'[b] \\ & \text{iff} \quad \tau[f] = 1 \text{ implies that } \xi[f] = 1 \\ & \text{iff} \quad \theta[f] = 1. \end{aligned}$$

This finishes the proof of (*).

Applying (*) to χ, we get $\overline{A} \models \chi'[b]$, i.e., $\overline{A} \models \psi[b]$. □

. . . .

A language for the structure $(\omega, <)$ is the quadruple $(\{11\}, \emptyset, \emptyset, rnk)$, where rnk is the function with domain $\{11\}$ such that $rnk(11) = 2$.

A language for the set A (no individual constants, function symbols, or relation symbols) is the quadruple $(\emptyset, \emptyset, \emptyset, \emptyset)$. Note that the last $\emptyset$ is the empty function.

Proposition 2.10. $+\bullet v_0 v_0 v_1$ *is a term in the language for* $(\mathbb{R}, +, \cdot, 0, 1, <)$.

Proof. $\langle v_0, \bullet v_0 v_0, v_1, + \bullet v_0 v_0 v_1\rangle$. □

Proposition 2.11. *In any first-order language, the sequence $\langle v_0, v_0\rangle$ is not a term.*

Proof. Suppose that $\langle v_0, v_0\rangle$ is a term. This contradicts Proposition 2.2(ii). □

Proposition 2.12. *In the language for $(\omega, S, 0, +, \cdot)$, the sequence $\langle +, v_0, v_1, v_2\rangle$ is not a term. Here $S(i) = i + 1$ for any $i \in \omega$.*

Proof. Suppose it is a term. By Proposition 2.2(ii)(c), there are terms σ, τ such that $\langle +, v_0, v_1, v_2\rangle$ is $\langle +\rangle^\frown\sigma^\frown\tau$. Thus $\langle v_0, v_1, v_2\rangle = \sigma^\frown\tau$. So the term v_0 is an initial segment of the term σ. By Proposition 2.2(iii) it follows that $v_0 = \sigma$. Hence $\langle v_1, v_2\rangle = \tau$. This contradicts Proposition 2.2(ii). □

The structure $(\omega, S, 0, +, \cdot)$ can be put in the general framework of structures as follows. It can be considered to be the structure $(\omega, Rel', Fcn', Cn')$ where $Rel' = \emptyset$, Cn' is the function with domain $\{8\}$ such that $Cn'(8) = 0$, and Fcn' is the function with domain $\{6, 7, 9\}$ such that $Fcn'(6) = S$, $Fcn'(7) = +$, and $Fcn'(9) = \cdot$.

Proposition 2.13. *In the language for the structure $(\omega, +)$, a term has length m iff m is odd.*

Proof. First we show by induction on terms that every term has odd length. This is true for variables. Suppose that it is true for terms σ and τ. Then also $\sigma + \tau$ has odd length. Hence every term has odd length.

Second we prove by induction on m that for all m, there is a term of length $2m + 1$. A variable has length 1, so our assertion holds for $m = 0$. Assume that there is a term σ of length $2m + 1$. Then $\sigma + v_0$ has length $2m + 3$. This finishes the inductive proof. □

Proposition 2.14. *In the language for $(\mathbf{Q}, +, \cdot)$ the formula $\varphi \stackrel{\text{def}}{=} \forall v_1[v_0 \cdot v_1 = v_1]$ is such that for any $a : \omega \to \mathbb{Q}$, $(\mathbb{Q}, +, \cdot) \models \varphi[a]$ iff $a_0 = 1$.* □

Proposition 2.15. *The following formula φ holds in a structure, under any assignment, iff the structure has at least 3 elements.*

$$\exists v_0 \exists v_1 \exists v_2 (\neg(v_0 = v_1) \wedge \neg(v_0 = v_2) \wedge \neg(v_1 = v_2)).$$ □

Proposition 2.16. *The following formula φ holds in a structure, under any assignment, iff the structure has exactly 4 elements.*

$$\exists v_0 \exists v_1 \exists v_2 \exists v_3 (\neg(v_0 = v_1) \wedge \neg(v_0 = v_2) \wedge \neg(v_0 = v_3) \wedge \neg(v_1 = v_2) \wedge \neg(v_1 = v_3) \wedge \neg(v_2 = v_3)$$
$$\wedge \forall v_4 (v_0 = v_4 \vee v_1 = v_4 \vee v_2 = v_4 \vee v_3 = v_4)).$$ □

Proposition 2.17. *The following formula φ in the language for $(\omega, <)$ is such that for any assignment a, $(\omega, <) \models \varphi[a]$ iff $a_0 < a_1$ and there are exactly two integers between a_0 and a_1.*

$$\begin{aligned} &v_0 < v_1 \wedge \exists v_2 \exists v_3 [v_0 < v_2 \wedge v_2 < v_3 \wedge v_3 < v_1 \\ &\wedge \forall v_4 [v_0 < v_4 \wedge v_4 < v_1 \to v_4 = v_2 \vee v_4 = v_3]]. \end{aligned}$$ □

Proposition 2.18. *The formula*

$$v_0 = v_1 \to (\mathbf{R}v_0v_2 \to \mathbf{R}v_1v_2)$$

is universally valid, where $\mathbf{R}$ *is a binary relation symbol.*

Proof. Let $\overline{A}$ be a structure and $a : \omega \to A$ an assignment. Suppose that $\overline{A} \models (v_0 = v_1)[a]$. Then $a_0 = a_1$. Also suppose that $\overline{A} \models \mathbf{R}v_0v_2[a]$. Then $(a_0, a_2) \in \mathbf{R}^{\overline{A}}$. Hence $(a_1, a_2) \in \mathbf{R}^{\overline{A}}$. Hence $\overline{A} \models \mathbf{R}v_1v_2[a]$, as desired. $\square$

Proposition 2.19. *The formula*

$$v_0 = v_1 \to \forall v_0(v_0 = v_1)$$

is not universally valid.

Proof. Consider the structure $\overline{A} \stackrel{\text{def}}{=} (\omega, <)$, and let $a : \omega \to \omega$ be defined by $a(i) = 0$ for all $i \in \omega$. Then $\overline{A} \models (v_0 = v_1)[a]$. Now $\overline{A} \not\models (v_0 = v_1)[a_1^0]$ since $1 \neq 0$, so $\overline{A} \not\models \forall v_0(v_0 = v_1)[a]$. Therefore $\overline{A} \not\models (v_0 = v_1 \to \forall v_0(v_0 = v_1))[a]$. $\square$

Proposition 2.20. $\exists v_0 \forall v_1 \varphi \to \forall v_1 \exists v_0 \varphi$ *is universally valid.*

Proof. Assume that $a : \omega \to A$ and $\overline{A} \models \exists v_0 \forall v_1 \varphi[a]$. Choose $u \in A$ so that $\overline{A} \models \forall v_1 \varphi[a_u^0]$. In order to show that $\overline{A} \models \forall v_1 \exists v_0 \varphi[a]$, let $w \in A$ be given. Then $\overline{A} \models \varphi_{uw}^{01}$. It follows that $\overline{A} \models \exists v_0 \varphi[u_w^1]$. Hence $\overline{A} \models \forall v_1 \exists v_0 \varphi[a]$, as desired. $\square$

3. Proofs

The purpose of this chapter is give the definition of a mathematical proof, and give the simplest proofs which will be needed in proving the completeness theorem in the next chapter. Given a set Γ of formulas in a first-order language, and a formula φ in that language, we explain what it means to have a proof of φ from Γ.

The following formulas are the *logical axioms*. Here φ, ψ, χ are arbitrary formulas unless otherwise indicated.

(L1a) $\varphi \to (\psi \to \varphi)$.
(L1b) $[\varphi \to (\psi \to \chi)] \to [(\varphi \to \psi) \to (\varphi \to \chi)]$.
(L1c) $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$.
(L2) $\forall v_i(\varphi \to \psi) \to (\forall v_i\varphi \to \forall v_i\psi)$, for any $i \in \omega$.
(L3) $\varphi \to \forall v_i\varphi$ for any $i \in \omega$ such that v_i does not occur in φ.
(L4) $\exists v_i(v_i = \sigma)$ if σ is a term and v_i does not occur in σ.
(L5) $\sigma = \tau \to (\sigma = \rho \to \tau = \rho)$, where σ, τ, ρ are terms.
(L6) $\sigma = \tau \to (\rho = \sigma \to \rho = \tau)$, where σ, τ, ρ are terms.
(L7) $\sigma = \tau \to \mathbf{F}\xi_0 \ldots \xi_{i-1}\sigma\xi_{i+1} \ldots \xi_{m-1} = \mathbf{F}\xi_0 \ldots \xi_{i-1}\tau\xi_{i+1} \ldots \xi_{m-1}$, where $\mathbf{F}$ is an m-ary function symbol, $i < m$, and $\sigma, \tau, \xi_0, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots \xi_{m-1}$ are terms.
(L8) $\sigma = \tau \to (\mathbf{R}\xi_0 \ldots \xi_{i-1}\sigma\xi_{i+1} \ldots \xi_{m-1} \to \mathbf{R}\xi_0 \ldots \xi_{i-1}\tau\xi_{i+1} \ldots \xi_{m-1})$, where $\mathbf{R}$ is an m-ary relation symbol, $i < m$, and $\sigma, \tau, \xi_0, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots \xi_{m-1}$ are terms.

Theorem 3.1. *Every logical axiom is universally valid.*

Proof. (L1a–c): Universally valid by Theorem 2.9.

(L2): Assume that

(1) $\overline{A} \models \forall v_i(\varphi \to \psi)[a]$ and
(2) $\overline{A} \models \forall v_i\varphi[a]$;

We want to show that $\overline{A} \models \forall v_i\psi[a]$. To this end, take any $b \in A$; we want to show that $\overline{A} \models \varphi[a^i_b]$. Now by (1) we have $\overline{A} \models (\varphi \to \psi)[a^i_b]$, hence $\overline{A} \models \varphi[a^i_b]$ implies that $\overline{A} \models \psi[a^i_b]$. Now by (2) we have $\overline{A} \models \varphi[a^i_b]$, so $\overline{A} \models \psi[a^i_b]$.

(L3): We prove by induction on φ that if v_i does not occur in φ, and if $a, b : \omega \to A$ are such that $a(j) = b(j)$ for all $j \neq i$, then $\overline{A} \models \varphi[a]$ iff $\overline{A} \models \varphi[b]$. This will imply that (L3) is universally valid.

• φ is $\sigma = \tau$. Thus v_i does not occur in σ or in τ. Then

$$\begin{aligned} \overline{A} \models (\sigma = \tau)[a] \quad &\text{iff} \quad \sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a) \\ &\text{iff} \quad \sigma^{\overline{A}}(b) = \tau^{\overline{A}}(b) \quad \text{by Proposition 2.4} \\ &\text{iff} \quad \overline{A} \models (\sigma = \tau)[b]. \end{aligned}$$

• φ is $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$ for some m-ary relation symbol and some terms $\sigma_0, \ldots, \sigma_{m-1}$. We are assuming that v_i does not occur in $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$; hence it does not occur in any term σ_i.

$$\overline{A} \models (\mathbf{R}\sigma_0 \ldots \sigma_{m-1})[a] \quad \text{iff} \quad \langle \sigma_0^{\overline{A}}(a), \ldots, \sigma_{m-1}^{\overline{R}}(a) \rangle \in \mathbf{R}^{\overline{A}}$$

$$\begin{aligned} &\text{iff} \quad \langle \sigma_0^{\overline{A}}(b), \ldots, \sigma_{m-1}^{\overline{R}}(b)\rangle \in \mathbf{R}^{\overline{A}} \\ &\qquad \text{(by Proposition 2.4)} \\ &\text{iff} \quad \overline{A} \models (\mathbf{R}\sigma_0 \ldots \sigma_{m-1})[b]. \end{aligned}$$

• φ is $\neg\psi$ (inductively).

$$\begin{aligned} \overline{A} \models \varphi[a] \quad &\text{iff} \quad \text{not}(\overline{A} \models \psi[a]) \\ &\text{iff} \quad \text{not}(\overline{A} \models \psi[b]) \quad \text{(inductive hypothesis)} \\ &\text{iff} \quad \overline{A} \models \varphi[b]. \end{aligned}$$

• φ is $\psi \to \chi$ (inductively).

$$\begin{aligned} \overline{A} \models \varphi[a] \quad &\text{iff} \quad (\overline{A} \models \psi[a] \text{ implies that } \overline{A} \models \chi[a]) \\ &\text{iff} \quad (\overline{A} \models \psi[b] \text{ implies that } \overline{A} \models \chi[b]) \\ &\qquad \text{(inductive hypothesis)} \\ &\text{iff} \quad \overline{A} \models \varphi[b]. \end{aligned}$$

• φ is $\forall v_k\psi$ (inductively). By symmetry it suffices to prove just one direction. Suppose that $\overline{A} \models \varphi[a]$; we want to show that $\overline{A} \models \varphi[b]$. To this end, suppose that $u \in A$; we want to show that $\overline{A} \models \psi[b^k_u]$. Since $\overline{A} \models \varphi[a]$, we have $\overline{A} \models \psi[a^k_u]$. Now $k \neq i$, since v_i does not occur in φ. Hence $(a^k_u)(j) = (b^k_u)(j)$ for all $j \neq i$. Hence $\overline{A} \models \psi[b^k_u]$ by the inductive hypothesis, as desired.

This finishes our proof by induction of the statement made above. Now assume that $\overline{A} \models \varphi[a]$ and $u \in A$; we want to show that $\overline{A} \models \varphi[a^i_u]$. This holds by the statement above.

This finishes the proof of (L3).

(L4): Suppose that σ is a term and v_i does not occur in σ. To prove that $\overline{A} \models (\exists v_i(v_i = \sigma))[a]$, we want to find $u \in A$ such that $\overline{A} \models (v_i = \sigma)[a^i_u]$. Let $u = \sigma^{\overline{A}}(a)$. Then

$$(v_i)^{\overline{A}}[a^i_u] = u = \sigma^{\overline{A}}(a) = \sigma^{\overline{A}}(a^i_u)$$

by Proposition 2.4 (since v_i does not occur in σ, hence $a(j) = a^i_u(j)$ for all j such that v_j occurs in σ). Hence $\overline{A} \models (v_i = \sigma)[a^i_u]$.

(L5): Assume that $\overline{A} \models (\sigma = \tau)[a]$ and $\overline{A} \models (\sigma = \rho)[a]$. Then $\sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a)$ and $\sigma^{\overline{A}}(a) = \rho^{\overline{A}}(a)$, so $\tau^{\overline{A}}(a) = \rho^{\overline{A}}(a)$, hence $\overline{A} \models (\tau = \rho)[a]$.

(L6): Assume that $\overline{A} \models (\sigma = \tau)[a]$ and $\overline{A} \models (\rho = \sigma)[a]$. Then $\sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a)$ and $\rho^{\overline{A}}(a) = \sigma^{\overline{A}}(a)$, so $\rho^{\overline{A}}(a) = \tau^{\overline{A}}(a)$, hence $\overline{A} \models (\rho = \tau)[a]$.

(L7): Assume that $\overline{A} \models (\sigma = \tau)[a]$. Then $\sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a)$, and so

$$\begin{aligned} (\mathbf{F}\xi_0 \ldots \xi_{i-1}\sigma\xi_{i+1} \ldots \xi_{m-1})^{\overline{A}}(a) &= \mathbf{F}^{\overline{A}}(\xi_0^{\overline{A}}(a), \ldots, \xi_{i-1}^{\overline{A}}(a), \sigma^{\overline{A}}(a), \xi_{i+1}^{\overline{A}}(a), \ldots, \xi_{m-1}^{\overline{A}}(a)) \\ &= \mathbf{F}^{\overline{A}}(\xi_0^{\overline{A}}(a), \ldots, \xi_{i-1}^{\overline{A}}(a), \tau^{\overline{A}}(a), \xi_{i+1}^{\overline{A}}(a), \ldots, \xi_{m-1}^{\overline{A}}(a)) \\ &= (\mathbf{F}\xi_0 \ldots \xi_{i-1}\tau\xi_{i+1} \ldots \xi_{m-1})^{\overline{A}}(a); \end{aligned}$$

it follows that $\overline{A} \models (\mathbf{F}\xi_0 \ldots \xi_{i-1}\sigma\xi_{i+1} \ldots \xi_{m-1} = \mathbf{F}\xi_0 \ldots \xi_{i-1}\tau\xi_{i+1} \ldots \xi_{m-1})[a]$, hence (L7) is universally valid.

(L8): Assume that $\overline{A} \models (\sigma = \tau)[a]$. Then $\sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a)$. Assume that

$$\begin{aligned}
&\overline{A} \models (\mathbf{R}\xi_0 \ldots \xi_{i-1}\sigma\xi_{i+1} \ldots \xi_{m-1})[a];\text{ hence}\\
&\langle \xi_0^{\overline{A}}(a), \ldots, \xi_{i-1}^{\overline{A}}(a), \sigma^{\overline{A}}(a), \xi_{i+1}^{\overline{A}}(a), \ldots, \xi_{m-1}^{\overline{A}}(a)\rangle \in \mathbf{R}^{\overline{A}};\text{ hence}\\
&\langle \xi_0^{\overline{A}}(a), \ldots, \xi_{i-1}^{\overline{A}}(a), \tau^{\overline{A}}(a), \xi_{i+1}^{\overline{A}}(a), \ldots, \xi_{m-1}^{\overline{A}}(a)\rangle \in \mathbf{R}^{\overline{A}};\text{ hence}\\
&\overline{A} \models (\mathbf{R}\xi_0 \ldots \xi_{i-1}\tau\xi_{i+1} \ldots \xi_{m-1})[a];
\end{aligned}$$

hence (L8) is universally valid.

Now let Γ be a set of formulas. A Γ-*proof* is a finite sequence $\langle \varphi_0, \ldots, \varphi_{m-1}\rangle$ of formulas such that for each $i < m$ one of the following conditions holds:

(I1) φ_i is a logical axiom
(I2) $\varphi_i \in \Gamma$.
(I3) (modus ponens) There are $j, k < i$ such that φ_j is the formula $\varphi_k \to \varphi_i$.
(I4) (generalization) There exist $j < i$ and $k \in \omega$ such that φ_i is the formula $\forall v_k \varphi_j$.

Then we say that Γ *proves* φ, in symbols $\Gamma \vdash \varphi$, provided that φ is an entry in some Γ-proof. We write $\vdash \varphi$ in place of $\emptyset \vdash \varphi$.

Theorem 3.2. *If* $\Gamma \vdash \varphi$, *then* $\Gamma \models \varphi$.

Proof. Recall the notion $\Gamma \models \varphi$ from Chapter 2: it says that for every structure $\overline{A}$ for the implicit language we are dealing with, if $\overline{A} \models \psi[a]$ for all $\psi \in \Gamma$ and all $a : \omega \to A$, then $\overline{A} \models \varphi[a]$ for every $a : \omega \to A$. Now it suffices to take a Γ-proof $\langle \psi_0, \ldots, \psi_{m-1}\rangle$ and prove by complete induction on i that $\Gamma \models \psi_i$ for each $i < m$.

Case 1. ψ_i is a logical axiom. Then the result follows by Theorem 3.1.

Case 2. $\psi_i \in \Gamma$. Obviously then $\Gamma \models \psi_i$.

Case 3. There are $j, k < i$ such that φ_j is $\varphi_k \to \varphi_i$. Suppose that $\overline{A}$ is a model of Γ and $a : \omega \to A$. Then $\overline{A} \models \varphi_k[a]$ by the inductive hypothesis, and also $\overline{A} \models (\varphi_k \to \varphi_i)[a]$ by the inductive hypothesis. Thus $\overline{A} \models \varphi_k[a]$ implies that $\overline{A} \models \varphi_i[a]$, so $\overline{A} \models \varphi_i[a]$.

Case 3. There exist $j < i$ and $k \in \omega$ such that φ_i is $\forall v_k \varphi_j$. Given $u \in A$, we want to show that $\overline{A} \models \varphi_j[a^k_u]$; but this follows from the inductive hypothesis. □

One form of the completeness theorem, proved in the next chapter, is that, conversely, $\Gamma \models \varphi$ implies that $\Gamma \vdash \varphi$.

In this chapter we will show that many definite formulas φ are such that $\vdash \varphi$. We begin with tautologies.

Lemma 3.3. $\vdash \varphi$ *for any first-order tautology* φ.

Proof. Let χ be a sentential tautology, and let $\langle \psi_0, \psi_1, \ldots\rangle$ be a sequence of first-order formulas such that φ is obtained from χ by replacing each sentential variable S_i by ψ_i. For each sentential formula θ, let θ' be obtained from θ by replacing each sentential variable

S_i by ψ_i. By Theorem 1.16, $\vdash \chi$ (in the sentential sense). Hence there is a sentential proof $\langle \theta_0, \ldots, \theta_m \rangle$ with $\theta_m = \chi$. We claim that $\langle \theta'_0, \ldots, \theta'_m \rangle$ is a first-order proof. Since $\theta'_m = \chi' = \varphi$, this will prove the lemma. If $i \leq m$ and θ_i is a (sentential) axiom, then θ'_i is the corresponding first-order axiom:

$$
\begin{aligned}
&[\rho \to (\sigma \to \rho)]' = [\rho' \to (\sigma' \to \rho')];\\
&[[\rho \to (\sigma \to \tau] \to [(\rho \to \sigma) \to (\rho \to \tau)]]' =\\
&\qquad [[\rho' \to (\sigma' \to \tau')] \to [(\rho' \to \sigma') \to (\rho' \to \tau')]];\\
&[(\neg\rho \to \neg\sigma) \to (\sigma \to \rho)]' = [(\neg\rho' \to \neg\sigma') \to (\sigma' \to \rho')].
\end{aligned}
$$

If $j, k < i$ and θ_k is $\theta_j \to \theta_i$, then θ'_k is $\theta'_j \to \theta'_i$. □

We proceed with simple theorems concerning equality.

Proposition 3.4. $\vdash \sigma = \sigma$ *for any term* σ.

Proof. The following is a $\emptyset$-proof; on the left is the entry number, and on the right a justification. Let v_i be a variable not occurring in σ.

(1)	$v_i = \sigma \to (v_i = \sigma \to \sigma = \sigma)$	(L5)
(2)	$[v_i = \sigma \to (v_i = \sigma \to \sigma = \sigma) \to [\neg(\sigma = \sigma) \to \neg(v_i = \sigma)]$	(taut.)
(3)	$\neg(\sigma = \sigma) \to \neg(v_i = \sigma)$	((1), (2), MP)
(4)	$\forall v_i[\neg(\sigma = \sigma) \to \neg(v_i = \sigma)]$	((3), gen.)
(5)	$\forall v_i[\neg(\sigma = \sigma) \to \neg(v_i = \sigma)] \to [\forall v_i \neg(\sigma = \sigma) \to \forall v_i \neg(v_i = \sigma)]$	(L2)
(6)	$\forall v_i \neg(\sigma = \sigma) \to \forall v_i \neg(v_i = \sigma)$	(4), (5), MP
(7)	$\neg(\sigma = \sigma) \to \forall v_i \neg(\sigma = \sigma)$	(L3)
(8)	$(7) \to [(6) \to [\neg(\sigma = \sigma) \to \forall v_i \neg(v_i = \sigma)]$	(taut.)
(9)	$(6) \to [\neg(\sigma = \sigma) \to \forall v_i \neg(v_i = \sigma)]$	(7), (8), MP
(10)	$\neg(\sigma = \sigma) \to \forall v_i \neg(v_i = \sigma)$	(6), (9), MP
(11)	$(10) \to [\exists v_i(v_i = \sigma) \to \sigma = \sigma]$	(taut.)
(12)	$\exists v_i(v_i = \sigma) \to \sigma = \sigma$	(10), (11), MP
(13)	$\exists v_i(v_i = \sigma)$	(L4)
(14)	$(13) \to [(12) \to \sigma = \sigma]$	(L1)
(15)	$(12) \to \sigma = \sigma$	((13), (14), MP)
(16)	$\sigma = \sigma$	((12), (15), MP)

□

Proposition 3.5. $\vdash \sigma = \tau \to \tau = \sigma$ *for any terms* σ, τ.

Proof. By (L5) we have

$$\vdash \sigma = \tau \to (\sigma = \sigma \to \tau = \sigma);$$

and by Proposition 3.4 we have $\vdash \sigma = \sigma$. Now

$$\sigma = \sigma \to ([\sigma = \tau \to (\sigma = \sigma \to \tau = \sigma)] \to (\sigma = \tau \to \tau = \sigma))$$

is a tautology, so $\vdash \sigma = \tau \to \tau = \sigma$. $\square$

Proposition 3.6. *$\vdash \sigma = \tau \to (\tau = \rho \to \sigma = \rho)$ for any terms σ, τ, ρ.*

Proof. By (L5), $\vdash \tau = \sigma \to (\tau = \rho \to \sigma = \rho)$. By Proposition 3.5, $\vdash \sigma = \tau \to \tau = \sigma$. Now

$$(\sigma = \tau \to \tau = \sigma) \to ([\tau = \sigma \to (\tau = \rho \to \sigma = \rho)] \to [\sigma = \tau \to (\tau = \rho \to \sigma = \rho)])$$

is a tautology, so $\vdash \sigma = \tau \to (\tau = \rho \to \sigma = \rho)$. $\square$

We now give several results expressing the principle of substitution of equals for equals. The main fact is expressed in Theorem 3.16, which says that under certain conditions the formula $\sigma = \tau \to (\varphi \leftrightarrow \psi)$ is provable, where ψ is obtained from φ by replacing some occurrences of σ by τ.

Lemma 3.7. *If σ and τ are terms, φ and ψ are formulas, v_i is a variable not occurring in σ or τ, and $\vdash \sigma = \tau \to (\varphi \to \psi)$, then $\vdash \sigma = \tau \to (\forall v_i \varphi \to \forall v_i \psi)$.*

Proof.

(1)	$\vdash \forall v_i[\sigma = \tau \to (\varphi \to \psi)]$	(hypothesis, gen.)
(2)	$\vdash \forall v_i(\sigma = \tau) \to \forall v_i(\varphi \to \psi)]$	(from (1), using (L2))
(3)	$\vdash \forall v_i(\varphi \to \psi) \to (\forall v_i \varphi \to \forall v_i \psi)$	((L2))
(4)	$\vdash \sigma = \tau \to \forall v_i(\sigma = \tau)$.	((L3))

Now putting (2)–(4) together with a tautology gives the lemma. $\square$

To proceed further we need to discuss the notion of free and bound occurrences of variables and terms. This depends on the notion of a subformula. Recall that a formula is just a finite sequence of positive integers, subject to certain conditions. Atomic equality formulas have the form $\sigma = \tau$ for some terms σ, τ, and $\sigma = \tau$ is defined to be $\langle 3 \rangle^\frown \sigma^\frown \tau$. Atomic non-equality formulas have the form $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$ for some m, some m-ary relation symbol $\mathbf{R}$, and some terms $\sigma_0, \ldots, \sigma_{m-1}$. $\mathbf{R}$ is actually some positive integer k greater than 5 and not divisible by 5, and $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$ is the sequence $\langle k \rangle^\frown \sigma_0^\frown \cdots ^\frown \sigma_{m-1}$. Non-atomic formulas have the form

$\neg\varphi = \langle 1 \rangle^\frown \varphi$,
$\varphi \to \psi = \langle 2 \rangle^\frown \varphi^\frown \psi$, or
$\forall v_s \varphi = \langle 4, 5(s+1) \rangle^\frown \varphi$.

Thus every formula begins with one of the integers 1,2,3,4 or some positive integer greater than 5 not divisible by 5 which is a relation symbol. This helps motivate the following propositions.

Proposition 3.8. *If $\sigma = \langle \sigma_0, \ldots, \sigma_{k-1} \rangle$ is a term, then each σ_i is either of the form $5m$ with m a positive integer, or it is an odd integer greater than 5 which is a function symbol or individual constant.*

Proof. We prove this by induction on σ, thus using Proposition 2.1. The proposition is obvious if σ is a variable or individual constant. Suppose that $\mathbf{F}$ is a function symbol of rank m, $\tau_0, \ldots, \tau_{m-1}$ are terms, and σ is $\mathbf{F}\tau_0 \ldots \tau_{m-1}$, where we assume the truth of the proposition for $\tau_0, \ldots, \tau_{m-1}$. Suppose that $i < k$. If $i = 0$, then σ_i is $\mathbf{F}$, a function symbol. If $i > 0$, then σ_i is an entry in some τ_j, and the desired conclusion follows by the inductive hypothesis. $\square$

Proposition 3.9. *Let $\varphi = \langle \varphi_0, \ldots \varphi_{k-1} \rangle$ be a formula, suppose that $i < k$, and φ_i is one of the integers 1,2,3,4 or a positive integer greater than 5 which is a relation symbol. Then there is a unique segment $\langle \varphi_i, \varphi_{i+1}, \ldots, \varphi_j \rangle$ of φ which is a formula.*

Proof. We prove this by induction on φ, thus using Proposition 2.5. We assume the hypothesis of the proposition. First suppose that φ is an atomic equality formula $\sigma = \tau$ with σ and τ terms. Thus $\sigma = \tau$ is the sequence $\langle 1 \rangle^\frown \sigma^\frown \tau$. Now by Proposition 2.2(ii), no entry of a term is among the integers $1, 2, 3, 4$ or is a positive integer greater than 5 which is a relation symbol. It follows from the assumption about i that $i = 0$, and hence the desired segment of φ is φ itself. It is unique by Proposition 2.6(iii). Second suppose that φ is an atomic non-equality formula $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$ with $\mathbf{R}$ an m-ary relation symbol and $\sigma_0, \ldots, \sigma_{m-1}$ terms. This is very similar to the first case. $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$ is the sequence $\langle \mathbf{R} \rangle^\frown \sigma_0^\frown \cdots ^\frown \sigma_{m-1}$. By Proposition 2.2(ii) i must be 0, and hence the desired segment of φ is φ itself. It is unique by Proposition 2.6(iii).

Now assume inductively that φ is $\neg\psi$; so φ is $\langle 1 \rangle^\frown \psi$. If $i = 0$, then φ itself is the desired segment, unique by Proposition 2.6(iii). If $i > 0$, then $\varphi_i = \psi_{i-1}$, where $\psi = \langle \psi_0, \ldots, \psi_{k-1} \rangle$. By the inductive hypothesis there is a segment $\langle \psi_{i-1}, \psi_i, \ldots, \psi_j \rangle$ of ψ which is a formula. This gives a segment $\langle \varphi_i, \varphi_{i+1}, \ldots, \varphi_{j+1} \rangle$ of φ which is a formula; it is unique by Proposition 2.6(iii).

Assume inductively that φ is $\psi \to \chi$ for some formulas ψ, χ. So φ is $\langle 2 \rangle^\frown \psi^\frown \chi$. If $i = 0$, then φ itself is the required segment, unique by Proposition 2.6(iii). Now suppose that $i > 0$. Now we have $\psi = \langle \varphi_1, \ldots, \varphi_m \rangle$ and $\chi = \langle \varphi_{m+1}, \ldots, \varphi_{k-1} \rangle$ for some m. If $1 \le i \le m$, then by the inductive assumption there is a segment $\langle \varphi_i, \varphi_{i+1}, \ldots, \varphi_n \rangle$ of ψ which is a formula. This is also a segment of φ, and it is unique by Proposition 2.6(iii). If $m + 1 \le i \le k - 1$, a similar argument with χ gives the desired result.

Finally, assume inductively that φ is $\forall v_s \psi$ with ψ some formula and $s \in \omega$. If $i = 0$ then φ itself is the desired segment, unique by Proposition 2.6(iii). If $i > 0$ then actually $i > 1$ so that φ_i is within ψ, and the inductive hypothesis applies. $\square$

The segment of φ asserted to exist in Proposition 3.9 is called the *subformula of* φ beginning at i. For example, consider the formula $\varphi \stackrel{\text{def}}{=} \forall v_0[v_0 = v_2 \to v_0 = v_2]$. The formula $v_0 = v_2$ occurs in two places in φ. In detail, φ is the sequence $\langle 4, 5, 2, 3, 5, 15, 3, 5, 15 \rangle$. Thus

$\varphi_0 = 4$;
$\varphi_1 = 5$;
$\varphi_2 = 2$;
$\varphi_3 = 3$;
$\varphi_4 = 5$;
$\varphi_5 = 15$;

$\varphi_6 = 3$;
$\varphi_7 = 5$;
$\varphi_8 = 15$;

On the other hand, $v_0 = v_2$ is the formula $\langle 3, 5, 15\rangle$. It occurs in φ beginning at 3, and also beginning at 6.

Now a variable v_s is said to *occur bound* in φ at the j-th position iff with $\varphi = \langle \varphi_0, \ldots, \varphi_{m-1}\rangle$, we have $\varphi_j = v_s$ and there is a subformula of φ of the form $\forall v_s \psi = \langle \varphi_i, \varphi_{i+1}, \ldots, \varphi_m\rangle$ with $i+1 \le j \le m$. If a variable v_s occurs at the j-th position of φ but does not occur bound there, then that occurrence is said to be *free*. We give some examples. Let φ be the formula $v_0 = v_1 \to v_1 = v_2$. All the occurrences of v_0, v_1, v_2 are free occurrences in φ. Note that as a sequence φ is $\langle 2, 3, 5, 10, 3, 10, 15\rangle$; so $\varphi_0 = 2$, $\varphi_1 = 3$, $\varphi_2 = 5$, $\varphi_3 = 10$, $\varphi_4 = 3$, $\varphi_5 = 10$, and $\varphi_6 = 15$. The variable v_0, which is the integer 5, occurs free at the 2-nd position. The variable v_1, which is the integer 10, occurs free at the 3rd and 5th positions. The variable v_2, which is the integer 15, occurs free at the 6th position.

Now let ψ be the formula $v_0 = v_1 \to \forall v_1(v_1 = v_2)$. Then the first occurence of v_1 is free, but the other two occurrences are bound. As a sequence, ψ is $\langle 2, 3, 5, 10, 4, 10, 3, 10, 15\rangle$. The variable v_1 occurs free at the 3rd position, and bound at the 5th and 7th positions.

We also need the notion of a term occurring in another term, or in a formula. The following two propositions are proved much like 3.9.

Proposition 3.10. *If $\sigma = \langle \sigma_0, \ldots, \sigma_{m-1}\rangle$ is a term and $i < m$, then there is a unique term τ which is a segment of σ beginning at i.*

Proof. We prove this by induction on σ. For σ a variable or individual constant, we have $m = 1$ and so $i = 0$, and σ itself is the only possibility for τ. Now suppose that the proposition is true for terms $\tau_0, \ldots \tau_{n-1}$, $\mathbf{F}$ is an n-ary function symbol, and σ is $\mathbf{F}\tau_0 \ldots \tau_{n-1}$. If $i = 0$, then σ itself begins at i, and it is the only term beginning at i by Proposition 2.2(iii). If $i > 0$, then i is inside some term τ_k, and so by the inductive assumption there is a term which is a segment of τ_k beginning there; this term is a segment of σ too, and it is unique by Proposition 2.2(iii). □

Under the assumptions of Proposition 3.10, we say that τ occurs in σ beginning at i.

Proposition 3.11. *If $\varphi = \langle \varphi_0, \ldots, \varphi_{m-1}\rangle$ is a formula, $i < m$, and φ_i is a variable, an individual constant, or a function symbol, then there is a unique segment of φ beginning at i which is a term.*

Proof. We prove this by induction on φ. First suppose that φ is an atomic equality formula $\sigma = \tau$ for some terms σ, τ. Thus φ is $\langle 3\rangle^\frown\sigma^\frown\tau$. So $i > 0$, and hence i is inside σ or τ. If i is inside σ, then by Proposition 3.10, there is a term which is a segment of σ beginning at i; it is also a segment of φ, and it is unique by Proposition 2.2(iii). Similarly for τ.

Suppose inductively that φ is $\neg\psi$. Thus φ is $\langle 1\rangle^\frown\psi$. It follows that $i > 0$, so that φ_i appears in ψ; then the inductive hypothesis applies.

Suppose inductively that φ is $\psi \to \chi$. Thus φ is $\langle 2\rangle^\frown\psi^\frown\chi$. It follows that $i > 0$, so that φ_i appears in ψ or χ; then the inductive hypothesis applies.

Finally, suppose that φ is $\forall v_s\psi$ with ψ a formula and $s \in \omega$. Thus φ is $\langle 4, 5(s+1)\rangle^\frown\psi$. Hence $i > 0$. If $i = 1$, then $\langle 5(s+1)\rangle$ is the desired segment, unique by Proposition 2.6(iii). Suppose that $i > 1$. So φ_i is an entry in ψ and hence by the inductive assumption, there is a segment $\langle \varphi_i, \varphi_{i+1}, \ldots \varphi_m\rangle$ which is a term; this is also a segment of φ, and it is unique by Proposition 2.6(iii). $\square$

Under the assumptions of Proposition 3.11, we say that the indicated segment occurs in φ beginning at i.

We now extend the notions of free and bound occurrences to terms. Let σ be a term which occurs as a segment in a formula φ. Say that $\varphi = \langle \varphi_0, \ldots, \varphi_{m-1}\rangle$ and $\sigma = \langle \varphi_i, \ldots \varphi_k\rangle$. We say that this occurrence of σ in φ is *bound* iff there is a variable v_s which occurs bound in φ at some place t with $i \le t \le k$; the occurrence of σ is *free* iff there is no such variable.

We give some examples. The term $v_0 + v_1$ is bound in its only occurrence in the formula $\forall v_0(v_0 + v_1 = v_2)$. The same term is bound in its first occurrence and free in its second occurrence in the formula $\forall v_0(v_0 + v_1 = v_2) \wedge v_0 + v_1 = v_0$.

Suppose that σ, τ, ρ are terms, and τ occurs in σ beginning at i. By the *result of replacing that occurrence of τ by ρ* we mean the following sequence ξ. Say σ, τ, ρ have domains (lengths) m, n, p respectively. Then ξ is the sequence

$$\langle \sigma_0, \ldots, \sigma_{i-1}, \rho_0, \ldots, \rho_{p-1}, \sigma_{i+n}, \ldots, \sigma_{m-1}\rangle.$$

Put another way, if σ is $\theta^\frown\tau^\frown\eta$ with θ of length i, then ξ is $\theta^\frown\rho^\frown\eta$.

Proposition 3.12. *Suppose that σ, τ, ρ are terms, and the sequence ξ is obtained from ρ by replacing one occurrence of σ by τ. Then ξ is a term.*

Proof. We prove this by induction on ρ, thus by using Proposition 2.1. If ρ is a variable or an individual constant, then σ must be ρ itself, and ξ is τ, which is a term. Now suppose that ρ is $\mathbf{F}\eta_0 \ldots \eta_{m-1}$ for some m-ary function symbol $\mathbf{F}$ and some terms $\eta_0, \ldots, \eta_{m-1}$, and the proposition holds for $\eta_0, \ldots, \eta_{m-1}$. Say the occurrence of σ in ρ begins at i. If $i = 0$, then σ equals ρ, and hence ξ equals τ, which is a term. If $i > 0$, then i is inside some η_j, and hence the occurrence of σ is actually an occurrence in η_j by Proposition 2.2(iii). Replacing this occurrence of σ in η_j by τ we obtain a term by the inductive hypothesis; call this term η'_j. It follows that ξ is $\mathbf{F}\eta_0 \ldots \eta_{j-1}\eta'_j, \eta_{j+1} \ldots \eta_{m-1}$, which is a term. $\square$

As an example, consider the term $v_0 \bullet (v_1 + v_2)$ in the language for $(\mathbb{Q}, +, \cdot)$. Replacing the occurrence of v_1 by $v_0 \bullet v_1$ we obtain the term $v_0 \bullet ((v_0 \bullet v_1) + v_2)$. Writing this out in detail, assuming that $\bullet$ corresponds to 9 and $+$ corresponds to 7, we start with the sequence $\langle 9, 5, 7, 10, 15\rangle$ and end with the sequence $\langle 9, 5, 7, 9, 5, 10, 15\rangle$.

Our first form of subsitution of equals for equals only involves terms:

Theorem 3.13. *If σ, τ, ρ are terms, and ξ is a sequence obtained from ρ by replacing an occurrence of σ in ρ by τ, then ξ is a term and $\vdash \sigma = \tau \to \rho = \xi$.*

Proof. ξ is a term by Proposition 3.12. Now we proceed by induction on ρ. If ρ is a variable or an individual constant, then σ must be the same as ρ, since ρ has length 1 and σ occurs in ρ. Then ξ is τ, and $\sigma = \tau \rightarrow \rho = \xi$ is $\sigma = \tau \rightarrow \sigma = \tau$, a tautology. So the proposition is true in this case.

Now assume inductively that ρ is $\mathbf{F}\eta_0 \ldots \eta_{m-1}$ with $\mathbf{F}$ an m-ary function symbol and $\eta_0, \ldots, \eta_{m-1}$ terms. There are two possibilities for the occurrence of σ. First, possibly σ is the same as ρ. Then ξ is τ, and again we have the tautology $\sigma = \tau \rightarrow \sigma = \tau$, Second, the occurrence of σ is within some η_i. Then by the inductive hypothesis, $\vdash \sigma = \tau \rightarrow \eta_i = \eta'_i$, where η'_i is obtained from η_i by replacing the indicated occurrence of σ by τ. Now an instance of (L7) is

$$\eta_i = \eta'_i \rightarrow \mathbf{F}\eta_0 \ldots \eta_{i-1} \ldots \eta_i\eta_{i+1} \ldots \eta_{m-1} = \mathbf{F}\eta_0 \ldots \eta_{i-1} \ldots \eta'_i\eta_{i+1} \ldots \eta_{m-1}.$$

Putting this together with $\vdash \sigma = \tau \rightarrow \eta_i = \eta'_i$ and a tautology gives $\vdash \sigma = \tau \rightarrow \rho = \xi$. □

Proposition 3.14. *Suppose that φ is a formula and σ, τ are terms. Suppose that σ occurs at the i-th place in φ, and if $i > 0$ and $\varphi_{i-1} = \forall$, then τ is a variable. Let the sequence ψ be obtained from φ by replacing that occurrence of σ by τ. Then ψ is a formula.*

Proof. Induction on φ. Suppose that φ is $\rho = \xi$. Then by Proposition 3.13, σ occurs in ρ or ξ. Suppose that it occurs in ρ. Let ρ' be obtained from ρ by replacing that occurrence of σ by τ. Then ρ' is a term by Proposition 3.14. Since ψ is $\rho' = \xi$, ψ is a formula. The case in which σ occurs in ξ is similar. Now suppose that φ is $\mathbf{R}\eta_0 \ldots \eta_{m-1}$ with $\mathbf{R}$ an m-ary relation symbol and $\eta_0, \ldots, \eta_{m-1}$ are terms. Then the occurrence of σ is within some η_i. Let η'_i be obtained from η_i by replacing that occurrence by τ. Now ψ is $\mathbf{R}\eta_0 \ldots \eta_{i-1}\eta'_i \ldots \eta_{m-1}$, so ψ is a formula.

Now suppose that the result holds for φ', and φ is $\neg\varphi'$. Then σ occurs in φ', so if ψ' is obtained from φ' by replacing the occurrence of σ by τ, then ψ' is a formula by the inductive assumption. Since ψ is $\neg\psi'$ also ψ is a formula.

Next, suppose that the result holds for φ' and φ'', and φ is $\varphi' \rightarrow \varphi''$. Then the occurrence of σ is within φ' or is within φ''. If it is within φ', let ψ' be obtained from φ' by replacing that occurrence of σ by τ. Then ψ' is a formula by the inductive hypothesis. Since ψ is $\psi' \rightarrow \varphi''$, also ψ is a formula. If the occurrence is within φ'', let ψ'' be obtained from φ'' by replacing that occurrence of σ by τ. Then ψ'' is a formula by the inductive hypothesis. Since ψ is $\varphi' \rightarrow \psi''$, also ψ is a formula.

Finally, suppose that the result holds for φ', and φ is $\forall v_k\varphi'$. If $i = 1$, then σ is v_k, and by hypothesis τ is some variable v_l. Then ψ is $\forall v_l\varphi'$, which is a formula. If $i > 1$, then σ occurs in φ', so if ψ' is obtained from φ' by replacing the occurrence of σ by τ, then ψ' is a formula by the inductive assumption. Since ψ is $\forall v_k\psi'$ also ψ is a formula. □

For the exact definition of ψ see the description before Proposition 3.12.

Lemma 3.15. *Suppose that σ and τ are terms, φ is a formula, and ψ is obtained from φ by replacing one free occurrence of σ in φ by τ, such that the occurrence of τ that results is free in ψ. Then $\vdash \sigma = \tau \rightarrow (\varphi \leftrightarrow \psi)$.*

Proof. We proceed by induction on φ. First suppose that φ is an atomic equality formula $\rho = \xi$. If the occurrence of σ that is replaced is in ρ, let ρ' be the resulting term. Then by Proposition 3.13, $\vdash \sigma = \tau \to \rho = \rho'$. Now (L5) gives $\vdash \rho = \rho' \to (\rho = \xi \to \rho' = \xi)$. Putting these two together with a tautology gives $\vdash \sigma = \tau \to (\rho = \xi \to \rho' = \xi)$. By symmetry, $\vdash \sigma = \tau \to (\rho' = \xi \to \rho = \xi)$. Hence $\vdash \sigma = \tau \to (\rho = \xi \leftrightarrow \rho' = \xi)$.

If the occurrence of σ that is replaced is in ξ, a similar argument using (L6) works.

Second, suppose that φ is an atomic non-equality formula $\mathbf{R}\rho_0 \ldots \rho_{m-1}$, with $\mathbf{R}$ an m-ary relation symbol and $\rho_0, \ldots, \rho_{m-1}$ terms. Say that the occurrence of σ that is replaced by τ is in ρ_i, the resulting term being ρ_i'. Then by Proposition 3.13, $\vdash \sigma = \tau \to \rho_i = \rho_i'$. By (L8) we have

$$\vdash \rho_i = \rho_i' \to (\mathbf{R}\rho_0 \ldots \rho_{m-1} \to \mathbf{R}\rho_0 \ldots \rho_{i-1}\rho_i'\rho_{i+1} \ldots \rho_{m-1}),$$

so by a tautology we get from these two facts

$$\vdash \sigma = \tau \to (\mathbf{R}\rho_0 \ldots \rho_{m-1} \to \mathbf{R}\rho_0 \ldots \rho_{i-1}\rho_i'\rho_{i+1} \ldots \rho_{m-1}),$$

and by symmetry

$$\vdash \sigma = \tau \to (\mathbf{R}\rho_0 \ldots \rho_{i-1}\rho_i'\rho_{i+1} \ldots \rho_{m-1} \to \mathbf{R}\rho_0 \ldots \rho_{m-1}),$$

and then another tautology gives

$$\vdash \sigma = \tau \to (\mathbf{R}\rho_0 \ldots \rho_{m-1} \leftrightarrow \mathbf{R}\rho_0 \ldots \rho_{i-1}\rho_i'\rho_{i+1} \ldots \rho_{m-1}),$$

This finishes the atomic cases. Now suppose inductively that φ is $\neg\chi$. The occurrence of σ in φ that is replaced actually occurs in χ; let χ' be the result of replacing that occurrence of σ by τ. Now the occurrence of σ in χ is free in χ. In fact, suppose that $\forall v_i\theta$ is a subformula of χ which has as a segment the indicated occurrence of σ, and v_i occurs in σ. Then $\forall v_i\theta$ is also a subformula of φ, contradicting the assumption that the occurrence of σ is free in φ. Similarly the occurrence of τ in χ' which replaced the occurrence of σ is free. So by the inductive hypothesis, $\vdash \sigma = \tau \to (\chi \leftrightarrow \chi')$, and hence a tautology gives $\vdash \sigma = \tau \to (\neg\chi \leftrightarrow \neg\chi')$, i.e., $\vdash \sigma = \tau \to (\varphi \leftrightarrow \psi)$.

Suppose inductively that φ is $\chi \to \theta$.

Case 1. The occurrence of σ in φ is within χ. Let χ' be obtained from χ by replacing that occurrence by τ, such that that occurrence is free in ψ, hence free in χ'. By the inductive hypothesis, $\vdash \sigma = \tau \to (\chi \leftrightarrow \chi')$. Since ψ is $\chi' \to \theta$, a tautology gives the desired result.

Case 2. The occurrence of σ in φ is within θ. Let θ' be obtained from θ by replacing that occurrence by τ, such that that occurrence is free in ψ, hence free in θ'. By the inductive hypothesis, $\vdash \sigma = \tau \to (\theta \leftrightarrow \theta')$. Since ψ is $\chi \to \theta'$, a tautology gives the desired result.

Finally, suppose that φ is $\forall v_i\rho$. Then the occurrence of σ in φ that is replaced is in ρ. Let ρ' be obtained from ρ by replacing that occurrence of σ by τ. The occurrence of σ in ρ must be free since it is free in φ, as in the treatment of $\neg$ above; similarly for τ and ρ'. Hence by the inductive hypothesis, $\vdash \sigma = \tau \to (\rho \leftrightarrow \rho')$. Now since the occurrence of σ in φ is free, the variable v_i does not occur in σ. Similarly, it does not occur in τ. Hence by Proposition 3.7 and tautologies we get $\vdash \sigma = \tau \to (\forall v_i\rho \leftrightarrow \forall v_i\rho')$, i.e., $\vdash \sigma = \tau \to (\varphi \leftrightarrow \psi)$. □

The hypothesis that the term τ is still free in the result of the replacement in this proposition is necessary for the truth of the proposition. This hypothesis is equivalent to saying that the occurrence of σ which is replaced is not inside a subformula of φ of the form $\forall v_i\chi$ with v_i a variable occurring in τ.

Theorem 3.16. (Substitution of equals for equals) *Suppose that φ is a formula, σ is a term, and σ occurs freely in φ starting at indices $i(0) < \cdots < i(m-1)$. Also suppose that τ is a term. Let ψ be obtained from φ by replacing each of these occurrences of σ by τ, and each such occurrence of τ is free in ψ. Then $\vdash \sigma = \tau \to (\varphi \leftrightarrow \psi)$.*

Proof. We prove this by induction on m. If $m = 0$, then φ is the same as ψ, and the conclusion is clear. Now assume the result for m, for any φ. Now assume that σ occurs freely in φ starting at indices $i(0) < \cdots < i(m)$, and no such occurrence is inside a subformula of φ of the form $\forall v_j\chi$ with v_j a variable occurring in τ. Let θ be obtained from φ by replacing the last occurrence of σ, the one beginning at $i(m)$, by τ. By Proposition 3.15, $\vdash \sigma = \tau \to (\varphi \leftrightarrow \theta)$. Now we apply the inductive hypothesis to θ and the occurrences of σ starting at $i(0), \ldots, i(m-1)$; this gives $\vdash \sigma = \tau \to (\theta \leftrightarrow \psi)$. Hence a tautology gives $\vdash \sigma = \tau \to (\varphi \leftrightarrow \psi)$, finishing the inductive proof. □

Proposition 3.17. *Suppose that φ, ψ, χ are formulas, and the sequence θ is obtained from φ by replacing an occurrence of ψ in φ by χ. Then θ is a formula.*

Proof. Induction on φ. If φ is atomic, then ψ is equal to φ, and θ is equal to χ and hence is a formula. Suppose the result is true for φ' and φ is $\neg\varphi'$. If $\psi = \varphi$, again the desired conclusion is clear. Otherwise the occurrence of ψ is within the subformula φ'. If θ' is obtained from φ' by replacing that occurrence by χ, then θ' is a formula by the inductive hypothesis. Since θ is $\neg\theta'$, also θ is a formula.

Now suppose the result is true for φ' and φ'', and φ is $\varphi' \to \varphi''$. If $\psi = \varphi$, again the desired conclusion is clear. Otherwise the occurrence of ψ is within the subformula φ' or is within the subformula φ''. If it is within φ' and θ' is obtained from φ' by replacing that occurrence by χ, then θ' is a formula by the inductive hypothesis. Since θ is $\theta' \to \varphi''$, also θ is a formula. If it is within φ'' and θ'' is obtained from φ'' by replacing that occurrence by χ, then θ'' is a formula by the inductive hypothesis. Since θ is $\varphi' \to \theta''$, also θ is a formula.

Finally, suppose the result is true for φ' and φ is $\forall v_i\varphi'$. If $\psi = \varphi$, again the desired conclusion is clear. Otherwise the occurrence of ψ is within the subformula φ'. If θ' is obtained from φ' by replacing that occurrence by χ, then θ' is a formula by the inductive hypothesis. Since θ is $\forall v_i\theta'$, also θ is a formula. □

For the exact meaning of θ see the description before Proposition 3.12.

Another form of the substitution of equals by equals principle is as follows:

Theorem 3.18. *Let φ, χ, ρ be formulas, and let ψ be obtained from φ by replacing an occurrence of χ in φ by ρ. Suppose that $\vdash \chi \leftrightarrow \rho$. Then $\vdash \varphi \leftrightarrow \psi$.*

Proof. Induction on φ. If φ is atomic, then ψ is the same as ρ, and the conclusion is clear. Suppose inductively that φ is $\neg\varphi'$. If χ is equal to φ, then ψ is equal to ρ and the conclusion is clear. Suppose that χ occurs within φ', and let ψ' be obtained from φ' by replacing that occurrence by ρ. Assume that $\vdash \chi \leftrightarrow \rho$. Then by the inductive hypothesis $\vdash \varphi' \leftrightarrow \psi'$, so $\vdash \neg\varphi' \leftrightarrow \neg\psi'$, as desired.

The case in which φ is $\varphi' \to \varphi''$ is similar. Finally, suppose that φ is $\forall v_i\varphi'$, and χ occurs within φ'. Let ψ' be obtained from φ' by replacing that occurrence by ρ. Assume that $\vdash \chi \leftrightarrow \rho$. Then $\vdash \varphi' \leftrightarrow \psi'$ by the inductive assumption. So by a tautology, $\vdash \varphi' \to \psi'$, and then by generalization $\vdash \forall v_i(\varphi' \to \psi')$. Using (L2) we then get $\vdash \forall v_i\varphi' \to \forall v_i\psi'$. Similarly, $\vdash \forall v_i\psi' \to \forall v_i\varphi'$. Hence using a tautology, $\vdash \forall v_i\varphi' \leftrightarrow \forall v_i\psi'$. □

Now we work to prove two important logical principles: changing bound variables, and dropping a universal quantifier in favor of a term.

For any formula φ, $i \in \omega$, and term σ by $\mathrm{Subf}^{v_i}_{\sigma}\varphi$ we mean the result of replacing each free occurrence of v_i in φ by σ. We now work towards showing that under suitable conditions, the formula $\forall v_i\varphi \to \mathrm{Subf}^{v_i}_{\sigma}\varphi$ is provable. The supposition expressed in the first sentence of the following lemma will be eliminated later on.

Lemma 3.19. *Suppose that v_i does not occur bound in φ, and does not occur in the term σ.*

Assume that no free occurrence of v_i in φ is within a subformula of φ of the form $\forall v_j\chi$ with v_j a variable occurring in σ. Then $\vdash \forall v_i\varphi \to \mathrm{Subf}^{v_i}_{\sigma}\varphi$.

Proof.

(1)	$\vdash v_i = \sigma \to (\varphi \to \mathrm{Subf}^{v_i}_{\sigma}\varphi)$	(by Proposition 3.16 and a tautology)
(2)	$\vdash \varphi \to (\neg\mathrm{Subf}^{v_i}_{\sigma}\varphi \to \neg(v_i = \sigma))$	(using a tautology)
(3)	$\vdash \forall v_i[\varphi \to (\neg\mathrm{Subf}^{v_i}_{\sigma}\varphi \to \neg(v_i = \sigma))]$	(generalization)
(4)	$\vdash \forall v_i\varphi \to \forall v_i(\neg\mathrm{Subf}^{v_i}_{\sigma}\varphi \to \neg(v_i = \sigma))$	(using (L2))
(5)	$\vdash \forall v_i(\neg\mathrm{Subf}^{v_i}_{\sigma}\varphi \to \neg(v_i = \sigma)) \to (\forall v_i\neg\mathrm{Subf}^{v_i}_{\sigma}\varphi \to \forall v_i\neg(v_i = \sigma))$	((L2))
(6)	$\vdash \forall v_i\varphi \to (\forall v_i\neg\mathrm{Subf}^{v_i}_{\sigma}\varphi \to \forall v_i\neg(v_i = \sigma))$	((4), (5), a tautology)
(7)	$\vdash \neg\forall v_i\neg(v_i = \sigma) \to (\forall v_i\varphi \to \neg\forall v_i\neg\mathrm{Subf}^{v_i}_{\sigma}\varphi)$	((6), a tautology)
(8)	$\vdash \neg\forall v_i\neg(v_i = \sigma)$	((L4))
(9)	$\vdash \forall v_i\varphi \to \neg\forall v_i\neg\mathrm{Subf}^{v_i}_{\sigma}\varphi)$	((7), (8), modus ponens)
(10)	$\vdash \neg\mathrm{Subf}^{v_i}_{\sigma}\varphi \to \forall v_i\neg\mathrm{Subf}^{v_i}_{\sigma}\varphi$	((L3))
(11)	$\vdash \forall v_i\varphi \to \mathrm{Subf}^{v_i}_{\sigma}\varphi$	((9), (10), a tautology)

□

Lemma 3.20. *If $i \neq j$, φ is a formula, v_i does not occur bound in φ, and v_j does not occur in φ at all, then $\vdash \forall v_i\varphi \to \forall v_j\mathrm{Subf}^{v_i}_{v_j}\varphi$.*

Proof.

$\vdash \forall v_i\varphi \to \mathrm{Subf}^{v_i}_{v_j}\varphi$	(by Lemma 3.19)
$\vdash \forall v_j\forall v_i\varphi \to \forall v_j\mathrm{Subf}^{v_i}_{v_j}\varphi$	(using (L2) and a tautology)

$\vdash \forall v_i\varphi \to \forall v_j\forall v_i\varphi$ (by (L3))
$\vdash \forall v_i\varphi \to \forall v_j\mathrm{Subf}^{v_i}_{v_j}\varphi$

$\square$

Lemma 3.21. *If $i \neq j$, φ is a formula, v_i does not occur bound in φ, and v_j does not occur in φ at all, then $\vdash \forall v_i\varphi \leftrightarrow \forall v_j\mathrm{Subf}^{v_i}_{v_j}\varphi$.*

Proof. By Proposition 3.20 we have $\vdash \forall v_i\varphi \to \forall v_j\mathrm{Subf}^{v_i}_{v_j}\varphi$. Now v_j does not occur bound in $\mathrm{Subf}^{v_i}_{v_j}\varphi$ and v_i does not occur in $\mathrm{Subf}^{v_i}_{v_j}\varphi$ at all. Hence by Proposition 3.20 again, $\vdash \forall v_j\mathrm{Subf}^{v_i}_{v_j}\varphi \to \forall v_i\mathrm{Subf}^{v_j}_{v_i}\mathrm{Subf}^{v_i}_{v_j}\varphi$. Now $\mathrm{Subf}^{v_j}_{v_i}\mathrm{Subf}^{v_i}_{v_j}\varphi$ is actually just φ itself; so $\vdash \forall v_j\mathrm{Subf}^{v_i}_{v_j}\varphi \to \forall v_i\varphi$. Hence the proposition follows. $\square$

For $i, j \in \omega$ and φ a formula, by $\mathrm{Subb}^{v_i}_{v_j}\varphi$ we mean the result of replacing all bound occurrences of v_i in φ by v_j. By Proposition 3.14 this gives another formula.

Proposition 3.22. *If v_i occurs bound in a formula φ, then there is a subformula $\forall v_i\psi$ of φ such that v_i does not occur bound in ψ.*

Proof. Induction on φ. Note that the statement to be proved is an implication. If φ is atomic, then v_i cannot occur bound in φ; thus the hypothesis of the implication is false, and so the implication itself is true. Now suppose inductively that φ is $\neg\chi$, and v_i occurs bound in φ. Then it occurs bound in χ, and so by the inductive hypothesis, χ has a subformula $\forall v_i\psi$ such that v_i does not occur bound in ψ. This is also a subformula of φ. The implication case is similar. Finally, suppose that φ is $\forall v_k\chi$, and v_i occurs bound in φ. If it occurs bound in χ, then by the inductive hypothesis χ has a subformula $\forall v_i\psi$ such that v_i does not occur bound in ψ; this is also a subformula of φ. If v_i does not occur bound in χ, then we must have $i = k$ since v_i occurs bound in φ, and then φ itself is the desired subformula. $\square$

Theorem 3.23. (Change of bound variables) *If ψ_j does not occur in φ, then $\vdash \varphi \leftrightarrow \mathrm{Subb}^{v_i}_{v_j}\varphi$.*

Proof. We proceed by induction on the number m of bound occurrences of v_i in φ. If $m = 0$, then $\mathrm{Subb}^{v_i}_{v_j}\varphi$ is just φ itself, and the conclusion is clear. Now assume that $m > 0$ and the conclusion is known for all formulas with fewer than m bound occurrences of v_i. By Proposition 3.22, let $\forall v_i\psi$ be a formula occurring in φ such that v_i does not occur bound in ψ. Let k be such that v_k does not occur in φ, and hence also does not occur in ψ, and with $k \neq j$. Note that $k \neq i$ since v_k does not occur in φ while v_i does. Then by Proposition 3.21 we have

$$\vdash \forall v_i\psi \leftrightarrow \forall v_k\mathrm{Subf}^{v_i}_{v_k}\psi. \tag{1}$$

Let φ' be obtained from φ by replacing an occurrence of $\forall v_i\psi$ by $\forall v_k\mathrm{Subf}^{v_i}_{v_k}\psi$. By Theorem 3.18,

$$\vdash \varphi \leftrightarrow \varphi'. \tag{2}$$

Now v_j does not occur in φ', and φ' has fewer than m bound occurrences of v_i. Hence by the inductive hypothesis,

$$\vdash \varphi' \leftrightarrow \mathrm{Subb}^{v_i}_{v_j}\varphi'. \tag{3}$$

Now $k \neq i, j$ and v_k does not occur bound in $\mathrm{Subf}^{v_i}_{v_k}\psi$. Moreover, v_j does not occur in $\mathrm{Subf}^{v_i}_{v_k}\psi$ at all. Hence by Proposition 3.20,

$$\vdash \forall v_k \mathrm{Subf}^{v_i}_{v_k}\psi \leftrightarrow \forall v_j \mathrm{Subf}^{v_k}_{v_j}\mathrm{Subf}^{v_i}_{v_k}\psi.$$

Now clearly $\mathrm{Subf}^{v_k}_{v_j}\mathrm{Subf}^{v_i}_{v_k}\psi = \mathrm{Subf}^{v_i}_{v_j}\psi$; so

$$\vdash \forall v_k \mathrm{Subf}^{v_i}_{v_k}\psi \leftrightarrow \forall v_j \mathrm{Subf}^{v_i}_{v_j}\psi. \tag{4}$$

Now $\mathrm{Subb}^{v_i}_{v_j}\varphi$ can be obtained from $\mathrm{Subb}^{v_i}_{v_j}\varphi'$ by replacing an occurrence of $\forall v_k \mathrm{Subf}^{v_i}_{v_k}\psi$ by $\forall v_j \mathrm{Subf}^{v_i}_{v_j}\psi$. Hence by (4) and Theorem 3.18 we get

$$\vdash \mathrm{Subb}^{v_i}_{v_j}\varphi \leftrightarrow \mathrm{Subb}^{v_i}_{v_j}\varphi'. \tag{5}$$

(2), (3), and (5) now give the desired result, finishing the inductive proof. □

We can now strengthen Lemma 3.19 by eliminating one of its hypotheses; the remaining inessential hypothesis will be eliminated next.

Lemma 3.24. *Suppose that v_i does not occur in the term σ.*

Assume that no free occurrence of v_i in a formula φ is within a subformula of φ of the form $\forall v_j \chi$ with v_j a variable occurring in σ. Then $\vdash \forall v_i \varphi \rightarrow \mathrm{Subf}^{v_i}_{\sigma}\varphi$.

Proof. Choose j so that v_j does not occur in φ or in σ, with $i \neq j$. Then by the change of bound variables theorem 3.23, $\vdash \varphi \leftrightarrow \mathrm{Subb}^{v_i}_{v_j}\varphi$. From this, using generalization and (L2) we obtain

$$\vdash \forall v_i \varphi \leftrightarrow \forall v_i \mathrm{Subb}^{v_i}_{v_j}\varphi. \tag{1}$$

Now v_i does not occur bound in $\mathrm{Subb}^{v_i}_{v_j}\varphi$, and no free occurrence of v_i in $\mathrm{Subb}^{v_i}_{v_j}\varphi$ is in a subformula of $\mathrm{Subb}^{v_i}_{v_j}\varphi$ of the form $\forall v_k \psi$, with v_k a variable occurring in σ. This is true since it is true of φ, and v_j does not occur in σ. Hence by Lemma 3.19 we get

$$\vdash \forall v_i \mathrm{Subb}^{v_i}_{v_j}\varphi \rightarrow \mathrm{Subf}^{v_i}_{\sigma}\mathrm{Subb}^{v_i}_{v_j}\varphi. \tag{2}$$

Now v_i does not occur at all in $\mathrm{Subf}^{v_i}_{\sigma}\mathrm{Subb}^{v_i}_{v_j}\varphi$, so by change of bound variable,

$$\vdash \mathrm{Subf}^{v_i}_{\sigma}\mathrm{Subb}^{v_i}_{v_j}\varphi \leftrightarrow \mathrm{Subb}^{v_j}_{v_i}\mathrm{Subf}^{v_i}_{\sigma}\mathrm{Subb}^{v_i}_{v_j}\varphi. \tag{3}$$

But clearly $\mathrm{Subb}^{v_j}_{v_i}\mathrm{Subf}^{v_i}_{\sigma}\mathrm{Subb}^{v_i}_{v_j}\varphi = \mathrm{Subf}^{v_i}_{\sigma}\varphi$. Hence from (1)–(3) and tautologies we get the result of the lemma. □

Theorem 3.25. (Universal specification) *Assume that no free occurrence of v_i in a formula φ is within a subformula of φ of the form $\forall v_j\chi$ with v_j a variable occurring in a term σ. Then $\vdash \forall v_i\varphi \to \mathrm{Subf}^{v_i}_{\sigma}\varphi$.*

Proof. Choose j so that v_j does not occur in φ or in σ, with $j \neq i$. Then by Lemma 3.24, $\vdash \forall v_i\varphi \to \mathrm{Subf}^{v_i}_{v_j}\varphi$. Hence using (L2) we easily get

(1) $$\vdash \forall v_j\forall v_i\varphi \to \forall v_j\mathrm{Subf}^{v_i}_{v_j}\varphi.$$

By (L3) we have

(2) $$\vdash \forall v_i\varphi \to \forall v_j\forall v_i\varphi.$$

Now no free occurrence of v_j in $\mathrm{Subf}^{v_i}_{v_j}\varphi$ is within a subformula of $\mathrm{Subf}^{v_i}_{v_j}\varphi$ of the form $\forall v_k\psi$ with v_k occurring in σ; this is true because it holds for φ. Also, v_j does not occur in σ. Hence by Lemma 3.24 we have

(3) $$\vdash \forall v_j\mathrm{Subf}^{v_i}_{v_j}\varphi \to \mathrm{Subf}^{v_j}_{\sigma}\mathrm{Subf}^{v_i}_{v_j}\varphi.$$

Clearly $\mathrm{Subf}^{v_j}_{\sigma}\mathrm{Subf}^{v_i}_{v_j}\varphi = \mathrm{Subf}^{v_i}_{\sigma}\varphi$, so from (1)–(3) the desired result follows. □

This finishes the fundamental things that can be proved. We now give various corollaries.

Corollary 3.26. $\vdash \forall v_i\varphi \to \varphi$. □

Proposition 3.27. *If v_i does not occur free in φ, then $\vdash \varphi \leftrightarrow \forall v_i\varphi$.*

Proof. By Corollary 3.26 we have

(1) $$\vdash \forall v_i\varphi \to \varphi.$$

Now let v_j be a variable not occurring in φ. Then by a change of bound variable,

(2) $$\vdash \varphi \leftrightarrow \mathrm{Subb}^{v_i}_{v_j}\varphi.$$

Hence using (L2) we easily get

(3) $$\vdash \forall v_i\mathrm{Subb}^{v_i}_{v_j}\varphi \to \forall v_i\varphi.$$

Now note that v_i does not occur in $\mathrm{Subb}^{v_i}_{v_j}\varphi$. Hence by (L3) we get

(4) $$\vdash \mathrm{Subb}^{v_i}_{v_j}\varphi \to \forall v_i\mathrm{Subb}^{v_i}_{v_j}\varphi.$$

Now from (1)–(4) the desired result easily follows. □

Proposition 3.28. $\vdash \forall v_i\forall v_j\varphi \leftrightarrow \forall v_j\forall v_i\varphi$*, for any formula φ and any $i, j \in \omega$.*

Proof.

$$\begin{array}{lr}
\vdash \forall v_i \forall v_j \varphi \to \varphi & \text{by Corollary 3.26 twice} \\
\vdash \forall v_i \forall v_i \forall v_j \varphi \to \forall v_i \varphi & \text{by (L2)} \\
\vdash \forall v_i \forall v_j \varphi \to \forall v_i \forall v_i \forall v_j \varphi & \text{using Prop. 3.27} \\
\vdash \forall v_i \forall v_j \varphi \to \forall v_i \varphi & \\
\vdash \forall v_j \forall v_i \forall v_j \varphi \to \forall v_j \forall v_i \varphi & \text{by (L2)} \\
\vdash \forall v_i \forall v_j \varphi \to \forall v_j \forall v_i \forall v_j \varphi & \text{using Prop. 3.27} \\
\vdash \forall v_i \forall v_j \varphi \to \forall v_j \forall v_i \varphi & \\
\vdash \forall v_j \forall v_i \varphi \to \forall v_i \forall v_j \varphi & \text{similarly} \\
\vdash \forall v_i \forall v_j \varphi \leftrightarrow \forall v_j \forall v_i \varphi & \square
\end{array}$$

Recall that $\exists v_i \varphi$ is defined to be the formula $\neg \forall v_i \neg \varphi$. The following simple propositions expand on this.

Proposition 3.29. $\vdash \neg \forall v_i \varphi \leftrightarrow \exists v_i \neg \varphi$ *for any formula* φ *and any* $i \in \omega$.

Proof. Proof. By definition, $\exists v_i \neg \varphi$ is $\neg \forall v_i \neg\neg \varphi$. Now $\vdash \varphi \leftrightarrow \neg\neg\varphi$ by a tautology. Hence using generalization and (L2) we get $\vdash \forall v_i \varphi \leftrightarrow \forall v_i \neg\neg\varphi$. Hence another tautology yields $\vdash \neg\forall v_i \varphi \leftrightarrow \neg\forall v_i \neg\neg\varphi$, i.e., $\vdash \neg\forall v_i \varphi \leftrightarrow \exists v_i \neg\varphi$. $\square$

Proposition 3.30. $\vdash \neg \exists v_i \varphi \leftrightarrow \forall v_i \neg \varphi$ *for any formula* φ *and any* $i \in \omega$.

Proof. $\neg\exists v_i \varphi$ is the formula $\neg\neg\forall v_i \neg\varphi$, so a simple tautology gives the result. $\square$

Some important results concerning $\exists$ are as follows.

Theorem 3.31. *If no free occurrence of* v_i *in a formula* φ *is within a subformula of the form* $\forall v_k \psi$ *with* v_k *occurring in a term* σ, *then* $\vdash \mathrm{Subf}^{v_i}_{\sigma} \varphi \to \exists v_i \varphi$.

Proof. By Theorem 3.25 we have $\vdash \forall v_i \neg\varphi \to \mathrm{Subf}^{v_i}_{\sigma}(\neg\varphi)$. Since clearly $\mathrm{Subf}^{v_i}_{\sigma}(\neg\varphi)$ is the same as $\neg\mathrm{Subf}^{v_i}_{\sigma}\varphi$, a tautology gives $\vdash \mathrm{Subf}^{v_i}_{\sigma}\varphi \to \exists v_i \varphi$. $\square$

Corollary 3.32. $\vdash \varphi \to \exists v_i \varphi$ *for any formula* φ. $\square$

Corollary 3.33. $\vdash \forall v_i \varphi \to \exists v_i \varphi$.

Proof. By Corollary 3.26 and Corollary 3.32. $\square$

Proposition 3.34. *If* v_i *does not occur free in* φ, *then* $\vdash \varphi \leftrightarrow \exists v_i \varphi$.

Proof. $\vdash \neg\varphi \leftrightarrow \forall v_i \neg\varphi$. Now use a tautology. $\square$

Theorem 3.35. $\vdash \exists v_i \forall v_j \varphi \to \forall v_j \exists v_i \varphi$ *for any formula* φ.

Proof.

$$\begin{array}{lr}
\vdash \varphi \to \exists v_i \varphi & \text{by Corollary 3.32} \\
\vdash \forall v_j \varphi \to \forall v_j \exists v_i \varphi & \text{generalization, (L2)} \\
\vdash \neg\forall v_j \exists v_i \varphi \to \neg\forall v_j \varphi & \text{tautology} \\
\vdash \forall v_i[\neg\forall v_j \exists v_i \varphi \to \neg\forall v_j \varphi] & \text{generalization} \\
\vdash \forall v_i[\neg\forall v_j \exists v_i \varphi \to \neg\forall v_j \varphi] \to [\forall v_i \neg\forall v_j \exists v_i \varphi \to \forall v_i \neg\forall v_j \varphi] & \text{(L2)} \\
\vdash \forall v_i \neg\forall v_j \exists v_i \varphi \to \forall v_i \neg\forall v_j \varphi & \\
\vdash \neg\forall v_j \exists v_i \varphi \to \forall v_i \neg\forall v_j \varphi & \text{by Proposition 3.27} \\
\vdash \exists v_i \forall v_j \varphi \to \forall v_j \exists v_i \varphi & \text{tautology}
\end{array}$$

□

Now we prove several results involving two formulas φ and ψ, and some variable v_i which is not free in one of them.

Proposition 3.36. *If v_i does not occur free in the formula φ, and ψ is any formula, then $\vdash \forall v_i(\varphi \to \psi) \to (\varphi \to \forall v_i \psi)$.*

Proof. By Proposition 3.27,

$$\vdash \varphi \to \forall v_i \varphi. \tag{1}$$

By (L2) we have $\vdash \forall v_i(\varphi \to \psi) \to (\forall v_i \varphi \to \forall v_i \psi)$, and hence by a tautology

$$\vdash \forall v_i \varphi \to [\forall v_i(\varphi \to \psi) \to \forall v_i \psi] \tag{2}$$

By a tautology, from (1) and (2) we get

$$\vdash \varphi \to [\forall v_i(\varphi \to \psi) \to \forall v_i \psi],$$

and then another tautology gives the desired result. □

Proposition 3.37. *If v_i does not occur free in the formula ψ, then $\vdash \forall v_i(\varphi \to \psi) \to (\exists v_i \varphi \to \psi)$.*

Proof.

$$\begin{array}{lll}
(1) & \vdash (\varphi \to \psi) \to (\neg\psi \to \neg\varphi) & \text{(taut.)} \\
(2) & \vdash \forall v_i(\varphi \to \psi) \to \forall v_i(\neg\psi \to \neg\varphi) & \text{((1), gen., (L2))} \\
(3) & \vdash \forall v_i(\neg\psi \to \neg\varphi) \to (\neg\psi \to \forall v_i \neg\varphi) & \text{(Prop. 3.36)} \\
(4) & \vdash (\neg\psi \to \forall v_i \neg\varphi) \to (\exists v_i \varphi \to \psi) & \text{(taut.)} \\
 & \vdash \forall v_i(\varphi \to \psi) \to (\exists v_i \varphi \to \psi) & \text{((2)–(4), taut.)}
\end{array}$$

□

Lemma 3.38. *If φ and ψ are formulas and v_i does not occur free in ψ, then $\vdash \forall v_i \varphi \vee \psi \leftrightarrow \forall v_i(\varphi \vee \psi)$.*

Proof.

(1)	$\vdash \forall v_i\varphi \vee \psi \leftrightarrow (\neg\psi \rightarrow \forall v_i\varphi)$	taut.
(2)	$\vdash \forall v_i\varphi \rightarrow \varphi$	Cor. 3.26
(3)	$\vdash (\neg\psi \rightarrow \forall v_i\varphi) \rightarrow (\neg\psi \rightarrow \varphi)$	(2), taut.
(4)	$\vdash (\neg\psi \rightarrow \varphi) \rightarrow (\varphi \vee \psi)$	taut.
(5)	$\vdash \forall v_i(\neg\psi \rightarrow \varphi) \rightarrow \forall v_i(\varphi \vee \psi)$	(4), gen., (L2)
(6)	$\vdash \forall v_i(\neg\psi \rightarrow \forall v_i\varphi) \rightarrow \forall v_i(\neg\psi \rightarrow \varphi)$	(3), gen., (L2)
(7)	$(\neg\psi \rightarrow \forall v_i\varphi) \rightarrow \forall v_i(\neg\psi \rightarrow \forall v_i\varphi)$	Prop. 3.27
(8)	$\vdash \forall v_i\varphi \vee \psi \rightarrow \forall v_i(\varphi \vee \psi)$	(1), (7), (6), (5)
(9)	$\vdash \varphi \vee \psi \rightarrow (\neg\psi \rightarrow \varphi)$	taut.
(10)	$\vdash \forall v_i(\varphi \vee \psi) \rightarrow \forall v_i(\neg\psi \rightarrow \varphi)$	(9), gen., (L2)
(11)	$\forall v_i(\neg\psi \rightarrow \varphi) \rightarrow (\neg\psi \rightarrow \forall v_i\varphi)$	Prop. 3.36
(12)	$\vdash (\neg\psi \rightarrow \forall v_i\varphi) \rightarrow \forall v_i\varphi \vee \psi$	taut.

The desired conclusion now follows from (8) and (10)–(12). $\square$

Proposition 3.39. $\vdash \forall v_i(\varphi \wedge \psi) \leftrightarrow \forall v_i\varphi \wedge \forall v_i\psi$, *for any formulas* φ, ψ.

Proof.

	$\vdash \forall v_i(\varphi \wedge \psi) \rightarrow \varphi \wedge \psi$	by Corollary 3.26
	$\vdash \forall v_i(\varphi \wedge \psi) \rightarrow \varphi$	using a tautology
	$\vdash \forall v_i\forall v_i(\varphi \wedge \psi) \rightarrow \forall v_i\varphi$	using (L2)
	$\vdash \forall v_i(\varphi \wedge \psi) \rightarrow \forall v_i\varphi$	using Proposition 3.27
	$\vdash \forall v_i(\varphi \wedge \psi) \rightarrow \forall v_i\psi$	similarly
(1)	$\vdash \forall v_i(\varphi \wedge \psi) \rightarrow \forall v_i\varphi \wedge \forall v_i\psi$	a tautology
	$\vdash \forall v_i\varphi \rightarrow \varphi$	by Corollary 3.26
	$\vdash \forall v_i\psi \rightarrow \psi$	by Corollary 3.26
	$\vdash \forall v_i\varphi \wedge \forall v_i\psi \rightarrow \varphi \wedge \psi$	by a tautology
	$\vdash \forall v_i(\forall v_i\varphi \wedge \forall v_i\psi) \rightarrow \forall v_i(\varphi \wedge \psi)$	using (L2)
	$\vdash \forall v_i\varphi \wedge \forall v_i\psi \rightarrow \forall v_i(\varphi \wedge \psi)$.	using Proposition 3.27

Now the desired result follows using (1) and a tautology. $\square$

Proposition 3.40. *If* φ *and* ψ *are formulas and* v_i *does not occur free in* ψ, *then* $\vdash \exists v_i\varphi \wedge \psi \leftrightarrow \exists v_i(\varphi \wedge \psi)$.

Proof.

$\vdash \neg\exists v_i\varphi \vee \neg\psi \leftrightarrow \forall v_i\neg\varphi \vee \neg\psi$	by Prop. 3.30
$\vdash \forall v_i\neg\varphi \vee \neg\psi \leftrightarrow \forall v_i(\neg\varphi \vee \neg\psi)$	by Prop. 3.38
$\vdash (\neg\varphi \vee \neg\psi) \leftrightarrow \neg(\varphi \wedge \psi)$	taut.
$\vdash \forall v_i(\neg\varphi \vee \neg\psi) \leftrightarrow \forall v_i\neg(\varphi \wedge \psi)$	gen., (L2)
$\vdash \forall v_i\neg(\varphi \wedge \psi) \leftrightarrow \neg\exists v_i(\varphi \wedge \psi)$.	taut.

From these facts we get $\vdash \neg\exists v_i\varphi \vee \neg\psi \leftrightarrow \neg\exists v_i(\varphi \wedge \psi)$. The proposition follows by a tautology. $\square$

Proposition 3.41. *If* $\vdash \varphi \leftrightarrow \psi$, *then* $\vdash \forall v_i\varphi \leftrightarrow \forall v_i\psi$.

Proof. Assume $\vdash \varphi \leftrightarrow \psi$. By a tautology, $\vdash \varphi \rightarrow \psi$. Hence by generalization and (L2), $\vdash \forall v_i\varphi \rightarrow \forall v_i\psi$. Similarly, $\vdash \forall v_i\psi \rightarrow \forall v_i\varphi$. The proposition follows by a tautology. $\square$

Proposition 3.42. *If* $\vdash \varphi \leftrightarrow \psi$, *then* $\vdash \exists v_i\varphi \leftrightarrow \exists v_i\psi$.

Proof. Assume $\vdash \varphi \leftrightarrow \psi$. By a tautology, $\vdash \neg\varphi \leftrightarrow \neg\psi$. Hence by exercise E3.18, $\vdash \forall v_i\neg\varphi \leftrightarrow \forall v_i\neg\psi$. Now a tautology gives the desired result. $\square$

Proposition 3.43. $\vdash \exists v_i(\varphi \vee \psi) \leftrightarrow \exists v_i\varphi \vee \exists v_i\psi$ *for any formulas* φ, ψ.

Proof.

$$\begin{aligned}
&\vdash \neg(\varphi \vee \psi) \leftrightarrow \neg\varphi \wedge \neg\psi && \text{a tautology}\\
&\vdash \forall v_i\neg(\varphi \vee \psi) \leftrightarrow \forall v_i(\neg\varphi \wedge \neg\psi) && \text{by Proposition 3.41}\\
&\vdash \forall v_i(\neg\varphi \wedge \neg\psi) \leftrightarrow \forall v_i\neg\varphi \wedge \forall v_i\neg\psi && \text{by Proposition 3.39}\\
&\vdash \neg\forall v_i\neg(\varphi \vee \psi) \leftrightarrow \neg\forall v_i\neg\varphi \vee \neg\forall v_i\neg\psi; && \text{a tautology}
\end{aligned}$$

this gives the desired result. $\square$

. . . .

Proposition 3.44. *Consider the following formulas.*

$\exists v_0(v_0 < v_1) \wedge \forall v_1(v_0 = v_1)$.
$v_4 + v_2 = v_0 \wedge \forall v_3(v_0 = v_1)$.
$\exists v_2(v_4 + v_2 = v_0)$.

In the first formula: the first and second occurrences of v_0 *are bound, and the third one is free. The first occurrence of* v_1 *is free, and the other two are bound.*

In the second formula: the occurrence of v_3 *is bound. All other occurrences of variables are free.*

In the third formula: the two occurrences of v_2 *are bound. The other occurrences of variables are free.* $\square$

Proposition 3.45. *In the formula* $v_0 = v_1 + v_1 \rightarrow \exists v_2(v_0 + v_2 = v_1)$,

v_0 *is free in both of its occcurrences.*
v_1 *is free in all three of its occurrences.*
v_2 *is bound in both of its occurrences.*
$v_1 + v_1$ *is free in its occurrence.*
$v_0 + v_2$ *is bound in its occurrence.* $\square$

Proposition 3.46. *The condition in Lemma 3.15 that the resulting occurrence of τ is free is necessary.*

Proof. Consider the language for (ω, S), and the formula

$$v_0 = v_1 \to (\exists v_1(\mathbf{S}v_0 = v_1) \leftrightarrow \exists v_1(\mathbf{S}v_1 = v_1)).$$

Taking an assignment $a : \omega \to \omega$ with $a_0 = a_1$ makes this sentence false; hence it is not provable, by Theorem 3.2. $\square$

Proposition 3.47. *The hypothesis of Theorem 3.25 is necessary.*

Proof. Consider the formula

$$\forall v_0 \exists v_1(v_0 < v_1) \to \exists v_1(v_1 < v_1).$$

This formula is not universally valid; it fails to hold in $(\omega, <)$, for example. $\square$

Proposition 3.48.

$$\vdash \forall v_0 \forall v_1(v_0 = v_1) \to \forall v_0(v_0 = v_1 \vee v_0 = v_2).$$

Proof.

$\vdash \forall v_0 \forall v_1(v_0 = v_1) \to v_0 = v_1;$ Cor. 3.26 twice, taut. (1)

$\vdash \forall v_1(v_0 = v_1) \to v_0 = v_2;$ Thm. 3.25 (2)

$\vdash \forall v_0 \forall v_1(v_0 = v_1) \to v_0 = v_2;$ (2), Cor. 3.26, taut. (3)

$\vdash \forall v_0 \forall v_1(v_0 = v_1) \to v_0 = v_1 \vee v_0 = v_2;$ (1), (3), taut. (4)

$\vdash \forall v_0 \forall v_0 \forall v_1(v_0 = v_1) \to \forall v_0(v_0 = v_1 \vee v_0 = v_2);$ (4), (L2), taut. (5)

$\vdash \forall v_0 \forall v_1(v_0 = v_1) \to \forall v_0(v_0 = v_1 \vee v_0 = v_2).$ (5), Prop. 3.27, taut.

Proposition 3.49.

$$\vdash \exists v_0(\neg v_0 = v_1 \wedge \neg v_0 = v_2) \to \exists v_0 \exists v_1(\neg v_0 = v_1).$$

$\vdash \neg\forall v_0(v_0 = v_1 \vee v_0 = v_2) \to \neg\forall v_0 \forall v_1(v_0 = v_1);$ E3.20, taut. (1)

$\vdash \neg\forall v_0(v_0 = v_1 \vee v_0 = v_2) \leftrightarrow \exists v_0 \neg(v_0 = v_1 \vee v_0 = v_2);$ Prop. 3.29 (2)

$\vdash \neg(v_0 = v_1 \vee v_0 = v_2) \leftrightarrow (\neg(v_0 = v_1) \wedge \neg(v_0 = v_2));$ taut. (3)

$\vdash \exists v_0 \neg(v_0 = v_1 \vee v_0 = v_2) \leftrightarrow \exists v_0(\neg(v_0 = v_1) \wedge \neg(v_0 = v_2));$ (3), Prop. 3.42 (4)

$\vdash \neg\forall v_0(v_0 = v_1 \vee v_0 = v_2) \leftrightarrow \exists v_0(\neg(v_0 = v_1) \wedge \neg(v_0 = v_2));$ (2), (4), taut. (5)

$\vdash \neg\forall v_1(v_0 = v_1) \leftrightarrow \exists v_1 \neg(v_0 = v_1);$ Prop. 3.29 (6)

$\vdash \exists v_0 \neg\forall v_1(v_0 = v_1) \leftrightarrow \exists v_0 \exists v_1 \neg(v_0 = v_1);$ (6), Prop. 3.42 (7)

$\vdash \neg\forall v_0 \forall v_1(v_0 = v_1) \leftrightarrow \exists v_0 \neg\forall v_1(v_0 = v_1);$ Prop. 3.29 (8)

$\vdash \neg\forall v_0 \forall v_1(v_0 = v_1) \leftrightarrow \exists v_0 \exists v_1 \neg(v_0 = v_1)$ (7), (8), taut. (9)

$\vdash \exists v_0(\neg v_0 = v_1 \wedge \neg v_0 = v_2) \to \exists v_0 \exists v_1(\neg v_0 = v_1).$ (1), (5), (9), taut. $\square$

4. The completeness theorem

The completeness theorem, in its simplest form, says that for any formula φ, $\vdash \varphi$ iff $\models \varphi$. We already know the direction $\Rightarrow$, in Theorem 3.2.

A more general form of the completeness theorem is that $\Gamma \vdash \varphi$ iff $\Gamma \models \varphi$, for any set $\Gamma \cup \{\varphi\}$ of formulas. Again the direction $\Rightarrow$ is given in Theorem 3.2.

Basic for the proof of the completeness theorem is the notion of consistency. A set Γ of formulas is *consistent* iff there is a formula φ such that $\Gamma \nvdash \varphi$.

Lemma 4.1. *For any set Γ of formulas the following conditions are equivalent:*

(i) Γ is inconsistent.

(ii) There is a formula φ such that $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$.

(iii) $\Gamma \vdash \neg(v_0 = v_0)$.

Proof. (i)$\Rightarrow$(ii): Assume (i). Since $\Gamma \vdash \psi$ for every formula ψ, (ii) is clear.

(ii)$\Rightarrow$(iii): Assume (ii). Then the following is a Γ-proof:

A Γ-proof of φ.
A Γ-proof of $\neg\varphi$.
A $\emptyset$-proof of $\varphi \to (\neg\varphi \to \neg(v_0 = v_0)$. (This is a tautology; see Lemma 3.3.)
$\neg\varphi \to \neg(v_0 = v_0)$.
$\neg(v_0 = v_0)$.

(iii)$\Rightarrow$(i): By (iii) we have $\Gamma \vdash \neg(v_0 = v_0)$, while by Proposition 3.4 we have $\Gamma \vdash v_0 = v_0$. Then for any formula φ, the following is a Γ-proof of φ:

A $\emptyset$-proof of $v_0 = v_0$
A Γ-proof of $\neg(v_0 = v_0)$
A $\emptyset$-proof of $v_0 = v_0 \to (\neg(v_0 = v_0) \to \varphi)$. (This is a tautology; see Lemma 3.3.)
$\neg(v_0 = v_0) \to \varphi$
φ. $\square$

A *sentence* is a formula which has no variable occurring free in it. A set Γ of sentences *has a model* iff there is a structure $\overline{A}$ for the language in question such that $\overline{A} \models \varphi[a]$ for every $\varphi \in \Gamma$ and every $a : \omega \to A$.

The following first-order version of the deduction theorem, Theorem 1.8, will be useful.

Theorem 4.2. (First-order deduction theorem) *If $\Gamma \cup \{\psi\}$ is a set of formulas, φ is a sentence, and $\Gamma \cup \{\varphi\} \vdash \psi$, then $\Gamma \vdash \varphi \to \psi$.*

Proof. Let $\langle \chi_0, \ldots, \chi_{m-1} \rangle$ be a $(\Gamma \cup \{\varphi\})$-proof with $\chi_i = \psi$ for some $i < m$. We modify this proof, replacing each χ_j by one or more formulas, converting the proof to a Γ-proof, in such a way that $\varphi \to \chi_j$ is in the new proof for every $j < m$. If χ_j is a logical axiom or a member of Γ, we replace it by the three formulas

$$\chi_j \to (\varphi \to \chi_j)$$
$$\chi_j$$
$$\varphi \to \chi_j.$$

If χ_j is φ, we replace it by the five formulas giving a little proof of $\varphi \to \varphi$; see Lemma 1.7. If there exist $k, l < j$ such that χ_k is $\chi_l \to \chi_j$, we replace χ_j by the formulas

$$\begin{aligned}
&(\varphi \to \chi_k) \to [(\varphi \to \chi_l) \to (\varphi \to \chi_j)]\\
&(\varphi \to \chi_l) \to (\varphi \to \chi_j)\\
&\varphi \to \chi_j.
\end{aligned}$$

If there exist $k < j$ and $l \in \omega$ such that χ_j is $\forall v_l \chi_k$, we replace χ_j by the formulas

$$\begin{aligned}
&\forall v_l(\varphi \to \chi_k)\\
&\text{a proof of } \forall v_l(\varphi \to \chi_k) \to (\varphi \to \forall v_l \chi_k) \qquad \text{see Proposition 3.36}\\
&\varphi \to \chi_j.
\end{aligned}$$

□

Theorem 4.3. *Suppose that every consistent set of sentences has a model. Then* $\Gamma \vdash \varphi$ *iff* $\Gamma \models \varphi$*, for every set* $\Gamma \cup \{\varphi\}$ *of formulas.*

Proof. Assume that every consistent set of sentences has a model. Note again that $\Gamma \vdash \varphi$ implies that $\Gamma \models \varphi$, by Theorem 3.2. We prove the converse by proving its contrapositive. Thus suppose that $\Gamma \cup \{\varphi\}$ is a set of formulas such that $\Gamma \nvdash \varphi$. We want to show that $\Gamma \not\models \varphi$, i.e., there is a model of Γ which is not a model of φ. For any formula ψ, let $[\![\psi]\!]$ be the *closure* of ψ, i.e., the sentence

$$\forall v_{i(0)} \ldots \forall v_{i(m-1)} \psi,$$

where $i(0) < \cdots < i(m-1)$ lists all the integers j such that v_j occurs free in ψ. Let $\Gamma' = \{[\![\psi]\!] : \psi \in \Gamma\}$. We claim that $\Gamma' \cup \{\neg[\![\varphi]\!]\}$ is consistent. Suppose not. Then $\Gamma' \cup \{\neg[\![\varphi]\!]\} \vdash \neg(v_0 = v_0)$. Hence by the deduction theorem, $\Gamma' \vdash \neg[\![\varphi]\!] \to \neg(v_0 = v_0)$, so $\Gamma' \vdash v_0 = v_0 \to [\![\varphi]\!]$. Hence, using Proposition 3.4, $\Gamma' \vdash [\![\varphi]\!]$. Now in a Γ'-proof that has $[\![\varphi]\!]$ as a member, replace each formula

$$\forall v_{i(0)} \ldots \forall v_{i(m-1)} \psi,$$

with $\psi \in \Gamma$, by the sequence

$$\begin{aligned}
&\psi\\
&\forall v_{i(m-1)} \psi\\
&\cdots\cdots\cdots\\
&\forall v_{i(0)} \ldots \forall v_{i(m-1)} \psi.
\end{aligned}$$

This converts the proof into a Γ-proof one of whose members is $[\![\varphi]\!]$. Thus $\Gamma \vdash [\![\varphi]\!]$. Using Corollary 3.26, it follows that $\Gamma \vdash \varphi$, contradiction.

Hence $\Gamma' \cup \{\neg[\![\varphi]\!]\}$ is consistent. Since this is a set of sentences, by supposition it has a model $\overline{M}$. Clearly $\overline{M}$ is a model of Γ. Since $\overline{M}$ is a model of $\neg[\![\varphi]\!]$, clearly there is an $a \in {}^{\omega}M$ such that $\overline{M} \models \neg\varphi[a]$. Thus $\overline{M}$ is not a model of φ. This shows that $\Gamma \not\models \varphi$. □

To prove that every consistent set of sentences has a model, we need several lemmas, starting with some additional facts about structures and satisfaction.

Lemma 4.4. *Suppose that $\overline{A}$ is a structure, a and b map ω into A, φ is a formula, and $a_i = b_i$ for every i such that v_i occurs free in φ. Then $\overline{A} \models \varphi[a]$ iff $\overline{A} \models \varphi[b]$.*

Proof. Induction on φ. For φ an atomic equality formula $\sigma = \tau$, the hypothesis means that $a_i = b_i$ for all i such that v_i occurs in σ or τ. Hence, using Proposition 2.4,

$$\overline{A} \models \varphi[a] \text{ iff } \sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a) \text{ iff } \sigma^{\overline{A}}(b) = \tau^{\overline{A}}(b) \text{ iff } \overline{A} \models \varphi[b].$$

For φ an atomic non-equality formula $\mathbf{R}\eta_0 \ldots \eta_{m-1}$, the hypothesis means that $a_i = b_i$ for all i such that v_i occurs in one of the terms η_j. Hence, again using Proposition 2.4,

$$\begin{aligned}
\overline{A} \models \varphi[a] \quad &\text{iff} \quad \langle \eta_0^{\overline{A}}(a), \ldots, \eta_{m-1}^{\overline{A}}(a) \rangle \in \mathbf{R}^{\overline{A}} \\
&\text{iff} \quad \langle \eta_0^{\overline{A}}(b), \ldots, \eta_{m-1}^{\overline{A}}(b) \rangle \in \mathbf{R}^{\overline{A}} \\
&\text{iff} \quad \overline{A} \models \varphi[b].
\end{aligned}$$

Assume inductively that φ is $\neg\psi$. The hypothesis implies that $a_i = b_i$ for all i such that v_i occurs free in ψ. Hence

$$\begin{aligned}
\overline{A} \models \varphi[a] \quad &\text{iff} \quad \text{not}(\overline{A} \models \psi[a]) \\
&\text{iff} \quad \text{not}(\overline{A} \models \psi[b]) \quad \text{(induction hypothesis)} \\
&\text{iff} \quad \overline{A} \models \varphi[b].
\end{aligned}$$

Assume inductively that φ is $\psi \to \chi$. The hypothesis implies that $a_i = b_i$ for all i such that v_i occurs free in ψ or in χ. Hence

$$\begin{aligned}
\overline{A} \models \varphi[a] \quad &\text{iff} \quad \text{not}(\overline{A} \models \psi[a]) \text{ or } \overline{A} \models \chi[a] \\
&\text{iff} \quad \text{not}(\overline{A} \models \psi[b]) \text{ or } \overline{A} \models \chi[b] \quad \text{(induction hypothesis)} \\
&\text{iff} \quad \overline{A} \models \varphi[b].
\end{aligned}$$

Now assume inductively that φ is $\forall v_k \psi$. By symmetry it suffices to show that $\overline{A} \models \varphi[a]$ implies that $\overline{A} \models \varphi[b]$. So, assume that $\overline{A} \models \varphi[a]$. Take any $u \in A$. Then $\overline{A} \models \psi[a^k_u]$. We claim that $(a^k_u)_i = (b^k_u)_i$ for every i such that v_i occurs free in ψ. If $i \neq k$ this is true since v_i also occurs free in φ, so that $a_i = b_i$; and $(a^k_u)_i = a_i = b_i = (b^k_u)_i$. If $i = k$, then $(a^k_u)_i = u = (b^k_u)_i$. It follows now by the inductive hypothesis that $\overline{A} \models \psi[b^k_u]$. Since u is arbitrary, $\overline{A} \models \varphi[b]$. □

As in the case of terms (see Proposition 2.4 and the comments after it), Lemma 4.4 enables us to simplify the notation $\overline{A} \models \varphi[a]$. Instead of a full assignment $a : \omega \to A$, it suffices to take a function $a : \{0, \ldots, m\} \to A$ such that every variable v_i occurring free in φ is such that $i \leq m$. Then $\overline{A} \models \varphi[a]$ means that $\overline{A} \models \varphi[b]$ for any b (or some b) such that b extends a. If φ is a sentence, thus with no free variables, then $\overline{A} \models \varphi$ means that $\overline{A} \models \varphi[b]$ for any, or some, $b : \omega \to A$.

Lemma 4.5. *Suppose that τ, ρ, and ν are terms, and ρ is obtained from τ by replacing all occurrences of v_i in τ by ν. Then for any structure $\overline{A}$ and any assignment $a : \omega \to A$, $\rho^{\overline{A}}(a) = \tau^{\overline{A}}\left(a^i_{\nu^{\overline{A}}(a)}\right)$.*

Proof. By induction on τ. If τ is v_k with $k \neq i$, then ρ is the same as τ, and both sides of the above equation are equal to a_k. If τ is v_i, then ρ is ν, and $\rho^{\overline{A}}(a) = \nu^{\overline{A}}(a) = v_i^{\overline{A}}\left(a^i_{\nu^{\overline{A}}(a)}\right) = \tau^{\overline{A}}\left(a^i_{\nu^{\overline{A}}(a)}\right)$. If τ is an individual constant $\mathbf{k}$, then ρ is equal to τ, and both sides of the equation in the lemma are equal to $\mathbf{k}^{\overline{A}}$.

Now suppose inductively that τ is $\mathbf{F}\eta_0 \ldots \eta_{m-1}$. Let μ_i be obtained from η_i by replacing all occurrences of v_i by ν. Then

$$\begin{aligned}
\rho^{\overline{A}}(a) &= (\mathbf{F}\mu_0 \ldots \mu_{m-1})^{\overline{A}}(a) \\
&= \mathbf{F}^{\overline{A}}(\mu_0^{\overline{A}}(a), \ldots, \mu_{m-1}^{\overline{A}}(a)) \\
&= \mathbf{F}^{\overline{A}}\left(\eta_0\left(a^i_{\nu^{\overline{A}}(a)}\right), \ldots, \eta_{m-1}\left(a^i_{\nu^{\overline{A}}(a)}\right)\right) \\
&= (\mathbf{F}\eta_0 \ldots \eta_{m-1})\left[a^i_{\nu^{\overline{A}}(a)}\right] \\
&= \tau^{\overline{A}}\left(a^i_{\nu^{\overline{A}}(a)}\right).
\end{aligned}$$

□

Lemma 4.6. *Suppose that φ is a formula, ν is a term, no free occurrence of v_i in φ is within a subformula of the form $\forall v_k \mu$ with v_k a variable occurring in ν, and $\overline{A}$ is a structure. Then $\overline{A} \models \mathrm{Subf}^{v_i}_{\nu}\varphi[a]$ iff $\overline{A} \models \varphi\left[a^i_{\nu^{\overline{A}}(a)}\right]$.*

Proof. By induction on φ. For φ a formula $\sigma = \tau$, let ρ and η be obtained from σ and τ by replacing all occurrences of v_i by ν. Then by Lemma 4.5,

$$\begin{aligned}
\overline{A} \models \mathrm{Subf}^{v_i}_{\nu}\varphi[a] \quad &\text{iff} \quad \overline{A} \models (\rho = \eta)[a] \\
&\text{iff} \quad \rho^{\overline{A}}(a) = \eta^{\overline{A}}(a) \\
&\text{iff} \quad \sigma^{\overline{A}}\left(a^i_{\nu^{\overline{A}}(a)}\right) = \tau^{\overline{A}}\left(a^i_{\nu^{\overline{A}}(a)}\right) \\
&\text{iff} \quad \overline{A} \models (\sigma = \tau)\left(a^i_{\nu^{\overline{A}}(a)}\right) \\
&\text{iff} \quad \overline{A} \models \varphi\left(a^i_{\nu^{\overline{A}}(a)}\right).
\end{aligned}$$

For φ a formula $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$, let η_i be obtained from σ_i by replacing all occurrences of v_i by ν. Then

$$\begin{aligned}
\overline{A} \models \mathrm{Subf}^{v_i}_{\nu}\varphi[a] \quad &\text{iff} \quad \overline{A} \models (\mathbf{R}\eta_0 \ldots \eta_{m-1})[a] \\
&\text{iff} \quad \langle \eta_0^{\overline{A}}(a), \ldots, \eta_{m-1}^{\overline{A}}(a) \in \mathbf{R}^{\overline{A}} \\
&\text{iff} \quad \left\langle \sigma_0^{\overline{A}}\left(a^i_{\nu^{\overline{A}}(a)}\right), \ldots \sigma_{m-1}^{\overline{A}}\left(a^i_{\nu^{\overline{A}}(a)}\right)\right\rangle \in \mathbf{R}^{\overline{A}} \\
&\text{iff} \quad \overline{A} \models (\mathbf{R}\sigma_0 \ldots \sigma_{m-1})\left[a^i_{\nu^{\overline{A}}(a)}\right] \\
&\text{iff} \quad \overline{A} \models \varphi\left[a^i_{\nu^{\overline{A}}(a)}\right].
\end{aligned}$$

Now suppose inductively that φ is $\neg\psi$. Then

$$\begin{aligned}
\overline{A} \models \mathrm{Subf}^{v_1}_{\nu}\varphi[a] \quad &\text{iff} \quad \overline{A} \models (\neg\mathrm{Subf}^{v_1}_{\nu}\psi)\,[a] \\
&\text{iff} \quad \text{not}\left(\overline{A} \models (\mathrm{Subf}^{v_1}_{\nu}\psi)\right)[a] \\
&\text{iff} \quad \text{not}\left(\overline{A} \models \psi\left[a^i_{\nu^{\overline{A}}(a)}\right]\right) \\
&\text{iff} \quad \overline{A} \models \varphi\left[a^i_{\nu^{\overline{A}}(a)}\right].
\end{aligned}$$

Suppose inductively that φ is $\psi \to \chi$. Then

$$\begin{aligned}
\overline{A} \models \mathrm{Subf}^{v_1}_{\nu}\varphi[a] \quad &\text{iff} \quad \text{not}\left(\overline{A} \models \mathrm{Subf}^{v_1}_{\nu}\psi[a]\right) \text{ or } \overline{A} \models \mathrm{Subf}^{v_1}_{\nu}\chi[a] \\
&\text{iff} \quad \text{not}\left(\overline{A} \models \psi\left[a^i_{\nu^{\overline{A}}(a)}\right]\right) \text{ or } \overline{A} \models \chi\left[a^i_{\nu^{\overline{A}}(a)}\right] \\
&\text{iff} \quad \overline{A} \models \varphi\left[a^i_{\nu^{\overline{A}}(a)}\right].
\end{aligned}$$

Finally, suppose inductively that φ is $\forall v_k\psi$. Now if v_i does not occur free in φ, then $\mathrm{Subf}^{v_i}_{\nu}\varphi$ is just φ itself, and $\overline{A} \models \varphi[a]$ iff $\overline{A} \models \varphi[a^i_{\nu^{\overline{A}}(a)}]$ by Lemma 4.4. Hence we may assume that v_i occurs free in φ.

If $k = i$, then $\mathrm{Subf}^{v_i}_{\nu}\varphi$ is φ, and by Lemma 4.4, $\overline{A} \models \varphi\left[a^i_{\nu^{\overline{A}}(a)}\right]$ iff $\overline{A} \models \varphi[a]$; so the theorem holds in this case. Now suppose that $k \neq i$. Then $\mathrm{Subf}^{v_i}_{\nu}\varphi$ is $\forall v_k\mathrm{Subf}^{v_i}_{\nu}\psi$. Suppose that $\overline{A} \models \mathrm{Subf}^{v_i}_{\nu}\varphi[a]$. Take any $u \in A$. Then $\overline{A} \models \mathrm{Subf}^{v_i}_{\nu}\psi[a^k_u]$. Now no free occurrence of v_i in ψ is within a subformula of the form $\forall v_s\mu$ with v_s occurring in ν. Hence by the inductive hypothesis $\overline{A} \models \psi\left[(a^k_u)^i_{\nu^{\overline{A}}(a^k_u)}\right]$. Now since φ is $\forall v_k\psi$ and v_i occurs free in φ, the assumption of the lemma says that v_k does not occur in ν. Hence $\nu^{\overline{A}}(a) = \nu^{\overline{A}}(a^k_u)$ by Proposition 2.4. Hence $\overline{A} \models \psi\left[(a^k_u)^i_{\nu^{\overline{A}}(a)}\right]$. Since $(a^k_u)^i_{\nu^{\overline{A}}(a)} = \left(a^i_{\nu^{\overline{A}}(a)}\right)^k_u$, it follows that $\overline{A} \models \varphi\left[a^i_{\nu^{\overline{A}}(a)}\right]$.

Conversely, suppose that $\overline{A} \models \varphi\left[a^i_{\nu^{\overline{A}}(a)}\right]$. Take any $u \in A$. Then $\overline{A} \models \psi\left[\left(a^i_{\nu^{\overline{A}}(a)}\right)^k_u\right]$. Since $\left(a^i_{\nu^{\overline{A}}(a)}\right)^k_u = (a^k_u)^i_{\nu^{\overline{A}}(a)}$, and $\nu^{\overline{A}}(a) = \nu^{\overline{A}}(a^k_u)$ (see above), by the inductive hypothesis we get $\overline{A} \models \mathrm{Subf}^{v_i}_{\nu}\psi[a^k_u]$. It follows that $\overline{A} \models \mathrm{Subf}^{v_i}_{\nu}\varphi[a]$. $\square$

A set Γ of sentences is *complete* iff for every sentence φ, $\Gamma \vdash \varphi$ or $\Gamma \vdash \neg\varphi$. Γ is *rich* iff for every sentence of the form $\exists v_i\varphi$ there is an individual constant $\mathbf{c}$ such that $\Gamma \vdash \exists v_i\varphi \to \mathrm{Subf}^{v_i}_{\mathbf{c}}(\varphi)$.

The main lemma for the completeness proof is as follows.

Lemma 4.7. *If Γ is a complete, rich, consistent set of sentences, then Γ has a model.*

Proof. Let $B = \{\sigma : \sigma \text{ is a term in which no variable occurs}\}$. We define $\equiv$ to be the set

$$\{(\sigma, \tau) : \sigma, \tau \in B \text{ and } \Gamma \vdash \sigma = \tau\}.$$

By Propositions 3.4–3.6, $\equiv$ is an equivalence relation on B. Let π be the function which assigns to each $\sigma \in B$ the equivalence class $[\sigma]_\equiv$, and let A be the set of all equivalence classes.

We recall some basic facts about equivalence relations. An *equivalence relation* on a set M is a set R of ordered pairs (a, b) with $a, b \in M$ satisfying the following conditions:

(reflexivity) $(a, a) \in R$ for all $a \in M$.

(symmetry) For all $(a, b) \in R$ we have $(b, a) \in R$.

(transitivity) For all a, b, c, if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.

Given an equivalence relation R on a set M, for each $a \in M$ we let $[a]_R = \{b \in M : (a, b) \in R\}$; this is the *equivalence class* of a. Some basic facts:

(a) For any $a, b \in M$, $(a, b) \in R$ iff $[a]_R = [b]_R$.

Proof. $\Rightarrow$: suppose that $(a, b) \in R$. Suppose also that $x \in [a]_R$. Thus $(a, x) \in R$. Since R is symmetric, $(b, a) \in R$. Since R is transitive, $(b, x) \in R$. Hence $x \in [b]_R$. This proves that $[a]_R \subseteq [b]_R$. Suppose that $x \in [b]_R$. Thus $(b, x) \in R$. Since also $(a, b) \in R$, by transitivity we get $(a, x) \in R$. So $x \in [a]_R$. This proves that $[b]_R \subseteq [a]_R$, and completes the proof that $[a]_R = [b]_R$.

$\Leftarrow$: Assume that $[a]_R = [b]_R$. Since R is reflexive on M, we have $(b, b) \in R$, and hence $b \in [b]_R$. Now $[a]_R = [b]_R$, so $b \in [a]_R$. Hence $(a, b) \in R$. □

(b) For any $a, b \in M$, $[a]_R = [b]_R$ or $[a]_R \cap [b]_R = \emptyset$.

Proof. Suppose that $[a]_R \cap [b]_R \neq \emptyset$; say $x \in [a]_R \cap [b]_R$. Thus $(a, x) \in R$ and $(b, x) \in R$. By symmetry, $(x, b) \in R$. By transitivity, $(a, b) \in R$. By (a), $[a]_R = [b]_R$. □

We are now going to define a structure with universe A. If $\mathbf{k}$ is an individual constant, let $\mathbf{k}^{\overline{A}} = [\mathbf{k}]_\equiv$.

(1) If $\mathbf{F}$ is an m-ary function symbol and $\sigma_0, \ldots, \sigma_{m-1}, \tau_0, \ldots, \tau_{m-1}$ are members of B such that $\sigma_i \equiv \tau_i$ for all $i < m$, then $\mathbf{F}\sigma_0 \ldots \sigma_{m-1} \equiv \mathbf{F}\tau_0 \ldots \tau_{m-1}$.

In fact, the hypothesis implies that $\Gamma \vdash \sigma_i = \tau_i$ for all $i < m$. Now we claim

(2) $\mathbf{F}\sigma_0 \ldots \sigma_{m-1} \equiv \mathbf{F}\sigma_0 \ldots \sigma_{m-i}\tau_{m-i+1} \ldots \tau_{m-1}$ for every positive integer $i \leq m + 1$.

We prove (2) by induction on i. For $i = 1$ the statement is $\mathbf{F}\sigma_0 \ldots \sigma_{m-1} \equiv \mathbf{F}\sigma_0 \ldots \sigma_{m-1}$, which holds by Proposition 3.4. Now assume that $1 \leq i \leq m$ and $\mathbf{F}\sigma_0 \ldots \sigma_{m-1} \equiv \mathbf{F}\sigma_0 \ldots \sigma_{m-i}\tau_{m-i+1} \ldots \tau_{m-1}$. By logical axiom (L7) we also have

$$\mathbf{F}\sigma_0 \ldots \sigma_{m-i}\tau_{m-i+1} \ldots \tau_{m-1} \equiv \mathbf{F}\sigma_0 \ldots \sigma_{m-i-1}\tau_{m-i} \ldots \tau_{m-1},$$

so Proposition 3.6 yields

$$\mathbf{F}\sigma_0 \ldots \sigma_{m-1} \equiv \mathbf{F}\sigma_0 \ldots \sigma_{m-i-1}\tau_{m-i} \ldots \tau_{m-1}.$$

This finishes the inductive proof of (2). The case $i = m + 1$ in (2) gives (1).

(3) If $\mathbf{F}$ is an m-ary function symbol, then there is a function $\mathbf{F}^{\overline{A}}$ mapping m-tuples of members of A into A, such that for any $\sigma_0, \ldots, \sigma_{m-1} \in B$, $\mathbf{F}^{\overline{A}}([\sigma_0]_\equiv, \ldots, [\sigma_{m-1}]_\equiv) = [\mathbf{F}\sigma_0 \ldots \sigma_{m-1}]_\equiv$.

In fact, we can define $\mathbf{F}^{\overline{A}}$ as a set of ordered pairs:

$$\begin{aligned}\mathbf{F}^{\overline{A}} = \{(x, y) :& \text{there are } \sigma_0, \ldots, \sigma_{m-1} \in B \text{ such that} \\ & x = \langle [\sigma_0]_\equiv, \ldots, [\sigma_{m-1}]_\equiv \rangle \text{ and } y = [\mathbf{F}\sigma_0 \ldots \sigma_{m-1}]_\equiv\}\end{aligned}$$

Then $\mathbf{F}^{\overline{A}}$ is a function. For, suppose that $(x, y), (x, z) \in \mathbf{F}^{\overline{A}}$. Accordingly choose elements $\sigma_0, \ldots \sigma_{m-1} \in B$ and $\tau_0, \ldots \tau_{m-1} \in B$ such that $x = \langle [\sigma_0]_\equiv, \ldots, [\sigma_{m-1}]_\equiv \rangle = \langle [\tau_0]_\equiv, \ldots, [\tau_{m-1}]_\equiv$, $y = [\mathbf{F}\sigma_0 \ldots \sigma_{m-1}]_\equiv$, and $z = [\mathbf{F}\tau_0 \ldots \varphi_{m-1}]_\equiv$. Thus for any $i < m$ we have $[\sigma_i]_\equiv = [\tau_i]_\equiv$, hence $\sigma_i \equiv \tau_i$. From (1) it then follows that $\mathbf{F}\sigma_0 \ldots \sigma_{m-1} \equiv \mathbf{F}\tau_0 \ldots \varphi_{m-1}$, hence $y = z$. So $\mathbf{F}^{\overline{A}}$ is a function. Clearly then (3) holds.

For $\mathbf{R}$ an m-ary relation symbol we define

$$\mathbf{R}^{\overline{A}} = \{x : \exists \sigma_0, \ldots \sigma_{m-1} \in B[x = \langle [\sigma_0]_\equiv, \ldots, [\sigma_{m-1}]_\equiv \rangle \text{ and } \Gamma \vdash \mathbf{R}\sigma_0 \ldots \sigma_{m-1}]\}.$$

(4) If $\mathbf{R}$ is an m-ary relation symbol and $\sigma_i \equiv \tau_i$ for all $i < m$, then for any positive integer $i < m + 1$, $\vdash \mathbf{R}\sigma_0 \ldots \sigma_{m-1} \leftrightarrow \mathbf{R}\sigma_0 \ldots \sigma_{m-i}\tau_{m-i+1} \ldots \tau_{m-1}$.

We prove (4) by induction on i. For $i = 1$ the conclusion is $\vdash \mathbf{R}\sigma_0 \ldots \sigma_{m-1} \leftrightarrow \mathbf{R}\sigma_0 \ldots \sigma_{m-1}$, so this holds by a tautology. Now assume our statement for $i < m$. Then by logical axiom (L8),

$$\vdash \mathbf{R}\sigma_0 \ldots \sigma_{m-i}\tau_{m-i+1} \ldots \tau_{m-1} \rightarrow \mathbf{R}\sigma_0 \ldots \sigma_{m-i-1}\tau_{m-i} \ldots \tau_{m-1};$$

using Proposition 3.5 we can easily get

$$\vdash \mathbf{R}\sigma_0 \ldots \sigma_{m-i}\tau_{m-i+1} \ldots \tau_{m-1} \leftrightarrow \mathbf{R}\sigma_0 \ldots \sigma_{m-i-1}\tau_{m-i} \ldots \tau_{m-1}.$$

This finishes the inductive proof of (4). Now we have

(5) If $\mathbf{R}$ is an m-ary relation symbol and $\sigma_0, \ldots, \sigma_{m-1} \in B$, then $\langle [\sigma_0]_\equiv, \ldots, [\sigma_{m-1}]_\equiv \rangle \in \mathbf{R}^{\overline{A}}$ iff $\Gamma \vdash \mathbf{R}\sigma_0 \ldots \sigma_{m-1}$.

In fact, $\Leftarrow$ follows from the definition. Now suppose that $\langle [\sigma_0]_\equiv, \ldots, [\sigma_{m-1}]_\equiv \rangle \in \mathbf{R}^{\overline{A}}$. Then by definition there exist $\tau_0, \ldots, \tau_{m-1} \in B$ such that

$$\langle [\sigma_0]_\equiv, \ldots, [\sigma_{m-1}]_\equiv \rangle = \langle [\tau_0]_\equiv, \ldots, [\tau_{m-1}]_\equiv \rangle \text{ and } \Gamma \vdash \mathbf{R}\tau_0 \ldots \tau_{m-1}.$$

Thus $[\sigma_i]_\equiv = [\tau_i]_\equiv$, hence $\sigma_i \equiv \tau_i$, hence $\Gamma \vdash \sigma_i = \tau_i$, for each $i < m$. Now by (4), $\vdash \bigwedge_{i<m}(\sigma_i = \tau_i) \rightarrow (\mathbf{R}\sigma_0 \ldots \sigma_{m-1} \leftrightarrow \mathbf{R}\tau_0 \ldots \tau_{m-1})$. It follows that $\Gamma \vdash \mathbf{R}\sigma_0 \ldots \sigma_{m-1}$, as desired; so (5) holds.

(6) For any $\sigma \in B$ we have $\sigma^{\overline{A}} = [\sigma]_\equiv$.

We prove (6) by induction on σ. If σ is an individual constant $\mathbf{k}$, then by definition $\mathbf{k}^{\overline{A}} = [\mathbf{k}]_{\equiv}$. Now suppose that (6) is true for $\tau_0, \ldots, \tau_{m-1} \in B$ and σ is $\mathbf{F}\tau_0 \ldots \tau_{m-1}$. Then

$$\sigma^{\overline{A}} = \mathbf{F}^{\overline{A}}([\tau_0]_{\equiv}, \ldots, [\tau_{m-1}]_{\equiv}) = [\mathbf{F}\tau_0 \ldots \tau_{m-1}]_{\equiv} = [\sigma]_{\equiv},$$

proving (6).

The following claim is the heart of the proof.

(7) For any sentence φ, $\Gamma \vdash \varphi$ iff $\overline{A} \models \varphi$.

We prove (7) by induction on the number m of the symbols $=$, relation symbols, $\neg$, $\rightarrow$, and $\forall$ in φ. For $m = 1$, φ is atomic, and we have

$$\begin{array}{rll}
\Gamma \vdash \sigma = \tau & \text{iff} & \sigma \equiv \tau \\
& \text{iff} & [\sigma]_{\equiv} = [\tau]_{\equiv} \\
& \text{iff} & \sigma^{\overline{A}} = \tau^{\overline{A}} \quad \text{by (6)} \\
& \text{iff} & \overline{A} \models \sigma = \tau; \\
\Gamma \vdash \mathbf{R}\sigma_0 \ldots \sigma_{m-1} & \text{iff} & \langle [\sigma_0]_{\equiv}, \ldots, [\sigma_{m-1}]_{\equiv} \rangle \in \mathbf{R}^{\overline{A}} \quad \text{by (5)} \\
& \text{iff} & \langle \sigma_0^{\overline{A}}, \ldots, \sigma_{m-1}^{\overline{A}} \rangle \in \mathbf{R}^{\overline{A}} \quad \text{by (6)} \\
& \text{iff} & \overline{A} \models \mathbf{R}\sigma_0 \ldots \sigma_{m-1}.
\end{array}$$

Now we take the inductive steps.

$$\begin{array}{rll}
\Gamma \vdash \neg\psi & \text{iff} & \text{not}(\Gamma \vdash \psi) \\
& \text{iff} & \text{not}(\overline{A} \models \psi) \\
& \text{iff} & \overline{A} \models \neg\psi; \\
\Gamma \vdash \psi \rightarrow \chi & \text{iff} & \text{not}(\Gamma \vdash \psi) \text{ or } \Gamma \vdash \chi \\
& \text{iff} & \text{not}(\overline{A} \models \psi) \text{ or } \overline{A} \models \chi \\
& \text{iff} & \overline{A} \models \psi \rightarrow \chi.
\end{array}$$

Finally, suppose that φ is $\forall v_i \psi$. First suppose that $\Gamma \vdash \varphi$. We want to show that $\overline{A} \models \varphi$, so take any $\sigma \in B$ and let $u = [\sigma]_{\equiv}$; we want to show that $\overline{A} \models \psi[a_u^i]$, where $a : \omega \rightarrow A$. Let χ be the sentence $\mathrm{Subf}_{\sigma}^{v_i}\psi$. Then by Theorem 3.25 we have $\Gamma \vdash \chi$, and hence by the inductive assumption $\overline{A} \models \chi$. By (6) we have $\sigma^{\overline{A}} = [\sigma]_{\equiv}$. Hence by Lemma 4.6 we get $\overline{A} \models \psi[a_u^i]$.

Second suppose that $\Gamma \not\vdash \varphi$. Then by completeness $\Gamma \vdash \neg\varphi$, and hence $\Gamma \vdash \exists v_i \neg\psi$. Hence by richness there is an individual constant $\mathbf{c}$ such that $\Gamma \vdash \exists v_i \neg\psi \rightarrow \mathrm{Subf}_{\mathbf{c}}^{v_i}(\neg\psi)$, hence $\Gamma \vdash \neg\mathrm{Subf}_{\mathbf{c}}^{v_i}\psi$, and so $\Gamma \not\vdash \mathrm{Subf}_{\mathbf{c}}^{v_i}\psi$. By the inductive assumption, $\overline{A} \not\models \mathrm{Subf}_{\mathbf{c}}^{v_i}\psi$, and so by (6) and Lemma 4.6, $\overline{A} \not\models \psi[a_u^i]$, where $a : \omega \rightarrow A$ and $u = [\mathbf{c}]_{\equiv}$. So $\overline{A} \not\models \varphi$.

This finishes the proof of (7). Applying (7) to members φ of Γ we see that $\overline{A}$ is a model of Γ. $\square$

The following rather technical lemma will be used in a few places below.

Lemma 4.8. *Suppose that Γ is a set of formulas in $\mathscr{L}$, and $\langle\psi_0,\ldots,\psi_{m-1}\rangle$ is a Γ-proof in $\mathscr{L}$. Suppose that C is a set of individual constants such that no member of C occurs in any member of Γ. Let v_j be a variable not occurring in any formula ψ_k, and for each k let ψ'_k be obtained from ψ_k by replacing each member of C by v_j. Similarly, for each term σ let σ' be obtained from σ by replacing each member of C by v_j. Then $\langle\psi'_0,\psi'_1,\ldots,\psi'_{m-1}\rangle$ is a Γ-proof in $\mathscr{L}$.*

Proof. Assume the hypotheses. We need to show that if ψ_k is a logical axiom, then so is ψ'_k. We consider the possibilities one by one:

$$
\begin{aligned}
&(\varphi\to(\psi\to\varphi))' \text{ is } \varphi'\to(\psi'\to\varphi');\\
&((\varphi\to(\psi\to\chi)\to((\varphi\to\psi)\to(\varphi\to\chi)))' \text{ is}\\
&\quad(\varphi'\to(\psi'\to\chi')\to((\varphi'\to\psi')\to(\varphi'\to\chi'));\\
&((\neg\varphi\to\neg\psi)\to(\psi\to\varphi))' \text{ is } (\neg\varphi'\to\neg\psi')\to(\psi'\to\varphi');\\
&(\forall v_k(\varphi\to\psi)\to(\forall v_k\varphi\to\forall v_k\psi))' \text{ is } \forall v_k(\varphi'\to\psi')\to(\forall v_k\varphi'\to\forall v_k\psi');\\
&(\varphi\to\forall v_k\varphi)' \text{ is } \varphi'\to\forall v_k\varphi' \qquad\qquad \text{if } v_k \text{ does not occur in } \varphi;\\
&(\exists v_k(v_k=\sigma))' \text{ is } \exists v_k(v_k=\sigma') \qquad\qquad \text{if } v_k \text{ does not occur in } \sigma;\\
&(\sigma=\tau\to(\sigma=\rho\to\tau=\rho))' \text{ is } (\sigma'=\tau'\to(\sigma'=\rho'\to\tau'=\rho');\\
&(\sigma=\tau\to(\rho=\sigma\to\rho=\tau))' \text{ is } (\sigma'=\tau'\to(\rho'=\sigma'\to\rho'=\tau');\\
&(\sigma=\tau\to\mathbf{F}\xi_0\ldots\xi_{i-1}\sigma\xi_{i+1}\ldots\xi_{m-1}=\mathbf{F}\xi_0\ldots\xi_{i-1}\tau\xi_{i+1}\ldots\xi_{m-1})' \text{ is}\\
&\quad\sigma'=\tau'\to\mathbf{F}\xi'_0\ldots\xi'_{i-1}\sigma'\xi'_{i+1}\ldots\xi'_{m-1}=\mathbf{F}\xi'_0\ldots\xi'_{i-1}\tau'\xi'_{i+1}\ldots\xi'_{m-1};\\
&(\sigma=\tau\to(\mathbf{R}\xi_0\ldots\xi_{i-1}\sigma\xi_{i+1}\ldots\xi_{m-1}\to\mathbf{R}\xi_0\ldots\xi_{i-1}\tau\xi_{i+1}\ldots\xi_{m-1}))' \text{ is}\\
&\quad\sigma'=\tau'\to(\mathbf{R}\xi'_0\ldots\xi'_{i-1}\sigma'\xi'_{i+1}\ldots\xi'_{m-1}\to\mathbf{R}\xi'_0\ldots\xi'_{i-1}\tau'\xi'_{i+1}\ldots\xi'_{m-1}).
\end{aligned}
$$

Now back to our claim that $\langle\psi'_0,\ldots,\psi'_{m-1}\rangle$ is a Γ-proof. If ψ_k is a logical axiom, then by the above, ψ'_k is a logical axiom. If $\psi_k\in\Gamma$, then no member of C occurs in ψ_k, and hence $\psi'_k=\psi_k$. Suppose that $s,t<k$ and ψ_s is $\psi_t\to\psi_k$. Then ψ'_s is $\psi'_t\to\psi'_k$. If $s<k$ and $t\in\omega$, and ψ_k is $\forall v_t\psi_s$, then ψ'_k is $\forall v_t\psi'_s$. Thus our claim holds. $\square$

Lemma 4.9. *Suppose that* $\mathbf{c}$ *is an individual constant not occurring in any formula in $\Gamma\cup\{\varphi\}$. Suppose that $\Gamma\vdash\mathrm{Subf}^{v_i}_{\mathbf{c}}\varphi$. Then $\Gamma\vdash\varphi$.*

Proof. Let $\langle\psi_0,\ldots,\psi_{m-1}\rangle$ be a Γ-proof with $\psi_j=\mathrm{Subf}^{v_i}_{\mathbf{c}}\varphi$. Let v_j and the sequence $\langle\psi'_0,\ldots,\psi'_{m-1}\rangle$ be as in Lemma 4.8, with $C=\{\mathbf{c}\}$. Then by Lemma 4.8, $\langle\psi'_0,\ldots,\psi'_{m-1}\rangle$ is a Γ-proof. Note that ψ'_j is $\mathrm{Subf}^{v_i}_{v_j}\varphi$. Thus $\Gamma\vdash\mathrm{Subf}^{v_i}_{v_j}\varphi$. Hence $\Gamma\vdash\forall v_j\mathrm{Subf}^{v_i}_{v_j}\varphi$, and so by Theorem 3.25, $\Gamma\vdash\varphi$. $\square$

A first-order language $\mathscr{L}$ is *finite* iff $\mathscr{L}$ has only finitely many non-logical symbols. Note that in a finite language there are infinitely many integers which are not symbols of the language. We prove the main completeness theorem only for finite languages. This is not an essential restriction. With an expanded notion of first-order language the present proof still goes through.

Lemma 4.10. *Let $\mathscr{L}$ be a finite first-order language. Let $\mathscr{L}'$ extend $\mathscr{L}$ by adding individual constants $\mathbf{c}_0, \mathbf{c}_1, \ldots$. Suppose that Γ is a consistent set of formulas in $\mathscr{L}$. Then it is also consistent as a set of formulas in $\mathscr{L}'$.*

Suppose not. Let $\langle \psi_0, \ldots, \psi_{m-1} \rangle$ be a Γ-proof in the $\mathscr{L}'$ sense with ψ_i the formula $\neg(v_0 = v_0)$. Let C be the set of all constants $\mathbf{c}_i$ which appear in some formula ψ_k. Let v_j and $\langle \psi'_0, \psi'_1, \ldots, \psi'_{m-1} \rangle$ be as in Lemma 4.8. Then by Lemma 4.8, $\langle \psi'_0, \psi'_1, \ldots, \psi'_{m-1} \rangle$ is a Γ-proof. Clearly each ψ'_k is a $\mathscr{L}$ formula. Note that $\psi'_i = \psi_i = \neg(v_0 = v_0)$. So Γ is inconsistent in $\mathscr{L}$, contradiction. $\square$

Lemma 4.11. *Let $\mathscr{L}$ be a finite first-order language. Let $\mathscr{L}'$ extend $\mathscr{L}$ by adding individual constants $\mathbf{c}_0, \mathbf{c}_1, \ldots$.*

Then there is an enumeration $\langle \varphi_0, \varphi_1, \ldots \rangle$ of all of the sentences of $\mathscr{L}'$, and also an enumeration $\langle \psi_0, \psi_1, \ldots \rangle$ of all the sentences of $\mathscr{L}'$ of the form $\exists v_i \chi$.

Proof. Recall that a formula is a certain finite sequence of positive integers. First we describe how to list all finite sequences of positive integers. Given positive integers m and n, we can list all sequences of members of $\{1, \ldots, m\}$ of length n by just listing them in dictionary order. For example, with $m = 3$ and $n = 2$ our list is

$$\langle 1,1 \rangle$$
$$\langle 1,2 \rangle$$
$$\langle 1,3 \rangle$$
$$\langle 2,1 \rangle$$
$$\langle 2,2 \rangle$$
$$\langle 2,3 \rangle$$
$$\langle 3,1 \rangle$$
$$\langle 3,2 \rangle$$
$$\langle 3,3 \rangle$$

To list all finite sequences, we then do the following:

(1) List all sequences of members of $\{1\}$ of length 1. (There is only one such, namely $\langle 1 \rangle$.)

(2) List all sequences of members of $\{1, 2\}$ of length 1 or 2. Here they are:

$$\langle 1 \rangle$$
$$\langle 2 \rangle$$
$$\langle 1,1 \rangle$$
$$\langle 1,2 \rangle$$
$$\langle 2,1 \rangle$$
$$\langle 2,2 \rangle$$

(3) List all sequences of members of $\{1, 2, 3\}$ of length 1,2, or 3.

(4) General step: list all members of $\{1, \ldots, m\}$ of length $1, 2, \ldots m$.

Let $\langle \psi_0, \psi_1, \ldots \rangle$ be the listing described. Now we go through this list and select the ones which are sentences of $\mathscr{L}'$, giving the desired list $\langle \varphi_0, \varphi_1, \ldots \rangle$. Similarly for the list $\langle \psi_0, \psi_1, \ldots \rangle$ of all sentences of the form $\exists v_i \chi$. $\square$

Lemma 4.12. *Let $\mathscr{L}$ be a finite first-order language. Let $\mathscr{L}'$ extend $\mathscr{L}$ by adding individual constants $\mathbf{c}_0, \mathbf{c}_1, \ldots$.*

Suppose that Γ is a consistent set of sentences of $\mathscr{L}'$. Then there is a rich consistent set Δ of sentences with $\Gamma \subseteq \Delta$.

Proof. By Lemma 4.11, let $\langle \psi_0, \psi_1, \ldots \rangle$ enumerate all the sentences of $\mathscr{L}'$ of the form $\exists v_i \chi$; say that ψ_k is $\exists v_{t(k)} \psi'_k$ for all $k \in \omega$. Now we define an increasing sequence $\langle j(k) : k \in \omega \rangle$ by recursion. Suppose that $j(k)$ has been defined for all $k < l$. Let $j(l)$ be the smallest natural number not in the set

$$\{j(k) : k < l\} \cup \{s : \mathbf{c}_s \text{ occurs in some formula } \psi_k \text{ with } k \leq l\}.$$

Again we justify this definition. Let M be the set of all functions f defined on some set $m' = \{i \in \omega : i < m\}$ with $m \in \omega$ such that for all $l < m$, $f(l)$ is the smallest number not in the set

$$\{f(k) : k < l\} \cup \{s : \mathbf{c}_s \text{ occurs in some formula } \psi_k \text{ with } k \leq l\}.$$

(1) If $f, g \in M$, say with domains s', t' respectively, with $s \leq t$, then $f(k) = g(k)$ for all $k < s$.

We prove this by complete induction on k. Assume that it is true for all $k' < k$. Then $f(k)$ is the smallest number not in the set

$$\begin{aligned}&\{f(k'') : k'' < k\} \cup \{u : \mathbf{c}_u \text{ occurs in some formula } \psi''_k \text{ with } k'' \leq k\} = \\ &\{g(k'') : k'' < k\} \cup \{u : \mathbf{c}_u \text{ occurs in some formula } \psi''_k \text{ with } k'' \leq k\},\end{aligned}$$

and this is the same as $g(k)$. So (1) holds.

(2) For each $m \in \omega$ there is a member of M with domain m'.

We prove this by induction on m. For $m = 0$ we take the empty function. Assume that $f \in M$ has domain m'. Define the extension g of f with domain $(m+1)'$ by letting $g(m)$ be the smallest number not in the set

$$\{f(k) : k < m\} \cup \{s : \mathbf{c}_s \text{ occurs in some formula } \psi_k \text{ with } k \leq m\}.$$

This proves (2).

Now we define $f(l)$ to be $g(l)$ for any $g \in M$ with l in the domain of g.

For each $l \in \omega$ let

$$\Theta_l = \Gamma \cup \{\exists v_{t(k)} \psi'_k \rightarrow \mathrm{Subf}^{v_{t(k)}}_{\mathbf{c}_{j(k)}} \psi'_k : k < l\}.$$

We claim that each set Θ_l is consistent. We prove this by induction on l. Note that $\Theta_0 = \Gamma$, which is given as consistent. Now suppose that we have shown that Θ_l is consistent. Now $\Theta_{l+1} = \Theta_l \cup \{\exists v_{t(l)}\psi'_l \to \mathrm{Subf}^{v_{t(l)}}_{\mathbf{c}_{j(l)}}\psi'_l\}$. Assume that Θ_{l+1} is inconsistent. Then $\Theta_{l+1} \vdash \neg(v_0 = v_0)$. By the deduction theorem 4.2, it follows that

$$\Theta_l \vdash (\exists v_{t(l)}\psi'_l \to \mathrm{Subf}^{v_{t(l)}}_{\mathbf{c}_{j(l)}}\psi'_l) \to \neg(v_0 = v_0),$$

hence easily

$$\Theta_l \vdash \neg(\exists v_{t(l)}\psi'_l \to \mathrm{Subf}^{v_{t(l)}}_{\mathbf{c}_{j(l)}}\psi'_l),$$

so that using tautologies

$$\begin{aligned} &\Theta_l \vdash \exists v_{t(l)}\psi'_l \quad \text{and} \\ &\Theta_l \vdash \neg\mathrm{Subf}^{v_{t(l)}}_{\mathbf{c}_{j(l)}}\psi'_l. \end{aligned}$$

Now by the definition of the sequence $\langle j(k) : k \in \omega\rangle$, it follows that $\mathbf{c}_{j(l)}$ does not occur in any formula in $\Theta_l \cup \{\psi'_l\}$. Hence by Lemma 4.9 we get $\Theta_l \vdash \neg\psi'_l$, and so $\Theta_l \vdash \forall v_{t(l)}\neg\psi'_l$. But we also have $\Theta_l \vdash \exists v_{t(l)}\psi'_l$, so that Θ_l is inconsistent, contradiction.

Now let $\Delta = \bigcup_{l\in\omega}\Theta_l$. We claim that Δ is consistent. Suppose not. Then $\Delta \vdash \neg(v_0 = v_0)$. Let $\langle\varphi_0, \ldots \varphi_{m-1}\rangle$ be a Δ-proof with $\varphi_i = \neg(v_0 = v_0)$. For each $k < m$ such that $\varphi_k \in \Delta$, choose $s(k) \in \omega$ such that $\varphi_k \in \Theta_{s(k)}$. Let l be such that $s(l)$ is largest among all $k < m$ such that $\varphi_k \in \Theta_{s(k)}$. Then $\langle\varphi_0, \ldots \varphi_{m-1}\rangle$ is a $\Theta_{s(l)}$-proof, and hence $\Theta_{s(l)}$ is inconsistent, contradiction.

Now clearly $\Gamma \subseteq \Delta$, since $\Theta_0 = \Gamma$. We claim that Δ is rich. For, let $\exists v_l\chi$ be a sentence. Say $\exists v_l\chi$ is ψ_m. Then $\exists v_l\chi$ is $\exists v_{t(m)}\psi'_m$, so that $l = t(m)$ and $c = \psi'_m$. Now the formula

$$\exists v_{t(m)}\psi'_m \to \mathrm{Subf}^{v_{t(m)}}_{\mathrm{c}_{j(m)}}\psi'_m$$

is a member of Θ_{m+1}, and hence is a member of Δ. This formula is $\exists v_l\chi \to \mathrm{Subf}^{v_l}_{\mathrm{c}_{j(m)}}\chi$. Hence Δ is rich. $\square$

Lemma 4.13. *Let $\mathscr{L}$ be a finite first-order language. Let $\mathscr{L}'$ extend $\mathscr{L}$ by adding individual constants $\mathbf{c}_0, \mathbf{c}_1, \ldots$.*

Suppose that Γ is a consistent set of sentences of $\mathscr{L}'$. Then there is a consistent complete set Δ of sentences with $\Gamma \subseteq \Delta$.

Proof. By Lemma 4.11, let $\langle\varphi_0, \varphi_1, \ldots\rangle$ be an enumeration of all the sentences of $\mathscr{L}'$. We now define by recursion sets Θ_i of sentences. Let $\Theta_0 = \Gamma$. Suppose that Θ_i has been defined so that it is consistent. If $\Theta_i \cup \{\varphi_i\}$ is consistent, let $\Theta_{i+1} = \Theta_i \cup \{\varphi_i\}$. Otherwise let $\Theta_{i+1} = \Theta_i \cup \{\neg\varphi_i\}$. We claim that in this otherwise case, still Θ_{i+1} is consistent. Suppose not. Then $\Theta_{i+1} \vdash \neg(v_0 = v_0)$, i.e., $\Theta_i \cup \{\neg\varphi_i\} \vdash \neg(v_0 = v_0)$. By the deduction theorem, $\Theta_i \vdash \neg\varphi_i \to \neg(v_0 = v_0)$, and then by Proposition 3.4 and a tautology $\Theta_i \vdash \varphi_i$. It follows that $\Theta_i \cup \{\varphi_i\}$ is consistent; otherwise $\Theta_i \cup \{\varphi_i\} \vdash \neg(v_0 = v_0)$, hence by the deduction theorem $\Theta_i \vdash \varphi_i \to \neg(v_0 = v_0)$, so by Proposition 4.3 and a tautology $\Theta_i \vdash \neg\varphi_i$. Together with $\Theta_i \vdash \varphi_i$, this shows that Θ_i is inconsistent, contradiction. So, $\Theta_i \cup \{\varphi_i\}$ is consistent. But this contradicts our "otherwise" condition. So, Θ_{i+1} is consistent. So the recursion continues.

Once more we give details on the recursion. Let M be the set of all functions f such that the domain of f is $m' = \{0, \ldots, m-1\}$ for some $m \in \omega$, and for all $i < m$ one of the following holds:

(1) $i = 0$ and $f(0) = \Gamma$.

(2) $i = j+1$ for some $j \in \omega$, $f(j)$ is a set of sentences, $f(j) \cup \{\varphi_j\}$ is consistent, and $f(i) = f(j) \cup \{\varphi_j\}$.

(3) $i = j+1$ for some $j \in \omega$, $f(j)$ is a set of sentences, $f(j) \cup \{\varphi_j\}$ is not consistent, and $f(i) = f(j) \cup \{\neg\varphi_j\}$.

We claim:

(4) If $f, g \in M$, say with domains m', n' respectively, with $m \le n$, then $f(i) = g(i)$ for all $i < m$.

We prove this by induction on i. For $i = 0$ we have $f(0) = \Gamma = g(0)$. Suppose it is true for i, with $i + 1 < m$. Then by the definition of M we have two cases.

Case 1. $f(i)$ is a set of sentences, $f(i) \cup \{\varphi_i\}$ is consistent, and $f(i+1) = f(i) \cup \{\varphi_i\}$. Since $f(i) = g(i)$ by the inductive assumption, the definition of M gives $g(i+1) = g(i) \cup \{\varphi_i\} = f(i) \cup \{\varphi_i\} = f(i+1)$.

Case 2. $f(i)$ is a set of sentences, $f(i) \cup \{\varphi_i\}$ is not consistent, and $f(i+1) = f(i) \cup \{\neg\varphi_i\}$. Since $f(i) = g(i)$ by the inductive assumption, the definition of M gives $g(i+1) = g(i) \cup \{\neg\varphi_i\} = f(i) \cup \{\neg\varphi_i\} = f(i+1)$.

This finishes the inductive proof of (4).

(5) For all $f \in M$ and all i in the domain of f, $f(i)$ is a set of sentences.

This is easily proven by induction on i.

(6) For each $m \in \omega$ there is an $f \in M$ with domain m'.

We prove (5) by inducation on m. For $m = 0$ we can let f be the empty function. Suppose $f \in M$ with the domain of f equal to m'. If $m = 0$ we can let g be the function with domain $\{0\}$ and $g(0) = \Gamma$. Assume that $m > 0$. By (5), $f(m-1)$ is a set of sentences. Then we define g to be the extension of f such that

$$g(m) = \begin{cases} f(m-1) \cup \{\varphi_{m-1}\} & \text{if this set is consistent,} \\ f(m-1) \cup \{\neg\varphi_{m-1}\} & \text{otherwise.} \end{cases}$$

Thus (6) holds.

Now we define $\Theta_i = f(i)$ for any $f \in M$ which has i in its domain. Then by (5), each Θ_i is a set of sentences, $\Theta_0 = \Gamma$, and

$$\Theta_{i+1} = \begin{cases} \Theta_i \cup \{\varphi_i\} & \text{if this set is consistent,} \\ \Theta_i \cup \{\neg\varphi_i\} & \text{otherwise.} \end{cases}$$

Now we show by induction that each Θ_i is consistent. Since $\Theta_0 = \Gamma$, Θ_0 is consistent by assumption. Now suppose that Θ_i is consistent. If $\Theta_i \cup \{\varphi_i\}$ is consistent, then $\Theta_{i+1} = \Theta_i \cup \{\varphi_i\}$ and hence Θ_{i+1} is consistent. Suppose that $\Theta_i \cup \{\varphi_i\}$ is not consistent. Then $\Theta_i \cup \{\varphi_i\} \vdash \neg(v_0 = v_0)$, and hence an easy argument which we have used before gives $\Theta_i \vdash \neg\varphi_i$. Now $\Theta_{i+1} = \Theta_i \cup \{\neg\varphi_i\}$, so if Θ_{i+1} is not consistent we easily get $\Theta_i \vdash \varphi_i$. Hence Θ_i is inconsistent, contradiction. This completes the inductive proof.

Now let $\Delta = \bigcup_{i\in\omega} \Theta_i$. Then Δ is consistent. In fact, suppose not. Then $\Delta \vdash \neg(v_0 = v_0)$. Let $\langle \psi_0, \ldots, \psi_{m-1}\rangle$ be a Δ-proof with $\psi_i = \neg(v_0 = v_0)$. Let $\langle \chi_0, \ldots, \chi_{n-1}\rangle$ enumerate all of the members of Δ which are in the proof. Say $\chi_j \in \Theta_{s(j)}$ for each $j < n$. Let t be maximum among all the $s(j)$ for $j < n$. Then each χ_k is in Θ_t, so that $\langle \psi_0, \ldots, \psi_{m-1}\rangle$ is a Θ_t-proof. It follows that Θ_t is inconsistent, contradiction.

So Δ is consistent. Since $\Theta_0 = \Gamma$, we have $\Gamma \subseteq \Delta$. Finally, Δ is complete, since every sentence is equal to some φ_i, and our construction assures that $\varphi_i \in \Delta$ or $\neg\varphi_i \in \Delta$. $\square$

Lemma 4.14. *Let $\mathscr{L}$ be a first-order language. Let $\mathscr{L}'$ extend $\mathscr{L}$ by adding new non-logical symbols Suppose that $\overline{M}$ is an $\mathscr{L}'$-structure, and $\overline{N}$ is the $\mathscr{L}$-structure obtained from $\overline{M}$ by removing the denotations of the new non-logical symbols. Suppose that φ is a formula of $\mathscr{L}$, and $a : \omega \to M$. Then $\overline{M} \models \varphi[a]$ iff $\overline{N} \models \varphi[a]$.*

Proof. First we prove the following similar statement for terms:

(1) If σ is a term of $\mathscr{L}$, then $\sigma^{\overline{M}}(a) = \sigma^{\overline{N}}(a)$.

We prove this by induction on σ:

$$\begin{aligned}
v_i^{\overline{M}}(a) &= a_i = v_i^{\overline{N}}(a);\\
\mathbf{k}^{\overline{M}}(a) &= \mathbf{k}^{\overline{M}} = \mathbf{k}^{\overline{N}} = \mathbf{k}^{\overline{N}}(a) \quad \text{for } \mathbf{k} \text{ an individual constant of } \mathscr{L}\\
(\mathbf{F}\sigma_0 \ldots \sigma_{m-1})^{\overline{M}}(a) &= \mathbf{F}^{\overline{M}}(\sigma_0^{\overline{M}}(a), \ldots \sigma_{m-1}^{\overline{M}}(a))\\
&= \mathbf{F}^{\overline{N}}(\sigma_0^{\overline{N}}(a), \ldots \sigma_{m-1}^{\overline{N}}(a))\\
&= (\mathbf{F}\sigma_0 \ldots \sigma_{m-1})^{\overline{N}}(a).
\end{aligned}$$

Here $\mathbf{F}$ is a function symbol of $\mathscr{L}$. Thus (1) holds.

Now we prove the lemma itself by induction on φ:

$$\begin{aligned}
\overline{M} \models (\sigma = \tau)[a] \quad &\text{iff} \quad \sigma^{\overline{M}}(a) = \tau^{\overline{M}}(a)\\
&\text{iff} \quad \sigma^{\overline{N}}(a) = \tau^{\overline{N}}(a)\\
&\text{iff} \quad \overline{N} \models (\sigma = \tau)[a];\\
\overline{M} \models (\mathbf{R}\sigma_0 \ldots \sigma_{m-1})[a] \quad &\text{iff} \quad \langle \sigma_0^{\overline{M}}(a), \ldots, \sigma_{m-1}^{\overline{M}}(a)\rangle \in \mathbf{R}^{\overline{M}}\\
&\text{iff} \quad \langle \sigma_0^{\overline{N}}(a), \ldots, \sigma_{m-1}^{\overline{N}}(a)\rangle \in \mathbf{R}^{\overline{N}}\\
&\text{iff} \quad \overline{N} \models (\mathbf{R}\sigma_0 \ldots \sigma_{m-1})[a];\\
\overline{M} \models (\neg\varphi)[a] \quad &\text{iff} \quad \text{not}(\overline{M} \models \varphi[a])\\
&\text{iff} \quad \text{not}(\overline{N} \models \varphi[a])
\end{aligned}$$

$$\begin{array}{rll}
 & \text{iff} & \overline{N} \models (\neg\varphi)[a]; \\
\overline{M} \models (\varphi \to \psi)[a] & \text{iff} & \text{not}(\overline{M} \models \varphi[a]) \text{ or } \overline{M} \models \psi[a] \\
 & \text{iff} & \text{not}(\overline{N} \models \varphi[a]) \text{ or } \overline{N} \models \psi[a] \\
 & \text{iff} & \overline{N} \models (\varphi \to \psi)[a]; \\
\overline{M} \models (\forall v_i\varphi)[a] & \text{iff} & \text{for all } u \in M(\overline{M} \models \varphi[a^i_u]) \\
 & \text{iff} & \text{for all } u \in N(\overline{N} \models \varphi[a^i_u]) \\
 & \text{iff} & \overline{N} \models (\forall v_i\varphi)[a].
\end{array}$$ □

Theorem 4.15. (Completeness Theorem 1) *Every consistent set of sentences in a finite language has a model.*

Proof. Let Γ be a consistent set of sentences in the finite language $\mathscr{L}$. Let $\mathscr{L}'$ be obtained from $\mathscr{L}$ by adjoining individual constants $\mathbf{c}_i$ for each $i \in \omega$. By Lemmas 4.12 and 4.13 let Δ be a consistent rich complete set of sentences in $\mathscr{L}'$ such that $\Gamma \subseteq \Delta$. By Lemma 4.7, let $\overline{M}$ be a model of Δ. Let $\overline{N}$ be the $\mathscr{L}$-structure obtained from $\overline{M}$ by removing the denotations of the constants $\mathbf{c}_i$ for $i \in \omega$. By Lemma 4.14, $\overline{N}$ is a model of Γ. □

Theorem 4.16. (Completeness Theorem 2) *Let $\Gamma \cup \{\varphi\}$ be a set of formulas in a finite language. Then $\Gamma \vdash \varphi$ iff $\Gamma \models \varphi$.*

Proof. By Theorems 4.3 and 4.15. □

Theorem 4.17. (Completeness Theorem 3) *For any formula φ, $\vdash \varphi$ iff $\models \varphi$.*

Proof. Note that the implicit language $\mathscr{L}$ here is arbitrary, not necessarily finite. $\Rightarrow$ holds by Theorem 4.3. Now suppose that $\models \varphi$ in the sense of $\mathscr{L}$: for every $\mathscr{L}$-structure $\overline{M}$ and every $a : \omega \to M$ we have $\overline{M} \models \varphi[a]$. Let $\mathscr{L}'$ be the language whose non-logical symbols are those occurring in φ. There are finitely many such symbols, so $\mathscr{L}'$ is a finite language. By Lemma 4.14 we have $\models \varphi$ in the sense of $\mathscr{L}'$. Hence by Theorem 4.16, $\vdash \varphi$ in the sense of $\mathscr{L}'$. But every $\mathscr{L}'$-proof is also an $\mathscr{L}$-proof; so $\vdash \varphi$ in the sense of $\mathscr{L}$. □

As the final topic of this chapter we consider the role of definitions. To formulate the results we need another elementary logical notion. We define $\exists! v_i\varphi$ to be the formula $\exists v_i[\varphi \wedge \forall v_j[\mathrm{Subf}^{v_i}_{v_j}\varphi \to v_i = v_j]]$, where j is minimum such that $j \neq i$ and v_j does not occur in φ.

Theorem 4.18. *$\overline{A} \models \exists! v_i\varphi[a]$ iff there is a unique $u \in A$ such that $\overline{A} \models \varphi[a^i_u]$.*

Proof. $\Rightarrow$: Assume that $\overline{A} \models \exists! v_i\varphi[a]$. Choose $u \in A$ such that

(1) $$\overline{A} \models (\varphi \wedge \forall v_j[\mathrm{Subf}^{v_i}_{v_j}\varphi \to v_i = v_j])[a^i_u].$$

In particular, $\overline{A} \models \varphi[a^i_u]$. Suppose that also $\overline{A} \models \varphi[a^i_w]$. By Lemma 4.4, $\overline{A} \models \varphi[(a^j_w)^i_w]$, i.e.,

$$\overline{A} \models \varphi[(a^j_w)^i_{v_j(a^j_w)}].$$

Now we apply Lemma 4.6, with a replaced by a^j_w and obtain

$$\overline{A} \models \mathrm{Subf}^{v_i}_{v_j}\varphi[a^j_w].$$

Since v_i does not occur free in $\mathrm{Subf}^{v_i}_{v_j}\varphi$, this implies that

$$(2) \qquad \overline{A} \models \mathrm{Subf}^{v_i}_{v_j}\varphi[(a^i_u)^j_w].$$

Now by (1) we have

$$\overline{A} \models (\mathrm{Subf}^{v_i}_{v_j}\varphi \rightarrow v_i = v_j)[(a^i_u)^j_w],$$

so by (2) we have $u = w$.

$\Leftarrow$: Suppose that $u \in A$ is unique such that $\overline{A} \models \varphi[a^i_u]$. To check the other part of $\exists! v_i\varphi$, suppose that $w \in A$ and $\overline{A} \models \mathrm{Subf}^{v_i}_{v_j}\varphi[(a^i_u)^j_w]$. Since v_i does not occur free in $\mathrm{Subf}^{v_i}_{v_j}\varphi$, it follows by Proposition 4.4 that $\overline{A} \models \mathrm{Subf}^{v_i}_{v_j}\varphi[a^j_w]$. Applying Lemma 4.6 with a replaced by a^j_w we obtain $\overline{A} \models \varphi[(a^j_w)^i_{v_j(a^j_w)}]$, i.e., $\overline{A} \models \varphi[(a^j_w)^i_w]$. By Proposition 4.4 again this yields $\overline{A} \models \varphi[a^i_w]$. Hence by supposition $u = w$, as desired. $\square$

By a *theory* we mean a pair $(\mathscr{L}, \Gamma)$ such that $\mathscr{L}$ is a first-order language and Γ is a set of formulas in $\mathscr{L}$. A theory $(\mathscr{L}', \Gamma')$ is a *simple definitional expansion* of a theory $(\mathscr{L}, \Gamma)$ provided that the following conditions hold:

(1) $\mathscr{L}'$ is obtained from $\mathscr{L}$ by adding one new non-logical symbol.

(2) If the new symbol of $\mathscr{L}'$ is an m-ary relation symbol $\mathbf{R}$, then there is a formula φ of $\mathscr{L}$ with free variables among $v_0, \ldots, v_{m-1}$ such that

$$\Gamma' = \Gamma \cup \{\mathbf{R}v_0 \ldots v_{m-1} \leftrightarrow \varphi\}.$$

(3) If the new symbol of $\mathscr{L}'$ is an individual constant $\mathbf{c}$, then there is a formula φ of $\mathscr{L}$ with free variables among v_0 such that $\Gamma \vdash \exists! v_0\varphi$ and

$$\Gamma' = \Gamma \cup \{\mathbf{c} = v_0 \leftrightarrow \varphi\}.$$

(4) If the new symbol of $\mathscr{L}'$ is an m-ary function symbol $\mathbf{F}$, then there is a formula φ of $\mathscr{L}$ with free variables among $v_0, \ldots, v_m$ such that $\Gamma \vdash \forall v_0 \ldots \forall v_{m-1} \exists! v_m\varphi$ and

$$\Gamma' = \Gamma \cup \{\mathbf{F}v_0 \ldots v_{m-1} = v_m \leftrightarrow \varphi\}.$$

The basic facts about definitions are that the defined terms can always be eliminated, and adding a definition does not change what is is provable in the original language. In order to prove these two facts, we first show that any formula can be put in a certain standard form, which is interesting in its own right. A formula φ is *standard* provided that every atomic subformula of φ has one of the following forms:

$v_i = v_j$ for some $i, j \in \omega$.
$\mathbf{c} = v_0$ for some individual constant $\mathbf{c}$.

$\mathbf{R}v_0 \ldots v_{m-1}$ for some m-ary relation symbol $\mathbf{R}$.
$\mathbf{F}v_0 \ldots v_{m-1} = v_m$ for some m-ary function symbol $\mathbf{F}$.

Lemma 4.19. *If* $\mathbf{c}$ *is an individual constant and* $i \neq 0$*, then* $\vdash \mathbf{c} = v_i \leftrightarrow \exists v_0(v_0 = v_i \wedge \mathbf{c} = v_0)$.

Proof. We argue model-theoretically. Suppose that $\overline{A}$ is a structure and $a : \omega \to A$. If $\overline{A} \models (\mathbf{c} = v_i)[a]$, then $\mathbf{c}^{\overline{A}} = a_i$. Then $v_0^{\overline{A}}(a^0_{a_i}) = a_i$ and $v_i^{\overline{A}}(a^0_{a_i}) = a_i$. Hence $\overline{A} \models (v_0 = v_i \wedge \mathbf{c} = v_0)[a^0_{a_i}]$, and so $\overline{A} \models \exists v_0(v_0 = v_i \wedge \mathbf{c} = v_0)[a]$. Thus $\overline{A} \models (\vdash \mathbf{c} = v_i)[a]$ implies that $\overline{A} \models \exists v_0(v_0 = v_i \wedge \mathbf{c} = v_0)[a]$.

Conversely, suppose that $\overline{A} \models \exists v_0(v_0 = v_i \wedge \mathbf{c} = v_0)[a]$. Choose $x \in A$ such that $\overline{A} \models (v_0 = v_i \wedge \mathbf{c} = v_0)[a^0_x]$. Then $x = v_0^{\overline{A}}(a^0_x) = v_i^{\overline{A}}(a^0_x) = a_i$ and $\mathbf{c}^{\overline{A}} = v_0^{\overline{A}}(a^0_x) = a_i = v_i^{\overline{A}}(a)$. Hence $\overline{A} \models (\vdash \mathbf{c} = v_i)[a]$.

So we have shown that $\overline{A} \models (\vdash \mathbf{c} = v_i)[a]$ iff $\overline{A} \models \exists v_0(v_0 = v_i \wedge \mathbf{c} = v_0)[a]$. It follows that $\models \mathbf{c} = v_i \leftrightarrow \exists v_0(v_0 = v_i \wedge \mathbf{c} = v_0)$. Hence by the completeness theorem, $\vdash \mathbf{c} = v_i \leftrightarrow \exists v_0(v_0 = v_i \wedge \mathbf{c} = v_0)$. □

Lemma 4.20. *Suppose that* $\mathbf{R}$ *is an* m*-ary relation symbol and* $\langle i(0), \ldots, i(m-1)\rangle$ *is a sequence of natural numbers such that* $m \leq i(j)$ *for all* $j < m$*. Also assume that* $k < m$*. Then*

$$\vdash \mathbf{R}v_0 \ldots v_{k-1}v_{i(k)} \ldots v_{i(m-1)} \leftrightarrow \exists v_k[v_k = v_{i(k)} \wedge \mathbf{R}v_0 \ldots v_k v_{i(k+1)} \ldots v_{i(m-1)}].$$

Proof. Again we argue model-theoretically. Suppose that $\overline{A}$ is a structure and $a : \omega \to A$. First suppose that

$$\overline{A} \models \mathbf{R}v_0 \ldots v_{k-1}v_{i(k)} \ldots v_{i(m-1)}[a]$$

Thus

$$\begin{aligned}
&\langle a_0, \ldots, a_{k-1}, a_{i(k)}, \ldots, a_{i(m-1)}\rangle \in \mathbf{R}^{\overline{A}} \quad \text{hence}\\
&\langle (a^k_{a_{i(k)}})_0, \ldots, (a^k_{a_{i(k)}})_{k-1}, (a^k_{a_{i(k)}})_{i(k)}, \ldots, (a^k_{a_{i(k)}})_{i(m-1)}\rangle \in \mathbf{R}^{\overline{A}} \quad \text{hence}\\
&\langle (a^k_{a_{i(k)}})_0, \ldots, (a^k_{a_{i(k)}})_{k-1}, (a^k_{a_{i(k)}})_{k}, \ldots, (a^k_{a_{i(k)}})_{i(m-1)}\rangle \in \mathbf{R}^{\overline{A}} \quad \text{hence}\\
&\overline{A} \models [v_k = v_{i(k)} \wedge \mathbf{R}v_0 \ldots v_k v_{i(k+1)} \ldots v_{i(m-1)}][a^k_{a_{i(k)}}] \quad \text{hence}\\
(*)\qquad &\overline{A} \models \exists v_k[v_k = v_{i(k)} \wedge \mathbf{R}v_0 \ldots v_k v_{i(k+1)} \ldots v_{i(m-1)}][a].
\end{aligned}$$

Second, suppose that $(*)$ holds. Choose $s \in A$ such that

$$\overline{A} \models [v_k = v_{i(k)} \wedge \mathbf{R}v_0 \ldots v_k v_{i(k+1)} \ldots v_{i(m-1)}][a^k_s]$$

Then $s = a_{i(k)}$ and $\langle a_0, \ldots, a_{k-1}, s, a_{i(k+1)}, \ldots, a_{i(m-1)}\rangle \in \mathbf{R}^{\overline{A}}$, so

$$\langle a_0, \ldots, a_{k-1}, a_{i(k)}, a_{i(k+1)}, \ldots, a_{i(m-1)}\rangle \in \mathbf{R}^{\overline{A}},$$

so $\overline{A} \models \mathbf{R}v_0 \ldots v_{k-1}v_{i(k)} \ldots v_{i(m-1)}[a]$.

Now the Lemma follows by the completeness theorem. $\square$

Lemma 4.21. *Suppose that* $\mathbf{R}$ *is an* m*-ary relation symbol and* $\langle i(0), \ldots, i(m-1)\rangle$ *is a sequence of natural numbers such that* $m \le i(j)$ *for all* $j < m$*. Then there is a standard formula* φ *with free variables* $v_{i(j)}$ *for* $j < m$ *such that* $\vdash \mathbf{R}v_{i(0)} \ldots v_{i(m-1)} \leftrightarrow \varphi$.

Proof. This follows by an easy induction from Lemma 4.20. $\square$

The proof of the following lemma is very similar to the proof of Lemma 4.20.

Lemma 4.22. *Suppose that* $\mathbf{F}$ *is an* m*-ary function symbol and* $\langle i(0), \ldots, i(m)\rangle$ *is a sequence of natural numbers such that* $m+1 \le i(j)$ *for all* $j \le m$*. Also assume that* $k < m$ *Then*

$$\begin{aligned}&\vdash \mathbf{F}v_0 \ldots v_{k-1}v_{i(k)} \ldots v_{i(m-1)} = v_{i(m)} \leftrightarrow \\ &\quad \exists v_k[v_k = v_{i(k)} \wedge \mathbf{F}v_0 \ldots v_k v_{i(k+1)} \ldots v_{i(m-1)} = v_{i(m)}].\end{aligned}$$

Proof. Again we argue model-theoretically. Suppose that $\overline{A}$ is a structure and $a : \omega \to A$. First suppose that $\overline{A} \models (\mathbf{F}v_0 \ldots v_{k-1}v_{i(k)} \ldots v_{i(m-1)} = v_{i(m)})[a]$. Then

$$\begin{aligned}
&\mathbf{F}^{\overline{A}}(a_0, \ldots, a_{k-1}, a_{i(k)}, \ldots, a_{i(m-1)} = a_{i(m)}, \quad \text{hence}\\
&\mathbf{F}^{\overline{A}}((a^k_{i(k)})_0, \ldots, (a^k_{i(k)})_{k-1}, (a^k_{i(k)})_{i(k)}, \ldots, (a^k_{i(k)})_{i(m-1)} = (a^k_{i(k)})_{i(m)}, \quad \text{hence}\\
&\mathbf{F}^{\overline{A}}((a^k_{i(k)})_0, \ldots, (a^k_{i(k)})_{k-1}, (a^k_{i(k)})_k, \ldots, (a^k_{i(k)})_{i(m-1)} = (a^k_{i(k)})_{i(m)}, \quad \text{hence}\\
&\overline{A} \models (v_k = v_{i(k)} \wedge \mathbf{F}v_0 \ldots v_k v_{i(k+1)} \ldots v_{i(m-1)} = v_{i(m)})[a^k_{i(k)}], \quad \text{hence}\\
(*) \quad &\overline{A} \models \exists v_k[v_k = v_{i(k)} \wedge \mathbf{F}v_0 \ldots v_k v_{i(k+1)} \ldots v_{i(m-1)} = v_{i(m)}][a].
\end{aligned}$$

Second, suppose that $(*)$ holds. Choose s so that

$$\overline{A} \models [v_k = v_{i(k)} \wedge \mathbf{F}v_0 \ldots v_k v_{i(k+1)} \ldots v_{i(m-1)} = v_{i(m)}][a^k_s].$$

Then $s = a_{i(k)}$ and $\mathbf{F}^{\overline{A}}(a_0, \ldots, s, a_{i(k+1)} \ldots a_{i(m-1)}) = a_{i(m)}$. Hence

$$\mathbf{F}^{\overline{A}}(a_0, \ldots, a_{k-1}, a_{i(k)}, a_{i(k+1)} \ldots a_{i(m-1)}) = a_{i(m)},$$

and so $\overline{A} \models (\mathbf{F}v_0 \ldots v_{k-1}v_{i(k)} \ldots v_{i(m-1)} = v_{i(m)})[a]$.

The Lemma now follows by the completeness theorem. $\square$

Lemma 4.23. *Suppose that* $\mathbf{F}$ *is an* m*-ary function symbol and* $\langle i(0), \ldots, i(m)\rangle$ *is a sequence of natural numbers such that* $m+1 \le i(j)$ *for all* $j \le m$*. Then there is a standard formula* φ *with free variables* $v_{i(j)}$ *for* $j \le M$ *such that* $\vdash \mathbf{F}v_{i(0)} \ldots v_{i(m-1)} = v_{i(m)} \leftrightarrow \varphi$.

Proof. By an easy induction using Lemma 4.22 there is a formula ψ with free variables $v_{i(j)}$ for $j \le m$ such that the only nonlogical atomic formula which is a segment of ψ is

$\mathbf{F}v_0 \ldots v_{m-1} = v_{i(m)}$ and $\vdash \mathbf{F}v_{i(0)} \ldots v_{i(m-1)} = v_{i(m)} \leftrightarrow \varphi$. Now for any structure $\overline{A}$ and any $a : \omega \to A$ we have

$$(*) \qquad \overline{A} \models (\mathbf{F}v_0 \ldots v_{m-1} = v_{i(m)} \leftrightarrow \exists v_m[v_m = v_{i(m)} \wedge \mathbf{F}v_0 \ldots v_{m-1} = v_m)][a].$$

To prove $(*)$, first suppose that $\overline{A} \models (\mathbf{F}v_0 \ldots v_{m-1} = v_{i(m)})[a]$. Thus $\mathbf{F}(a_0, \ldots, a_{m-1}) = a_{i(m)}$. Hence

$$\begin{aligned}
&\mathbf{F}(a_0, \ldots, a_{m-1}) = a_{i(m)} \quad \text{hence}\\
&\mathbf{F}((a^m_{a_{i(m)}})_0, \ldots, (a^m_{a_{i(m)}})_{m-1}) = (a^m_{a_{i(m)}})_{i(m)} \quad \text{hence}\\
&\overline{A} \models (v_m = v_{i(m)} \wedge \mathbf{F}v_0 \ldots v_{m-1} = v_m)[a^m_{a_{i(m)}}] \quad \text{hence}\\
(**) \qquad &\overline{A} \models \exists v_m[v_m = v_{i(m)} \wedge \mathbf{F}v_0 \ldots v_{m-1} = v_m)][a]
\end{aligned}$$

Second, assume $(**)$. Choose $s \in A$ such that $\overline{A} \models [v_m = v_{i(m)} \wedge \mathbf{F}v_0 \ldots v_{m-1} = v_m)][a^m_s]$. It follows that $s = a_{i(m)}$ and $\mathbf{F}^{\overline{A}}(a_0, \ldots, a_{m-1} = s$, so $\mathbf{F}^{\overline{A}}(a_0, \ldots, a_{m-1} = a_{i(m)}$, hence $\overline{A} \models (\mathbf{F}v_0 \ldots v_{m-1} = v_{i(m)})[a]$.

This proves $(*)$. From $(*)$ the Lemma is clear. □

Lemma 4.24. *Suppose that* $\mathbf{F}$ *is an* m*-ary function symbol,* $\sigma_0, \ldots, \sigma_{m-1}$ *are terms, the integers* $i(0), \ldots, i(m)$ *are all greater than* m *and do not appear in any of the terms* σ_j*, and* $k < m$*. Then*

$$\begin{aligned}
&\vdash \mathbf{F}v_{i(0)} \ldots v_{i(k-1)}\sigma_k \ldots \sigma_{m-1} = v_{i(m)}\\
&\quad \leftrightarrow \exists v_{i(k)}[\sigma_k = v_{i(k)} \wedge \mathbf{F}v_{i(0)} \ldots v_{i(k)}\sigma_{k+1} \ldots \sigma_{m-1} = v_{i(m)}].
\end{aligned}$$

Proof. Arguing model-theoretically, let $\overline{A}$ be a structure and $a : \omega \to A$. Let $b = a^{i(k)}_{\sigma^{\overline{A}}_k(a)}$. First suppose that $\overline{A} \models (\mathbf{F}v_{i(0)} \ldots v_{i(k-1)}\sigma_k \ldots \sigma_{m-1} = v_{i(m)})[a]$. Thus

$$\begin{aligned}
&\mathbf{F}^{\overline{A}}(a_{i(0)}, \ldots, a_{i(k-1)}, \sigma^{\overline{A}}_k(a), \ldots, \sigma^{\overline{A}}_{m-1}(a)) = v^{\overline{A}}_{i(m)}(a) \quad \text{hence}\\
&\mathbf{F}^{\overline{A}}(b_{i(0)}, \ldots, b_{i(k-1)}, b_{i(k)}, \sigma^{\overline{A}}_{k+1}(b), \ldots, \sigma^{\overline{A}}_{m-1}(b)) = v^{\overline{A}}_{i(m)}(b) \quad \text{hence}\\
&\overline{A} \models (\sigma_k = v_{i(k)} \wedge \mathbf{F}v_{i(0)} \ldots v_{i(k)}\sigma_{k+1} \ldots \sigma_{m-1} = v_{i(m)})[b] \quad \text{hence}\\
(*) \qquad &\overline{A} \models \exists v_{i(k)}(\sigma_k = v_{i(k)} \wedge \mathbf{F}v_{i(0)} \ldots v_{i(k)}\sigma_{k+1} \ldots \sigma_{m-1} = v_{i(m)})[a].
\end{aligned}$$

Second, suppose that $(*)$ holds. Choose $s \in A$ so that

$$\overline{A} \models (\sigma_k = v_{i(k)} \wedge \mathbf{F}v_{i(0)} \ldots v_{i(k)}\sigma_{k+1} \ldots \sigma_{m-1} = v_{i(m)})[a^{i(k)}_s].$$

Hence $\sigma^{\overline{A}}_k(a^{i(k)}_s) = s$ and

$$\begin{aligned}
&\mathbf{F}^{\overline{A}}((a^{i(k)}_s)_{i(0)}, \ldots, (a^{i(k)}_s)_{i(k)}, \sigma^{\overline{A}}_{k+1}(a^{i(k)}_s), \ldots \sigma^{\overline{A}}(a^{i(k)}_s) = (a^{i(k)}_s)_{i(m)}, \quad \text{hence}\\
&\mathbf{F}^{\overline{A}}((a^{i(k)}_s)_{i(0)}, \ldots, \sigma^{\overline{A}}_k(a^{i(k)}_s), \sigma^{\overline{A}}_{k+1}(a^{i(k)}_s), \ldots \sigma^{\overline{A}}(a^{i(k)}_s) = (a^{i(k)}_s)_{i(m)}, \quad \text{hence}\\
&\mathbf{F}^{\overline{A}}(a_{i(0)}, \ldots, a_{i(k-1)}, \sigma^{\overline{A}}_k(a), \ldots, \sigma^{\overline{A}}_{m-1}(a)) = v^{\overline{A}}_{i(m)}(a) \quad \text{hence}\\
&\overline{A} \models (\mathbf{F}v_{i(0)} \ldots v_{i(k-1)}\sigma_k \ldots \sigma_{m-1} = v_{i(m)})[a]
\end{aligned}$$

□

Lemma 4.25. *Suppose that* $\mathbf{F}$ *is an* m*-ary function symbol,* $\sigma_0, \ldots, \sigma_{m-1}$ *are terms, the integers* $i(0), \ldots, i(m)$ *are all greater than* m *and do not appear in any of the terms* σ_j*.*

Then there is a formula φ *with free variables among* $v_{i(0)}, \ldots, v_{i(m)}$ *such that the atomic subformulas of* φ *are the formulas* $\sigma_k = v_{i(k)}$ *for* $k < m$ *along with the formula* $\mathbf{F}v_{i(0)} \ldots v_{i(m-1)} = v_{i(m)}$*, and* $\vdash \mathbf{F}\sigma_0 \ldots \sigma_{m-1} = v_{i(m)} \leftrightarrow \varphi$*.*

Lemma 4.26. *Suppose that* τ *is a term and* $i \in \omega$ *is greater than* m *for each* m *such that a function symbol of rank* m *occurs in* τ*, and such that* v_m *does not occur in* τ*.*

Then there is a standard formula φ *with the same free variables occurring in* $\tau = v_m$*, such that* $\vdash \tau = v_m \leftrightarrow \varphi$*.*

Proof. We go by induction on τ. If τ is v_i, then we can take φ to be $v_i = v_m$. If τ is an individual constant $\mathbf{c}$, then Lemma 4.19 gives the desired result. Finally, suppose inductively that τ is $\mathbf{F}\sigma_0 \ldots \sigma_{m-1}$. Then the desired result follows by Lemma 4.26, the inductive hypothesis, and Lemma 4.23. □

Lemma 4.27. *For any terms* σ, τ *there is a standard formula* φ *with the same free variables as* $\sigma = \tau$ *such that* $\vdash \sigma = \tau \leftrightarrow \varphi$*.*

Proof. Let i be greater than each m such that there is a function symbol of rank m appearing in $\sigma = \tau$, and also such that v_i does not occur in $\sigma = \tau$. Then

$$\vdash \sigma = \tau \leftrightarrow \exists v_i(\sigma = v_i \wedge \tau = v_i). \tag{1}$$

We prove (1) model-theoretically. First suppose that $\overline{A} \models (\sigma = \tau)[a]$. Thus $\sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a)$. By Proposition 2.4 we then have

$$\begin{aligned}
&\sigma^{\overline{A}}(a^i_{\sigma^{\overline{A}}(a)}) = \tau^{\overline{A}}(a^i_{\alpha^{\overline{A}}(a)}) \quad \text{hence}\\
&\overline{A} \models (\sigma = v_i \wedge \tau = v_i)[a^i_{\sigma^{\overline{A}}(a)}] \quad \text{hence}\\
(*) \qquad &\overline{A} \models \exists v_i(\sigma = v_i \wedge \tau = v_i)[a].
\end{aligned}$$

Second, suppose that $(*)$ holds. Choose $s \in A$ such that $\overline{A} \models (\sigma = v_i \wedge \tau = v_i)[a^i_s]$. Thus $\sigma^{\overline{A}}(a^i_s) = s = \tau^{\overline{A}}(a^i_s)$, hence $\sigma^{\overline{A}}(a) = \tau^{\overline{A}}(a)$ by Proposition 2.4. That is, $\overline{A} \models (\sigma = \tau)[a]$. This finishes the proof of (1).

Now by (1) and Lemma 4.26 our lemma follows. □

The proof of the following lemma is very similar to that of Lemma 4.24.

Lemma 4.28. *Suppose that* $\mathbf{R}$ *is an* m*-ary relation symbol,* $\sigma_0, \ldots, \sigma_{m-1}$ *are terms, the integers* $i(0), \ldots, i(m)$ *are all greater than* m *and do not appear in any of the terms* σ_j*, and* $k < m$*. Then*

$$\begin{aligned}
\vdash &\mathbf{R}v_{i(0)} \ldots v_{i(k-1)}\sigma_k \ldots \sigma_{m-1}\\
&\leftrightarrow \exists v_{i(k)}[\sigma_k = v_{i(k)} \wedge \mathbf{R}v_{i(0)} \ldots v_{i(k)}\sigma_{k+1} \ldots \sigma_{m-1}].
\end{aligned}$$

Proof. Arguing model-theoretically, let $\overline{A}$ be a structure and $a : \omega \to A$. Let $b = a^{i(k)}_{\sigma_k^{\overline{A}}(a)}$. First suppose that $\overline{A} \models (\mathbf{R}v_{i(0)} \ldots v_{i(k-1)}\sigma_k \ldots \sigma_{m-1}[a]$. Thus

$$\begin{aligned}
&\langle a_{i(0)}, \ldots, a_{i(k-1)}, \sigma_k^{\overline{A}}(a), \ldots, \sigma_{m-1}^{\overline{A}}(a) \in \mathbf{R}^{\overline{A}}, \quad \text{hence}\\
&\langle b_{i(0)}, \ldots, b_{i(k-1)}, b_{i(k)}, \sigma_{k+1}^{\overline{A}}(b), \ldots, \sigma_{m-1}^{\overline{A}}(b) \in \mathbf{R}^{\overline{A}}, \quad \text{hence}\\
&\overline{A} \models (\sigma_k = v_{i(k)} \wedge \mathbf{R}v_{i(0)} \ldots v_{i(k)}\sigma_{k+1} \ldots \sigma_{m-1})[b] \quad \text{hence}\\
(*) \qquad &\overline{A} \models \exists v_{i(k)}(\sigma_k = v_{i(k)} \wedge \mathbf{R}v_{i(0)} \ldots v_{i(k)}\sigma_{k+1} \ldots \sigma_{m-1})[a].
\end{aligned}$$

Second, suppose that $(*)$ holds. Choose $s \in A$ so that

$$\overline{A} \models (\sigma_k = v_{i(k)} \wedge \mathbf{R}v_{i(0)} \ldots v_{i(k)}\sigma_{k+1} \ldots \sigma_{m-1})[a^{i(k)}_s].$$

Hence $\sigma_k^{\overline{A}}(a^{i(k)}_s) = s$ and

$$\begin{aligned}
&\langle (a^{i(k)}_s)_{i(0)}, \ldots, (a^{i(k)}_s)_{i(k)}, \sigma_{k+1}^{\overline{A}}(a^{i(k)}_s), \ldots \sigma^{\overline{A}}(a^{i(k)}_s)\rangle \in \mathbf{R}^{\overline{A}}, \quad \text{hence}\\
&\langle (a^{i(k)}_s)_{i(0)}, \ldots, \sigma_k^{\overline{A}}(a^{i(k)}_s), \sigma_{k+1}^{\overline{A}}(a^{i(k)}_s), \ldots \sigma^{\overline{A}}(a^{i(k)}_s)\rangle \in \mathbf{R}^{\overline{A}}, \quad \text{hence}\\
&\langle a_{i(0)}, \ldots, a_{i(k-1)}, \sigma_k^{\overline{A}}(a), \ldots \sigma_{m-1}^{\overline{A}}(a)\rangle \in \mathbf{R}^{\overline{A}}, \quad \text{hence}\\
&\overline{A} \models (\mathbf{R}v_{i(0)} \ldots v_{i(k-1)}\sigma_k \ldots \sigma_{m-1})[a]
\end{aligned}$$

□

Theorem 4.29. *For any formula φ there is a standard formula ψ with the same free variables as φ such that $\vdash \varphi \leftrightarrow \psi$.*

Proof. We proceed by induction on φ. For φ an atomic equality formula $\sigma = \tau$ the desired result is given by Lemma 4.27. Now suppose that φ is an atomic nonequality formul $\mathbf{R}\sigma_0 \ldots \sigma_{m-1}$. Using induction we see from Lemma 4.28 that there is a formula φ whose atomic parts are of the form $\sigma_k = v_{i(k)}$ and $\mathbf{R}v_{i(0} \ldots v_{i(m-1)}$ such that $\vdash \mathbf{R}\sigma_0 \ldots \sigma_{m-1} \leftrightarrow \varphi$, and each $i(k)$ is greater than each n such that a function symbol of rank n occurs in some σ_l, and also is such that no $v_{i(k)}$ occurs in any σ_s, and eack $i(k) > m$. Now by Lemmas 4.21 and 4.26, $\vdash \varphi \leftrightarrow \psi$ for some standard formula ψ. The condition on free variables holds in each of these steps. Thus the atomic cases of the induction hold.

The induction steps are easy:

Suppose that $\vdash \varphi \leftrightarrow \psi$ with ψ standard. Then $\vdash \neg\varphi \leftrightarrow \neg\psi$ and $\neg\psi$ is standard.

Suppose that $\vdash \varphi \leftrightarrow \psi$ with ψ standard and $\vdash \varphi' \leftrightarrow \psi'$ with ψ' standard. Then $\vdash (\varphi \to \psi) \leftrightarrow (\psi \to \psi')$ and $\psi \to \psi'$ is standard.

Suppose that $\vdash \varphi \leftrightarrow \psi$ with ψ standard. Then $\vdash \forall v_i\varphi \leftrightarrow \forall v_i\psi$ with $\forall v_i\psi$ standard.

□

The following theorem expresses that defined notions can be eliminated.

Theorem 4.30. *Let $(\mathscr{L}', \Gamma')$ be a simple definitional expansion of $(\mathscr{L}, \Gamma)$, and let φ be a formula of $\mathscr{L}'$. Then there is a formula ψ of $\mathscr{L}$ with the same free variables as φ such that $\Gamma' \vdash \varphi \leftrightarrow \psi$.*

(Note here that $\vdash$ is in the sense of $\mathscr{L}'$.)

Proof. Let χ be a standard formula (of $\mathscr{L}'$) such that $\vdash \varphi \leftrightarrow \chi$, such that χ has the same free variables as φ. Now we consider cases depending on what the new symbol s of $\mathscr{L}'$ is. Let θ be as in the definition of simple definitional expansion, with θ instead of φ.

Case 1. s is an individual constant $\mathbf{c}$. Then we let ψ be obtained from χ by replacing every subformula $\mathbf{c} = v_0$ of χ by θ.

Case 2. s is an m-ary relation symbol $\mathbf{R}$. Then we let ψ be obtained from χ by replacing every subformula $\mathbf{R}v_0 \ldots v_{m-1}$ of χ by θ.

Case 3. s is an m-ary function symbol $\mathbf{F}$. Then we let ψ be obtained from χ by replacing every subformula $\mathbf{F}v_0 \ldots v_{m-1} = v_m$ of χ by θ. $\square$

The following theorem expresses that a simple definitional expansion does not increase the set of old formulas which are provable.

Theorem 4.31. *Let $(\mathscr{L}', \Gamma')$ be a simple definitional expansion of $(\mathscr{L}, \Gamma)$ with $\mathscr{L}$ finite, and let φ be a formula of $\mathscr{L}$. Suppose that $\Gamma' \vdash \varphi$. Then $\Gamma \vdash \varphi$.*

Proof. By the completeness theorem we have $\Gamma' \models \varphi$, and it suffices to show that $\Gamma \models \varphi$. So, suppose that $\overline{A} \models \psi$ for each $\psi \in \Gamma$. In order to show that $\overline{A} \models \varphi$, suppose that $a : \omega \to A$; we want to show that $\overline{A} \models \varphi[a]$. We define an $\mathscr{L}'$-structure $\overline{A}'$ by defining the denotation of the new symbol s of $\mathscr{L}'$. The three cases are treated similarly, but we give full details for each of them.

Case 1. s is $\mathbf{c}$, an individual constant. By the definition of simple definitional expansion, there is a formula χ of $\mathscr{L}$ with free variables among v_0 such that $\Gamma \vdash \exists! v_0 \chi$, and $\Gamma' = \Gamma \cup \{\mathbf{c} = v_0 \leftrightarrow \chi\}$. Then $\Gamma \models \exists! v_0 \chi$. Since $\overline{A} \models \Gamma$, it follows that $\overline{A} \models \chi[a^0_x]$ for a unique $x \in A$. Let $\mathbf{c}^{\overline{A}'} = x$. We claim that $\overline{A}' \models (\mathbf{c} = v_0 \leftrightarrow \chi)$. In fact, suppose that $b : \omega \to A$. If $\overline{A}' \models (\mathbf{c} = v_0)[b]$, then $b_0 = \mathbf{c}^{\overline{A}'} = x$. Then a^0_x and b agree at 0, so by Lemma 4.4, since the free variables of χ are among v_0, we have $\overline{A} \models \chi[b]$. By Lemma 4.14, $\overline{A}' \models \chi[b]$. Conversely, suppose that $\overline{A}' \models \chi[b]$. Then b and $a^0_{b(0)}$ agree on 0, so $\overline{A}' \models \chi[a^0_{b(0)}]$. Hence $\overline{A} \models \chi[a^0_{b(0)}]$ by Lemma 4.14. Since also $\overline{A} \models \chi[a^0_x]$ and $\overline{A} \models \exists! v_0 \chi$, it follows that $b(0) = x$. Hence $\overline{A}' \models (\mathbf{c} = v_0)[b]$. This proves the claim.

By the claim, $\overline{A}'$ is a model of Γ'. Hence it is a model of φ. By Lemma 4.14, $\overline{A}$ is a model of φ, as desired.

Case 2. s is $\mathbf{F}$, an m-ary function symbol. By the definition of simple definitional expansion, there is a formula χ of $\mathscr{L}$ with free variables among $v_0, \ldots, v_m$ such that $\Gamma \vdash \forall v_0 \ldots \forall v_{m-1} \exists! v_m \chi$, and $\Gamma' = \Gamma \cup \{\mathbf{F}v_0 \ldots v_{m-1} = v_m \leftrightarrow \chi\}$. Then $\Gamma \models \forall v_0 \ldots \forall v_{m-1} \exists! v_m \chi$. Let $x(0), \ldots, x(m-1) \in A$. Since $\overline{A} \models \Gamma$, it follows that $\overline{A} \models \chi[(\cdots(a^0_{x(0)})^1_{x(1)})\cdots)^{m-1}_{x(m-1)})^m_y]$ for a unique $y \in A$. Let $\mathbf{F}^{\overline{A}'}(x(0), \ldots, x((m-1)) = y$. We claim that $\overline{A}' \models (\mathbf{F}v_0 \ldots v_{m-1} = v_m \leftrightarrow \chi)$. In fact, suppose that $b : \omega \to A$. If $\overline{A}' \models (\mathbf{F}v_0 \ldots v_{m-1} = v_m)[b]$, then $\mathbf{F}^{\overline{A}'}(b_0, \ldots, b_{m-1}) = b_m$. Now b and $(\cdots(a^0_{b_0})^1_{b_1})\cdots)^m_{b_m}$ and b agree on $\{0, \ldots, m\}$, so by the definition of $\mathbf{F}^{\overline{A}'}$ we get $\overline{A} \models \chi[(\cdots(a^0_{b_0})^1_{b_1})\cdots)^m_{b_m}]$, and hence also $\overline{A} \models \chi[b]$, and by Lemma 4.14 $\overline{A}' \models \chi[b]$.

Conversely, suppose that $\overline{A}' \models \chi[b]$. Then $\overline{A} \models [(\cdots(a^0_{b_0})^1_{b_1})\cdots)^m_{b_m}]$, and therefore $\mathbf{F}^{\overline{A}'}(b_0,\ldots,b_{m-1}) = b_m$. This proves the claim.

By the claim, $\overline{A}'$ is a model of Γ'. Hence it is a model of φ. By Lemma 4.14, $\overline{A}$ is a model of φ, as desired.

Case 3. s is $\mathbf{R}$, an m-ary relation symbol. By the definition of simple definitional expansion, there is a formula χ of $\mathscr{L}$ with free variables among $v_0,\ldots,v_{m-1}$ such that $\Gamma' = \Gamma \cup \{\mathbf{R}v_0\ldots v_{m-1} \leftrightarrow \chi\}$. Let

$$\begin{aligned}\mathbf{R}^{\overline{A}'} = \{\langle a_0,\ldots,a_{m-1}\rangle : \overline{A} \models \chi[a] \\ \text{for some } a:\omega\to A \text{ which extends } \langle a_0,\ldots,a_{m-1}\rangle\}.\end{aligned}$$

We claim that $\overline{A}' \models (\mathbf{R}v_0\ldots v_{m-1} \leftrightarrow \chi)$. In fact, suppose that $b:\omega\to A$. If $\overline{A}' \models (\mathbf{R}v_0\ldots v_{m-1}[b]$, then $\langle b_0,\ldots,b_{m-1}\rangle \in \mathbf{R}^{\overline{A}'}$, and so there is an extension $a:\omega\to A$ of $\langle b_0,\ldots,b_{m-1}\rangle$ such that $\overline{A} \models \chi[a]$. Since a and b agree on all k such that v_k occurs in χ, it follows that $\overline{A} \models \chi[b]$, and hence $\overline{A}' \models \chi[b]$.

Conversely, suppose that $\overline{A}' \models \chi[b]$. Then $\overline{A} \models \chi[b]$ by Lemma 4.14, and it follows that $\langle b_0,\ldots,b_{m-1}\rangle \in \mathbf{R}^{\overline{A}'}$. This proves the claim.

By the claim, $\overline{A}'$ is a model of Γ'. Hence it is a model of φ. By Lemma 4.14, $\overline{A}$ is a model of φ, as desired. □

Theorem 4.32. *Let m be an integer ≥ 2, and suppose that $(\mathscr{L}_{i+1},\Gamma_{i+1})$ is a simple definitional expansion of $(\mathscr{L}_i,\Gamma_i)$ for each $i<m$. Suppose that φ is an $\mathscr{L}_m$ formula. Then there is an $\mathscr{L}_0$ formula ψ with the same free variables as φ such that $\Gamma_m \vdash \varphi \leftrightarrow \psi$.*

Proof. By induction on m. If $m=2$, the conclusion follows from Theorem 4.30. now assume the result for m and suppose that $(\mathscr{L}_{i+1},\Gamma_{i+1})$ is a simple definitional expansion of $(\mathscr{L}_i,\Gamma_i)$ for each $i\leq m$. Let φ be a formula of $\mathscr{L}_{m+1}$. Then by Theorem 4.30 there is a formula ψ of $\mathscr{L}$ with the same free variables as φ such that $\Gamma_{m+1} \vdash \varphi \leftrightarrow \psi$. By the inductive hypothesis, there is a formula χ with the same free variables as ψ such that $\Gamma_m \vdash \psi \leftrightarrow \chi$. Then $\Gamma_{m+1} \vdash \varphi \leftrightarrow \chi$. □

Theorem 4.33. *Let m be an integer ≥ 2, and suppose that $(\mathscr{L}_{i+1},\Gamma_{i+1})$ is a simple definitional expansion of $(\mathscr{L}_i,\Gamma_i)$ for each $i<m$. Also assume that $\mathscr{L}_0$ is finite. Suppose that φ is an $\mathscr{L}_0$ formula and $\Gamma_m \vdash \varphi$. Then $\Gamma_0 \vdash \varphi$.*

Proof. By induction on m. If $m=2$, the conclusion follows from Theorem 4.31. now assume the result for m and suppose that $(\mathscr{L}_{i+1},\Gamma_{i+1})$ is a simple definitional expansion of $(\mathscr{L}_i,\Gamma_i)$ for each $i\leq m$. Suppose that φ is an $\mathscr{L}_0$ formula and $\Gamma_{m+1}\vdash\varphi$. Then by Theorem 4.31, $\Gamma_m \vdash \varphi$, and so by the inductive assumption, $\Gamma_0\vdash\varphi$. □

Proposition 4.34. *Suppose that $\Gamma\vdash\varphi\to\psi$, $\Gamma\vdash\varphi\to\neg\psi$, and $\Gamma\vdash\neg\varphi\to\varphi$. Then Γ is inconsistent.*

Proof. The formula $(\neg\varphi\to\varphi)\to\varphi$ is a tautology. Hence by Lemma 3.3, $\Gamma\vdash(\neg\varphi\to\varphi)\to\varphi$. Since also $\Gamma\vdash\neg\varphi\to\varphi$, it follows that $\Gamma\vdash\varphi$. Hence $\Gamma\vdash\psi$ and $\Gamma\vdash\neg\psi$. Hence by Lemma 4.1, Γ is inconsistent. □

Proposition 4.35. *Let $\mathscr{L}$ be a language with just one non-logical constant, a binary relation symbol* $\mathbf{R}$. *Let* Γ *consist of all sentences of the form* $\exists v_1 \forall v_0[\mathbf{R}v_0v_1 \leftrightarrow \varphi]$ *with* φ *a formula with only* v_0 *free. Then* Γ *is inconsistent.*

Proof.
By Theorem 3.25 we have

$$\Gamma \vdash \forall v_0[\mathbf{R}v_0v_1 \leftrightarrow \neg\mathbf{R}v_0v_0] \rightarrow [\mathbf{R}v_1v_1 \leftrightarrow \neg\mathbf{R}v_1v_1]. \tag{1}$$

Now $[\mathbf{R}v_1v_1 \leftrightarrow \neg\mathbf{R}v_1v_1] \rightarrow \neg(v_0 = v_0)$ is a tautology, so from (1) we obtain

$$\Gamma \vdash \forall v_0[\mathbf{R}v_0v_1 \leftrightarrow \neg\mathbf{R}v_0v_0] \rightarrow \neg(v_0 = v_0);$$

then generalization gives

$$\Gamma \vdash \forall v_1[\forall v_0[\mathbf{R}v_0v_1 \leftrightarrow \neg\mathbf{R}v_0v_0] \rightarrow \neg(v_0 = v_0)].$$

Then by Proposition 3.37 we get

$$\Gamma \vdash \exists v_1 \forall v_0[\mathbf{R}v_0v_1 \leftrightarrow \neg\mathbf{R}v_0v_0] \rightarrow \neg(v_0 = v_0).$$

But the hypothesis here is a member of Γ, so we get $\Gamma \vdash \neg(v_0 = v_0)$. Hence by Lemma 4.1, Γ is inconsistent. $\square$

Proposition 4.36. *The first-order deduction theorem fails if the condition that* φ *is a sentence is omitted.*

Proof. Take $\Gamma = \emptyset$, let φ be the formula $v_0 = v_1$, and let ψ be the formula $v_0 = v_2$. Then

$$\begin{aligned}
&\{v_0 = v_1\} \vdash v_0 = v_1\\
&\{v_0 = v_1\} \vdash \forall v_1(v_0 = v_1)\\
&\{v_0 = v_1\} \vdash \forall v_1(v_0 = v_1) \rightarrow v_0 = v_2 \quad \text{by Theorem 3.25}\\
&\{v_0 = v_1\} \vdash v_0 = v_2.
\end{aligned}$$

On the other hand, let $\overline{A}$ be the structure with universe ω and define $a = \langle 0, 0, 1, 1, \ldots\rangle$. Clearly $\overline{A} \not\models [v_0 = v_1 \rightarrow v_0 = v_2][a]$. Hence $\not\vdash v_0 = v_1 \rightarrow v_0 = v_2$ by Theorem 3.2. $\square$

. . . .

Proposition 4.37. *In the language for* $\overline{A} \stackrel{\text{def}}{=} (\omega, S, 0, +, \cdot)$, *let* τ *be the term* $v_0 + v_1 \cdot v_2$ *and* ν *the term* $v_0 + v_2$. *Let* a *be the sequence* $\langle 0, 1, 2, \ldots\rangle$. *Let* ρ *be obtained from* τ *by replacing the occurrence of* v_1 *by* ν. *Then*

(a) ρ *is* $v_0 + (v_0 + v_2) \cdot v_2$; *as a sequence of integers it is* $\langle 7, 5, 9, 7, 5, 15, 15\rangle$.

(b) $\rho^{\overline{A}}(a) = 0 + (0 + 2) \cdot 2 = 4$.

(c) $\nu^{\overline{A}}(a) = 0 + 2 = 2$.

(d) $a^1_{\nu^{\overline{A}}(a)} = \langle 0, 2, 2, 3, \ldots \rangle$. *integers.*

(e) $\rho^{\overline{A}}(a) = \tau^{\overline{A}}(a^1_{\nu^{\overline{A}}(a)})$ *(cf. Lemma 4.4.)* $\square$

Proposition 4.38. *In the language for* $\overline{A} \stackrel{\text{def}}{=} (\omega, S, 0, +, \cdot)$, *let* φ *be the formula* $\forall v_0(v_0 \cdot v_1 = v_1)$, *let* ν *be the formula* $v_1 + v_1$, *and let* $a = \langle 1, 0, 1, 0, \ldots \rangle$. *Then*

(a) $\mathrm{Subf}^{v_1}_{\nu}\varphi$ *is* $\forall v_0(v_0 \cdot (v_1 + v_1) = v_1 + v_1$; *as a sequence of integers it is*

$$\langle 4, 5, 3, 9, 5, 7, 10, 10, 7, 10, 10 \rangle.$$

(b) $\nu^{\overline{A}}(a) = (v_1 + v_1)^{\overline{A}}(\langle 1, 0, 1, 0, \ldots \rangle) = 0 + 0 = 0.$

(c) $a^1_{\nu^{\overline{A}}(a)} = \langle 1, 0, 1, 0, \ldots \rangle$.

(d) $\overline{A} \models \mathrm{Subf}^{v_1}_{\nu}\varphi[a]$

(e) $\overline{A} \models \varphi[a^1_{\nu^{\overline{A}}(a)}]$

Proof. Only (d) and (e) need a proof.

(d): $\overline{A} \models \mathrm{Subf}^{v_1}_{\nu}\varphi[a]$ iff $\overline{A} \models [\forall v_0(v_0 \cdot (v_1 + v_1) = v_1 + v_1][\langle 1, 0, 1, 0, \ldots \rangle]$ iff for all $a \in \omega$, $a \cdot (0 + 0) = 0 + 0$, which is true.

(e): $\overline{A} \models \varphi[a^1_{\nu^{\overline{A}}(a)}]$ iff $\overline{A} \models [\forall v_0(v_0 \cdot v_1 = v_1][\langle 1, 0, 1, 0, \ldots \rangle]$ iff for all $a \in \omega$, $a \cdot 0 = 0$, which is true. $\square$

Proposition 4.39. *The condition in Lemma 4.6 that*

no free occurrence of v_i *in* φ *is within a subformula of the form* $\forall v_k \mu$ *with* v_k *a variable occurring in* ν

is necessary for the conclusion of the lemma.

Proof. In the language for $\overline{A} = (\omega, S, 0, +, \cdot)$, let φ be the formula $\exists v_1[Sv_1 = v_0]$, $\nu = v_1$, and $a = \langle 1, 1, \ldots \rangle$. Note that the condition on v_0 fails. Now $\mathrm{Subf}^{v_0}_{v_1}\varphi$ is the formula $\exists v_1[Sv_1 = v_1]$, and there is no $a \in \omega$ such that $Sa = a$, and hence $\overline{A} \not\models \mathrm{Subf}^{v_0}_{v_1}\varphi[a]$. Also, $\nu^{\overline{A}}(a) = v_1^{\overline{A}}(a) = a_1 = 1$, and hence $a^0_{\nu^{\overline{A}}(a)} = \langle 1, 1, \ldots \rangle$. Since $S0 = 1$, it follows that $\overline{A} \models \varphi[a^0_{\nu^{\overline{A}}(a)}]$. $\square$

Proposition 4.40. *Let* $\overline{A}$ *be an* $\mathscr{L}$*-structure, with* $\mathscr{L}$ *arbitrary. Define* $\Gamma = \{\varphi : \varphi$ *is a sentence and* $\overline{A} \models \varphi[a]$ *for some* $a : \omega \to A\}$. *Then* Γ *is complete and consistent.*

Proof. Note by Lemma 4.4 that $\overline{A} \models \varphi[a]$ for some $a : \omega \to A$ iff $\overline{A} \models \varphi[a]$ for every $a : \omega \to A$. Let φ be any sentence. Take any $a : \omega \to A$. If $\overline{A} \models \varphi[a]$, then $\varphi \in \Gamma$ and hence $\Gamma \vdash \varphi$. Suppose that $\overline{A} \not\models \varphi[a]$. Then $\overline{A} \models \neg\varphi[a]$, hence $\neg\varphi \in \Gamma$, hence $\Gamma \vdash \neg\varphi$.

This shows that Γ is complete. Suppose that Γ is not consistent. Then $\Gamma \vdash \neg(v_0 = v_0)$ by Lemma 4.1. Then $\Gamma \models \neg(v_0 = v_0)$ by Theorem 3.2. Since $\overline{A}$ is a model of Γ, it is also a model of $\neg(v_0 = v_0)$, contradiction. $\square$

Proposition 4.41. *Call a set* Γ *strongly complete iff for every formula* φ, $\Gamma \vdash \varphi$ *or* $\Gamma \vdash \neg\varphi$. *If* Γ *is strongly complete, then* $\Gamma \vdash \forall v_0 \forall v_1(v_0 = v_1)$.

Proof. Assume that Γ is strongly complete. Then $\Gamma \vdash v_0 = v_1$ or $\Gamma \vdash \neg(v_0 = v_1)$. If $\Gamma \vdash v_0 = v_1$, then by generalization, $\Gamma \vdash \forall v_0 \forall v_1 (v_0 = v_1)$. Suppose that $\Gamma \vdash \neg(v_0 = v_1)$. Then by generalization, $\Gamma \vdash \forall v_0 \neg(v_0 = v_1)$. By Theorem 3.25, $\Gamma \vdash \forall v_0 \neg(v_0 = v_1) \to \neg(v_1 = v_1)$. Hence $\Gamma \vdash \neg(v_1 = v_1)$. But also $\Gamma \vdash v_1 = v_1$ by Proposition 3.4, so Γ is inconsistent by Lemma 4.1, and hence again $\Gamma \vdash \forall v_0 \forall v_1 (v_0 = v_1)$. $\square$

Proposition 4.42. *If Γ is rich, then for every term σ with no variables occurring in σ there is an individual constant $\mathbf{c}$ such that $\Gamma \vdash \sigma = \mathbf{c}$.*

Proof. By richness we have $\Gamma \vdash \exists v_0(v_0 = \sigma) \to \mathbf{c} = \sigma$ for some individual constant $\mathbf{c}$. Then using (L4) it follows that $\Gamma \vdash \mathbf{c} = \sigma$. $\square$

Proposition 4.43. *If Γ is rich, then for every sentence φ there is a sentence ψ with no quantifiers in it such that $\Gamma \vdash \varphi \leftrightarrow \psi$.*

Proof. We proceed by induction on the number m of symbols $\neg, \to, \forall$ in φ. (More exactly, by the number of the integers 1,2,4 that occur in the sequence φ.) If $m = 0$, then φ is atomic and we can take $\psi = \varphi$. Assume the result for m and suppose that φ has $m+1$ integers 1,2,4 in it. Then there are three possibilities. First, $\varphi = \neg\varphi'$. Let ψ' be a quantifier-free sentence such that $\Gamma \vdash \varphi' \leftrightarrow \psi'$. Then $\Gamma \vdash \varphi \leftrightarrow \neg\psi'$. Second, $\varphi = (\varphi' \to \varphi'')$. Choose quantifier-free sentences ψ' and ψ'' such that $\Gamma \vdash \varphi' \leftrightarrow \psi'$ and $\Gamma \vdash \varphi'' \leftrightarrow \psi''$. Then $\Gamma \vdash \varphi \leftrightarrow (\psi' \to \psi'')$. Third, $\varphi = \forall v_i \varphi'$. By richness, let c be an individual constant such that $\Gamma \vdash \exists v_i \neg\varphi' \to \mathrm{Subf}_c^{v_i} \neg\varphi'$. Then by Theorem 3.31 we get

(1) $\Gamma \vdash \exists v_i \neg\varphi' \leftrightarrow \mathrm{Subf}_c^{v_i} \neg\varphi'$.

Now $\mathrm{Subf}_c^{v_i} \varphi'$ has only m integers 1,2,4 in it, so by the inductive hypothesis there is a sentence ψ with no quantifiers in it such that $\Gamma \vdash \mathrm{Subf}_c^{v_i} \varphi' \leftrightarrow \psi$ and hence

(2) $\Gamma \vdash \mathrm{Subf}_c^{v_i} \neg\varphi' \leftrightarrow \neg\psi$.

From (1) and (2) and a tautology we get $\Gamma \vdash \neg\exists v_i \neg\varphi' \leftrightarrow \psi$. Then by Proposition 3.31, $\Gamma \vdash \forall v_i \varphi' \leftrightarrow \psi$, finishing the inductive proof. $\square$

Proposition 4.44. *The following set Γ of sentences says that $<$ is a linear ordering and there are infinitely many elements. This set Γ is not complete.*

$$\neg\exists v_0(v_0 < v_0);$$
$$\forall v_0 \forall v_1 \forall v_2 [v_0 < v_1 \wedge v_1 < v_2 \to v_0 < v_2];$$
$$\forall v_0 \forall v_1 [v_0 < v_1 \vee v_0 = v_1 \vee v_1 < v_0];$$
$$\bigwedge_{i<j<n} \neg(v_i = v_j) \quad \textit{for every positive integer } n.$$

Proof. The following sentence φ holds in $(\mathbb{Q}, <)$ but not in $(\omega, <)$:

$$\forall v_0 \forall v_1 [v_0 < v_1 \to \exists v_2 (v_0 < v_2 \wedge v_2 < v_1)].$$

Since φ does not hold in $(\omega, <)$, we have $\Gamma \nvdash \varphi$, by Theorem 4.2. But since φ holds in $(\mathbb{Q}, <)$, we also have $\Gamma \nvdash \neg\varphi$ by Theorem 4.2. So Γ is not complete. □

Proposition 4.45. *If a sentence φ holds in every infinite model of a set Γ of sentences, then there is an $m \in \omega$ such that it holds in every model of Γ with at least m elements.*

Proof. Suppose that φ holds in every infinite model of a set Γ of sentences, but for every $m \in \omega$ there is a model $\overline{M}$ of Γ with at least m elements such that φ does not hold in $\overline{M}$. Let Δ be the following set:

$$\Gamma \cup \left\{ \bigwedge_{i<j<n} \neg(v_i = v_j) : n \text{ a positive integer} \right\} \cup \{\neg\varphi\}.$$

Our hypothesis implies that every finite subset Δ' of Δ has a model; for if m is the maximum of all n such that the above big conjunction is in Δ', then the hypothesis yields a model of Δ'. By the compactness theorem we get a model $\overline{N}$ of Δ. Thus $\overline{N}$ is an infinite model of Γ in which φ does not hold, contradiction. □

Proposition 4.46. *Let $\mathscr{L}$ be the language of ordering. There is no set Γ of sentences whose models are exactly the well-ordering structures.*

Proof. Suppose there is such a set. Let us expand the language $\mathscr{L}$ to a new one $\mathscr{L}'$ by adding an infinite sequence $\mathbf{c}_m$, $m \in \omega$, of individual constants. Then consider the following set Θ of sentences: all members of Γ, plus all sentences $\mathbf{c}_{m+1} < \mathbf{c}_m$ for $m \in \omega$. Clearly every finite subset of Θ has a model, so let $\overline{A} = (A, <, a_i)_{i<\omega}$ be a model of Θ itself. (Here a_i is the 0-ary function, i.e., element of A, corresponding to $\mathbf{c}_i$.) Then $a_0 > a_1 > \cdots$; so $\{a_i : i \in \omega\}$ is a nonempty subset of A with no least element, contradiction. □

Proposition 4.47. *Suppose that Γ is a set of sentences, and φ is a sentence. If $\Gamma \models \varphi$, then $\Delta \models \varphi$ for some finite $\Delta \subseteq \Gamma$.*

Proof. We prove the contrapositive: Suppose that for every finite subset Δ of Γ, $\Delta \not\models \varphi$. Thus every finite subset of $\Gamma \cup \{\neg\varphi\}$ has a model, so $\Gamma \cup \{\neg\varphi\}$ has a model, proving that $\Gamma \not\models \varphi$. □

Proposition 4.48. *Suppose that f is a function mapping a set M into a set N. Let $R = \{(a, b) : a, b \in M$ and $f(a) = f(b)\}$. Then R is an equivalence relation on M.*

Proof. If $a \in M$, then $f(a) = f(a)$, so $(a, a) \in R$. Thus R is reflexive on M. Suppose that $(a, b) \in R$. Then $f(a) = f(b)$, so $f(b) = f(a)$ and hence $(b, a) \in R$. Thus R is symmetric. Suppose that $(a, b) \in R$ and $(b, c) \in R$. Then $f(a) = f(b)$ and $f(b) = f(c)$, so $f(a) = f(c)$ and hence $(a, c) \in R$. □

Proposition 4.49. *Suppose that R is an equivalence relation on a set M. Then there is a function f mapping M into some set N such that $R = \{(a, b) : a, b \in M$ and $f(a) = f(b)\}$.*

Let N be the collection of all equivalence classes under R. For each $a \in M$ let $f(a) = [a]_R$. Then $(a, b) \in R$ iff $a, b \in M$ and $[a]_R = [b]_R$ iff $a, b \in M$ and $f(a) = f(b)$. □

Proposition 4.50. *Let Γ be a set of sentences in a first-order language, and let Δ be the collection of all sentences holding in every model of Γ. Then $\Delta = \{\varphi : \varphi$ is a sentence and $\Gamma \vdash \varphi\}$.*

Proof. For $\subseteq$, suppose that $\varphi \in \Delta$. To prove that $\Gamma \vdash \varphi$ we use the compactness theorem, proving that $\Gamma \models \varphi$. Let $\overline{A}$ be any model of Γ. Since $\varphi \in \Delta$, it follows that $\overline{A}$ is a model of Γ, as desired.

For $\supseteq$, suppose that φ is a sentence and $\Gamma \vdash \varphi$. Then by the easy direction of the completeness theorem, $\Gamma \models \varphi$. That is, every model of Γ is a model of φ. Hence $\varphi \in \Delta$. □

5. The axioms of set theory

ZFC, the axioms of set theory, are formulated in a language which has just one nonlogical constant, a binary relation symbol $\in$. The development of set theory can be considered as taking place entirely within this language, or in various finite definitional extensions of it.

Before introducing any set-theoretic axioms at all, we can introduce some more abbreviations.

$x \subseteq y$ abbreviates $\forall z(z \in x \to z \in y)$.
$x \subset y$ abbreviates $x \subseteq y \wedge x \neq y$.

For $x \subseteq y$ we say that x is *included* or *contained* in y, or that x is a subset of y. Then $x \subset y$ means *proper* inclusion, containment, or subset.

Now we introduce the axioms of ZFC set theory. We give both a formal and informal description of them. The informal versions will suffice for much of these notes.

Axiom 1. (Extensionality) If two sets have the same members, then they are equal. Formally:

$$\forall x \forall y[\forall z(z \in x \leftrightarrow z \in y) \to x = y].$$

Note that the other implication here holds on the basis of logic.

Axiom 2. (Comprehension) Given any set z and any property φ, there is a subset of z consisting of those elements of z with the property φ.

Formally, for any formula φ with free variables among $x, z, w_1, \ldots, w_n$ we have an axiom

$$\forall z \forall w_1 \ldots \forall w_n \exists y \forall x(x \in y \leftrightarrow x \in z \wedge \varphi).$$

Note that the variable y is *not* free in φ.

From these first two axioms the existence of a set with no members, the *empty set* $\emptyset$, , follows:

Proposition 5.1. *There is a unique set with no members.*

Proof. On the basis of logic, there is at least one set z. By the comprehension axiom, let y be a set such that $\forall x(x \in y \leftrightarrow x \in z \wedge x \neq x)$. Thus y does not have any elements. By the extensionality axiom, such a set y is unique. $\square$

Proposition 5.1 is written in usual mathematical fashion. More formally we would write

$$\text{ZFC} \vdash \exists v_0[\forall v_1[\neg(v_1 \in v_0)] \wedge \forall v_2[\forall v_1[\neg(v_1 \in v_2)] \to v_0 = v_2]].$$

The same applies to most of the results which we will state. But some results are metatheorems, describing a whole collection of results of this sort.

In general, the set asserted to exist in the comprehension axiom is unique; we denote it by $\{x \in z : \varphi\}$. We sometimes write $\{x : \varphi\}$ if a suitable z is evident. Note that this notation cannot be put into the framework of definitional extensions. But it is clear that uses of it can be eliminated, if necessary.

Axiom 3. (Pairing) For any sets x, y there is a set which has them as members (possibly along with other sets). Formally:

$$\forall x \forall y \exists z (x \in z \wedge y \in z).$$

The *unordered pair* $\{x, y\}$ is by definition the set $\{u \in z : u = x \text{ or } u = y\}$, where z is as in the pairing axiom. The definition does not depend on the particular such z that is chosen. This same remark can be made for several other definitions below. We define the *singleton* $\{x\}$ to be $\{x, x\}$.

Axiom 4. (Union) For any family $\mathscr{A}$ of sets, we can form a new set A which has as elements all elements which are in at least one member of $\mathscr{A}$ (maybe A has even more elements). Formally:

$$\forall \mathscr{A} \exists A \forall Y \forall x (x \in Y \wedge Y \in \mathscr{A} \to x \in A).$$

With A as in this axiom, we define $\bigcup \mathscr{A} = \{x \in A : \exists Y \in \mathscr{A}(x \in Y)\}$. We call $\bigcup \mathscr{A}$ the *union* of $\mathscr{A}$. Also, let $x \cup y = \bigcup\{x, y\}$. This is the *union* of x and y.

Axiom 5. (Power set) For any set x, there is a set which has as elements all subsets of x, and again possibly has more elements. Formally:

$$\forall x \exists y \forall z (z \subseteq x \to z \in y).$$

Axiom 6. (Infinity) There is a set which intuitively has infinitely many elements:

$$\exists x [\emptyset \in x \wedge \forall y \in x (y \cup \{y\} \in x)].$$

If we take the smallest set x with these properties we get the natural numbers, as we will see later.

Axiom 7. (Replacement) If a function has domain a set, then its range is also a set. Here we use the intuitive notion of a function. Later we define the rigorous notion of a function. The present intuitive notion is more general, however; it is expressed rigorously as a formula with a function-like property. The rigorous version of this axiom runs as follows.

For each formula with free variables among $x, y, A, w_1, \ldots, w_n$, the following is an axiom.

$$\forall A \forall w_1 \ldots \forall w_n [\forall x \in A \exists! y \varphi \to \exists Y \forall x \in A \exists y \in Y \varphi].$$

For the next axiom, we need another definition. For any sets x, y, let $x \cap y = \{z \in x : z \in y\}$. This is the *intersection* of x and y.

Axiom 8. (Foundation) Every nonempty set x has a member y which has no elements in common with x. This is a somewhat mysterious axiom which rules out such anti-intuitive situations as $a \in a$ or $a \in b \in a$.

$$\forall x [x \neq \emptyset \to \exists y \in x (x \cap y = \emptyset)].$$

Axiom 9. (Choice) This axiom will be discussed carefully later; it allows one to pick out elements from each of an infinite family of sets. A convenient form of the axiom to start with is as follows. For any family $\mathscr{A}$ of nonempty sets such that no two members of $\mathscr{A}$ have an element in common, there is a set B having exactly one element in common with each member of $\mathscr{A}$.

$$\forall \mathscr{A} [\forall x \in \mathscr{A} (x \neq \emptyset) \wedge \forall x \in \mathscr{A} \forall y \in \mathscr{A} (x \neq y \to x \cap y = \emptyset) \to \exists \mathscr{B} \forall x \in \mathscr{A} \exists ! y (y \in x \wedge y \in \mathscr{B}).$$

The axiom of choice will not be used until later, where we will give several equivalent forms of it.

6. Elementary set theory

Here we will see how the axioms are used to develop very elementary set theory. To some extent the main purpose of this chapter is to establish common notation.

The proof of the following theorem shows what happens to Russell's paradox in our axiomatic development. Russell's paradox runs as follows, working in naive, non-axiomatic set theory. Let $x = \{y : y \notin y\}$. If $x \in x$, then $x \notin x$; but also if $x \notin x$, then $x \in x$. Contradiction.

Theorem 6.1. $\neg\exists z\forall x(x \in z)$.

Proof. Suppose to the contrary that $\forall x(x \in z)$. Let $y = \{x \in z : x \notin x\}$. Then $(y \in y \leftrightarrow y \notin y)$, contradiction. □

Lemma 6.2. *If $\{x, y\} = \{u, v\}$, then one of the following conditions holds:*
(i) $x = u$ and $y = v$;
(ii) $x = v$ and $y = u$.

Proof. Since $x \in \{x, y\} = \{u, v\}$, we have $x = u$ or $x = v$.

Case 1. $x = u$. Since $y \in \{x, y\} = \{u, v\}$, we have $y = u$ or $y = v$. If $y = v$, that is as desired. If $y = u$, then $x = y$ too, and $v \in \{u, v\} = \{x, y\}$, so $v = x = y$. In any case, $y = v$.

Case 2. $x = v$. By symmetry to case 1, $y = u$. □

Now we can define the notion of an *ordered pair*: $(x, y) = \{\{x\}, \{x, y\}\}$.

Lemma 6.3. *If $(x, y) = (u, v)$, then $x = u$ and $y = v$.*

Proof. Assume that $(x, y) = (u, v)$. Thus $\{\{x\}, \{x, y\}\} = \{\{u\}, \{u, v\}\}$. By Lemma 6.1, this gives two cases.

Case 1. $\{x\} = \{u\}$ and $\{x, y\} = \{u, v\}$. Then $x \in \{x\} = \{u\}$, so $x = u$. By Lemma 6.1 again, $\{x, y\} = \{u, v\}$ implies that either $y = v$, or else $x = v$ and $y = u$; in the latter case, $y = u = x = v$. So $y = v$ in any case.

Case 2. $\{x\} = \{u, v\}$ and $\{x, y\} = \{u\}$. Then $u \in \{u, v\} = \{x\}$, so $u = x$. Similarly $v = x$. Now $y \in \{x, y\} = \{u\}$, so $y = u = x = v$. □

This lemma justifies the following definition:

$$1^{\text{st}}(a, b) = a \quad \text{and} \quad 2^{\text{nd}}(a, b) = b.$$

These are the *first and second coordinates* of the ordered pair.

The notion of intersection is similar to that of union, but there is a minor problem concerning what to define the intersection of the empty set to be. We have decided to let it be the empty set.

Theorem 6.4. *For any set $\mathscr{F}$ there is a set y such that if $\mathscr{F} \neq \emptyset$ then $\forall x[x \in y \leftrightarrow \forall z \in \mathscr{F}[x \in z]]$, while $y = \emptyset$ if $\mathscr{F} = \emptyset$.*

Proof. Let $\mathscr{F}$ be given. If $\mathscr{F} = \emptyset$, let $y = \emptyset$. Otherwise, choose $w \in \mathscr{F}$ and let $y = \{x \in w : \forall z \in \mathscr{F}[x \in z]\}$. $\square$

The set y in Theorem 6.4 is clearly unique, and we denote it by $\bigcap \mathscr{F}$. This is the *intersection* of $\mathscr{F}$. We already introduced in Chapter 2 the notations $\bigcup$, $\cup$, and $\cap$. To round out the simple Boolean operations we define

$$A \backslash B = \{x \in A : x \notin B\}.$$

This is the *relative complement* of B in A.

Sets a, b are *disjoint* iff $a \cap b = \emptyset$.

The replacement schema will almost always be used in connection with the comprehension schema. Namely, under the assumption $\forall x \in A \exists! y \varphi(x, y)$, we choose Y by the replacement axiom, so that $\forall x \in A \exists y \in Y \varphi(x, y)$; then we form

$$\{y \in Y : \exists x \in A \varphi(x, y)\}.$$

Lemma 6.5. $\forall A \forall B \exists Z \forall z (z \in Z \leftrightarrow \exists x \in A \exists y \in B (z = (x, y)))$.

Proof. Define

$$Z = \{z \in \mathscr{P}(\mathscr{P}(A \cup B)) : \exists a \in A \exists b \in B[z = (a, b)]\}.$$

Thus if $z \in Z$ then $\exists a \in A \exists b \in B[z = (a, b)]$. Now suppose that $a \in A$, $b \in B$, and $z = (a, b)$. Then $a, b \in A \cup B$, so $\{a\}, \{a, b\} \in \mathscr{P}(A \cup B)$, and so $z = (a, b) = \{\{a\}, \{a, b\}\} \in \mathscr{P}(\mathscr{P}(A \cup B))$, and hence $z \in Z$. $\square$

We now define $A \times B$ to be the unique Z of Lemma 6.5; this is the *cartesian product* of A and B. Normally we would define $A \times B$ as follows:

$$A \times B = \{(x, y) : x \in A \wedge y \in B\}.$$

This notation means

$$\{u : \exists x, y (u = (x, y) \wedge x \in A \wedge y \in B)\},$$

which is justified by the lemma.

An important informal notation is

$$(*) \qquad \{\tau(x, y) \in A : \varphi(x, y)\},$$

where $\tau(x, y)$ is some set determined by x and y. That is, there is a formula $\psi(w, x, y)$ in our set theoretic language such that $\forall x, y \exists! w \psi(w, x, y)$, and $\tau(x, y)$ is this w. For example $\tau(x, y)$ might be $x \cup y$, or (x, y). Then $(*)$ really means

$$(**) \qquad \{w \in A : \exists x, y[\psi(w, x, y) \wedge \varphi(x, y)]\}.$$

A *relation* is a set of ordered pairs.

Lemma 6.6. *If* $(x,y) \in R$ *then* $x, y \in \bigcup\bigcup R$.

Proof. $x \in \{x\} \in \{\{x\},\{x,y\}\} = (x,y) \in R$, so $x \in \bigcup\bigcup R$. Similarly $y \in \bigcup\bigcup R$. □

This lemma justifies the following definitions of the *domain* and *range* of a set R (we think of R as a relation, but the definitions apply to any set):

$$\begin{aligned} \mathrm{dmn}(R) &= \{x : \exists y((x,y) \in R)\}; \\ \mathrm{rng}(R) &= \{y : \exists x((x,y) \in R)\}. \end{aligned}$$

Now we define, using the notation above,

$$R^{-1} = \{(x,y) \in \mathrm{rng}(R) \times \mathrm{dmn}(R) : (y,x) \in R\}.$$

This is the *inverse* or *converse* of R. Note that R^{-1} is a relation, even if R is not. Clearly $(x,y) \in R^{-1}$ iff $(y,x) \in R$, for any x, y, R. Usually we use this notation only when R is a function (defined shortly as a special kind of relation), and even then it is more general than one might expect, since the function in question does not have to be 1-1 (another notion defined shortly).

A *function* is a relation f such that

$$\forall x \in \mathrm{dmn}(f) \exists! y \in \mathrm{rng}(f)[(x,y) \in f].$$

Some common notation and terminology is as follows. $f : A \to B$ means that f is a function, $\mathrm{dmn}(f) = A$, and $\mathrm{rng}(f) \subseteq B$. We say then that f *maps* A *into* B. If $f : A \to B$ and $x \in A$, then $f(x)$ is the unique y such that $(x,y) \in f$. This is the *value of* f *with the argument* x. We may write other things like f_x, f^x in place of $f(x)$. Note that if $f, g : A \to B$, then $f = g$ iff $\forall a \in A[f(a) = g(a)]$. If $f : A \to B$ and $C \subseteq A$, the *restriction* of f to C is $f \cap (C \times B)$; it is denoted by $f \restriction C$. The *image* of a subset C of A is $f[C] \stackrel{\text{def}}{=} \mathrm{rng}(f \restriction C)$. Note that $f[C] = \{f(c) : c \in C\}$. If $D \subseteq B$ then the *preimage* of D under f is $f^{-1}[D] \stackrel{\text{def}}{=} \{x \in A : f(x) \in D\}$. $x \mapsto \tau$ means $(x, \tau) \in f$.

For any sets f, g we define

$$f \circ g = \{(a,b) : \exists c[(a,c) \in g \text{ and } (c,b) \in f]\}.$$

This is the *composition* of f and g. We usually apply this notation when there are sets A, B, C such that $g : A \to B$ and $f : B \to C$.

Lemma 6.7. *(i) If* $g : A \to B$ *and* $f : B \to C$, *then* $(f \circ g) : A \to C$ *and* $\forall a \in A[(f \circ g)(a) = f(g(a))]$.

(ii) If $g : A \to B$, $f : B \to C$, *and* $h : C \to D$, *then* $h \circ (f \circ g) = (h \circ f) \circ g$.

Proof. (i): First we show that $f \circ g$ is a function. Suppose that $(a,b), (a,b') \in (f \circ g)$. Accordingly choose c, c' so that $(a,c) \in g$, $(c,b) \in f$, $(a,c') \in g$, and $(c',b') \in f$. Then $g(a) = c$, $f(c) = b$, $g(a) = c'$, and $f(c') = b'$. So $c = c'$ and hence $b = b'$. This shows that $f \circ g$ is a function. Clearly $\text{dmn}(f \circ g) = A$ and $\text{rng}(f \circ g) \subseteq C$. For any $a \in A$ we have $(a, g(a)) \in g$ and $(g(a), f(g(a))) \in f$, and hence $(a, f(g(a))) \in (f \circ g)$, so that $(f \circ g)(a) = f(g(a))$.

(ii): By (i), both functions map A into D. For any $a \in A$ we have

$$(h \circ (f \circ g))(a) = h((f \circ g)(a)) = h(f(g(a))) = (h \circ f)(g(a)) = ((h \circ f) \circ g)(a).$$

Hence the equality holds. □

Given $f : A \to B$, we call f *injective*, or 1-1, if f^{-1} is a function; we call f *surjective*, or *onto*, if $\text{rng}(f) = B$; and we call f *bijective* if it is both injective and surjective.

A function f will sometimes be written in the form $\langle f(i) : i \in I \rangle$, where $I = \text{dmn}(f)$. As an informal usage, we will even define functions in the form $\langle \ldots x \ldots : x \in I \rangle$, meaning the function f with domain I such that $f(x) = \ldots x \ldots$ for all $x \in I$.

If A is a function with domain I, we define

$$\bigcup_{i \in I} A_i = \bigcup \text{rng}(A) \quad \text{and} \quad \bigcap_{i \in I} A_i = \bigcap \text{rng}(A).$$

Proposition 6.8. *If $f : A \to B$ and $\langle C_i : i \in I \rangle$ is a system of subsets of A, then $f\left[\bigcup_{i \in I} C_i\right] = \bigcup_{i \in I} f[C_i]$.*

Proof.

$$\begin{aligned}
x \in f\left[\bigcup_{i \in I} C_i\right] \quad & \text{iff} \quad x \in \text{rng}\left(f \restriction \bigcup_{i \in I} C_i\right) \\
& \text{iff} \quad \exists y \in \bigcup_{i \in I} C_i [f(y) = x] \\
& \text{iff} \quad \exists i \in I \exists y \in C_i [f(y) = x] \\
& \text{iff} \quad \exists i \in I [x \in \text{rng}(f \restriction C_i)] \\
& \text{iff} \quad \exists i \in I [x \in f[C_i]] \\
& \text{iff} \quad x \in \bigcup_{i \in I} f[C_i].
\end{aligned}$$

□

Proposition 6.9. *If $f : A \to B$ and $C, D \subseteq A$, then $f[C \cap D] \subseteq f[C] \cap f[D]$.*

Proof. Take any $x \in f[C \cap D]$. Choose $y \in C \cap D$ such that $x = f(y)$. Since $y \in C$, we have $x \in f[C]$. Similarly, $x \in f[D]$. So $x \in f[C] \cap f[D]$. Since x is arbitrary, this shows that $f[C \cap D] \subseteq f[C] \cap f[D]$. □

Proposition 6.10. *There are $f : A \to B$ and $C, D \subseteq A$ such that $f[C \cap D] \neq f[C] \cap f[D]$.*

Proof. Let $\text{dmn}(f) = \{a, b\}$ with $a \neq b$ and with $f(a) = a = f(b)$. Let $C = \{a\}$ and $D = \{b\}$. Then $C \cap D = \emptyset$, so $f[C \cap D] = \emptyset$, while $f[C] = \{a\} = f[D]$ and hence $f[C] \cap f[D] = \{a\} \neq \emptyset$. So $f[C \cap D] \neq f[C] \cap f[D]$. □

Proposition 6.11. *If $f : A \to B$ and $C, D \subseteq A$, then $f[C] \backslash f[D] \subseteq f[C \backslash D]$.*

Proof. Suppose that $x \in f[C] \backslash f[D]$. Choose $c \in C$ such that $x = f(c)$. Since $x \notin f[D]$, we have $c \notin D$. So $c \in C \backslash D$ and hence $x \in f[C \backslash D]$, □

Proposition 6.12. *There are $f : A \to B$ and $C, D \subseteq A$, such that $f[C] \backslash f[D] \neq f[C \backslash D]$.*

Proof. Take the same f, C, D as for Proposition 6.10. Then $C \backslash D = \{a\}$ and so $f[C \backslash D] \neq \emptyset$. But $f[C] = \{a\} = f[D]$, so $f[C] \backslash f[D] = \emptyset$. □

Proposition 6.13. *If $f : A \to B$ and $\langle C_i : i \in I \rangle$ is a system of subsets of B, then $f^{-1}\left[\bigcup_{i \in I} C_i\right] = \bigcup_{i \in I} f^{-1}[C_i]$.*

Proof. For any $b \in B$ we have

$$\begin{aligned} b \in f^{-1}\left[\bigcup_{i \in I} C_i\right] \quad & \text{iff} \quad f(b) \in \bigcup_{i \in I} C_i \\ & \text{iff} \quad \exists i \in I[f(b) \in C_i] \\ & \text{iff} \quad \exists i \in I[b \in f^{-1}[C_i]] \\ & \text{iff} \quad b \in \bigcup_{i \in I} f^{-1}[C_i]. \end{aligned}$$

□

Proposition 6.14. *If $f : A \to B$ and $\langle C_i : i \in I \rangle$ is a system of subsets of B, then $f^{-1}\left[\bigcap_{i \in I} C_i\right] = \bigcap_{i \in I} f^{-1}[C_i]$.*

Proof. For any a,

$$\begin{aligned} a \in f^{-1}\left[\bigcap_{i \in I} C_i\right] \quad & \text{iff} \quad f(a) \in \bigcap_{i \in I} C_i \\ & \text{iff} \quad \forall i \in I[f(a) \in C_i] \\ & \text{iff} \quad \forall i \in I[a \in f^{-1}[C_i]] \\ & \text{iff} \quad a \in \bigcap_{i \in I} f^{-1}[C_i]. \end{aligned}$$

□

Proposition 6.15. *If $f : A \to B$ and $C, D \subseteq B$, then $f^{-1}[C \backslash D] = f^{-1}[C] \backslash f^{-1}[D]$.*

Proof. For any a,

$$\begin{aligned} a \in f^{-1}[C \backslash D] \quad & \text{iff} \quad f(a) \in C \backslash D \\ & \text{iff} \quad f(a) \in C \text{ and } f(a) \notin D \\ & \text{iff} \quad a \in f^{-1}[C] \text{ and } a \notin f^{-1}[D] \\ & \text{iff} \quad a \in f^{-1}[C] \backslash f^{-1}[D]. \end{aligned}$$

□

Proposition 6.16. *If $f : A \to B$ and $C \subseteq A$, then*

$$\{b \in B : f^{-1}[\{b\}] \subseteq C\} = B \backslash f[A \backslash C].$$

Proof. First suppose that b is in the left side; but suppose also, aiming for a contradiction, that $b \in f[A\backslash C]$. Say $b = f(a)$, with $a \in A\backslash C$. Then $a \in f^{-1}[\{b\}]$, so $a \in C$, contradiction.

Second, suppose that b is in the right side. Take any $a \in f^{-1}[\{b\}]$. Then $f(a) = b$, and it follows that $a \in C$, as desired. □

. . . .

Proposition 6.17. *For any sets A, B define $A \triangle B = (A\backslash B) \cup (B\backslash A)$; this is called the symmetric difference of A and B. If A, B, C are given sets, then $A\triangle(B\triangle C) = (A\triangle B)\triangle C$.*

Proof. Let $D = A \cup B \cup C$, $A' = D\backslash A$, $B' = D\backslash B$, and $C' = D\backslash C$. Then

$$\begin{aligned}
A\triangle B &= (A \cap B') \cup (B \cap A');\\
(A\triangle B)' &= ((A \cap B') \cup (B \cap A'))'\\
&= (A \cap B')' \cap (B \cap A')'\\
&= (A' \cup B) \cap (B' \cup A)\\
&= (A' \cap B') \cup (A \cap B).
\end{aligned}$$

These equations hold for any sets A, B. Now

$$\begin{aligned}
A\triangle(B\triangle C) &= (A \cap (B\triangle C)') \cup ((B\triangle C) \cap A'\\
&= (A \cap ((B' \cap C') \cup (B \cap C))) \cup (((B \cap C') \cup (C \cap B')) \cap A')\\
&= (A \cap B' \cap C') \cup (A \cap B \cap C) \cup (A' \cap B \cap C') \cup (A' \cap B' \cap C).
\end{aligned}$$

This holds for any sets A, B, C. Hence

$$\begin{aligned}
(A\triangle B)\triangle C &= C\triangle(A\triangle B)\\
&= (C \cap A' \cap B') \cup (C \cap A \cap B) \cup (C' \cap A \cap B') \cup (C' \cap A' \cap B)\\
&= A\triangle(B\triangle C).
\end{aligned}$$

□

For any set A let

$$\mathrm{Id}_A = \{\langle x, x\rangle : x \in A\}.$$

Thus

$$\mathrm{Id}_A = \{y \in A \times A : \exists x \in A[y = \langle x, x\rangle]\}.$$

Proposition 6.18. *Suppose that $f : A \to B$. Then f is surjective iff there is a $g : B \to A$ such that $f \circ g = Id_B$.*

Proof. $\Leftarrow$: given $b \in B$, we have $b = (f \circ g)(b) = f(g(b))$; so f is surjective.

$\Rightarrow$: Assume that f is surjective. Let

$$\mathscr{A} = \{\{(b,a) : a \in A, f(a) = b\} : b \in B\}.$$

Each member of $\mathscr{A}$ is nonempty; for let $x \in \mathscr{A}$. Choose $b \in B$ such that $x = \{(b,a) : a \in A, f(a) = b\}$. Choose $a \in A$ such that $f(a) = b$. So $(b,a) \in x$.

The members of $\mathscr{A}$ are pairwise disjoint: suppose $x, y \in \mathscr{A}$ with $x \neq y$. Choose b, c so that $x = \{(b,a) : a \in A, f(a) = b\}$ and $y = \{(b,a) : a \in A, f(a) = c\}$. If $u \in x \cap y$, then there exist $a, a' \in A$ such that $u = (b,a)$, $f(a) = b$, and also $(u = (c,a')$, $f(a') = c$. So by Theorem 6.3, $b = c$. But then $x = y$, contradiction.

Now by the axiom of choice, let C have exactly one element in common with each member of $\mathscr{A}$. Then define

$$g = \{(b,a) \in C : a \in A, f(a) = b\}.$$

Now g is a function. For, suppose that $(b,a), (b,a') \in g$. Let $x = \{(b,a'') : a'' \in A, f(a'') = b\}$. Then $(b,a), (b,a') \in C \cap x$, so $(b,a) = (b,a')$. Hence $a = a'$.

Clearly $g \subseteq B \times A$. Next, $\mathrm{dmn}(g) = B$, for suppose that $b \in B$. Choose $x \in C \cap \{(b,a'') : a'' \in A, f(a'') = b\}$; say $x = (b,a)$ with $a \in A$, $f(a) = b$. Then $x \in g$ and so $b \in \mathrm{dmn}(g)$.

Thus $g : B \to A$. Take any $b \in B$, and let $g(b) = a$. So $(b,a) \in g$ and hence $f(a) = b$. So $f \circ g = \mathrm{Id}_B$. $\square$

Proposition 6.19. *Let A be a nonempty set. Suppose that $f : A \to B$. Then f is injective iff there is a $g : B \to A$ such that $g \circ f = Id_A$.*

Proof. First suppose that f is injective. Fix $a \in A$, and let

$$g = f^{-1} \cup \{(b,a) : b \in B \backslash \mathrm{rng}(f)\}.$$

Then g is a function. In fact, suppose that $(b,c), (b,d) \in g$. If both are in f^{-1}, then $(c, b(, (d,b) \in f$, so $f(c) = b = f(d)$ and hence $c = d$ since f is injective. If $(b,c) \in f^{-1}$ and $b \in B \backslash \mathrm{rng}(f)$, the $(c,b) \in f$, so $b \in \mathrm{rng}(f)$, contradiction. If $(b,c), (b,d) \notin f^{-1}$, then $c = d = a$.

Clearly then $g : B \to A$. For any $a \in A$ we have $(a, f(a)) \in f$, hence $(f(a), a) \in f^{-1} \subseteq g$, and so $g(f(a)) = a$.

Second, suppose that $g : B \to A$ and $g \circ f = \mathrm{Id}_A$. Suppose that $f(a) = f(a')$. Then $a = (g \circ f)(a) = g(f(a)) = g(f(a')) = (g \circ f)(a') = a'$. $\square$

Proposition 6.20. *Suppose that $f : A \to B$. f is a bijection iff there is a $g : B \to A$ such that $f \circ g = Id_B$ and $g \circ f = Id_A$.*

Proof. $\Rightarrow$: Assume that f is a bijection. By E6.11 there is a $g : B \to A$ such that $g \circ f = \mathrm{Id}_A$. We claim that $f \circ g = \mathrm{Id}_B$. Since f is a bijection, the relation f^{-1} is also a bijection. Now for any $b \in B$,

$$(f \circ g)(b) = f(g(b)) = f(g(f(f^{-1}(b)))) = f((g \circ f)(f^{-1}(b))) = f(f^{-1}(b)) = b.$$

So $f \circ g = \mathrm{Id}_B$, as desired.

$\Leftarrow$: Assume that g is as indicated. Then f is injective, since $f(a) = f(b)$ implies that $a = g(f(a)) = g(f(a')) = a'$. And f is surjective, since for a given $b \in B$ we have $f(g(b)) = b$. $\square$

For any sets R, S define

$$R|S = \{(x,z) : \exists y((x,y) \in R \wedge (y,z) \in S)\}.$$

Thus

$$R|S = \{(x,z) \in \mathrm{dmn}(R) \times \mathrm{rng}(S) : \exists y((x,y) \in R \wedge (y,z) \in S)\}.$$

Proposition 6.21. *Suppose that $f, g : A \to A$. Then*

$$(A \times A)\backslash[((A \times A)\backslash f)|((A \times A)\backslash g)]$$

is a function.

Proof. Suppose that $(x,y),(x,z)$ are in the indicated set, with $y \neq z$. By symmetry say $f(x) \neq y$. Then $(x,y) \in [(A \times A)\backslash f]$, so it follows that $(y,z) \in g$, as otherwise $(x,z) \in [((A \times A)\backslash f)|((A \times A)\backslash g)]$. Hence $(y,y) \notin g$, so $(x,y) \in [((A \times A)\backslash f)|((A \times A)\backslash g)]$, contradiction. $\square$

Proposition 6.22. *Suppose that $f : A \to B$ is a surjection, $g : A \to C$, and $\forall x, y \in A[f(x) = f(y) \to g(x) = g(y)]$. Then there is a function $h : B \to C$ such that $h \circ f = g$.*

Proof. Let $h = \{(f(a), g(a)) : a \in A\}$. Then h is a function, for suppose that $(x,y),(x,z) \in h$. Choose $a, a' \in A$ so that $x = f(a)$, $y = g(a)$, $x = f(a')$, and $y = g(a')$. Thus $f(a) = f(a')$, so $g(a) = g(a')$, as desired.

Since f is a surjection it is clear that $\mathrm{dmn}(h) = B$. Clearly $\mathrm{rng}(h) \subseteq C$. So $h : B \to C$. If $a \in A$, then $(f(a), b(a)) \in h$, hence $h(f(a)) = g(a)$. This shows that $h \circ f = g$. $\square$

Proposition 6.23. *There are sets $\mathscr{A}$ and $\mathscr{B}$ for which the statement*

$\forall A \in \mathscr{A} \forall B \in \mathscr{B}(A \subseteq B)$ *implies that* $\bigcup \mathscr{A} \subseteq \bigcap \mathscr{B}$

is wrong.

Proof. If $\mathscr{A}$ has a nonempty member and $\mathscr{B}$ is empty, the implication does not hold. $\square$

Proposition 6.24. *If $\mathscr{B} \neq \emptyset$, then the statement in Proposition 6.23 holds.*

Proof. Suppose that $a \in \bigcup A$ and $B \in \mathscr{B}$; we want to show that $a \in B$. Choose $A \in \mathscr{A}$ such that $a \in A$. Since $A \subseteq B$, we have $a \in B$. $\square$

Proposition 6.25. *Suppose that $\forall A \in \mathscr{A} \exists B \in \mathscr{B}(A \subseteq B)$. Then $\bigcup \mathscr{A} \subseteq \bigcup \mathscr{B}$.*

Proof. Suppose that $a \in \bigcup \mathscr{A}$; we want to show that $a \in \mathscr{B}$. Choose $A \in \mathscr{A}$ such that $a \in A$. Then choose $B \in \mathscr{B}$ such that $A \subseteq B$. Then $a \in B$. Hence $a \in \bigcup \mathscr{B}$. $\square$

Proposition 6.26. *There are sets* $\mathscr{A}$ *and* $\mathscr{B}$ *such that the statement*

$\forall A \in \mathscr{A} \exists B \in \mathscr{B}(B \subseteq A)$ *implies that* $\bigcap \mathscr{B} \subseteq \bigcap \mathscr{A}$.

is wrong.

Proof. If $\mathscr{A}$ is empty and $\bigcap \mathscr{B}$ is nonempty, the statement is false. $\square$

Proposition 6.27. *If* $\mathscr{A} \neq \emptyset$, *then the statement in Proposition 6.26 holds.*

Proof. Suppose that $b \in \bigcap \mathscr{B}$ and $A \in \mathscr{A}$; we want to show that $b \in A$. Choose $B \in \mathscr{B}$ such that $B \subseteq A$. Now $b \in B$ since $b \in \bigcap \mathscr{B}$, so $b \in A$. $\square$

7. Ordinals, I

In this chapter we introduce the ordinals and give basic facts about them.

A set A is *transitive* iff $\forall x \in A \forall y \in x(y \in A)$; in other words, iff every element of A is a subset of A. This is a very important notion in the foundations of set theory, and it is essential in our definition of ordinals. An *ordinal number*, or simply an *ordinal*, is a transitive set of transitive sets. We use the first few Greek letters to denote ordinals. If α, β, γ are ordinals and $\alpha \in \beta \in \gamma$, then $\alpha \in \gamma$ since γ is transitive. This partially justifies writing $\alpha < \beta$ instead of $\alpha \in \beta$ when α and β are ordinals. This helps the intuition in thinking of the ordinals as kinds of numbers. We also define $\alpha \leq \beta$ iff $\alpha < \beta$ or $\alpha = \beta$.

By a vacuous implication we have:

Proposition 7.1. *$\emptyset$ is an ordinal.* □

Because of this proposition, the empty set is a number; it will turn out to be the first nonnegative integer, the first ordinal, and the first cardinal number. For this reason, we will use 0 and $\emptyset$ interchangably, trying to use 0 when numbers are involved, and $\emptyset$ when they are not.

Proposition 7.2. *If α is an ordinal, then so is $\alpha \cup \{\alpha\}$.*

Proof. If $x \in y \in \alpha \cup \{\alpha\}$, then $x \in y \in \alpha$ or $x \in y = \alpha$. Since α is transitive, $x \in \alpha$ in either case. So $\alpha \cup \{\alpha\}$ is transitive. Clearly every member of $\alpha \cup \{\alpha\}$ is transitive. □

We denote $\alpha \cup \{\alpha\}$ by $\alpha +' 1$. After introducing addition of ordinals, it will turn out that $\alpha + 1 = \alpha +' 1$ for every ordinal α, so that the prime can be dropped. This ordinal $\alpha +' 1$ is the *successor* of α. We define $1 = 0 +' 1$, $2 = 1 +' 1$, etc. (up through 9; no further since we do not want to try to justify decimal notation).

Proposition 7.3. *If A is a set of ordinals, then $\bigcup A$ is an ordinal.*

Proof. Suppose that $x \in y \in \bigcup A$. Choose $z \in A$ such that $y \in z$. Then z is an ordinal, and $x \in y \in z$, so $x \in z$; hence $x \in \bigcup A$. Thus $\bigcup A$ is transitive.

If $u \in \bigcup A$, choose $v \in A$ such that $u \in v$. then v is an ordinal, so u is transitive. □

We sometimes write $\sup(A)$ for $\bigcup A$. In fact, $\bigcup A$ is the least ordinal $\geq$ each member of A. We prove this shortly.

Proposition 7.4. *Every member of an ordinal is an ordinal.*

Proof. Let α be an ordinal, and let $x \in \alpha$. Then x is transitive since all members of α are transitive. Suppose that $y \in x$. Then $y \in \alpha$ since α is transitive. So y is transitive, since all members of α are transitive. □

Theorem 7.5. *$\forall x(x \notin x)$.*

Proof. Suppose that x is a set such that $x \in x$. Let $y = \{x\}$. By the foundation axiom, choose $z \in y$ such that $z \cap y = \emptyset$. But $z = x$, so $x \in z \cap y$, contradiction. □

Theorem 7.6. *There does not exist a set which has every ordinal as a member.*

Proof. Suppose to the contrary that A is such a set. Let $B = \{x \in A : x \text{ is an ordinal}\}$. Then B is a set of transitive sets and B itself is transitive. Hence B is an ordinal. So $B \in A$. It follows that $B \in B$. contradicting Theorem 7.5. □

Theorem 7.6 is what happens in our axiomatic framework to the Burali-Forti paradox.

Theorem 7.7. *If α and β are ordinals, then $\alpha = \beta$, $\alpha \in \beta$, or $\beta \in \alpha$.*

Proof. Suppose that this is not true, and let α and β be ordinals such that $\alpha \neq \beta$, $\alpha \notin \beta$, and $\beta \notin \alpha$. Let $A = (\alpha +' 1) \cup (\beta +' 1)$. Define $B = \{\gamma \in A : \exists \delta \in A[\gamma \neq \delta, \gamma \notin \delta, \text{ and } \delta \notin \gamma]\}$. Thus $\alpha \in B$, since we can take $\delta = \beta$. So $B \neq \emptyset$. By the foundation axiom, choose $\gamma \in B$ such that $\gamma \cap B = \emptyset$. Let $C = \{\delta \in A : \gamma \neq \delta, \gamma \notin \delta, \text{ and } \delta \notin \gamma\}$. So $C \neq \emptyset$ since $\gamma \in B$. By the foundation axiom choose $\delta \in C$ such that $\delta \cap C = \emptyset$. We will now show that $\gamma = \delta$, which is a contradiction.

Suppose that $\varepsilon \in \gamma$. Then $\varepsilon \notin B$. Clearly $\varepsilon \in A$, so it follows that $\forall \varphi \in A[\varepsilon = \varphi$ or $\varepsilon \in \varphi$ or $\varphi \in \varepsilon]$. Since $\delta \in A$ we thus have $\varepsilon = \delta$ or $\varepsilon \in \delta$ or $\delta \in \varepsilon$. If $\varepsilon = \delta$ then $\delta \in \gamma$, contradiction. If $\delta \in \varepsilon$, then $\delta \in \gamma$ since γ is transitive, contradiction. So $\varepsilon \in \delta$. This proves that $\gamma \subseteq \delta$.

Suppose that $\varepsilon \in \delta$. Then $\varepsilon \notin C$. It follows that $\gamma = \varepsilon$ or $\gamma \in \varepsilon$ or $\varepsilon \in \gamma$. If $\gamma = \varepsilon$ then $\gamma \in \delta$, contradiction. If $\gamma \in \varepsilon$ then $\gamma \in \delta$ since δ is transitive, contradiction. So $\varepsilon \in \gamma$. This proves that $\delta \subseteq \gamma$.

Hence $\delta = \gamma$, contradiction. □

Proposition 7.8. *$\alpha \leq \beta$ iff $\alpha \subseteq \beta$.*

Proof. $\Rightarrow$: Assume that $\alpha \leq \beta$ and $x \in \alpha$. Then $x < \alpha \leq \beta$, so $x < \beta$ since β is transitive. Hence $x \in \beta$. Thus $\alpha \subseteq \beta$.

$\Leftarrow$: Assume that $\alpha \subseteq \beta$. If $\beta < \alpha$, then $\beta < \beta$, hence $\beta \in \beta$, contradicting Theorem 7.5. Hence $\alpha \leq \beta$ by Theorem 7.7. □

Proposition 7.9. *$\alpha < \beta$ iff $\alpha \subset \beta$.*

Proof. $\alpha < \beta$ iff ($\alpha \leq \beta$ and $\alpha \neq \beta$) iff ($\alpha \subseteq \beta$ and $\alpha \neq \beta$) (by Proposition 7.8) iff $\alpha \subset \beta$. □

Proposition 7.10. *$\alpha < \beta$ iff $\alpha +' 1 \leq \beta$.*

Proof. $\Rightarrow$: Assume that $\alpha < \beta$. If $\beta < \alpha +' 1$, then $\beta \in \alpha \cup \{\alpha\}$, so $\beta \in \alpha$ or $\beta = \alpha$. Since $\alpha \in \beta$, this implies that $\beta \in \beta$, contradicting Theorem 7.5. Hence by Theorem 7.7, $\alpha +' 1 \leq \beta$.

$\Leftarrow$: Assume that $\alpha +' 1 \leq \beta$. Then $\alpha < \alpha +' 1 \leq \beta$, so $\alpha < \beta$. □

Proposition 7.11. *There do not exist ordinals α, β such that $\alpha < \beta < \alpha +' 1$.* □

Theorem 7.12. *If A is a set of ordinals, then $\alpha \leq \bigcup A$ for each $\alpha \in A$, and if β is an ordinal such that $\alpha \leq \beta$ for all $\alpha \in A$ then $\bigcup A \leq \beta$.*

Proof. Suppose that A is a set of ordinals. If $\alpha \in A$, then $\alpha \subseteq \bigcup A$, and so $\alpha \leq \bigcup A$ by Proposition 7.8.

Now suppose that β is an ordinal such that $\alpha \leq \beta$ for all $\alpha \in A$. Take any $x \in \bigcup A$. Choose $y \in A$ such that $x \in y$. Then $y \leq \beta$. Also $x < y$, so $x < \beta$. Hence $x \in \beta$. This proves that $\bigcup A \subseteq \beta$. Hence $\bigcup A \leq \beta$ by Proposition 7.8. $\square$

Theorem 7.13. *If Γ is a nonempty set of ordinals, then $\bigcap \Gamma$ is an ordinal, $\bigcap \Gamma \in \Gamma$, and $\bigcap \Gamma \leq \alpha$ for every $\alpha \in \Gamma$.*

Proof. The members of $\bigcap \Gamma$ are clearly ordinals, so for the first statement it suffices to show that $\bigcap \Gamma$ is transitive. Suppose that $\alpha \in \beta \in \bigcap \Gamma$; and suppose that $\gamma \in \Gamma$. Then $\beta \in \gamma$, and hence $\alpha \in \gamma$ since γ is transitive. This argument shows that $\alpha \in \bigcap \Gamma$. Since α is arbitrary, it follows that $\bigcap \Gamma$ is transitive, and hence is an ordinal.

For every $\alpha \in \Gamma$ we have $\bigcap \Gamma \subseteq \alpha$, and hence $\bigcap \Gamma \leq \alpha$ by Proposition 7.8.

Suppose that $\bigcap \Gamma \notin \Gamma$. For any $\alpha \in \Gamma$ we have $\bigcap \Gamma \subseteq \alpha$, hence $\bigcap \Gamma \leq \alpha$, hence $\bigcap \Gamma < \alpha$ since $\alpha \in \Gamma$ but we are assuming that $\bigcap \Gamma \notin \Gamma$. But this means that $\forall \alpha \in \Gamma[\bigcap \Gamma \in \alpha]$. So $\bigcap \Gamma \in \bigcap \Gamma$, contradiction. $\square$

Ordinals are divided into three classes as follows. First there is 0, the empty set. An ordinal α is a *successor ordinal* if $\alpha = \beta +' 1$ for some β. Finally, α is a *limit ordinal* if it is nonzero and is not a successor ordinal. Thus 1, 2, etc. are successor ordinals.

To prove the existence of limit ordinals, we need the infinity axiom. Let x be as in the statement of the infinity axiom. Thus $0 \in x$, and $y \cup \{y\} \in x$ for all $y \in x$. We define

$$\omega = \bigcap \{z \subseteq x : 0 \in z \text{ and } y \cup \{y\} \in z \text{ for all } y \in z\}.$$

This definition does not depend on the choice of x. In fact, suppose that also $0 \in x'$, and $y \cup \{y\} \in x'$ for all $y \in x'$; we want to show that

$$\begin{aligned} &\bigcap \{z \subseteq x : 0 \in z \text{ and } y \cup \{y\} \in z \text{ for all } y \in z\} \\ &= \bigcap \{z \subseteq x' : 0 \in z \text{ and } y \cup \{y\} \in z \text{ for all } y \in z\}. \end{aligned}$$

Let $\mathscr{A} = \{z \subseteq x : 0 \in z \text{ and } y \cup \{y\} \in z \text{ for all } y \in z\}$ and $\mathscr{A}' = \{z \subseteq x' : 0 \in z \text{ and } y \cup \{y\} \in z \text{ for all } y \in z\}$. Suppose that $w \in \bigcap \mathscr{A}$, and suppose that $z \in \mathscr{A}'$. Clearly $z \cap x \in \mathscr{A}$, so $w \in z \cap x$, so $w \in z$. This shows that $w \in \bigcap \mathscr{A}'$. Hence $\bigcap \mathscr{A} \subseteq \bigcap \mathscr{A}'$. The other inclusion is proved in the same way.

The members of ω are *natural numbers.*

Theorem 7.14. *If $A \subseteq \omega$, $0 \in A$, and $y \cup \{y\} \in A$ for all $y \in A$, then $A = \omega$.*

Proof. With x as in the definition of ω, we clearly have $x \cap A \in \mathscr{A}$ where $\mathscr{A}$ is as above. Hence $\omega \subseteq x \cap A \subseteq A$, so $A = \omega$. $\square$

Proposition 7.15. $0 \in \omega$, *and for all $y \in \omega$, also $y +' 1 \in \omega$.*

Proof. With $\mathscr{A}$ as above, if $z \in \mathscr{A}$, then $0 \in z$. So $0 \in \bigcap \mathscr{A} = \omega$. Now suppose that $y \in \omega$ and $z \in \mathscr{A}$. Then $y \in z$, and it follows that $y +' 1 \in z$. Hence $y +' 1 \in \omega$. $\square$

Theorem 7.16. *ω is the first limit ordinal.*

Proof. Let $A = \{y \in \omega : y \text{ is an ordinal}\}$. Then $0 \in A$ by Propositions 7.1 and 7.15. Suppose that $y \in A$. Then $y \in \omega$, so $y +' 1 \in \omega$ by Proposition 7.15. Also, y is an ordinal, so $y +' 1$ is an ordinal by Proposition 7.2. This shows that $y +' 1 \in A$. It follows that $A = \omega$, by Theorem 7.17. Hence every member of ω is an ordinal, and hence is transitive.

Next, let $B = \{y \in \omega : y \subseteq \omega\}$. Then $0 \in B$ by Proposition 7.15. Suppose that $y \in B$. Then $y \in \omega$, so $y +' 1 \in \omega$ by Proposition 7.15. Also, $y \subseteq \omega$. Since $y \in \omega$, it follows that $y \cup \{y\} \subseteq \omega$. So $y +' 1 \in B$. Hence $B = \omega$ by Theorem 7.17. This shows that ω is transitive, and hence is an ordinal.

Next, let $C = \{y \in \omega : y \text{ is not a limit ordinal}\}$. $0 \in \omega$ by Theorem 7.15, and by definition 0 is not a limit ordinal, so $0 \in C$. Suppose that $y \in C$. Then $y \in \omega$, so $y +' 1 \in \omega$. Also, by definition $y +' 1$ is not a limit ordinal. So $y +' 1 \in C$. It follows that $C = \omega$, and hence for every $\alpha < \omega$, α is not a limit ordinal.

Since $0 \in \omega$, $\omega \neq 0$. If $\omega = y +' 1$, then $y \in \omega$ and hence $\omega = y +' 1 \in \omega$, contradiction. Thus ω is a limit ordinal. $\square$

Proposition 7.17. *The following conditions are equivalent:*

(i) α is a limit ordinal;

(ii) $\alpha \neq 0$, and for every $\beta < \alpha$ there is a γ such that $\beta < \gamma < \alpha$.

(iii) $\alpha = \bigcup \alpha \neq 0$.

Proof. (i)⇒(ii): suppose that α is a limit ordinal. So $\alpha \neq 0$, by definition. Suppose that $\beta < \alpha$. Then $\beta +' 1 \leq \alpha$ by Proposition 7.10. Hence $\beta +' 1 < \alpha$ since α is not a successor ordinal. Thus $\beta < \beta +' 1 < \alpha$.

(ii)⇒(iii): if $\beta \in \bigcup \alpha$, choose $\gamma \in \alpha$ such that $\beta \in \gamma$. Then $\beta \in \alpha$ since α is an ordinal. This shows that $\bigcup \alpha \subseteq \alpha$.

If $\beta \in \alpha$, choose γ with $\beta < \gamma < \alpha$. Thus $\beta \in \bigcup \alpha$. This proves that $\alpha = \bigcup \alpha$, and $\alpha \neq 0$ is given.

(iii)⇒(i): suppose that $\alpha = \beta +' 1$. Then $\beta \in \alpha = \bigcup \alpha$, so choose $\gamma \in \alpha$ such that $\beta \in \gamma$. Thus $\beta < \gamma \leq \beta$, so $\beta < \beta$, contradiction. $\square$

Proposition 7.18. *If $\alpha = \beta +' 1$, then $\bigcup \alpha = \beta$.*

Proof. Assume that $\alpha = \beta +' 1$. Suppose that $\gamma \in \bigcup \alpha$. Choose $\delta \in \alpha$ such that $\gamma \in \delta$. Thus $\gamma < \delta < \alpha$, so $\delta \leq \beta$, hence $\gamma \in \beta$. This shows that $\bigcup \alpha \subseteq \beta$.

If $\gamma \in \beta$, then $\gamma \in \beta \in \alpha$, so $\gamma \in \bigcup \alpha$. So $\bigcup \alpha = \beta$. $\square$

We now give some equivalent definitions of ordinals. A *well-ordered set* is a pair $(A, <)$ such that A is a set, $<$ is a relation included in $A \times A$, $<$ is irreflexive on A ($a \not< a$ for all $a \in A$), $<$ is transitive ($a < b < c$ implies that $a < c$), $<$ is linear on A (for all $a, b \in A$, either $a = b$, $a < b$, or $b < a$), and any nonempty subset X of A has a least element (an element $a \in X$ such that $a \leq b$ for all $b \in X$).

Theorem 7.19. *The following conditions are equivalent:*

(i) x is an ordinal.

(ii) x is transitive and $(x, \{(y, z) \in x \times x : y \in z\})$ is a well-ordered set.

(iii) x is transitive, and for all $y, z \in x$, $y = z$ or $y \in z$ or $z \in y$.

(iv) For all y, if $y \subset x$ and y is transitive, then $y \in x$.

(v) The following two conditions hold:

(a) For all $y \in x$, either $y \cup \{y\} = x$ or $y \cup \{y\} \in x$.

(b) For all $y \subseteq x$, either $\bigcup y = x$ or $\bigcup y \in x$.

Proof. (i)⇒(ii): Assume (i). By definition, x is transitive. Let $R = \{(y, z) \in x \times x : y \in z\})$. Obviously R is a relation. By definition, $R \subseteq x \times x$. R is irreflexive on x by Theorem 7.5. R is transitive since x is transitive. R is linear on x by Theorem 7.7. The final well-ordering property follows from Theorem 7.13.

(ii)⇒(iii): Assume (ii). Then (iii) is obvious.

(iii)⇒(iv): Assume (iii), and suppose that $y \subset x$ and y is transitive. Choose $z \in x \backslash y$ such that $z \cap (x \backslash y) = \emptyset$. If $u \in y$, then $u \in x$ since $y \subset x$. So $u, z \in x$, so by hypothesis we have $u \in z$, $u = z$, or $z \in u$. Now $u \neq z$ since $z \notin y$ and $u \in y$. And $z \notin u$, since $z \in u$ would imply, because y is transitive and $u \in y$, that $z \in y$, which is not true. Hence $u \in z$. This is true for any $u \in y$. So $y \subseteq z$. Clearly also $z \subseteq y$, so $y = z \in x$.

(iv)→(i): Assume (iv). Let $y = \{z \in x : z \text{ is an ordinal}\}$. So $y \subseteq x$. Suppose that $y \subset x$. Now y is transitive, for assume that $z \in y$. Thus $z \in x$ and z is an ordinal. Suppose that $w \in z$. Then $w \in x$ since x is transitive, and w is an ordinal since z is an ordinal. So $w \in y$. Thus, indeed, y is transitive. So by assumption $y \in x$. Now y is a transitive set of transitive sets, so y is an ordinal. It follows that $y \in y$, contradiction. This proves that $x = y$. So x is a transitive set of transitive sets, and hence x is an ordinal.

(i)⇒(v): Assume (i). (a) holds by Proposition 7.10. Now suppose that $y \subseteq x$. If $z \in \bigcup y$, choose $w \in y$ such that $z \in w$. Then also $w \in x$, so $z \in x$ since x is transitive. This shows that $\bigcup y \subseteq x$. By Proposition 7.3, $\bigcup y$ is an ordinal. Hence by Proposition 7.8, $\bigcup y \leq x$.

(v)⇒(i): Assume (v). By Theorem 7.6 there is an ordinal α not in x. Then by Theorem 7.13 there is a least $\beta \in \alpha \cup \{\alpha\}$ such that $\beta \notin x$. Now we have two possibilities:

Case 1. $\beta = \bigcup \beta$. Now $\beta \subseteq x$, so by (ii) second clause, since $\bigcup \beta = \beta \notin x$ we have $x = \bigcup \beta$, hence x is an ordinal, as desired.

Case 2. $\beta = (\bigcup \beta) + 1$. Thus $\bigcup \beta$ is an ordinal smaller than β, so it is in x. By (i), since $\beta = \bigcup \beta + 1 \notin x$ we have $x = (\bigcup \beta) + 1$, hence x is an ordinal. □

8. Recursion

In this chapter we prove a general recursion theorem which will be used many times in these notes. The theorem involves classes, so we begin with a few remarks about classes and sets.

Classes and sets

Although expressions like $\{x : x = x\}$ and $\{\alpha : \alpha \text{ is an ordinal}\}$ are natural, they cannot be put into the framework of our logic for set theory. These "collections" are "too big". It is intuitively indispensible to continue using such expressions. One should understand that when this is done, there is a rigorous way of reformulating what is said. These big collections are called *classes*; their rigorous counterparts are simply formulas of our set theoretic language. We can also talk about *class functions*, *class relations*, the *domain* of class functions, etc. Most of the notions that we have introduced so far have class counterparts. In particular, we have the important classes $\mathbf{V}$, the class of all sets, and $\mathbf{On}$, the class of all ordinals. They correspond to the formulas "$x = x$" and "α is an ordinal". We attempt to use bold face letters for classes; in some cases the classes in question are actually sets. A class which is not a set is called a *proper class.*

Well-founded class relations

If $\mathbf{A}$ is a class, a class relation $\mathbf{R}$ is *well-founded on* $\mathbf{A}$ iff $\mathbf{R} \subseteq \mathbf{A} \times \mathbf{A}$ and for every nonempty subset X of $\mathbf{A}$ there is an $x \in X$ such that for all $y \in X$ it is not the case that $(y, x) \in \mathbf{R}$. Such a set x is called $\mathbf{R}$-*minimal.*

This notion is important even if $\mathbf{A}$ and $\mathbf{R}$ are mere sets. Two important examples of well-founded class relations are as follows.

Proposition 8.1. *The class relation $\{(x, y) : x \in y\}$ is well-founded on $\mathbf{V}$.*

Proof. Let X be a nonempty subset of $\mathbf{V}$. This just means that X is a nonempty set. By the foundation axiom, choose $x \in X$ such that $x \cap X = \emptyset$. Then for all $y \in X$ it is not the case that $y \in x$. □

Proposition 8.2. *The class relation $\{(\alpha, \beta) : \alpha < \beta\}$ is well-founded on $\mathbf{On}$.*

Proof. Let X be a nonempty subset of $\mathbf{On}$. Thus X is a nonempty set of ordinals. By Theorem 4.14 we have $\bigcap X \in X$ and for all $y \in X$ it is not the case that $y \in \bigcap X$. □

On the other hand, the class relation $\mathbf{R} = \{(x, y) : y \in x\}$ is not well-founded on $\mathbf{V}$. In fact the set ω does not have an $\mathbf{R}$-minimal element, since if $m \in \omega$ then also $m +' 1 \in \omega$ and $(m +' 1, m) \in \mathbf{R}$.

Recall that our intuitive notion of class is made rigorous by using formulas instead. Thus we would talk about a formula $\varphi(x, y)$ being well-founded on another formula $\psi(x)$. In the case of $\in$, we are really looking at the formula $x \in y$ being well-founded on the formula $x = x$. So, rigorously we are associating with two formulas $\varphi(x, y)$ and $\psi(x)$ another formula "$\varphi(x, y)$ is well-founded on $\psi(x)$", namely the following formula:

$$\forall x \forall y[\varphi(x, y) \to \psi(x) \wedge \psi(y)] \wedge \forall X[\forall x \in X \psi(x) \wedge X \neq \emptyset \to \exists x \in X \forall y \in X \neg\varphi(y, x)].$$

Let $\mathbf{A}$ be a class and $\mathbf{R}$ a class relation with $\mathbf{R} \subseteq \mathbf{A} \times \mathbf{A}$. For any $x \in \mathbf{A}$ we define $\mathrm{pred}_{\mathbf{AR}}(x) = \{y \in \mathbf{A} : (y, x) \in \mathbf{R}\}$. We say that $\mathbf{R}$ is *set-like* on $\mathbf{A}$ iff $\mathbf{R} \subseteq \mathbf{A} \times \mathbf{A}$ and $\mathrm{pred}_{\mathbf{AR}}(x)$ is a set for all $x \in \mathbf{A}$.

For example, for $\mathbf{R} = \{(x, y) : x \in y\}$ we have $\mathrm{pred}_{\mathbf{VR}}(x) = x$ for any set x, and $\mathbf{R}$ is set-like on $\mathbf{V}$. For $\mathbf{R} = \{(\alpha, \beta) : \alpha < \beta\}\}$ we have $\mathrm{pred}_{\mathbf{OnR}}(\alpha) = \alpha$ for any ordinal α, and $\mathbf{R}$ is set-like on $\mathbf{On}$.

On the other hand, $\mathbf{R} = \{(\alpha, \beta) : \alpha > \beta\}$ is not set-like on $\mathbf{On}$, since for example $\mathrm{pred}_{\mathbf{OnR}}(0) = \{\alpha : \alpha > 0\}$ and this is not a set, as otherwise $\mathbf{On} = \{0\} \cup \mathrm{pred}_{\mathbf{OnR}}(0)$ would be a set.

Formally we are dealing with formulas $\varphi(x, y)$ and $\psi(x)$, such that $\forall x, y[\varphi(x, y) \rightarrow \psi(x) \wedge \psi(y)]$. Then $\mathrm{pred}_{\varphi\psi}$ is the formula $\varphi(y, x)$, and "φ is set-like on ψ" is the formula

$$\forall x[\psi(x) \rightarrow \exists z \forall y[y \in z \leftrightarrow \varphi(y, x)]].$$

Now let $\mathbf{R}$ be a class relation. We define

$$\begin{aligned}\mathbf{R}^* = \{(a, b) : \exists n \in \omega \backslash 1 \exists f[&f \text{ is a function with domain } n +' 1 \text{ and} \\ &\forall i < n[(f(i), f(i +' 1)) \in \mathbf{R} \text{ and } f(0) = a \text{ and } f(n) = b]]\}.\end{aligned}$$

This is called the *transitive closure* of $\mathbf{R}$.

Formally, given a formula $\varphi(x, y)$, we define another formula φ^*:

$$\begin{aligned}&\exists n \in \omega \backslash 1 \exists f[f \text{ is a function with domain } n +' 1 \text{ and} \\ &\forall i < n[\varphi(f(i), f(i +' 1)) \text{ and } f(0) = x \text{ and } f(n) = y]]\}.\end{aligned}$$

The actual formula in our set-theoretical language is long, since we have to replace the definitions of ω, function, ordered pair, etc. by formulas involving $\in$ alone.

We actually do not need the fact that $\mathbf{R}^*$ is transitive, but we do need the following facts. If $\mathbf{R} \subseteq \mathbf{A} \times \mathbf{A}$ and $x \in \mathbf{A}$, let $\mathrm{pred}'_{\mathbf{AR}}(x) = \{x\} \cup \mathrm{pred}_{\mathbf{AR}}(x)$.

Theorem 8.3. *Let* $\mathbf{R}$ *be a class relation.*

(i) $\mathbf{R} \subseteq \mathbf{R}^*$.

(ii) If $\mathbf{R} \subseteq \mathbf{A} \times \mathbf{A}$, $x \in \mathbf{A}$, $(u, v) \in \mathbf{R}$, *and* $v \in \mathrm{pred}'_{\mathbf{AR}^*}(x)$, *then* $u \in \mathrm{pred}'_{\mathbf{AR}^*}(x)$.

Proof. (i): Suppose that $(a, b) \in \mathbf{R}$. Let f be the function with domain 2 such that $f(0) = a$ and $f(1) = b$. This function shows that $(a, b) \in \mathbf{R}^*$.

(ii): Assume the hypotheses. There are two cases.

Case 1. $v = x$. Then $(u, x) \in \mathbf{R}$, so by (i), $(u, x) \in \mathbf{R}^*$. Hence $u \in \mathrm{pred}_{\mathbf{AR}}(x) \subseteq \mathrm{pred}'_{\mathbf{AR}}(x)$.

Case 2. $v \in \mathrm{pred}_{\mathbf{AR}^*}(x)$. Choose n and f correspondingly. Let

$$g = \{(0, u)\} \cup \{(i +' 1, f(i)) : i \leq n\}.$$

Then g is a function with domain $n+'2$, $g(0) = u$, $g(n+'1) = x$, and $\forall i < n+'1[(g(i), g(i+' 1)) \in \mathbf{R}]$. Hence $u \in \mathrm{pred}_{\mathbf{AR}^*}(x) \subseteq \mathrm{pred}'_{\mathbf{AR}^*}(x)$. $\square$

Theorem 8.4. *If* $\mathbf{R}$ *is set-like on* $\mathbf{A}$*, then also* $\mathbf{R}^*$ *is set-like on* $\mathbf{A}$*.*

Proof. Let $x \in \mathbf{A}$; we want to show that $\mathrm{pred}_{\mathbf{AR}^*}(x)$ is a set. For each $n \in \omega\backslash 1$ let

$$\begin{aligned} D_n = \{y \in \mathbf{A} : &\text{there is a function } f \text{ with domain } n +' 1 \text{ such that} \\ &f(0) = y, f(n) = x \text{ and } \forall i < n[(f(i), f(i +' 1)) \in \mathbf{R}]\} \end{aligned}$$

We will prove by induction on n that each D_n is a set. First take $n = 1$. Now $D_1 = \{y \in \mathbf{A}$:there is a function f with domain 2 such that $f(0) = y$, $f(1) = x$, and $(y, x) \in \mathbf{R}\} = \mathrm{pred}_{\mathbf{AR}}(x)$, so D_1 is a set by hypothesis. Now assume that D_n is a set. Let $\mathbf{F}(y) = \mathrm{pred}_{\mathbf{AR}}(y)$ for each $y \in D_n$. This makes sense, since by hypothesis each class $\mathrm{pred}_{\mathbf{AR}}(y)$ is a set. So $\mathbf{F}$ is a function whose domain is the set D_n. By the replacement and comprehension axioms, its range is a set. That is, $\{\mathrm{pred}_{\mathbf{AR}}(y) : y \in D_n\}$ is a set. Now we claim

$$(*) \qquad D_{n+'1} = \bigcup\{\mathrm{pred}_{\mathbf{AR}}(y) : y \in D_n\}.$$

This claim shows that $D_{n+'1}$ is a set, completing the inductive proof.

To prove the claim, first suppose that $z \in D_{n+'1}$. Let f be a function with domain $n +' 2$ such that $f(0) = z$, $f(n +' 1) = x$, and $\forall i < n +' 1[(f(i), f(i +' 1) \in \mathbf{R}$. Define g with domain $n +' 1$ by setting $g(i) = f(i +' 1)$ for all $i < n +' 1$. Then $g(0) = f(1)$, $g(n) = f(n +' 1) = x$. and for all $i < n$, $(g(i), g(i +' 1)) = (f(i +' 1), f(i +' 2)) \in \mathbf{R}$. Hence $g(0) \in D_n$. Clearly $(z, g(0)) \in \mathbf{R}$, so $z \in \mathrm{pred}_{\mathbf{AR}}(g(0))$. Thus z is in the right side of $(*)$.

Second, suppose that z is in the right side of $(*)$. Say $z \in \mathrm{pred}_{\mathbf{AR}}(y)$ with $y \in D_n$. So $(z, y) \in \mathbf{R}$, and there is a function f with domain $n +' 1$ such that $f(0) = y$, $f(n) = x$, and $\forall i < n[(f(i), f(i+'1)) \in \mathbf{R}]$. Define g with domain $n+'2$ by setting $g(0) = z$ and $g(i+'1) = f(i)$ for all $i < n +' 1$. Then $g(n +' 1) = f(n) = x$ and $\forall i < n +' 1[(g(i), g(i +' 1)) \in \mathbf{R}]$. Hence $z \in D_{n+'1}$, and the claim is proved.

Now for each $n \in \omega\backslash 1$ let $\mathbf{G}(n) = D_n$. Then $\mathbf{G}$ is a function whose domain is the set $\omega\backslash 1$, so by replacement and comprehension, it range is a set. Thus $\{D_n : n \in \omega\backslash 1\}$ is a set. Now we claim

$$\mathrm{pred}_{\mathbf{AR}^*}(x) = \bigcup\{D_n : n \in \omega\backslash 1\}.$$

This claim will finish the proof. We have

$$\begin{aligned} \bigcup\{D_n : n \in \omega\backslash 1\} = \{y \in \mathbf{A} : &\exists n \in \omega\backslash 1 \exists f[f \text{ is a function with domain } n +' 1 \\ &\text{such that } f(0) = y,\ f(n) = x, \text{ and } \forall i < n[(f(i), f(i +' 1)) \in \mathbf{R}]] \\ = &\ \mathrm{pred}_{\mathbf{AR}^*}(x). \end{aligned}$$

□

Theorem 8.5. *If* $\mathbf{R}$ *is well-founded and set-like on a class* $\mathbf{A}$*, then every nonempty subclass of* $\mathbf{A}$ *has an* $\mathbf{R}$*-minimal element.*

Proof. Suppose that $\mathbf{X}$ is a nonempty subclass of $\mathbf{A}$. Take any $x \in \mathbf{X}$. Now $\mathrm{pred}'_{\mathbf{AR}^*}(x) \cap \mathbf{X}$ is a nonempty subset of $\mathbf{A}$, by Theorem 8.4 and the comprehension axioms. Let y be an $\mathbf{R}$-minimal element of $\mathrm{pred}'_{\mathbf{AR}^*}(x) \cap \mathbf{X}$. In particular, $y \in \mathbf{X}$. Suppose that

$(z, y) \in \mathbf{R}$. Then $z \in \text{pred}_{\mathbf{AR}}(y)$. By Theorem 8.3(ii) it follows that $z \in \text{pred}'_{\mathbf{AR}^*}(x)$. Hence $z \notin \mathbf{X}$ by the choice of y. so y is the desired $\mathbf{R}$-minimal element of $\mathbf{X}$. $\square$

Theorem 8.6. *If* $\mathbf{F}$ *is a class function and* a *is a set contained in the domain of* $\mathbf{F}$*, then there is a (set) function* f *with domain* a *such that* $f(x) = \mathbf{F}(x)$ *for all* $x \in a$*.*

Proof. Let $\mathbf{G}(x) = (x, \mathbf{F}(x))$ for all $x \in \text{dmn}(\mathbf{F}$. By the replacement and comprehension axioms, the class $\{\mathbf{G}(x) : x \in a\}$ is a set. This class is $\{(x, \mathbf{F}(x)) : x \in a\}$. Thus it is the desired function f. $\square$

In terms of formulas, $\mathbf{F}$ corresponds to a formula $\varphi(x, y)$ such that for all x there is at most one y such that $\varphi(x, y)$. Then $\mathbf{G}$ corresponds to the formula $\psi(x, y) \overset{\text{def}}{=} \exists z[\varphi(x, z) \wedge y = (x, z)]$. Clearly for all x there is at most one y such that $\psi(x, y)$, and if $\psi(x, y)$ then $y = (x, z)$ where $\varphi(x, z)$ holds.

The function asserted to exist in Theorem 8.6 will be denoted by $\mathbf{F} \restriction a$.

Theorem 8.7. (The recursion theorem) *Suppose that* $\mathbf{R}$ *is a class relation which is well-founded and set like on a class* $\mathbf{A}$*, and* $\mathbf{G}$ *is a class function mapping* $\mathbf{A} \times \mathbf{V}$ *into* $\mathbf{V}$*. Then there is a class function* $\mathbf{F}$ *mapping* $\mathbf{A}$ *into* $\mathbf{V}$ *such that* $\mathbf{F}(a) = \mathbf{G}(a, \mathbf{F} \restriction \text{pred}_{\mathbf{AR}}(a))$ *for all* $a \in \mathbf{A}$*.*

Proof. We say that a function f is an *approximation to* $\mathbf{F}$ iff $\text{dmn}(f) \subseteq \mathbf{A}$ and for every $a \in \text{dmn}(f)$ we have $\text{pred}_{\mathbf{AR}}(a) \subseteq \text{dmn}(f)$ and $f(a) = \mathbf{G}(a, f \restriction \text{pred}_{\mathbf{AR}}(a))$.

(1) If f and f' are approximations to $\mathbf{F}$ and $a \in \text{dmn}(f) \cap \text{dmn}(f')$, then $f(a) = f'(a)$.

In fact, suppose that this is not true. Then the set $X = \{a \in \text{dmn}(f) \cap \text{dmn}(f') : f(a) \neq f'(a)\}$ is nonempty. Let a be an R-minimal element of X. Now if $b \in \text{pred}_{\mathbf{AR}}(a)$ then $b \in \text{dmn}(f) \cap \text{dmn}(f')$ and $(b, a) \in \mathbf{R}$, hence $b \notin X$; so $f(b) = f'(b)$. Thus $f \restriction \text{pred}_{\mathbf{AR}}(a) = f' \restriction \text{pred}_{\mathbf{AR}}(a)$. It follows that

$$f(a) = \mathbf{G}(a, f \restriction \text{pred}_{\mathbf{AR}}(a)) = \mathbf{G}(a, f' \restriction \text{pred}_{\mathbf{AR}}(a)) = f'(a),$$

contradiction. So (1) holds.

(2) If f is an approximation to $\mathbf{F}$, $x \in \text{dmn}(f)$, n is a positive integer, g is a function with domain $n +' 1$, $g(n) = x$, and $\forall i < n[(g(i), g(i +' 1)) \in \mathbf{R}]$, then $g(0) \in \text{dmn}(f)$.

To prove this, assuming that f is an approximation to $\mathbf{F}$ and $x \in \text{dmn}(f)$, we prove by induction on $n \geq 1$ that if g is a function with domain $n +' 1$, $g(n) = x$, and $\forall i < n[(g(i), g(i +' 1)) \in \mathbf{R}]$, then $g(0) \in \text{dmn}(f)$. For $n = 1$ we have $(g(0), x) \in \mathbf{R}$, so $g(0) \in \text{pred}_{\mathbf{AR}}(x)$ and hence $g(0) \in \text{dmn}(f)$. Assume that it is true for n, and now assume that g is a function with domain $n +' 2$, $g(n +' 1) = x$, and $\forall i < n +' 1[(g(i), g(i +' 1)) \in \mathbf{R}]$. Define $h(i) = g(i +' 1)$ for all $i < n$. Then $h(n) = g(n +' 1) = x$ and $\forall i < n[(h(i), h(i +' 1)) = (g(i +' 1), g(i +' 2)) \in \mathbf{R}]$. Hence $h(0) \in \text{dmn}(f)$ by the inductive hypothesis. Since $(g(0), h(0)) = (g(0), g(1))$ we have $(g(0), h(0)) \in \mathbf{R}$ and hence $g(0) \in \text{dmn}(f)$. This finishes the inductive proof of (2).

(3) If f is an approximation to $\mathbf{F}$ and $x \in \text{dmn}(f)$, then $\text{pred}'_{\mathbf{AR}^*}(x) \subseteq \text{dmn}(f)$.

This is clear from (2) and the definition of $\mathrm{pred}'_{\mathbf{AR}^*}(x)$.

(4) If f is an approximation to $\mathbf{F}$ and $x \in \mathrm{dmn}(f)$, then $f \restriction \mathrm{pred}'_{\mathbf{AR}^*}(x)$ is an approximation to $\mathbf{F}$.

In fact, if $a \in \mathrm{pred}'_{\mathbf{AR}^*}(x)$ then $\mathrm{pred}_{AR}(a) \subseteq \mathrm{pred}'_{\mathbf{AR}^*}(x)$ by Theorem 8.3(ii). Suppose that $a \in \mathrm{pred}'_{\mathbf{AR}^*}(x)$. Then

$$\begin{aligned}(f \restriction \mathrm{pred}'_{\mathbf{AR}^*}(x))(a) &= f(a) = \mathbf{G}(a, f \restriction \mathrm{pred}_{\mathbf{AR}}(a)) \\ &= \mathbf{G}(a, (f \restriction \mathrm{pred}'_{\mathbf{AR}^*}(x)) \restriction \mathrm{pred}_{\mathbf{AR}}(a)).\end{aligned}$$

This proves (4).

(5) For all $x \in \mathbf{A}$ there is an approximation f to $\mathbf{F}$ such that $x \in \mathrm{dmn}(f)$.

Suppose not. Let $\mathbf{X} = \{x \in \mathbf{A} :$ there does not exist an approximation f to $\mathbf{F}$ such that $x \in \mathrm{dmn}(f)\}$. So $\mathbf{X}$ is a nonempty subclass of $\mathbf{A}$. By Theorem 8.5, let x be an R-minimal element of $\mathbf{X}$. Now if $(y, x) \in \mathbf{R}$ then $y \notin \mathbf{X}$, and so there is an approximation f to $\mathbf{F}$ such that $y \in \mathrm{dmn}(f)$. Then by (4), also $f \restriction \mathrm{pred}'_{\mathbf{AR}^*}(y)$ is an approximation to $\mathbf{F}$. If also g is an approximation to $\mathbf{F}$ such that $y \in \mathrm{dmn}(g)$, then by (4) $g \restriction \mathrm{pred}'_{\mathbf{AR}^*}(y)$ is an approximation to $\mathbf{F}$. By (1), $f \restriction \mathrm{pred}'_{\mathbf{AR}^*}(y) = g \restriction \mathrm{pred}'_{\mathbf{AR}^*}(y)$. Thus there is a unique approximation to $\mathbf{F}$ whose domain is $\mathrm{pred}'_{\mathbf{AR}^*}(y)$. This is true for all $y \in \mathrm{pred}_{\mathbf{AR}}(x)$, so by replacement and comprehension there is a set

$$\begin{aligned}\mathscr{A} \stackrel{\text{def}}{=} \{f : f &\text{ is an approximation to } \mathbf{F} \text{ with domain} \\ &\mathrm{pred}'_{\mathbf{AR}^*}(y), \text{ for some } y \in \mathrm{pred}_{\mathbf{AR}}(x)\}.\end{aligned}$$

Let $g = \bigcup \mathscr{A}$. We claim that g is an approximation to $\mathbf{F}$. We prove this in several steps.

First, g is a function. For suppose that $(a, b), (a, c) \in g$. Choose $f, f' \in \mathscr{A}$ such that $(a, b) \in f$ and $(a, c) \in f'$. Since both f and f' are approximations to $\mathbf{F}$ and $a \in \mathrm{dmn}(f) \cap \mathrm{dmn}(f')$, it follows from (1) that $b = c$.

Second, the domain of g is $\bigcup\{\mathrm{pred}'_{\mathbf{AR}^*}(y) : y \in \mathrm{pred}_{\mathbf{AR}}(x)\}$. In fact, if $a \in \mathrm{dmn}(g)$ then there is an $f \in \mathscr{A}$ such that $a \in \mathrm{dmn}(f)$, and $\mathrm{dmn}(f) = \mathrm{pred}'_{\mathbf{AR}^*}(y)$ for some $y \in \mathrm{pred}_{\mathbf{AR}}(x)$; so a is in the indicated union. If $y \in \mathrm{pred}_{\mathbf{AR}}(x)$, then $\mathrm{pred}'_{\mathbf{AR}^*}(y)$ is the domain of some $f \in \mathscr{A}$, and hence $\mathrm{pred}'_{\mathbf{AR}^*}(y) \subseteq \mathrm{dmn}(g)$. So the domain of g is as indicated.

Next, if $a \in \mathrm{dmn}(g)$ then $\mathrm{pred}_{\mathbf{AR}}(a) \subseteq \mathrm{dmn}(g)$. For, suppose that $b \in \mathrm{pred}_{\mathbf{AR}}(a)$. Then $a \in \mathrm{dmn}(f)$ for some $f \in \mathscr{A}$, hence $a \in \mathrm{pred}'_{\mathbf{AR}^*}(y)$ for some $y \in \mathrm{pred}_{\mathbf{AR}}(x)$, so $b \in \mathrm{pred}'_{\mathbf{AR}^*}(y)$ by Theorem 8.3(ii), and it follows that $b \in \mathrm{dmn}(g)$. This proves that $\mathrm{pred}_{\mathbf{AR}}(a) \subseteq \mathrm{dmn}(g)$.

The final condition for g to be an approximation to $\mathbf{F}$ is shown as follows. Suppose that $a \in \mathrm{dmn}(g)$. Choose $y \in \mathrm{pred}_{\mathbf{AR}}(x)$ such that $a \in \mathrm{dmn}(f)$, where f is an approximation to $\mathbf{F}$ with domain $\mathrm{pred}'_{\mathbf{AR}}(y)$. Then

$$g(a) = f(a) = \mathbf{G}(a, f \restriction \mathrm{pred}'_{\mathbf{AR}}(y)) = \mathbf{G}(a, g \restriction \mathrm{pred}'_{\mathbf{AR}}(y)).$$

Now let $h = g \cup \{(x, \mathbf{G}(x, g \restriction \mathrm{pred}_{\mathbf{AR}}(x)))\}$. We claim that h is an approximation to $\mathbf{F}$. Since $x \in \mathrm{dmn}(h)$, this is a contradiction, proving (5).

To prove the claim, first note that h is a function, since $x \notin \text{dmn}(g)$ by the choice of x, since g is an approximation. Clearly $\text{dmn}(h) \subseteq \mathbf{A}$. Suppose that $a \in \text{dmn}(h)$. if $a \in \text{dmn}(g)$, then $\text{pred}_{\mathbf{AR}}(a) \subseteq \text{dmn}(g) \subseteq \text{dmn}(h)$. If $a = x$ and $y \in \text{pred}_{\mathbf{AR}}(x)$, then $y \in \text{pred}'_{\mathbf{AR}^*} \subseteq \text{dmn}(g) \subseteq \text{dmn}(h)$. Hence $\text{pred}_{\mathbf{AR}}(x) \subseteq \text{dmn}(h)$. Finally, if $a \in \text{dmn}(g)$, then

$$h(a) = g(a) = \mathbf{G}(a, g \restriction \text{pred}_{\mathbf{AR}}(a)) = \mathbf{G}(a, h \restriction \text{pred}_{\mathbf{AR}}(a)).$$

If $a = x$, then

$$h(a) = h(x) = \mathbf{G}(x, g \restriction \text{pred}_{\mathbf{AR}}(x)) = \mathbf{G}(x, h \restriction \text{pred}_{\mathbf{AR}}(x)).$$

So we have proved (5).

Now by (5), for all $a \in \mathbf{A}$ there is a b such that there is an f such that f is an approximation to $\mathbf{F}$ and $b \in \text{dmn}(f)$. By (1), this b is uniquely determined by a. Hence there is a class function $\mathbf{F}'$ such that for all $a \in \mathbf{A}$, $\mathbf{F}'(a)$ is equal to such a b. Moreover, if f is as indicated and $b \in \text{pred}_{\mathbf{AR}}(a)$, then $\mathbf{F}'(b) = f(b)$. Thus $\mathbf{F}' \restriction \text{pred}_{\mathbf{AR}}(a) = f \restriction \text{pred}_{\mathbf{AR}}(a)$. It follows that

$$\mathbf{F}'(a) = f(a) = \mathbf{G}(a, f \restriction \text{pred}_{\mathbf{AR}}(a)) = \mathbf{G}(a, \mathbf{F}' \restriction \text{pred}_{\mathbf{AR}}(a)). \qquad \square$$

Theorem 8.8. *Suppose that* $\mathbf{R}$ *is a class relation which is well-founded and set like on a class* $\mathbf{A}$*, and* $\mathbf{G}$ *is a class function mapping* $\mathbf{A} \times \mathbf{V}$ *into* $\mathbf{V}$*. Suppose that* $\mathbf{F}$ *and* $\mathbf{F}'$ *are class functions mapping* $\mathbf{A}$ *into* $\mathbf{V}$ *such that* $\mathbf{F}(a) = \mathbf{G}(a, \mathbf{F} \restriction \text{pred}_{\mathbf{AR}}(a))$ *and* $\mathbf{F}'(a) = \mathbf{G}(a, \mathbf{F}' \restriction \text{pred}_{\mathbf{AR}}(a))$ *for all* $a \in \mathbf{A}$*. Then* $\mathbf{F} = \mathbf{F}'$*.*

Proof. Suppose not. Then $\mathbf{X} \stackrel{\text{def}}{=} \{a \in \mathbf{A} : \mathbf{F}(a) \neq \mathbf{F}'(a)\}$ is a nonempty subclass of $\mathbf{A}$. Hence by Theorem 8.5 let a be an $\mathbf{R}$-minimal element of $\mathbf{X}$. If $(b, a) \in \mathbf{R}$, then $b \notin \mathbf{X}$, and hence $\mathbf{F}(b) = \mathbf{F}'(b)$. Thus $\mathbf{F} \restriction \text{pred}_{\mathbf{AR}}(a) = \mathbf{F}' \restriction \text{pred}_{\mathbf{AR}}(a)$. So

$$\mathbf{F}(a) = \mathbf{G}(a, \mathbf{F} \restriction \text{pred}_{\mathbf{AR}}(a)) = \mathbf{G}(a, \mathbf{F}' \restriction \text{pred}_{\mathbf{AR}}(a)) = \mathbf{F}'(a),$$

contradiction. $\square$

We make some remarks about the rigorous formulation of Theorems 8.7 and 8.8. In Theorem 8.7 we are given formulas $\varphi(x, y)$, $\psi(x)$, and $\chi(x, y, z)$ corresponding to $\mathbf{R}$, $\mathbf{A}$, and $\mathbf{G}$. We assume that φ is well-founded and set-like on ψ. The assumption on χ is

$$\begin{aligned} &\forall x \forall y[\psi(x) \to \exists! z \chi(x, y, z)] \\ &\wedge \forall x \forall y \forall z[\chi(x, y, z) \to \psi(x)]. \end{aligned}$$

The conclusion is that there is a formula $\theta(x, y)$ such that

$$\begin{aligned} (*) \qquad &\forall x[\psi(x) \to \exists! y \theta(x, y)] \wedge \forall x \forall y[\theta(x, y) \to \psi(x)] \\ &\wedge \forall x \exists y \exists f[f \text{ is a function } \wedge \forall u \forall v[(u, v) \in f \leftrightarrow \varphi(u, x) \wedge \theta(u, v)] \wedge \chi(x, f, y)]] \end{aligned}$$

The proof defines the formula θ explicitly. Namely, let $\mu(f)$ be the following formula, the rigorous version of "f is an approximation to $\mathbf{F}$":

$$\begin{aligned}
&f \text{ is a function } \wedge \forall\, x[x \in \mathrm{dmn}(f) \to \varphi(x)] \wedge \\
&\forall x \forall y[x \in \mathrm{dmn}(f) \wedge \varphi(y, x) \to y \in \mathrm{dmn}(f)] \wedge \\
&\forall a \forall g[a \in \mathrm{dmn}(f) \wedge g \text{ is a function } \wedge \forall y[y \in \mathrm{dmn}(g) \leftrightarrow \varphi(y, x)] \wedge \\
&\forall y \in \mathrm{dmn}(g)[g(y) = f(y)] \to \chi(a, g, f(a))]
\end{aligned}$$

Then $\theta(x, y)$ is the formula $\exists f[\mu(f) \wedge x \in \mathrm{dmn}(f) \wedge f(x) = y]$.

The rigorous version of Theorem 8.8 is that if $\theta'(x, y)$ is another formula satisfying $(*)$ (with θ replaced by θ'), then $\forall x \forall y[\theta(x, y) \leftrightarrow \theta'(x, y)]$.

. . . .

Proposition 8.9. *There are* $\mathbf{A}, \mathbf{R}$ *such that* $\mathbf{R}$ *is not well-founded on* $\mathbf{A}$ *and is not set-like on* $\mathbf{A}$.

Proof. We take $\mathbf{On}$ and $\mathbf{R}$, where $\mathbf{R} = \{(\alpha, \beta) : \alpha > \beta\}$. As shown after 8.2, $\mathbf{R}$ is not set-like on $\mathbf{On}$. It is also not well-founded on $\mathbf{On}$, since ω is a nonempty set of ordinals, but if $m \in \omega$ then $(m +' 1, m) \in \mathbf{R}$, so that ω does not have an $\mathbf{R}$-minimal element. $\square$

Proposition 8.10. *There are proper classes* $\mathbf{A}, \mathbf{R}$ *such that* $\mathbf{R}$ *is not well-founded on* $\mathbf{A}$ *but is set-like on* $\mathbf{A}$.

Proof. Let $\mathbf{A} = \mathbf{On}$ and let

$$\mathbf{R} = \{(m, n) : m, n \in \omega \text{ and } m > n\} \cup \{(\alpha, \beta) : \alpha < \beta\}. \qquad \square$$

Proposition 8.11. *There are sets* $\mathbf{A}, \mathbf{R}$ *such that* $\mathbf{R}$ *is not well-founded on* $\mathbf{A}$ *but is set-like on* $\mathbf{A}$.

Proof. Let $\mathbf{A} = \omega$ and $\mathbf{R} = \{(m, n) : m, n \in \omega$ and $m > n$. Then ω does not have an $\mathbf{R}$-minimal element, since for any $m \in \omega$ we have $(m +' 1, m) \in \mathbf{R}$. $\square$

Proposition 8.12. *There are* $\mathbf{A}, \mathbf{R}$ *such that* $\mathbf{R}$ *is well-founded on* $\mathbf{A}$ *but is not set-like on* $\mathbf{A}$.

Proof. Let $\mathbf{A} = \mathbf{V}$ and $\mathbf{R} = \{(a, \emptyset) : a \in \mathbf{V}, a \neq \emptyset\}$. Thus $\mathrm{pred}_{\mathbf{AR}}(\emptyset) = \mathbf{V}$, so $\mathbf{R}$ is not set-like on $\mathbf{V}$. Now let X be a nonempty set. If $X = \{\emptyset\}$, then $\emptyset \in X$ and $\forall a \in X[(a, \emptyset) \notin \mathbf{R}]$. If $X \neq \{\emptyset\}$, take any $a \in X \backslash \{\emptyset\}$. Then $\forall b \in X[(b, a) \notin \mathbf{R}]$.

Proposition 8.13. *Suppose that* $\mathbf{R}$ *is a class relation contained in* $\mathbf{A} \times \mathbf{A}$, $x \in \mathbf{A}$, *and* $v \in \mathrm{pred}_{\mathbf{AR}^*}(x)$. *Then if* $n \in \omega \backslash 1$, f *is a function with domain* $n +' 1$, $\forall i < n[(f(i), f(i +' 1)) \in \mathbf{R}]$ *and* $f(n) = v$, *then* $f(0) \in \mathrm{pred}_{\mathbf{AR}^*}(x)$.

Proof. Suppose that $\mathbf{R}$ is a class relation contained in $\mathbf{A} \times \mathbf{A}$, $x \in \mathbf{A}$, and $v \in \mathrm{pred}_{\mathbf{AR}^*}(x)$. We take $n = 1$ in the condition to be proved. So, suppose that f is a function with domain 2 such that $\forall i < 1[(f(i), f(i +' 1)) \in \mathbf{R}]$ and $f(1) = v$. Thus $(f(0), v) \in \mathbf{R}$, so $f(0) \in \mathrm{pred}_{\mathbf{AR}}(v)$. By Lemma 8.3(ii), $f(0) \in \mathrm{pred}_{\mathbf{AR}^*}(x)$.

Now suppose that if $n \in \omega\backslash 1$, f is a function with domain $n +' 1$, $\forall i < n[(f(i), f(i +' 1)) \in \mathbf{R}]$ and $f(n) = v$, then $f(0) \in \mathrm{pred}_{\mathbf{AR}^*}(x)$. Suppose also now that f is a function with domain $n +' 2$, $\forall i < n +' 1[(f(i), f(i +' 1)) \in \mathbf{R}]$ and $f(n +' 1) = v$. Define g with domain $n +' 1$ by setting $g(i) = f(i +' 1)$ for all $i < n +' 1$. Then $\forall i < n[(g(i), g(i +' 1)) = (f(i +' 1), f(i +' 2)) \in \mathbf{R}]$ and $g(n) = f(n +' 1) = v$. Hence by the inductive assumption, $f(1) = g(0) \in \mathrm{pred}_{\mathbf{AR}^*}(x)$. We also have $(f(0), f(1)) \in \mathbf{R}$, so by Lemma 8.3(ii), $f(0) \in \mathrm{pred}_{\mathbf{AR}^*}(x)$. □

Proposition 8.14. *Suppose that* $\mathbf{R}$ *is a class relation contained in* $\mathbf{A} \times \mathbf{A}$, $(u, v) \in \mathbf{R}^*$, *and* $(v, w) \in \mathbf{R}^*$. *Then* $(u, w) \in \mathbf{R}^*$.

Proof. Assume that $\mathbf{R}$ is a class relation contained in $\mathbf{A} \times \mathbf{A}$, $(u, v) \in \mathbf{R}^*$, and $(v, w) \in \mathbf{R}^*$. Since $(u, v) \in \mathbf{R}^*$, there exist $n \in \omega\backslash 1$ and a function f with domain $n +' 1$ such that $\forall i < n[(f(i), f(i +' 1)) \in \mathbf{R}$, $f(0) = u$, and $f(n) = v$. From exercise E8.4 it follows that $(u, w) \in \mathbf{R}^*$. □

Proposition 8.15. *There is a proper class* $\mathbf{X}$ *which has a proper class of* $\in$*-minimal elements.*

Proof. Let $\mathbf{X} = \{\{\alpha\} : \alpha \geq 2\}$. We claim that all elements of $\mathbf{X}$ are $\in$-minimal. Suppose that $\alpha, \beta \geq 2$ and $\{\alpha\} \in \{\beta\}$. Then $\{\alpha\} = \beta$, Since $\beta \geq 2$ we have $0, 1 \in \beta$, so $0 = \alpha = 1$, contradiction. □

Proposition 8.16. *There is a proper class relation* $\mathbf{R}$ *contained in* $\mathbf{A} \times \mathbf{A}$ *for some proper class* $\mathbf{A}$, *and a class function* $\mathbf{G}$ *mapping* $\mathbf{A} \times \mathbf{V}$ *into* $\mathbf{V}$ *such that* $\mathbf{R}$ *is set-like on* $\mathbf{A}$ *but not well-founded on* $\mathbf{A}$ *and there is no class function* $\mathbf{F}$ *mapping* $\mathbf{A}$ *into* $\mathbf{V}$ *such that* $\mathbf{F}(a) = \mathbf{G}(a, \mathbf{F}\langle \mathrm{pred}_{\mathbf{AR}}(a))$ *for all* $a \in \mathbf{A}$.

Proof. Let $\mathbf{A} = \mathbf{On}$ and

$$\mathbf{R} = \{(m, n) : m, n \in \omega \text{ and } m > n\} \cup \{(\alpha, \beta) : \omega \leq \alpha < \beta\}.$$

Thus $\mathbf{R}$ is a proper class relation contained in $\mathbf{A} \times \mathbf{A}$. Clearly $\mathbf{R}$ is set-like on $\mathbf{On}$ but it is not well-founded on $\mathbf{On}$. Define $\mathbf{G} : \mathbf{On} \times \mathbf{V} \to \mathbf{V}$ by setting

$$\mathbf{G}(\alpha, a) = \begin{cases} \{a(\alpha +' 1)\} & \text{if } \alpha \in \omega \text{ and } a \text{ is a function with domain } \{m \in \omega : m > \alpha\}, \\ \emptyset & \text{otherwise.} \end{cases}$$

Suppose that $\mathbf{F} : \mathbf{A} \to \mathbf{V}$ is such that $\mathbf{F}(\alpha) = \mathbf{G}(\alpha, \mathbf{F} \restriction \mathrm{pred}_{\mathbf{OnR}}(\alpha))$ for all $\alpha \in \mathbf{On}$. Let $f = \mathbf{F} \restriction \omega$. Choose $b \in \mathrm{rng}(f)$ such that $b \cap \mathrm{rng}(f) = \emptyset$. Say $b = f(m)$ with $m \in \omega$. Now

$$\begin{aligned} f(m) = \mathbf{F}(m) &= \mathbf{G}(m, \mathbf{F} \restriction \mathrm{pred}_{\mathbf{OnR}}(m)) = \mathbf{G}(m, \mathbf{F} \restriction \{n : n \in \omega, n > m\}) \\ &= \{\mathbf{F}(m +' 1)\} = \{f(m +' 1)\}, \end{aligned}$$

so that $f(m +' 1) \in f(m) \cap \mathrm{rng}(f)$, contradiction. □

Proposition 8.17. *There is a proper class relation* $\mathbf{R}$ *contained in some* $\mathbf{A}\times\mathbf{A}$ *for some proper class* $\mathbf{A}$ *and a class function* $\mathbf{G}$ *mapping* $\mathbf{A}\times\mathbf{V}$ *into* $\mathbf{V}$ *such that* $\mathbf{R}$ *is set-like on* $\mathbf{A}$ *but not well-founded on* $\mathbf{A}$ *but still there is a class function* $\mathbf{F}$ *mapping* $\mathbf{A}$ *into* $\mathbf{V}$ *such that* $\mathbf{F}(a)=\mathbf{G}(a,\mathbf{F}\langle\mathrm{pred}_{\mathbf{AR}}(a))$ *for all* $a\in\mathbf{A}$*.*

Proof. Let $\mathbf{A}$ and $\mathbf{R}$ be as in Proposition 8.16, but define $\mathbf{G}(\alpha,a)=\alpha$ for all $\alpha\in\mathbf{On}$ and all $a\in\mathbf{V}$. Then the function $\mathbf{F}:\mathbf{On}\to\mathbf{V}$ such that $\mathbf{F}(\alpha)=\alpha$ for all $\alpha\in\mathbf{On}$ is as desired. $\square$

9. Ordinals, II

Transfinite induction

The transfinite induction principles follow rather easily from the following generalization of Theorem 7.13.

Theorem 9.1. *Let* $\mathbf{A}$ *be an ordinal, or* $\mathbf{On}$*. Then every nonempty subclass of* $\mathbf{A}$ *has a least element.*

Proof. This follows from Theorem 8.5. □

There are two forms of the principle of transfinite induction, given in the following two theorems.

Theorem 9.2. *Let* $\mathbf{A}$ *be an ordinal or* $\mathbf{On}$*. Suppose that* $\mathbf{B} \subseteq \mathbf{A}$ *and the following condition holds:*

$$\forall \alpha \in \mathbf{A}[\alpha \subseteq \mathbf{B} \Rightarrow \alpha \in \mathbf{B}].$$

Then $\mathbf{B} = \mathbf{A}$.

Proof. Suppose not, and let α be the least element of $\mathbf{A}\backslash\mathbf{B}$. Thus $\alpha \subseteq \mathbf{B}$, so by hypothesis $\alpha \in \mathbf{B}$, contradiction. □

Corollary 9.3. *Suppose that* $\mathbf{B}$ *is a class of ordinals and the following condition holds:*

$$\forall \alpha[\alpha \subseteq \mathbf{B} \Rightarrow \alpha \in \mathbf{B}].$$

Then $\mathbf{B} = \mathbf{On}$. □

Corollary 9.4. *Suppose that* β *is an ordinal,* $X \subseteq \beta$*, and the following condition holds:*

$$\forall \alpha < \beta[\alpha \subseteq X \Rightarrow \alpha \in X].$$

Then $X = \beta$. □

Theorem 9.5. *Suppose that* $\mathbf{A}$ *is an ordinal or* $\mathbf{On}$*,* $\mathbf{B} \subseteq \mathbf{A}$*, and the following conditions hold:*

(i) If $0 \in \mathbf{A}$*, then* $0 \in \mathbf{B}$.
(ii) If $\alpha +' 1 \in \mathbf{A}$ *and* $\alpha \in \mathbf{B}$*, then* $\alpha +' 1 \in \mathbf{B}$.
(iii) If α *is a limit ordinal,* $\alpha \in \mathbf{A}$*, and* $\alpha \subseteq \mathbf{B}$*, then* $\alpha \in \mathbf{B}$.

Then $\mathbf{B} = \mathbf{A}$.

Proof. Suppose not, and let α be the least element of $\mathbf{A}\backslash\mathbf{B}$. Then $\alpha \neq 0$ by (i). If $\alpha = \beta +' 1$ for some β, then $\beta < \alpha$, so $\beta \in \mathbf{B}$, and then $\alpha \in \mathbf{B}$ by (ii), contradiction. Finally, suppose that α is a limit ordinal. Then $\alpha \subseteq \mathbf{B}$, and so $\alpha \in \mathbf{B}$ by (iii), conradiction. □

Corollary 9.6. *Suppose that* $\mathbf{B} \subseteq \mathbf{On}$ *and the following conditions hold:*

(i) $0 \in \mathbf{B}$.
(ii) If $\alpha \in \mathbf{B}$, *then* $\alpha +' 1 \in \mathbf{B}$.
(iii) If α *is a limit ordinal and* $\alpha \subseteq \mathbf{B}$, *then* $\alpha \in \mathbf{B}$.

Then $\mathbf{B} = \mathbf{On}$. $\square$

Transfinite recursion

Theorem 9.7. *Suppose that* $\mathbf{G}$ *is a class function mapping* $\mathbf{On} \times \mathbf{V}$ *into* $\mathbf{V}$. *Then there is a unique class function* $\mathbf{F}$ *mapping* $\mathbf{On}$ *into* $\mathbf{V}$ *such that* $\mathbf{F}(\alpha) = \mathbf{G}(\alpha, \mathbf{F} \restriction \alpha)$ *for every ordinal* α.

Proof. We apply Theorems 8.7 and 8.8 with $\mathbf{R} = \{(\alpha, \beta) : \alpha < \beta\}$. $\square$

Well-order

A *partial order* is a pair $(P, <)$ such that P is a set, $<$ is a relation contained in $P \times P$, $<$ is irreflexive ($x \not< x$ for all $x \in P$), and $<$ is transitive (for all $x, y, z \in P$, $x < y < z$ implies that $x < z$). For $(P, <)$ a partial order, we define $p_1 \leq p_2$ iff $p_1 < p_2$ or $p_1 = p_2$. A *linear order* is a partial order $(P, <)$ such that for all $x, y \in P$, either $x < y$, $x = y$, or $y < x$. A *well-order* is a linear order $(P, <)$ such that for every nonempty $X \subseteq P$ there is an $x \in X$ such that $\forall y \in X[y \not< x]$. This element x is called the $<$-*least element* of X.

Proposition 9.8. *For any ordinal* α, $(\alpha, <)$ *is a well-order.* $\square$

Proposition 9.9. *If* $(P, <)$ *is a well-order, then* $<$ *is well-founded.* $\square$

If $(P, <)$ and $(Q, \prec)$ are partial orders, then a function $f : P \to Q$ is *strictly increasing* iff $\forall p_1, p_2 \in P[p_1 < p_2 \Rightarrow f(p_1) \prec f(p_2)]$.

Proposition 9.10. *If* $(A, <)$ *and* $(B, \prec)$ *are linearly orders and* $f : A \to B$ *is strictly increasing, then* $\forall a_0, a_1 \in A[a_0 < a_1 \Leftrightarrow f(a_0) \prec f(a_1)]$.

Proof. The direction $\Rightarrow$ is given by the definition. Now suppose that it is not true that $a_0 < a_1$. Then $a_1 \leq a_0$, so $f(a_1) \leq f(a_0)$. So $f(a_0) < f(a_1)$ is not true. $\square$

Proposition 9.11. *If* $(A, <)$ *is a well-ordered set and* $f : A \to A$ *is strictly increasing, then* $x \leq f(x)$ *for all* $x \in A$.

Proof. Suppose not. Then then set $B \stackrel{\text{def}}{=} \{x \in A : f(x) < x\}$ is nonempty. Let b be the least element of B. Thus $f(b) < b$. Hence by the choice of b, we have $f(b) \leq f(f(b))$. Hence by Proposition 9.10, $b \leq f(b)$, contradiction. $\square$

Let $(A, <)$ and $(B, \prec)$ be partial orders. An *isomorphism* from $(A, <)$ onto $(B, \prec)$ is a function f mapping A onto B such that $\forall a_1, a_2 \in A[a_1 < a_2$ iff $f(a_1) \prec f(a_2)]$.

Proposition 9.12. *If* $(A, <)$ *and* $(B, \prec)$ *are isomorphic well-orders, then there is a unique isomorphism* f *mapping* A *onto* B.

Proof. The existence of f follows from the definition. Suppose that both f and g are isomorphisms from A onto B. Then $f^{-1} \circ g$ is a strictly increasing function from A into A, so by Proposition 9.11 we get $x \leq (f^{-1} \circ g)(x)$ for every $x \in A$; so $f(x) \leq g(x)$ for every $x \in A$. Similarly, $g(x) \leq f(x)$ for every $x \in A$, so $f = g$. □

Corollary 9.13. *If $\alpha \neq \beta$, then $(\alpha, <)$ and $(\beta, <)$ are not isomorphic.*

Proof. Suppose to the contrary that f is an isomorphism from $(\alpha, <)$ onto $(\beta, <)$, with $\beta < \alpha$. Then f is a strictly increasing function mapping α into α. Hence $\beta \leq f(\beta) < \beta$ by Proposition 9.11, contradiction. □

The following theorem is fundamental. The proof is also of general interest; it can be followed in outline form in many other situations.

Theorem 9.14. *Every well-order is isomorphic to an ordinal.*

Proof. Let $(A, \prec)$ be a well-order. We may assume that $A \neq \emptyset$. We define a class function $\mathbf{G} : \mathbf{On} \times \mathbf{V} \to \mathbf{V}$ as follows. For any ordinal α and set x,

$$\mathbf{G}(\alpha, x) = \begin{cases} \prec\text{-least element of } A \backslash \operatorname{rng}(x) & \text{if } x \text{ is a function and this set is nonempty,} \\ A & \text{otherwise.} \end{cases}$$

Now by Theorem 9.7 let $\mathbf{F} : \mathrm{On} \to \mathbf{V}$ be such that $\mathbf{F}(\beta) = \mathbf{G}(\beta, \mathbf{F} \restriction \beta)$ for each ordinal β.

(1) If $\beta < \gamma$ and $\mathbf{F}(\beta) = A$, then $\mathbf{F}(\gamma) = A$.

For, $A \backslash \operatorname{rng}(\mathbf{F} \restriction \gamma) \subseteq A \backslash \operatorname{rng}(\mathbf{F} \restriction \beta)$, so $A \backslash \operatorname{rng}(\mathbf{F} \restriction \beta)$ empty implies that $A \backslash \operatorname{rng}(\mathbf{F} \restriction \gamma)$ is empty, giving (1).

(2) if $\beta < \gamma$ and $\mathbf{F}(\gamma) \neq A$, then $\mathbf{F}(\beta) \neq A$ and $\mathbf{F}(\beta) \prec \mathbf{F}(\gamma)$.

The first assertion follows from (1). For the second assertion, note that $A \backslash \operatorname{rng}(\mathbf{F} \restriction \gamma) \subseteq A \backslash \operatorname{rng}(\mathbf{F} \restriction \beta)$, hence $\mathbf{F}(\gamma) \in A \backslash \operatorname{rng}(\mathbf{F} \restriction \beta)$, so $\mathbf{F}(\beta) \preceq \mathbf{F}(\gamma)$ by definition. Also $\mathbf{F}(\beta) \in \operatorname{rng}(\mathbf{F} \restriction \gamma)$, and $\mathbf{F}(\gamma) \notin \operatorname{rng}(\mathbf{F} \restriction \gamma)$, so $\mathbf{F}(\beta) \prec \mathbf{F}(\gamma)$, as desired in (2).

(3) There is an ordinal γ such that $\mathbf{F}(\gamma) = A$.

In fact, suppose not. Let $B = \{a \in A : \exists \alpha[\mathbf{F}(\alpha) = a]\}$. Then $\mathbf{F}^{-1}$ maps B onto $\mathbf{On}$, so by the replacement axiom, $\mathbf{On}$ is a set, contradiction.

Choose γ minimum such that $\mathbf{F}(\gamma) = A$. (Note that $\mathbf{F}(0) \neq A$, since A is nonempty and so has a least element.) By (2), $\mathbf{F} \restriction \gamma$ is strictly inceasing and maps onto A. Hence $\mathbf{F} \restriction \gamma$ is the desired isomorphism, using Proposition 9.10. □

Ordinal class functions

We say that $\mathbf{F}$ is an *ordinal class function* iff $\mathbf{F}$ is a class function whose domain is an ordinal, or the whole class $\mathbf{On}$, and whose range is contained in $\mathbf{On}$. We consider three properties of an ordinal class function $\mathbf{F}$ with domain $\mathbf{A}$:

- $\mathbf{F}$ is *strictly increasing* iff for any ordinals $\alpha, \beta \in \mathbf{A}$, if $\alpha < \beta$ then $\mathbf{F}(\alpha) < \mathbf{F}(\beta)$.

• $\mathbf{F}$ is *continuous* iff for every limit ordinal $\alpha \in \mathbf{A}$, $\mathbf{F}(\alpha) = \bigcup_{\beta<\alpha} \mathbf{F}(\beta)$.

• $\mathbf{F}$ is *normal* iff it is continuous and strictly increasing.

The following is a version of Proposition 9.11, with essentially the same proof.

Proposition 9.15. *If $\mathbf{F}$ is a strictly increasing ordinal class function with domain $\mathbf{A}$, then $\alpha \leq \mathbf{F}(\alpha)$ for every ordinal $\alpha \in \mathbf{A}$.*

Proof. Suppose not, and let α be the least member of $\mathbf{A}$ such that $\mathbf{F}(\alpha) < \alpha$. Then $\mathbf{F}(\mathbf{F}(\alpha)) < \mathbf{F}(\alpha)$, so that $\mathbf{F}(\alpha)$ is an ordinal β smaller than α such that $\mathbf{F}(\beta) < \beta$, contradiction. □

Proposition 9.16. *If $\mathbf{F}$ is a continuous ordinal class function with domain $\mathbf{A}$, and $\mathbf{F}(\alpha) < \mathbf{F}(\alpha +' 1)$ for every ordinal α such that $\alpha +' 1 \in \mathbf{A}$, then $\mathbf{F}$ is strictly increasing.*

Proof. Fix an ordinal $\gamma \in \mathbf{A}$, and suppose that there is an ordinal $\delta \in \mathbf{A}$ with $\gamma < \delta$ and $\mathbf{F}(\delta) \leq \mathbf{F}(\gamma)$; we want to get a contradiction. Take the least such δ.

Case 1. $\delta = \theta +' 1$ for some θ. Thus $\gamma \leq \theta$. If $\gamma = \theta$, then $\mathbf{F}(\gamma) < \mathbf{F}(\delta)$ by the hypothesis of the proposition, contradicting our supposition. Hence $\gamma < \theta$. Hence $\mathbf{F}(\gamma) < \mathbf{F}(\theta)$ by the minimality of δ, and $\mathbf{F}(\theta) < \mathbf{F}(\delta)$ by the assumption of the proposition, so $\mathbf{F}(\gamma) < \mathbf{F}(\delta)$, contradiction.

Case 2. δ is a limit ordinal. Then there is a $\theta < \delta$ with $\gamma < \theta$, and so by the minimality of δ we have

$$\mathbf{F}(\gamma) < \mathbf{F}(\theta) \leq \bigcup_{\varepsilon<\delta} \mathbf{F}(\varepsilon) = \mathbf{F}(\delta),$$

contradiction. □

Proposition 9.17. *Suppose that $\mathbf{F}$ is a normal ordinal class function with domain $\mathbf{A}$, and $\xi \in \mathbf{A}$ is a limit ordinal. Then $\mathbf{F}(\xi)$ is a limit ordinal too.*

Proof. Suppose that $\gamma < \mathbf{F}(\xi)$. Thus $\gamma \in \bigcup_{\eta<\xi} \mathbf{F}(\eta)$, so there is a $\eta < \xi$ such that $\gamma < \mathbf{F}(\eta)$. Now $\mathbf{F}(\eta) < \mathbf{F}(\xi)$. Hence $\mathbf{F}(\xi)$ is a limit ordinal. □

Proposition 9.18. *Suppose that $\mathbf{F}$ and $\mathbf{G}$ are normal ordinal class functions, with domains $\mathbf{A}, \mathbf{B}$ respectively, and the range of $\mathbf{F}$ is contained in $\mathbf{B}$. Then also $\mathbf{G} \circ \mathbf{F}$ is normal.*

Proof. Clearly $\mathbf{G} \circ \mathbf{F}$ is strictly increasing. Now suppose that $\xi \in \mathbf{A}$ is a limit ordinal. Then $\mathbf{F}(\xi)$ is a limit ordinal by Proposition 9.17.

Suppose that $\rho < \xi$. Then $\mathbf{F}(\rho) < \mathbf{F}(\xi)$, so $\mathbf{G}(\mathbf{F}(\rho)) \leq \bigcup_{\eta<\mathbf{F}(\xi)} \mathbf{G}(\eta) = \mathbf{G}(\mathbf{F}(\xi))$. Thus

$$(*) \qquad \bigcup_{\rho<\xi} \mathbf{G}(\mathbf{F}(\rho)) \leq \mathbf{G}(\mathbf{F}(\xi)).$$

Now if $\eta < \mathbf{F}(\xi)$, then by the continuity of $\mathbf{F}$, $\eta < \bigcup_{\rho<\xi} \mathbf{F}(\rho)$, and hence there is a $\rho < \xi$ such that $\eta < \mathbf{F}(\rho)$; so $\mathbf{G}(\eta) < \mathbf{G}(\mathbf{F}(\rho))$. So for any $\eta < \mathbf{F}(\xi)$ we have $\mathbf{G}(\eta) \leq \bigcup_{\rho<\xi} \mathbf{G}(\mathbf{F}(\rho))$. Hence

$$\mathbf{G}(\mathbf{F}(\xi)) = \bigcup_{\eta<\mathbf{F}(\xi)} \mathbf{G}(\eta) \leq \bigcup_{\rho<\xi} \mathbf{G}(\mathbf{F}(\rho));$$

together with $(*)$ this gives the continuity of $\mathbf{G} \circ F$. $\square$

Ordinal addition

We use the general recursion theorem to define ordinal addition:

Theorem 9.19. *There is a unique function $+$ mapping $\mathbf{On} \times \mathbf{On}$ into $\mathbf{On}$ such that the following conditions hold for any α:*

(i) $\alpha + 0 = \alpha$;

(ii) $\alpha + (\beta +' 1) = (\alpha + \beta) +' 1$;

(iii) $\alpha + \gamma = \bigcup_{\beta<\gamma}(\alpha + \beta)$ for γ a limit ordinal.

Proof. For the existence we use the main recursion theorem, Theorem 8.7. Let $\mathbf{A} = \mathbf{On} \times \mathbf{On}$, and let $\mathbf{R} = \{((\alpha, \beta), (\alpha, \gamma) : \beta < \gamma$. Then $\mathrm{pred}_{\mathbf{AR}}(\alpha, \beta) = \{\alpha\} \times \beta$, a set. Thus $\mathbf{R}$ is set-like. Given a nonempty subset X of $\mathbf{A}$, choose $(\alpha, \gamma) \in X$, and then choose β minimum such that $(\alpha, \beta) \in X$. Clearly (α, β) is an $\mathbf{R}$-minimal element of X. Thus $\mathbf{R}$ is well-founded on $\mathbf{A}$.

Now we define $\mathbf{G} : \mathbf{A} \times \mathbf{V} \to \mathbf{V}$. For any α, β and any set x, let

$$\mathbf{G}((\alpha, \beta), x) = \begin{cases} \alpha & \text{if } \beta = 0, \\ x(\alpha, \gamma) +' 1 & \text{if } x \text{ is a function with domain } \{\alpha\} \times \beta \\ & \text{and } \beta = \gamma +' 1, \\ \bigcup_{\gamma<\beta} x(\alpha, \gamma) & \text{if } x \text{ is a function with domain } \{\alpha\} \times \beta \\ & \text{and } \beta \text{ is a limit ordinal,} \\ \emptyset & \text{otherwise.} \end{cases}$$

Then by Theorem 8.7 let $\mathbf{F} : \mathbf{A} \to \mathbf{V}$ be such that $\mathbf{F}(y) = \mathbf{G}(y, \mathbf{F} \restriction \mathrm{pred}_{\mathbf{AR}}(y))$ for any $y \in \mathbf{A}$. Then

$$\begin{aligned} \mathbf{F}(\alpha, 0) &= \mathbf{G}((\alpha, 0), \mathbf{F} \restriction \mathrm{pred}_{\mathbf{AR}}((\alpha, 0))) = \alpha; \\ \mathbf{F}(\alpha, \beta +' 1) &= \mathbf{G}((\alpha, \beta +' 1), \mathbf{F} \restriction \mathrm{pred}_{\mathbf{AR}}((\alpha, \beta +' 1))) \\ &= \mathbf{F}(\alpha, \beta) +' 1; \\ \mathbf{F}(\alpha, \beta) &= \mathbf{G}((\alpha, \beta), \mathbf{F} \restriction \mathrm{pred}_{\mathbf{AR}}((\alpha, \beta))) \\ &= \bigcup_{\gamma<\beta} \mathbf{F}(\alpha, \gamma) \quad \text{if } \beta \text{ is a limit ordinal.} \end{aligned}$$

Thus writing $\alpha + \beta$ instead of $\mathbf{F}(\alpha, \beta)$ we see that $\mathbf{F}$ is as desired.

Now suppose that $+^o$ also satisfies the conditions of the theorem. We show that $\alpha + \beta = \alpha +^o \beta$ for all α, β, by fixing α and going by induction on β, using Corollary 9.9. We have $\alpha + 0 = \alpha = \alpha +^o \beta$. Assume that $\alpha + \beta = \alpha +^o \beta$. Then $\alpha + (\beta +' 1) = (\alpha + \beta) +' 1 = (\alpha +^o \beta) +' 1 = \alpha +^o (\beta +' 1)$. Assume that β is a limit ordinal and $\alpha + \gamma = \alpha +^o \gamma$ for every $\gamma < \beta$. Then $\alpha + \beta = \bigcup_{\gamma<\beta} \alpha + \gamma = \bigcup_{\gamma<\beta} \alpha +^o \gamma = \alpha +^o \beta$. $\square$

Proposition 9.20. $\alpha + 1 = \alpha +' 1$ *for any ordinal α.*

Proof. $\alpha + 1 = \alpha + (0 +' 1) = (\alpha + 0) +' 1 = \alpha +' 1$. $\square$

Now we can stop using the notation $\alpha +' 1$, using $\alpha + 1$ instead.

We state the simplest properties of ordinal addition in the following theorem., *Weakly increasing* means that $\alpha < \beta$ implies that $\mathbf{F}(\alpha) \leq \mathbf{F}(\beta)$.

Theorem 9.21. *(i) If $m, n \in \omega$, then $m + n \in \omega$.*

(ii) For any ordinal α, the class function $\mathbf{F}$ which takes each ordinal β to $\alpha + \beta$ is a normal function.

(iii) For any ordinal β, the class function $\mathbf{F}$ which takes each ordinal α to $\alpha + \beta$ is weakly increasing.

(iv) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.

(v) $\beta \leq \alpha + \beta$.

(vi) $0 + \alpha = \alpha$.

(vii) $\alpha \leq \beta$ iff there is a δ such that $\alpha + \delta = \beta$.

(viii) $\alpha < \beta$ iff there is a $\delta > 0$ such that $\alpha + \delta = \beta$.

Proof. (i): with m fixed we use induction on n, thus appealing to Theorem 7.14. We have $m+0 = m \in \omega$. Assume that $n \in \omega$ and $m+n \in \omega$. then $m+(n+1) = (m+n)+1 \in \omega$, completing the induction.

(ii): by Proposition 9.19.

(iv): Fix α and β; we proceed by induction on γ. The case $\gamma = 0$ is obvious. Assume that $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$. Then

$$\begin{aligned}
\alpha + (\beta + (\gamma + 1)) &= \alpha + ((\beta + \gamma) + 1) \\
&= (\alpha + (\beta + \gamma)) + 1 \\
&= ((\alpha + \beta) + \gamma) + 1 \\
&= (\alpha + \beta) + (\gamma + 1).
\end{aligned}$$

Finally, suppose that γ is a limit ordinal and we know our result for all $\delta < \gamma$. Let $\mathbf{F}, \mathbf{G}, \mathbf{H}$ be the ordinal class functions such that, for any ordinal δ,

$$\begin{aligned}
\mathbf{F}(\delta) &= \alpha + \delta; \\
\mathbf{G}(\delta) &= (\alpha + \beta) + \delta; \\
\mathbf{H}(\delta) &= \beta + \delta.
\end{aligned}$$

Thus according to (ii), all three of these functions are normal. Hence, using Proposition 9.18,

$$\begin{aligned}
\alpha + (\beta + \gamma) &= \mathbf{F}(\mathbf{H}(\gamma)) \\
&= \bigcup_{\delta < \gamma} \mathbf{F}(\mathbf{H}(\delta)) \\
&= \bigcup_{\delta < \gamma} (\alpha + (\beta + \delta)) \\
&= \bigcup_{\delta < \gamma} ((\alpha + \beta) + \delta
\end{aligned}$$

$$
\begin{aligned}
&= \bigcup_{\delta<\gamma} \mathbf{G}(\delta)\\
&= \mathbf{G}(\gamma)\\
&= (\alpha+\beta)+\gamma.
\end{aligned}
$$

(v): by (ii) and Proposition 9.15.

(vi): induction on α. $0+0=0$. If $0+\alpha=\alpha$, then $0+(\alpha+1)=(0+\alpha)+1=\alpha+1$. If α is limit and $0+\beta=\beta$ for all $\beta<\alpha$, then $0+\alpha=\bigcup_{\beta<\alpha}(0+\beta)=\bigcup_{\beta<\alpha}\beta=\alpha$.

(vii): In the $\Rightarrow$ direction, assume that $\alpha\le\beta$. Now $\beta\le\alpha+\beta$ by (v). Let δ be minimum such that $\beta\le\alpha+\delta$. Suppose that $\beta<\alpha+\delta$. If $\delta=\varepsilon+1$ for some ε, then $\beta<(\alpha+\varepsilon)+1$ and hence $\beta\le\alpha+\varepsilon$ using Proposition 7.10. This contradicts the choice of δ. A similar contradiction is reached if δ is a limit ordinal. So $\beta=\alpha+\delta$.

For the $\Leftarrow$ direction, we prove that $\alpha\le\alpha+\delta$ for all δ by induction on δ. It is clear for $\delta=0$. Assume that $\alpha\le\alpha+\delta$. Now $\alpha+\delta<(\alpha+\delta)+1=\alpha+(\delta+1)$, so $\alpha\le\alpha+(\delta+1)$. Finally, suppose that δ is a limit ordinal and $\alpha\le\alpha+\gamma$ for all $\gamma<\delta$. Clearly then $\alpha\le\bigcup_{\gamma<\chi}(\alpha+\gamma)=\alpha+\delta$.

(viii): If $\alpha<\beta$, choose δ by (vii) so that $\alpha+\delta=\beta$. Since $\alpha\ne\beta$ we have $\delta>0$. For the other direction, if $\alpha+\delta=\beta$ with $\delta>0$, then $\alpha=\alpha+0<\alpha+\delta=\beta$, using (ii).

(iii): Suppose that $\gamma<\alpha$. By (viii), choose $\delta>0$ such that $\gamma+\delta=\alpha$. Then $\beta\le\delta+\beta$ by (v), and so by (ii) and (iv), $\gamma+\beta\le\gamma+(\delta+\beta)=(\gamma+\delta)+\beta=\alpha+\beta$. □

Note that $+$ is not commutative. In fact, $1+\omega=\omega<\omega+1$. The ordinal class function $\mathbf{F}$, which for a fixed β takes each ordinal α to $\alpha+\beta$, is not continuous. For example, $\omega+1$ is not equal to $\bigcup_{m\in\omega}(m+1)$, as the latter is equal to ω.

Ordinal multiplication

Theorem 9.22. *There is a unique function $\cdot$ mapping* **On** $\times$ **On** *into* **On** *such that the following conditions hold:*

$$
\begin{aligned}
\alpha\cdot 0 &= 0;\\
\alpha\cdot(\beta+1) &= \alpha\cdot\beta+\alpha;\\
\alpha\cdot\beta &= \bigcup_{\gamma<\beta}(\alpha\cdot\gamma) \quad \textit{for } \beta \textit{ limit.}
\end{aligned}
$$

Proof. The proof is very similar to the proof of Theorem 9.19. We start with **A** and **R** as in that proof.

Now we define $\mathbf{G}:\mathbf{A}\times\mathbf{V}\to\mathbf{V}$. For any α,β and any set x, let

$$
\mathbf{G}((\alpha,\beta),x)=\begin{cases}
0 & \text{if } \beta=0,\\
x(\alpha,\gamma)+\alpha & \text{if } x \text{ is a function with domain } \{\alpha\}\times\beta\\
 & \text{and } \beta=\gamma+' 1,\\
\bigcup_{\gamma<\beta}x(\alpha,\gamma) & \text{if } x \text{ is a function with domain } \{\alpha\}\times\beta\\
 & \text{and } \beta \text{ is a limit ordinal,}\\
\emptyset & \text{otherwise.}
\end{cases}
$$

Then by Theorem 8.7 let $\mathbf{F} : \mathbf{A} \to \mathbf{V}$ be such that $\mathbf{F}(y) = \mathbf{G}(y, \mathbf{F} \restriction \mathrm{pred}_{\mathbf{AR}}(y))$ for any $y \in \mathbf{A}$. Then

$$\begin{aligned}
\mathbf{F}(\alpha, 0) &= \mathbf{G}((\alpha, 0), \mathbf{F} \restriction \mathrm{pred}_{\mathbf{AR}}((\alpha, 0))) = 0;\\
\mathbf{F}(\alpha, \beta +' 1) &= \mathbf{G}((\alpha, \beta +' 1), \mathbf{F} \restriction \mathrm{pred}_{\mathbf{AR}}((\alpha, \beta +' 1)))\\
&= \mathbf{F}(\alpha, \beta) + \alpha;\\
\mathbf{F}(\alpha, \beta) &= \mathbf{G}((\alpha, \beta), \mathbf{F} \restriction \mathrm{pred}_{\mathbf{AR}}((\alpha, \beta)))\\
&= \bigcup_{\gamma<\beta} \mathbf{F}(\alpha, \gamma) \quad \text{if } \beta \text{ is a limit ordinal.}
\end{aligned}$$

Thus writing $\alpha \cdot \beta$ instead of $\mathbf{F}(\alpha, \beta)$ we see that $\mathbf{F}$ is as desired.

Now suppose that $\cdot^o$ also satisfies the conditions of the theorem. We show that $\alpha \cdot \beta = \alpha \cdot^o \beta$ for all α, β, by fixing α and going by induction on β, using Corollary 9.9. We have $\alpha \cdot 0 = 0 = \alpha \cdot^o \beta$. Assume that $\alpha \cdot \beta = \alpha \cdot^o \beta$. Then $\alpha \cdot (\beta + 1) = (\alpha \cdot \beta) + \alpha = (\alpha \cdot^o \beta) + 1 = \alpha \cdot^o (\beta + 1)$. Assume that β is a limit ordinal and $\alpha \cdot \gamma = \alpha \cdot^o \gamma$ for every $\gamma < \beta$. Then $\alpha \cdot \beta = \bigcup_{\gamma<\beta}(\alpha \cdot \gamma) = \bigcup_{\gamma<\beta}(\alpha \cdot^o \gamma) = \alpha \cdot^o \beta$. □

Here are some basic properties of ordinal multiplication:

Theorem 9.23. *(i) If $m, n \in \omega$, then $m \cdot n \in \omega$.*

(ii) If $\alpha \neq 0$, then $\alpha \cdot \beta < \alpha \cdot (\beta + 1)$;

(iii) If $\alpha \neq 0$, then the class function assigning to each ordinal β the product $\alpha \cdot \beta$ is normal.

(iv) $0 \cdot \alpha = 0$*;*

(v) $\alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma)$*;*

(vi) $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$*;*

(vii) If $\alpha \neq 0$, then $\beta \leq \alpha \cdot \beta$;

(viii) If $\alpha < \beta$ then $\alpha \cdot \gamma \leq \beta \cdot \gamma$;

(ix) $\alpha \cdot 1 = \alpha$*.*

(x) $\alpha \cdot 2 = \alpha + \alpha$*.*

(xi) If $\alpha, \beta \neq 0$ then $\alpha \cdot \beta \neq 0$.

Proof. (i): Induction on n, with m fixed. $m \cdot 0 = 0 \in \omega$. Assume that $m \cdot n \in \omega$. Then $m \cdot (n+1) = m \cdot n + m$; this is in ω by the inductive hypothesis and Theorem 9.21(i).

(ii): Using 9.21(ii), $\alpha \cdot \beta = \alpha \cdot \beta + 0 < \alpha \cdot \beta + \alpha = \alpha \cdot (\beta + 1)$.

(iii): this follows from (ii) and Proposition 9.16.

(iv): We prove this by induction on α. $0 \cdot 0 = 0$. Assuming that $0 \cdot \alpha = 0$, we have $0 \cdot (\alpha + 1) = 0 \cdot \alpha + 0 = 0 + 0 = 0$. Assuming that α is a limit ordinal and $0 \cdot \gamma = 0$ for all $\gamma < \alpha$, we have $0 \cdot \alpha = \bigcup_{\gamma<\alpha}(0 \cdot \gamma) = \bigcup_{\gamma<\alpha} 0 = 0$.

(v) Fix α and β. By (iv) we may assume that $\alpha \neq 0$; we then proceed by induction on γ. We define some ordinal class functions $\mathbf{F}, \mathbf{F}', \mathbf{G}$: for any γ, $\mathbf{F}(\gamma) = \beta + \gamma$; $\mathbf{F}'(\gamma) = \alpha \cdot \beta + \gamma$; $\mathbf{G}(\gamma) = \alpha \cdot \gamma$. These are normal functions by (iii) and Theorem 9.21(ii).

First of all,

$$\alpha \cdot (\beta + 0) = \alpha \cdot \beta = (\alpha \cdot \beta) + 0 = (\alpha \cdot \beta) + (\alpha \cdot 0),$$

so (v) holds for $\gamma = 0$. Now assume that (v) holds for γ. Then

$$\begin{aligned}\alpha\cdot(\beta+(\gamma+1)) &= \alpha\cdot((\beta+\gamma)+1)\\ &= \alpha\cdot(\beta+\gamma)+\alpha\\ &= (\alpha\cdot\beta)+(\alpha\cdot\gamma)+\alpha\\ &= (\alpha\cdot\beta)+(\alpha\cdot(\gamma+1)),\end{aligned}$$

as desired.

Finally, suppose that δ is a limit ordinal and we know (v) for all $\gamma < \delta$. Then

$$\begin{aligned}\alpha\cdot(\beta+\delta) &= \mathbf{G}(\mathbf{F}(\delta))\\ &= (\mathbf{G}\circ\mathbf{F})(\delta)\\ &= \bigcup_{\gamma<\delta}(\mathbf{G}\circ\mathbf{F})(\gamma)\\ &= \bigcup_{\gamma<\delta}(\alpha\cdot(\beta+\gamma))\\ &= \bigcup_{\gamma<\delta}((\alpha\cdot\beta)+(\alpha\cdot\gamma))\\ &= \bigcup_{\gamma<\delta}\mathbf{F}'(\mathbf{G}(\gamma))\\ &= \bigcup_{\gamma<\delta}(\mathbf{F}'\circ\mathbf{G})(\gamma)\\ &= (\mathbf{F}'\circ\mathbf{G})(\delta)\\ &= (\alpha\cdot\beta)+(\alpha\cdot\delta),\end{aligned}$$

as desired. This completes the proof of (v).

(vi): For $\alpha = 0$, $0\cdot(\beta\cdot\gamma) = 0$ by (iv), and by (iv) again, $(0\cdot\beta)\cdot\gamma = 0\cdot\gamma = 0$. For $\beta = 0$, $\alpha\cdot(0\cdot\gamma) = \alpha\cdot 0 = 0$ using (iv), and $(\alpha\cdot 0)\cdot\gamma = 0\cdot\gamma = 0$, using (iv) again.

So we assume that $\alpha, \beta \neq 0$. With fixed α, β we now proceed by induction on γ. Let $\mathbf{F}$ and $\mathbf{G}$ be the class functions defined by $\mathbf{F}(\delta) = \beta\cdot\delta$ and $\mathbf{G}(\delta) = \alpha\cdot\delta$ for all δ. These are normal functions by (iii). Then $\alpha\cdot(\beta\cdot 0) = \alpha\cdot 0 = 0 = (\alpha\cdot\beta)\cdot 0$. Assuming that $\alpha\cdot(\beta\cdot\gamma) = (\alpha\cdot\beta)\cdot\gamma$, we have

$$\begin{aligned}\alpha\cdot(\beta\cdot(\gamma+1)) &= \alpha\cdot(\beta\cdot\gamma+\beta)\\ &= \alpha\cdot(\beta\cdot\gamma)+\alpha\cdot\beta\\ &= (\alpha\cdot\beta)\cdot\gamma+\alpha\cdot\beta\\ &= (\alpha\cdot\beta)\cdot(\gamma+1).\end{aligned}$$

Finally, for δ limit, assuming that $\alpha\cdot(\beta\cdot\gamma) = (\alpha\cdot\beta)\cdot\gamma$ for all $\gamma < \delta$, we have

$$\alpha\cdot(\beta\cdot\delta) = \mathbf{G}(\mathbf{F}(\delta))$$

$$
\begin{aligned}
&= (\mathbf{G} \circ \mathbf{F})(\delta) \\
&= \bigcup_{\gamma<\delta} \mathbf{G}(\mathbf{F}(\gamma)) \\
&= \bigcup_{\gamma<\delta} \alpha \cdot (\beta \cdot \gamma) \\
&= \bigcup_{\gamma<\delta} (\alpha \cdot \beta) \cdot \gamma \\
&= (\alpha \cdot \beta) \cdot \delta.
\end{aligned}
$$

(vii): follows from (ii) and Proposition 9.15.

(viii): Fix $\alpha < \beta$. We prove that $\alpha\cdot\gamma \leq \beta\cdot\gamma$ by induction on γ. We have $\alpha\cdot 0 = 0 = \beta\cdot 0$, so $\alpha \cdot 0 \leq \beta \cdot 0$. Suppose that $\alpha \cdot \gamma \leq \beta \cdot \gamma$. Then

$$
\begin{aligned}
\alpha \cdot (\gamma + 1) &= \alpha \cdot \gamma + \alpha \\
&\leq \beta \cdot \gamma + \alpha \quad \text{induction hypothesis, Theorem 9.21(iii)} \\
&< \beta \cdot \gamma + \beta \quad \text{Theorem 9.21(ii)} \\
&= \beta \cdot (\gamma + 1).
\end{aligned}
$$

Finally, suppose that γ is a limit ordinal and $\alpha \cdot \delta \leq \beta \cdot \delta$ for every $\delta < \gamma$. Then

$$
\begin{aligned}
\alpha \cdot \gamma &= \bigcup_{\delta<\gamma} (\alpha \cdot \delta) \\
&\leq \bigcup_{\delta<\gamma} (\beta \cdot \delta) \quad \text{induction hypothesis, Proposition 7.8} \\
&= \beta \cdot \gamma.
\end{aligned}
$$

(ix): $\alpha \cdot 1 = \alpha \cdot (0 + 1) = \alpha \cdot 0 + \alpha = 0 + \alpha = \alpha$ using Proposition 9.21(vi).

(x): $\alpha \cdot 2 = \alpha \cdot (1 + 1) = \alpha \cdot 1 + \alpha = \alpha + \alpha$.

(xi): With $\alpha \neq 0$ fixed we go by induction on β, proving that $\beta \neq 0$ implies that $\alpha \cdot \beta \neq 0$. This is vacuously true for $\beta = 0$. Assume that the implication holds for β, and assume that $\beta + 1 \neq 0$. Then $\alpha \cdot (\beta + 1) = \alpha \cdot \beta + \alpha > \alpha \cdot \beta + 0 = \alpha \cdot \beta$ using (iii); so $\alpha \cdot (\beta + 1) \neq 0$. Finally, suppose that β is a limit ordinal and the implication holds for all $\gamma < \beta$. Then $\alpha \cdot \beta = \bigcup_{\gamma<\beta} (\alpha \cdot \gamma) \geq \alpha \cdot 1 \neq 0$. □

The commutative law for multiplication fails in general. For example, $2 \cdot \omega = \omega$ while $\omega \cdot 2 = \omega + \omega > \omega$. Also the distributive law $(\alpha + \beta) \cdot \gamma = \alpha \cdot \gamma + \beta \cdot \gamma$ fails in general. For example, $(1 + 1) \cdot \omega = 2 \cdot \omega = \omega$, while $1 \cdot \omega + 1 \cdot \omega = \omega + \omega > \omega$. Here we use the fact that $1 \cdot \omega = \omega$. In fact, $1 \cdot \alpha = \alpha$ for any ordinal α, as is easily shown by induction on α.

Ordinal exponentiation

Theorem 9.24. *There is a unique function mapping* **On** $\times$ **On** *into* **On** *such that the following conditions hold, where we write the value of the function at an argument* (α, β)

as α^β*:*

$$\begin{aligned}\alpha^0 &= 1;\\ \alpha^{\beta+1} &= \alpha^\beta \cdot \alpha;\\ \alpha^\beta &= \bigcup_{\gamma<\beta}(\alpha^\gamma) \quad \textit{for } \beta \textit{ limit.}\end{aligned}$$

Proof. The proof is very similar to the proofs of Theorem 9.19 and 9.22. We start with $\mathbf{A}$ and $\mathbf{R}$ as in those proofs.

Now we define $\mathbf{G} : \mathbf{A} \times \mathbf{V} \to \mathbf{V}$. For any α, β and any set x, let

$$\mathbf{G}((\alpha,\beta),x) = \begin{cases} 1 & \text{if } \beta = 0,\\ x(\alpha,\gamma)\cdot\alpha & \text{if } x \text{ is a function with domain } \{\alpha\}\times\beta\\ & \text{and } \beta = \gamma +' 1,\\ \bigcup_{\gamma<\beta} x(\alpha,\gamma) & \text{if } x \text{ is a function with domain } \{\alpha\}\times\beta\\ & \text{and } \beta \text{ is a limit ordinal,}\\ \emptyset & \text{otherwise.}\end{cases}$$

Then by Theorem 8.7 let $\mathbf{F} : \mathbf{A} \to \mathbf{V}$ be such that $\mathbf{F}(y) = \mathbf{G}(y, \mathbf{F} \restriction \mathrm{pred}_{\mathbf{AR}}(y))$ for any $y \in \mathbf{A}$. Then

$$\begin{aligned}\mathbf{F}(\alpha,0) &= \mathbf{G}((\alpha,0), \mathbf{F}\restriction \mathrm{pred}_{\mathbf{AR}}((\alpha,0))) = 1;\\ \mathbf{F}(\alpha,\beta+'1) &= \mathbf{G}((\alpha,\beta+'1), \mathbf{F}\restriction \mathrm{pred}_{\mathbf{AR}}((\alpha,\beta+'1)))\\ &= \mathbf{F}(\alpha,\beta)\cdot\alpha;\\ \mathbf{F}(\alpha,\beta) &= \mathbf{G}((\alpha,\beta), \mathbf{F}\restriction \mathrm{pred}_{\mathbf{AR}}((\alpha,\beta)))\\ &= \bigcup_{\gamma<\beta}\mathbf{F}(\alpha,\gamma) \quad \text{if } \beta \text{ is a limit ordinal.}\end{aligned}$$

Thus writing α^β instead of $\mathbf{F}(\alpha,\beta)$ we see that $\mathbf{F}$ is as desired.

Now suppose that $\mathbf{F}'$ also satisfies the conditions of the theorem. We show that $\alpha^\beta = \mathbf{F}'(\alpha,\beta)$ for all α,β, by fixing α and going by induction on β, using Corollary 9.9. We have $\alpha^0 = 1 = \mathbf{F}'(\alpha,\beta)$. Assume that $\alpha^\beta = \mathbf{F}'(\alpha,\beta)$. Then $\alpha^{\beta+1} = (\alpha^\beta)\cdot\alpha = \mathbf{F}'(\alpha,\beta)\cdot\alpha = \mathbf{F}'(\alpha,\beta+1)$. Assume that β is a limit ordinal and $\alpha^\gamma = \mathbf{F}'(\alpha,\gamma)$ for every $\gamma < \beta$. Then $\alpha^\beta = \bigcup_{\gamma<\beta}\alpha^\gamma = \bigcup_{\gamma<\beta}\mathbf{F}'(\alpha,\gamma) = \mathbf{F}'(\alpha,\beta)$. □

Now we give the simplest properties of exponentiation.

Theorem 9.25. *(i) If* $m,n\in\omega$*, then* $m^n \in \omega$*.*

(ii) $0^0 = 1$*;*

(iii) $0^{\beta+1} = 0$*;*

(iv) $0^\beta = 1$ *for* β *a limit ordinal;*

(v) $1^\beta = 1$*;*

(vi) If $\alpha \neq 0$*, then* $\alpha^\beta \neq 0$*;*

(vii) If $\alpha > 1$ *then* $\alpha^\beta < \alpha^{\beta+1}$*;*

(viii) If $\alpha > 1$, then the ordinal class function assigning to each ordinal β the value α^β is normal;

(ix) If $\alpha > 1$, then $\beta \leq \alpha^\beta$;

(x) If $0 < \alpha < \beta$, then $\alpha^\gamma \leq \beta^\gamma$;

(xi) For $\alpha \neq 0$, $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$;

(xii) For $\alpha \neq 0$, $(\alpha^\beta)^\gamma = \alpha^{\beta\cdot\gamma}$.

Proof. (i): With m fixed we go by induction on n. $m^0 = 1 \in \omega$. Assume that $m^n \in \omega$. Then $m^{n+1} = m^n \cdot m \in \omega$ by the induction hypothesis and Theorem 9.23(i).

(ii): Obvious.

(iii): $0^{\beta+1} = 0^\beta \cdot 0 = 0$.

(iv): We prove by induction on β that

$$0^\beta = \begin{cases} 1 & \text{if } \beta = 0, \\ 0 & \text{if } \beta \text{ is a successor ordinal,} \\ 1 & \text{if } \beta \text{ is a limit ordinal.} \end{cases}$$

This is clearly true for $\beta = 0$, and if it is true for γ then it is true for $\gamma + 1$ by (iii). Now suppose that β is a limit ordinal and it is true for all $\gamma < \beta$. Thus 0^γ is 0 or 1 for each $\gamma < \beta$, and $0^0 = 1$ with $0 < \beta$, so $0^\beta = \bigcup_{\gamma<\beta} 0^\gamma = 1$.

(v): we prove this by induction on β. $1^0 = 1$. Assume that $1^\beta = 1$. Then $1^{\beta+1} = 1^\beta \cdot 1 = 1 \cdot 1 = 1$. Assume that β is a limit ordinal and $1^\gamma = 1$ for all $\gamma < \beta$. Then $1^\beta = \bigcup_{\gamma<\beta} 1^\gamma = 1$.

(vi) With $\alpha \neq 0$ fixed, we go by induction on β. $\alpha^0 = 1 \neq 0$. Assume that $\alpha^\beta \neq 0$. Then $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha \neq 0$ by the inductive hypothesis and Theorem 9.23(xi). Assume that β is a limit ordinal and $\alpha^\gamma \neq 0$ for all $\gamma < \beta$. Then $\alpha^\beta = \bigcup_{\gamma<\beta} \alpha^\gamma \neq 0$ by the inductive hypothesis, since $0 < \beta$ and $\alpha^0 \neq 0$.

(vii): We have $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha > \alpha^\beta \cdot 1 = \alpha^\beta$ using (vi) and Theorem 9.23(iii),(ix).

(viii): by (vii) and Theorem 9.16.

(ix): by (viii) and Theorem 9.15.

(x): With $0 < \alpha < \beta$, induction on γ. $\alpha^0 = 1 = \beta^0$. Assume that $\alpha^\gamma \leq \beta^\gamma$. Then $\alpha^{\gamma+1} = \alpha^\gamma \cdot \alpha \leq \beta^\gamma \cdot \alpha$ (by the inductive hypothesis and Theorem 9.23(viii)) $< \beta^\gamma \cdot \beta$ (by Theorem 9.23 (iii)) $= \beta^{\gamma+1}$. Now assume that $\alpha^\gamma \leq \alpha^\beta$ for all $\gamma < \delta$, where δ is a limit ordinal. Then $\alpha^\delta = \bigcup_{\gamma<\delta} \alpha^\gamma \leq \bigcup_{\gamma<\delta} \beta^\gamma = \beta^\delta$, using Proposition 7.12.

(xi): By (v) we may assume that $\alpha > 1$. Define $\mathbf{F}(\delta) = \beta+\delta$, $\mathbf{G}(\delta) = \alpha^\delta$, $\mathbf{H}(\delta) = \alpha^\beta\cdot\delta$. These are normal functions by Theorem 9.21(ii), Theorem 9.23(iii) and (vi), and (viii).

Now we go by induction on γ. $\alpha^{\beta+0} = \alpha^\beta = \alpha^\beta \cdot 1 = \alpha^\beta \cdot \alpha^0$. Assume that $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$. Then $\alpha^{\beta+\gamma+1} = \alpha^{\beta+\gamma} \cdot \alpha = \alpha^\beta \cdot \alpha^\gamma \cdot \alpha + a^\beta \cdot \alpha^{\gamma+1}$. Finally, suppose that δ is limit and $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$ for every $\gamma < \delta$. Then

$$\begin{aligned} \alpha^{\beta+\delta} &= \mathbf{G}(\mathbf{F}(\delta)) \\ &= \bigcup_{\gamma<\delta} \mathbf{G}(\mathbf{F}(\gamma)) \\ &= \bigcup_{\gamma<\delta} \alpha^{\beta+\gamma} \end{aligned}$$

$$
\begin{aligned}
&= \bigcup_{\gamma<\delta} (\alpha^\beta \cdot \alpha^\gamma) \\
&= \bigcup_{\gamma<\delta} \mathbf{H}(\gamma) \\
&= \mathbf{H}(\delta) \\
&= \alpha^\beta \cdot \alpha^\delta.
\end{aligned}
$$

(xii): First note that it holds for $\beta = 0$, since $(\alpha^0)^\gamma = 1^\gamma = 1$ and $\alpha^{0\cdot\gamma} = \alpha^0 = 1$. Similarly, it holds for $\alpha = 1$. Now assume that $\alpha > 1$ and $\beta > 0$. Let $\mathbf{F}(\delta) = \alpha^\delta$ for any δ, and $G(\delta) = \beta \cdot \delta$ for any δ. Then $\mathbf{F}$ and $\mathbf{G}$ are normal functions. Now we prove the result by induction on γ. First, $(\alpha^\beta)^0 = 1 = \alpha^0 = \alpha^{\beta\cdot 0}$. Now assume that $(\alpha^\beta)^\gamma = \alpha^{\beta\cdot\gamma}$. Then

$$(\alpha^\beta)^{\gamma+1} = (\alpha^\beta)^\gamma \cdot \alpha^\beta = \alpha^{\beta\cdot\gamma} \cdot \alpha^\beta = \alpha^{\beta\cdot\gamma+\beta} = \alpha^{\beta\cdot(\gamma+1)}.$$

Finally, suppose that δ is a limit ordinal and $(\alpha^\beta)^\gamma = \alpha^{\beta\cdot\gamma}$ for all $\gamma < \delta$. Then

$$
\begin{aligned}
(\alpha^\beta)^\delta &= \bigcup_{\gamma<\delta} (\alpha^\beta)^\gamma \\
&= \bigcup_{\gamma<\delta} \alpha^{\beta\cdot\gamma} \\
&= \bigcup_{\gamma<\delta} \mathbf{F}(\mathbf{G}(\gamma)) \\
&= \mathbf{F}(\mathbf{G}(\delta)) \\
&= \alpha^{\beta\cdot\delta}
\end{aligned}
$$

□

Theorem 9.26. (division algorithm) *Suppose that α and β are ordinals, with $\beta \neq 0$. Then there are unique ordinals ξ, η such that $\alpha = \beta \cdot \xi + \eta$ with $\eta < \beta$.*

Proof. First we prove the existence. Note that $\alpha < \alpha + 1 \leq \beta \cdot (\alpha + 1)$. Thus there is an ordinal number ρ such that $\alpha < \beta \cdot \rho$; take the least such ρ. Obviously $\rho \neq 0$. If ρ is a limit ordinal, then because $\beta \cdot \rho = \bigcup_{\sigma<\rho}(\beta \cdot \sigma)$, it follows that there is a $\sigma < \rho$ such that $\alpha < \beta \cdot \sigma$, contradicting the minimality of ρ. Thus ρ is a successor ordinal $\xi + 1$. By the definition of ρ we have $\beta \cdot \xi \leq \alpha$. Hence there is an ordinal η such that $\beta \cdot \xi + \eta = \alpha$. We claim that $\eta < \beta$. Otherwise, $\alpha = \beta \cdot \xi + \eta \geq \beta \cdot \xi + \beta = \beta \cdot (\xi + 1) = \beta \cdot \rho$, contradicting the definition of ρ. This finishes the proof of existence.

For uniqueness, suppose that also $\alpha = \beta \cdot \xi' + \eta'$ with $\eta' < \beta$. Suppose that $\xi \neq \xi'$. By symmetry, say $\xi < \xi'$. Then

$$\alpha = \beta \cdot \xi + \eta < \beta \cdot \xi + \beta = \beta \cdot (\xi + 1) \leq \beta \cdot \xi' \leq \beta \cdot \xi' + \eta' = \alpha,$$

contradiction. Hence $\xi = \xi'$. Hence also $\eta = \eta'$. □

Theorem 9.27. (extended division algorithm)*Let α and β be ordinals, with $\alpha \neq 0$ and $1 < \beta$. Then there exist unique ordinals $\gamma, \delta, \varepsilon$ such that the following conditions hold:*

(i) $\alpha = \beta^\gamma \cdot \delta + \varepsilon$.
(ii) $\gamma \le \alpha$.
(iii) $0 < \delta < \beta$,
(iv) $\varepsilon < \beta^\gamma$.

Proof. We have $\alpha < \alpha + 1 \le \beta^{\alpha+1}$; so there is an ordinal φ such that $\alpha < \beta^\varphi$. We take the least such φ. Clearly φ is a successor ordinal $\gamma + 1$. So we have $\beta^\gamma \le \alpha < \beta^{\gamma+1}$. Now $\beta^\gamma \ne 0$, since $\beta > 1$. Hence by the division algorithm there are ordinals δ, ε such that $\alpha = \beta^\gamma \cdot \delta + \varepsilon$, with $\varepsilon < \beta^\gamma$. Now $\delta < \beta$; for if $\beta \le \delta$, then

$$\alpha = \beta^\gamma \cdot \delta + \varepsilon \ge \beta^\gamma \cdot \beta = \beta^{\gamma+1} > \alpha,$$

contradiction. We have $\delta \ne 0$, as otherwise $\alpha = \varepsilon < \beta^\gamma$, contradiction.. Also, $\gamma \le \alpha$, since

$$\alpha = \beta^\gamma \cdot \delta + \varepsilon \ge \beta^\gamma \ge \gamma.$$

This proves the existence of $\gamma, \delta, \varepsilon$ as called for in the theorem.

Suppose that $\gamma', \delta', \varepsilon'$ also satisfy the indicated conditions; thus

(1) $\alpha = \beta^{\gamma'} \cdot \delta' + \varepsilon'$,

(2) $\gamma' \le \alpha$,

(3) $0 < \delta' < \beta$,

(4) $\varepsilon' < \beta^{\gamma'}$.

Suppose that $\gamma \ne \gamma'$; by symmetry, say that $\gamma < \gamma'$. Then

$$\alpha = \beta^\gamma \cdot \delta + \varepsilon < \beta^\gamma \cdot \delta + \beta^\gamma = \beta^\gamma \cdot (\delta + 1) \le \beta^\gamma \cdot \beta = \beta^{\gamma+1} \le \beta^{\gamma'} \le \alpha,$$

contradiction. Hence $\gamma = \gamma'$. Hence by the ordinary division algorithm we also have $\delta = \delta'$ and $\varepsilon = \varepsilon'$. □

We can obtain an interesting normal form for ordinals by re-applying Theorem 9.27 to the "remainder" ε over and over again. That is the purpose of the following definitions and results. This generalizes the ordinary decimal and binary systems of notation, by taking $\beta = 10$ or $\beta = 2$ and restricting to natural numbers. For infinite ordinals it is useful to take $\beta = \omega$; this gives the *Cantor normal form.*

To abbreviate some long expressions, we let $N(\beta, m, \gamma, \delta)$ stand for the following statement:

β is an ordinal > 1, m is a positive integer, γ and δ are sequences of ordinals each of length m, and: *(1)* $\gamma(0) > \gamma(1) > \cdots > \gamma(m-1)$;   *(2)* $0 < \delta(i) < \beta$ *for each* $i < m$.

If $N(\beta, m, \gamma, \delta)$, then we define

$$k(\beta, m, \gamma, \delta) = \beta^{\gamma(0)} \cdot \delta(0) + \beta^{\gamma(1)} \cdot \delta(1) + \cdots + \beta^{\gamma(m-1)} \cdot \delta(m-1).$$

Lemma 9.28. *Assume that $N(\beta, m, \gamma, \delta)$ and $N(\beta, n, \gamma', \delta')$. Then:*

(i) $k(\beta, m, \gamma, \delta) \geq \gamma(0)$.

(ii) If $m > 1$ and $N(\beta, m, \gamma, \delta)$, then $N(\beta, m-1, \langle\gamma(1), \ldots, \gamma(m-1)\rangle, \langle\delta(1), \ldots, \delta(m-1)\rangle)$.

(iii) If $m > 1$, then $k(\beta, m, \gamma, \delta) = \beta^{\gamma(0)} \cdot \delta(0) + k(\beta, m-1, \langle\gamma(1), \ldots, \gamma(m-1)\rangle, \langle\delta(1), \ldots, \delta(m-1)\rangle)$.

(iv) $k(\beta, m, \gamma, \delta) < \beta^{\gamma(0)} \cdot (\delta(0) + 1) \leq \beta^{\gamma(0)+1}$.

(v) If $\gamma(0) \neq \gamma'(0)$, then $k(\beta, m, \gamma, \delta) < k(\beta, n, \gamma', \delta')$ iff $\gamma(0) < \gamma'(0)$.

(vi) If $\gamma(0) = \gamma'(0)$ and $\delta(0) \neq \delta'(0)$, then $k(\beta, m, \gamma, \delta) < k(\beta, n, \gamma', \delta')$ iff $\delta(0) < \delta'(0)$.

(vii) If $\gamma(j) = \gamma'(j)$ and $\delta(j) = \delta'(j)$ for all $j < i$, while $\gamma(i) \neq \gamma'(i)$, then $k(\beta, m, \gamma, \delta) < k(\beta, n, \gamma', \delta')$ iff $\gamma(i) < \gamma'(i)$.

(viii) If $\gamma(j) = \gamma'(j)$ and $\delta(j) = \delta'(j)$ for all $j < i$, while $\gamma(i) = \gamma'(i)$ and $\delta(i) \neq \delta'(i)$, then $k(\beta, m, \gamma, \delta) < k(\beta, n, \gamma', \delta')$ iff $\delta(i) < \delta'(i)$.

(ix) If $\gamma \subseteq \gamma'$, $\delta \subseteq \delta'$, and $m < n$, then $k(\beta, m, \gamma, \delta) < k(\beta, n, \gamma', \delta')$.

Proof. (i): $k(\beta, m, \gamma, \delta) \geq \beta^{\gamma(0)} \geq \gamma(0)$.

(ii), (iii): Clear.

(iv): Induction on m. It is clear for $m = 1$. Now suppose inductively that $m > 1$. Then

$$\begin{aligned}
&\beta^{\gamma(0)} \cdot \delta(0) + \beta^{\gamma(1)} \cdot \delta(1) + \cdots + \beta^{\gamma(m-1)} \cdot \delta(m-1) \\
&\quad = \beta^{\gamma(0)} \cdot \delta(0) + k(\beta, m-1, \langle\gamma(1), \ldots, \gamma(m-1)\rangle, \langle\delta(1), \ldots, \delta(m-1)\rangle) \quad \text{by (iii)} \\
&< \beta^{\gamma(0)} \cdot \delta(0) + \beta^{\gamma(1)} \cdot (\delta(1) + 1) \quad \text{(inductive hypothesis)} \\
&\leq \beta^{\gamma(0)} \cdot \delta(0) + \beta^{\gamma(1)+1} \\
&\leq \beta^{\gamma(0)} \cdot \delta(0) + \beta^{\gamma(0)} \\
&= \beta^{\gamma(0)} \cdot (\delta(0) + 1) \\
&\leq \beta^{\gamma(0)} \cdot \beta \\
&= \beta^{\gamma(0)+1}.
\end{aligned}$$

For (v), assume the hypothesis, and suppose that $\gamma(0) < \gamma'(0)$. Then

$$\begin{aligned}
k(\beta, m, \gamma, \delta) < \beta^{\gamma(0)} \cdot (\delta(0) + 1) &\leq \beta^{\gamma(0)+1} \\
&\leq \beta^{\gamma'(0)} \\
&\leq k(\beta, n, \gamma', \delta').
\end{aligned}$$

By symmetry (v) now follows.

For (vi), assume the hypothesis, and suppose that $\delta(0) < \delta'(0)$. Then

$$\begin{aligned}
k(\beta, m, \gamma, \delta) < \beta^{\gamma(0)} \cdot (\delta(0) + 1) &= \beta^{\gamma'(0)} \cdot (\delta(0) + 1) \\
&\leq \beta^{\gamma'(0)} \cdot \delta'(0) \\
&\leq k(\beta, n, \gamma', \delta')
\end{aligned}$$

By symmetry (vi) now follows.

(vii) is clear from (v), by deleting the first i summands of the sums.

(viii) is clear from (vi), by deleting the first i summands of the sums.

(ix) is clear. $\square$

Theorem 9.29. (expansion theorem) *Let α and β be ordinals, with $\alpha \neq 0$ and $1 < \beta$. Then there exist a unique $m \in \omega$ and finite sequences $\langle \gamma(i) : i < m \rangle$ and $\langle \delta(i) : i < m \rangle$ of ordinals such that the following conditions hold:*

(i) $\alpha = \beta^{\gamma(0)} \cdot \delta(0) + \beta^{\gamma(1)} \cdot \delta(1) + \cdots + \beta^{\gamma(m-1)} \cdot \delta(m-1)$.

(ii) $\alpha \geq \gamma(0) > \gamma(1) > \cdots > \gamma(m-1)$.

(iii) $0 < \delta(i) < \beta$ *for each* $i < m$.

Proof. For the existence, with $\beta > 1$ fixed we proceed by induction on α. Assume that the theorem holds for every $\alpha' < \alpha$ such that $\alpha' \neq 0$, and suppose that $\alpha \neq 0$. By Theorem 9.27, let φ, ψ, θ be such that

(1) $\alpha = \beta^\varphi \cdot \psi + \theta$,

(2) $\varphi \leq \alpha$,

(3) $0 < \psi < \beta$,

(4) $\theta < \beta^\varphi$.

If $\theta = 0$, then we can take our sequences to be $\langle \gamma(0) \rangle$ and $\langle \delta(0) \rangle$, with $\gamma(0) = \varphi$ and $\delta(0) = \psi$. Now assume that $\theta > 0$. Then

$$\theta < \beta^\varphi \leq \beta^\varphi \cdot \psi + \theta = \alpha;$$

so $\theta < \alpha$. Hence by the inductive assumption we can write

$$\theta = \beta^{\gamma(0)} \cdot \delta(0) + \beta^{\gamma(1)} \cdot \delta(1) + \cdots + \beta^{\gamma(m-1)} \cdot \delta(m-1)$$

with

(5) $\theta \geq \gamma(0) > \gamma(1) > \cdots > \gamma(m-1)$.

(6) $0 < \delta(i) < \beta$ for each $i < m$.

Then our desired sequences for α are

$$\langle \varphi, \gamma(0), \gamma(1), \ldots, \gamma(m-1) \rangle \quad \text{and} \quad \langle \psi, \delta(0), \delta(1), \ldots, \delta(m-1) \rangle.$$

To prove this, we just need to show that $\varphi > \gamma(0)$. If $\varphi \leq \gamma(0)$, then

$$\beta^\varphi \leq \beta^{\gamma(0)} \leq \theta,$$

contradiction.

This finishes the existence part of the proof.

For the uniqueness, we use the notation introduced above, and proceed by induction on α. Suppose the uniqueness statement holds for all nonzero $\alpha' < \alpha$, and now we have $N(\beta, m, \gamma, \delta)$, $N(\beta, n, \gamma', \delta')$, and

$$\alpha = k(\beta, m, \gamma, \delta) = k(\beta, n, \gamma', \delta').$$

We suppose that the uniqueness fails. Say $m \leq n$. Then there is an $i < m$ such that $\gamma(i) \neq \gamma'(i)$ or $\delta(i) \neq \delta'(i)$; we take the least such i. Then we have a contradiction of Lemma 9.28. □

Lemma 9.30. *(i) If $\omega \leq \alpha$, then $1 + \alpha = \alpha$.*
(ii) If $\delta \neq 0$, then $\omega \leq \omega^\delta$ and $1 + \omega^\delta = \omega^\delta$.

Proof. (i): By Theorem 9.21(vii) there is a β such that $\omega + \beta = \alpha$. Hence $1 + \alpha = 1 + \omega + \beta = \omega + \beta = \alpha$.

(ii) $\omega = \omega^1 \leq \omega^\delta$, and $1 + \omega^\delta = \omega^\delta$ by (i). □

Lemma 9.31. *If $\alpha < \omega^\beta$ then $\alpha + \omega^\beta = \omega^\beta$.*

Proof. First we prove

(1) If $\gamma < \beta$, then $\omega^\gamma + \omega^\beta = \omega^\beta$.

In fact, suppose that $\gamma < \beta$. Then there is a nonzero δ such that $\gamma + \delta = \beta$. Then

$$\omega^\gamma + \omega^\beta = \omega^\gamma + \omega^{\gamma+\delta} = \omega^\gamma + \omega^\gamma \cdot \omega^\delta = \omega^\gamma \cdot (1 + \omega^\delta) = \omega^\gamma \cdot \omega^\delta = \omega^\beta.$$

By an easy ordinary induction, we obtain from (1)

(2) If $\gamma < \beta$ and $m \in \omega$, then $\omega^\gamma \cdot m + \omega^\beta = \omega^\beta$.

Now we turn to the general case. If $\beta = 0$ or $\alpha < \omega$, the desired conclusion is clear. So suppose that $\omega \leq \alpha$ and $\beta > 0$. Then we can write $\alpha = \omega^\gamma \cdot m + \delta$ with $m \in \omega$ and $\delta < \omega^\gamma$. Then

$$\omega^\beta \leq \alpha + \omega^\beta = \omega^\gamma \cdot m + \delta + \omega^\beta \leq \omega^\gamma \cdot (m+1) + \omega^\beta = \omega^\beta$$ □

Theorem 9.32. *The following conditions are equivalent:*
(i) $\beta + \alpha = \alpha$ for all $\beta < \alpha$. (Absorption under addition)
(ii) For all $\beta, \gamma < \alpha$, also $\beta + \gamma < \alpha$.
(iii) $\alpha = 0$, or $\alpha = \omega^\beta$ for some β.

Proof. (i)⇒(ii): Assuming (i), if $\beta, \gamma < \alpha$, then $\beta + \gamma < \beta + \alpha = \alpha$.

(ii)⇒(iii): Assume (ii). If $\alpha = 0$ or $\alpha = 1$, condition (iii) holds, so suppose that $2 \leq \alpha$. Then clearly (ii) implies that $\alpha \geq \omega$. Choose β, m, γ such that $m \in \omega$, $\alpha = \omega^\beta \cdot m + \gamma$, and $\gamma < \omega^\beta$. If $\gamma \neq 0$, then $\omega^\beta \cdot m < \omega^\beta \cdot m + \gamma = \alpha$, and also $\gamma < \omega^\beta < \alpha$, so that (ii) is contradicted. So $\gamma = 0$. If $m > 1$, write $m = n + 1$ with $n \neq 0$. Then

$$\alpha = \omega^\beta \cdot m = \omega^\beta \cdot (n+1) = \omega^\beta \cdot n + \omega^\beta,$$

and $\omega^\beta \cdot n, \omega^\beta < \alpha$, again contradicting (ii). Hence $m = 1$, as desired in (iii).

Finally, (iii)⇒(i) by Lemma 9.31. □

Lemma 9.33. *If $\alpha \neq 0$ and m is a positive integer, then $m \cdot \omega^\alpha = \omega^\alpha$.*

Proof. Induction on α. It is clear for $\alpha = 1$. Assuming it true for α, we have $m \cdot \omega^{\alpha+1} = m \cdot \omega^\alpha \cdot \omega = \omega^\alpha \cdot \omega = \omega^{\alpha+1}$. Assuming it is true for every $\beta < \alpha$ with α a limit ordinal, we have $1 + \alpha = \alpha$, and so $m \cdot \omega^\alpha = m \cdot \omega \cdot \omega^\alpha = \omega \cdot \omega^\alpha = \omega^\alpha$. □

Theorem 9.34. *The following conditions are equivalent:*

(i) For all β, if $0 < \beta < \alpha$ then $\beta \cdot \alpha = \alpha$. (absorption under multiplication)

(ii) For all $\beta, \gamma < \alpha$, also $\beta \cdot \gamma < \alpha$.

(iii) $\alpha \in \{0, 1, 2\}$ or there is a β such that $\alpha = \omega^{(\omega^\beta)}$.

Proof. (i)⇒(ii): Assume (i), and suppose that $\beta, \gamma < \alpha$. If $\beta = 0$, then $\beta \cdot \gamma = 0 < \alpha$. If $\beta \neq 0$, then $\beta \cdot \gamma < \beta \cdot \alpha = \alpha$.

(ii)⇒(iii): Assume (ii), and suppose that $\alpha \notin \{0, 1, 2\}$. Clearly then $\omega \leq \alpha$. Now if $\beta, \gamma < \alpha$, then $\beta + \gamma < \alpha$. In fact, if $\beta \leq \gamma$, then $\beta + \gamma \leq \gamma + \gamma = \gamma \cdot 2 < \alpha$ by (ii); and if $\gamma < \beta$ then $\beta + \gamma < \beta + \beta = \beta \cdot 2 < \alpha$. Hence by Theorem 9.32 there is a γ such that $\alpha = \omega^\gamma$. Now if $\delta, \varepsilon < \gamma$, then $\omega^\delta, \omega^\varepsilon < \omega^\gamma = \alpha$, and hence $\omega^{\delta+\varepsilon} = \omega^\delta \cdot \omega^\varepsilon < \alpha = \omega^\gamma$, so that $\delta + \varepsilon < \gamma$. Hence by Theorem 9.32, $\gamma = \omega^\beta$ for some β.

(iii)⇒(i): Assume (iii). Clearly $0, 1, 2$ satisfy (i), so assume that $\alpha = \omega^{(\omega^\beta)}$. Take any $\gamma < \alpha$ with $\gamma \neq 0$. If $\gamma < \omega$, then $\gamma \cdot \alpha = \alpha$ by Lemma 9.33. So assume that $\omega \leq \gamma$. Write $\gamma = \omega^\delta \cdot m + \varepsilon$ with $m \in \omega$ and $\varepsilon < \omega^\delta$. Then $\delta < \beta$, and so

$$\begin{aligned}
\alpha = \omega^{(\omega^\beta)} \leq \gamma \cdot \omega^{(\omega^\beta)} &= (\omega^\delta \cdot m + \varepsilon) \cdot \omega^{(\omega^\beta)} \\
&\leq (\omega^\delta \cdot m + \omega^\delta) \cdot \omega^{(\omega^\beta)} \\
&= \omega^\delta \cdot (m+1) \cdot \omega^{(\omega^\beta)} \\
&\leq \omega^{\delta+1} \cdot \omega^{(\omega^\beta)} \\
&= \omega^{\delta+1+\omega^\beta} \\
&= \omega^{(\omega^\beta)} \\
&= \alpha
\end{aligned}$$

□

Proposition 9.35. $1 + m = m + 1$ *for any* $m \in \omega$.

Proof. (Ordinary) induction on m. $0+1 = 1 = 1+0$ using Theorem 9.21(vi). Assume that $1 + m = m + 1$. Then $1 + (m + 1) = (1 + m) + 1 = (m + 1) + 1$. □

Proposition 9.36. $m + n = n + m$ *for any* $m, n \in \omega$.

Proof. With m fixed, induction on n. $0 + m = m = m + 0$ using Theorem 9.21(vi). Assume that $m+n = n+m$. Then $(n+1)+m = n+(1+m) = n+(m+1)$ (by Proposition 9.35) $= (n + m) + 1 = (m + n) + 1 = m + (n + 1)$. □

Proposition 9.37. *$\omega \leq \alpha$ iff $1+\alpha = \alpha$.*

Proof. $\Rightarrow$ holds by Lemma 9.30(i). For $\Leftarrow$, if $\alpha < \omega$, then $1+\alpha = \alpha+1 > \alpha$ by Proposition 9.35. $\square$

Proposition 9.38. *For any ordinals α, β let*

$$\alpha \oplus \beta = (\alpha \times \{0\}) \cup (\beta \times \{1\}).$$

We define a relation $\prec$ as follows. For any $x, y \in \alpha \oplus \beta$, $x \prec y$ iff one of the following three conditions holds:

(i) There are $\xi, \eta < \alpha$ such that $x = (\xi, 0)$, $y = (\eta, 0)$, and $\xi < \eta$.

(ii) There are $\xi, \eta < \beta$ such that $x = (\xi, 1)$, $y = (\eta, 1)$, and $\xi < \eta$.

(ii) There are $\xi < \alpha$ and $\eta < \beta$ such that $x = (\xi, 0)$ and $y = (\eta, 1)$.

Then $(\alpha \oplus \beta, \prec)$ is a well order which is isomorphic to $\alpha + \beta$.

Proof. Clearly $\prec$ is a well-order. We show by transfinite induction on β, with α fixed, that $(\alpha \oplus \beta, \prec)$ is order isomorphic to $\alpha + \beta$. For $\beta = 0$ we have $\alpha + \beta = \alpha + 0 = \alpha$, while $\alpha \oplus \beta = \alpha \oplus 0 = \alpha \times \{0\}$. Clearly $\xi \mapsto (\xi, 0)$ defines an order-isomorphism from α onto $(\alpha \times \{0\}, \prec)$. So our result holds for $\beta = 0$. Assume it for β, and suppose that f is an order-isomorphism from $\alpha + \beta$ onto $(\alpha \oplus \beta, \prec)$. Now the last element of $\alpha \oplus (\beta + 1)$ is $(\beta, 1)$, and the last element of $\alpha + (\beta + 1)$ is $\alpha + \beta$, so the function

$$f \cup \{(\alpha + \beta, (\beta, 1))\}$$

is an order-isomorphism from $\alpha + (\beta + 1)$ onto $\alpha \oplus (\beta + 1)$.

Now assume that β is a limit ordinal, and for each $\gamma < \beta$, the ordinal $\alpha + \gamma$ is isomorphic to $\alpha \oplus \gamma$. For each such γ let f_γ be the unique isomorphism from $\alpha + \gamma$ onto $\alpha \oplus \gamma$. Note that if $\gamma < \delta < \beta$, then $f_\delta \restriction \gamma$ is an isomorphism from $\alpha + \gamma$ onto $\alpha \oplus \gamma$; hence $f_\delta \restriction \gamma = f_\gamma$. It follows that

$$\bigcup_{\gamma < \beta} f_\gamma$$

is an isomorphism from $\alpha + \beta$ onto $\alpha \oplus \beta$, finishing the inductive proof. $\square$

Proposition 9.39. *Given ordinals α, β, we define the following relation $\prec$ on $\alpha \times \beta$:*

$$\begin{aligned}(\xi, \eta) \prec (\xi', \eta') \quad \textit{iff} \quad & ((\xi, \eta) \textit{ and } (\xi', \eta') \textit{ are in } \alpha \times \beta \textit{ and:} \\ & \eta < \eta', \textit{ or } (\eta = \eta' \textit{ and } \xi < \xi').\end{aligned}$$

We may say that this is the anti-dictionary or anti-lexicographic order.

Then the set $\alpha \times \beta$ under the anti-lexicographic order is a well order which is isomorphic to $\alpha \cdot \beta$.

Proof. We may assume that $\alpha \neq 0$. It is straightforward to check that $\prec$ is a well-order.

Now we define, for any $(\xi, \eta) \in \alpha \times \beta$,

$$f(\xi, \eta) = \alpha \cdot \eta + \xi.$$

We claim that f is the desired order-isomorphism from $\alpha \times \beta$ onto $\alpha \cdot \beta$. If $(\xi, \eta) \in \alpha \times \beta$, then

$$f(\xi, \eta) = \alpha \cdot \eta + \xi < \alpha \cdot \eta + \alpha = \alpha \cdot (\eta + 1) \leq \alpha \cdot \beta.$$

Thus f maps into $\alpha \cdot \beta$.

To show that f is one-one, suppose that $(\xi, \eta), (\xi', \eta') \in \alpha \times \beta$ and $f(\xi, \eta) = f(\xi', \eta')$. Then by Theorem 9.26, $(\xi, \eta) = (\xi', \eta')$. So f is one-one.

To show that f maps onto $\alpha \cdot \beta$, let $\gamma < \alpha \cdot \beta$. Choose ξ and η so that $\gamma = \alpha \cdot \eta + \xi$ with $\xi < \alpha$. Now $\eta < \beta$, as otherwise

$$\gamma = \alpha \cdot \eta + \xi \geq \alpha \cdot \eta \geq \alpha \cdot \beta.$$

It follows that $f(\xi, \eta) = \alpha \cdot \eta + \xi = \gamma$. so f is onto.

Finally, we show that the order is preserved. Suppose that $(\xi, \eta) \prec (\xi', \eta')$. Then one of these cases holds:

Case 1. $\eta < \eta'$. Then

$$f(\xi, \eta) = \alpha \cdot \eta + \xi < \alpha \cdot \eta + \alpha = \alpha \cdot (\eta + 1) \leq \alpha \cdot \eta' \leq \alpha \cdot \eta' + \xi' = f(\xi', \eta'),$$

as desired.

Case 2. $\eta = \eta'$ and $\xi < \xi'$. Then $f(\xi, \eta) < f(\xi', \eta')$.

Now it follows that f is the desired isomorphism. □

Proposition 9.40. *Suppose that α and β are ordinals, with $\beta \neq 0$. We define*

$${}^{\alpha}\beta^{\mathrm{w}} = \{f \in {}^{\alpha}\beta : \{\xi < \alpha : f(\xi) \neq 0\} \textit{ is finite}\}.$$

For $f, g \in {}^{\alpha}\beta^{\mathrm{w}}$ we write $f \prec g$ iff $f \neq g$ and $f(\xi) < g(\xi)$ for the **greatest** *$\xi < \alpha$ for which $f(\xi) \neq g(\xi)$.*

Then $({}^{\alpha}\beta^{\mathrm{w}}, \prec)$ is a well-order which is order-isomorphic to the ordinal exponent β^{α}. (A set X is finite iff there is a bijection from some natural number onto X.)

Proof. If $\alpha = 0$, then $\beta^{\alpha} = 1$, and ${}^{\alpha}\beta^{\mathrm{w}}$ also has only one element, the empty function (= the emptyset). So, assume that $\alpha \neq 0$. If $\beta = 1$, then ${}^{\alpha}\beta^{\mathrm{w}}$ has only one member, namely the function with domain α whose value is always 0. This is clearly order-isomorphic to 1, as desired. So, suppose that $\beta > 1$.

Now we define a function f mapping β^{α} into ${}^{\alpha}\beta^{\mathrm{w}}$. Let $f(0)$ be the member of ${}^{\alpha}\beta^{\mathrm{w}}$ which takes only the value 0. Now suppose that $0 < \varepsilon < \beta^{\alpha}$. By Theorem 9.29 write

$$\varepsilon = \beta^{\gamma(0)} \cdot \delta(0) + \beta^{\gamma(1)} \cdot \delta(1) + \cdots + \beta^{\gamma(m-1)} \cdot \delta(m-1),$$

where $\varepsilon \geq \gamma(0) > \gamma(1) > \cdots > \gamma(m-1)$ and $0 < \delta(i) < \beta$ for each $i < m$. Note that $\beta^{\gamma(0)} \leq \varepsilon < \beta^\alpha$, so $\gamma(0) < \alpha$. Then we define, for any $\zeta < \alpha$,

$$(f(\varepsilon))(\zeta) = \begin{cases} 0 & \text{if } \zeta \notin \{\gamma(0), \ldots, \gamma(m-1)\}, \\ \delta(i) & \text{if } \zeta = \gamma(i) \text{ with } i < m. \end{cases}$$

Clearly $f(\varepsilon) \in {}^\alpha\beta^{\mathrm{w}}$. To see that f maps onto ${}^\alpha\beta^{\mathrm{w}}$, suppose that $x \in {}^\alpha\beta^{\mathrm{w}}$. If x takes only the value 0, then $f(0) = x$. Suppose that x takes on some nonzero value. Let

$$\{\xi < \alpha : x(\xi) \neq 0\} = \{\gamma(0), \gamma(1), \ldots, \gamma(m-1)\},$$

where $\gamma(0) > \gamma(1) > \cdots > \gamma(m-1)$. Let $\delta(i) = x(\gamma(i))$ for each $i < m$, and let

$$\varepsilon = \beta^{\gamma(0)} \cdot \delta(0) + \beta^{\gamma(1)} \cdot \delta(1) + \cdots + \beta^{\gamma(m-1)} \cdot \delta(m-1).$$

Clearly then $f(\varepsilon) = x$.

Now we complete the proof by showing that for any $\varepsilon, \theta < \beta^\alpha$, $\varepsilon < \theta$ iff $f(\varepsilon) < f(\theta)$. This equivalence is clear if one of ε, θ is 0, so suppose that both are nonzero. Write

$$\varepsilon = \beta^{\gamma(0)} \cdot \delta(0) + \beta^{\gamma(1)} \cdot \delta(1) + \cdots + \beta^{\gamma(m-1)} \cdot \delta(m-1),$$

where $\alpha \geq \gamma(0) > \gamma(1) > \cdots > \gamma(m-1)$ and $0 < \delta(i) < \beta$ for each $i < m$, and

$$\theta = \beta^{\gamma'(0)} \cdot \delta'(0) + \beta^{\gamma'(1)} \cdot \delta'(1) + \cdots + \beta^{\gamma'(n-1)} \cdot \delta'(n-1),$$

where $\alpha \geq \gamma'(0) > \gamma'(1) > \cdots > \gamma'(n-1)$ and $0 < \delta'(i) < \beta$ for each $i < n$.

By symmetry we may suppose that $m \leq n$. Note that $N(\beta, m, \gamma, \delta)$, $k(\beta, m, \gamma, \delta) = \varepsilon$, $N(\beta, n, \gamma', \delta')$, and $k(\beta, n, \gamma', \delta') = \theta$. We now consider several possibilities.

Case 1. $\varepsilon = \theta$. Then clearly $f(\varepsilon) = f(\theta)$.

Case 2. $\gamma \subseteq \gamma'$, $\delta \subseteq \delta'$, and $m < n$. Thus $\varepsilon < \theta$. Also, $\gamma'(m)$ is the largest $\xi < \alpha$ such that $(f(\varepsilon))(\xi) \neq (f(\theta))(\xi)$, and $(f(\varepsilon))(\xi) = 0 < \delta'(m) = (f(\theta))(\gamma'(m))$, so $f(\varepsilon) < f(\theta)$.

Case 3. There is an $i < m$ such that $\gamma(j) = \gamma'(j)$ and $\delta(j) = \delta'(j)$ for all $j < i$, while $\gamma(i) \neq \gamma'(i)$. By symmetry, say that $\gamma(i) < \gamma'(i)$. Then we have $\varepsilon < \theta$. Since $\gamma'(i)$ is the largest $\xi < \alpha$ such that $(f(\varepsilon))(\xi) \neq (f(\theta))(\xi)$, and $(f(\varepsilon))(\gamma'(i)) = 0 < \delta'(i) = (f(\theta))(\gamma'(i))$, we also have $f(\varepsilon) < f(\theta)$.

Case 4. There is an $i < m$ such that $\gamma(j) = \gamma'(j)$ and $\delta(j) = \delta'(j)$ for all $j < i$, while $\gamma(i) = \gamma'(i)$ and $\delta(i) \neq \delta'(i)$. By symmetry, say that $\delta(i) < \delta'(i)$. Then we have $\varepsilon < \theta$. Since $\gamma(i)$ is the largest $\xi < \alpha$ such that $(f(\varepsilon))(\xi) \neq (f(\theta))(\xi)$, and $(f(\varepsilon))(\gamma'(i)) = \delta(i) < \delta'(i) = (f(\theta))(\gamma'(i))$, we also have $f(\varepsilon) < f(\theta)$. $\square$

We have observed that ordinal addition is not commutative; see before Theorem 9.22. We now introduce an addition of ordinals which is commutative; this is the *Hessenberg* natural sum.

Given two nonzero ordinal numbers α and β, we write them in Cantor normal form:

$$\begin{aligned} &\alpha = \omega^{\gamma(0)} \cdot \delta(0) + \omega^{\gamma(1)} \cdot \delta(1) + \cdots + \omega^{\gamma(m-1)} \cdot \delta(m-1) \quad \text{with} \\ &\quad \alpha \geq \gamma(0) > \gamma(1) > \cdots > \gamma(m-1) \quad \text{and} \quad 0 < \delta(i) < \omega \quad \text{for each } i < m; \\ &\quad \beta = \omega^{\gamma'(0)} \cdot \delta'(0) + \omega'^{\gamma(1)} \cdot \delta'(1) + \cdots + \omega'^{\gamma(n-1)} \cdot \delta'(n-1) \text{with} \\ &\quad \alpha \geq \gamma'(0) > \gamma'(1) > \cdots > \gamma'(n-1) \quad \text{and} \quad 0 < \delta'(i) < \omega \quad \text{for each } i < n. \end{aligned}$$

Now enumerate $\{\gamma(i) : i < m\} \cup \{\gamma'(i) : i < n\}$ as $\{\gamma''(i) : i < p\}$ with $\gamma''(0) > \gamma''(1) > \cdots > \gamma''(p)$. For $i < p$ define

$$\delta''(i) = \begin{cases} \delta(j) & \text{if } \delta(j) = \delta''(i) \text{ for some } j, \\ 0 & \text{otherwise;} \end{cases}$$
$$\delta'''(i) = \begin{cases} \delta'(j) & \text{if } \delta'(j) = \delta''(i) \text{ for some } j, \\ 0 & \text{otherwise;.} \end{cases}$$

Then we have

$$\begin{aligned} &\alpha = \omega^{\gamma'(0)} \cdot \delta''(0) + \omega^{\gamma'(1)} \cdot \delta''(1) + \cdots + \omega^{\gamma'(p-1)} \cdot \delta''(p-1) \quad \text{with} \\ &\ \alpha \geq \gamma'(0) > \gamma'(1) > \cdots > \gamma'(p-1) \quad \text{and} \quad 0 \leq \delta(i) < \omega \quad \text{ for each } i < p; \\ &\ \beta = \omega^{\gamma'(0)} \cdot \delta'''(0) + \omega'^{\gamma'(1)} \cdot \delta'''(1) + \cdots + \omega'^{\gamma(n-1)} \cdot \delta'''(p-1) \text{with} \\ &\ \alpha \geq \gamma'(0) > \gamma'(1) > \cdots > \gamma'(p-1) \quad \text{and} \quad 0 \leq \delta'''(i) < \omega \quad \text{ for each } i < p; \end{aligned}$$

moreover, for each $k < p$ $\delta''_k \neq 0$ or $\delta'''(k) \neq 0$. Now for any ordinal α we define $\alpha\#0 = 0\#\alpha = \alpha$; and for $\alpha, \beta \neq 0$ we define

$$\alpha\#\beta = \sum_{k<p} \omega^{\gamma'(k)} \cdot (\delta''(k) + \delta'''(k)).$$

Proposition 9.41. *(i)* $\alpha\#0 = \alpha = 0\#\alpha$.
(ii) $\alpha\#\beta = \beta\#\alpha$.
(iii) $\alpha\#(\beta\#\gamma) = (\alpha\#\beta)\#\gamma$.
(iv) $\alpha \leq \alpha'$ *and* $\beta \leq \beta'$ *imply that* $\alpha\#\beta \leq \alpha'\#\beta'$.
(v) $\alpha \leq \alpha'$ *and* $\beta < \beta'$ *imply that* $\alpha\#\beta < \alpha'\#\beta'$.
(vi) $\alpha < \alpha'$ *and* $\beta \leq \beta'$ *imply that* $\alpha\#\beta < \alpha'\#\beta'$.
(vii) $\alpha + \beta \leq \alpha\#\beta$. □

Note that # is not continuous in either argument. For example, $\omega\#(\omega+1) = (\omega+1)\#\omega = \omega \cdot 2 + 1$.

Proposition 9.42. *Let* $(A, <)$ *be a well order. Suppose that* $B \subset A$ *and* $\forall b \in B \forall a \in A[a < b \rightarrow a \in B]$. *Then there is an element* $a \in A$ *such that* $B = \{b \in A : b < a\}$.

Proof. Let a be the least element of $A \backslash B$. We claim that a is as desired. For, if $b \in B$, then it cannot happen that $a \leq b$, since this would imply that $a \in B$; so $b < a$. And if $b < a$, then $b \in B$ by the minimality of a. □

Proposition 9.43. *Let* $(A, <)$ *be a well order. Suppose that* $B \subset A$ *and* $\forall b \in B \forall a \in A[a < b \rightarrow a \in B]$. *Then* $(A, <)$ *is not isomorphic to* $(B, <)$.

Proof. Suppose that f is such an isomorphism from $(A, <)$ onto $(B, <)$. By Proposition 9.35, let $a \in A$ be such that $B = \{x \in A : x < a\}$. By Proposition 9.11, $a \leq f(a)$, contradicting the assumption that f maps into B. □

Proposition 9.44. *Suppose that f is a one-one function mapping an ordinal α onto a set A. Then there is a relation $\prec$ which is a subset of $A \times A$ such that $(A,<)$ is a well-order and f is an isomorphism of $(\alpha,<)$ onto $(A,\prec)$.*

Proof. Define $\prec = \{(a,b) \in A \times A : f^{-1}(a) < f^{-1}(b)\}$. We check that $(A,<)$ is a well-order. If $a \in A$ and $a \prec a$, then $f^{-1}(a) < f^{-a}(a)$, contradiction. So $\prec$ is irreflexive. Suppose that $a \prec b \prec c$. Then $f^{-1}(a) < f^{-1}(b) < f^{-1}(c)$, so $f^{-1}(a) < f^{-1}(c)$ and hence $a \prec c$. So $\prec$ is transitive. Now given $a, b \in A$, either $f^{-1}(a) < f^{-1}(b)$ or $f^{-1}(a) = f^{-1}(b)$ or $f^{-1}(b) < f^{-1}(a)$, so $a \prec b$ or $a = b$ or $b \prec a$. Thus $(A,\prec)$ is a linear order. Finally, suppose that $\emptyset \neq X \subseteq A$. Then $\emptyset \neq f^{-1}[X]$, so let ξ be the least element of $f^{-1}[X]$. Then $f(\xi) \in X$. Suppose that $b \in X$. Then $f^{-1}(b) \in f^{-1}[X]$, so $\xi \le f^{-1}(b)$. Hence $f(\xi) \preceq b$. This shows that $f(\xi)$ is the $\prec$-least element of X. We have shown that $(A,\prec)$ is a well-order.

We are given that f is a bijection from α onto A. If $\xi, \eta \in \alpha$ and $\xi < \eta$, then $f(\xi) \prec f(\eta)$. If $f(\xi) \prec f(\eta)$, then $\xi < \eta$. Thus f is an isomorphism. □

. . . .

Proposition 9.45. *For every nonzero ordinal α there are only finitely many ordinals β such that $\alpha = \gamma \cdot \beta$ for some γ.*

Proof. Suppose there are infinitely many such β; let $\langle \beta_i : i \in \omega \rangle$ be a one-one enumeration of infinitely many of them. For each $i \in \omega$ let γ_i be such that $\alpha = \gamma_i \cdot \beta_i$. Clearly $\beta_i < \beta_j$ iff $\gamma_j < \gamma_i$. We define $\langle i_j : j \in \omega \rangle$ by recursion. Let i_0 be such that β_{i_0} is the smallest element of $\{\beta_k : k \in \omega\}$. Having defined $i_0, \ldots, i_s$, let i_{s+1} be such that $\beta_{i_{s+1}}$ is the smallest element of

$$\{\beta_k : k \in \omega\} \backslash \{\beta_{i_k} : k \le s\}$$

Clearly $\beta_{i_0} < \beta_{i_1} < \cdots$, and hence $\gamma_{i_0} > \gamma_{i_1} > \cdots$, contradiction.

Proposition 9.46. $n^{(\omega^\omega)} = \omega^{(\omega^\omega)}$ *for every natural number $n > 1$.*

Proof. Note that $n^\omega = \omega$ by an easy argument. Hence

$$\begin{aligned}\omega^{(\omega^\omega)} &= (n^\omega)^{(\omega^\omega)}\\ &= n^{(\omega\cdot(\omega^\omega))}\\ &= n^{(\omega^\omega)}. \quad \text{by Theorem 9.32}\end{aligned}$$

□

Proposition 9.47. *The following conditions are equivalent for any ordinals α, β:*

(i) $\alpha + \beta = \beta + \alpha$.

(ii) There exist an ordinal γ and natural numbers k, l such that $\alpha = \gamma \cdot k$ and $\beta = \gamma \cdot l$.

Proof. $\Rightarrow$: Assume that $\alpha + \beta = \beta + \alpha$. The desired conclusion is clear if $\alpha = 0$ or $\beta = 0$, so assume that $\alpha, \beta \neq 0$. Write $\alpha = \omega^\delta \cdot k + \varepsilon$ with $\delta \le \alpha$, $0 < k \in \omega$, and $\varepsilon < \omega^\delta$, and write $\beta = \omega^\rho \cdot l + \sigma$ with $\rho \le \beta$, $0 < l \in \omega$, and $\sigma < \omega^\rho$. If $\delta < \rho$, then

$$\alpha + \beta = \beta < \beta + \alpha,$$

contradiction. A similar contradiction is reached if $\rho < \delta$. So $\delta = \rho$. Now

$$\alpha + \beta = \omega^\delta \cdot (k + l) + \sigma = \beta + \alpha = \omega^\delta \cdot (k + l) + \varepsilon,$$

so $\sigma = \varepsilon$. Hence $\alpha = (\omega^\delta + \varepsilon) \cdot k$ and $\beta = (\omega^\delta + \varepsilon) \cdot l$, as desired.

$\Leftarrow$: Obvious. □

Proposition 9.48. *Suppose that $\alpha < \omega^\gamma$. Then $\alpha + \beta + \omega^\gamma = \beta + \omega^\gamma$.*

Proof. Suppose that α, β, γ are ordinals and $\alpha < \omega^\gamma$. If also $\beta < \omega^\gamma$, then $\alpha + \beta < \omega^\gamma$ by Theorem 9.31, and also by Theorem 9.31 $\alpha + \beta + \omega^\gamma = \omega^\gamma$ and $\beta + \omega^\gamma = \omega^\gamma$.

Now suppose that $\omega^\gamma \leq \beta$. Write $\beta = \omega^\gamma \cdot \delta + \varepsilon$ with $\delta > 0$ and $\varepsilon < \omega^\gamma$.

(1) $\alpha + \omega^\gamma \cdot \varphi = \omega^\gamma \cdot \varphi$ for every positive φ.

We prove (1) by induction on φ. It is true for $\varphi = 1$ by Theorem 9.31. Assume that it holds for φ. Then

$$\alpha + \omega^\gamma \cdot (\varphi + 1) = \alpha + \omega^\gamma \cdot \varphi + \omega^\gamma = \omega^\gamma \cdot \varphi + \omega^\gamma = \omega^\gamma \cdot (\varphi + 1),$$

as desired. Finally, assume that φ is limit and (1) holds for all $\psi < \varphi$. Let $F(\varphi) = \alpha + \varphi$ for all φ, and $G(\varphi) = \omega^\gamma \cdot \varphi$. Both of these are normal functions. Hence

$$\alpha + \omega^\gamma \cdot \varphi = F(G(\varphi)) = \bigcup_{\psi < \varphi} F(G(\psi)) = \bigcup_{\psi < \varphi} (\alpha + \omega^\gamma \cdot \psi) = \bigcup_{\psi < \varphi} (\omega^\gamma \cdot \psi) = \omega^\gamma \cdot \varphi,$$

finishing the inductive proof of (1).

Now by (1) we have

$$\alpha + \beta + \omega^\gamma = \alpha + \omega^\gamma \cdot \delta + \varepsilon + \omega^\gamma = \omega^\gamma \cdot \delta + \varepsilon + \omega^\gamma = \beta + \omega^\gamma.$$ □

Proposition 9.49. *The following conditions are equivalent:*

(i) α is a limit ordinal

(ii) $\alpha = \omega \cdot \beta$ for some $\beta \neq 0$.

(iii) For every $m \in \omega \backslash 1$ we have $m \cdot \alpha = \alpha$, and $\alpha \neq 0$.

Proof. (i)$\Rightarrow$(ii): Assume (i). By Theorem 9.26 write $\alpha = \omega \cdot \beta + n$ with $n < \omega$. If $\beta = 0$, then $\alpha = n$, contradiction. If $n \neq 0$, then $\alpha = \omega \cdot \beta + (n - 1) + 1$, contradiction.

(ii)$\Rightarrow$(iii): Assume (ii). By Theorem 9.23(iii), $\alpha \neq 0$. Suppose that $m \in \omega \backslash 1$. Then $m \cdot \omega = \bigcup_{n \in \omega} (m \cdot n) = \omega$ by Theorem 9.23(iii), so $m \cdot \alpha = \alpha$.

(iii)$\Rightarrow$(i): Assume (iii), but suppose that $\alpha = \beta + 1$. Then $\alpha = 2 \cdot \alpha = 2 \cdot (\beta + 1) = 2 \cdot \beta + 2 > \alpha$, contradiction.qed

Proposition 9.50. $(\alpha + \beta) \cdot \gamma \leq \alpha \cdot \gamma + \beta \cdot \gamma$ *for any ordinals α, β, γ.*

Proof. Assume that $\alpha, \beta, \gamma \neq 0$. Write $\alpha = \omega^\delta \cdot k + \varepsilon$ with $\delta \leq \alpha$, $0 \neq k \in \omega$, $\varepsilon < \omega^\delta$, and $\beta = \omega^\rho \cdot l + \sigma$ with $\rho \leq \beta$, $0 \neq l \in \omega$, $\sigma < \omega^\rho$. Also, write $\gamma = \omega \cdot \xi + m$ with $m \in \omega$. Now we consider some cases.

Case 1. $\delta < \rho$. Then $\alpha + \beta = \beta$, and the desired conclusion follows.

Case 2. $\delta = \rho$. Note that if $m > 0$, then

$$\begin{aligned}\alpha \cdot m &= \omega^\delta \cdot k \cdot m + \varepsilon;\\ \beta \cdot m &= \omega^\delta \cdot l \cdot m + \sigma;\\ (\alpha + \beta) \cdot m &= \omega^\delta \cdot (k + l) \cdot m + \sigma.\end{aligned}$$

If $\xi = 0$ it is then clear that $(\alpha + \beta) \cdot \gamma = \alpha \cdot \gamma + \beta \cdot \gamma$. Hence assume that $\xi > 0$. Then

$$\begin{aligned}\alpha \cdot \gamma &= \alpha \cdot \omega \cdot \xi + \alpha \cdot m\\ &= \omega^{\delta+1} \cdot \xi + \alpha \cdot m;\\ \beta \cdot \gamma &= \omega^{\delta+1} \cdot \xi + \beta \cdot m;\\ \alpha \cdot \gamma + \beta \cdot \gamma &= \omega^{\delta+1} \cdot \xi \cdot 2 + \beta \cdot m;\\ (\alpha + \beta) \cdot \gamma &= \omega^{\delta+1} \cdot \xi + (\alpha + \beta) \cdot m.\end{aligned}$$

Now clearly $(\alpha + \beta) \cdot m = \omega^\delta \cdot (k + l) \cdot m + \sigma < \omega^{\delta+1} \cdot \sigma + \beta \cdot m$, so the desired conclusion follows.

Case 3. $\rho < \delta$. Then if $m > 0$ we have

$$\begin{aligned}\alpha \cdot m &= \omega^\delta \cdot k \cdot m + \varepsilon;\\ \beta \cdot m &= \omega^\rho \cdot l \cdot m + \sigma;\\ \alpha \cdot m + \beta \cdot m &= \omega^\delta \cdot k \cdot m + \varepsilon + \omega^\rho \cdot l \cdot m + \sigma;\\ (\alpha + \beta) \cdot m &= \omega^\delta \cdot k \cdot m + \varepsilon + \omega^\rho \cdot l + \sigma.\end{aligned}$$

Hence the desired conclusion follows if $\xi = 0$. Assume now that $\xi \neq 0$. Then

$$\begin{aligned}\alpha \cdot \gamma &= \omega^{\delta+1} \cdot \xi + \alpha \cdot m;\\ \beta \cdot \gamma &= \omega^{\rho+1} \cdot \xi + \beta \cdot m;\\ \alpha \cdot \gamma + \beta \cdot \gamma &= \omega^{\delta+1} \cdot \xi + \alpha \cdot m + \omega^{\rho+1} \cdot \xi + \beta \cdot m;\\ (\alpha + \beta) \cdot \gamma &= \omega^{\delta+1} \cdot \xi + \omega^\delta \cdot k \cdot m + \varepsilon + \omega^\rho \cdot l + \sigma\\ &= \omega^{\delta+1} \cdot \xi + \alpha \cdot m + \omega^\rho \cdot l + \sigma.\end{aligned}$$

Again the desired conclusion holds. $\square$

10. The axiom of choice

We give a small number of equivalent forms of the axiom of choice; these forms should be sufficient for most mathematical purposes. The axiom of choice has been investigated a lot, and we give some references for this after proving the main theorem of this chapter.

The set of axioms of ZFC with the axiom of choice removed is denoted by ZF; so we work in ZF in this chapter.

The two main equivalents to the axiom of choice are as follows.

Zorn's Lemma. *If $(A, <)$ is a partial order such that $A \neq \emptyset$ and every subset of A simply ordered by $<$ has an upper bound, then A has a maximal element under $<$, i.e., an element a such that there is no element $b \in A$ such that $a < b$.*

Well-ordering principle. *For every set A there is a well-ordering of A, i.e., there is a relation $<$ such that $(A, <)$ is a well-order.*

In addition, the following principle, usually called the axiom of choice, is equivalent to the actual form that we have chosen:

Choice-function principle. *If A is a family of nonempty sets, then there is a function f with domain A such that $f(a) \in a$ for every $a \in A$. Such a function f is called a choice function for A.*

Lemma 10.1. *Suppose that $(A, <)$ is a partial order and $a \in A$. Then $A \not< a$.*

Proof. Suppose to the contrary that $A < a$. Then $(A, a) \in <\subseteq A \times A$, so $A \in A$, contradicting Theorem 7.5. □

Theorem 10.2. *In ZF the following four statements are equivalent:*
(i) the axiom of choice;
(ii) the choice-function principle;
(iii) Zorn's lemma.
(iv) the well-ordering principle.

Proof. Axiom of choice $\Rightarrow$ choice-function principle. Assume the axiom of choice, and let A be a family of nonempty sets. Let

$$\mathscr{A} = \{X : \exists a \in A[X = \{(a, x) : x \in a\}]\}.$$

Since each member of A is nonempty, also each member of $\mathscr{A}$ is nonempty. Given $X, Y \in \mathscr{A}$ with $X \neq Y$, choose $a, b \in A$ such that $X = \{(a, x) : x \in a\}$ and $Y = \{(b, x) : x \in b\}$. Thus $a \neq b$ since $A \neq B$. The basic property of ordered pairs then implies that $A \cap B = \emptyset$.

So, by the axiom of choice, let $\mathscr{B}$ have exactly one element in common with each element of $\mathscr{A}$. Define $f = \{b \in \mathscr{B} : \text{there exist } a \in A \text{ and } x \text{ such that } b = (a, x)\}$. Clearly f is the desired choice function for A.

Choice-function principle $\Rightarrow$ Zorn's lemma. Assume the choice-function principle, and also assume the hypotheses of Zorn's lemma. Let f be a choice function for

$\mathscr{P}(A)\backslash\{\emptyset\}$. Define $\mathbf{G} : \mathbf{On} \times \mathbf{V} \to \mathbf{V}$ by setting, for any ordinal α and any set x,

$$\mathbf{G}(\alpha, x) = \begin{cases} f(\{a \in A : x(\beta) < a \text{ for all } \beta < \alpha\}) & \text{if } x \text{ is a function with domain } \alpha \\ & \text{and this set is nonempty,} \\ A & \text{otherwise.} \end{cases}$$

Let $\mathbf{F}$ be obtained from $\mathbf{G}$ by the recursion theorem 6.7; thus for any ordinal α,

$$\mathbf{F}(\alpha) = \mathbf{G}(\alpha, \mathbf{F} \restriction \alpha) = \begin{cases} f(\{a \in A : \mathbf{F}(\beta) < a \text{ for all } \beta < \alpha\}) & \text{if this set is nonempty,} \\ A & \text{otherwise.} \end{cases}$$

(1) If $\alpha < \beta \in \mathbf{On}$ and $\mathbf{F}(\beta) \neq A$, then $\mathbf{F}(\alpha) \neq A$, and $\mathbf{F}(\alpha) < \mathbf{F}(\beta)$.

In fact, suppose that $\mathbf{F}(\alpha) = A$. Now by the definition of $\mathbf{F}(\beta)$, the set $\{a \in A : \mathbf{F}(\gamma) < a$ for all $\gamma < \beta\}$ is nonempty. Let a be a member of this set. Now $\alpha < \beta$, so $A = \mathbf{F}(\alpha) < a$, contradicting Lemma 10.1.

Since $\mathbf{F}(\beta) = f(\{a \in A : \mathbf{F}(\gamma) < a \text{ for all } \gamma < \beta\})$ and $\alpha < \beta$, it follows that $\mathbf{F}(\alpha) < \mathbf{F}(\beta)$.

(2) There is an ordinal α such that $\mathbf{F}(\alpha) = A$.

Otherwise, by (1), $\mathbf{F}$ is a one-one function from $\mathbf{On}$ into A. So by the comprehension axioms, $\text{rng}(\mathbf{F})$ is a set, and hence by the replacement axioms, $\mathbf{On} = \mathbf{F}^{-1}[\text{rng}\mathbf{F}]$ is a set, contradicting Theorem 7.6.

Let α be minimum such that $\mathbf{F}(\alpha) = A$. Now $\mathbf{F}[\alpha]$ is linearly ordered by (1), so by the hypothesis of Zorn's lemma, there is an $a \in A$ such that $\mathbf{F}(\beta) \leq a$ for all $\beta < \alpha$. Now the set $\{a \in A : f(\beta) < f(0) \text{ for all } \beta < 0\}$ is trivially nonempty, since A is nonempty, so $\mathbf{F}(0) \neq A$. Hence $\alpha > 0$. If α is a limit ordinal, then for any $\beta < \alpha$ we have $\mathbf{F}(\beta) < \mathbf{F}(\beta + 1) \leq a$, and hence $\mathbf{F}(\alpha) \neq A$, contradiction. Hence α is a successor ordinal $\beta + 1$, and so $\mathbf{F}(\beta)$ is a maximal element of A.

Zorn's lemma $\Rightarrow$ well-ordering principle. Assume Zorn's lemma, and let A be any set. We may assume that A is nonempty. Let

$$P = \{(B, <) : B \subseteq A \text{ and } (B <) \text{ is a well-ordering structure}\}.$$

We partially order P as follows: $(B, <) \prec (C, \ll)$ iff $B \subseteq C$, $\forall a, b \in B[a < b \text{ iff } a \ll b]$, and $\forall b \in B \forall c \in C\backslash B[b \ll c]$. Clearly this does partially order P. $P \neq \emptyset$, since $(\{a\}, \emptyset) \in P$ for any $a \in A$. Now suppose that Q is a nonempty subset of P simply ordered by $\prec$. Let

$$D = \bigcup_{(B,<) \in Q} B,$$

$$<_D = \bigcup_{(B,<) \in Q} < .$$

Clearly $(D, <_D)$ is a linear order with $D \subseteq A$. Suppose that X is a nonempty subset of D. Fix $z \in X$, and choose $(B, <) \in Q$ such that $z \in B$. Then $X \cap B$ is a nonempty subset of B; let x be its least element under $<$. Suppose that $y \in X$ and $y <_D x$. Choose $(C, \ll) \in Q$ such that $x, y \in C$ and $y \ll x$. Since Q is simply ordered by $\prec$, we have two cases.

Case 1. $(C, \ll) \preceq (B, <)$. Then $y \in C \subseteq B$ and $y \in X$. so $y < x$, contradicting the choice of x.

Case 2. $(B, <) \prec (C, \ll)$. If $y \in B$, then $y < x$, contradicting the choice of x. So $y \in C \backslash B$. But then $x \ll y$, contradiction.

Thus we have shown that x is the $<_D$-least element of X. So $(D, <_D)$ is the desired upper bound for Q.

Having verified the hypotheses of Zorn's lemma, we get a maximal element $(B, <)$ of P. Suppose that $B \neq A$. Choose any $a \in A \backslash B$, and let

$$C = B \cup \{a\},$$
$$<_C = < \cup \{(b, a) : b \in B\}.$$

Clearly this gives an element $(C, <_C)$ of P such that $(B, <) \prec (C, <_C)$, contradiction.

Well-ordering principle $\Rightarrow$ Axiom of choice. Assume the well-ordering principle, and let $\mathscr{A}$ be a family of pairwise disjoint nonempty sets. Let $C = \bigcup \mathscr{A}$, and let $\prec$ be a well-order of C. Define $B = \{c \in C : c$ is the $\prec$-least element of the $P \in A$ for which $c \in P\}$. Clearly B has exactly one element in common with each member of A. $\square$

There are many statements which are equivalent to the axiom of choice on the basis of ZF. We list some striking ones. A fairly complete list is in

Rubin, H.; Rubin, J. **Equivalents of the axiom of choice.** North-Holland (1963), 134pp.

(About 100 forms are listed, with proofs of equivalence.)

1. For every relation R there is a function $f \subseteq R$ such that $\text{dmn}(f) = \text{dmn}(R)$.

2. For any sets A, B, either there is an injection of A into B or one of B into A.

3. For any transitive relation R there is a maximal $S \subseteq R$ which is a linear ordering.

4. Every product of compact spaces is compact.

5. Every formula having a model of size ω also has a model of any infinite size.

6. If A can be well-ordered, then so can $\mathscr{P}(A)$.

There are also statements which follow from the axiom of choice but do not imply it on the basis of ZF. A fairly complete list of such statement is in

Howard, P.; Rubin, J. **Consequences of the axiom of choice.** Amer. Math. Soc. (1998), 432pp.

(383 forms are listed)

Again we list some striking ones:

1. Every Boolean algebra has a maximal ideal.

2. Any product of compact Hausdorff spaces is compact.

3. The compactness theorem of first-order logic.

4. Every commutative ring has a prime ideal.

5. Every set can be linearly ordered.

6. Every linear ordering has a cofinal well-ordered subset.

7. The Hahn-Banach theorem.

8. Every field has an algebraic closure.

9. Every family of unordered pairs has a choice function.

10. Every linearly ordered set can be well-ordered.

. . . .

Proposition 10.3. *Any vector space over a field has a basis (possibly infinite).*

Proof. Let V be any vector space over F. Let $A = \{X \subseteq V : X$ is linearly independent$\}$, partially ordered by $\subseteq$. Then $A \neq \emptyset$, since trivially $\emptyset \in A$. Now suppose that B is a subset of A simply ordered by $\subseteq$. We claim that $\bigcup B \in A$; this will verify the hypothesis of Zorn's lemma. Suppose that $v_1, \ldots, v_n \in \bigcup B$, $a_1, \ldots, a_n \in F$, and $a_1v_1 + \cdots + a_nv_n = 0$; we want to show that all a_i are 0. For each $i = 1, \ldots, n$ choose $X_i \in B$ such that $v_i \in X_i$. Now $\{X_i : i = 1, \ldots, n\}$ has a largest member X_j under $\subseteq$, since B is simply ordered. [Easy proof by induction on n.] Clearly $v_i \in X_j$ for all $i = 1, \ldots, n$. Since X_j is linearly independent, it follows that each $a_i = 0$, as desired.

Now we apply Zorn's lemma to obtain a maximal member Y of A under $\subseteq$. We claim that Y is a basis for V. Since Y is linearly independent, it suffices to show that Y spans V. Suppose that $w \in V$. If $w \in Y$, then obviously w is in the span of Y. Suppose that $w \notin Y$. Then $Y \subset Y \cup \{w\}$ so by the maximality of Y, $Y \cup \{w\}$ is linearly dependent. Hence there is a natural number n, distinct elements $v_1, \ldots, v_n \in Y \cup \{w\}$, and elements $a_1, \ldots, a_n \in F$, not all 0, such that $a_1v_1 + \cdots + a_nv_n = 0$. Since Y is linearly independent, not all v_i are in Y; say that $v_j = w$. Then again because Y is linearly independent, we must have $a_j \neq 0$. So

$$w = \left(-\frac{a_1}{a_j}v_1\right) + \cdots + \left(-\frac{a_{j-1}}{a_j}v_{j-1}\right) + \left(-\frac{a_{j+1}}{a_j}v_{j+1}\right) + \cdots + \left(-\frac{a_n}{a_j}v_n\right),$$

so that w is in the span of Y, as desired. □

Proposition 10.4. *A subset C of $\mathbb{R}$ is closed iff the following condition holds:*

For every sequence $f \in {}^{\omega}C$, if f converges to a real number x, then $x \in C$.

Here to say that f converges to x means that

$$\forall \varepsilon > 0 \exists M \forall m \geq M[|f_m - x| < \varepsilon].$$

If $\langle C_m : m \in \omega\rangle$ is a sequence of nonempty closed subsets of $\mathbb{R}$, $\forall m \in \omega \forall x, y \in C_m[|x-y| < 1/(m+1)]$, and $C_m \supseteq C_n$ for $m < n$, then $\bigcap_{m\in\omega} C_m$ is nonempty.

Proof. Let c be a choice function for $\mathscr{P}(\mathbb{R})\backslash\{\emptyset\}$. For each $m \in \omega$ let $f_m = c(C_m)$. We claim that f is a Cauchy sequence, and hence it converges to some point x. For, let $\varepsilon > 0$ be given. Choose $m \in \omega$ such that $\frac{1}{m+1} < \varepsilon$. Then for any $n, p \geq m$ we have $f_n, f_p \in C_m$ and hence by a hypothesis of the exercise, $|f_n - f_p| < \frac{1}{m+1} < \varepsilon$, as desired. Now for any $m \in \omega$ we have $f_n \in C_m$ for all $n \geq m$, and hence $x \in C_m$. Thus $x \in \bigcap_{m\in\omega} C_m$. □

Proposition 10.5. *Every nontrivial commutative ring with identity has a maximal ideal. Nontrivial means that* $0 \neq 1$.

Proof. Let R be a nontrivial commutative ring with identity. Let $\mathscr{A}$ be the collection of all proper ideals, partially ordered under $\subset$. Obviously $\mathscr{A} \neq \emptyset$. Suppose that $\mathscr{B}$ is a nonempty subset of $\mathscr{A}$ simply ordered by $\subset$. Let $I = \bigcup \mathscr{B}$. We claim that I is a proper ideal, so that it is an upper bound for $\mathscr{B}$. In fact, if $a, b \in I$, choose $J, K \in \mathscr{B}$ such that $a \in J$ and $b \in K$. Since $\mathscr{B}$ is simply ordered by $\subset$, by symmetry say $J \subset K$. Then $a, b \in K$, hence $a + b$ and $a - b$ are also in K, and hence they are in I too. Also, if $a \in I$ and $b \in R$, then $a \cdot b \in I$ by an even easier argument. Thus I is an ideal. Clearly $1 \notin I$, so I is proper.

By Zorn's lemma, $\mathscr{A}$ has a maximal element L. Clearly L is a maximal ideal. □

Proposition 10.6. *A function $g : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$ iff for every sequence $f \in {}^{\omega}\mathbb{R}$ which converges to a, the sequence $g \circ f$ converges to $g(a)$. g is continuous at a iff the following condition holds:*

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x \in \mathbb{R}[|x - a| < \delta \to |g(x) - g(a)| < \varepsilon].$$

$\to$: Suppose that g is continuous at a but the indicated condition fails. Thus

$$(*) \qquad \exists \varepsilon > 0 \forall \delta > 0 \exists x \in \mathbb{R}[|x - a| < \delta \text{ and } |g(x) - g(a)| \geq \varepsilon].$$

Let c be a choice function for $\mathbb{R}$. For each $m \in \omega$ let

$$f_m = c\left\{x \in \mathbb{R}\left[|x - a| < \frac{1}{m+1} \text{ and } |g(x) - g(a)| \geq \varepsilon\right]\right\}.$$

Then f converges to a. In fact, given $\xi > 0$, choose M such that $\frac{1}{M-1} < \xi$. Then for any $m \geq M$, $|f_m - a| < \frac{1}{m+1} \leq \frac{1}{M-1} < \xi$. Since f converges to a and g is continuous at a, it follows that $g \circ f$ converges to $g(a)$. Hence we can choose N such that $\forall n \geq N[|g(f_m) - g(a)| < \varepsilon]$. But by the definition of f, $|g(f_N) - g(a)| \geq \varepsilon$, contradiction.

$\leftarrow$: Assume the indicated condition, and suppose that $f \in {}^{\omega}R$ converges to a. In order to show that $g \circ f$ also converges to a, let $\varepsilon > 0$ be given. By the condition, choose $\delta > 0$ such that $\forall x \in \mathbb{R}[|x - a| < \delta \to |g(x) - g(a)| < \varepsilon]$. Since f converges to a, choose M such that $\forall m \geq M[|f_m - a| < \delta]$. Then for any $m \geq M$ we have $|g(f_m) - g(a)| < \varepsilon$, as desired. □

Proposition 10.7. (In ZF) *If $m \in \omega$ and $\langle A_i : i \in m\rangle$ is a system of nonempty sets, then there is a function f with domain m such that $f(i) \in A_i$ for all $i \in m$.*

Proof. For $m = 0$, the system itself is empty, and the desired function f is the empty set.

Now suppose that $\langle A_i : i \in m+1\rangle$ is a system of nonempty sets, and we know our result for a system of m nonempty sets. So, let f be a function with domain m such that $f(i) \in A_i$ for all $i \in m$. Pick $a \in A_m$, and let $g = f \cup \{(m, a)\}$. Clearly g is as desired, completing the inductive proof. □

Proposition 10.8. *If A is a nonempty set, R is a relation, $R \subseteq A \times A$, and for every $a \in A$ there is a $b \in A$ such that aRb, then there is a function $f : \omega \to A$ such that $f(i)Rf(i+1)$ for all $i \in \omega$.*

This is called the *Principle of Dependent Choice*; it is weaker than the axiom of choice, but cannot be proved in ZF).

Proof. Let c be a choice function for nonempty subsets of A. We define $f : \omega \to A$ by recursion, as follows. Fix $a \in A$. For any $m \in \omega$ let

$$f(m) = \begin{cases} a & \text{if } m = 0, \\ c(\{x \in A : f(n)Rx\}) & \text{if } m = n+1 \text{ and } \{x \in A : f(n)Rx\} \neq 0, \\ a & \text{otherwise.} \end{cases}$$

By induction, $f(i)Rf(i+1)$ for every $i \in \omega$, as desired. □

Proposition 10.9. *In ZF, for any set A there is an ordinal α such that there is no one-one function mapping α into A.*

Proof. Let X be the set of all well-orderings contained in $A \times A$. Now each $\prec \in X$ is isomorphic to an ordinal $\beta_\prec$. Let $\alpha = \bigcup_{\prec \in X}(\beta_\prec + 1)$. Suppose that f is a one-one function mapping α into A. Let $\prec = \{(f(\xi), f(\eta)) : \xi < \eta\}$. Then $\prec$ is a well-ordering contained in $A \times A$, and so $\beta_X = \alpha$; consequently $\alpha \in \alpha$, contradiction. □

Proposition 10.10. (ZF) *The following are equivalent:*

(i) The axiom of choice.

(ii) If $<$ is a partial ordering and $\prec$ is a simple ordering which is a subset of $<$, then there is a maximal (under $\subseteq$) simple ordering $\ll$ such that $\prec$ is a subset of $\ll$, which in turn is a subset of $<$.

(iii) For any two sets A and B, either there is a one-one function mapping A into B or there is a one-one function mapping B into A.

(iv) For any two nonempty sets A and B, either there is a function mapping A onto B or there is a function mapping B onto A.

(v) Every family of finite character has a maximal element under $\subseteq$. Here a family $\mathscr{F}$ of subsets of a set A has finite character if for all $X \subseteq A$, $X \in \mathscr{F}$ iff every finite subset of X is in $\mathscr{F}$.

(vi) For any relation R there is a function $f \subseteq R$ such that $\operatorname{dmn} R = \operatorname{dmn} f$.

Proof. (i)⇒(ii): Assume (i). Let

$$\mathscr{A} = \{\ll : \ll \text{ is a simple ordering and } \prec \subseteq \ll \subseteq <\}.$$

Note that $\mathscr{A}$ is nonempty, since $\prec \in \mathscr{A}$. We partially order $\mathscr{A}$ by inclusion. To check the hypothesis of Zorn's lemma, suppose that $\mathscr{B}$ is a nonempty subset of $\mathscr{A}$ simply ordered by inclusion. We claim that $\bigcup \mathscr{B} \in \mathscr{A}$; this is clear, by checking all the necessary conditions. For example, $\bigcup \mathscr{B}$ is transitive since if $(a, b), (b, c) \in \bigcup \mathscr{B}$, then there are $R, S \in \mathscr{B}$ with $(a, b) \in R$ and $(b, c) \in S$; by symmetry $R \subseteq S$, hence $(a, b), (b, c) \in S$, hence $(a, c) \in S$, hence $(a, c) \in \bigcup \mathscr{B}$.

So we apply Zorn's lemma to obtain a maximal member $\ll$ of $\mathscr{A}$; this is as desired.

(ii)⇒(iii): Assume (ii). Given sets A and B, define $f < g$ iff f and g are one-one functions which are subsets of $A \times B$, and $f \subset g$. Apply (ii) to $<$ and the empty simple ordering. We get a maximal simple ordering $\prec$ such that $\prec \subseteq <$. Let $f = \bigcup(\prec)$. Since $\prec$ is a simply ordered collection of one-one functions, it is clear that f is a one-one function. It suffices to show that $\mathrm{dmn}(f) = A$ or $\mathrm{rng}(f) = B$. Suppose that this is not true, and choose $a \in A \backslash \mathrm{dmn}(f)$ and $b \in B \backslash \mathrm{rng}(f)$. Let $g = f \cup \{(a, b)\}$. Clearly g is a one-one function contained in $A \times B$. Thus if we define $\prec'$ as an extension of $\prec$ with $g \prec' f$ for all g in the domain of $\prec$, we get a proper extension of $\prec$, contradiction.

(iii)⇒(iv): Assume (iii). Let A and B be nonempty sets. By (iii) and symmetry, say that f is a one-one function mapping A into B. Fix $a \in A$, and define g with domain B by setting, for each $b \in B$,

$$g(b) = \begin{cases} f^{-1}(b) & \text{if } b \in \mathrm{rng}(f), \\ a & \text{otherwise.} \end{cases}$$

Clearly g maps B onto A, as desired.

(iv)⇒(i): Assume (iv) We show that any set A can be well-ordered, as follows. Use Proposition 10.9 to find an ordinal α which cannot be mapped one-one into $\mathscr{P}(A)$.Suppose that $f : A \to \alpha$ maps onto α. Let $g = \langle f^{-1}[\{\beta\}] : \beta < \alpha\rangle$. Clearly g maps α into $\mathscr{P}(A)$. Suppose that $g(\beta) = g(\gamma)$. Thus $f^{-1}[\{\beta\}] = f^{-1}[\{\gamma\}]$. Choose $a \in A$ such that $f(a) = \beta$; this is possible because f maps onto α. thus $a \in f^{-1}[\{\beta\}] = f^{-1}[\{\gamma\}]$, so $f(a) \in \{\gamma\}$, hence $\beta = f(a) = \gamma$. So g is one-one, contradicting the choice of α.

Now it follows from (iv) that there is a function f mapping α onto A. Define $a \prec b$ iff the least element of $f^{-1}[\{a\}]$ is less than the first element of $f^{-1}[\{b\}]$. Clearly $\prec$ is a linear order on A. To show that it is a well-order, let B be a nonempty subset of A. Let β be the least element of $f^{-1}[B]$. Then $f(\beta)$ is clearly the $\prec$-least element of B.

(i)⇒(v): Assume (i). Let $\mathscr{F}$, a nonempty family of subsets of A, have finite character. We consider $\mathscr{F}$ as a partially ordered set under inclusion. It is nonempty by assumption. Now suppose that $\mathscr{G}$ is a nonempty subset of $\mathscr{F}$ linearly ordered by inclusion. To show that $\bigcup \mathscr{G} \in \mathscr{F}$, it suffices to show that every finite F subset of it is in $\mathscr{F}$, by the definition of finite character. For each $a \in F$ choose $X_a \in \mathscr{G}$ such that $a \in X_a$. Since $\mathscr{G}$ is linearly ordered by inclusion, choose $a \in F$ such that $X_b \subseteq X_a$ for all $b \in F$. Now $X_a \in \mathscr{F}$ since $\mathscr{G} \subseteq \mathscr{F}$, and F is a finite subset of X_a, so $F \in \mathscr{F}$ by the definition of finite character.

Thus we have verified the hypotheses of Zorn's lemma, and it gives the desired maximal element.

(v)⇒(vi): Assume (v). Given a relation R, let $\mathscr{F}$ consist of all functions contained in R. We verify that $\mathscr{F}$ has finite character. It is obviously nonempty, since $\emptyset \in \mathscr{F}$. Of course, if $f \in \mathscr{F}$, then every finite subset of f is in $\mathscr{F}$. Now suppose that $f \subseteq R$ and every finite subset of f is in $\mathscr{F}$. We just need to show that f is a function. Suppose that $(a, b), (a, c) \in f$. Then $\{(a, b), (a, c)\}$ is a finite subset of f, and so it is in $\mathscr{F}$, which means that it is a function, and so $b = c$. Thus f is a function.

Now by (v), let f be a maximal member of $\mathscr{F}$ under inclusion. So, f is a function included in R. Suppose that $a \in \mathrm{dmn}(R)\backslash\mathrm{dmn}(f)$. Choose b such that $(a, b) \in R$. Then $f \subset f \cup \{(a, b)\} \in \mathscr{F}$, contradiction. Therefore, $\mathrm{dmn}(R) = \mathrm{dmn}(f)$, as desired.

(vi)⇒(i): Assume (vi). Given a family $\langle A_i : i \in I\rangle$ of nonempty sets, let $R = \{(i, x) : i \in I \text{ and } x \in A_i\}$. Let f be a function such that $\mathrm{dmn}(f) = \mathrm{dmn}(R)$. Thus $\mathrm{dmn}(f) = I$ and $f(i) \in A_i$ for all $i \in I$. $\square$

11. Cardinals

This chapter is concerned with the basics of cardinal arithmetic.

Definition and basic properties

To abbreviate longer expressions, we say that sets A and B are *equipotent* iff there is a bijection between them. A *cardinal*, or *cardinal number*, is an ordinal α which is not equipotent with a smaller ordinal. We generally use Greek letters κ, λ, μ for cardinals. Obviously if κ and λ are distinct cardinals, then they are not equipotent.

Proposition 11.1. *For any set X there is an ordinal α equipotent with X.*

Proof. By the well-ordering principle, let $<$ be a well-ordering of X. Then X under $<$ is isomorphic to an ordinal. □

By this proposition, any set is equipotent with a cardinal—namely the least ordinal equipotent with it. This justifies the following definition. For any set X, the *cardinality*, or *size*, or *magnitude*, etc. of X is the unique cardinal $|X|$ equipotent with X. The basic property of this definition is given in the following theorem.

Theorem 11.2. *For any sets X and Y, the following conditions are equivalent:*
(i) $|X| = |Y|$.
(ii) X and Y are equipotent. □

The following proposition gives obvious facts about the particular way that we have defined the notion of cardinality.

Proposition 11.3. *(i) $|\alpha| \leq \alpha$.*
(ii) $|\alpha| = \alpha$ iff α is a cardinal. □

Lemma 11.4. *If $0 \neq m \in \omega$ then there is an $n \in \omega$ such that $m = n + 1$.*

Proof. Assume that $0 \neq m \in \omega$. By Theorem 7.16, m is a successor ordinal $\alpha + 1$. Since ω is transitive we have $\alpha \in \omega$. □

Proposition 11.5. *Every natural number is a cardinal.*

Proof. We prove by ordinary induction on n that for every natural number n and for every natural number m, if $m < n$ then there is no bijection from n to m. This is vacuously true for $n = 0$. Now assume it for n, but suppose that m is a natural number less than $n + 1$ and f is a bijection from $n + 1$ onto m. Since $n + 1 \neq 0$, obviously also $m \neq 0$. So $m = m' + 1$ for some natural number m', by Lemma 11.4. Let g be the bijection from m onto m which interchanges m' and $f(n)$ and leaves fixed all other elements of m. Then $g \circ f$ is a bijection from $n + 1$ onto m which takes n to m'. Hence $(g \circ f) \restriction n$ is a bijection from n onto m', and $m' < n$, contradicting the inductive hypothesis. □

Thus the natural numbers are the first cardinals, in the ordering of cardinals determined by the fact that they are special kinds of ordinals. A set is *finite* iff it is equipotent with some natural number; otherwise it is *infinite*. The following general lemma helps to prove that ω is the next cardinal.

Lemma 11.6. *If $(A,<)$ is a simple order, then every finite nonempty subset of A has a greatest element.*

Proof. We prove by induction on $m \geq 1$ that if $X \subseteq A$ and $|X| = m$ then X has a greatest element. For $m = 1$ this is obvious. Now assume the implication for m, and suppose that $X \subseteq A$ and $|X| = m+1$. Let f be a bijection from $m+1$ onto X, and let $X' = X\backslash\{f(m)\}$. So $|X'| = m$, and so X' has a largest element x. If $f(m) < x$, then x is the greatest element of X. If $x < f(m)$, then $f(m)$ is the greatest element of X. □

Theorem 11.7. *ω is a cardinal.* □

It is harder to find larger cardinals, but they exist; in fact the collection of cardinals is so big that, like the collection of ordinals, it does not exist as a set. We will see this a little bit later.

Note that a cardinal is infinite iff it is greater or equal ω. The following fact will be useful later.

Proposition 11.8. *Every infinite cardinal is a limit ordinal.*

Proof. Suppose not: κ is an infinite cardinal, and $\kappa = \alpha+1$. We define $f : \alpha \to \kappa$ as follows: $f(0) = \alpha$, $f(m+1) = m$ for all $m \in \omega$, and $f(\beta) = \beta$ for all $\beta \in \alpha\backslash\omega$. Clearly f is one-one and maps onto κ, contradiction. □

Lemma 11.9. *If κ and λ are cardinals and $f : \kappa \to \lambda$ is one-one, then $\kappa \leq \lambda$.*

Proof. We define $\alpha \prec \beta$ iff $\alpha, \beta \in \kappa$ and $f(\alpha) < f(\beta)$. Clearly $\prec$ well-orders κ. Let g be an isomorphism from $(\kappa, \prec)$ onto an ordinal γ. Thus $\kappa \leq \gamma$ by the definition of cardinals. If $\alpha < \beta < \gamma$, then $g^{-1}(\alpha) \prec g^{-1}(\beta)$, hence by definition of $\prec$, $f(g^{-1}(\alpha)) < f(g^{-1}(\beta))$. Thus $f \circ g^{-1} : \gamma \to \lambda$ is strictly increasing. Hence by Proposition 6.15, $\alpha \leq (f \circ g^{-1})(\alpha)$ for all $\alpha < \gamma$, so $\lambda \not< \gamma$, hence $\gamma \leq \lambda$. We already know that $\kappa \leq \gamma$, so $\kappa \leq \lambda$. □

The purpose of this lemma is to prove the following basic theorem.

Theorem 11.10. *If $A \subseteq B$, then $|A| \leq |B|$.*

Proof. Let $\kappa = |A|$, $\lambda = |B|$, and let f and g be one-one functions from κ onto A and of λ onto B, respectively. Then $g \circ f^{-1}$ is a one-one function from κ into λ, so $\kappa \leq \lambda$. □

Corollary 11.11. *For any sets A and B the following conditions are equivalent:*
(i) $|A| \leq |B|$.
(ii) There is a one-one function mapping A into B.
(iii) $A = \emptyset$, or there is a function mapping B onto A.

Proof. Let f be a bijection from $|A|$ to A, and g a bijection from $|B|$ to B,

(i)⇒(ii): Assume that $|A| \leq |B|$. Then $|A| \subseteq |B|$ by Proposition 4.8, and $g \circ f^{-1}$ is a one-one mapping from A into B.

(ii)⇒(iii): Assume that $h : A \to B$ is one-one and $A \neq \emptyset$. Fix $a \in A$, and define $k : B \to A$ by setting, for any $b \in B$,

$$k(b) = \begin{cases} h^{-1}(b) & \text{if } b \in \operatorname{rng}(h), \\ a & \text{otherwise.} \end{cases}$$

Clearly k maps B onto A.

(iii)⇒(i): Obviously $A = \emptyset$ implies that $0 = |A| \leq |B|$. Now suppose that h maps B onto A. Then for any $\alpha < |A|$ there is a $b \in B$ such that $h(b) = f(\alpha)$, and hence there is a $\beta < |B|$ such that $h(g(\beta)) = f(\alpha)$. For each $\alpha < |A|$ let $k(\alpha) = \min\{\beta < |B| : h(g(\beta)) = f(\alpha)\}$. Then $h \circ g \circ k = f$, so k is one-one. By Lemma 11.9, $|A| \leq |B|$. □

Corollary 11.12. *If there is a one-one function from A into B and a one-one function from B into A, then there is a one-one function from A onto B.* □

This corollary is called the Cantor-Bernstein, or Schröder-Bernstein theorem. Our proof, if traced back, involves the axiom of choice. It can be proved without the axiom of choice, and this is sometimes desirable when describing a small portion of set theory to students.

Proposition 11.13. *If $m \in \omega$, A is a set with $|A| = m+1$, and $a \in A$, then $|A \backslash \{a\}| = m$.*

Proof. Let $f : A \to m+1$ be a bijection. Let g be a bijection from $m+1$ onto $m+1$ which interchanges m and $f(a)$, leaving other elements fixed. Then $g \circ f$ is a bijection of A onto $m+1$, and $(g \circ f)(a) = m$. Hence $(g \circ f) \langle (A \backslash \{a\})$ is a bijection from $A \backslash \{a\}$ onto m. □

Theorem 11.14. *Suppose that $m \in \omega$ and A and B are sets of size m. Let $f : A \to B$. Then f is one-one iff f is onto.*

Proof. We prove the statement

$$\forall m \in \omega \forall A, B, f[(|A| = |B| = m \text{ and } f : A \to B) \Rightarrow (f \text{ is one-one } \Leftrightarrow f \text{ is onto})]$$

by induction on m. It is obvious for $m = 0$. Suppose it is true for m, and $|A| = |B| = m+1$ and $f : A \to B$.

First suppose that f is one-one. Pick $a \in A$. Then by Proposition 11.13, $|A \backslash \{a\}| = |B \backslash \{f(a)\}| = m$. Now $f \restriction (A \backslash \{a\})$ maps into $B \backslash \{f(a)\}$, since if $x \in A \backslash \{a\}$ and $f(x) = f(a)$ then f being one-one is contradicted. Now $f \restriction (A \backslash \{a\})$ is one-one, so by the inductive hypothesis $f \restriction (A \backslash \{a\})$ is onto. Clearly then f is onto.

Second suppose that f is onto. Let $g : m+1 \to A$ be a bijection. Now for any $b \in B$ there is an $a \in A$ such that $f(a) = b$, hence there is an $i \in m+1$ such that $f(g(i)) = b$. Let $h(b)$ be the least such i. Then $(f \circ g \circ h)(b) = b$ for all $b \in B$. It follows that $h : B \to m+1$ is one-one. Hence by the first step above, h is onto. To show that f is one-one, suppose that $f(a_0) = f(a_1)$. Choose $i_0, i_1 \in m+1$ such that $g(i_0) = a_0$ and $g(i_1) = a_1$. Since h is onto, choose $b_0, b_1 \in B$ such that $h(b_0) = i_0$ and $h(b_1) = i_1$. Then $b_0 = f(g(h(b_0))) = f(g(i_0)) = f(a_0) = f(a_1) = f(g(i_1)) = f(g(h(b_1))) = b_1$. Hence $i_0 = h(b_0) = h(b_1) = i_1$, and $a_0 = g(i_0) = g(i_1) = a_1$. □

Theorem 11.14 does not extend to infinite sets.

The following simple theorem is very important and basic for the theory of cardinals. It embodies in perhaps its simplest form the Cantor diagonal argument.

Theorem 11.15. *For any set A we have $|A| < |\mathscr{P}(A)|$.*

Proof. The function given by $a \mapsto \{a\}$ is a one-one function from A into $\mathscr{P}(A)$, and so $|A| \le |\mathscr{P}(A)|$. [Saying that $a \mapsto \{a\}$ is giving the value of the function at the argument a.] Suppose equality holds. Then there is a one-one function f mapping A onto $\mathscr{P}(A)$. Let $X = \{a \in A : a \notin f(a)\}$. Since f maps onto $\mathscr{P}(A)$, choose $a_0 \in A$ such that $f(a_0) = X$. Then $a_0 \in X$ iff $a_0 \notin X$, contradiction. □

By this theorem, for every ordinal α there is a larger cardinal, namely $|\mathscr{P}(\alpha)|$. Hence we can define α^+ to be the least cardinal $> \alpha$. Cardinals of the form κ^+ are called *successor cardinals*; other infinite cardinals are called *limit cardinals.* Is $\kappa^+ = |\mathscr{P}(\kappa)|$? The statement that this is true for every infinite cardinal κ is the famous *generalized continuum hypothesis* (GCH). The weaker statement that $\omega^+ = |\mathscr{P}(\omega)|$ is the *continuum hypothesis* (CH).

It can be shown that the generalized continuum hypothesis is consistent with our axioms. But also its negation is consistent; in fact, the negation of the weaker continuum hypothesis is consistent. All of this under the assumption that our axioms are consistent. (It is not possible to prove this consistency.)

Theorem 11.16. *If Γ is a set of cardinals, then $\bigcup\Gamma$ is also a cardinal.*

Proof. We know already that $\bigcup\Gamma$ is an ordinal. Suppose that $\kappa \stackrel{\text{def}}{=} |\bigcup\Gamma| < \bigcup\Gamma$. By definition of $\bigcup$, there is a $\lambda \in \Gamma$ such that $\kappa < \lambda$. (Membership is the same as $<$.) Now $\lambda \subseteq \bigcup\Gamma$. So $\lambda = |\lambda| \le |\bigcup\Gamma| = \kappa$, contradiction. □

We can now define the standard sequence of infinite cardinal numbers, by transfinite recursion.

Theorem 11.17. *There is a class ordinal function $\aleph$ with domain* **On** *such that the following conditions hold.*

(i) $\aleph_0 = \omega$.

(ii) $\aleph_{\beta+1} = \aleph_\beta^+$ *for any ordinal β.*

(iii) $\aleph_\beta = \bigcup_{\gamma<\beta} \aleph_\gamma$ *for every limit ordinal β.*

Proof. We define $\mathbf{G} : \mathbf{On} \times \mathbf{V} \to \mathbf{V}$ as follows. For any ordinal α and any set x,

$$\mathbf{G}(\alpha, x) = \begin{cases} \omega & \text{if } \alpha = 0, \\ (x(\beta))^+ & \text{if } \alpha = \beta + 1 \text{ for some ordinal } \beta \text{ and} \\ & x \text{ is a function with domain } \alpha \text{ and } x(\beta) \text{ is an ordinal} \\ \bigcup_{\beta<\alpha} x(\beta) & \text{if } \alpha \text{ is a limit ordinal and } x \text{ is a function} \\ & \text{with domain } \alpha \text{ and ordinal values} \\ \emptyset & \text{otherwise.} \end{cases}$$

Now we apply Theorem 6.7 and get a function $\mathbf{F} : \mathbf{On} \to \mathbf{V}$ such that $\mathbf{F}(\alpha) = \mathbf{G}(\alpha, \mathbf{F} \restriction \alpha)$ for every ordinal α. Then

$$\begin{aligned} \mathbf{F}(0) &= \mathbf{G}(0, \mathbf{F} \restriction 0) = \omega; \\ \mathbf{F}(\beta+1) &= \mathbf{G}(\beta+1, \mathbf{F} \restriction (\beta+1)) = (\mathbf{F}(\beta))^+ \\ \mathbf{F}(\alpha) &= \mathbf{G}(\alpha, \mathbf{F} \restriction \alpha) = \bigcup_{\beta<\alpha} \mathbf{F}(\beta) \quad \text{for } \alpha \text{ limit} \end{aligned}$$

□

For historical reasons, one sometimes writes ω_α in place of $\aleph_\alpha$. Now we get the following two results by Propositions 9.15 and 9.16.

Lemma 11.18. *If $\alpha < \beta$, then $\aleph_\alpha < \aleph_\beta$.* □

Lemma 11.19. *$\alpha \leq \aleph_\alpha$ for every ordinal α.* □

Theorem 11.20. *For every infinite cardinal κ there is an ordinal α such that $\kappa = \aleph_\alpha$.*

Proof. Let κ be any infinite cardinal. Then $\kappa \leq \aleph_\kappa < \aleph_{\kappa+1}$. Here $\kappa + 1$ refers to ordinal addition. This shows that there is an ordinal α such that $\kappa < \aleph_\alpha$; choose the least such α. Clearly $\alpha \neq 0$ and α is not a limit ordinal. Say $\alpha = \beta + 1$. Then $\aleph_\beta \leq \kappa < \aleph_{\beta+1}$, so $\kappa = \aleph_\beta$. □

We can now say a little more about the continuum hypothesis. Not only is it consistent that it fails, but it is even consistent that $|\mathscr{P}(\omega)| = \aleph_2$, or $|\mathscr{P}(\omega)| = \aleph_{17}$, or $|\mathscr{P}(\omega)| = \aleph_{\omega+1}$; the possibilities have been spelled out in great detail. Some impossible situations are $|\mathscr{P}(\omega)| = \aleph_\omega$ and $|\mathscr{P}(\omega)| = \aleph_{\omega+\omega}$; we will establish this later in this chapter.

Addition of cardinals

Let κ and λ be cardinals. We define

$$\kappa + \lambda = |\{(\alpha, 0) : \alpha \in \kappa\} \cup \{(\beta, 1) : \beta \in \lambda\}|.$$

The idea is to take disjoint copies $\kappa \times \{0\}$ and $\lambda \times \{1\}$ of κ and λ and count the number of elements in their union.

Two immediate remarks should be made about this definition. First of all, this is not, in general, the same as the ordinal sum $\kappa + \lambda$. We depend on the context to distinguish the two notions of addition. For example, $\omega + 1 = \omega$ in the cardinal sense, but not in the ordinal sense. In fact, we know that $\omega < \omega + 1$ in the ordinal sense. To show that $\omega + 1 = \omega$ in the cardinal sense, it suffices to define a one-one function from ω onto the set

$$\{(m, 0) : m \in \omega\} \cup \{(0, 1)\}.$$

Let $f(0) = (0, 1)$ and $f(m + 1) = (m, 0)$ for any $m \in \omega$.

Secondly, the definition is consistent with our definition of addition for natural numbers (as a special case of ordinal addition), and thus it does coincide with ordinal addition when restricted to ω; this will be proved shortly.

Proposition 11.21. *If $A \cap B = \emptyset$, then $|A \cup B| = |A| + |B|$.*

Proof. We have $|A| + |B| = |\{(\alpha, 0) : \alpha \in |A|\} \cup \{(\alpha, 1) : \alpha \in |B|\}|$. Now let $f : A \to |A|$ and $g : B \to |B|$ be bijections. Define h with domain $A \cup B$ by

$$\begin{aligned} h(a) &= (f(a), 0) \quad \text{for all } a \in A, \\ h(b) &= (g(b), 1) \quad \text{for all } b \in B. \end{aligned}$$

Then it is clear that h is a bijection from $A \cup B$ onto $\{(\alpha, 0) : \alpha \in |A|\} \cup \{(\alpha, 1) : \alpha \in |B|\}$. Hence $|A \cup B| = |\{(\alpha, 0) : \alpha \in |A|\} \cup \{(\alpha, 1) : \alpha \in |B|\}| = |A| + |B|$. □

Proposition 11.22. *If m and n are natural numbers, then addition in the ordinal sense and in the cardinal number sense are the same.*

Proof. For this proof we denote ordinal addition by $+'$ and cardinal addition by $+$. With $m \in \omega$ fixed we prove that $m +' n = m + n$ by induction on n. The case $n = 0$ is clear. Now suppose that $m +' n = m + n$. Then $m +' (n +' 1) = (m +' n) +' 1 = (m +' n) \cup \{m +' n\}$. On the other hand,

$$\begin{aligned} m + (n +' 1) &= |\{(i, 0) : i \in m\} \cup \{(i, 1) : i \in n +' 1\}| \\ &= |\{(i, 0) : i \in m\} \cup \{(i, 1) : i \in n \cup \{n\}\}| \\ &= |\{(i, 0) : i \in m\} \cup \{(i, 1) : i \in n\} \cup \{(n, 1)\}| \\ &= |\{(i, 0) : i \in m\} \cup \{(i, 1) : i \in n\}| + 1 \\ &= (m +' n) +' 1 = m +' (n +' 1). \end{aligned}$$
□

Aside from simple facts about addition, there is the remarkable fact that $\kappa + \kappa = \kappa$ for every infinite cardinal κ. We shall prove this as a consequence of the similar result for multiplication.

The definition of cardinal addition can be extended to infinite sums, and very elementary properties of the binary sum are then special cases of more general results; so we proceed with the general definition. Let $\langle \kappa_i : i \in I \rangle$ be a system of cardinals (this just means that κ is a function with domain I whose values are always cardinals). Then we define

$$\sum_{i \in I} \kappa_i = \left| \bigcup_{i \in I} (\kappa_i \times \{i\}) \right|.$$

This is a generalization of summing two cardinals, as is immediate from the definitions:

Proposition 11.23. *If $\langle \kappa_i : i \in 2 \rangle$ is a system of cardinals (meaning that κ is a function with domain 2 such that both κ_0 and κ_1 are cardinals), then $\sum_{i \in 2} \kappa_i = \kappa_0 + \kappa_1$.* □

The following is easily proved by induction on $|I|$:

Proposition 11.24. *If $\langle m_i : i \in I \rangle$ is a system of natural numbers, with I finite, then $\sum_{i \in I} m_i$ is a natural number.* □

We mention some important but easy facts concerning the cardinalities of unions:

Proposition 11.25. *If $\langle A_i : i \in I \rangle$ is a system of pairwise disjoint sets, then $\left|\bigcup_{i\in I} A_i\right| = \sum_{i\in I} |A_i|$.* □

Proposition 11.26. *If $\langle A_i : i \in I \rangle$ is any system of sets, then $\left|\bigcup_{i\in I} A_i\right| \le \sum_{i\in I} |A_i|$.*

Proof. For each $i \in I$ let f_i be a bijection from $|A_i|$ onto A_i. (We use the axiom of choice here.) For any $i \in I$ and $\alpha \in |A_i|$ let $g((\alpha, i)) = f_i(\alpha)$. Then g maps $\bigcup_{i\in I}(|A_i| \times \{i\})$ onto $\bigcup_{i\in I} A_i$. Hence by Corollary 11.11,

$$\left|\bigcup_{i\in I} A_i\right| \le \left|\bigcup_{i\in I}(|A_i| \times \{i\})\right| = \sum_{i\in I} |A_i|. \qquad \square$$

Corollary 11.27. *If $\langle \kappa_i : i \in I \rangle$ is a system of cardinals, then $\bigcup_{i\in I} \kappa_i \le \sum_{i\in I} \kappa_i$.* □

Finally, we gather together some simple arithmetic of infinite sums:

Proposition 11.28. *(i) $\sum_{i\in I} 0 = 0$.*
(ii) $\sum_{i\in 0} \kappa_i = 0$.
(iii) $\sum_{i\in I} \kappa_i = \sum_{i\in I, \kappa_i \ne 0} \kappa_i$.
(iv) If $I \subseteq J$, then $\sum_{i\in I} \kappa_i \le \sum_{i\in J} \kappa_i$.
(v) If $\kappa_i \le \lambda_i$ for all $i \in I$, then $\sum_{i\in I} \kappa_i \le \sum_{i\in I} \lambda_i$.
(vi) $\sum_{i\in I} 1 = |I|$.
(vii) If κ is infinite, then $\kappa + 1 = \kappa$.

Proof. (i): $\sum_{i\in I} 0 = \left|\bigcup_{i\in I}(0 \times \{i\})\right| = |\emptyset| = 0$.
(ii): $\sum_{i\in 0} \kappa_i = \left|\bigcup_{i\in 0}(\kappa_i \times \{i\})\right| = |\emptyset| = 0$.
(iii): $\sum_{i\in I} \kappa_i = \left|\bigcup_{i\in I}(\kappa_i \times \{i\})\right| = \left|\bigcup_{i\in I, \kappa_i \ne 0}(\kappa_i \times \{i\}\right| = \sum_{i\in I, \kappa_i \ne 0} \kappa_i$.
(iv): Assume that $I \subseteq J$. Then $\bigcup_{i\in I}(\kappa_i \times \{i\}) \subseteq \bigcup_{i\in J}(\kappa_i \times \{i\})$ and so the desired conclusion follows by Theorem 11.10.
(v): Assume that $\kappa_i \le \lambda_i$ for all $i \in I$. Then $\bigcup_{i\in I}(\kappa_i \times \{i\}) \subseteq \bigcup_{i\in I}(\lambda_i \times \{i\})$, and Theorem 11.10 applies.
(vi): We have $\sum_{i\in I} 1 = \left|\bigcup_{i\in I}(1 \times \{i\})\right|$. Now the mapping $i \mapsto (0, i)$ is a bijection from I to $\bigcup_{i\in I}(1 \times \{i\})$, so the desired conclusion follows.
(vii) We define a function f mapping κ into $\{(\alpha, 0) : \alpha < \kappa\} \cup \{(0, 1)\}$ as follows. For any $\alpha < \kappa$,

$$f(\alpha) = \begin{cases} (0,1) & \text{if } \alpha = 0, \\ (\beta, 0) & \text{if } \alpha = \beta + 1 \in \omega, \\ (\alpha, 0) & \text{if } \omega \le \alpha < \kappa. \end{cases}$$

It is clear that f is a bijection, as desired. □

By definition,

$$\kappa \cdot \lambda = |\kappa \times \lambda|.$$

Again this is different from ordinal multiplication, and we depend on the context to distinguish between them. For example, in the ordinal sense $\omega \cdot 2 > \omega \cdot 1 = \omega$ but in the cardinal sense $\omega \cdot 2 = \omega$. One can see the latter by using the following function f from ω to $\omega \times 2$: $f(2m) = (m, 0)$ and $f(2m+1) = (m, 1)$ for any $m \in \omega$.

The following simple result can be used in verifying many simple facts concerning products.

Proposition 11.29. *If A is equipotent with C and B is equipotent with D, then $A \times B$ is equipotent with $C \times D$.*

Proof. Assume the hypothesis. Say $f : A \to C$ is a bijection, and $g : B \to D$ is a bijection. Define $h : A \times B \to C \times D$ by setting $h(a, b) = (f(a), g(b))$. Clearly h is a bijection from $A \times B$ onto $C \times D$. □

Proposition 11.30. *(i) $\kappa \cdot \lambda = \lambda \cdot \kappa$;*
(ii) $\kappa \cdot (\lambda \cdot \mu) = (\kappa \cdot \lambda) \cdot \mu$;
(iii) $\kappa \cdot (\lambda + \mu) = \kappa \cdot \lambda + \kappa \cdot \mu$;
(iv) $\kappa \cdot 0 = 0$;
(v) $\kappa \cdot 1 = \kappa$;
(vi) $\kappa \cdot 2 = \kappa + \kappa$;
(vii) $\sum_{i \in I} \kappa = \kappa \cdot |I|$;
(viii) If $\kappa \le \mu$ and $\lambda \le \nu$, then $\kappa \cdot \lambda \le \mu \cdot \nu$.

Proof. (i): For any $\alpha \in \kappa$ and $\beta \in \lambda$ let $f(\alpha, \beta) = (\beta, \alpha)$. Clearly f is a bijection from $\kappa \times \lambda$ onto $\lambda \times \kappa$.

(ii): We have $\kappa \cdot (\lambda \cdot \mu) = |\kappa \times (\lambda \cdot \mu)| = |\kappa \times |\lambda \times \mu||$. Let f be a bijection from $\lambda \cdot \mu$ onto $\lambda \times \mu$. Define $g : \kappa \times |\lambda \times \mu| \to (\kappa \times \lambda) \times \mu$ by setting, for $\alpha \in \kappa$ and $\beta \in |\lambda \times \mu|$, $g(\alpha, \beta) = ((\alpha, 1^{\text{st}} f(\beta)), 2^{\text{nd}} f(\beta))$. Clearly g is a bijection.

We have $(\kappa \cdot \lambda) \cdot \mu = |(\kappa \cdot \lambda) \times \mu| = ||\kappa \times \lambda| \times \mu|$. Now let $h : \kappa \cdot \lambda \to \kappa \times \lambda$ be a bijection. Define $k : |\kappa \times \lambda| \times \mu \to (\kappa \times \lambda) \times \mu$ by setting, for $\alpha \in |\kappa \times \lambda|$ and $\beta \in \mu$, $k(\alpha, \beta) = (h(\alpha), \beta)$. Clearly h is a bijection.

Now $h^{-1} \circ g$ is a bijection from $\kappa \times |\lambda \times \mu|$ onto $|\kappa \times \lambda| \times \mu$, as desired.

(iii): We have

$$\begin{aligned}
\kappa \cdot (\lambda + \mu) &= |\kappa \times (\lambda + \mu)| \\
&= |\kappa \times |(\lambda \times \{0\}) \cup (\mu \times \{1\})|| \\
&= |\kappa \times (\lambda \times \{0\}) \cup (\mu \times \{1\}))| \quad \text{using Proposition 11.29;} \\
\kappa \cdot \lambda + \kappa \cdot \mu &= |(\kappa \cdot \lambda) \times \{0\}) \cup (\kappa \cdot \mu) \times \{1\})| \\
&= |(|\kappa \times \lambda| \times \{0\}) \cup (|\kappa \times \mu| \times \{1\}|.
\end{aligned}$$

Take bijections $g : |\kappa \times \lambda| \to \kappa \times \lambda$ and $h : |\kappa \times \mu| \to \kappa \times \mu$. Now we define

$$h : \kappa \times ((\lambda \times \{0\}) \cup (\mu \times \{1\})| \to (|\kappa \times \lambda| \times \{0\}) \cup (|\kappa \times \mu| \times \{1\}.$$

Let $\alpha \in \kappa$, $\beta \in \lambda$ and $\gamma \in \mu$. Then we define

$$h((\alpha, (\beta, 0))) = (g^{-1}((\alpha, \beta)), 0);$$
$$h((\alpha, (\gamma, 1))) = (g^{-1}(\alpha, \gamma)), 1)$$

It suffices to show that h is a bijection. Clearly it is one-one. For ontoness, given $\alpha \in |\kappa \times \lambda|$ we have

$$h((1^{\mathrm{st}}(g(\alpha), (2^{\mathrm{nd}}(g(\alpha)), 0)) = (g^{-1}((1^{\mathrm{st}}(g(\alpha)), 2^{\mathrm{nd}}(g(\alpha)))), 0) = (\alpha, 0),$$

and similarly for $\alpha \in |\kappa \times \mu|$.

(iv): $\kappa \cdot 0 = |\kappa \times 0| = |0| = 0$.

(v): $\kappa \cdot 1 = |\kappa \times 1| = \kappa$, since $\alpha \mapsto (\alpha, 0)$ is a bijection from κ to $\kappa \times 1$.

(vi): $\kappa \cdot 2 = |\kappa \times 2|$ and $\kappa + \kappa = |(\kappa \times \{0\}) \cup (\kappa \times \{1\})| = |\kappa \times 2|$.

(vii): $\sum_{i \in I} \kappa = \left|\bigcup_{i \in I} (\kappa \cdot \{i\})\right|$ and $\kappa \cdot |I|| = |\kappa \times |I||$. Let f be a bijection from I to $|I|$. For any $\alpha \in \kappa$ and $i \in I$ let $g((\alpha, i)) = (\alpha, f(i))$. Clearly g is a bijection from $\bigcup_{i \in I} (\kappa \cdot \{i\})$ onto $\kappa \times |I|$.

(viii): Assume that $\kappa \leq \mu$ and $\lambda \leq \nu$. Then $\kappa \times \lambda \subseteq \mu \times \nu$, so the desired conclusion follows by Theorem 11.10. □

Proposition 11.31. *Multiplication of natural numbers means the same in the cardinal number sense as in ordinal sense.*

Proof. For this proof we use $\circ$ for ordinal multiplication and $\cdot$ for cardinal multiplication. We prove with fixed $m \in \omega$ that $m \circ n = m \cdot n$ for all $n \in \omega$. We have $m \circ 0 = 0$ and $m \cdot 0 = |m \times 0| = |0| = 0$. Assume that $m \circ n = m \cdot n$. Then $m \circ (n+1) = m \circ n + m = m \cdot n + m = m \cdot n + m \cdot 1 = m \cdot (n+1)$ using Proposition 11.30(iii). □

The basic theorem about multiplication of infinite cardinals is as follows.

Theorem 11.32. *$\kappa \cdot \kappa = \kappa$ for every infinite cardinal κ.*

Proof. Suppose not, and let κ be the least infinite cardinal such that $\kappa \cdot \kappa \neq \kappa$. Then $\kappa = \kappa \cdot 1 \leq \kappa \cdot \kappa$, and so $\kappa < \kappa \cdot \kappa$. We now define a relation $\prec$ on $\kappa \times \kappa$. For all $\alpha, \beta, \gamma, \delta \in \kappa$,

$$\begin{aligned}(\alpha, \beta) \prec (\gamma, \delta) \text{ iff } & \max(\alpha, \beta) < \max(\gamma, \delta) \\ \text{or } & \max(\alpha, \beta) = \max(\gamma, \delta) \text{ and } \alpha < \gamma \\ \text{or } & \max(\alpha, \beta) = \max(\gamma, \delta) \text{ and } \alpha = \gamma \text{ and } \beta < \delta.\end{aligned}$$

Clearly this is a well-order. It follows that $(\kappa \times \kappa, \prec)$ is isomorphic to an ordinal α; let f be the isomorphism. We have $|\alpha| = |\kappa \times \kappa| = \kappa \cdot \kappa > \kappa$ by the remark at the beginning of this proof. So $\kappa < \alpha$. Therefore there exist $\beta, \gamma \in \kappa$ such that $f(\beta, \gamma) = \kappa$. Now

$$f[\{(\delta, \varepsilon) \in \kappa \times \kappa : (\delta, \varepsilon) \prec (\beta, \gamma)\}] = \kappa,$$

so, with $\varphi = \max(\beta, \gamma){+}1$,

$$\begin{aligned}\kappa &= |\{(\delta, \varepsilon) \in \kappa \times \kappa : (\delta, \varepsilon) \prec (\beta, \gamma)\}| \\ &\leq |\varphi \times \varphi| = |\varphi| \cdot |\varphi|.\end{aligned}$$

But $\varphi < \kappa$, so either φ is finite, and $|\varphi| \cdot |\varphi|$ is then also finite, or else φ is infinite, and $|\varphi| \cdot |\varphi| = |\varphi|$ by the minimality of κ. In any case, $|\varphi| \cdot |\varphi| < \kappa$, contradiction. □

Corollary 11.33. *If κ and λ are nonzero cardinals and at least one of them is infinite, then $\kappa + \lambda = \kappa \cdot \lambda = \max(\kappa, \lambda)$.*

Proof. Say wlog $\kappa \leq \lambda$. Then $\kappa + \lambda \leq \lambda + \lambda = \lambda \cdot 2 \leq \lambda \cdot \lambda = \lambda \leq \kappa + \lambda$. □

Corollary 11.34. *If $\langle A_i : i \in I\rangle$ is any system of sets, then*

$$\left|\bigcup_{i\in I} A_i\right| \leq |I| \cdot \bigcup_{i\in I} |A_i|.$$

Proof. For each $i \in I$ let $g_i : A_i \to |A_i|$ be a bijection (using the axiom of choice). Moreover, let c be a choice function for nonempty subsets of I. Now we define a function f mapping $\bigcup_{i\in I} A_i$ into $I \times \bigcup_{i\in I} |A_i|$. Take any $a \in \bigcup_{i\in I} A_i$, and let $j = c(\{i \in I : a \in A_i\})$. Then we set $f(a) = (j, g_j(a))$. Clearly f is one-one, and hence

$$\begin{aligned}\left|\bigcup_{i\in I} A_i\right| &\leq \left|I \times \bigcup_{i\in I} |A_i|\right| && \text{by Corollary 11.11} \\ &= \left||I| \times \bigcup_{i\in I} |A_i|\right| && \text{by Proposition 11.29} \\ &= |I| \cdot \bigcup_{i\in I} |A_i|.\end{aligned}$$

A set A is *countable* if $|A| \leq \omega$. So another corollary is

Corollary 11.35. *A countable union of countable sets is countable.* □

Proposition 11.36. *If $\langle \kappa_i : i \in I\rangle$ is a system of nonzero cardinals, and either I is infinite or some κ_i is infinite, then $\sum_{i\in I} \kappa_i = |I| \cdot \bigcup_{i\in I} \kappa_i$.*

Proof. We have

$$\begin{aligned}\sum_{i\in I} \kappa_i &\leq \sum_{i\in I} \bigcup_{j\in I} \kappa_j && \text{by Proposition 11.28(v)} \\ &= |I| \cdot \bigcup_{j\in I} \kappa_j && \text{by Proposition 11.30(vi)}\end{aligned}$$

This proves $\leq$ in the proposition.

Next, $\bigcup_{i\in I}\kappa_i \leq \sum_{i\in I}\kappa_i$ by Proposition 11.27, and $|I| = \sum_{i\in I} 1$ (by Proposition 11.28(vi)) $\leq \sum_{i\in I}\kappa_i$ (by Proposition 11.28(v)). Now the direction $\geq$ of the proposition follows from Corollary 11.33. $\square$

By the above results, the binary operations of addition and multiplication of cardinals are trivial when applied to infinite cardinals; and the infinite sum is also easy to calculate. We now introduce infinite products which, as we shall see, are not so trivial. We need the following standard elementary notion: for $\langle A_i : i \in I\rangle$ a family of sets, we define

$$\prod_{i\in I} A_i = \{f : f \text{ is a function, } \operatorname{dmn}(f) = I, \text{ and } \forall i \in I[f(i) \in A_i]\}.$$

This is the *cartesian product* of the sets A_i. Now if $\langle \kappa_i : i \in I\rangle$ is a system of cardinals, we define

$$\prod_{i\in I}^{c} \kappa_i = \left|\prod_{i\in I}\kappa_i\right|.$$

Some elementary properties of this notion are summarized in the following proposition.

Proposition 11.37. *(i)* $\left|\prod_{i\in I} A_i\right| = \prod_{i\in I}^c |A_i|$.

(ii) If $\kappa_i = 0$ *for some* $i \in I$, *then* $\prod_{i\in I}^c \kappa_i = 0$.

(iii) $\prod_{i\in 0}^c \kappa_i = 1$.

(iv) $\prod_{i\in I}^c \kappa_i = \prod_{i\in I,\kappa_i\neq 1}^c \kappa_i$.

(v) $\prod_{i\in I}^c 1 = 1$.

(vi) If $\kappa_i \leq \lambda_i$ *for all* $i \in I$, *then* $\prod_{i\in I}^c \kappa_i \leq \prod_{i\in I}^c \lambda_i$.

(vii) $\prod_{i\in 2}^c \kappa_i = \kappa_0 \cdot \kappa_1$.

Proof. (i): For each $i \in I$, let f_i be a one-one function mapping A_i onto $|A_i$. (We are using the axiom of choice here.) Note that $\prod_{i\in I}^c |A_i| = \left|\prod_{i\in I}|A_i|\right|$. Thus we want to find a bijection from $\prod_{i\in I} A_i$ onto $\prod_{i\in I}|A_i|$. For each $x \in \prod_{i\in I} A_i$ and $j \in I$ let $(g(x))_j = f_j(x_j)$. Thus $g : \prod_{i\in I} A_i \to \prod_{i\in I}|A_i|$. Suppose that $g(x) = g(y)$. Then for any $j \in I$ we have $f_j(x_j) = ((g(x))_j = ((g(y))_j = f_j(y_j)$, and hence $x_j = y_j$; so $x = y$. Thus g is one-one. Given $y \in \prod_{i\in I}|A_i|$, define $x_j = f_j^{-1}(y_j)$ for any $j \in I$. Then $x \in \prod_{i\in I} A_i$ and $(g(x))_j = f_j(x_j) = f_j(f_j^{-1}(y_j)) = y_j$; so $g(x) = y$. This shows that g is onto.

(ii): If $\kappa_i = 0$ for some $i \in I$, then $\prod_{j\in I} A_j = \emptyset$, and hence $\prod_{j\in I}^c A_j = \left|\prod_{j\in I} A_j\right| = |\emptyset| = 0$.

(iii): We have $\prod_{i\in 0}^c \kappa_i = \left|\prod_{i\in 0}\kappa_i\right| = |\{\emptyset\}| = 1$.

(iv): We have $\prod_{i\in I}^c \kappa_i = \left|\prod_{i\in I}\kappa_i\right|$ and $\prod_{i\in I,\kappa_i\neq 1}^c \kappa_i = \left|\prod_{i\in I,\kappa_i\neq 1}\kappa_i\right|$, so we want a bijection from $\prod_{i\in I}\kappa_i$ onto $\prod_{i\in I,\kappa_i\neq 1}\kappa_i$. For each $x \in \prod_{i\in I}\kappa_i$ let $f(x) = x \restriction \{i \in I : \kappa_i \neq 1\}$. If $f(x) = f(y)$, then for any $i \in I$,

$$\begin{aligned} x(i) &= \begin{cases} (f(x))(i) & \text{if } \kappa_i \neq 1 \\ 0 & \text{if } \kappa_1 = 1 \end{cases} \\ &= \begin{cases} (f(y))(i) & \text{if } \kappa_i \neq 1 \\ 0 & \text{if } \kappa_1 = 1 \end{cases} \\ &= y(i). \end{aligned}$$

Thus f is one-one. Clearly it is onto.

(v): We have $\prod^c_{i\in I} 1 = \left|\prod_{i\in I} 1\right|$. Hence it suffices to show, using (iii), that if $I \neq \emptyset$ then $\prod_{i\in I} 1$ has only one element. This is clear.

(vi): Assume that $\kappa_i \le \lambda_i$ for all $i \in I$. Then $\prod_{i\in I} \kappa_i \subseteq \prod_{i\in I} \lambda_i$, so (vi) follows from Theorem 11.10.

(vii): We have $\prod^c_{i\in 2} \kappa_i = \left|\prod_{i\in 2} \kappa_i\right|$, and $\kappa_0 \cdot \kappa_1 = |\kappa_0 \times \kappa_1|$. Hence it suffices to describe a bijection from $\prod_{i\in 2} \kappa_i$ onto $\kappa_0 \times \kappa_1$. For each $x \in \prod_{i\in 2} \kappa_i$ let $f(x) = (x(0), x(1))$. Clearly f is as desired. □

General commutative, associative, and distributive laws hold also:

Proposition 11.38. (Commutative law) *If $\langle \kappa_i : i \in I\rangle$ is a system of cardinals and $f : I \to I$ is one-one and onto, then*

$$\prod^c_{i\in I} \kappa_i = \prod^c_{i\in I} \kappa_{f(i)}.$$

Proof. For each $x \in \prod_{i\in I} \kappa_i$ define $g(x) \in \prod_{i\in I} \kappa_{f(i)}$ by setting $(g(x))_i = x_{f(i)}$. Clearly g is a bijection, and the proposition follows. □

Proposition 11.39. (Associative law) *If $\langle \kappa_{ij} : (i,j) \in I \times J\rangle$ is a system of cardinals, then*

$$\prod^c_{i\in I}\left(\prod^c_{j\in J} \kappa_{ij}\right) = \prod^c_{(i,j)\in I\times J} \kappa_{ij}.$$

Proof. Note that $\prod^c_{i\in I}\left(\prod^c_{j\in J} \kappa_{ij}\right) = \left|\prod_{i\in I}\left|\prod_{j\in J} \kappa_{ij}\right|\right|$. For each $i \in I$ let f_i be a bijection from $\left|\prod_{i\in J} \kappa_{ij}\right|$ onto $\prod_{i\in J} \kappa_{ij}$ (using the axiom of choice). Now we define g mapping $\prod_{i\in I}\left|\prod_{j\in J} \kappa_{ij}\right|$ to $\prod_{(i,j)\in I\times J} \kappa_{ij}$ by setting, for any $x \in \prod_{i\in I}\left|\prod_{j\in J} \kappa_{ij}\right|$ and any $(i,j) \in I \times J$, $(g(x))_{ij} = (f_i(x_i))_j$. To show that g is one-one, suppose that $g(x) = g(y)$. Take any $(i,j) \in I \times J$. Then $(f_i(x_i))_j = (g(x))_{ij} = (g(y))_{ij} = (f_i(y_i))_j$. Since j is arbitrary, $f_i(x_i) = f_i(y_i)$. Since f_i is one-one, $x_i = y_i$. Since i is arbitrary, $x = y$. Thus g is one-one. To show that g is onto, let $z \in \prod_{(i,j)\in I\times J} \kappa_{ij}$. Define $x \in \prod_{i\in I}\left|\prod_{j\in J} \kappa_{ij}\right|$ by setting $x_i = f_i^{-1}(\langle z_{ij} : j \in J\rangle)$. Then $(g(x))_{ij} = (f_i(x_i))_j = z_{ij}$; so $g(x) = z$. □

Proposition 11.40. (Distributive law) *If $\langle \lambda_i : i \in I\rangle$ is a system of cardinals, then*

$$\kappa \cdot \sum_{i\in I} \lambda_i = \sum_{i\in I} (\kappa \cdot \lambda_i).$$

Proof. We have

$$\kappa \cdot \sum_{i\in I} \lambda_i = \left|\kappa \times \left|\bigcup_{i\in I} (\lambda_i \times \{i\})\right|\right|;$$

$$\sum_{i\in I} (\kappa \cdot \lambda_i) = \left|\bigcup_{i\in I} ((\kappa \cdot \lambda_i) \times \{i\})\right|.$$

Let f be a bijection from $\left|\bigcup_{i\in I}(\lambda_i \times \{i\})\right|$ onto $\bigcup_{i\in I}(\lambda_i \times \{i\})$. For each $i \in I$ let g_i be a bijection from $\kappa \cdot \lambda_i$ onto $\kappa \times \lambda_i$ (using the axiom of choice). Now we define a function

$$h : \kappa \times \left|\bigcup_{i\in I}(\lambda_i \times \{i\})\right| \to \bigcup_{i\in I}((\kappa \cdot \lambda_i) \times \{i\}).$$

Let $(\alpha, \beta) \in \kappa \times \left|\bigcup_{i\in I}(\lambda_i \times \{i\})\right|$. Say $f(\beta) = (\gamma, i)$ with $i \in I$ and $\gamma \in \lambda_i$. Then we set $h((\alpha, \beta)) = (g_i^{-1}(\alpha, \gamma), i)$.

To show that h is one-one, suppose that $h((\alpha, \beta)) = h((\alpha', \beta'))$. Say $f(\beta) = (\gamma, i)$ and $f(\beta') = (\gamma', j)$. Then

$$(g_i^{-1}(\alpha, \gamma), i) = h((\alpha, \beta)) = h((\alpha', \beta')) = (g_j^{-1}(\alpha', \gamma'), j).$$

It follows that $i = j$ and $g_i^{-1}(\alpha, \gamma) = g_j^{-1}(\alpha', \gamma')$, hence $(\alpha, \gamma) = (\alpha', \gamma')$. So $\alpha = \alpha'$ and $\gamma = \gamma'$. Therefore $f(\beta) = f(\beta')$, so $\beta = \beta'$. We have shown that $(\alpha, \beta) = (\alpha', \beta')$. Hence h is one-one.

To show that h is onto, let $z \in \bigcup_{i\in I}((\kappa \cdot \lambda_i) \times \{i\})$; say $i \in I$ and $z = (\alpha, i)$ with $\alpha \in \kappa \cdot \lambda_i$. Let $g(\alpha) = (\beta, \gamma)$ with $\beta \in \kappa$ and $\gamma \in \lambda_i$. Then $(\gamma, i) \in \lambda_i \times \{i\}$. Let $\delta = f^{-1}(\gamma, i)$. Then we claim that $h((\beta, \delta)) = z$. For, we have $f(\delta) = (\gamma, i)$, and hence $h((\beta, \delta)) = (g_i^{-1}(\beta, \gamma), i) = (\alpha, i) = z$. □

Theorem 11.41. (König) *Suppose that $\langle \kappa_i : i \in I\rangle$ and $\langle \lambda_i : i \in I\rangle$ are systems of cardinals such that $\lambda_i < \kappa_i$ for all $i \in I$. Then*

$$\sum_{i\in I} \lambda_i < \prod_{i\in I}^{c} \kappa_i.$$

Proof. The proof is another instance of Cantor's diagonal argument. Suppose that this is not true; thus $\prod_{i\in I}^{c} \kappa_i \le \sum_{i\in I} \lambda_i$. It follows that there is a one-one function f mapping $\prod_{i\in I}^{c} \kappa_i$ into $\{(\alpha, i) : i \in I, \alpha < \lambda_i\}$. For each $i \in I$ let

$$K_i = \{(f^{-1}(\alpha, i))_i : \alpha < \lambda_i, (\alpha, i) \in \mathrm{rng}(f)\}.$$

Clearly $K_i \subseteq \kappa_i$. Now $|K_i| \le \lambda_i < \kappa_i$, so we can choose $x_i \in \kappa_i \backslash K_i$ (using the axiom of choice). Say $f(x) = (\alpha, i)$. Then $x_i = (f^{-1}(\alpha, i))_i \in K_i$, contradiction. □

Exponentiation of cardinals

We define

$$\kappa^\lambda = |{}^\lambda\kappa|.$$

The following simple proposition will be useful.

Proposition 11.42. *If $|A| = |A'|$ and $|B| = |B'|$, then $|{}^AB| = |{}^{A'}B'|$.*

Proof. Let $f : A \to A'$ and $g : B \to B'$ be bijections. For any $x \in {}^AB$ let $F(x) = g \circ x \circ f^{-1}$. Thus $F(x) \in {}^{A'}B'$. If $F(x) = F(y)$, then $x = g^{-1} \circ F(x) \circ f = g^{-1} \circ F(y) \circ f = y$. So F is one-one. It is onto, since given $z \in {}^{A'}B'$ we have $g^{-1} \circ z \circ f \in {}^AB$, and $F(g^{-1} \circ z \circ f) = z$. $\square$

The elementary arithmetic of exponentiation is summarized in the following proposition:

Proposition 11.43. *(i)* $\kappa^0 = 1$.

(ii) If $\kappa \neq 0$, *then* $0^\kappa = 0$.

(iii) $\kappa^1 = \kappa$.

(iv) $1^\kappa = 1$.

(v) $\kappa^2 = \kappa \cdot \kappa$.

(vi) $\kappa^\lambda \cdot \kappa^\mu = \kappa^{\lambda+\mu}$.

(vii) $(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$.

(viii) $(\kappa^\lambda)^\mu = \kappa^{\lambda \cdot \mu}$.

(ix) If $\kappa \leq \lambda \neq 0$ *and* $\mu \leq \nu$, *then* $\kappa^\mu \leq \lambda^\nu$.

(x) $\prod^c_{i \in I} \kappa = \kappa^{|I|}$.

(xi) $\kappa^{\sum_{i \in I} \lambda_i} = \prod^c_{i \in I} \kappa^{\lambda_i}$.

(xii) $\left(\prod^c_{i \in I} \kappa_i\right)^\lambda = \prod^c_{i \in I} \kappa_i^\lambda$.

Proof. (i): $\kappa^0 = |{}^0\kappa|$. Now ${}^0\kappa = \{\emptyset\}$, so $\kappa^0 = 1$.

(ii): if $\kappa \neq 0$, then $0^\kappa = |{}^\kappa 0|$ and ${}^\kappa 0 = \emptyset$, so $\kappa^0 = 0$.

(iii): $\kappa^1 = |{}^1\kappa|$, and ${}^1\kappa = \{\{(0, \alpha)\} : \alpha < \kappa\}$. The mapping $\alpha \mapsto \{(0, \alpha)\}$ is a bijection from κ onto $\{\{(0, \alpha)\} : \alpha < \kappa\}$.

(iv): $1^\kappa = |{}^\kappa 1|$, and ${}^\kappa 1$ has only one member, the function with domain κ and value always 0.

(v): $\kappa^2 = |{}^2\kappa|$ and $\kappa \cdot \kappa = |\kappa \times \kappa|$. For any $x \in {}^2\kappa$ let $f(x) = (x(0), x(1))$. Clearly f is a bijection from ${}^2\kappa$ onto $\kappa \times \kappa$.

(vi): $\kappa^\lambda \cdot \kappa^\mu = ||{}^\lambda\kappa| \times |{}^\mu\kappa|| = |{}^\lambda\kappa \times {}^\mu\kappa|$, using Proposition 11.29. Also, $\kappa^{\lambda+\mu} = |{}^{|(\lambda \times \{0\}) \cup (\mu \times \{1\})|}\kappa| = |{}^{(\lambda \times \{0\}) \cup (\mu \times \{1\})}\kappa|$, using Proposition 11.42. Hence it suffices to define a bijection from ${}^\lambda\kappa \times {}^\mu\kappa$ to ${}^{(\lambda \times \{0\}) \cup (\mu \times \{1\})}\kappa$. If $x \in {}^\lambda\kappa$ and $y \in {}^\mu\kappa$, define $(h(x, y))((\alpha, 0)) = x(\alpha)$ for any $\alpha \in \lambda$, and $(h(x, y))((\alpha, 1)) = y(\alpha)$ for any $\alpha \in \mu$. To show that h is one-one, suppose that $x, x' \in {}^\lambda\kappa$, $y, y' \in {}^\mu\kappa$, and $h(x, y) = h(x', y')$. To show that $x = x'$, take any $\alpha \in \lambda$. Then $x(\alpha) = (h(x, y))((\alpha, 0)) = (h(x', y'))((\alpha, 0)) = x'(\alpha)$. So $x = x'$. Similarly $y = y'$, so h is one-one. To show that h is onto, take any $z \in {}^{(\lambda \times \{0\}) \cup (\mu \times \{1\})}\kappa$. Define $x \in {}^\lambda\kappa$ by setting $x(\alpha) = z((\alpha, 0))$ for any $\alpha \in \lambda$, and define $y \in {}^\mu\kappa$ by setting $y(\alpha) = z((\alpha, 1))$ for any $\alpha \in \mu$. Then $h(x, y) = z$, since for any $\alpha \in \lambda$ we have $(h(x, y))((\alpha, 0)) = x(\alpha) = z((\alpha, 0))$ and for any $\alpha \in \mu$ we have $(h(x, y))((\alpha, 1)) = y(\alpha) = z((\alpha, 1))$.

(vii): $(\kappa \cdot \lambda)^\mu = |{}^\mu|\kappa \times \lambda|| = |{}^\mu(\kappa \times \lambda)|$ using Proposition 11.42. $\kappa^\mu \cdot \lambda^\mu = ||{}^\mu\kappa| \times |{}^\mu\lambda|| = |({}^\mu\kappa) \times ({}^\mu\lambda)|$ using Proposition 11.29. Hence it suffices to define a bijection from ${}^\mu(\kappa \times \lambda)$ onto $({}^\mu\kappa) \times ({}^\mu\lambda)$. For any $x \in \mu(\kappa \times \lambda)$, define $f(x) = (g(x), h(x))$, where $g(x)$ is the member of ${}^\mu\kappa$ such that $(g(x))(\alpha) = 1^{\text{st}}(x(\alpha))$ for any $\alpha \in \mu$, and $h(x)$ is the member of ${}^\mu\lambda$ such that $(h(x))(\alpha) = 2^{\text{nd}}(x(\alpha))$ for any $\alpha \in \mu$. To show that f is one-one, suppose that $f(x) = f(y)$. Then $g(x) = g(y)$, so for any $\alpha \in \mu$ we have $1^{\text{st}}(x(\alpha)) = (g(x))(\alpha) = (g(y))(\alpha) = 1^{\text{st}}(y(\alpha))$. Similarly, $2^{\text{nd}}(x(\alpha)) = 2^{\text{nd}}(y(\alpha))$ for any $\alpha \in \mu$. Hence $x(\alpha) = y(\alpha)$ for any $\alpha \in \mu$. Thus

$x = y$. So f is one-one. To show that f is onto, suppose that $(u, v) \in ({}^{\mu}\kappa) \times ({}^{\mu}\lambda)$. Define $x \in {}^{\mu}(\kappa \times \lambda)$ by setting $x(\alpha) = (u(\alpha), v(\alpha))$ for any $\alpha \in \mu$. Say $f(x) = (g(x), h(x))$. Then $(g(x))(\alpha) = 1^{\rm st}(x(\alpha)) = u(\alpha)$ for any $\alpha \in \mu$; so $g(x) = u$. Similarly, $h(x) = v$. So $f(x) = (u, v)$, as desired.

(viii): $(\kappa^{\lambda})^{\mu} = |{}^{\mu}|{}^{\lambda}\kappa|| = |{}^{\mu}({}^{\lambda}\kappa)|$, using Proposition 11.42. $\kappa^{\lambda\cdot\mu} = |{}^{|\lambda\times\mu|}\kappa| = |{}^{\lambda\times\mu}\kappa|$ using Proposition 11.42. Hence it suffices to define a bijection from ${}^{\mu}({}^{\lambda}\kappa)$ onto ${}^{\lambda\times\mu}\kappa$. For any $x \in {}^{\mu}({}^{\lambda}\kappa)$ and any $\alpha \in \lambda$ and $\beta \in \mu$, let $(f(x))(\alpha, \beta) = (x(\alpha))(\beta)$. To show that f is one-one, suppose that $x, y \in {}^{\mu}({}^{\lambda}\kappa)$ and $f(x) = f(y)$. Take any $\alpha \in \lambda$ and $\beta \in \mu$. Then $(x(\alpha))(\beta) = (f(x))(\alpha, \beta) = (f(y))(\alpha, \beta) = (y(\alpha))(\beta)$. This being true for all $\beta \in \mu$, it follows that $x(\alpha) = y(\alpha)$. This is true for all $\alpha \in \lambda$, so $x = y$.

To see that f is onto, suppose that $z \in {}^{\lambda\times\mu}\kappa$. Define $x \in {}^{\lambda}({}^{\mu}\kappa)$ by setting $(x(\alpha))(\beta) = z(\alpha, \beta)$ for any $\alpha \in \lambda$ and $\beta \in \mu$. Then for any $\alpha \in \lambda$ and $\beta \in \mu$ we have $(f(x))(\alpha, \beta) = (x(\alpha))(\beta) = z(\alpha, \beta)$. So $f(x) = z$.

(ix): Assume that $\kappa \leq \lambda \neq 0$ and $\mu \leq \nu$. For every $x \in {}^{\mu}\kappa$ let $x+ \in {}^{\nu}\lambda$ be an extension of x. Then the mapping $x \mapsto x+$ is a one-one function from ${}^{\mu}\kappa$ into $\nu\lambda$. So (ix) follows.

(x): $\prod^{c}_{i\in I} \kappa = \left|\prod_{i\in i} \kappa\right|$ and $\kappa^{|I|} = |{}^{|I|}\kappa| = |{}^{I}\kappa|$ using Proposition 11.42. Note that actually $\prod_{i\in i} \kappa = {}^{I}\kappa$.

(xi): $\kappa^{\sum_{i\in I}\lambda_i} = \left|{}^{|\bigcup_{i\in I}(\lambda_i\times\{i\})|}\kappa\right| = \left|{}^{\bigcup_{i\in I}(\lambda_i\times\{i\})}\kappa\right|$, using Proposition 11.42. Also, $\prod^{c}_{i\in I} \kappa^{\lambda_i} = |\prod_{i\in I} |{}^{\lambda_i}\kappa|| = |\prod_{i\in I} {}^{\lambda_i}\kappa|$, using Proposition 11.37(i). Hence it suffices to define a bijection from ${}^{\bigcup_{i\in I}(\lambda_i\times\{i\})}\kappa$ onto $\prod_{i\in I} {}^{\lambda_i}\kappa$. Take any $x \in {}^{\bigcup_{i\in I}(\lambda_i\times\{i\})}\kappa$, $i \in I$, and $\alpha \in \lambda_i$. Define $(f(x))_i(\alpha) = x(\alpha, i)$. Then f is one-one. For, suppose that $f(x) = f(y)$. Take any $i \in I$ and $\alpha \in \lambda_i$. Then $x(\alpha, i) = (f(x))_i(\alpha) = (f(y))_i(\alpha) = y(\alpha, i)$. Hence $x = y$. To show that f is onto, let $z \in \prod_{i\in I} {}^{\lambda_i}\kappa$. Define $x \in {}^{\bigcup_{i\in I}(\lambda_i\times\{i\})}\kappa$ by setting, for any $i \in I$ and $\alpha \in \lambda_i$, $x(\alpha, i) = (z(i))(\alpha)$. Then for any $i \in I$ and $\alpha \in \lambda_i$, $(f(x))_i(\alpha) = x(\alpha, i) = (z(i))(\alpha)$. So $f(x) = z$.

(xii): $\left(\prod^{c}_{i\in I} \kappa_i\right)^{\lambda} = \left|{}^{\lambda}\left|\prod_{i\in I}\kappa_i\right|\right| = \left|{}^{\lambda}\prod_{i\in I}\kappa_i\right|$, using Proposition 11.42. Also, we have $\prod^{c}_{i\in I} \kappa_i^{\lambda} = \left|\prod_{i\in I} |{}^{\lambda}\kappa_i|\right| = \left|\prod_{i\in I} {}^{\lambda}\kappa_i\right|$, using Proposition 11.37(i). Hence it suffices to define a bijection from ${}^{\lambda}\prod_{i\in I}\kappa_i$ onto $\prod_{i\in I} {}^{\lambda}\kappa_i$. For any $x \in {}^{\lambda}\prod_{i\in I}\kappa_i$, $i \in I$, and $\alpha \in \lambda$, let $(f(x))_i(\alpha) = (x(\alpha))_i$. Then f is one-one. For, assume that $f(x) = f(y)$. Then for any $\alpha \in \lambda$ and $i \in I$ we have $(x(\alpha))_i = (f(x))_i(\alpha) = (f(y))_i(\alpha) = (y(\alpha))_i$. So $x = y$. Also, f is onto. For, suppose that $z \in \prod_{i\in I} {}^{\lambda}\kappa_i$. Define $x \in {}^{\lambda}\prod_{i\in I}\kappa_i$ by setting $(x(\alpha))_i = z_i(\alpha)$ for any $\alpha \in \lambda$ and $i \in I$. Then for any $i \in I$ and $\alpha \in \lambda$ we have $(f(x))_i(\alpha) = (x(\alpha))_i = z_i(\alpha)$. So $f(x) = z$. □

Proposition 11.44. *If $m, n \in \omega$, then $m^n \in \omega$, and m^n has the same meaning in the ordinal or cardinal sense.*

Proof. For this proof, denote ordinal exponentiation by $\exp(m, n)$. With m fixed, we show that $m^n \in \omega$ and $m^n = \exp(m, n)$ by induction on n. We have $m^0 = 1$ by Proposition 11.43(i), and $\exp(m, 0) = 1$ also. Now assume that $m^n \in \omega$ and $m^n = \exp(m, n)$. Then $m^{n+1} = m^n \cdot m^1 = m^n \cdot m$ by Proposition 11.43(vi),(iii). We also have $\exp(m, n + 1) = \exp(m, n) \cdot m$, so the inductive hypothesis gives the desired conclusion. □

Proposition 11.45. $|\mathscr{P}(A)| = 2^{|A|}$.

For each $X \subseteq A$ define $\chi_X \in {}^A 2$ by setting

$$\chi_X(a) = \begin{cases} 1 & \text{if } a \in X, \\ 0 & \text{otherwise.} \end{cases}$$

[This is the *characteristic function* of X.] It is easy to check that χ is a bijection from $\mathscr{P}(A)$ onto ${}^A 2$. $\square$

The calculation of exponentiation is not as simple as that for addition and multiplication. The following result gives one of the most useful facts about exponentiation, however.

Theorem 11.46. *If $2 \leq \kappa \leq \lambda \geq \omega$, then $\kappa^\lambda = 2^\lambda$.*

Proof. Note that each function $f : \lambda \to \lambda$ is a subset of $\lambda \times \lambda$. Hence ${}^\lambda\lambda \subseteq \mathscr{P}(\lambda \times \lambda)$, and so $\lambda^\lambda \leq |\mathscr{P}(\lambda \times \lambda)|$. Therefore,

$$2^\lambda \leq \kappa^\lambda \leq \lambda^\lambda \leq |\mathscr{P}(\lambda \times \lambda)| = 2^{\lambda \cdot \lambda} = 2^\lambda;$$

so all the entries in this string of inequalities are equal, and this gives $\kappa^\lambda = 2^\lambda$. $\square$

Cofinality, and regular and singular cardinals

Further cardinal arithmetic depends on the notion of cofinality. For later purposes we define a rather general version of this notion. Let $(P, <)$ be a partial order. A subset X of P is *dominating* iff for every $p \in P$ there is an $x \in X$ such that $p \leq x$. The *cofinality* of P is the smallest cardinality of a dominating subset of P. We denote this cardinal by $\mathrm{cf}(P)$.

A subset X of P is *unbounded* iff there does not exist a $p \in P$ such that $x \leq p$ for all $x \in X$. If P is simply ordered without largest element, then these notions—dominating and unbounded—coincide. In fact, suppose that X is dominating but not unbounded. Since X is not unbounded, choose $p \in P$ such that $x \leq p$ for all $x \in X$. Since P does not have a largest element, choose $q \in P$ such that $p < q$. Then because X is dominating, choose $x \in X$ such that $q \leq x$. Then $q \leq x \leq p < q$, contradiction. Thus X dominating implies that X is unbounded. Now suppose that Y is unbounded but not dominating. Since Y is not dominating, there is a $p \in P$ such that $p \not\leq x$, for all $x \in Y$. Since P is a simple order, it follows that $x < p$ for all $x \in Y$. This contradicts Y being unbounded.

We apply these notions to infinite cardinals, which are simply ordered sets with no last element. Obviously any infinite cardinal κ is a dominating subset of itself; so $\mathrm{cf}(\kappa) \leq \kappa$. A cardinal κ is *regular* iff κ is infinite and $\mathrm{cf}(\kappa) = \kappa$. An infinite cardinal that is not regular is called *singular*.

Theorem 11.47. *For every infinite cardinal κ, the cardinal κ^+ is regular.*

Proof. Suppose that $\Gamma \subseteq \kappa^+$, Γ is unbounded in κ^+, and $|\Gamma| < \kappa^+$. Hence

$$\kappa^+ = \left|\bigcup_{\gamma \in \Gamma} \gamma\right| \leq \sum_{\gamma \in \Gamma} |\gamma| \leq \sum_{\gamma \in \Gamma} \kappa = \kappa \cdot \kappa = \kappa,$$

contradiction. The first equality here holds because Γ is unbounded in κ^+ and κ^+ is a limit ordinal. □

This theorem almost tells the full story about when a cardinal is regular. Examples of singular cardinals are $\aleph_{\omega+\omega}$ and $\aleph_{\omega_1}$. But it is conceivable that there are regular cardinals not covered by Theorem 11.47. An uncountable regular limit cardinal is said to be *weakly inaccessible.* A cardinal κ is said to be *inaccessible* if it is regular, uncountable, and has the property that for any cardinal $\lambda < \kappa$, also $2^\lambda < \kappa$. Clearly every inaccessible cardinal is also weakly inaccessible. Under GCH, the two notions coincide. If it is consistent that there are weak inaccessibles, then it is consistent that 2^ω is weakly inaccessible; but of course it definitely is not inaccessible. It is consistent with ZFC that there are no uncountable weak inaccessibles at all. These consistency results will be proved later in these notes. It is reasonable to postulate the existence of inaccessibles, and they are useful in some situations. In fact, the subject of *large cardinals* is one of the most studied in contemporary set theory, with many spectacular results.

Theorem 11.48. *Suppose that $(A,<)$ is a simple order with no largest element. Then there is a strictly increasing function $f : \mathrm{cf}(A) \to A$ such that $\mathrm{rng}(f)$ is unbounded in A.*

Proof. Let X be a dominating subset of A of size $\mathrm{cf}(A)$, and let g be a bijection from $\mathrm{cf}(A)$ onto X. We define a function $f : \mathrm{cf}(A) \to X$ by recursion, as follows. If $f(\beta) \in X$ has been defined for all $\beta < \alpha$, where $\alpha < \mathrm{cf}(A)$, then $\{f(\beta) : \beta < \alpha\}$ has size less than $\mathrm{cf}(A)$, and hence it is not dominating. Hence there is an $a \in A$ such that $f(\beta) < a$ for all $\beta < \alpha$. We let $f(\alpha)$ be an element of X such that $a, g(\alpha) \leq f(\alpha)$.

Clearly f is strictly increasing. If $a \in A$, choose $\alpha < \mathrm{cf}(A)$ such that $a \leq g(\alpha)$. Then $a \leq f(\alpha)$. □

Proposition 11.49. *Suppose that $(A,<)$ is a simple ordering with no largest element. Then $\mathrm{cf}(\mathrm{cf}(A)) = \mathrm{cf}(A)$.*

Proof. Clearly $\mathrm{cf}(\alpha) \leq \alpha$ for any ordinal α; in particular, $\mathrm{cf}(\mathrm{cf}(A)) \leq \mathrm{cf}(A)$. Now by Theorem 11.48, let $f : \mathrm{cf}(A) \to A$ be strictly increasing with $\mathrm{rng}(f)$ unbounded in A. Now $\mathrm{cf}(A)$ is an infinite cardinal, and hence it is a limit ordinal by Proposition 11.8. Hence Theorem 11.48 again applies, and we can let $g : \mathrm{cf}(\mathrm{cf}(A)) \to \mathrm{cf}(A)$ be strictly increasing with $\mathrm{rng}(g)$ unbounded in $\mathrm{cf}(A)$. Clearly $f \circ g : \mathrm{cf}(\mathrm{cf}(A)) \to A$ is strictly increasing. We claim that $\mathrm{rng}(f \circ g)$ is unbounded in A. For, given $a \in A$, choose $\alpha < \mathrm{cf}(A)$ such that $a \leq f(\alpha)$, and then choose $\beta < \mathrm{cf}(\mathrm{cf}(A))$ such that $\alpha \leq g(\beta)$. Then $a \leq f(\alpha) \leq f(g(\beta))$, proving the claim. It follows that $\mathrm{cf}(A) \leq \mathrm{cf}(\mathrm{cf}(A))$. □

Proposition 11.50. *If κ is a regular cardinal, $\Gamma \subseteq \kappa$, and $|\Gamma| < \kappa$, then $\bigcup \Gamma < \kappa$.*

Proof. Since $\mathrm{cf}(\kappa) = \kappa$, from the definition of cf it follows that Γ is bounded in κ. Hence there is an $\alpha < \kappa$ such that $\gamma \leq \alpha$ for all $\gamma \in \Gamma$. So $\bigcup \Gamma \leq \alpha < \kappa$. □

Proposition 11.51. *If A is a linearly ordered set with no greatest element, κ is a regular cardinal, and $f : \kappa \to A$ is strictly increasing with $\mathrm{rng}(f)$ unbounded in A, then $\kappa = \mathrm{cf}(A)$.*

Proof. By the definition of cf we have $\mathrm{cf}(A) \leq \kappa$. Suppose that $\mathrm{cf}(A) < \kappa$. By Theorem 11.48 let $g : \mathrm{cf}(A) \to A$ be strictly increasing with $\mathrm{rng}(g)$ unbounded in A. For each $\alpha < \mathrm{cf}(\alpha)$ choose $\beta_\alpha < \kappa$ such that $g(\alpha) \leq f(\beta_\alpha)$. Then $\{\beta_\alpha : \alpha < \mathrm{cf}(A)\} \subseteq \kappa$ and $|\{\beta_\alpha : \alpha < \mathrm{cf}(A)\}| < \kappa$, so by Proposition 11.50, $\bigcup_{\alpha<\mathrm{cf}(A)} \beta_\alpha < \kappa$. Let $\gamma < \kappa$ be such that $\beta_\alpha < \gamma$ for all $\alpha < \mathrm{cf}(A)$. Then $f(\gamma)$ is a bound for $\mathrm{rng}(g)$, contradiction. □

Proposition 11.52. *A cardinal κ is regular iff for every system $\langle \lambda_i : i \in I\rangle$ of cardinals less than κ, with $|I| < \kappa$, one also has $\sum_{i\in I} \lambda_i < \kappa$.*

Proof. $\Rightarrow$: Assume that κ is regular, $\langle \lambda_i : i \in I\rangle$ is a system of cardinals less than κ, and $|I| < \kappa$. We have $\{\lambda_i : i \in I\} \subseteq \kappa$ and $|\{\lambda_i : i \in I\}| \leq |I|$, so by Proposition 11.50, $\bigcup_{i\in I} \lambda_i < \kappa$. Hence

$$\sum_{i\in I} \lambda_i \leq \sum_{i\in I} \bigcup_{i\in I} \lambda_i = |I| \cdot \bigcup_{i\in I} \lambda_i < \kappa.$$

$\Leftarrow$: Assume the indicated condition. Suppose that $\Gamma \subseteq \kappa$ and $|\Gamma| < \kappa$. Then $\langle |\alpha| : \alpha \in \Gamma\rangle$ is a system of cardinals less than κ, and $|\Gamma| < \kappa$. Hence $|\bigcup \Gamma| \leq \sum_{\lambda\in\Gamma} |\lambda| < \kappa$, so also $\bigcup \Gamma < \kappa$. Thus κ is regular. □

Proposition 11.53. *If κ is an infinite singular cardinal, then there is a strictly increasing sequence $\langle \lambda_\alpha : \alpha < \mathrm{cf}(\kappa)\rangle$ of infinite successor cardinals such that $\kappa = \sum_{\alpha<\mathrm{cf}(\kappa)} \lambda_\alpha$.*

Proof. By Theorem 11.48, let $f : \mathrm{cf}(\kappa) \to \kappa$ be strictly increasing such that $\mathrm{rng}(f)$ is unbounded in κ. We define the desired sequence by recursion. Suppose that $\lambda_\beta < \kappa$ has been defined for all $\beta < \alpha$, with $\alpha < \mathrm{cf}(\kappa)$. Then $\bigcup_{\beta<\alpha} \lambda_\beta < \kappa$ by the definition of cofinality. So also

$$\left(\max\left(f(\alpha), \bigcup_{\beta<\alpha} \lambda_\beta\right)\right)^+ < \kappa,$$

and we define λ_α to be this cardinal.

Now $f(\delta) \leq \sum_{\alpha<\mathrm{cf}(\kappa)} \lambda_\alpha$ for each $\delta < \mathrm{cf}(\kappa)$, so

$$\kappa = \bigcup_{\delta\in\mathrm{cf}(\kappa)} f(\delta) \leq \sum_{\alpha<\mathrm{cf}(\kappa)} \lambda_\alpha \leq \sum_{\alpha<\mathrm{cf}(\kappa)} \kappa = \kappa \cdot \mathrm{cf}(\kappa) = \kappa.$$ □

The main theorem of cardinal arithmetic

Now we return to the general treatment of cardinal arithmetic.

Theorem 11.54. (König) *If κ is infinite and* $\mathrm{cf}(\kappa) \leq \lambda$, *then $\kappa^\lambda > \kappa$.*

Proof. If κ is regular, then $\kappa^\lambda \geq \kappa^\kappa = 2^\kappa > \kappa$. So, assume that κ is singular. Then by Theorem 11.53 there is a system $\langle \mu_\alpha : \alpha < \mathrm{cf}(\kappa)\rangle$ of nonzero cardinals such that each μ_α is less than κ, and $\sum_{\alpha<\mathrm{cf}(\kappa)} \mu_\alpha = \kappa$. Hence, using Theorem 11.41,

$$\kappa = \sum_{\alpha<\mathrm{cf}(\kappa)} \mu_\alpha < \prod_{\alpha<\mathrm{cf}(\kappa)} \kappa = \kappa^{\mathrm{cf}(\kappa)} \leq \kappa^\lambda.$$ □

Corollary 11.55. *For λ infinite we have* $\mathrm{cf}(2^\lambda) > \lambda$.

Proof. Suppose that $\mathrm{cf}(2^\lambda) \leq \lambda$. Then by Theorem 11.54, $(2^\lambda)^\lambda > 2^\lambda$. But $(2^\lambda)^\lambda = 2^{\lambda\cdot\lambda} = 2^\lambda$, contradiction. □

We can now verify a statement made earlier about possibilities for $|\mathscr{P}(\omega)|$. Since $|\mathscr{P}(\omega)| = 2^\omega$, the corollary says that $\mathrm{cf}(|\mathscr{P}(\omega)) > \omega$. So this implies that $|\mathscr{P}(\omega)|$ cannot be $\aleph_\omega$ or $\aleph_{\omega+\omega}$. Here $\omega+\omega$ is the ordinal sum of ω with ω. It rules out many other possibilities of this sort.

We now prove a lemma needed for the last major theorem of this subsection, which says how to compute exponents (in a way).

Lemma 11.56. *If κ is a limit cardinal and $\lambda \geq \mathrm{cf}(\kappa)$, then*

$$\kappa^\lambda = \left(\bigcup_{\substack{\mu<\kappa \\ \mu \text{ a cardinal}}} \mu^\lambda\right)^{\mathrm{cf}(\kappa)}.$$

Proof. By Theorem 11.48, let $\gamma : \mathrm{cf}(\kappa) \to \kappa$ be strictly increasing with $\mathrm{rng}(\gamma)$ unbounded in κ, and with $0 < \gamma_0$. We define $F : {}^\lambda\kappa \to \prod_{\alpha<\mathrm{cf}(\kappa)} {}^\lambda\gamma_\alpha$ as follows. If $f \in {}^\lambda\kappa$, $\alpha < \mathrm{cf}(\kappa)$, and $\beta < \lambda$, then

$$((F(f))_\alpha)_\beta = \begin{cases} f(\beta) & \text{if } f(\beta) < \gamma_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Now F is a one-one function. For, if $f, g \in {}^\lambda\kappa$ and $f \neq g$, say $\beta < \lambda$ and $f(\beta) \neq g(\beta)$. Choose $\alpha < \mathrm{cf}\kappa$ such that $f(\beta)$ and $g(\beta)$ are both less than γ_α. Then $((F(f))_\alpha)_\beta = f(\beta) \neq g(\beta) = ((F(g))_\alpha)_\beta$, from which it follows that $F(f) \neq F(g)$. Since F is one-one,

$$\begin{aligned}
\kappa^\lambda = |{}^\lambda\kappa| &\leq \left|\prod_{\alpha<\mathrm{cf}(\kappa)} {}^\lambda\gamma_\alpha\right| \\
&\leq \left|\prod_{\alpha<\mathrm{cf}(\kappa)} \left(\bigcup_{\substack{\mu<\kappa \\ \mu \text{ a cardinal}}} {}^\lambda\mu\right)\right| \\
&= \left(\bigcup_{\substack{\mu<\kappa \\ \mu \text{ a cardinal}}} \mu^\lambda\right)^{\mathrm{cf}(\kappa)} \\
&\leq (\kappa^\lambda)^{\mathrm{cf}(\kappa)} = \kappa^{\lambda\cdot\mathrm{cf}(\kappa)} = \kappa^\lambda,
\end{aligned}$$

and the lemma follows. □

The following theorem is not needed for the main result, but it is a classical result about exponentiation.

Theorem 11.57. (Hausdorff) *If κ and λ are infinite cardinals, then $(\kappa^+)^\lambda = \kappa^\lambda \cdot \kappa^+$.*

Proof. If $\kappa^+ \le \lambda$, then both sides are equal to 2^λ. Suppose that $\lambda < \kappa^+$. Then

$$\begin{aligned}(\kappa^+)^\lambda = |{}^\lambda(\kappa^+)| &= \left|\bigcup_{\alpha<\kappa^+} {}^\lambda\alpha\right| \\ &\le \sum_{\alpha<\kappa^+} |\alpha|^\lambda \le \kappa^\lambda \cdot \kappa^+ \le (\kappa^+)^\lambda,\end{aligned}$$

as desired. □

Here is the promised theorem giving computation rules for exponentiation. It essentially reduces the computation of κ^λ to two special cases: 2^λ, and $\kappa^{\mathrm{cf}(\kappa)}$. Generalizations of the results mentioned about the continuum hypothesis give a pretty good picture of what can happen to 2^λ. The case of $\kappa^{\mathrm{cf}(\kappa}$ is more complicated, and there is still work being done on what the possibilities here are. Shelah used his PCF theory to prove that $\aleph_\omega^{\aleph_0} \le 2^{\aleph_0} + \aleph_{\omega_4}$.

Theorem 11.58. (main theorem of cardinal arithmetic) *Let κ and λ be cardinals with $2 \le \kappa$ and $\lambda \ge \omega$. Then*

(i) If $\kappa \le \lambda$, then $\kappa^\lambda = 2^\lambda$.

(ii) If κ is infinite and there is a $\mu < \kappa$ such that $\mu^\lambda \ge \kappa$, then $\kappa^\lambda = \mu^\lambda$.

(iii) Assume that κ is infinite and $\mu^\lambda < \kappa$ for all $\mu < \kappa$. Then $\lambda < \kappa$, and:

(a) if $\mathrm{cf}(\kappa) > \lambda$, then $\kappa^\lambda = \kappa$;

(b) if $\mathrm{cf}(\kappa) \le \lambda$, then $\kappa^\lambda = \kappa^{\mathrm{cf}(\kappa)}$.

Proof. (i) has already been noted, in Theorem 11.46. Under the hypothesis of (ii),

$$\kappa^\lambda \le (\mu^\lambda)^\lambda = \mu^\lambda \le \kappa^\lambda,$$

as desired.

Now assume the hypothesis of (iii). In particular, $2^\lambda < \kappa$, so of course $\lambda < \kappa$. Next, assume the hypothesis of (iii)(a): $\mathrm{cf}(\kappa) > \lambda$. Then

$$\begin{aligned}\kappa^\lambda = |{}^\lambda\kappa| &= \left|\bigcup_{\alpha<\kappa} {}^\lambda\alpha\right| \quad (\text{since } \lambda < \mathrm{cf}(\kappa)) \\ &\le \sum_{\alpha<\kappa} |\alpha|^\lambda \le \kappa,\end{aligned}$$

giving the desired result.

Finally, assume the hypothesis of (iii)(b): $\mathrm{cf}(\kappa) \le \lambda$. Since $\lambda < \kappa$, it follows that κ is singular, so in particular it is a limit cardinal. Then by Lemma 11.56,

$$\kappa^\lambda = \left(\bigcup_{\substack{\mu<\kappa \\ \mu \text{ a cardinal}}} \mu^\lambda\right)^{\mathrm{cf}(\kappa)} \le \kappa^{\mathrm{cf}(\kappa)} \le \kappa^\lambda.$$

□

In theory one can now compute κ^λ for infinite κ, λ as follows. If $\kappa \le \lambda$, then $\kappa^\lambda = 2^\lambda$. Suppose that $\kappa > \lambda$. Let κ' be minimum such that $(\kappa')^\lambda = \kappa^\lambda$. Then $\forall \mu < \kappa'[\mu^\lambda < \kappa']$. In fact, if $\mu < \kappa'$ and $\mu^\lambda \ge \kappa'$, then $(\kappa')^\lambda \le (\mu^\lambda)^\lambda = \mu^{\lambda \cdot \lambda} = \mu^\lambda < \kappa^\lambda = (\kappa')^\lambda$, contradiction. Now $(\kappa')^\lambda$ is computed by 11.58(iii).

Under the generalized continuum hypothesis the computation of exponents is very simple:

Corollary 11.59. *Assume GCH, and suppose that κ and λ are cardinals with $2 \le \kappa$ and λ infinite. Then:*

(i) If $\kappa \le \lambda$, then $\kappa^\lambda = \lambda^+$.

(ii) If $\mathrm{cf}(\kappa) \le \lambda < \kappa$, then $\kappa^\lambda = \kappa^+$.

(iii) If $\lambda < \mathrm{cf}(\kappa)$, then $\kappa^\lambda = \kappa$.

Proof. (i) is immediate from Theorem 11.58(i). For (ii), assume that $\mathrm{cf}(\kappa) \le \lambda < \kappa$. Then κ is a limit cardinal, and so for each $\mu < \kappa$ we have $\mu^\lambda \le (\max(\mu, \lambda))^+ < \kappa$; hence by Theorem 11.58(iii)(b) and Theorem 11.54 we have $\kappa^\lambda = \kappa^{\mathrm{cf}(\kappa)} > \kappa$; since $\kappa^{\mathrm{cf}(\kappa)} \le \kappa^\kappa = \kappa^+$, it follows that $\kappa^\lambda = \kappa^+$. For (iii), assume that $\lambda < \mathrm{cf}(\kappa)$. If there is a $\mu < \kappa$ such that $\mu^\lambda \ge \kappa$, then by Theorem 11.58(ii), $\kappa^\lambda = \mu^\lambda \le (\max(\lambda, \mu))^+ \le \kappa$, as desired. If $\mu^\lambda < \kappa$ for all $\mu < \kappa$, then $\kappa^\lambda = \kappa$ by Theorem 11.58(iii)(a). $\square$

Proposition 11.60. *There are sets A, B, and a one-one function $f : A \to B$ such that $|A| = |B|$ and f is not onto.*

Proof. Let $A = B = \omega$ and $f(m) = m + 1$ for all $m \in \omega$. $\square$

Proposition 11.61. *There are sets A, B, and an onto function $f : A \to B$ such that $|A| = |B|$ and f is not one-one.*

Proof. Let $A = B = \omega$ and $f(0) = 0$, $f(m + 1) = m$ for all $m \in \omega$. $\square$

Proposition 11.62. *The restriction $\lambda \ne 0$ is necessary in Proposition 11.43(ix).*

Proof. Take $\kappa = \lambda = 0$, $\mu = 0$, $\nu = 1$. Then $\kappa^\mu = 0^0 = 1$ and $\lambda^\nu = 0^1 = 0$. $\square$

Proposition 11.63. (ZF) *Let $F : \mathscr{P}(A) \to \mathscr{P}(A)$, and assume that for all $X, Y \subseteq A$, if $X \subseteq Y$, then $F(X) \subseteq F(Y)$. Let $\mathscr{A} = \{X : X \subseteq A$ and $X \subseteq F(X)\}$, and set $X_0 = \bigcup_{X \in \mathscr{A}} X$. Then $X_0 \subseteq F(X_0)$.*

Proof. For any $Y \in \mathscr{A}$ we have $Y \subseteq X_0$, and hence $Y \subseteq F(Y) \subseteq F(X_0))$, so $X_0 \subseteq F(X_0)$. $\square$

Proposition 11.64. (ZF) *Under the assumptions of Proposition 11.63 we actually have $X_0 = F(X_0)$.*

Proof. By Proposition 11.63, $X_0 \subseteq F(X_0)$, so $F(X_0) \subseteq F(F(X_0))$, hence $F(X_0) \in \mathscr{A}$, hence $F(X_0) \subseteq X_0$; together with Proposition 11.63 this proves that $X_0 = F(X_0)$. $\square$

Proposition 11.65. (ZF) *Suppose that $f : A \to B$ is one-one and $g : B \to A$ is also one-one. For every $X \subseteq A$ let $F(X) = A\backslash g[B\backslash f[X]]$. Then for all $X, Y \subseteq A$, if $X \subseteq Y$ then $F(X) \subseteq F(Y)$.*

Proof. We have $f[X] \subseteq f[Y]$, so $B\backslash f[Y] \subseteq B\backslash f[X]$, hence $g[B\backslash f[Y]] \subseteq g[B\backslash f[X]]$, hence $F(X) = A\backslash g[B\backslash f[X]] \subseteq A\backslash g[B\backslash f[Y]] = F(Y)$. □

Proposition 11.66. (ZF) (Cantor-Schröder-Bernstein theorem) *Assume that f and g are as in Proposition 11.65, and choose F as in that proposition. Let X_0 be as in Proposition 11.63. Then $A\backslash X_0 \subseteq rng(g)$. Then define $h : A \to B$ by setting, for any $a \in A$,*

$$h(a) = \begin{cases} f(a) & \text{if } a \in X_0, \\ g^{-1}(a) & \text{if } a \in A\backslash X_0. \end{cases}$$

Then that h is one-one and maps onto B.

Proof. $A\backslash X_0 = A\backslash F(X_0) = g[B\backslash f[X]] \subseteq \mathrm{rng}(g)$. Now note that $h \restriction X_0$ is a bijection from X_0 to $f[X_0]$, and $h \restriction (A\backslash X_0)$ is a bijection from $A\backslash X_0$ to $g^{-1}[A\backslash X_0] = g^{-1}[g[B\backslash f[X_0]]] = B\backslash f[X_0]$. So h is the union of two functions with disjoint domains and disjoint ranges, so h is a one-one function, and it maps A onto B. □

Proposition 11.67. *If α and β are ordinals, then $|\alpha \dotplus \beta| = |\alpha| + |\beta|$, where $\dotplus$ is ordinal addition and $+$ is cardinal addition.*

Proof. Clear by Proposition 9.41. □

Proposition 11.68. *If α and β are ordinals, then $|\alpha \odot \beta| = |\alpha| + |\beta|$, where $\odot$ is ordinal multiplication and $+$ is cardinal addition.*

Proof. Clear by Proposition 9.42. □

Proposition 11.69. *If α and β are ordinals, $2 \leq \alpha$, and $\omega \leq \beta$, then $|{}^{\cdot}\alpha^{\beta}| = |\alpha| \cdot |\beta|$. Here the dot to the left of the first exponent indicates that ordinal exponentiation is involved.*

Proof. First note that $|\alpha| \leq \alpha \leq {}^{\cdot}\alpha^{\beta}$ and $|\beta| \leq \beta \leq {}^{\cdot}\alpha^{\beta}$, so $|\alpha| \cdot |\beta| \leq |{}^{\cdot}\alpha^{\beta}|$. Hence it suffices to prove the other direction, which we do by induction on β, starting with $\beta = \omega$. First, $\beta = \omega$: If $\alpha < \omega$, then $|{}^{\cdot}\alpha^{\omega}| = \left|\bigcup_{m\in\omega} {}^{\cdot}\alpha^{m}\right| = \omega = |\alpha| \cdot |\omega|$. If $\alpha \geq \omega$, then

$$|{}^{\cdot}\alpha^{\omega}| = |\bigcup_{m\in\omega} {}^{\cdot}\alpha^{m}| \leq \sum_{m\in\omega} |{}^{\cdot}\alpha^{m}| \leq \sum_{m\in\omega} |\alpha| \leq \omega \cdot |\alpha|,$$

as desired.

Now we assume the result for $\beta \geq \omega$. Then

$$|{}^{\cdot}\alpha^{\beta+1}| = |{}^{\cdot}\alpha^{\beta} \odot \alpha| = |{}^{\cdot}\alpha^{\beta}| \cdot |\alpha| = |\alpha| \cdot |\beta| \cdot |\alpha| = |\alpha| \cdot |\beta|.$$

using the inductive hypothesis. Finally, if β is a limit ordinal $> \omega$ and the result is true for all $\gamma < \beta$, then

$$\begin{aligned}
|{}^{\cdot}\alpha^{\beta}| = \left|\bigcup_{\gamma<\beta} {}^{\cdot}\alpha^{\gamma}\right| = \left|\bigcup_{\omega\le\gamma<\beta} {}^{\cdot}\alpha^{\gamma}\right| &\le \sum_{\omega\le\gamma<\beta} |{}^{\cdot}\alpha^{\gamma}| \\
= \sum_{\omega\le\gamma<\beta} |\alpha|\cdot|\gamma| \le \sum_{\omega\le\gamma<\beta} |\alpha|\cdot|\beta| &\le |\alpha|\cdot|\beta|\cdot|\beta| = |\alpha|\cdot|\beta|. \qquad \square
\end{aligned}$$

Proposition 11.70. *If* $|A| \le |B|$ *then* $|\mathscr{P}(A)| \le |\mathscr{P}(B)|$.

Proof. Let f be a one-one function mapping A into B. For each $X \in \mathscr{P}(A)$ let $g(X) = f[X]$. So g maps $\mathscr{P}(A)$ into $\mathscr{P}(B)$. We claim that g is one-one. For, suppose that $X, Y \in \mathscr{P}(A)$ and $X \ne Y$. Say by symmetry that $x \in X \backslash Y$. Then $f(x) \in f[X]$ but $f(x) \notin f[Y]$ by one-oneness. $\square$

Proposition 11.71.

$$\prod_{i\in I}^{c} \sum_{j\in J_i} \kappa_{ij} = \sum_{f\in P} \prod_{i\in I}^{c} \kappa_{i,f(i)},$$

where $P = \prod_{i\in I} J_i$.

Proof. The left side is the number of elements of

$$\prod_{i\in I}\left(\bigcup_{j\in J_i}(\kappa_{ij}\times\{j\})\right), \tag{1}$$

and the right side is the number of elements of

$$\bigcup_{f\in P}\left(\prod_{i\in I}\kappa_{i,f(i)}\times\{f\}\right). \tag{2}$$

For each $f \in P$ let F_f be a bijection from $\prod_{i\in I}\kappa_{i,f(i)}$ onto $\prod_{i\in I}\kappa_{i,f(i)}$. Now given x in (1) we define $G(x)$ in (2) as follows. For each $i \in I$ we have $x_i \in \bigcup_{j\in J_i}(\kappa_{ij}\times\{j\})$, and so there is a unique $j \in J_i$ such that $x \in \kappa_{ij}\times\{j\}$; let $f_x(i)$ be this j. Thus $f_x \in P$. Now $1^{\rm st}(x_i) \in \kappa_{i,f_x(i)}$ for all $i \in I$, so $\langle 1^{\rm st}(x_i) : i \in I\rangle \in \prod_{i\in I}\kappa_{i,f_x(i)}$. Now we define

$$G(x) = (F_{f_x}(\langle 1^{\rm st}(x_i) : i \in I\rangle), f_x).$$

Clearly $G(x)$ is in (2).

Suppose that $G(x) = G(y)$. Now $f_x = 2^{\rm nd}(G(x)) = 2^{\rm nd}(G(y) = f_y$. Write $f_x = g$. Then for any $i \in I$,

$$\begin{aligned}
1^{\rm st}(x_i) &= (F_g^{-1}(1^{\rm st}(G(x))))_i \\
&= (F_g^{-1}(1^{\rm st}(G(y))))_i \\
&= 1^{\rm st}(y_i),
\end{aligned}$$

and $2^{\text{nd}}(x_i) = g(i) = 2^{\text{nd}}(y_i)$. So $x_i = y_i$. Hence $x = y$. So G is one-one.

To show that G maps onto (2), suppose that z is a member of (2). Choose $h \in P$ such that $z \in (\prod_{i\in I} \kappa_{i,h(i)}) \times \{h\}$. Now $F_h^{-1}(1^{\text{st}}(z)) \in \prod_{i\in I} \kappa_{i,h(i)}$, so for each $i \in I$ we can let

$$x_i = ((F_h^{-1}(1^{\text{st}}(z))_i, h(i)).$$

Then x is in (1). Moreover, clearly $f_x = h$. Then $1^{\text{st}}(x_i) = (F_h^{-1}(1^{\text{st}}(z))_i$, hence $\langle 1^{\text{st}}(x_i) : i \in I\rangle = F_h^{-1}(1^{\text{st}}(z))$, and so

$$\begin{aligned} G(x) &= (F_{f_x}(\langle 1^{\text{st}}(x_i) : i \in I\rangle), f_x) \\ &= (F_h(\langle 1^{\text{st}}(x_i) : i \in I\rangle), h) \\ &= (F_h(F_h^{-1}(1^{\text{st}}(z))), h) \\ &= (1^{\text{st}}(z), 2^{\text{nd}}(z)) \\ &= z, \end{aligned}$$

as desired. □

Proposition 11.72. *For any cardinal κ we have $\kappa^+ = \{\alpha : \alpha$ is an ordinal and $|\alpha| \le \kappa\}$.*

Proof. First suppose that $\alpha < \kappa^+$. Then $|\alpha| \le \alpha$, so $|\alpha| \le \kappa$. Now suppose that $|\alpha| \le \kappa$. Thus there is a one-one function from α into κ. If $\kappa^+ \le \alpha$, then we could also get a one-one function from κ^+ into κ, so $\kappa^+ = |\kappa^+| \le |\kappa| = \kappa$, contradiction. So $\alpha < \kappa^+$, as desired. □

Proposition 11.73. *For every infinite cardinal λ there is a cardinal $\kappa > \lambda$ such that $\kappa^\lambda = \kappa$.*

Proof. Let $\kappa = (2^\lambda)^+$. Then by Hausdorff's theorem,

$$\kappa^\lambda = ((2^\lambda)^+)^\lambda = (2^\lambda)^\lambda \cdot (2^\lambda)^+ = 2^\lambda \cdot (2^\lambda)^+ = (2^\lambda)^+ = \kappa. \qquad \square$$

Proposition 11.74. *For every infinite cardinal λ there is a cardinal $\kappa > \lambda$ such that $\kappa^\lambda > \kappa$.*

Proof. Let $\lambda = \aleph_\alpha$. Note that $\mathrm{cf}(\aleph_{\alpha+\omega}) = \omega \le \lambda$. Let $\kappa = \aleph_{\alpha+\omega}$. Then $\kappa^\lambda > \kappa$. □

Proposition 11.75. *For every $n \in \omega$, and every infinite cardinal κ, $\aleph_n^\kappa = 2^\kappa \cdot \aleph_n$.*

Proof. We prove this by induction on n. $n = 0$: $\aleph_0^\kappa = 2^\kappa = 2^\kappa \cdot \aleph_0$. Assume it for n. Then by Hausdorff's theorem,

$$\aleph_{n+1}^\kappa = ((\aleph_n)^+)^\kappa = \aleph_n^\kappa \cdot (\aleph_n)^+ = 2^\kappa \cdot \aleph_n \cdot \aleph_{n+1} = 2^\kappa \cdot \aleph_{n+1}. \qquad \square$$

Proposition 11.76. $\aleph_\omega^{\aleph_1} = 2^{\aleph_1} \cdot \aleph_\omega^{\aleph_0}$.

Proof. Note that $\aleph_\omega = \sum_{m\in\omega} \aleph_m$, since $\aleph_n \le \sum_{m\in\omega} \aleph_m$ for each $n \in \omega$, hence

$$\aleph_\omega \le \sum_{m\in\omega} \aleph_m \le \sum_{m\in\omega} \aleph_\omega = \omega \cdot \aleph_\omega = \aleph_\omega.$$

Hence by Theorem 11.41 we have $\aleph_\omega < \prod_{m\in\omega} \aleph_{m+1} \le \prod_{m\in\omega} \aleph_m$. So

$$\begin{aligned}
\aleph_\omega^{\aleph_1} &\le \left(\prod_{m\in\omega} \aleph_m\right)^{\aleph_1} \\
&= \prod_{m\in\omega} \aleph_m^{\aleph_1} \\
&= \prod_{m\in\omega} (2^{\aleph_1} \cdot \aleph_m) \quad \text{by Proposition 11.75} \\
&= 2^{\aleph_1} \cdot \prod_{m\in\omega} \aleph_m \\
&\le 2^{\aleph_1} \cdot \prod_{m\in\omega} \aleph_\omega \\
&= 2^{\aleph_1} \cdot \aleph_\omega^{\aleph_0} \\
&\le \aleph_\omega^{\aleph_1}.
\end{aligned}$$

□

Proposition 11.77. $\aleph_\omega^{\aleph_0} = \prod^c_{n\in\omega} \aleph_n$.

Proof. By the argument at the beginning of the proof of Proposition 11.76, $\aleph_\omega = \sum_{m\in\omega} \aleph_m < \prod_{m\in\omega} \aleph_m$. Hence

$$\aleph_\omega^{\aleph_0} \le \left(\prod_{m\in\omega} \aleph_m\right)^{\aleph_0} = \prod_{m\in\omega} \aleph_m^{\aleph_0} = \prod_{m\in\omega} (2^{\aleph_0} \cdot \aleph_m) = 2^{\aleph_0} \cdot \prod_{m\in\omega} \aleph_m = \prod_{m\in\omega} \aleph_m \le \prod_{m\in\omega} \aleph_\omega = \aleph_\omega^{\aleph_0}.$$

□

Proposition 11.78. *For any infinite cardinal κ, $(\kappa^+)^\kappa = 2^\kappa$.*

Proof. By Hausdorff's theorem, $(\kappa^+)^\kappa = \kappa^\kappa \cdot \kappa^+ = 2^\kappa \cdot \kappa^+ = 2^\kappa$. □

Proposition 11.79. *If κ is an infinite cardinal and C is the collection of all cardinals less than κ, then $|C| \le \kappa$.*

Proof. Let $\kappa = \aleph_\alpha$. Then $|C| = \omega + |\alpha| \le \omega + \alpha \le \aleph_\alpha = \kappa$. □

Proposition 11.80. *If κ is an infinite cardinal and C is the collection of all cardinals less than κ, then*

$$2^\kappa = \left(\sum_{\nu\in C} 2^\nu\right)^{\mathrm{cf}(\kappa)}.$$

Proof. First suppose that κ is a successor cardinal λ^+. Then

$$2^\kappa \leq \left(\sum_{\nu\in C} 2^\nu\right)^\kappa = \left(\sum_{\nu\in C} 2^\nu\right)^{\mathrm{cf}(\kappa)} \leq \left(\sum_{\nu\in C} 2^\lambda\right)^\kappa = (|C|\cdot 2^\lambda)^\kappa \leq (2^\kappa)^\kappa = 2^\kappa,$$

as desired.

Now suppose that κ is a limit cardinal. Let $\langle\mu_\xi : \xi < \mathrm{cf}(\kappa)\rangle$ be a strictly increasing sequence of cardinals with supremum κ. Then

$$2^\kappa = 2^{\sum_{\xi<\mathrm{cf}(\kappa)}\mu_\xi} = \prod_{\xi<\mathrm{cf}(\kappa)} 2^{\mu_\xi} \leq \left(\sum_{\nu\in C} 2^\lambda\right)^{\mathrm{cf}(\kappa)} \leq (2^\kappa)^{\mathrm{cf}(\kappa)} = 2^\kappa. \qquad \square$$

Proposition 11.81. *For any limit ordinal τ, $\prod^c_{\xi<\tau} 2^{\aleph_\xi} = 2^{\aleph_\tau}$.*

Proof.

$$2^{\aleph_\tau} = 2^{\sum_{\xi<\tau}\aleph_\xi} = \prod_{\xi<\tau} 2^{\aleph_\xi}. \qquad \square$$

Proposition 11.82. *Assume that κ is an infinite cardinal, and $2^\lambda < \kappa$ for every cardinal $\lambda < \kappa$. Then $2^\kappa = \kappa^{\mathrm{cf}(\kappa)}$.*

Proof. If κ is a successor cardinal, then $\mathrm{cf}(\kappa) = \kappa$ and the desired conclusion is clear. Suppose that κ is a limit cardinal. Let $\langle\mu_\xi : \xi < \mathrm{cf}(\kappa)\rangle$ be a strictly increasing sequence of cardinals with supremum κ. Then

$$2^\kappa = 2^{\sum_{\xi<\mathrm{cf}(\kappa)}\mu_\xi} = \prod_{\xi<\mathrm{cf}(\kappa)} 2^{\mu_\xi} \leq \prod_{\xi<\mathrm{cf}(\kappa)} \kappa = \kappa^{\mathrm{cf}(\kappa)} \leq \kappa^\kappa = 2^\kappa. \qquad \square$$

Theorem 11.83. (Cf. the proof of Theorem 11.43.) *For all ordinals $\alpha, \beta, \gamma, \delta$ define*

$$\begin{aligned}(\alpha,\beta) \prec (\gamma,\delta) \text{ iff } & \max(\alpha,\beta) < \max(\gamma,\delta)\\ \text{or } & \max(\alpha,\beta) = \max(\gamma,\delta) \text{ and } \alpha < \gamma\\ \text{or } & \max(\alpha,\beta) = \max(\gamma,\delta) \text{ and } \alpha = \gamma \text{ and } \beta < \delta.\end{aligned}$$

Clearly this is a well-order. Let $\Gamma(\gamma,\delta)$ be the ordinal which is the order type of $\{(\alpha,\beta) : (\alpha,\beta) < (\gamma,\delta)\}$. Then the following conditions are equivalent, for any ordinal α:

(i) $\Gamma(\alpha,\alpha) = \alpha$.

(ii) $\alpha = 0$ or there is an ordinal ξ such that $\alpha = \omega^{\omega^\xi}$.

Proof. The proof goes by a series of lemmas.

Lemma 1. $\Gamma(\alpha+1,\alpha+1) = \Gamma(\alpha,\alpha) + \alpha\cdot 2 + 1.$ $\square$

Lemma 2. *$\Gamma(n,n)) = n^2$ for all $n \in \omega$.*

Proof. Induction on n, using Lemma 1. □

Lemma 3. *$\Gamma(\omega, \omega) = \omega$.* □

Lemma 4. *If $\alpha < \beta$, then $(\alpha, \gamma) < (\beta, \gamma)$.*

Proof. *Case 1.* $\gamma \leq \alpha$. Then $\max(\alpha, \gamma) = \alpha < \beta = \max(\beta, \gamma)$, so $(\alpha, \gamma) < (\beta, \gamma)$.
Case 2. $\alpha < \gamma < \beta$. Then $\max(\alpha, \gamma) = \gamma < \beta = \max(\beta, \gamma)$, so $(\alpha, \gamma) < (\beta, \gamma)$.
Case 3. $\beta \leq \gamma$. Then $\max(\alpha, \gamma) = \gamma = \max(\beta, \gamma)$ and $\alpha < \beta$, so $(\alpha, \gamma) < (\beta, \gamma)$. □

Lemma 5. *If $\alpha < \beta$, then $(\gamma, \alpha) < (\gamma, \beta)$.*

Proof. *Case 1.* $\gamma \leq \alpha$. Then $\max(\gamma, \alpha) = \alpha < \beta = \max(\gamma, \beta)$. So $(\gamma, \alpha) < (\gamma, \beta)$.
Case 2. $\alpha < \gamma < \beta$. Then $\max(\alpha, \gamma) = \gamma < \beta = \max(\gamma, \beta)$, so $(\gamma, \alpha) < (\gamma, \beta)$.
Case 3. $\beta \leq \gamma$. Then $\max(\gamma, \alpha) = \gamma = \max(\gamma, \beta)$ and $\alpha < \beta$, so $(\gamma, \alpha) < (\gamma, \beta)$. □

Lemma 6. *$\beta \leq \Gamma(\beta, 0)$.*

Proof. In fact, $(0,0) < (1,0) < (2,0) < \cdots < (\xi, 0) \cdots$ for $\xi < \beta$. □

Lemma 7. *$\beta + \gamma \leq \Gamma(\beta, \gamma)$.*

Proof. Induction on γ. It holds for $\gamma = 0$ by Lemma 6. Assume it for γ. Then $\Gamma(\beta, \gamma + 1) > \Gamma(\beta, \gamma) \geq \beta + \gamma$, so $\Gamma(\beta, \gamma + 1) \geq \beta + \gamma + 1$. Now assume that it holds for all $\delta < \gamma$. Then $\Gamma(\beta, \gamma) \geq \sup_{\delta < \gamma} \Gamma(\beta, \delta) \geq \sup_{\delta < \gamma} (\beta + \delta) = \beta + \gamma$. □

Lemma 8. *If $\Gamma(\alpha, \alpha) = \alpha$ and $\beta, \gamma < \alpha$, then $\beta + \gamma < \alpha$.*

Proof. By Lemma 7, $\beta + \gamma \leq \Gamma(\beta, \gamma) < \Gamma(\alpha, \alpha) = \alpha$. □

Lemma 9. *If $\Gamma(\alpha, \alpha) = \alpha$ and $\beta, \gamma < \alpha$, then $\beta \cdot \gamma \leq \Gamma(\beta + \gamma, \beta + \gamma)$.*

Proof. Induction on γ. It is clear for $\gamma = 0$. Assume that $\gamma + 1 < \alpha$ and $\beta \cdot \gamma \leq \Gamma(\beta + \gamma, \beta + \gamma)$. Then

$$\begin{aligned}\Gamma(\beta + \gamma + 1, \beta + \gamma + 1) &= \Gamma(\beta + \gamma, \beta + \gamma) + (\beta + \gamma) \cdot 2 + 1 \\ &\geq \beta \cdot \gamma + \beta = \beta \cdot (\gamma + 1).\end{aligned}$$

Now suppose that γ is limit and $\forall \delta < \gamma [\beta \cdot \delta \leq \Gamma(\beta + \delta, \beta + \delta)]$. Then

$$\begin{aligned}\Gamma(\beta + \gamma, \beta + \gamma) &= \bigcup_{\delta < \gamma} \Gamma(\beta + \delta, \beta + \delta) \\ &\geq \bigcup_{\delta < \gamma} \beta \cdot \delta = \beta \cdot \gamma.\end{aligned}$$ □

Lemma 10. *If $\Gamma(\alpha, \alpha) = \alpha$, then $\alpha = 0$ or there is a β such that $\alpha = \omega^{\omega^\beta}$.* □

Lemma 11. *If $\alpha = 0$ or $\alpha = \omega^{\omega^\xi}$ for some ξ, then $\Gamma(\alpha,\alpha) = \alpha$.*

Proof. We may assume that $\alpha = \omega^{\omega^\xi}$ with $\xi > 0$.

Case 1. $\xi = \eta + 1$ for some η. Note that

$$\omega^{\omega^{\eta+1}} = \omega^{\omega^\eta\cdot\omega} = \bigcup_{n\in\omega} \omega^{\omega^\eta\cdot n}.$$

(1) $\forall n \in \omega\backslash 1 \forall \beta < \omega^{\omega^\eta\cdot n}[\Gamma(\beta,\beta) \le \omega^{\omega^{\eta\cdot n}}\cdot\beta]$.

For a fixed $n \in \omega\backslash 1$ we prove this by induction on β. It is clear for $\beta = 0$. Assume it for β. Then

$$\begin{aligned}\Gamma(\beta+1,\beta+1) &= \Gamma(\beta,\beta) + \beta\cdot 2 + 1 \le \omega^{\omega^{\eta\cdot n}}\cdot\beta + \beta\cdot 2 + 1\\ &< \omega^{\omega^{\eta\cdot n}}\cdot\beta + \omega^{\omega^{\eta\cdot n}} = \omega^{\omega^{\eta\cdot n}}\cdot(\beta+1).\end{aligned}$$

Now assume that $\beta < \omega^{\omega^\eta\cdot n}$ is limit, and for all $\delta < \beta$, $\Gamma(\delta,\delta) \le \omega^{\omega^{\eta\cdot n}}\cdot\delta$. Then

$$\Gamma(\beta,\beta) = \bigcup_{\delta<\beta}\Gamma(\delta,\delta) \le \bigcup_{\delta<\beta}(\omega^{\omega^{\eta\cdot n}}\cdot\delta) = \omega^{\eta\cdot n}\cdot\beta.$$

This proves (1). If follows that $\forall\beta < \alpha[\Gamma(\beta,\beta) < \alpha]$.

Case 2. ξ is limit.

(2) $\forall\eta \in \xi\backslash 1\forall\beta < \omega^{\omega^\eta}[\Gamma(\beta,\beta) \le \omega^{\omega^\eta}\cdot\beta]$.

The proof is like that for (1). ☐

Now Theorem 11.83 follows from Lemmas 10 and 11. ☐

For each $n \in \omega\backslash 1$ define

$$\begin{aligned}s <_n t \quad\text{iff}\quad & s,t \in {}^n\mathbf{On} \wedge \max(\mathrm{rng}(s)) < \max(\mathrm{rng}(t)) \text{ or}\\ & \max(\mathrm{rng}(s)) = \max(\mathrm{rng}(t)) \text{ and } s_i < t_i,\\ & \text{where } i < n \text{ is minimum such that } s_i \ne t_i\end{aligned}$$

Lemma 11.84. *If $n \in \omega\backslash 1$, then $<_n$ is a well-order, and for each ordinal α,* $\mathrm{o.t.}({}^n\omega_\alpha) = \omega_\alpha$.

Proof. Clearly $<_n$ is a well-order. Now let α be any ordinal. If $\xi < \omega_\alpha$, then $\langle\langle\eta,\eta,\ldots\rangle : \eta < \xi\rangle$ is a strictly increasing sequence of members of ${}^n\omega_\alpha$, so $\omega_\alpha \le \mathrm{o.t.}({}^n\omega_\alpha)$. Suppose that $\omega_\alpha < \mathrm{o.t.}({}^n\omega_\alpha)$. Then there is an $s \in {}^n\omega_\alpha$ such that $\mathrm{o.t.}(\{t \in {}^n\mathbf{On} : t <_n s\}) = \omega_\alpha$. Now $|\{t \in {}^n\mathbf{On} : t <_n s\}| = \prod_{i<n}(|s_i|+1) < \omega_\alpha$, contradiction. ☐

Lemma 11.85. *There is a well-order $<'$ of ${}^{<\omega}\omega$ of order type ω.*

Proof. Clearly $|{}^{<\omega}\omega| = \omega$. Let f be a bijection from ${}^{<\omega}\omega$ onto ω. For $s,t \in {}^{<\omega}\omega$ define $s < t$ iff $f(s) < f(t)$. ☐

Theorem 11.86. *There is a well-order $<$ of ${}^{<\omega}\mathbf{On}$ such that for every ordinal α, ${}^{<\omega}\omega_\alpha$ has order type ω_α.*

Proof. For each $n \in \omega\backslash 1$ let $s^n_\xi : \xi \in \mathbf{On}$ enumerate ${}^n\mathbf{On}$ in increasing order according to $<_n$. Now for any $u, v \in {}^{<\omega}\mathbf{On}$ we define

$u < v$ iff (1) $u, v \in {}^{<\omega}\omega$ and $u <' v$, or
(2) $u \in {}^{<\omega}\omega$ and $\exists\beta \in \mathrm{rng}(v)[\omega \le \beta]$, or
(3) $\exists\beta \in \mathrm{rng}(u)[\omega \le \beta]$ and $\exists\beta \in \mathrm{rng}(v)[\omega \le \beta]$ and $\mathrm{dmn}(u) < \mathrm{dmn}(v)$ and $\exists\xi[u = s^{\mathrm{dmn}(u)}_\xi$ and $v = s^{\mathrm{dmn}(v)}_\xi]$, or
(4) $\exists\beta \in \mathrm{rng}(u)[\omega \le \beta]$ and $\exists\beta \in \mathrm{rng}(v)[\omega \le \beta]$ and $\exists\xi, \eta[\xi < \eta$ and $u = s^{\mathrm{dmn}(u)}_\xi$ and $v = s^{\mathrm{dmn}(v)}_\eta]$.

Clearly under $<$, ${}^{<\omega}\omega$ is the same as ${}^{<\omega}\omega$ under $<'$, and so it has order type ω. For $\alpha > 0$, we have

$$ {}^{<\omega}\omega_\alpha = {}^{<\omega}\omega \cup \bigcup_{n\in\omega\backslash 1} \bigcup_{\xi<\omega_\alpha} s^n_\xi, $$

and this has order type $\omega \cdot \omega_\alpha = \omega_\alpha$. $\square$

The following theorem and corollaries are due to Galvin and Hajnal.

Theorem 11.87. *Let κ and λ be uncountable regular cardinals such that $\forall\delta < \lambda[\delta^\kappa < \lambda]$. Assume that $\langle \mu_\alpha : \alpha < \kappa\rangle$ is a sequence of cardinals such that $\forall\alpha < \kappa[\prod^c_{\beta<\alpha} \mu_\beta < \aleph_\lambda]$.*

Then $\prod^c_{\alpha<\kappa} \mu_\alpha < \aleph_\lambda$.

The proof will be given after stating some corollaries and lemmas.

Corollary 11.88. *Let κ and λ be uncountable regular cardinals such that $\forall\delta < \lambda[\delta^\kappa < \lambda]$. Suppose that $\forall\sigma < \kappa[\tau^\sigma < \aleph_\lambda]$.*

Then $\tau^\kappa < \aleph_\lambda$.

Proof. Assume the hypotheses. For each $\alpha < \kappa$ let $\mu_\alpha = \tau$. Then for any $\alpha < \kappa$, $\prod^c_{\beta<\alpha} \mu_\beta = \tau^{|\alpha|} < \aleph_\lambda$. Hence by Theorem 11.87, $\tau^\kappa = \prod^c_{\alpha<\kappa} \mu_\alpha < \aleph_\lambda$. $\square$

Corollary 11.89. *Let κ and λ be uncountable regular cardinals such that $\forall\delta < \lambda[\delta^\kappa < \lambda]$. Suppose that τ is a cardinal such that $\mathrm{cf}(\tau) = \kappa$, and $\forall\sigma < \tau[2^\sigma < \aleph_\lambda]$.*

Then $2^\tau < \aleph_\lambda$.

Proof.

(1) There is a sequence $\langle \nu_\xi : \xi < \kappa\rangle$ of cardinals such that $\forall\xi, \eta[\xi < \eta < \kappa \to \nu_\xi \le \nu_\eta < \tau]$ and $\sum_{\xi<\kappa} \nu_\xi = \tau$.

In fact, if $\tau > \kappa$ this follows from Proposition 11.53. If $\tau = \kappa$, let each $\nu_\xi = 1$; then use Proposition 11.30(vi). So (1) holds.

Now for each $\xi < \kappa$ let $\mu_\xi = 2^{\nu_\xi}$. Now suppose that $\alpha < \kappa$. Let $\sigma = \sum_{\beta<\alpha} \nu_\beta$. Then

$$ \sigma \le |\alpha| \cdot \nu_\alpha = \begin{cases} |\alpha| < \tau & \text{if } \kappa = \tau, \\ < \tau & \text{if } \kappa < \tau. \end{cases} $$

Hence

$$\prod_{\beta<\alpha}^{c} \mu_\beta = \prod_{\beta<\alpha}^{c} 2^{\nu_\beta} = 2^\sigma < \aleph_\lambda,$$

using Proposition 11.43(xi) and Theorem 11.87. Hence by Theorem 11.87 and Proposition 11.43(xi).

$$2^\tau = 2^{\sum_{\xi<\kappa}\nu_\xi} = \prod_{\xi<\kappa}^{c} 2^{\nu_\xi} = \prod_{\xi<\kappa}^{c} \mu_\xi < \aleph_\lambda. \qquad \square$$

Corollary 11.90. *Let κ be an uncountable regular cardinal and let ρ and τ be cardinals such that $2 \le \rho$ and $\forall\sigma < \kappa[\tau^\sigma < \aleph_{(\rho^\kappa)^+}]$*

Then $\tau^\kappa < \aleph_{(\rho^\kappa)^+}$.

Proof. Let $\lambda = (\rho^\kappa)^+$. Then for all $\delta < \lambda$, $\delta^\kappa \le (\rho^\kappa)^\kappa = \rho^\kappa > \lambda$. Also, $\forall\sigma < \kappa[\tau^\sigma < \aleph_{(\rho^\kappa)^+} = \aleph_\lambda$. Hence by Corollary 11.89, $\tau^\kappa < \aleph_\lambda = \aleph_{(\rho^\kappa)^+}$. $\square$

Corollary 11.91. *Suppose that ρ and τ are cardinals, $\rho \ge 2$, $\mathrm{cf}(\tau) = \kappa > \omega$, and $\forall\sigma < \tau[2^\sigma < \aleph_{(\rho^\kappa)^+}]$.*

Then $2^\tau < \aleph_{(\rho^\kappa)^+}$.

Proof. Let $\lambda = (\rho^\kappa)^+$. If $\delta < \lambda$, then $\delta \le \rho^\kappa$ and so $\delta^\kappa < \lambda$. If $\sigma < \tau$, then $2^\sigma < \aleph_\lambda$. Hence by Corollary 11.90, $2^\tau < \aleph_{(\rho^\kappa)^+}$. $\square$

Corollary 11.92. *Let ξ be an ordinal with $\mathrm{cf}(\xi) > \omega$. Assume that $\forall\alpha < \xi[2^{\aleph_\alpha} < \aleph_{(|\xi|^{\mathrm{cf}(\xi)})^+}]$.*

Then $2^{\aleph_\xi} < \aleph_{(|\xi|^{\mathrm{cf}(\xi)})^+}$.

Proof. Let $\rho = |\xi|$, $\tau = \aleph_\xi$, and $\kappa = \mathrm{cf}(\xi)$. Then $|\xi| \ge 2$ and $\mathrm{cf}(\tau) = \mathrm{cf}(\xi) = \kappa > \omega$

(1) $\forall\sigma < \tau[2^\sigma < \aleph_{(\rho^\kappa)^+}]$.

In fact, suppose that $\sigma < \tau$.

Case 1. $\sigma < \omega$. Obiously then $2^\sigma < \aleph_{(\rho^\kappa)^+}$.

Case 2. $\sigma = \aleph_\alpha$ for some α. Then $\alpha < \xi$, so

$$2^\sigma = 2^{\aleph_\alpha} < \aleph_{(|\xi|^{\mathrm{cf}(\xi)})^+} = \aleph_{(\rho^\kappa)^+}.$$

Thus $\forall\sigma < \tau[2^\sigma < \aleph_{(\rho^\kappa)^+}]$. It follows from Corollary 5 that $2^{\aleph_\xi} = 2^\tau < \aleph_{(\rho^\kappa)^+} = \aleph_{(|\xi|^{\mathrm{cf}(\xi)})^+}$. $\square$

Corollary 11.93. *If $\aleph_\alpha$ is strong limit singular with $\mathrm{cf}(\alpha) > \omega$, then $2^{\aleph_\alpha} < \aleph_{(2^{|\alpha|})^+}$.*

Proof. Assume that $\aleph_\alpha$ is strong limit singular with $\mathrm{cf}(\alpha) > \omega$. Let $\rho = 2$, $\kappa = |\alpha|^+$, and $\tau = \aleph_\alpha$. If $\sigma < \tau$, then $2^\sigma < \aleph_\alpha \le \aleph_{(2^{|\alpha|})^+}$. Hence by Corollary 11.92, $2^{\aleph_\alpha} < \aleph_{(2^{|\alpha|})^+}$. $\square$

Corollary 11.94. *Let ξ be an ordinal with* $\mathrm{cf}(\xi) > \omega$. *Assume* $\forall\sigma < \mathrm{cf}(\xi)\forall\alpha < \xi[\aleph_\alpha^\sigma < \aleph_{(|\xi|^{\mathrm{cf}(\xi)})^+}]$.

Then $\aleph_\xi^{\mathrm{cf}(\xi)} < \aleph_{(|\xi|^{\mathrm{cf}(\xi)})^+}]$.

Proof. Let $\rho = |\xi|$, $\kappa = \mathrm{cf}(\xi)$, and $\tau = \aleph_\xi$. Then $\rho \geq 2$ and for all $\sigma < \kappa$,

$$\tau^\sigma = \aleph_\xi^\sigma = |{}^\sigma\aleph_\xi| = \left|{}^\sigma\left(\bigcup_{\alpha<\xi}\aleph_\alpha\right)\right| = \left|\bigcup_{\alpha<\xi}({}^\sigma\aleph_\alpha)\right| \leq \sum_{\alpha<\xi}\aleph_\alpha^\sigma$$
$$\leq |\xi|\cdot\aleph_{|\xi|^{\mathrm{cf}(\xi)}} = \aleph_{|\xi|^{\mathrm{cf}(\xi)}} < \aleph_{(|\xi|^{\mathrm{cf}(\xi)})^+} = \aleph_{(\rho^\kappa)^+}$$

Hence by Corollary 11.92, $\aleph_\xi^{\mathrm{cf}(\xi)} = \tau^\kappa < \aleph_{(\rho^\kappa)^+} = \aleph_{(|\xi|^{\mathrm{cf}(\xi)})^+}$. □

Corollary 11.95. *If* $\forall\alpha < \omega_1[2^{\aleph_\alpha} < \aleph_{(2^{\aleph_1})^+}]$, *then* $2^{\aleph_{\omega_1}} < \aleph_{(2^{\aleph_1})^+}$.

Proof. Take $\xi = \omega_1$ in Corollary 11.94. □

Corollary 11.96. *If* $\forall\alpha < \omega_1[\aleph_\alpha^\omega < \aleph_{(2^{\aleph_1})^+}]$, *then* $\aleph_{\omega_1}^{\aleph_1} < \aleph_{(2^{\aleph_1})^+}$.

Proof. Take $\xi = \omega_1$ in Corollary 11.95. □

If $A = \langle A_\alpha : \alpha < \kappa\rangle$ is a system of sets, an *almost disjoint transversal for* A, a.d.t., is a set $F \subseteq \prod_{\alpha<\kappa} A_\alpha$ such that $\forall f, g \in F[f \neq g \to |\{\alpha < \kappa : f(\alpha) = g(\alpha)\}| < \kappa]$.

Lemma 11.97. *Let $\langle\kappa_\alpha : \alpha < \lambda\rangle$ be a system of cardinals, with λ a cardinal. For each $\alpha < \lambda$ let $A_\alpha = \prod_{\beta<\alpha}\kappa_\beta$. Then there is an a.d.t. F for A with $|F| = \prod^c_{\alpha<\lambda}\kappa_\alpha$.*

Proof. Let $\tau = \prod^c_{\alpha<\lambda}\kappa_\alpha$, and let $\langle g_\xi : \xi < \tau\rangle$ enumerate $\prod_{\alpha<\lambda}\kappa_\alpha$ without repetitions. For each $\xi < \tau$ and $\alpha < \lambda$ let $f_\xi(\alpha) = g_\xi \restriction \alpha$. Thus $f_\xi \in \prod_{\alpha<\lambda} A_\alpha$ for each $\xi < \tau$. If $\xi, \eta < \tau$ and $\xi \neq \eta$, then $g_\xi \neq g_\eta$; choose β minimum so that $g_\xi(\beta) \neq g_\eta(\beta)$. Then for any $\alpha < \lambda$, $f_\xi(\alpha) = f_\eta(\alpha)$ iff $g_\xi \restriction \alpha = g_\eta \restriction \alpha$ iff $\alpha \leq \beta$. So $|\{\alpha < \lambda : f_\xi(\alpha) = f_\eta(\alpha)\}| < \kappa$. Hence $\mathrm{rng}(f)$ is the required a.d.t. □

Lemma 11.98. *Let λ be an uncountable regular cardinals, and κ a cardinal. Assume that $\forall\delta < \lambda[\delta^\kappa < \lambda]$. Let $A = \langle A_\alpha : \alpha < \kappa\rangle$ be a system of sets such that $\forall\alpha < \kappa[|A_\alpha| < \aleph_\lambda]$. Suppose that F is an a.d.t. for A,*

Then $|F| < \aleph_\lambda$.

The proof will be given after the following proof of the theorem.

Proof of Theorem 11.87. Assume the hypotheses, and for all $\alpha < \kappa$ let $A_\alpha = \prod_{\beta<\alpha}\mu_\beta$. By Lemma 98 there is an a.d.t. F for A with $|F| = \prod^c_{\alpha<\lambda}\mu_\alpha$. Now for each $\alpha < \kappa$, $|A_\alpha| = \prod^c_{\beta<\alpha}\mu_\beta < \alpha_\lambda$. Then by Lemma 11.98 $\prod^c_{\alpha<\lambda}\mu_\alpha = |F| < \aleph_\lambda$. □

Proof of Lemma 11.98. First note:

(1) If $|A_\alpha| = |B_\alpha|$ for all $\alpha < \kappa$, and let τ be a cardinal. Then (there is an a.d.t. F for A with $|F| = \tau$) iff (there is an a.d.t. G for B with $|G| = \tau$).

In fact, assume that $|A_\alpha| = |B_\alpha|$ for all $\alpha < \kappa$. By symmetry it suffices to assume that F is an a.d.t. for A and find an a.d.t. G for B such that $|F| = |G|$. For each $\alpha < \kappa$ let $f_\alpha : A_\alpha \to B_\alpha$ be a bijection. For each $g \in F$ define $g' \in \prod_{\alpha<\kappa} B_\alpha$ by setting $g'(\alpha) = f_\alpha(g(\alpha))$. Let $G = \{g' : g \in F\}$. If $g, h \in F$ and $g \neq h$, then

$$g(\alpha) = h(\alpha) \quad \text{iff} \quad f_\alpha(g(\alpha)) = f_\alpha(h(\alpha)) \quad \text{iff} \quad g'(\alpha) = h'(\alpha).$$

It follows that $|F| = |G|$ and G is an a.d.t. for B. So (1) holds.

Let ${}^\kappa\mathbf{ON}$ be the class of ordinal-valued functions with domain κ. For $\varphi, \psi \in {}^\kappa\mathbf{ON}$ define $\varphi \prec \psi$ iff $|\{\alpha < \kappa : \varphi(\alpha) \geq \psi(\alpha)\}| < \kappa$.

(2) $\prec$ is a well-founded partial order on ${}^\kappa\mathbf{ON}$.

In fact, clearly $\prec$ is irreflexive. Suppose that $\varphi \prec \psi \prec \theta$. Then

$$\{\alpha < \kappa : \varphi(\alpha) < \psi(\alpha)\} \cap \{\alpha < \kappa : \psi(\alpha) < \theta(\alpha)\} \subseteq \{\alpha < \kappa : \varphi(\alpha) < \theta(\alpha)\},$$

so

$$\{\alpha < \kappa : \varphi(\alpha) \geq \theta(\alpha)\} \subseteq \{\alpha < \kappa : \varphi(\alpha) \geq \psi(\alpha)\} \cup \{\alpha < \kappa :: \psi(\alpha) \geq \theta(\alpha)\},$$

and hence $\varphi \prec \theta$.

Now suppose that $\cdots\varphi_{n+1} \prec \varphi_n \prec \cdots \prec \varphi_0$. For all $n \in \omega$ let $X_n = \{\alpha < \kappa : \varphi_{n+1}(\alpha) \geq \varphi_n(\alpha)\}$. Then $|X_n| < \kappa$ for all $n \in \omega$. Then $Y \stackrel{\text{def}}{=} \bigcup_{n\in\omega} X_n$ has size less than κ. Choose $\alpha \in \kappa\backslash Y$. Then $\forall n[\varphi_{n+1}(\alpha) < \varphi_n(\alpha)]$, contradiction. So (2) holds.

For each $\varphi \in {}^\kappa\mathbf{ON}$ let

$$T(\varphi) = \sup\{|F| : F \text{ is an a.d.t. for } \varphi\}.$$

(3) It suffices to show that $\forall\varphi \in {}^\kappa\lambda[T(\aleph \circ \varphi) < \aleph_\lambda]$.

In fact, assume the statement in (3), and suppose that $\forall\delta < \lambda[\delta^\kappa < \lambda]$, $A = \langle A_\alpha : \alpha < \kappa\rangle$, $\forall\alpha < \kappa[|A_\alpha| < \aleph_\lambda]$, and F is an a.d.t. for A. For each $\alpha < \kappa$ let

$$A'_\alpha = \begin{cases} A_\alpha & \text{if } A_\alpha \text{ is infinite,} \\ B_\alpha & \text{with } A_\alpha \subseteq B_\alpha \text{ and } |B_\alpha| = \omega \text{ otherwise.} \end{cases}$$

Clearly F is an a.d.t. for B. Now let $\varphi \in {}^\kappa\lambda$ be such that $|B_\alpha| = \aleph_{\varphi(\alpha)}$ for all $\alpha < \kappa$. By (1) there is an a.d.t. G for $\aleph \circ \varphi$ such that $|F| = |G|$. Thus by (3), $|F| < \aleph_\lambda$.

Now we prove (3) by contradiction: suppose it does not hold, and let $\varphi \in {}^\kappa\lambda$ be minimal such that $\aleph_\lambda \leq T(\aleph \circ \varphi)$. We define

$$I = \{X \subseteq \kappa : \exists\psi \in {}^\kappa\lambda[\forall\alpha \in X[\psi(\alpha) < \varphi(\alpha) \text{ or } \psi(\alpha) = 0] \text{ and } T(\aleph \circ \psi) \geq \aleph_\lambda]\}.$$

Obviously

(4) If $Y \subseteq X \in I$ then $Y \in I$.

(5) $[\kappa]^{<\kappa} \subseteq I$.

In fact, let $X \in [\kappa]^{<\kappa}$. For any $\alpha \in \kappa$ define

$$\psi(\alpha) = \begin{cases} 0 & \text{if } \alpha \in X, \\ \varphi(\alpha) & \text{if } \alpha \notin X. \end{cases}$$

We claim that ψ shows that $X \in I$. For this it suffices to show that $T(\aleph \circ \psi) \geq \aleph_\lambda$. Let F be an a.d.t. for $\aleph \circ \varphi$ such that $|F| \geq \aleph_\lambda$. For each $f \in F$ define $f' \in \prod_{\alpha<\kappa} \aleph_{\psi(\alpha)}$ by setting for any $\alpha \in \kappa$

$$f'(\alpha) = \begin{cases} 0 & \text{if } \alpha \in X, \\ f(\alpha) & \text{if } \alpha \notin X. \end{cases}$$

If $f, g \in F$ and $f' = g'$, then $\{\alpha < \kappa : f(\alpha) = g(\alpha)\} \supseteq (\kappa \backslash X)$ and $\kappa \backslash X$ has size κ, so $f = g$. Let $G = \{f' : f \in F\}$. Thus $|G| = |F|$. If $f, g \in F$ and $f \neq g$, then

$$\{\alpha < \kappa : f'(\alpha) = g'(\alpha)\} = X \cup \{\alpha \in \kappa \backslash X : f(\alpha) = g(\alpha)\},$$

and this set has size less than κ. It follows that G is an a.d.t. for $\aleph \circ \psi$ of size $\geq \aleph_\lambda$, proving (5).

(6) I is κ-complete.

For, suppose that $0 < \delta < \kappa$ and $X_\mu \in I$ for all $\mu < \delta$. Say that for all $\mu < \delta$ we have $\psi_\mu \in {}^\kappa\lambda$ such that

$$\forall \alpha \in X_\mu[\psi_\mu(\alpha) < \varphi(\alpha) \text{ or } \psi_\mu(\alpha) = 0] \text{ and } T(\aleph \circ \psi_\mu) \geq \aleph_\lambda.$$

For all $\alpha < \kappa$ let $\chi(\alpha) = \min_{\mu<\delta} \psi_\mu(\alpha)$.

(7) There is a system $\langle S_\mu : \mu < \delta \rangle$ of pairwise disjoint subsets of κ such that $\bigcup_{\mu<\delta} S_\mu = \kappa$ and $\forall \mu < \delta \forall \alpha \in S_\mu[\chi(\alpha) = \psi_\mu(\alpha)]$.

In fact, for each $\mu < \delta$ let $S'_\mu = \{\alpha < \kappa : \chi(\alpha) = \psi_\mu(\alpha)\}$. Then let $S_\mu = S'_\mu \backslash \bigcup_{\nu<\mu} S'_\nu$. Clearly $\langle S_\mu : \mu < \delta \rangle$ is a pairwise disjoint system of subsets of κ. For any $\alpha \in \kappa$ let μ be minimum such that $\alpha \in S'_\mu$. Then $\alpha \in S_\mu$. So $\bigcup_{\mu<\delta} S_\mu = \kappa$. Clearly $\forall \mu < \delta \forall \alpha \in S_\mu[\chi(\alpha) = \psi_\mu(\alpha)]$. Thus (7) holds.

(8) If τ is a cardinal and $\forall \mu < \delta[\langle f_{\mu\xi} : \xi < \tau \rangle$ is an a.d.t. for $\aleph \circ \psi_\mu]$, and $\forall \xi < \tau[h_\xi = \bigcup_{\mu<\delta}(f_{\mu\xi} \upharpoonright S_\mu)]$, then $F \stackrel{\text{def}}{=} \{h_\xi : \xi < \tau\}$ is an a.d.t. for $\aleph \circ \chi$.

In fact, if $\xi < \tau$ then $\forall \mu < \delta[h_\xi \upharpoonright S_\mu \in \prod_{\alpha \in S_\mu} \aleph_{\psi_\mu}]$, and hence $h_\xi \in \prod_{\alpha\in\kappa} \aleph_{\chi(\alpha)}$. If $\xi, \eta < \tau$ and $\xi \neq \eta$, then

$$\{\alpha < \kappa : h_\xi(\alpha) = h_\eta(\alpha)\} = \bigcup_{\mu<\delta} \{\alpha \in S_\mu : f_{\mu\xi}(\alpha) = f_{\mu,\eta}(\alpha)\}.$$

Since $\delta < \kappa$ and $\forall \mu < \delta[|\{\alpha \in S_\mu : f_{\mu\xi}(\alpha) = f_{\mu,\eta}(\alpha)\}| < \kappa$ and κ is regular, we have $|\{\alpha < \kappa : h_\xi(\alpha) = h_\eta(\alpha)\}| < \kappa$. This proves (8).

Now clearly $\forall \alpha \in \bigcup_{\alpha<\delta} X_\mu[\chi(\alpha) < \varphi(\alpha) \text{ or } \chi(\alpha) = 0]$. Also, for each $\mu < \delta$ we have an a.d.t. G_μ for $\aleph \circ \psi_\mu$ with $|G_\mu| \geq \aleph_\lambda$. Choose $\mu < \delta$ with $|G_\mu|$ minimum, and now for any $\nu < \delta$ let G'_ν be a subset of G_ν of size $|G_\mu|$. Say $|G_\mu| = \tau$. Write $G'_\nu = \{f_{\mu\xi} : \xi < \tau\}$. Then by (8) we get an a.d.t. F for $\aleph \circ \chi$ of size $|G_\mu$. Hence $\bigcup_{\mu<\delta} X_\mu \in I$, proving (6).

Now let

$$X_0 = \{\alpha < \kappa : \varphi(\alpha) = 0\};$$
$$X_1 = \{\alpha < \kappa : \varphi(\alpha) \text{ is a limit ordinal}\};$$
$$X_2 = \{\alpha < \kappa : \varphi(\alpha) \text{ is a successor ordinal}\}.$$

(9) $|X_0| < \kappa$ (Hence $X_0 \in I$.)

For, suppose that $|X_0| = \kappa$. Then we claim

(10) $T(\aleph \circ \varphi) \leq \aleph_0^\kappa$.

For, suppose that F is an a.d.t for $\aleph \circ \varphi$ and $|F| > \aleph_0^\kappa$. Then there exist distinct $f, g \in F$ such that $f \restriction X = g \restriction X$. So $|\{\alpha < \kappa : f(\alpha) = g(\alpha)\}| = \kappa$, contradiction. So (10) holds.

But by an assumption of the lemma, $\forall \delta < \lambda[\delta^\kappa < \lambda]$. So $\aleph_0^\kappa < \lambda \leq \aleph_\lambda$. Hence (10) contradicts the choice of φ. Hence (9) holds.

(11) I is a proper ideal.

Suppose to the contrary that $\kappa \in I$. Choose $\psi \in {}^\kappa\lambda$ so that $\forall \alpha \in \kappa[\psi(\alpha) < \varphi(\alpha)$ or $\psi(\alpha) = 0]$ and $T(\aleph \circ \psi) \geq \aleph_\lambda]\}$. Then $\psi \prec \varphi$, since $\{\alpha < \kappa : \psi(\alpha) \geq \varphi(\alpha)\} = \{\alpha < \kappa : \varphi(\alpha) = 0\} = X_0$. Since $T(\aleph \circ \psi) \geq \aleph_\lambda$, this contradicts the minimality of φ. So (11) holds.

(12) $X_1 \in I$.

To prove this, first note that since λ is uncountable and regular and $\kappa < \lambda$, there is an ordinal $\rho < \lambda$ such that $\varphi \in {}^\kappa\rho$. Let Q be the set of all functions $\chi \in {}^\kappa\lambda$ such that $\forall \alpha \in X_1[\psi(\alpha) < \varphi(\alpha)]$ and $\forall \alpha \in (\kappa \backslash X_1)[\psi(\alpha) = 0]$. Then $|Q| \leq |\rho|^\kappa < \lambda$. Now since $T(\aleph \circ \varphi) \geq \aleph_\lambda$, for each $\mu < \lambda$ there is an a.d.t. F_μ for $\aleph \circ \varphi$ such that $|F_\mu| > \aleph_\mu$. For each $\mu < \lambda$ and $\psi \in Q$ let $F_\mu^\psi = F_\mu \cap \prod_{\alpha<\kappa} \aleph_{\psi(\alpha)}$.

(13) $\forall \mu < \lambda \forall \psi \in Q[F_\mu^\psi$ is an a.d.t. for $\aleph \circ \psi]$.

In fact, $F_\mu^\psi \subseteq \prod_{\alpha<\kappa} \aleph_{\psi(\alpha)}$. Suppose that $f, g \in F_\mu^\psi$ and $f \neq g$. Since $F_\mu^\psi \subseteq F_\mu$ it follows that $|\{\alpha < \kappa : f(\alpha) = g(\alpha)\}| < \kappa$. So (13) holds.

(14) $F_\mu = \bigcup_{\psi \in Q} F_\mu^\psi$.

In fact, $\supseteq$ is clear. Now if $f \in F_\mu$, then for all $\alpha \in X_1$, $\varphi(\alpha)$ is a limit ordinal, and $f(\alpha) \in \aleph_{\varphi(\alpha)}$, so there is a $\psi(\alpha) < \varphi(\alpha)$ such that $f(\alpha) \in \aleph_{\psi(\alpha)}$; and let $\psi(\alpha) = 0$ for $\alpha \in \kappa \backslash X_1$. Then $\psi \in Q$ and $f \in F_\mu^\psi$. This proves (14).

Now if $|Q| \leq \mu < \lambda$, then $|F_\mu| > \aleph_\mu$, and hence by (14) there is a $\psi_\mu \in Q$ such that $|F_\mu^{\psi_\mu}| > \aleph_\mu$. Now $\lambda \backslash |Q| = \bigcup_{\chi \in Q} \{\mu \in \lambda \backslash |Q| : \psi_\mu = \chi\}$ and λ is regular, so there is a $\chi \in Q$ such that $|\{\mu \in \lambda \backslash |Q| : \psi_\mu = \chi\}| = \lambda$. Thus for every $\mu < \lambda$ choose $\mu' \in \lambda \backslash |Q|$ such that $\mu < \mu'$ and $\psi_{\mu'} = \chi$. Then $F_{\mu'}^\chi = F_{\mu'} \cap \prod_{\alpha<\kappa} \aleph_{\chi(\alpha)}$ has size $> \aleph_{\mu'}$. Hence $T(\aleph \circ \chi) \geq \aleph_\lambda$. This proves (12).

For each $X \subseteq X_2$ define $\psi_X \in {}^\kappa\lambda$ as follows: for any $\alpha \in \kappa$ let

$$\psi_X(\alpha) = \begin{cases} \varphi(\alpha) - 1 & \text{if } \alpha \in X, \\ \varphi(\alpha) & \text{if } \alpha \notin X. \end{cases}$$

(15) For all $X \in \mathscr{P}(X_2)\backslash I$ there is a $\rho(X) < \lambda$ such that $T(\aleph \circ \psi_X) \leq \aleph_{\rho(X)}$.

In fact, clearly $\forall \alpha \in X[\psi_X(\alpha) < \varphi(\alpha)]$. Since $X \notin I$, it follows that $T(\aleph \circ \psi_X) < \aleph_\lambda$, so (15) follows.

Now clearly

(16) There is a $\rho < \lambda$ such that $2^\kappa \leq \aleph_\rho$ and $\forall X \in \mathscr{P}(X_2)\backslash I[\rho(X) \leq \rho]$.

Now let F be an a.d.t. for $\aleph \circ \varphi$ such that $|F| > \aleph_{\rho+1}$. For all $f \in F$ and $X \in \mathscr{P}(X_2)\backslash I$ let $H_X(f) = \{g \in F : \forall \alpha \in X[g(\alpha) < f(\alpha)]\}$. Now for all $f \in F$ and $X \in \mathscr{P}(X_2)\backslash I$ and all $\alpha < \kappa$ let

$$A_\alpha^{fX} = \begin{cases} f(\alpha) & \text{if } \alpha \in X, \\ \aleph_{\varphi(\alpha)} & \text{if } \alpha \notin X. \end{cases}$$

(17) For all $f \in F$ and $X \in \mathscr{P}(X_2)\backslash I$, $H_X(f)$ is an a.d.t. for A^{fX}.

In fact, if $g \in H_X(f)$, then $\alpha \in X \to [g(\alpha) < f(\alpha) = A_\alpha^{fX}]$ and $\alpha \in \kappa\backslash X \to [g(\alpha) \in \aleph_{\varphi(\alpha)} = A_\alpha^{fX}]$. Thus $H_X(f) \subseteq \prod_{\alpha<\kappa} A_\alpha^{fX}$. Now suppose that $g, h \in H_X(f)$ with $g \neq h$. Then $g, h \in F$, and hence $|\{\alpha < \kappa : g(\alpha) = h(\alpha)\}| < \kappa$. so (17) holds.

(18) $\forall f \in F \forall X \in \mathscr{P}(X_2)\backslash I \forall \alpha < \kappa[|A_\alpha^{fX}| \leq \aleph_{\psi_X(\alpha)}]$.

In fact, assume that $f \in F$, $X \in \mathscr{P}(X_2)\backslash I$, and $\alpha < \kappa$. If $\alpha \in X$, then $A_\alpha^{fX} = f(\alpha)$, and $f(\alpha) \in \aleph_{\varphi(\alpha)}$. Since $X \subseteq X_2$, we have $\varphi(\alpha) = (\varphi(\alpha) - 1) + 1$, and hence $|A_\alpha^{fX}| = |f(\alpha)| \leq \aleph_{\psi_X(\alpha)}$. If $\alpha \notin X$, then $A_\alpha^{fX} = \aleph_{f(\alpha)}$ and so $|A_\alpha^{fX}| = \aleph_{f(\alpha)} = \aleph_{\psi_X(\alpha)}$. So (18) holds.

(19) For all $f \in F$ and $X \in \mathscr{P}(X_2)\backslash I$ there is an a.d.t. G^{fX} for $\aleph \circ \psi_X$ such that $|G^{fX}| = |H_X(f)|$.

Assume that $f \in F$ and $X \in \mathscr{P}(X_2)\backslash I$. By (18) for each $\alpha < \kappa$ let h_α be an injection of A_α^{fX} into $\aleph_{\psi_X(\alpha)}$. By (1) and (17), there is an a.d.t. G^{fX} for $\langle h_\alpha[A_\alpha^{fx}] : \alpha < \kappa$ such that $|G^{fX}| = |H_X(f)|$. Clearly G^{fX} is an a.d.t. for $\aleph \circ \psi_X$. So (19) holds.

Thus

$$\forall f \in F \forall X \in \mathscr{P}(X_2)\backslash I[|H_X(f)| = |G^{fX}| \leq T(\aleph \circ \psi_X) \leq \aleph_{\rho(X)} \leq \aleph_\rho.$$

Now for any $f \in F$ let $H(f) = \bigcup\{H_X(f) : X \in \mathscr{P}(X_2)\backslash I\}$. Then for any $f \in F$, $|H(f)| \leq 2^\kappa \cdot \aleph_\rho = \aleph_\rho$. Recall that $F > \aleph_{\rho+1}$. Let $G \subseteq F$ with $|G| = \aleph_{\rho+1}$.

(20) $(F\backslash G)\backslash \bigcup_{g\in G} H(g) \neq \emptyset$.

In fact, $|F| > \aleph_{\rho+1}$, $|G| = \aleph_{\rho+1}$, and $\forall g \in H[|H(g)| \leq \aleph_\rho]$. So (20) is clear.

We choose $f_0 \in (F\backslash G)\backslash \bigcup_{g\in G} H(g)$. Clearly $G\backslash H(f_0) \neq \emptyset$; we choose $g_0 \in G\backslash H(f_0)$. Clearly

(21) $f_0, g_0 \in F$, $f_0 \neq g_0$, $f_0 \notin H(g_0)$ and $g_0 \notin H(f_0)$.

(22) $\{\alpha < \kappa : f_0(\alpha) = g_0(\alpha)\} \in I$.

This holds since $f, g \in F$ and $f \neq g$, by (5).

(23) $\{\alpha \in X_2 : f_0(\alpha) < g_0(\alpha)\} \in I$.

In fact, let $X = \{\alpha \in X_2 : f_0(\alpha) < g_0(\alpha)\}$. Then $f \in H_X(g)$. If $X \notin I$, then $f \in H(g)$, contradicting (21). So (23) holds. Similarly,

(24) $\{\alpha \in X_2 : g_0(\alpha) < f_0(\alpha)\} \in I$.

Now

$$\begin{aligned}\kappa = X_0 \cup X_1 \cup \{\alpha < \kappa : f_0(\alpha) = g_0(\alpha)\} \\ \cup \{\alpha \in X_2 : f_0(\alpha) < g_0(\alpha)\} \cup \{\alpha \in X_2 : g_0(\alpha) < f_0(\alpha)\},\end{aligned}$$

and all the sets on the right are in I, by (9), (12), (22), (23), and (24). This contradicts (6) and (11). $\square$

MODELS OF SET THEORY

12. The set-theoretical hierarchy

The hierarchy of sets is defined recursively as follows:

Theorem 12.3. *There is a class function $V : \mathbf{On} \to \mathbf{V}$ satisfying the following conditions:*

(i) $V_0 = \emptyset$.

(ii) $V_{\alpha+1} = \mathscr{P}(V_\alpha)$.

(iii) $V_\gamma = \bigcup_{\alpha<\gamma} V_\alpha$ *for* γ *limit.*

Proof. We apply Theorem 9.7. Define $\mathbf{G} : \mathbf{On} \times \mathbf{V} \to \mathbf{V}$ as follows. For any ordinal α and set x, let

$$\mathbf{G}(\alpha, x) = \begin{cases} \emptyset & \text{if } x = \emptyset, \\ \mathscr{P}(x(\beta)) & \text{if } x \text{ is a function with domain } \alpha = \beta + 1, \\ \bigcup_{\beta<\alpha} x(\beta) & \text{if } x \text{ is a function with domain } \alpha, \text{ and } \alpha \text{ is a limit ordinal,} \\ \emptyset & \text{otherwise.} \end{cases}$$

So we apply Theorem 9.7 to obtain a class function $\mathbf{F} : \mathbf{On} \to \mathbf{V}$ such that for every ordinal α, $\mathbf{F}(\alpha) = \mathbf{G}(\alpha, \mathbf{F} \restriction \alpha)$. Hence

$$\begin{aligned} \mathbf{F}(0) &= \mathbf{G}(0, \mathbf{F} \restriction 0) = \mathbf{G}(0, \emptyset) = \emptyset; \\ \mathbf{F}(\beta + 1) &= \mathbf{G}(\beta + 1, \mathbf{F} \restriction \beta + 1) = \mathscr{P}(\mathbf{F}(\beta)); \\ \mathbf{F}(\alpha) &= \mathbf{G}(\alpha, \mathbf{F} \restriction \alpha) = \bigcup_{\beta<\alpha} \mathbf{F}(\beta) \quad \text{for } \alpha \text{ limit.} \end{aligned}$$

□

Recall from chapter 7 the notion of a transitive set. We have used this notion only for defining ordinals so far. But the general notion will now play an important role in what follows.

Theorem 12.4. *For every ordinal α the following hold:*

(i) V_α *is transitive.*

(ii) $V_\beta \subseteq V_\alpha$ *for all* $\beta < \alpha$.

Proof. We prove these statements simultaneously by induction on α. They are clear for $\alpha = 0$. Assume that both statements hold for α; we prove them for $\alpha + 1$. First we prove

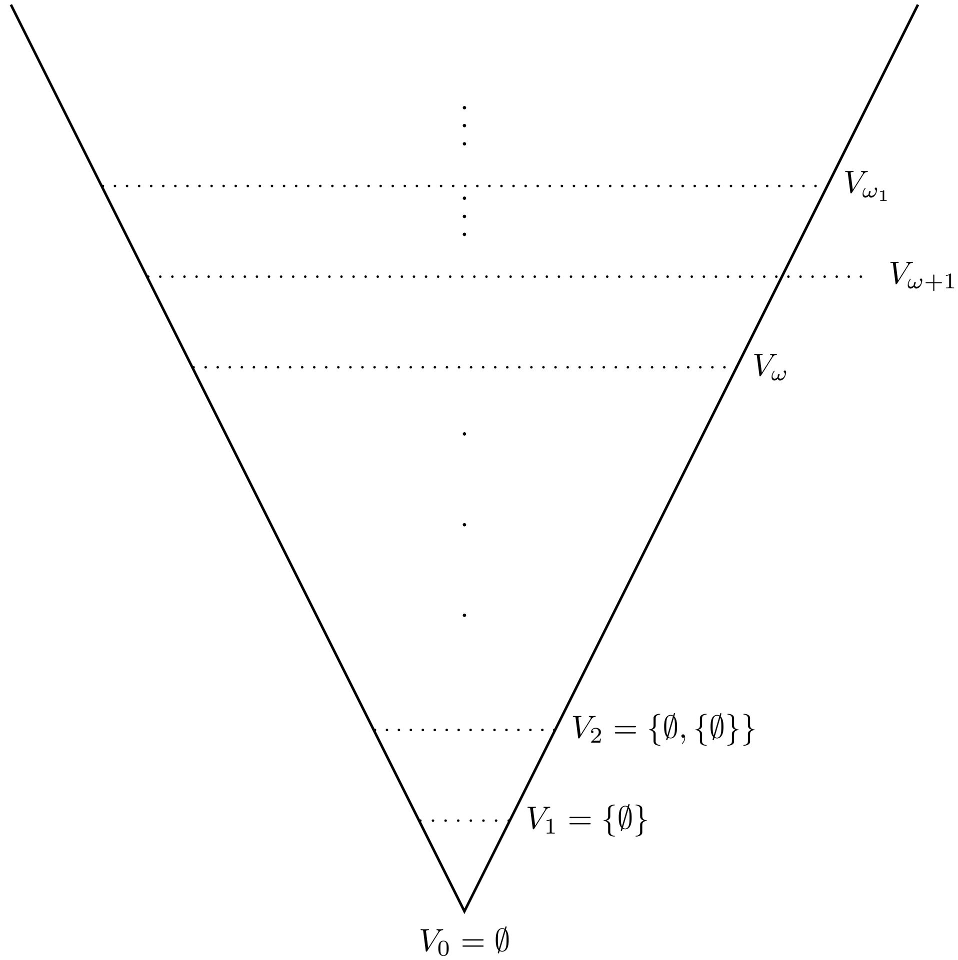
(1) $V_\alpha \subseteq V_{\alpha+1}$.

In fact, suppose that $x \in V_\alpha$. By (i) for α, the set V_α is transitive. Hence $x \subseteq V_\alpha$, so $x \in \mathscr{P}(V_\alpha) = V_{\alpha+1}$. So (1) holds.

Now (ii) follows. For, suppose that $\beta < \alpha + 1$. Then $\beta \leq \alpha$, so $V_\beta \subseteq V_\alpha$ by (ii) for α (or trivially if $\beta = \alpha$). Hence by (1), $V_\beta \subseteq V_{\alpha+1}$.

To prove (i) for $\alpha + 1$, suppose that $x \in y \in V_{\alpha+1}$. Then $y \in \mathscr{P}(V_\alpha)$, so $y \subseteq V_\alpha$, hence $x \in V_\alpha$. By (1), $x \in V_{\alpha+1}$, as desired.

For the final inductive step, suppose that γ is a limit ordinal and (i) and (ii) hold for all $\alpha < \gamma$. To prove (i) for γ, suppose that $x \in y \in V_\gamma$. Then by definition of V_γ, there is an $\alpha < \gamma$ such that $y \in V_\alpha$. By (i) for α we get $x \in V_\alpha$. So $x \in V_\gamma$ by the definition of V_γ. Condition (ii) for γ is obvious. □



A very important fact about this hierarchy is that every set is a member of some V_α. To prove this, we need the notion of transitive closure. We introduced and used this notion in Chapter 8, but we will prove the following independent of this.

Theorem 12.5. *For any set a there is a transitive set b with the following properties:*

(i) $a \subseteq b$.

(ii) For every transitive set c such that $a \subseteq c$ we have $b \subseteq c$.

Proof. We first make a definition by recursion. Define $\mathbf{G} : \mathbf{On} \times \mathbf{V} \to \mathbf{V}$ by setting, for an $\alpha \in \mathbf{On}$ and any $x \in \mathbf{V}$

$$\mathbf{G}(\alpha, x) = \begin{cases} a & \text{if } x = \emptyset, \\ x(m) \cup \bigcup x(m) & \text{if } x \text{ is a function with domain } m+1 \text{ with } m \in \omega, . \\ 0 & \text{otherwise} \end{cases}$$

By Theorem 9.7 let $\mathbf{F} : \mathbf{On} \to \mathbf{V}$ be such that $\mathbf{F}(\alpha) = \mathbf{G}(\alpha, \mathbf{F} \restriction \alpha)$ for any $\alpha \in \mathbf{On}$. Let $d = \mathbf{F} \restriction \omega$. Then $d_0 = \mathbf{F}(0) = \mathbf{G}(0, \mathbf{F} \restriction 0) = \mathbf{G}(0, \emptyset) = a$. For any $m \in \omega$ we have $d_{m+1} = \mathbf{F}(m+1) = \mathbf{G}(m+1, \mathbf{F} \restriction (m+1)) = \mathbf{F}(m) \cup \bigcup \mathbf{F}(m) = d_m \cup \bigcup d(m)$. Let $b = \bigcup_{m \in \omega} d_m$. Then $a = d_0 \subseteq b$. Suppose that $x \in y \in b$. Choose $m \in \omega$ such that $y \in d_m$. Then $x \in \bigcup d_m \subseteq d_{m+1} \subseteq b$. Thus b is transitive. Now suppose that c is a transitive set such that $a \subseteq c$. We show by induction that $d_m \subseteq c$ for every $m \in \omega$. First, $d_0 = a \subseteq c$, so this is true for $m = 0$. Now assume that it is true for m. Then $d_{m+1} = d_m \cup \bigcup d_m \subseteq c \cup \bigcup c = c$, completing the inductive proof.

Hence $b = \bigcup_{m \in \omega} d_m \subseteq c$. □

The set shown to exist in Theorem 12.5 is called the *transitive closure* of a, and is denoted by $\mathrm{trcl}(a)$.

Theorem 12.6. *Every set is a member of some V_α.*

Proof. Suppose that this is not true, and let a be a set which is not a member of any V_α. Let $A = \{x \in \mathrm{trcl}(a \cup \{a\}) : x \text{ is not in any of the sets } V_\alpha\}$. Then $a \in A$, so A is nonempty. By the foundation axiom, choose $x \in A$ such that $x \cap A = 0$. Suppose that $y \in x$. Then $y \in \mathrm{trcl}(a \cup \{a\})$, so y is a member of some V_α. Let α_y be the least such α. Let $\beta = \bigcup_{y \in x} \alpha_y$. Then by 12.1(ii), $x \subseteq V_\beta$. So $x \in V_{\beta+1}$, contradiction. □

Thus by Theorem 12.6 we have $\mathbf{V} = \bigcup_{\alpha \in \mathbf{On}} V_\alpha$. An important technical consequence of Theorem 12.6 is the following definition, known as *Scott's trick*:

• Let R be a class equivalence relation on a class A. For each $a \in A$, let α be the smallest ordinal such that there is a $b \in V_\alpha$ with $(a, b) \in R$, and define

$$\mathrm{type}_R(a) = \{b \in V_\alpha : (a, b) \in R\}.$$

This is the "reduced" equivalence class of a. It could be that the collection of b such that $(a, b) \in R$ is a proper class, but $\mathrm{type}_R(a)$ is always a set.

On the basis of our hierarchy we can define the important notion of *rank* of sets:

• For any set x, the rank of x, denoted by $\mathrm{rank}(x)$, is the smallest ordinal α such that $x \in V_{\alpha+1}$.

We take $\alpha + 1$ here instead of α just for technical reasons. Some of the most important properties of ranks are given in the following theorem.

Theorem 12.7. *Let x be a set and α an ordinal. Then*
(i) $V_\alpha = \{y : \mathrm{rank}(y) < \alpha\}$.
(ii) For all $y \in x$ we have $\mathrm{rank}(y) < \mathrm{rank}(x)$.
(iii) $\mathrm{rank}(y) \leq \mathrm{rank}(x)$ *for every $y \subseteq x$.*
(iv) $\mathrm{rank}(x) = \sup_{y \in x}(\mathrm{rank}(y) + 1)$.
(v) $\mathrm{rank}(\alpha) = \alpha$.
(vi) $V_\alpha \cap \mathbf{On} = \alpha$.

Proof. (i): Suppose that $y \in V_\alpha$. Then $\alpha \neq 0$. If α is a successor ordinal $\beta + 1$, then $\mathrm{rank}(y) \leq \beta < \alpha$. If α is a limit ordinal, then $y \in V_\beta$ for some $\beta < \alpha$, hence $y \in V_{\beta+1}$ also, so $\mathrm{rank}(y) \leq \beta < \alpha$. This proves $\subseteq$.

For $\supseteq$, suppose that $\beta \overset{\text{def}}{=} \text{rank}(y) < \alpha$. Then $y \in V_{\beta+1} \subseteq V_\alpha$, as desired.

(ii): Suppose that $x \in y$. Let $\text{rank}(y) = \alpha$. Thus $y \in V_{\alpha+1} = \mathscr{P}(V_\alpha)$, so $y \subseteq V_\alpha$ and hence $x \in V_\alpha$. Then by (i), $\text{rank}(x) < \alpha$.

(iii): Let $\text{rank}(x) = \alpha$. Then $x \in V_{\alpha+1}$, so $x \subseteq V_\alpha$. Let $y \subseteq x$. Then $y \subseteq V_\alpha$, and so $y \in V_{\alpha+1}$. Thus $\text{rank}(y) \leq \alpha$.

(iv): Let α be the indicated sup. Then $\geq$ holds by (ii). Now if $y \in x$, then $\text{rank}(y) < \alpha$, and hence $y \in V_{\text{rank}(y)+1} \subseteq V_\alpha$. This shows that $x \subseteq V_\alpha$, hence $x \in V_{\alpha+1}$, hence $\text{rank}(x) \leq \alpha$, finishing the proof of (iv).

(v): We prove this by transfinite induction. Suppose that it is true for all $\beta < \alpha$. Then by (iv),

$$\text{rank}(\alpha) = \sup_{\beta<\alpha}(\text{rank}(\beta) + 1) = \sup_{\beta<\alpha}(\beta + 1) = \alpha.$$

Finally, for (vi), using (i) and (v),

$$V_\alpha \cap \mathbf{On} = \{\beta \in \mathbf{On} : \beta \in V_\alpha\} = \{\beta \in \mathbf{On} : \text{rank}(\beta) < \alpha\} = \{\beta \in \mathbf{On} : \beta < \alpha\} = \alpha. \quad \square$$

We now define a sequence of cardinals by recursion:

Theorem 12.8. *There is a function* $\beth : \mathbf{On} \to \mathbf{V}$ *such that the following conditions hold:*

$$\begin{aligned} \beth_0 &= \omega; \\ \beth_{\alpha+1} &= 2^{\beth_\alpha}; \\ \beth_\gamma &= \bigcup_{\alpha<\gamma} \beth_\alpha \quad \textit{for } \gamma \textit{ limit.} \end{aligned}$$

Proof. Define $\mathbf{G} : \mathbf{On} \times \mathbf{V} \to \mathbf{V}$ by setting, for any ordinal α and any set x,

$$\mathbf{G}(\alpha, x) = \begin{cases} \omega & \text{if } x = \emptyset, \\ 2^{x(\beta)} & \text{if } x \text{ is a function with domain } \alpha = \beta + 1 \text{ and} \\ & \text{range a set of cardinals,} \\ \bigcup_{\beta<\alpha} x(\beta) & \text{if } x \text{ is a function with domain a limit ordinal } \alpha. \end{cases}$$

Then we obtain $\mathbf{F} : \mathbf{On} \to \mathbf{V}$ by Theorem 6.7: for any ordinal α, $\mathbf{F}(\alpha) = \mathbf{G}(\alpha, \mathbf{F} \restriction \alpha)$. Hence

$$\begin{aligned} \mathbf{F}(0) &= \mathbf{G}(0, \mathbf{F} \restriction 0) = \mathbf{G}(0, \emptyset) = \omega; \\ \mathbf{F}(\beta + 1) &= \mathbf{G}(\beta + 1, \mathbf{F} \restriction (\beta + 1)) = 2^{\mathbf{F}(\beta)}; \\ \mathbf{F}(\alpha) &= \mathbf{G}(\alpha, \mathbf{F} \restriction \alpha) = \bigcup_{\beta<\alpha} \mathbf{F}(\beta) \quad \text{for } \alpha \text{ limit.} \end{aligned} \quad \square$$

Thus under GCH, $\aleph_\alpha = \beth_\alpha$ for every ordinal α; in fact, this is just a reformulation of GCH.

Theorem 12.9. *(i)* $n \leq |V_n| \in \omega$ *for any* $n \in \omega$.

(ii) For any ordinal α, $|V_{\omega+\alpha}| = \beth_\alpha$.

Proof. (i) is clear by ordinary induction on n. We prove (ii) by the three-step transfinite induction (where γ is a limit ordinal below):

$$\begin{aligned}
|V_\omega| &= \left|\bigcup_{n\in\omega} V_n\right| = \omega = \beth_0 \quad \text{by (i);} \\
|V_{\omega+\alpha+1} &= |\mathscr{P}(V_{\omega+\alpha})| \\
&= 2^{|V_{\omega+\alpha}|} \\
&= 2^{\beth_\alpha} \quad \text{(inductive hypothesis)} \\
&= \beth_{\alpha+1}; \\
|V_{\omega+\gamma}| &= \left|\bigcup_{\beta<\gamma} V_{\omega+\beta}\right| \\
&\leq \sum_{\beta<\gamma} |V_{\omega+\beta}| \\
&= \sum_{\beta<\gamma} \beth_\beta \quad \text{(inductive hypothesis)} \\
&\leq \sum_{\beta<\gamma} \beth_\gamma \\
&= |\gamma| \cdot \beth_\gamma \\
&= \beth_\gamma \quad .
\end{aligned}$$

To finish this last inductive step, note that for each $\beta < \gamma$ we have $\beth_\beta = |V_{\omega+\beta}| \leq |V_{\omega+\gamma}|$, and hence $\beth_\gamma \leq |V_{\omega+\gamma}|$. □

. . . .

Lemma 12.10. (I.13.26) *If $\alpha \geq \omega^2$, then $|V_\alpha| = \beth_\alpha$.*

Proof. We have $|V_{\omega+\alpha}| = \beth_\alpha$ for all α. If $\alpha \geq \omega^2$, write $\alpha = \omega^2 + \beta$. Then $|V_\alpha| = |V_{\omega^2+\beta}| = |V_{\omega+\omega^2+\beta}| = \beth_{\omega^2+\beta} = \beth_\alpha$. □

We now give some further results about well-orders and related concepts; see Chapters 7 and 9.

If R well-orders A, we denote by $\text{type}(A, R)$ the unique ordinal isomorphic to (A, R).

Lemma 12.11. (I.8.5) *If R well-orders A and $X \subseteq A$, then R well-orders X, and* $\text{type}(X, R) \leq \text{type}(A, R)$.

Proof. Let f be an isomorphism from (A, R) onto an ordinal α, which by definition is $\text{type}(A, R)$. Clearly R well-orders X. Let g be an isomorphism from (X, R) onto an ordinal β, which by definition is $\text{type}(X, R)$. Then $f \circ g^{-1} : \beta \to \alpha$ is strictly increasing. By Proposition 9.15, $\beta \leq \alpha$. □

Let R be a relation and A a class. An *R-path of n steps on A* is a function s with domain $n+1$ such that $\mathrm{rng}(s) \subseteq A$ and $\forall j < n[s(j)Rs(j+1)]$. Then s is called a *path from $s(0)$ to $s(n)$*.

The *transitive closure of R on A* is the relation R^*_A defined by xR^*_Ay iff there is an R-path on A from x to y. Note that R^*, defined before Theorem 8.3, is R^*_V.

R is *acyclic on A* iff R^*_A is irreflexive on A.

Proposition 12.12. (I.9.8) *If R is well-founded on A, then R is acyclic on A.*

Proof. Assume that R is well-founded on A but R is cyclic on A; say aR^*_Aa. Let s be a path from a to a. Let $x \in \mathrm{rng}(s)$ be R-minimal. Say $x = s(i)$.

Case 1. $i > 0$. Then $s(i-1)Rx$, contradiction.

Case 2. $i = 0$. Then if s has domain $n+1$ we have $s(n-1)Rs(n) = a = x$, contradiction. □

Proposition 12.13. (I.9.8) *If A is finite and R is acyclic on A, then R is well-founded on A.*

Proof. Suppose that R is acyclic on A and A is finite. Suppose that $\emptyset \neq X \subseteq A$. If there are no R-paths in X, then any element of X is R-minimal. Suppose that there are R-paths in X. Let s be a longest R-path in X consisting of distinct elements. Then $s(0)$ is an R-minimal element of X, for if $bRs(0)$ and $b \in X$, then $b \neq s(i)$ for all $i < \mathrm{dmn}(s)$ by the acyclic condition, and this would give a longer R-path in X, contradiction. □

Lemma 12.14. (I.9.9) *Let R be a relation on a class A, and suppose that $\Phi : A \to \mathrm{ON}$ is such that $\forall x, y \in A[xRy \to \Phi(x) < \Phi(y)]$. Then R is well-founded.*

Proof. Suppose that $\emptyset \neq X \subseteq A$. Take $a \in X$ such that $\Phi(a)$ is the least element of $\{\Phi(x) : x \in X\}$. Then a is R-minimal. □

Theorem 12.15. $\mathrm{trcl}(a) = \{x : x \in^*_V a\}$.

Proof. A path of one step shows that $a \subseteq \{x : x \in^*_V a\}$. By induction on the number of steps, if c is transitive and $a \subseteq c$, then $\{x : x \in^*_V a\} \subseteq c$. □

Proposition 12.16. (I.9.13) *If R is well-founded and set-like on A, then there is a function $F : A \to V$ such that $F(a) = \bigcup\{F(b) \cup \{F(b)\} : bRa\}$ for any $a \in A$.*

Proof. We apply the recursion theorem 8.7. Define $G : A \times V \to V$ as follows. For any $a \in A$ and $f \in V$,

$$G(a, f) = \begin{cases} \bigcup\{f(b) \cup \{f(b)\} : bRa\} & \text{if } f \text{ is a function with domain } \mathrm{pred}_{AR}(a), \\ \emptyset & \text{othewise.} \end{cases}$$

Applying Theorem 8.7 we obtain $F : A \to V$ such that for all $a \in A$,

$$F(a) = G(a, F \restriction \mathrm{pred}_{AR}(a)) = \bigcup\{F(b) \cup \{F(b)\} : bRa\}$$ □

We denote the function of Proposition 12.16 by rank_{AR}.

Proposition 12.17. (I.9.14) *If R is well-founded and set-like on A, then:*
(i) $\forall a \in A[\mathrm{rank}_{AR}(a)$ *is an ordinal*$]$.
(ii) $\forall a, b \in A[aRb \to \mathrm{rank}_{AR}(a) < \mathrm{rank}_{AR}(b)]$.
(iii) $\forall a \in A[\mathrm{rank}_{AR}(a) = \sup_{yRx}(\mathrm{rank}_{AR}(y) + 1)]$.

Proof. (i): Suppose not, and let a be R-minimal such that $\mathrm{rank}_{AR}(a)$ is not an ordinal. Then $\forall b[bRa \to \mathrm{rank}_{AR}(b)$ is an ordinal$]$, and hence $\mathrm{rank}_{AR}(a) = \bigcup\{\mathrm{rank}_{AR}(b) \cup \{\mathrm{rank}_{AR}(b)\} : bRa\}$ is an ordinal, contradiction.

(ii) and (iii): clear. □

Lemma 12.18. (I.9.15) *For any relation R and class A, if R is well-founded and set-like on A, then R^*_A is well-founded on A.*

Proof. Suppose that xR^*y. Let s be a path from x to y in A. Thus there is a natural number n such that $\mathrm{dmn}(s) = n+1$, $s(0) = x$, $s(n) = y$, and $s(y)Rs(j+1)$ for all $j < n-1$. By induction, $\mathrm{rank}_{AR}(x) < \mathrm{rank}_{AR}(s(j))$ for all $j \in [1, n]$, so $\mathrm{rank}_{AR}(x) < \mathrm{rank}_{AR}(y)$.

Hence R^*_A is well-founded by Lemma 12.14 □

Lemma 12.19. (I.9.16) *Assume that R is well-founded and set-like on A. Assume that $b \in A$ and $\alpha < \mathrm{rank}_{AR}(b)$. Then there is an $a \in A$ such that aR^*_Ab and $\mathrm{rank}_{AR}(a) = \alpha$.*

Proof. Assume not, and let

$$X = \{b \in A : \exists\alpha[\alpha < \mathrm{rank}_{AR}(b) \text{ and } \neg\exists a \in A[\mathrm{rank}_{AR}(a) = \alpha \text{ and } aR^*b]]\}.$$

Thus $X \neq \emptyset$. Let b be an R-minimal element of X, and choose α accordingly. Now $\alpha < \mathrm{rank}_{AR}(b) = \bigcup\{\mathrm{rank}_{AR}(t) + 1 : t \in A \text{ and } tRb\}$, so we can choose $t \in A$ with tRb and $\alpha < \mathrm{rank}_{AR}(t) + 1$. Thus $\alpha \leq \mathrm{rank}_{AR}(t)$.

Case 1. $\mathrm{rank}_{AR}(t) = \alpha$. Since tRb, this contradicts $b \in X$.

Case 2. $\alpha < \mathrm{rank}_{AR}(t)$. Now $t \notin X$ since tRb, by the minimality of b. Hence there is an $a \in A$ such that $\mathrm{rank}_{AR}(a) = \alpha$ and aR^*t. Since tRb, we get aR^*b. This contradicts $b \in X$. □

Proposition 12.20. $\mathrm{rank}_{ON,<} = \mathrm{rank}$.

Proof. Assume not. Then $X \stackrel{\mathrm{def}}{=} \{a : \mathrm{rank}_{ON,<}(a) \neq \mathrm{rank}(a)\}$ is nonempty. By Theorem 8.5 choose $a \in X$ such that $\forall b \in a[b \notin X]$. Then

$$\begin{aligned}\mathrm{rank}_{ON,<}(a) &= \bigcup\{\mathrm{rank}_{ON,<}(b) \cup \{\mathrm{rank}_{ON,<}(b)\} : b < a\}\\ &= \bigcup\{\mathrm{rank}(b) \cup \{\mathrm{rank}(b)\} : b < a\} = \mathrm{rank}(a),\end{aligned}$$

contradiction. □

Lemma 12.21. (I.9.18) *If $A \subseteq B$ and R is well-founded and set-like on B, and if $b \in A$, then* $\mathrm{rank}_{AR}(b) \leq \mathrm{rank}_{BR}(b)$.

Proof. Suppose not, and let b be an R-mimimal element of $\{x \in A : \text{rank}_{AR}(x) > \text{rank}_{B,R}(x)$. Then

$$\begin{aligned}
\text{rank}_{AR}(b) &= \bigcup\{\text{rank}_{AR}(x)+1 : x \in A \text{ and } xRb\} \\
&\leq \bigcup\{\text{rank}_{BR}(x)+1 : x \in B \text{ and } xRb\} = \text{rank}_{BR}(b),
\end{aligned}$$

contradiction. □

Lemma 12.22. (I.9.18) *If $A \subseteq B$ and R is well-founded and set-like on B, and if $b \in A$ and* $\text{pred}_{BR^*}(b) \subseteq A$, *then* $\text{rank}_{AR}(b) = \text{rank}_{BR}(b)$.

Proof. Assume not, and let b be an R-mimimal element of

$$\{x \in A : \text{rank}_{AR}(x) \neq \text{rank}_{BR}(x) \text{ and } \text{pred}_{BR^*}(x) \subseteq A\}.$$

Note that xRb implies that $x \in A$ and $\text{pred}_{BR^*}(x) \subseteq A$, since $\text{pred}_{BR^*}(b) \subseteq A$. Hence

$$\begin{aligned}
\text{rank}_{AR}(b) &= \bigcup\{\text{rank}_{AR}(x)+1 : x \in A \text{ and } xRb\} \\
&= \bigcup\{\text{rank}_{BR}(x)+1 : x \in B \text{ and } xRb\} = \text{rank}_{BR}(b),
\end{aligned}$$

contradiction. □

Lemma 12.23. (I.9.26) $\text{rank}(\{x,y\}) = \max(\text{rank}(x), \text{rank}(y)) + 1$.

Proof.

$$\begin{aligned}
\text{rank}(\{x,y\}) &= \bigcup\{\text{rank}(z)+1 : z \in \{x,y\}\} \\
&= \max(\text{rank}(x)+1, \text{rank}(y)+1) \\
&= \max(\text{rank}(x), \text{rank}(y)) + 1. \quad \square
\end{aligned}$$

Lemma 12.24. (I.9.26) $\text{rank}((x,y)) = \max(\text{rank}(x), \text{rank}(y)) + 2$. □

Lemma 12.25. (I.9.26) $\text{rank}(\mathscr{P}(x)) = \text{rank}(x) + 1$.

Proof.

$$\text{rank}(\mathscr{P}(x)) = \bigcup\{\text{rank}(y)+1 : y \in \mathscr{P}(x)\} = \bigcup\{\text{rank}(y)+1 : y \subseteq x\} = \text{rank}(x)+1. \quad \square$$

Lemma 12.26. (I.9.26) $\text{rank}(\bigcup x) \leq \text{rank}(x)$.

Proof. If $y \in \bigcup x$, then there is a $z \in x$ such that $y \in z$; so $\text{rank}(y) < \text{rank}(z) < \text{rank}(x)$, hence $\text{rank}(y)+1 < \text{rank}(x)$. Hence $\text{rank}(\bigcup x) = \sup_{y \in \bigcup x}(\text{rank}(y)+1) \leq \text{rank}(x)$. □

Lemma 12.27. (I.9.26) $\text{rank}(x \cup y) = \max(\text{rank}(x), \text{rank}(y))$.

Proof.

$$\begin{aligned}\text{rank}(x \cup y) &= \bigcup\{\text{rank}(z) + 1 : z \in x \cup y\}\\ &= \bigcup\{\text{rank}(z) + 1 : z \in x\} \cup \bigcup\{\text{rank}(z) + 1 : z \in y\}\\ &= \max(\text{rank}(x), \text{rank}(y)). \qquad \square\end{aligned}$$

Lemma 12.28. (I.9.26) $\text{rank}(\text{trcl}(x)) = \text{rank}(x)$.

Proof. Clearly $\forall y \in \text{trcl}(x)[\text{rank}(y) < \text{rank}(x)]$, so $\text{rank}(\text{trcl}(x)) \leq \text{rank}(x)$; the other inequality follows since $x \subseteq \text{trcl}(x)$. $\square$

Suppose that R is well-founded and set-like on A. For each $y \in A$ we define

$$\text{mos}_{AR}(y) = \{\text{mos}_{AR}(x) : x \in A \text{ and } xRy\}.$$

Lemma 12.29. (I.9.32) *If R is well-founded and set-like on A, then $\text{mos}_{AR}[A]$ is transitive.*

Proof. Assume that R is well-founded and set-like on A, and $u \in v \in \text{mos}_{AR}[A]$. Say $v = \text{mos}_{AR}(y)$ with $y \in A$. Since $u \in v$, choose $x \in A$ with xRy and $u = \text{mos}_{AR}(x)$. Thus $u \in \text{mos}_{AR}[A]$. $\square$

A relation R is *extensional on A* iff $\forall x, y \in A[\{z \in A : zRx\} = \{z \in A : zRy\} \rightarrow x = y]$.

Lemma 12.30. (I.9.34) *If A is transitive, then $\in$ is extensional on A.* $\square$

Lemma 12.31. (I.9.35) *Suppose that R is well-founded and set-like on A. Then mos_{AR} is one-one iff R is extensional on A.*

Proof. If R is not extensional on A, then there exist $x, y \in A$ such that $\{z \in A : zRx\} = \{z \in A : zRy\}$ but $x \neq y$. Hence $\text{mos}_{AR}(x) = \text{mos}_{AR}(y)$, so mos is not one-one.

Now suppose that R is extensional on A; we show that mos_{AR} is one-one. Suppose that mos_{AR} is not one-one. So there exist distinct $a, b \in A$ such that $\text{mos}_{AR}(a) = \text{mos}_{AR}(b)$. Let $X = \{c \in A : \exists d \in A[c \neq d \text{ and } \text{mos}_{AR}(c) = \text{mos}_{AR}(d)]\}$. Thus $a \in X$, so $X \neq \emptyset$. Let c be an R-minimal element of X. By definition of X, let $d \in A$ be such that $c \neq d$ and $\text{mos}_{AR}(c) = \text{mos}_{AR}(d)$. Since R is extensional on A and $c \neq d$, there are two cases.

Case 1. There is a $z \in A$ such that zRc but $\text{not}(zRd)$. Then $\text{mos}_{AR}(z) \in \text{mos}_{AR}(c) = \text{mos}_{AR}(d) = \{\text{mos}_{AR}(x) : x \in A \text{ and } xRd\}$. Say $\text{mos}_{AR}(z) = \text{mos}_{AR}(x)$ with xRd and $x \in A$. Since c is an R-minimal element of X and zRc, it follows that $z \notin X$. Hence $\forall y \in A[\text{mos}_{AR}(z) = \text{mos}_{AR}(y) \rightarrow z = y]$. Since $\text{mos}_{AR}(z) = \text{mos}_{AR}(x)$, we thus have $z = x$. But $\text{not}(zRd)$ while xRd, contradiction.

Case 2. There is a $z \in A$ such that $\text{not}(zRc)$ but zRd. Then $\text{mos}_{AR}(z) \in \text{mos}_{AR}(d) = \text{mos}_{AR}(c) = \{\text{mos}_{AR}(x) : x \in A \text{ and } xRc\}$. Say $\text{mos}_{AR}(z) = \text{mos}_{AR}(x)$ with xRc and $x \in A$. Since c is an R-minimal element of X and xRc, it follows that $x \notin X$. Hence

$\forall y \in A[\text{mos}_{AR}(x) = \text{mos}_{AR}(y) \rightarrow x = y]$. Since $\text{mos}_{AR}(z) = \text{mos}_{AR}(x)$, we thus have $z = x$. But not(zRc) while xRc, contradiction. $\square$

Lemma 12.32. (I.9.35) *If R is well-founded, set-like, and extensional on A, then mos is an isomorphism from (A, R) onto $(\text{mos}_{AR}[A], \in)$.*

Proof. If $x, y \in A$ and xRy, then $\text{mos}_{AR}(x) \in \text{mos}_{AR}(y)$ by definition. Now suppose that $x, y \in A$ and $\text{mos}_{AR}(x) \in \text{mos}_{AR}(y)$. Choose zRy with $z \in A$ so that $\text{mos}_{AR}(x) = \text{mos}_{AR}(z)$. Then $x = z$, so xRy. $\square$

Lemma 12.33. (I.9.36) *Suppose that $\in$ is well-founded, set-like, and extensional on A. Also suppose that $T \subseteq A$ is transitive. Then $\text{mos}_{A\in}(y) = y$ for all $y \in T$.*

Proof. Suppose not, and let a be an $\in$-minimal element of $\{y \in T : \text{mos}_{A\in}(y) \neq y\}$. Then $\text{mos}_{A\in}(a) = \{\text{mos}_{A\in}(y) : y \in A \text{ and } y \in a\} = \{y : y \in A \text{ and } y \in a\} = a$, contradiction. $\square$

Lemma 12.34. (I.9.36) *Suppose that $\in$ is well-founded, set-like, and extensional on A, and A is transitive. Then $\text{mos}_{A\in} = \text{id}_A$.* $\square$

Lemma 12.35. (I.9.37) *Suppose that A and B are transitive classes and f is a bijection from A onto B such that $\forall a_0, a_1 \in A[a_0 \in a_i \leftrightarrow f(a_0) \in f(a_1)]$. Then $f = \text{id}_A$, and $A = B$.*

Proof. We claim that $f(a) = \text{mos}_{A\in}(a)$ for all $a \in A$. Suppose this is not true, and let a be an $\in$-minimal element of $\{y \in A : y \neq \text{mos}_{A\in}(y)\}$. If $y \in a$, then $y \in A$ since A is transitive, so $y = \text{mos}_{A\in}(y)$ by the minimality of a. Thus $a \subseteq \{\text{mos}_{A\in}(y) : y \in A \text{ and } y \in a\}$. Conversely, suppose that $y \in A$ and $y \in a$. Then by the minimality of a, $y = \text{mos}_{A\in}(y)$; so $\text{mos}_{A\in}(y) \in a$. This shows that $a = \text{mos}_{A\in}(a)$, contradiction. So the claim holds.

Now by Lemma 12.34 our lemma follows. $\square$

Lemma 12.36. (I.9.37) *If A is a transitive class and f is a permutation of A such that $\forall a_0, a_1 \in A[a_0 \in a_1 \leftrightarrow f(a_0) \in f(a_1)]$, then $f = \text{id}_A$.* $\square$

Lemma 12.37. (I.9.38) *If $(\text{trcl}(A) \cup \{A\}, \in) \cong (\text{trcl}(B) \cup \{B\}, \in)$, then $A = B$.*

Proof. By Lemma 12.35, $\text{trcl}(A) \cup \{A\} = \text{trcl}(B) \cup \{B\}$. Hence $A = B$ since A is the $\in$-maximal element of $\text{trcl}(A) \cup \{A\}$; similarly for B. $\square$

Proposition 12.38. (I.9.39) *If R is well-founded and set-like on A, then rank_{AR} is one-one iff R^* is a total order on A.*

Proof. $\Rightarrow$: assume that rank_{AR} is one-one, and $a, b \in A$ with $a \neq b$. Say $\text{rank}_{AR}(a) < \text{rank}_{AR}(b)$. By Lemma 12.18 choose $c \in A$ with cR^*b and $\text{rank}_{AR}(a) = \text{rank}_{AR}(c)$. Then $a = c$, so aR^*b.

$\Leftarrow$: assume that R^* is a total order, $a, b \in A$, and $a \neq b$. Say aR^*b. Then $\text{rank}_{AR}(a) < \text{rank}_{AR}(b)$. Thus rank is one-one. $\square$

Proposition 12.39. (I.9.40) *If $R_1 \subseteq R_2$ are both well-founded and set like on A, then $\forall y \in A[\mathrm{rank}_{AR_1}(y) \leq \mathrm{rank}_{AR_2}(y)]$.*

Proof. Suppose not, and let y be R_1-minimal such that $\mathrm{rank}_{AR_2}(y) < \mathrm{rank}_{AR_1}(y)$. Then

$$\begin{aligned}\mathrm{rank}_{AR_1}(y) &= \bigcup\{\mathrm{rank}_{AR_1}(x) + 1 : xR_1y\}\\ &\leq \bigcup\{\mathrm{rank}_{AR_2}(x) + 1 : xR_1y\}\\ &\leq \bigcup\{\mathrm{rank}_{AR_2}(x) + 1 : xR_2y\}\\ &= \mathrm{rank}_{AR_2}(y),\end{aligned}$$

contradiction. □

Proposition 12.40. (I.9.40) *If $R_1 \subseteq R_2$ are both well-founded and set like on A, and if $R_2 \subseteq R_1^*$, then $\forall y \in A[\mathrm{rank}_{AR_1}(y) = \mathrm{rank}_{AR_2}(y)]$.*

Proof. Assume the hypotheses. By Lemma 12.17 let y be R_1^*-minimal such that $\mathrm{rank}_{AR_1}(y) < \mathrm{rank}_{AR_2}(y)$. Then

$$\begin{aligned}\mathrm{rank}_{AR_2}(y) &= \bigcup\{\mathrm{rank}_{AR_2}(x) + 1 : xR_2y\}\\ &\leq \bigcup\{\mathrm{rank}_{AR_1}(x) + 1 : xR_1^*y\}\end{aligned}$$

Now xR_1^*y implies that $\mathrm{rank}_{AR_1}(x) + 1 \leq \mathrm{rank}_{AR_1}(y)$, so the above is $\leq \mathrm{rank}_{AR_1}(y)$, contradiction. □

Proposition 12.41. (I.9.41) *Assume that R is well-founded and set-like on A, and R is a transitive relation on A. Then $\forall a \in A[\mathrm{mos}_{AR}(a) = \mathrm{rank}_{AR}(a)]$.*

Proof. Suppose not, and let a be R-minimal such that $\mathrm{mos}_{AR}(a) \neq \mathrm{rank}_{AR}(a)$. Then

$$\mathrm{mos}_{AR}(a) = \{\mathrm{mos}_{AR}(x) : xRa\} = \{\mathrm{rank}_{AR}(x) : xRa\}$$

Call this set X. Thus X is a set of ordinals. We claim that it is transitive. For, suppose that $\alpha \in \mathrm{rank}(x)$ with xRa. By Lemma 12.18 there is a yR^*x such that $\alpha = \mathrm{rank}_{AR}(y)$. Since R is transitive, we have $R^* = R$. So yRa and hence $\alpha \in X$.

Thus X is a transitive set of transitive sets, so that X is an ordinal. Now $X \subseteq \mathrm{rank}_{AR}(a)$. So $X \neq \mathrm{rank}_{AR}(a)$. Choose $\beta \in \mathrm{rank}_{AR}(a)\backslash X$. Say $\beta \in \mathrm{rank}_{AR}(z) + 1$ with zRa. Then $\beta \leq \mathrm{rank}_{AR}(z) \in X$, so $\beta \in X$, contradiction. □

Proposition 12.42. (I.9.42) *If R well-orders a set A, then $\mathrm{mos}_{AR} = \mathrm{rank}_{AR}$, and mos_{AR} is the isomorphism from (A, R) onto an ordinal.*

Proof. We have $\mathrm{mos}_{AR} = \mathrm{rank}_{AR}$ by Proposition 12.41. Clearly R is extensional, so by Lemma 12.32 mos is an isomorphism from (A, R) onto $(\mathrm{mos}_{AR}[A], \in) = (\mathrm{rank}_{AR}[A], \in)$.

Now $\text{rank}_{AR}[A]$ is a set of ordinals, and it is transitive since $\text{mos}_{AR}[A] = \text{rank}_{AR}[A]$; see Lemma 12.29. So $\text{rank}_{AR}[A]$ is an ordinal. $\square$

Proposition 12.43. (I.9.44) *For any relation R on a class A and any $a \in A$, R is well-founded on* $\text{pred}_{AR^*}(a)$ *iff R is well-founded on* $\{a\} \cup \text{pred}_{AR^*}(a)$.

Proof. $\Leftarrow$ is clear. Now assume that R is well-founded on $\text{pred}_{AR^*}(a)$, and $\emptyset \neq W \subseteq \{a\} \cup \text{pred}_{AR^*}(a)$. If $a \notin W$ the desired conclusion is clear. If $a \in W$ and is an R-minimal element of W, this is as desired. Suppose that $a \in W$ but it is not an R-minimal element of W. Let $W' = W \backslash \{a\}$, and let b be an R-minimal element of W'. Suppose that aRb. Choose $c \in W$ such that cRa. Then cRb, contradiction. Thus b is an R-minimal element of W. $\square$

Let R be set-like on A. The *well-founded initial segment* WF_{AR} is the set of all $a \in A$ such that R is well-founded on $\text{pred}_{AR^*}(a)$. For $a \in WF_{AR}$ define $\text{rank}_{WF}(a) = \text{rank}_{\text{pred}_{AR^*}(a),R}(a)$.

Proposition 12.44. (I.9.46) *Let R be set-like on A and let $W = WF_{AR}$. Then $A\backslash W$ does not have an R-minimal element.*

Proof. Suppose that a is an R-minimal element of $A\backslash W$. Thus $\text{pred}_{AR}(a) \subseteq W$. In fact, $\text{pred}_{AR^*}(a) \subseteq W$. For, if $b \in \text{pred}_{AR^*}(a)\backslash\text{pred}_{AR}(a)$, then there is a $c \in \text{pred}_{AR}(a)$ such that $b \in \text{pred}_{AR^*}(c)$. Since R is well-founded on $\text{pred}_{AR^*}(c)$, it is also well-founded on $\text{pred}_{AR^*}(b)$, so that $b \in W$.

Since $a \notin W$, there is a nonempty $X \subseteq \text{pred}_{AR^*}(a)$ with no R-minimal element. Take any $b \in X$. Then b is not an R-minimal element of X, so $X \cap \text{pred}_{AR}(b)$ is nonempty. Now R is well-founded on $\text{pred}_{AR^*}(b)$ by the previous paragraph, so let c be an R-minimal element of $X \cap \text{pred}_{AR^*}(b)$. Since X does not have an R-minimal element, choose $d \in X$ with dRc. But $d \in X \cap \text{pred}_{AR^*}(b)$, contradiction. $\square$

Proposition 12.45. (I.9.46) *Let R be set-like on A and let $W = WF_{AR}$. Then R is well-founded on W*

Proof. Suppose that X is a nonempty subset of W. Take any $a \in X$. If a is R-minimal in X, this is as desired. Otherwise, the set $X \cap \text{pred}_{AR}(a)$ is nonempty. Since $a \in W$, R is well founded on the set $\text{pred}_{AR^*}(a)$. Let b be an R-minimal element of $X \cap \text{pred}_{AR^*}(a)$. Suppose that $c \in X$ and cRb. Then also $c \in \text{pred}_{AR^*}(a)$, contradiction. $\square$

Proposition 12.46. (I.9.46) *Let R be set-like on A and let $W = WF_{AR}$. Then for $a \in W$,* $\text{rank}_{AR^*}(a) = \text{rank}_{WR}(a)$.

Proof. Suppose that $a \in W$. Let $A' = \{a\} \cup \text{pred}_{AR^*}(a)$. Then $A' \subseteq W$, and $\text{pred}_{W,R^*}(a) \subseteq A'$. Hence $\text{rank}_{AR}(a) = \text{rank}_{A',R}(a) = \text{rank}_{W,R}(a)$ by Lemma 12.22. $\square$

Proposition 12.47. (I.9.46) *Let R be set-like on A and let $W = WF_{AR}$. Then for any $a \in A$, $a \in W$ iff* $\text{pred}_{AR^*}(a) \subseteq W$ *iff* $\text{pred}_{AR}(a) \subseteq W$.

Proof. Suppose that $a \in A$. If $a \in W$, clearly $\text{pred}_{AR^*}(a) \subseteq W$. If $\text{pred}_{AR^*}(a) \subseteq W$, obviously $\text{pred}_{AR}(a) \subseteq W$. Suppose that $\text{pred}_{AR}(a) \subseteq W$. Suppose that $\emptyset \neq X \subseteq \text{pred}_{AR^*}(a)$. Take any $b \in X$. If bRa then $b \in \text{pred}(a) \subseteq W$. Hence either b is an R-minimal element of X, or $\emptyset \neq \{c \in X : cRb\} \subseteq \text{pred}_{AR^*}(b)$ and $\{c \in X : cR^*b\}$ has an R-minimal element d since $b \in W$; and clearly d is an R-minimal element of X. $\square$

Proposition 12.48. (I.9.47) *A relation R on a set A is* **anti-transitive** *iff* $\forall n \in \omega \forall s \in {}^{n+3}A[\forall i < n+2[s(i)Rs(i+1)] \rightarrow \text{not}(s(0)Rs(n+2))]$.

Suppose that R is a well-founded and set-like relation on a set A and all elements of A have finite rank_{AR}. *Then there is a unique anti-transitive $H \subseteq R$ such that $H^* = R^*$.*

Proof. Let

$$H = \{(a,b) : aRb \text{ and } \neg\exists n \in \omega \exists s \in {}^{n+3}A[s_0 = a \text{ and } s_{n+2} = b \text{ and } \forall i < n+2[s_i R s_{i+1}]]\}.$$

Clearly H is anti-transitive,

Now we show by induction on $\text{rank}_{AR}(b)$ that for all a, if aR^*b then aH^*b. This is clear for rank 0, by a vacuous implication. Now suppose it is true for all $i \leq m$, suppose that b has rank $m+1$, and aR^*b. Let n be maximum such that there is an $s \in {}^{n+1}A$ with $s_0 = a$, $s_n = b$, and $s_i R s_{i+1}$ for all $i < n$. If $n = 1$, then clearly aHb. If $n > 1$, then $s_{n-1}Hb$ and aH^*s_{n-1} by the inductive hypothesis, so aH^*b.

For uniqueness, suppose that also $H' \subseteq R$ is anti-transitive with $H'^* = R^*$, but $H \neq H'$.

Case 1. $H \not\subseteq H'$. Say aHb but $\text{not}(aH'b)$. Then aRb, so aR^*b, hence aH'^*b. Let n be maximum such that there is an $s \in {}^{n+2}A$ with $s(0) = a$, $s(n) = b$, and $\forall i < n+1[s(i)H's(i+1)]$. Since $\text{not}(aH'b)$, we have $n > 0$. Now $\forall i < n+1[s(i)Rs(i+1)]$. Since aRb, this contradicts aHb.

Case 2. $H' \not\subseteq H$. Say $aH'b$ but $\text{not}(aHb)$. Hence there exist $n \in \omega$ and $s \in {}^{n+3}A$ such that $s_0 = a$, $s_{n+2} = b$, and $\forall i < n+2[s_i R s_{i+1}]$. Thus $\forall i < n+2[s_i R^* s_{i+1}]$ and hence $\forall i < n+2[s_i H'^* s_{i+1}]$. It follows that there exist an $m \in \omega$ with $m \geq n$ and a $t \in {}^{m+3}A$ such that $t_0 = a$, $t_{m+2} = b$, and $\forall i < m+2[t_i H' t_{i+1}]$. Since $aH'b$, this contradicts H' being anti-transitive. $\square$

Proposition 12.49. (I.9.47) *Let $R = \{(i,j) : i,j \in \omega+\omega$ and $i < j\}$. Then there is no anti-transitive $H \subseteq R$ such that $H^* = R^*$.*

Proof. Suppose that $H \subseteq R$ is anti-transitive and $H^* = R^*$. Now $0R\omega$, so $0R^*\omega$, hence $0H^*\omega$. It follows that there is an $i \in \omega$ such that $iH\omega$. Also, $(i+1)R\omega$, so $(i+1)R^*\omega$, hence $(i+1)H^*\omega$. It follows that there is a $j \in \omega$ with $i < j$ such that $jH\omega$. Now for any $k \in \omega$ we have $kR(k+1)$, hence $kR^*(k+1)$, hence $kH^*(k+1)$, hence $kH(k+1)$. Then $iH(i+1)H \cdots HjH\omega$ and $iH\omega$, contradicting H anti-transitive. $\square$

Proposition 12.50. (I.9.49) *If R is well-founded and set-like on A, then there exist a class M and an isomorphism from (A,R) onto $(M,\in)$.*

Proof. For each $y \in A$ let $F(y) = \{F(x) : xRy\} \cup \{(0,y),(1,y),2,y)\}$. Suppose that $F(x) = F(y)$. Now $(0,x) \in F(x) = F(y)$. Since any element $F(z)$ has at least three elements, and $(0,x)$ has at most two elements, it follows that $(0,x) = (0,y)$ and hence $x = y$.

Clearly xRy implies that $F(x) \in F(y)$. Now suppose that $F(x) \in F(y)$. Again since $F(x)$ has at least three elements and each (ε, y) has at most two elements, it follows that there is a zRy such that $F(x) = F(z)$. Then $x = z$, and so xRy. $\square$

Proposition 12.51. (I.9.50) *If R is set-like on A but not well-founded on A, then there is a function $G : V \times V \to V$ such that there is no function $F : A \to V$ such that $\forall a \in A[F(a) = G(a, F \restriction \{x \in A : xRa\})$.*

Proof. Define

$$G(x,s) = \begin{cases} \bigcup\{s(y) \cup \{s(y)\} : y \in \mathrm{dmn}(s)\} & \text{if } s \text{ is a function,} \\ \emptyset & \text{otherwise} \end{cases}$$

Suppose that $F : A \to V$ is such that $\forall a \in A[F(a) = G(a, F \restriction \{x \in A : xRa\})]$. Let X be a nonempty subset of A with no R-minimal element. Choose $a \in F[X]$ such that $a \cap F[X] = \emptyset$. Say $a = F(y)$ with $y \in X$. Take any z such that zRy. Then $z \in \mathrm{dmn}(F \restriction \{x : xRy\})$, and so $F(z) \cup \{F(z)\} \subseteq G(y, F \restriction \{x : xRy\}) = F(y)$. Thus $F(z) \in F(z) \cup \{F(z)\} = F(y) = a$, contradiction. $\square$

(I.13.27) For any infinite cardinal κ, $H(\kappa) = \{x : |\mathrm{trcl}(x)| < \kappa\}$. Elements of $H(\omega)$ are *hereditarily finite*, and elements of $H(\omega_1)$ are *hereditarily countable.*

Lemma 12.52. (I.13.28) *For any infinite cardinal κ, $H(\kappa) \subseteq V_\kappa$.*

Proof. Suppose that $x \in H(\kappa)$. Let $\alpha = \mathrm{rank}(x)$. If $\xi < \alpha$, then there is a $z \in \mathrm{trcl}(x)$ such that $\mathrm{rank}(z) = \xi$, by Lemma 12.18. Clearly $\mathrm{rank}(z) < \mathrm{rank}(x)$ for all $z \in \mathrm{trcl}(x)$. Thus $\alpha = \{\mathrm{rank}(z) : z \in \mathrm{trcl}(x)\}$. Now $\langle \mathrm{rank}(z) : z \in \mathrm{trcl}(x)\rangle$ maps $\mathrm{trcl}(x)$ onto α. Hence $|\alpha| \le |\mathrm{trcl}(z)| < \kappa$, so $\alpha < \kappa$. $\square$

Lemma 12.53. (I.13.28) *For any infinite cardinal κ, $|H(\kappa)| = 2^{<\kappa}$,*

Proof. If $\lambda < \kappa$, then $\mathscr{P}(\lambda) \subseteq H(\kappa)$, so $2^\lambda \le |H(\kappa)|$. Hence $2^{<\kappa} \le |H(\kappa)|$.

Now we define a function $F : H(\kappa) \to \bigcup_{\lambda<\kappa} \mathscr{P}(\lambda \times \lambda)$. If $x \in H(\kappa)$, let $\lambda = |\mathrm{trcl}(x) \cup \{x\}|$; so $\lambda < \kappa$. Let $f : \mathrm{trcl}(x) \cup \{x\} \to \lambda$ be a bijection, and let $F(x) = \{(f(u), f(v)) : u, v \in \mathrm{trcl}(x) \cup \{x\}$ and $u \in v\}$. Then f is an isomorphism from $(\mathrm{trcl}(x) \cup \{x\}, \in)$ onto (λ, F). Hence F is one-one by Lemma 12.37. Hence

$$|H(\kappa)| \le \left|\bigcup_{\lambda<\kappa} \mathscr{P}(\lambda \times \lambda)\right| \le \sum_{\lambda<\kappa} 2^\lambda = 2^{<\kappa}. \qquad \square$$

Lemma 12.54. (I.13.29) *If $\kappa > \omega$, then $H(\kappa) = V_\kappa$ iff $\kappa = \beth_\kappa$.*

Proof. $\Rightarrow$: Assume that $H(\kappa) = V_\kappa$. If $\omega^2 \le \alpha < \kappa$, then $|V_\alpha| = \beth_\alpha$ by Lemma 12.10. Now $V_\alpha \in V_\kappa = H(\kappa)$, and V_α is transitive, so $|\mathrm{trcl}(V_\alpha)| = |V_\alpha| = \beth_\alpha < \kappa$. It follows that $\beth_\kappa \le \kappa$. But also $\kappa \le \beth_\kappa$, so $\kappa = \beth_\kappa$.

$\Leftarrow$: Assume that $\kappa = \beth_\kappa$. Take any $x \in V_\kappa$. Then $x \in V_\alpha$ for some α with $\omega^2 \le \alpha < \kappa$. Then, using Lemma 12.10 $|\text{trcl}(x)| \le |V_\alpha| = \beth_\alpha < \beth_\kappa = \kappa$. Thus $x \in H(\kappa)$. So we have shown that $V_\kappa \subseteq H(\kappa)$. Hence by Lemma 12.52, $V_\kappa = H(\kappa)$. □

Lemma 12.55. (I.13.32) *Suppose that κ is regular. If $y \subseteq H(\kappa)$ and $|y| < \kappa$, then $y \in H(\kappa)$.*

Proof. $\text{trcl}(y) = \bigcup_{z\in y}(\{z\} \cup \text{trcl}(z))$, so $|\text{trcl}(y)| \le \sum_{z\in y}(1 + |\text{trcl}(z)|) < \kappa$. □

Lemma 12.56. (I.13.32) *Suppose that κ is regular. If $f : z \to H(\kappa)$ and $z \in H(\kappa)$, then $f \in H(\kappa)$ and* $\text{rng}(f) \in H(\kappa)$.

Proof. If $(z, u) \in f$, then

$$\begin{aligned}\text{trcl}((z,u)) &= \text{trcl}(\{\{z\},\{z,u\}\}) = \{\{z\},\{z,u\}\} \cup \text{trcl}(\{z\}) \cup \text{trcl}(\{z,u\})\\ &= \{\{z\},\{z,u\}, z, u\} \cup \text{trcl}(z) \cup \text{trcl}(u),\end{aligned}$$

and hence $|\text{trcl}((z,u))| < \kappa$, i.e., $(z,u) \in H(\kappa)$. So $f \subseteq H(\kappa)$. Also, $|f| = |z| < \kappa$. Hence $f \in H(\kappa)$ by Lemma 12.55. If $u \in \text{rng}(f)$, choose $v \in z$ such that $f(z) = u$. Then $u \in \{v, u\} \in (v, u) \in f$, so $u \in \text{trcl}(f)$. Thus $\text{rng}(f) \subseteq \text{trcl}(f)$, hence $\text{trcl}(\text{rng}(f)) \subseteq \text{trcl}(f)$. So $\text{rng}(f) \in H(\kappa)$. □

Proposition 12.57. (I.13.33) $\beth_\omega^{\aleph_0} = \prod_{n\in\omega} \beth_n = \beth_{\omega+1}$.

Proof.

$$\beth_\omega^\omega \le \beth_\omega^{\beth_\omega} = 2^{\beth_\omega} = \beth_{\omega+1} = 2^{\beth_\omega} = 2^{\sum_{n\in\omega}\beth_n} = \prod_{n\in\omega} 2^{\beth_n} \le \prod_{n\in\omega} \beth_n \le \beth_\omega^\omega$$ □

For the next results we need a simple result from infinite combinatorics. For cardinals κ, λ, μ we write $\kappa \to (\lambda, \mu)^2$ iff whenever $[\kappa]^2 = A \cup B$ either there is an $X \in [\kappa]^\lambda$ such that $[X]^2 \subseteq A$ or there is an $X \in [\kappa]^\mu$ such that $[X]^2 \subseteq B$.

Theorem 12.58. (Dushnik, Miller) *For any infinite cardinal κ we have $\kappa \to (\kappa, \omega)^2$.*

Proof. Suppose that $f : [\kappa]^2 \to 2$; we want to find a set $X \in [\kappa]^\kappa$ such that $f[[X]^2] = \{0\}$, or a set $X \in [\kappa]^\omega$ such that $f[[X]^2] = \{1\}$.

For each $x \in \kappa$ let $B(x) = \{y \in \kappa\backslash\{x\} : f(\{x, y\}) = 1\}$. Now we claim:

Claim. Suppose that for every $X \in [\kappa]^\kappa$ there is an $x \in X$ such that $|B(x) \cap X| = \kappa]$. Then there is an infinite $X \subseteq \kappa$ such that $f[[X]^2] \subseteq \{1\}$.

Proof of claim. Assume the hypothesis. We define x_n, Y_n for $n \in \omega$ by recursion. Let $Y_0 = \kappa$. Assume that $Y_n \in [\kappa]^\kappa$ has been defined. Then by supposition there is an $x_n \in Y_n$ such that $|B(x_n) \cap Y_n| = \kappa$. Let $Y_{n+1} = B(x_n) \cap Y_n$. Now if $n < m < \omega$, then $x_m \in Y_{n+1} \subseteq B(x_n)$, and hence $f(\{x_n, x_m\}) = 1$. Thus $\{x_n : n \in \omega\}$ is an infinite subset of κ such that $f[[\{x_n : n \in \omega\}]^2] \subseteq \{1\}$, as desired. □

First suppose that κ is regular, and assume that there is no $X \in [\kappa]^\kappa$ such that $f[[X]^2] \subseteq \{0\}$. We will verify the hypothesis of the claim; this gives the desired conclusion. So, suppose that $X \in [\kappa]^\kappa$. By Zorn's lemma let $Y \subseteq X$ be maximal such that $f[[Y]^2] \subseteq \{0\}$. Thus $|Y| < \kappa$ by assumption. Now

$$X \backslash Y \subseteq \bigcup_{y \in Y} \{x \in X \backslash Y : f(\{x, y\}) = 1\} = \bigcup_{y \in Y} [B(y) \cap (X \backslash Y)].$$

Since $|Y| < \kappa$ and κ is regular, there is a $y \in Y$ such that $|B(y) \cap X| = \kappa$. This verifies the hypothesis of the claim.

Second suppose that κ is singular, and suppose that there is no infinite $X \subseteq \kappa$ such that $f[[X]^2] \subseteq \{1\}$. Then by the claim,

$$(*) \qquad \exists X \in [\kappa]^\kappa \forall x \in X[|B(x) \cap X| < \kappa].$$

Let $\langle \lambda_\xi : \xi < \mathrm{cf}(\kappa) \rangle$ be a strictly increasing sequence of regular cardinals with supremum κ and with $\mathrm{cf}(\kappa) < \lambda_0$, and let $\langle Y_\xi : \xi < \mathrm{cf}(\kappa) \rangle$ be a system of pairwise disjoint subsets of X such that $\forall \xi < \mathrm{cf}(\kappa)[|Y_\xi| = \lambda_\xi]$. By the regular case, $\lambda_\xi \to (\lambda_\xi, \omega)^2$ for each $\xi < \mathrm{cf}(\kappa)$. It follows that for each $\xi < \mathrm{cf}(\kappa)$ there is a $Z_\xi \in [Y_\xi]^{\lambda_\xi}$ such that $f[[Z_\xi]^2] \subseteq \{0\}$. Now for each $\xi < \mathrm{cf}(\kappa)$, by $(*)$,

$$Z_\xi = \bigcup_{\alpha < \mathrm{cf}(\kappa)} \{x \in Z_\xi : |B(x) \cap X| < \lambda_\alpha\}.$$

Since $|Z_\xi| = \lambda_\xi > \mathrm{cf}(\kappa)$ and λ_ξ is regular, there is an $h(\xi) < \mathrm{cf}(\kappa)$ such that

$$W_\xi \stackrel{\mathrm{def}}{=} \{x \in Z_\xi : |B(x) \cap X| < \lambda_{h(\xi)}\}$$

has size λ_ξ.

Now we define a sequence $\langle \alpha_\xi : \xi < \mathrm{cf}(\kappa) \rangle$ of ordinals less than κ by recursion. If α_η has been defined for all $\eta < \xi$, with $\xi < \mathrm{cf}(\kappa)$, then the set $\{\alpha_\eta : \eta < \xi\} \cup \{\lambda_{h(\eta)} : \eta < \xi\}$ is bounded below κ and so there is an $\alpha_\xi < \kappa$ greater than each member of this set. Thus if $\eta < \xi$ then $\alpha_\eta < \alpha_\xi$ and $\lambda_{h(\eta)} < \alpha_\xi$. Now for any $\xi < \mathrm{cf}(\kappa)$ let

$$S_\xi = W_{\alpha_\xi} \backslash \bigcup \left\{ B(x) \cap X : x \in \bigcup_{\eta < \xi} W_{\alpha_\eta} \right\}.$$

Note that if $\eta < \xi < \mathrm{cf}(\kappa)$ then $|W_{\alpha_\eta}| = \lambda_{\alpha_\eta} < \lambda_{\alpha_\xi}$ and $\xi < \mathrm{cf}(k) < \lambda_0 < \lambda_{\alpha_\xi}$, so $\left|\bigcup_{\eta<\xi} W_{\alpha_\eta}\right| < \lambda_{\alpha_\xi}$. Moreover, if $\eta < \xi$ and $x \in W_{\alpha_\eta}$, then $|B(x) \cap X| < \lambda_{h(\alpha_\eta)} < \lambda_\xi$. Hence for each $x \in \bigcup_{\eta<\xi} W_{\alpha_\eta}$ we have $|B(x) \cap X| < \lambda_{\alpha_\xi}$. Hence $|S_\xi| = \lambda_{\alpha_\xi}$. Let $T = \bigcup_{\xi < \mathrm{cf}(\kappa)} S_\xi$. So $|T| = \kappa$. We claim that $f[[T]^2] \subseteq \{0\}$. For, suppose that $x, y \in T$ with $x \neq y$.

Case 1. There is a $\xi < \mathrm{cf}(\kappa)$ such that $x, y \in S_\xi$. Now $S_\xi \subseteq W_{\alpha_\xi} \subseteq Z_{\alpha_\xi}$, so $f(\{x, y\}) = 0$.

Case 2. There exist $\eta < \xi < \mathrm{cf}(\kappa)$ such that $x \in S_\eta$ and $y \in S_\xi$. (The case $x \in S_\xi$ and $y \in S_\eta$ is treated similarly.) Then $x \in W_{\alpha_\eta}$, so $y \notin B(x)$, i.e. $f(\{x, y\}) = 0$. $\square$

Proposition 12.59. (I.13.37) *Let κ be an infinite cardinal and let $\lhd$ be a well-order of κ. Then there is an $X \in [\kappa]^\kappa$ such that $\lhd$ and $<$ agree on X.*

Proof. We use the Dushnik-Miller theorem $\kappa \to (\kappa, \omega)^2$. Let

$$[X]^2 = \{\{x, y\} : x < y \text{ and } x \lhd y\} \cup \{\{x, y\} : x < y \text{ and } y \lhd x\}.$$

If $Y \in [X]^\omega$ is homogeneous for the second member here, then we get $x_0 < x_1 < \cdots$ in Y such that $x_0 \rhd x_1 \rhd \cdots$, contradiction. So there is a homogeneous set of size κ for the first member, as desired. $\square$

Proposition 12.60. (I.13.38) (Milner-Rado paradox) *If κ is an infinite cardinal and $\kappa \leq \alpha < \kappa^+$, then there are $X_n \subseteq \alpha$ for $n < \omega$ such that $\bigcup_{n<\omega} X_n = \alpha$ and $\forall n \in \omega[\mathrm{type}(X_n) \leq \kappa^n]$.*

Proof. We prove this by induction on α, where $\kappa \leq \alpha < \kappa^+$. It is clear for $\alpha = \kappa$. Assume that it is true for α. Say $\bigcup_{n<\omega} X_n = \alpha$ and $\mathrm{type}(X_n) \leq \kappa^n$. Then $\bigcup_{n<\omega}(X_n \cup \{\alpha\}) = \alpha + 1$ and $\mathrm{type}(X_n \cup \{\alpha\}) \leq \kappa^{n+1}$.

Now suppose that α is limit and we know the result for all $\beta \in [\kappa, \alpha)$. For each $\beta < \alpha$ let $\langle X_{\beta n} : n \in \omega\rangle$ be such that $\beta = \bigcup_{n<\omega} X_{\beta n}$ and $\mathrm{type}(X_{\beta,n}) \leq \kappa^{n+1}$. Let $\langle \beta_\gamma : \gamma < \mathrm{cf}(\alpha)\rangle$ be strictly increasing, continuous, with $\beta_0 = 0$ and with union α. Define $X_n = \bigcup_{\gamma<\mathrm{cf}(\alpha)}(X_{\beta_{\gamma+1},n} \backslash \beta_\gamma)$. Then

$$\begin{aligned}
\bigcup_{n<\omega} X_n &= \bigcup_{n<\omega} \bigcup_{\gamma<\mathrm{cf}(\alpha)} (X_{\beta_{\gamma+1},n} \backslash \beta_\gamma) \\
&= \bigcup_{\gamma<\mathrm{cf}(\alpha)} \bigcup_{n\in\omega} (X_{\beta_{\gamma+1},n} \backslash \beta_\gamma) \\
&= \bigcup_{\gamma<\mathrm{cf}(\alpha)} (\beta_{\gamma+1} \backslash \beta_\gamma) \\
&= \alpha
\end{aligned}$$

Moreover, $\mathrm{type}(X_n) \leq \kappa^n \cdot \mathrm{cf}(\alpha) \leq \kappa^{n+1}$. $\square$

Proposition 12.61. *The elements of V_α for $\alpha < 5$ are as follows:*

$V_0 = \emptyset$.

$V_1 = \mathscr{P}(V_0) = \mathscr{P}(\emptyset) = \{\emptyset\} = 1$.

$V_2 = \mathscr{P}(V_1) = \mathscr{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\} = 2$.

$V_3 = \mathscr{P}(V_2) = \mathscr{P}(\{\emptyset, \{\emptyset\}\}) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$. *Note that V_3 has four elements, but it is not equal to 4, since, for example, $\{\{\emptyset\}\} \in V_3 \backslash 4$.*

For V_4, it helps to use the usual abbreviations for natural numbers. Thus $V_3 = \{0, 1, 2, \{1\}\}$. We list out the subsets of V_3 with 0,1,2,3,4 elements:

$$\begin{aligned}
V_4 = \mathscr{P}(V_3) = \{0 & \quad \textit{0 elements; 1 of these}\\
& \{0\}, \{1\}, \{2\}, \{\{1\}\} \quad \textit{1 element, 4 of these}\\
& \{0,1\}, \{0,2\}, \{0,\{1\}\}, \{1,2\}, \{1,\{1\}\}, \{2,\{1\}\} \quad \textit{2 elements, 6 of these}\\
& \{0,1,2\}, \{0,1,\{1\}\}, \{0,2,\{1\}\}, \{1,2,\{1\}\} \quad \textit{3 elements, 4 of these}\\
& \{0,1,2,\{1\}\} \quad \textit{4 elements, 1 of these.}\}
\end{aligned}$$

□

Proposition 12.62. *There is a class function* $\mathbf{S}$ *with domain* $\mathbf{On}$ *such that for any ordinal* α,

$$\mathbf{S}(\alpha) = \bigcup_{\beta<\alpha} \mathscr{P}(\mathbf{S}(\beta)).$$

Proof. We apply the recursion theorem 9.7. Define $\mathbf{G} : \mathbf{On} \times \mathbf{V} \to \mathbf{V}$ by setting, for any ordinal α and any set x,

$$\mathbf{G}(\alpha, x) = \begin{cases} \bigcup_{\beta<\alpha} \mathscr{P}(x(\beta)) & \text{if } x \text{ is a function with domain } \alpha,\\ \emptyset & \text{otherwise.}\end{cases}$$

Then obtain $\mathbf{F} : \mathbf{On} \to \mathbf{V}$ by Theorem 6.7: for any ordinal α, $\mathbf{F}(\alpha) = \mathbf{G}(\alpha, \mathbf{F} \restriction \alpha)$. Thus $\mathbf{F}(\alpha) = \bigcup_{\beta<\alpha} \mathscr{P}(\mathbf{F}(\beta))$. □

Proposition 12.63. *With* $\mathbf{S}$ *as in Proposition 12.62,* $\forall\alpha \in \mathbf{On}[\mathbf{S}(\alpha) = V_\alpha]$.

Proof. We prove that $V_\alpha = \mathbf{S}(\alpha)$ for all α by induction:

$$\begin{aligned}
\mathbf{S}(0) &= \bigcup_{\beta<0} \mathscr{P}(\mathbf{S}(\beta)) = \emptyset = V_0;\\
\mathbf{S}(\alpha+1) &= \bigcup_{\beta<\alpha+1} \mathscr{P}(\mathbf{S}(\beta))\\
&= \bigcup_{\beta<\alpha} \mathscr{P}(\mathbf{S}(\beta)) \cup \mathscr{P}(\mathbf{S}(\alpha))\\
&= V_\alpha \cup \mathscr{P}(V_\alpha) \quad \text{(inductive hypothesis)}\\
&= V_{\alpha+1} \quad \text{(using Theorem 12.4(ii));}\\
\mathbf{S}(\alpha) &= \bigcup_{\beta<\alpha} \mathscr{P}(\mathbf{S}(\beta)) \quad \text{(with } \alpha \text{ limit)}\\
&= \bigcup_{\gamma<\alpha} \bigcup_{\beta<\gamma} \mathscr{P}(\mathbf{S}(\beta))\\
&= \bigcup_{\gamma<\alpha} V_\gamma \quad \text{(inductive hypothesis)}\\
&= V_\alpha.
\end{aligned}$$

□

Proposition 12.64.

(i) $\text{rank}(x \cap y) \leq \min(\text{rank}(x), \text{rank}(y))$*;*

(ii) $\text{rank}(x \cap y)$ *can take any value* $\leq \min(\text{rank}(x), \text{rank}(y))$*;*

(iii) $\text{rank}(x \backslash y) \leq \text{rank}(x)$*;*

(iv) $\text{rank}(x \backslash y)$ *can take any value* $\leq \text{rank}(x)$*;*

(v) $\text{rank}(\text{dmn}(R)) \leq \bigcup\bigcup \text{rank}(R)$*;*

(vi) If $\beta \leq \bigcup\bigcup\alpha$*, then there is an* R *such that* $\text{rank}(R) = \alpha$ *and* $\text{rank}(\text{dmn}(R)) = \beta$*;*

(vii) For any ordinals α, β *the following are equivalent:*

(a) There is an R *such that* $\text{rank}(R) = \alpha$ *and* $\text{rank}(R^{-1}) = \beta$*.*

(b) $\beta \leq \alpha$ *and one of the following holds:*

(I) $\beta = \gamma + 3$ *for some ordinal* γ*;*

(II) β *is a limit ordinal.*

Proof. (i): Let α and β be as in (iv). We claim that $\text{rank}(x \cap y) \leq \min(\text{rank}(x), \text{rank}(y))$. To prove this, by symmetry assume that $\alpha \leq \beta$. Then $x \cap y \subseteq x \subseteq V_\alpha$, so $\text{rank}(x \cap y) \leq \alpha$.

(ii): Now suppose that γ is any ordinal, and suppose that $\delta \leq \gamma$. We define two sets x and y such that $\min(\text{rank}(x), \text{rank}(y)) = \gamma$ while $\text{rank}(x \cap y) = \delta$. Let $x = \delta \cup \{\gamma\}$ and $y = \gamma$. Then $\text{rank}(x) = \gamma + 1$, $\text{rank}(y) = \gamma$, and $\text{rank}(x \cap y) = \text{rank}(\delta) = \delta$.

(iii): Let $\alpha = \text{rank}(x)$. We claim that $\text{rank}(x \backslash y) \leq \alpha$. In fact, $x \backslash y \subseteq x \subseteq V_\alpha$, so $x \backslash y \in V_{\alpha+1}$, and so $\text{rank}(x \backslash y) \leq \alpha$.

(iv): Let $\beta \leq \alpha$; we define x, y so that $\text{rank}(x) = \alpha$ while $\text{rank}(x \backslash y) = \beta$. Let $x = \alpha$ and $y = \alpha \backslash \beta$.

(v): Let $\alpha = \text{rank}(R)$. We claim that $\text{rank}(\text{dmn}(R)) \leq \bigcup\bigcup\alpha$. For, take any $x \in \text{dmn}(R)$. Choose y such that $(x, y) \in R$. So $x \in \{x\} \in (x, y) \in R$; it follows that $x \in \bigcup\bigcup R$. So $\text{dmn}(R) \subseteq \bigcup\bigcup R$, and so $\text{rank}(\text{dmn}(R)) \leq \text{rank}(\bigcup\bigcup R) = \bigcup\bigcup\alpha$ by (ix).

(vi): We claim that if $\beta \leq \bigcup\bigcup\alpha$, then there is a set R such that $\text{rank}(R) = \alpha$ while $\text{rank}(\text{dmn}(R)) = \beta$. To give the examples here, we consider two cases.

Case 1. $\beta = 0$. Then we take $R = \alpha$. We use the easy fact that no ordinal is an ordered pair. [(a, b) has at most two elements, and is a nonempty set. So the only possibilities for (a, b) to be an ordinal are $(a, b) = 1$ or $(a, b) = 2$. Since $(a, b) = \{\{a\}, \{a, b\}\}$, neither case is really possible, since the members of (a, b) are nonempty.]

Case 2. $0 < \beta$. Let $R = \{(\xi, 0) : \xi < \beta\} \cup \alpha$. To see that this works, note that $\text{dmn}(R) = \beta = \text{rank}(\beta)$. But we also need to see that $\text{rank}(R) \leq \alpha$. For this we consider several subcases.

Subcase 2.1. α is a limit ordinal. Then $\bigcup\bigcup\alpha = \alpha$, $\text{rank}(\{(\xi, 0) : \xi < \beta\}) \leq \alpha$

Subcase 2.2. $\alpha = \gamma + 1$ for some limit ordinal γ. Since $\beta \leq \gamma$, clearly $\text{rank}(\{(\xi, 0) : \xi < \beta\}) \leq \gamma < \alpha$

Subcase 2.3. $\alpha = \gamma + 2$ for some limit ordinal γ. Similar to Subcase 2.2.

Subcase 2.4. $\alpha = \gamma + n$ for some limit ordinal γ and some $n \in \omega \backslash 3$. Then $\beta \leq \gamma + n - 2$, and so $\text{rank}(\{(\xi, 0) : \xi < \beta\}) \leq \gamma + n$

Subcase 2.5. $0 < \beta \leq \alpha - 2$, with $\alpha \in \omega \backslash 3$. Again clearly ok.

These are all of the possibilities.

(vii): We first note that if a and b have ranks ξ, η respectively, then (a, b) has rank $\max(\xi, \eta) + 2$ by (iii). Hence if S is a collection of ordered pairs, then $\text{rank}(S) =$

$\sup\{\text{rank}(s)+1 : s \in S\}$, and hence $\text{rank}(S)$ is either a limit ordinal (if $\{\text{rank}(s)+1 : s \in S\}$ does not have a greatest element) or it is of the form $\gamma + 3$. It follows that if $\text{rank}(R) = \alpha$ and $\text{rank}(R^{-1}) = \beta$, then $\beta \le \alpha$ and (I) or (II) holds.

Now suppose that $\beta \le \alpha$. If $\beta = \gamma + 3$ for some γ, let $R = \{(\gamma, \gamma)\} \cup \alpha$; then $\text{rank}(R) = \alpha$ and $\text{rank}(R^{-1}) = \beta$. If β is a limit ordinal, let $R = \{(\gamma, \gamma) : \gamma < \beta\} \cup \alpha$; then $\text{rank}(R) = \alpha$ and $\text{rank}(R^{-1}) = \beta$. □

Proposition 12.65. *Define* $x\mathbf{R}y$ *iff* $(x, 1) \in y$. *Then* $\mathbf{R}$ *is well-founded and set-like on* $\mathbf{V}$.

Proof. Suppose that X is a nonempty set. Choose $x \in X$ of smallest rank. Suppose that $y \in X$ and yRx. Thus $(y, 1) \in x$, so $y \in \{y\} \in (y, 1) \in x$, and hence $\text{rank}(y) < \text{rank}(x)$, contradiction. Hence $\mathbf{R}$ is well-founded on $\mathbf{V}$.

For any $y \in \mathbf{V}$ we have $\text{pred}_{\mathbf{VR}}(y) = \{x : (x, 1) \in y\}$. Note that if $(x, 1) \in y$ then $x \in \{x\} \in \{\{x\}, \{x, 1\}\} = (x, 1) \in y$, so $x \in \bigcup\bigcup y$. So $\text{pred}_{\mathbf{VR}}(y) = \{x \in \bigcup\bigcup y : (x, 1) \in y\}$, and hence $\text{pred}_{\mathbf{VR}}(y)$ is a set. Thus $\mathbf{R}$ is set-like on $\mathbf{V}$. □

Proposition 12.66. *Define* $x\mathbf{R}y$ *iff* $x \in \text{trcl}(y)$. *Then* $\mathbf{R}$ *is well-founded and set like on* $\mathbf{V}$.

Clearly $x\mathbf{R}y$ implies that $\text{rank}(x) < \text{rank}(\text{trcl}(y)) = \text{rank}(y)$, so $\mathbf{R}$ is well-founded. For any $a \in \mathbf{V}$, the class $\{b \in \mathbf{V} : b\mathbf{R}a\} = \{b : b \in \text{trcl}(a)\} = \text{trcl}(a)$, a set. □

13. Absoluteness

Roughly speaking, a formula is absolute provided that its meaning does not change in going from one set to a bigger one, or vice versa. The exact definition is as follows.

• Suppose that $\mathbf{M} \subseteq \mathbf{N}$ are classes and $\varphi(x_1, \ldots, x_n)$ is a formula of our set-theoretical language. We say that φ is *absolute for* $\mathbf{M}, \mathbf{N}$ iff

$$\forall x_1, \ldots, x_n \in \mathbf{M}[\varphi^{\mathbf{M}}(x_1, \ldots, x_n) \text{ iff } \varphi^{\mathbf{N}}(x_1, \ldots, x_n)].$$

An important special case of this notion occurs when $\mathbf{N} = \mathbf{V}$. Then we just say that φ is *absolute for* $\mathbf{M}$.

More formally, we associate with three formulas $\mu(y, w_1, \ldots, w_m)$, $\nu(y, w_1, \ldots, w_m)$, $\varphi(x_1, \ldots, x_n)$ another formula "φ is absolute for μ, ν", namely the following formula:

$$\forall x_1, \ldots, x_n \left[\bigwedge_{1 \le i \le n} \mu(x_i) \to [\varphi^{\mu}(x_1, \ldots, x_n) \leftrightarrow \varphi^{\nu}(x_1, \ldots, x_n)] \right].$$

In full generality, very few formulas are absolute, as we will see later. Usually we need to assume that the sets are transitive. Then there is an important set of formulas all of which are absolute; this class is defined as follows.

• The set of Δ_0*-formulas* is the smallest set Γ of formulas satisfying the following conditions:
(a) Each atomic formula is in Γ.
(b) If φ and ψ are in Γ, then so are $\neg\varphi$ and $\varphi \wedge \psi$.
(c) If φ is in Γ, then so are $\exists x \in y\varphi$ and $\forall x \in y\varphi$.

Recall here that $\exists x \in y\varphi$ and $\forall x \in y\varphi$ are abbreviations for $\exists x(x \in y \wedge \varphi)$ and $\forall x(x \in y \to \varphi)$ respectively.

Theorem 13.1. *If* $\mathbf{M}$ *is transitive and* φ *is* Δ_0*, then* φ *is absolute for* $\mathbf{M}$*.*

Proof. We show that the collection of formulas absolute for $\mathbf{M}$ satisfies the conditions defining the set Δ_0. Absoluteness is clear for atomic formulas. It is also clear that if φ and ψ are absolute for $\mathbf{M}$, then so are $\neg\varphi$ and $\varphi \wedge \psi$. Now suppose that φ is absolute for $\mathbf{M}$; we show that $\exists x \in y\varphi$ is absolute for $\mathbf{M}$. Implicitly, φ can involve additional parameters $w_1, \ldots, w_n$. Assume that $y, w_1, \ldots, w_n \in \mathbf{M}$. First suppose that $\exists x \in y\varphi(x, y, w_1, \ldots, w_n)$. Choose $x \in y$ so that $\varphi(x, y, w_1, \ldots, w_n)$. Since $\mathbf{M}$ is transitive, $x \in \mathbf{M}$. Hence by the "inductive assumption", $\varphi^{\mathbf{M}}(x, y, w_1, \ldots, w_n)$ holds. This shows that $(\exists x \in y\varphi(x, y, w_1, \ldots, w_n))^{\mathbf{M}}$. Conversely suppose that $(\exists x \in y\varphi(x, y, w_1, \ldots, w_n))^{\mathbf{M}}$. Thus $\exists x \in \mathbf{M}[x \in y \wedge \varphi^{\mathbf{M}}(x, y, w_1, \ldots, w_n)$. By the inductive assumption, $\varphi(x, y, w_1, \ldots, w_n)$. So this shows that $\exists x \in y\varphi(x, y, w_1, \ldots, w_n)$. The case $\forall x \in y\varphi$ is treated similarly. □

Ordinals and special kinds of ordinals are absolute since they could have been defined using Δ_0 formulas:

Theorem 13.2. *The following are absolute for any transitive class:*

(i) x is an ordinal *(iii) x is a successor ordinal* *(v) x is ω*
(ii) x is a limit ordinal *(iv) x is a finite ordinal* *(vi) x is i (each $i < 10$)*

Proof.

$$\begin{aligned}
x \text{ is an ordinal} &\leftrightarrow \forall y \in x \forall z \in y[z \in x] \wedge \forall y \in x \forall z \in y \forall w \in z[w \in y];\\
x \text{ is a limit ordinal} &\leftrightarrow \exists y \in x[y = y] \wedge x \text{ is an ordinal} \wedge \forall y \in x \exists z \in x(y \in z);\\
x \text{ is a successor ordinal} &\leftrightarrow x \text{ is an ordinal} \wedge x \neq \emptyset \wedge x \text{ is not a limit ordinal};\\
x \text{ is a finite ordinal} &\leftrightarrow \forall y[y \notin x] \vee (x \text{ is a successor ordinal}\\
&\quad \wedge \forall y \in x(\forall z[z \notin y] \vee y \text{ is a successor ordinal}));\\
x = \omega &\leftrightarrow x \text{ is a limit ordinal} \wedge \forall y \in x(y \text{ is a finite ordinal});
\end{aligned}$$

finally, we do (vi) by induction on i. The case $i = 0$ is clear. Then

$$y = i + 1 \leftrightarrow \exists x \in y[x = i \wedge \forall z \in y[z \in x \vee z = x] \wedge \forall z \in x[z \in y] \wedge x \in y]. \qquad \square$$

The following theorem, while obvious, will be very useful in what follows.

Theorem 13.3. *Suppose that S is a set of sentences in our set-theoretic language, and $\mathbf{M}$ and $\mathbf{N}$ are classes which are models of S. Suppose that*

$$S \models \forall x_1, \ldots, x_n[\varphi(x_1, \ldots, x_n) \leftrightarrow \psi(x_1, \ldots, x_n)].$$

Then φ is absolute for $\mathbf{M}, \mathbf{N}$ iff ψ is. $\square$

Of course we will usually apply this when S is a subset of ZFC.

Recall that all of the many definitions that we have made in our development of set theory are supposed to be eliminable in favor of our original language. To apply Theorem 13.3, we should note that the development of the very elementary set theory in Chapter 6 did not use the axiom of choice or the axiom of infinity. We let ZF be our axioms without the axiom of choice, and ZF − Inf the axioms ZF without the axiom of infinity.

The status of the functions that we have defined requires some explanation. Whenever we defined a function $\mathbf{F}$ of n arguments, we have implicitly assumed that there is an associated formula φ whose free variables are among the first $n+1$ variables, so that the following is derivable from the axioms assumed at the time of defining the function:

$$\forall v_0, \ldots, v_{n-1} \exists! v_n \varphi(v_0, \ldots, v_n).$$

Recall that "$\exists! v_n$" means "there is exactly one v_n". Now if we have a class model $\mathbf{M}$ in which this sentence holds, then we can define $\mathbf{F}^{\mathbf{M}}$ by setting, for any $x_0, \ldots, x_{n-1} \in \mathbf{M}$,

$$\mathbf{F}^{\mathbf{M}}(x_0, \ldots, x_{n-1}) = \text{ the unique } y \text{ such that } \varphi^{\mathbf{M}}(x_0, \ldots, x_{n-1}, y).$$

In case $\mathbf{M}$ satisfies the indicated sentence, we say that $\mathbf{F}$ *is defined in* $\mathbf{M}$. Given two class models $\mathbf{M} \subseteq \mathbf{N}$ in which $\mathbf{F}$ is defined, we say that $\mathbf{F}$ is *absolute for* $\mathbf{M}, \mathbf{N}$ provide that φ is. Note that for $\mathbf{F}$ to be absolute for $\mathbf{M}, \mathbf{N}$ it must be defined in both of them.

Proposition 13.4. *Suppose that* $\mathbf{M} \subseteq \mathbf{N}$ *are models in which* $\mathbf{F}$ *is defined. Then the following are equivalent:*

(i) $\mathbf{F}$ *is absolute for* $\mathbf{M}, \mathbf{N}$.

(ii) For all $x_0, \ldots, x_{n-1} \in \mathbf{M}$ *we have* $\mathbf{F}^{\mathbf{M}}(x_0, \ldots, x_{n-1}) = \mathbf{F}^{\mathbf{N}}(x_0, \ldots, x_{n-1})$.

Proof. Let φ be as above.

Assume (i), and suppose that $x_0, \ldots, x_{n-1} \in \mathbf{M}$. Let $y = \mathbf{F}^{\mathbf{M}}(x_0, \ldots, x_{n-1})$. Then $y \in \mathbf{M}$, and $\varphi^{\mathbf{M}}(x_0, \ldots, x_{n-1}, y)$, so by (i), $\varphi^{\mathbf{N}}(x_0, \ldots, x_{n-1}, y)$. Hence $\mathbf{F}^{\mathbf{N}}(x_0, \ldots, x_{n-1}) = y$.

Assume (ii), and suppose that $x_0, \ldots, x_{n-1}, y \in \mathbf{M}$. Then

$$\begin{aligned}
\varphi^{\mathbf{M}}(x_0, \ldots, x_{n-1}, y) \quad &\text{iff} \quad \mathbf{F}^{\mathbf{M}}(x_0, \ldots, x_{n-1}) = y \quad (\text{definition of } \mathbf{F}) \\
&\text{iff} \quad \mathbf{F}^{\mathbf{N}}(x_0, \ldots, x_{n-1}) = y \quad (\text{by (ii)}) \\
&\text{iff} \quad \varphi^{\mathbf{N}}(x_0, \ldots, x_{n-1}, y) \quad (\text{definition of } \mathbf{F}). \qquad \square
\end{aligned}$$

The following theorem gives many explicit absoluteness results, and will be used frequently along with some similar results below. Note that we do not need to be explicit about how the relations and functions were really defined in Chapter 6; in fact, we were not very explicit about that in the first place.

Theorem 13.5. *The following relations and functions were defined by formulas equivalent to* Δ_0*-formulas on the basis of* ZF$-$Inf, *and hence are absolute for all transitive class models of* ZF $-$ Inf*:*

(i) $x \in y$
(ii) $x = y$
(iii) $x \subseteq y$
(iv) $\{x, y\}$
(v) $\{x\}$
(vi) (x, y)
(vii) $\emptyset$
(viii) $x \cup y$
(ix) $x \cap y$
(x) $x \backslash y$
(xi) $x \cup \{x\}$
(xii) x *is transitive*
(xiii) $\bigcup x$
(xiv) $\bigcap x$ *(with* $\bigcap \emptyset = \emptyset$*)*

Note here, for example, that in (iv) we really mean the 2-place function assigning to sets x, y the unordered pair $\{x, y\}$.

Proof. (i) and (ii) are already Δ_0 formulas. (iii):

$$x \subseteq y \leftrightarrow \forall z \in x(z \in y).$$

(iv):

$$z = \{x, y\} \leftrightarrow \forall w \in z(w = x \vee w = y) \wedge x \in z \wedge y \in z.$$

(v): Similarly. (vi):

$$z = (x, y) \leftrightarrow \forall w \in z[w = \{x, y\} \vee w = \{x\}] \wedge \exists w \in z[w = \{x, y\}] \wedge \exists w \in z[w = \{x\}].$$

(vii):

$$x = \emptyset \leftrightarrow \forall y \in x(y \neq y).$$

(viii):

$$z = x \cup y \leftrightarrow \forall w \in z(w \in x \vee w \in y) \wedge \forall w \in x(w \in z) \wedge \forall w \in y(w \in z).$$

(ix):

$$z = x \cap y \leftrightarrow \forall w \in z(w \in x \wedge w \in y) \wedge \forall w \in x(w \in y \rightarrow w \in z).$$

(x):

$$z = x \backslash y \leftrightarrow \forall w \in z(w \in x \wedge w \notin y) \wedge \forall w \in x(x \notin y \rightarrow w \in z).$$

(xi):

$$y = x \cup \{x\} \leftrightarrow \forall w \in y(w \in x \vee w = x) \wedge \forall w \in x(w \in y) \wedge x \in y.$$

(xii):

$$x \text{ is transitive} \leftrightarrow \forall y \in x(y \subseteq x).$$

(xiii):

$$y = \bigcup x \leftrightarrow \forall w \in y \exists z \in x(w \in z) \wedge \forall w \in x(w \subseteq y).$$

(xiv):

$$\begin{aligned} y = \bigcap x \leftrightarrow & [x \neq \emptyset \wedge \forall w \in y \forall z \in x(w \in z) \\ & \wedge \forall w \in x \forall t \in w[\forall z \in x(t \in z) \rightarrow t \in y] \vee [x = \emptyset \wedge y = \emptyset]. \end{aligned}$$

□

A stronger form of Theorem 13.5. For each of the indicated relations and functions, we do not need the model to be all of ZF − Inf. In fact, we need only finitely many of the axioms of ZF − Inf: enough to prove the uniqueness condition for any functions involved, and enough to prove the equivalence of the formula with a Δ_0-formula, since Δ_0 formulas are absolute for any transitive class model. To be absolutely rigorous here, one would need an explicit definition for each relation and function symbol involved, and then an explicit proof of equivalence to a Δ_0 formula; given these, a finite set of axioms becomes clear. And since any of the relations and functions of Theorem 13.5 require only finitely many basic relations and functcions, this can always be done. For Theorem 13.5 it is easy enough to work this all out in detail. We will be interested, however, in using this fact for more complicated absoluteness results to come.

As an illustration, however, we do some details for the function $\{x, y\}$. The definition involved is naturally taken to be the following:

$$\forall x, y, z[z = \{x, y\} \leftrightarrow \forall w[w \in z \leftrightarrow w = x \vee x = y]].$$

The axioms involved are the pairing axiom and one instance of the comprehension axiom:

$$\begin{aligned} & \forall x, y \exists w[x \in w \wedge y \in w]; \\ & \forall x, y, w \exists z \forall u(u \in z \leftrightarrow u \in w \wedge (u = x \vee u = y)). \end{aligned}$$

$\{x, y\}$ is then absolute for any transitive class model of these three sentences, by the proof of (iv) in Theorem 13.5, for which they are sufficient.

For further absoluteness results we will not reduce to Δ_0 formulas. We need the following extensions of the absoluteness notion.

• Suppose that $\mathbf{M} \subseteq \mathbf{N}$ are classes, and $\varphi(w_1, \ldots, w_n)$ is a formula. Then we say that φ is *absolute upwards* for $\mathbf{M}, \mathbf{N}$ iff for all $w_1, \ldots, w_n \in \mathbf{M}$, if $\varphi^{\mathbf{M}}(w_1, \ldots, w_n)$, then $\varphi^{\mathbf{N}}(w_1, \ldots, w_n)$. It is *absolute downwards* for $\mathbf{M}, \mathbf{N}$ iff for all $w_1, \ldots, w_n \in \mathbf{M}$, if $\varphi^{\mathbf{N}}(w_1, \ldots, w_n)$, then $\varphi^{\mathbf{M}}(w_1, \ldots, w_n)$. Thus φ is absolute for $\mathbf{M}, \mathbf{N}$ iff it it is both absolute upwards for $\mathbf{M}, \mathbf{N}$ and absolute downwards for $\mathbf{M}, \mathbf{N}$.

Theorem 13.6. *Suppose that $\varphi(x_1, \ldots, x_n, w_1, \ldots, w_m)$ is absolute for $\mathbf{M}, \mathbf{N}$. Then*
(i) $\exists x_1, \ldots \exists x_n \varphi(x_1, \ldots, x_n, w_1, \ldots, w_m)$ is absolute upwards for $\mathbf{M}, \mathbf{N}$.
(ii) $\forall x_1, \ldots \forall x_n \varphi(x_1, \ldots, x_n, w_1, \ldots, w_m)$ is absolute downwards for $\mathbf{M}, \mathbf{N}$. □

Theorem 13.7. *Absoluteness is preserved under composition. In detail: suppose that $\mathbf{M} \subseteq \mathbf{N}$ are classes, and the following are absolute for $\mathbf{M}, \mathbf{N}$:*

$\varphi(x_1, \ldots, x_n)$;
$\mathbf{F}$, an n-ary function ;
For each $i = 1, \ldots, n$, an m-ary function $\mathbf{G}_i$.

Then the following are absolute:
(i) $\varphi(\mathbf{G}_1(x_1, \ldots, x_m), \ldots, \mathbf{G}_n(x_1, \ldots, x_m))$.
(ii) The m-ary function assigning to $x_1, \ldots, x_m$ the value

$$\mathbf{F}(\mathbf{G}_1(x_1, \ldots, x_m), \ldots, \mathbf{G}_n(x_1, \ldots, x_m)).$$

Proof. We use Theorem 13.6:

$$\begin{aligned}
\varphi(\mathbf{G}_1(x_1, \ldots, x_m), \ldots, \mathbf{G}_n(x_1, \ldots, x_m)) \leftrightarrow & \exists z_1, \ldots \exists z_n \Bigg[\varphi(z_1, \ldots, z_n) \\
& \wedge \bigwedge_{i=1}^{n} (z_i = \mathbf{G}_i(x_1, \ldots, x_m))\Bigg]; \\
\varphi(\mathbf{G}_1(x_1, \ldots, x_m), \ldots, \mathbf{G}_n(x_1, \ldots, x_m)) \leftrightarrow & \forall z_1, \ldots \forall z_n \Bigg[\bigwedge_{i=1}^{n} (z_i = \mathbf{G}_i(x_1, \ldots, x_m)) \\
& \rightarrow \varphi(z_1, \ldots, z_n)\Bigg]; \\
y = \mathbf{F}(\mathbf{G}_1(x_1, \ldots, x_m), \ldots, \mathbf{G}_n(x_1, \ldots, x_m)) \leftrightarrow & \exists z_1, \ldots \exists z_n \Bigg[(y = \mathbf{F}(z_1, \ldots, z_n)) \\
& \wedge \bigwedge_{i=1}^{n} (z_i = \mathbf{G}_i(x_1, \ldots, x_m))\Bigg]; \\
y = \mathbf{F}(\mathbf{G}_1(x_1, \ldots, x_m), \ldots, \mathbf{G}_n(x_1, \ldots, x_m)) \leftrightarrow & \forall z_1, \ldots \forall z_n \Bigg[\bigwedge_{i=1}^{n} (z_i = \mathbf{G}_i(x_1, \ldots, x_m)) \\
& \rightarrow (y = \mathbf{F}(z_1, \ldots, z_n))\Bigg].
\end{aligned}$$
□

Theorem 13.8. *Suppose that* $\mathbf{M} \subseteq \mathbf{N}$ *are classes,* $\varphi(y, x_1, \ldots, x_m, w_1, \ldots, w_n)$ *is absolute for* $\mathbf{M}, \mathbf{N}$*, and* $\mathbf{F}$ *and* $\mathbf{G}$ *are n-ary functions absolute for* $\mathbf{M}, \mathbf{N}$*. Then the following are also absolute for* $\mathbf{M}, \mathbf{N}$*:*

(i) $z \in \mathbf{F}(x_1, \ldots, x_m)$.
(ii) $\mathbf{F}(x_1, \ldots, x_m) \in z$.
(iii) $\exists y \in \mathbf{F}(x_1, \ldots, x_m)\varphi(y, x_1, \ldots, x_m, w_1, \ldots, w_n)$.
(iv) $\forall y \in \mathbf{F}(x_1, \ldots, x_m)\varphi(y, x_1, \ldots, x_m, w_1, \ldots, w_n)$.
(v) $\mathbf{F}(x_1, \ldots, x_m) = \mathbf{G}(x_1, \ldots, x_m)$.
(vi) $\mathbf{F}(x_1, \ldots, x_m) \in \mathbf{G}(x_1, \ldots, x_m)$.

Proof.

$$\begin{aligned} z \in \mathbf{F}(x_1, \ldots, x_m) &\leftrightarrow \exists w[z \in w \wedge w = \mathbf{F}(x_1, \ldots, x_m)]; \\ &\leftrightarrow \forall w[w = \mathbf{F}(x_1, \ldots, x_m) \to z \in w]; \\ \mathbf{F}(x_1, \ldots, x_m) \in z &\leftrightarrow \exists w \in z[w = \mathbf{F}(x_1, \ldots, x_m)]; \end{aligned}$$

$$\begin{aligned} &\exists y \in \mathbf{F}(x_1, \ldots, x_m)\varphi(y, x_1, \ldots, x_m, w_1, \ldots, w_n) \\ &\leftrightarrow \exists w \exists y \in w[w = \mathbf{F}(x_1, \ldots, x_m) \wedge \varphi(y, x_1, \ldots, x_m, w_1, \ldots, w_n)]; \\ &\leftrightarrow \forall w[w = \mathbf{F}(x_1, \ldots, x_m) \to \exists y \in w\varphi(y, x_1, \ldots, x_m, w_1, \ldots, w_n)]; \end{aligned}$$

(iv)–(vi) are proved similarly. □

We now give some more specific absoluteness results.

Theorem 13.9. *The following relations and functions are absolute for all transitive class models of* ZF − Inf*:*

(i) x is an ordered pair
(ii) $A \times B$
(iii) R is a relation
(iv) $\mathrm{dmn}(R)$
(v) $\mathrm{rng}(R)$
(vi) R is a function
(vii) $R(x)$
(viii) R is a one-one function
(ix) x is an ordinal

Note concerning (vii): This is supposed to have its natural meaning if R is a function and x is in its domain; otherwise, $R(x) = \emptyset$.

Proof.

$$\begin{aligned} x \text{ is an ordered pair} &\leftrightarrow \left(\exists y \in \bigcup x\right)\left(\exists z \in \bigcup x\right)[x = (y, z)]; \\ y = A \times B &\leftrightarrow (\forall a \in A)(\forall b \in B)[(a, b) \in y] \wedge \\ &\qquad (\forall z \in y)(\exists a \in A)(\exists b \in B)[z = (a, b)]; \\ R \text{ is a relation} &\leftrightarrow \forall x \in R[x \text{ is an ordered pair}]; \\ x = \mathrm{dmn}(R) &\leftrightarrow (\forall y \in x)\left(\exists z \in \bigcup\bigcup R\right)[(x, z) \in R] \wedge \\ &\qquad \left(\forall y \in \bigcup\bigcup R\right)\left(\forall z \in \bigcup\bigcup R\right)[(y, z) \in R \to y \in x]; \\ x = \mathrm{rng}(R) &\leftrightarrow (\forall y \in x)\left(\exists z \in \bigcup\bigcup R\right)[(z, x) \in R] \wedge \end{aligned}$$

$$\left(\forall y \in \bigcup\bigcup R\right)\left(\forall z \in \bigcup\bigcup R\right)[(y,z) \in R \to z \in x];$$

$$\begin{aligned}
R \text{ is a function} \leftrightarrow & R \text{ is a relation } \wedge \left(\forall x \in \bigcup\bigcup R\right)\left(\forall y \in \bigcup\bigcup R\right) \\
& \left(\forall z \in \bigcup\bigcup R\right)[(x,y) \in R \wedge (x,z) \in R \to y = z]; \\
y = R(x) \leftrightarrow & [R \text{ is a function} \wedge (x,y) \in R] \vee \\
& [R \text{ is not a function} \wedge (\forall z \in y)(z \neq z)] \vee \\
& [x \notin \mathrm{dmn}(R) \wedge (\forall z \in y)(z \neq z)]; \\
R \text{ is a one-one function} \leftrightarrow & R \text{ is a function} \wedge \\
& \forall x \in \mathrm{dmn}(R) \forall y \in \mathrm{dmn}(R)[R(x) = R(y) \to x = y]; \\
x \text{ is an ordinal} \leftrightarrow & x \text{ is transitive} \wedge (\forall y \in x)(y \text{ is transitive}).
\end{aligned}$$

□

Theorem 13.10. *If* $\mathbf{M}$ *is a transitive class model of* ZF*, then* $\mathbf{M}$ *is closed under the following set-theoretic operations:*

(i) $\cup$
(ii) $\cap$
(iii) $(a,b) \mapsto a \backslash b$
(iv) $(a,b) \mapsto \{a,b\}$
(v) $(a,b) \mapsto (a,b)$
(vi) $x \mapsto x \cup \{x\}$
(vii) $\bigcup$
(viii) $\bigcap$

Moreover, $[\mathbf{M}]^{<\omega} \subseteq \mathbf{M}$.

Proof. (i)–(viii) are all very similar, so we only treat (i). Let $a, b \in \mathbf{M}$. Then because $\mathbf{M} \models$ ZF, there is a $c \in \mathbf{M}$ such that $(c = a \cup b)^{\mathbf{M}}$. By absoluteness, $c = a \cup b$.

Now we prove that $x \in \mathbf{M}$ for all $x \in [\mathbf{M}]^{<\omega}$ by induction on $|x|$. If $|x| = 0$, then $x = \emptyset$. Now $\mathbf{M} \models \exists v \forall w[w \notin v]$ by Proposition 5.1. So choose $s \in \mathbf{M}$ such that $\mathbf{M} \models \forall w[w \notin s]$. By transitivity, $s = \emptyset$. Thus $\emptyset \in \mathbf{M}$. If $a \in \mathbf{M}$, then $\mathbf{M} \models \exists v \forall w[w \in v \leftrightarrow w = a]$. Choose $s \in \mathbf{M}$ such that $\mathbf{M} \models \forall w[w \in s \leftrightarrow w = a]$. By absoluteness, $s = \{a\}$. So $\{a\} \in \mathbf{M}$. So our statement holds for all x with $|x| = 1$. Now suppose that $x \in \mathbf{M}$ for all $x \subseteq \mathbf{M}$ such that $|x| = n$. Suppose that $y \subseteq \mathbf{M}$ and $|y| = n+1$. Take any $a \in y$. Then $|y \backslash \{a\}| = n$, so $y \backslash \{a\} \in M$. Hence by (i), $y = (y \backslash \{a\}) \cup \{a\} \in \mathbf{M}$. □

Our final abstract absoluteness result concerns recursive definitions.

Theorem 13.11. *Suppose that* $\mathbf{R}$ *is a class relation which is well-founded and set-like on* $\mathbf{A}$*, and* $\mathbf{G} : \mathbf{A} \times \mathbf{V} \to \mathbf{V}$*. Let* $\mathbf{F}$ *be given by Theorem 8.7: for all* $x \in \mathbf{A}$*,*

$$\mathbf{F}(x) = \mathbf{G}(x, \mathbf{F} \restriction \mathrm{pred}_{\mathbf{AR}}(x)).$$

Let $\mathbf{M}$ *be a transitive class model of* ZF*, and assume the following additional conditions hold:*

(i) $\mathbf{G}$*,* $\mathbf{R}$*, and* $\mathbf{A}$ *are absolute for* $\mathbf{M}$*.*
(ii) $(\mathbf{R}$ *is set-like on* $\mathbf{A})^{\mathbf{M}}$*.*
(iii) $\forall x \in \mathbf{M} \cap \mathbf{A}[\mathrm{pred}_{\mathbf{AR}}(x) \subseteq \mathbf{M}]$*.*

Conclusion: $\mathbf{F}$ *is absolute for* $\mathbf{M}$*.*

Proof. First we claim

(1) ($\mathbf{R}$ is well-founded on $\mathbf{A}$)$^{\mathbf{M}}$.

In fact, by absolutenss $\mathbf{R}^{\mathbf{M}} = \mathbf{R} \cap (\mathbf{M} \times \mathbf{M})$ and $\mathbf{A}^{\mathbf{M}} = \mathbf{A} \cap \mathbf{M}$, so it follows that in $\mathbf{M}$ every nonempty subset of $\mathbf{A}^{\mathbf{M}}$ has an $\mathbf{R}^{\mathbf{M}}$-minimal element. Hence we can apply the recursion theorem within $\mathbf{M}$ to define a function $\mathbf{H} : \mathbf{A}^{\mathbf{M}} \to \mathbf{M}$ such that for all $x \in \mathbf{A}^{\mathbf{M}}$,

$$\mathbf{H}(x) = \mathbf{G}^{\mathbf{M}}(x, \mathbf{H} \restriction \mathrm{pred}^{\mathbf{M}}_{\mathbf{A}^{\mathbf{M}}\mathbf{R}^{\mathbf{M}}}(x)).$$

We claim that $\mathbf{H} = \mathbf{F} \restriction \mathbf{A}^{\mathbf{M}}$, which will prove the theorem. In fact, suppose that x is $\mathbf{R}$-minimal such that $x \in \mathbf{A}^{\mathbf{M}}$ and $\mathbf{F}(x) \neq \mathbf{H}(x)$. Then using absoluteness again,

$$\mathbf{H}(x) = \mathbf{G}^{\mathbf{M}}(x, \mathbf{H} \restriction \mathrm{pred}^{\mathbf{M}}_{\mathbf{A}^{\mathbf{M}}\mathbf{R}^{\mathbf{M}}}(x)) = \mathbf{G}(x, \mathbf{H} \restriction \mathrm{pred}_{\mathbf{AR}}(x) = \mathbf{F}(x),$$

contradiction. □

Theorem 13.11 is needed for many deeper applications of absoluteness. We illustrate its use by the following result.

Theorem 13.12. *The following are absolute for transitive class models of* ZF*.*
(i) $\alpha + \beta$ *(ordinal addition)*
(ii) $\alpha \cdot \beta$ *(ordinal multiplication)*
(iii) α^β *(ordinal exponentiation)*
(iv) $\mathrm{rank}(x)$
(v) $\mathrm{trcl}(x)$

Proof. In each case it is mainly a matter of identifying $\mathbf{A}, \mathbf{R}, \mathbf{G}$ to which to apply Theorem 13.11; checking the conditions of that theorem are straightforward.

(i): $\mathbf{A} = \mathbf{On}$, $\mathbf{R} = \{(\alpha, \beta) : \alpha, \beta \in \mathbf{On}$, and $\alpha \in \beta\}$, and $\mathbf{G} : \mathbf{On} \times \mathbf{V} \to \mathbf{V}$ is defined as follows:

$$\mathbf{G}(\alpha, f) = \begin{cases} \alpha & \text{if } f = \emptyset, \\ f(\beta) \cup \{f(\beta)\} & \text{if } f \text{ is a function with domain an ordinal } \beta + 1, \\ \bigcup_{\gamma \in \beta} f(\gamma) & \text{if } f \text{ is a function with domain a limit ordinal } \beta, \\ \emptyset & \text{otherwise.} \end{cases}$$

(ii) and (iii) are treated similarly. For (iv), take $\mathbf{R} = \{(x, y) : x \in y\}$, $\mathbf{A} = \mathbf{V}$, and define $\mathbf{G} : \mathbf{V} \times \mathbf{V} \to \mathbf{V}$ by setting, for all $x, f \in \mathbf{V}$,

$$\mathbf{G}(x, f) = \begin{cases} \bigcup_{y \in x}(f(y) \cup \{f(y)\}) & \text{if } f \text{ is a function with domain } x, \\ \emptyset & \text{otherwise.} \end{cases}$$

For (v), let $\mathbf{R} = \{(i, j) : i, j \in \omega \text{ and } i < j\}$, $\mathbf{A} = \omega$, and define $\mathbf{G} : \omega \times \mathbf{V} \to \mathbf{V}$ by setting, for all $m \in \omega$ and $f \in \mathbf{V}$,

$$\mathbf{G}(m, f) = \begin{cases} x & \text{if } m = 0, \\ f(\bigcup m) \cup \bigcup f(\bigcup m) & \text{if } m > 0 \text{ and } f \text{ is a function with domain } m, \\ \emptyset & \text{otherwise} \end{cases}$$

Then the function $\mathbf{F}$ obtained from Theorem 13.11 is absolute for transitive class models of ZF, and $\mathrm{trcl}(x) = \bigcup_{m \in \omega} \mathbf{F}(m)$. □

Theorem 13.13. *If* $\mathbf{M}$ *is a transitive model of ZF, then:*
(i) $\mathscr{P}^{\mathbf{M}}(x) = \mathscr{P}(x) \cap \mathbf{M}$ *for any* $x \in \mathbf{M}$*;*
(ii) $V_\alpha^{\mathbf{M}} = V_\alpha \cap \mathbf{M}$ *for any* $\alpha \in \mathbf{M}$*.*

Proof. (i): Assume that $x \in \mathbf{M}$. Then for any set y,

$$\begin{aligned} y \in \mathscr{P}^{\mathbf{M}}(x) \quad &\text{iff} \quad y \in \mathbf{M} \text{ and } (y \subseteq x)^{\mathbf{M}} \\ &\text{iff} \quad y \in \mathbf{M} \text{ and } y \subseteq x \\ &\text{iff} \quad y \in \mathscr{P}(x) \cap \mathbf{M}. \end{aligned}$$

(ii): Assume that $\alpha \in \mathbf{M}$. Then for any set x,

$$\begin{aligned} x \in V_\alpha^{\mathbf{M}} \quad &\text{iff} \quad x \in \mathbf{M} \text{ and } \operatorname{rank}^{\mathbf{M}}(x) < \alpha \\ &\text{iff} \quad x \in \mathbf{M} \text{ and } \operatorname{rank}(x) < \alpha \\ &\text{iff} \quad x \in V_\alpha \cap \mathbf{M} \end{aligned}$$ □

Proposition 13.14. *"R well-orders A" is absolute for models of ZF.*

Proof. Let $\mathbf{M}$ be a model of ZF. Suppose that $R, A \in \mathbf{M}$. Clearly

$$\begin{aligned} (R \text{ well-orders } A) \quad \text{iff} \quad &\exists x \exists f[x \text{ is an ordinal} \wedge f : x \to A \text{ is a bijection} \\ &\wedge \forall \beta, \gamma \in x[\beta < \gamma \text{ iff } (f(\beta), f(\gamma)) \in R]]. \end{aligned}$$

From this and elementary absoluteness results it is clear that $(R \text{ well-orders } A)^{\mathbf{M}}$ implies that (R well-orders A). Now suppose that (R well-orders A). Let x and f be such that x is an ordinal, $f : x \to A$ is a bijection, and $\forall \beta, \gamma \in x[\beta < \gamma \text{ iff } (f(\beta), f(\gamma)) \in R]$. Since $\mathbf{M}$ is a model of ZF, let $y, g \in \mathbf{M}$ be such that in $\mathbf{M}$ we have: y is an ordinal, $g : y \to A$ is a bijection, and $\forall \beta, \gamma \in y[\beta < \gamma \text{ iff } (g(\beta), g(\gamma)) \in R]$. By simple absoluteness results, this is really true. Then $x = y$ and $f = g$ by the uniqueness conditions in 9.12–9.13. □

. . . .

Proposition 13.15. (I.16.6) *Let* $\operatorname{pow}(x, y)$ *be* $\forall z[z \in y \leftrightarrow z \subseteq x]$*. Let* γ *be a limit ordinal with* $a, b \in V_\gamma$*. Then* $V_\gamma \models \operatorname{pow}[a, b]$ *iff* $b = \mathscr{P}(a)$*.*

Proof. First suppose that $V_\gamma \models \operatorname{pow}[a, b]$. Thus $V_\gamma \models \forall z[z \in b \leftrightarrow z \subseteq a]$. Suppose that $x \in b$. Then $x \in V_\gamma$ since V_γ is transitive, so $V_\gamma \models x \subseteq a$. Hence $x \subseteq a$, so $x \in \mathscr{P}(a)$. Conversely, suppose that $x \subseteq a$. Now $\operatorname{rank}(x) \le \operatorname{rank}(a)$, so $x \in V_\gamma$. Hence $V_\gamma \models x \subseteq a$, so $x \in b$. This shows that $\mathscr{P}(a) = b$.

Second, suppose that $b = \mathscr{P}(a)$. Take any $z \in V_\gamma$. If $z \in b$, then $z \subseteq a$, hence $V_\gamma \models z \subseteq a$. If $V_\gamma \models z \subseteq a$, then $z \subseteq a$, and hence $z \in b$. Therefore, $V_\gamma \models \operatorname{pow}[a, b]$. □

Proposition 13.16. *Let* $\gamma > \omega_1$ *be a limit ordinal. Then there is a countable transitive* M *and ordinals* $\alpha, \beta \in M$ *such that* $M \equiv V_\gamma$, $\text{not}(\alpha \sim \beta)^M$*, but* $(\alpha \sim \beta)^{V_\gamma}$*. Here* $\alpha \sim \beta$ *means that there is a bijection from* α *to* β*.*

Proof. Let A be countable such that $\omega, \omega_1 \in A$ and $(A, \in) \preceq (V_\gamma, \in)$. Since $V_\gamma \models \neg\exists f[f$ is a bijection from ω onto $\omega_1]$, it follows that $A \models \neg\exists f[f$ is a bijection from ω onto $\omega_1]$. Now extensionality holds in V_γ, hence in A. It follows that $\mathrm{mos}_{A\in}$ is an isomorphism from A onto some set M. Now $V_\gamma \models [\omega$ is an ordinal$]$, so $A \models [\omega$ is an ordinal$]$ and $M \models [\mathrm{mos}_{A\in}(\omega)$ is an ordinal$]$. Let $\alpha = \mathrm{mos}_{A\in}(\omega)$. Also $V_\gamma \models [\omega$ is the first limit ordinal$]$, so $A \models [\omega$ is the first limit ordinal$]$ and $M \models [\mathrm{mos}_{AR}(\omega)$ is the first limit ordinal$]$. So $\alpha = \omega$. Similarly, $\mathrm{mos}_{A\in}(\omega_1)$ is an ordinal; call it β. $M \models \neg\exists f[f$ is a bijection from ω onto $\beta]$. Since M is countable and transitive, so is β. □

Proposition 13.17. (I.16.17) *Let $M = ON \cup \{\{\alpha, \beta\} : \alpha < \beta \in ON\}$. Then M is transitive, and $\cap^M$ is not defined.*

Proof. Suppose that $x \in y \in M$. If $y \in ON$, then also $x \in ON \subseteq M$. If $y = \{\alpha, \beta\}$ with $\alpha < \beta$, then $x = \alpha$ or $x = \beta$, hence $x \in ON \subseteq M$. Thus M is transitive.

Suppose that $\cap^M$ is defined. Choose $z \in M$ such that for all $x \in M$, $x \in z$ iff $x \in \{0, 1\}$ and $x \in \{1, 2\}$. Clearly this implies that $z = \{1\}$, contradiction. □

Proposition 13.18. *The formula $\exists x(x \in y)$ is not absolute for all nonempty sets, but it is absolute for all nonempty transitive sets.*

Proof. Let $A = \{\{\emptyset\}\}$. Then $\exists x(x \in \{\emptyset\})$ holds in V, but not in A, since there is no $a \in A$ such that $a \in \{\emptyset\}$.

Now suppose that B is a nonempty transitive set, and $y \in B$. Then y has an element iff it has an element in B, and so $\exists x(x \in y)$ iff $\exists x \in B(x \in y)$ iff $(\exists x(x \in y))^B$. So the formula is absolute for B. (Note that $\exists x(x \in y)$ is not quite a Δ_0 formula.) □

Proposition 13.19. *The formula $\exists z(x \in z)$ is not absolute for every nonempty transitive set.*

Proof. Take the transitive set 2. Then $\exists z(1 \in z)$, but this does not hold in 2, since there is no $z \in 2$ such that $1 \in z$. □

14. Checking the axioms

Now we give some simple facts which will be useful in checking the axioms of ZFC in the transitive classes which we will define. See Chapter 5 for the original form of the axioms.

Theorem 14.1. *The extensionality axiom holds in any nonempty transitive class.*

Proof. Let $\mathbf{M}$ be any transitive class. The relativized version of the extensionality axiom is

$$\forall x \in \mathbf{M} \forall y \in \mathbf{M}[\forall z \in \mathbf{M}(z \in x \leftrightarrow z \in y) \rightarrow x = y].$$

To prove this, assume that $x, y \in \mathbf{M}$, and suppose that for all $z \in \mathbf{M}$, $z \in x$ iff $z \in y$. Take any $z \in x$. Because $\mathbf{M}$ is transitive, we get $z \in \mathbf{M}$. Hence $z \in y$. Thus $z \in x$ implies that $z \in y$. The converse is similar. So $x = y$. $\square$

The following theorem reduces checking the comprehension axioms to checking a closure property.

Theorem 14.2. *Suppose that* $\mathbf{M}$ *is a nonempty class, and for each formula* φ *with with free variables among* $x, z, w_1, \ldots, w_n$*,*

$$\forall z, w_1, \ldots, w_n \in \mathbf{M}[\{x \in z : \varphi^{\mathbf{M}}(x, z, w_1, \ldots, w_n)\} \in \mathbf{M}].$$

Then the comprehension axioms hold in $\mathbf{M}$*.*

Proof. The straightforward relativization of an instance of the comprehension axioms is

$$\forall z \in \mathbf{M} \forall w_1 \in \mathbf{M} \ldots \forall w_n \in \mathbf{M} \exists y \in \mathbf{M} \forall x \in \mathbf{M}(x \in y \leftrightarrow x \in z \wedge \varphi^{\mathbf{M}}).$$

So, we take $z, w_1, \ldots, w_n \in \mathbf{M}$. Let

$$y = \{x \in z : \varphi^{\mathbf{M}}(x, z, w_1, \ldots, w_n)\};$$

by hypothesis, we have $y \in \mathbf{M}$. Then for any $x \in \mathbf{M}$,

$$x \in y \quad \text{iff} \quad x \in z \text{ and } \varphi^{\mathbf{M}}(x, z, w_1, \ldots, w_n). \qquad \square$$

The following theorems are obvious from the forms of the pairing axiom and union axioms:

Theorem 14.3. *Suppose that* $\mathbf{M}$ *is a nonempty class and*

$$\forall x, y \in \mathbf{M} \exists z \in \mathbf{M}(x \in z \textit{ and } y \in z).$$

Then the pairing axiom holds in $\mathbf{M}$*.* $\square$

Theorem 14.4. *Suppose that* $\mathbf{M}$ *is a nonempty class and*

$$\forall x \in \mathbf{M} \exists z \in \mathbf{M} \left(\bigcup x \subseteq z\right).$$

Then the union axiom holds in $\mathbf{M}$. □

For the next result, recall that $z \subseteq x$ is an abbreviation for $\forall w(w \in z \to w \in x)$.

Theorem 14.5. *Suppose that* $\mathbf{M}$ *is a nonempty transitive class. Then the following are equivalent:*

(i) The power set axiom holds in $\mathbf{M}$.

(ii) For every $x \in \mathbf{M}$ *there is a* $y \in \mathbf{M}$ *such that* $\mathscr{P}(x) \cap \mathbf{M} \subseteq y$.

Proof. (i)⇒(ii): Assume (i). Thus

$$\forall x \in \mathbf{M} \exists y \in \mathbf{M} \forall z \in \mathbf{M}[\forall w \in \mathbf{M}(w \in z \to w \in x) \to z \in y]. \tag{1}$$

To prove (ii), take any $x \in \mathbf{M}$. Choose $y \in \mathbf{M}$ as in (1). Suppose that $z \in \mathscr{P}(x) \cap \mathbf{M}$. Clearly then $\forall w \in \mathbf{M}(w \in z \to w \in x)$, so by (1), $z \in y$, as desired in (ii).

(ii)⇒(i): Assume (ii). This time we want to prove (1). So, suppose that $x \in \mathbf{M}$. Choose $y \in \mathbf{M}$ as in (ii). Now suppose that $z \in \mathbf{M}$ and $\forall w \in \mathbf{M}(w \in z \to w \in x)$. Then the transitivity of $\mathbf{M}$ implies that $\forall w(w \in z \to w \in x)$, i.e., $z \subseteq x$. So by (ii), $z \in y$, as desired. □

Theorem 14.6. *Suppose that* $\mathbf{M}$ *is a transitive class, and for every formula* φ *with free variables among* $x, y, A, w_1, \ldots, w_n$ *and for any* $A, w_1, \ldots, w_n \in \mathbf{M}$ *the following implication holds:*

$$\begin{aligned}&\forall x \in A \exists! y[y \in \mathbf{M} \wedge \varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n)] \quad \textit{implies that}\\ &\exists Y \in \mathbf{M}[\{y \in \mathbf{M} : \exists x \in A \varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n)\} \subseteq Y]].\end{aligned}$$

Then the replacement axioms hold in $\mathbf{M}$.

Proof. Assume the hypothesis of the theorem. We write out the relativized version of an instance of the replacement axiom in full, remembering to replace the quantifier $\exists!$ by its definition:

$$\begin{aligned}&\forall A \in \mathbf{M} \forall w_1 \in \mathbf{M} \ldots \forall w_n \in \mathbf{M}\\ &[\forall x \in \mathbf{M}[x \in A \to \exists y \in \mathbf{M}[\varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n) \wedge \forall u \in \mathbf{M}\\ &\qquad [\varphi^{\mathbf{M}}(x, u, A, w_1, \ldots, w_n) \to y = u]]] \to\\ &\exists Y \in \mathbf{M} \forall x \in \mathbf{M}[x \in A \to \exists y \in \mathbf{M}[y \in Y \wedge \varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n)]]].\end{aligned}$$

To prove this, assume that $A, w_1, \ldots, w_n \in \mathbf{M}$ and

$$\begin{aligned}&\forall x \in \mathbf{M}[x \in A \to \exists y \in \mathbf{M}[\varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n) \wedge \forall u \in \mathbf{M}\\ &\qquad [\varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n) \to y = u]]].\end{aligned}$$

Since $\mathbf{M}$ is transitive, we get

$$\forall x \in A \exists y \in \mathbf{M}[\varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n) \wedge \forall u \in \mathbf{M}[\varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n) \to y = u]],$$

so that

(1) $$\forall x \in A \exists ! y[y \in \mathbf{M} \wedge \varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n)].$$

Hence by the hypothesis of the theorem we get $Y \in \mathbf{M}$ such that

(2) $$\{y \in \mathbf{M} : \exists x \in A \varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n)\} \subseteq Y.$$

Suppose that $x \in \mathbf{M}$ and $x \in A$. By (1) we get $y \in \mathbf{M}$ such that $\varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n)$. Hence by (2) we get $y \in Y$, as desired. □

Theorem 14.7. *If* $\mathbf{M}$ *is a transitive class, then the foundation axiom holds in* $\mathbf{M}$*.*

Proof. The foundation axiom, with the defined notion $\emptyset$ eliminated, is

$$\forall x[\exists y(y \in x) \to \exists y[y \in x \wedge \forall z \in y(z \notin x)]].$$

Hence the relativized version is

$$\forall x \in \mathbf{M}[\exists y \in \mathbf{M}(y \in x) \to \exists y \in \mathbf{M}[y \in x \wedge \forall z \in \mathbf{M}[z \in y \to z \notin x]]].$$

So, take any $x \in \mathbf{M}$, and suppose that there is a $y \in \mathbf{M}$ such that $y \in x$. Choose $y \in x$ so that $y \cap x = \emptyset$. Then $y \in \mathbf{M}$ by transitivity. If $z \in \mathbf{M}$ and $z \in y$, then $z \notin x$, as desired. □

Theorem 14.8. *Suppose that* $\mathbf{M}$ *is a transitive class, and* $\omega \in \mathbf{M}$*. Then the infinity axiom holds in* $\mathbf{M}$*.*

Proof. As a sentence in the language of set theory we take the infinity axiom to be the following:

$$\exists x[\exists y \in x \forall z \in y[z \neq z] \wedge \forall y \in x \exists z \in x[\forall w \in z[w \in y \vee w = y] \wedge \forall w \in y[w \in z] \wedge y \in z]].$$

We claim that

$$\begin{aligned} \mathbf{M} \models & \exists y \in \omega \forall z \in y[z \neq z] \wedge \forall y \in \omega \exists z \in \omega \\ & [\forall w \in z[w \in y \vee w = y] \wedge \forall w \in y[w \in z] \wedge y \in z]]. \end{aligned}$$

In fact, $\emptyset \in \mathbf{M}$ since $\emptyset \in \omega \in \mathbf{M}$. Hence $\mathbf{M} \models \forall z \in \emptyset[z \neq z]$. So $\mathbf{M} \models \exists y \in \omega \forall z \in y[z \neq z]$.

Now suppose that $a \in M$ and $a \in \omega$. Then $a \cup \{a\} \in \omega \in \mathbf{M}$, so $a \cup \{a\} \in \mathbf{M}$. Suppose that $c \in \mathbf{M}$ and $c \in a \cup \{a\}$. Then $c \in a$ or $c = a$. Thus $\mathbf{M} \models \forall w \in a \cup \{a\}[w \in a \vee w = a]$. Suppose that $d \in \mathbf{M}$ and $d \in a$. Then $d \in a \cup \{a\}$. Thus $\mathbf{M} \models \forall w \in a[w \in a \cup \{a\}]$. Finally, $\mathbf{M} \models [a \in a \cup \{a\}]$. This shows that

$$\mathbf{M} \models \exists z \in \omega[\forall w \in z[w \in a \vee w = a] \wedge \forall w \in a[w \in z] \wedge a \in z]].$$

Hence

$$\mathbf{M} \models \forall y \exists z \in \omega[\forall w \in z[w \in y \vee w = y] \wedge \forall w \in y[w \in z] \wedge y \in z]].$$

Hence the claim holds, and so the infinity axiom holds in **M**. $\square$

Lemma 14.9. (II.4.20) *Suppose that B is infinite, and $(B, \in)$ satisfies extensionality. Let κ be an infinite cardinal, and let $S \subseteq B$ with $|S| \leq \kappa \leq |B|$.*

Then there is a transitive M such that $S \subseteq M$, $(M, \in) \equiv (B, \in)$, and $|M| = \kappa$.

Proof. By the downward Löwenheim-Skolem theorem let $A \preceq B$ be such that $S \subseteq A$ and $|A| = \kappa$. Then A satisfies extensionality, and $\in$ is well-founded on A, so $\text{mos}_{A\in}$ is an isomorphism from $(A, \in)$ onto some $(M, \in)$ with M transitive. Since S is transitive, $\text{mos}_{AR}(y) = y$ for all $y \in S$ by Lemma 12.33. Hence $S \subseteq M$. $\square$

Proposition 14.10. (II.4.26) *Let M be a transitive class in which extensionality, comprehension, pairing, and union hold. Suppose that $\omega \subseteq a \in M$. Then $\omega \in M$.*

Proof. By comprehension let $w \in M$ be such that

$$\begin{aligned} M \models &\forall y[y \in w \leftrightarrow y \in a \text{ and } [\forall z[\exists w \in z \forall v \in w[v \neq v] \\ &\text{and } \forall w \in z \exists v \in z \forall u[u \in v \leftrightarrow u \in w \text{ or } u = w] \rightarrow y \in z]] \end{aligned}$$

Clearly $\emptyset \in w$. If $b \in w$, clearly $b \cup \{b\} \in w$. Hence $\omega \subseteq w$. Now suppose that $\omega \neq w$. Choose $b \in w \backslash \omega$ such that $b \cap (w \backslash \omega) = \emptyset$. Then $\emptyset \neq b$ since $\emptyset \in \omega$ and $b \notin \omega$. So $\emptyset \in w \backslash \{b\}$. If $c \in w \backslash \{b\}$ then $c \cup \{c\} \in w$. If $b = c \cup \{c\}$, then $c \in b$, hence $c \in \omega$ since $b \cap (w \backslash \omega) = \emptyset$. Then $b = c \cup \{c\} \in \omega$, contradiction. So $c \cup \{c\} \in w \backslash \{b\}$. It follows that $w \subseteq w \backslash \{b\}$, contradicting $b \in w$. So $\omega = w$. $\square$

Proposition 14.11. (II.4.26) *Let M be a transitive class in which extensionality, comprehension, pairing, union, and infinity hold. Then $\omega \in M$.*

Proof. Recall from the proof of Theorem 14.8 the formulation of the infinity axiom in the language of set theory ($\in$ only). So let $a \in M$ be such that

$$\begin{aligned} M \models &\exists y \in a \forall z \in y[z \neq z] \wedge \forall y \in a \exists z \in a \\ &[\forall w \in z[w \in y \vee w = y] \wedge \forall w \in y[w \in z] \wedge y \in z]]. \end{aligned}$$

Since M models extensionality, comprehension, pairing, and union, it follows by absoluteness that $\emptyset \in M$ and $u \cup \{u\} \in M$ whenever $u \in M$. Hence $\emptyset \in a$ and for all $b \in a[b \cup \{b\} \in a]$. So $\omega \subseteq a$. Hence $\omega \in M$ by Proposition 14.10. $\square$

Lemma 14.12. (II.4.27) *Suppose that M is a transitive class such that the comprehension axioms hold in M, and suppose that for every subset $x \subseteq M$ there is a $y \in M$ such that $x \subseteq y$. Then all of the ZF axioms hold in M.*

Proof. By Theorems 14.1 and 14.7, extensionality and foundation hold in M. If $x, y \in M$, then $\{x, y\} \subseteq M$, and so there is a $z \in M$ such that $\{x, y\} \subseteq z$, hence $x \in z$ and $y \in z$. Thus pairing holds in M. If $\mathscr{F} \in M$, then $\bigcup \mathscr{F} \subseteq M$, hence there is a $y \in M$ such that $\bigcup \mathscr{F} \subseteq y$. So union holds in M by Theorem 14.4. For replacement, assume that the hypothesis of Theorem 14.6 holds. By replacement in the real world, choose Z such that

$$(*) \qquad \forall x[x \in A \to \exists z[z \in Z \text{ and } z \in M \text{ and } \varphi^M(x, z, A, w_1, \ldots, w_n)]]$$

Then let $W = \{y \in Z : y \in M \text{ and } \exists x \in A\varphi^M(x, y, A, w_1, \ldots, w_n)\}$. Then $W \subseteq M$, so there is a $Y \in M$ such that $W \subseteq Y$. Now suppose that $y \in M$, $x \in A$, and $\varphi^M(x, y, A, z_1, \ldots, z_n)$. Choose $z \in Z$ such that $z \in M$ and $\varphi^M(x, z, A, w_1, \ldots, w_n)$, by $(*)$. By the uniqueness condition in the hypothesis of Theorem 14.6, $y = z$. Hence $y \in W$, so $y \in Y$. It follows that $\{y \in \mathbf{M} : \exists x \in A\varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n)\} \subseteq Y$, as desired. The power set axiom holds by Theorem 14.5. By induction it is clear that $\omega \subseteq M$. Choose $y \in M$ so that $\omega \subseteq y$. Then by Proposition 14.11, $\omega \in M$. By Theorem 14.10, infinity holds in M. □

Theorem 14.13. (II.2.1) *If κ is uncountable and regular, then $H(\kappa) \models ZFC - P$.*

Proof. Clearly $H(\kappa)$ is transitive, so extensionality and foundation hold in $H(\kappa)$. Also, $\omega \in H(\kappa)$, so by Theorem 14.8, infinity holds in $H(\kappa)$. The comprehension axioms hold in $H(\kappa)$ by Theorem 14.2, since using the notation there, if $z \in H(\kappa)$ then $\{x \in z : \varphi^{H(\kappa)}(x, z, w_1, \ldots, w_n)\} \subseteq z$ and hence $\{x \in z : \varphi^{H(\kappa)}(x, z, w_1, \ldots, w_n)\} \in H(\kappa)$. Pairing and union hold in $H(\kappa)$ by Theorems 14.3 and 14.4. For the replacement axioms, we apply Theorem 14.6. Assume that φ is a formula with free variables among $x, y, A, w_1, \ldots, w_n$ and $A, w_1, \ldots, w_n \in H(\kappa)$ and

$$\forall x \in A\exists! y[y \in H(\kappa) \wedge \varphi^{H(\kappa)}(x, y, A, w_1, \ldots, w_n)].$$

For each $a \in A$ let $f(a) \in H(\kappa)$ be such that $\varphi^{H(\kappa)}(x, f(a), A, w_1, \ldots, w_n)$. Since $A \in H(\kappa)$, we have $|A| < \kappa$. Hence $\mathrm{trcl}(f[A]) = f[A] \cup \bigcup_{a \in A} \mathrm{trcl}(f(a))$ has size less than κ, since κ is regular. Clearly $\mathrm{trcl}(f[A])$ is as desired in the conclusion of 14.6. Clearly the axiom of choice holds in $H(\kappa)$. □

ZC is ZFC without replacement.

Theorem 14.14. (II.2.2) *If γ is a limit ordinal $> \omega$, then $V_\gamma \models ZC$.*

Proof. V_γ is transitive, so extensionality and foundation hold. For $a \in V_\gamma$ we have $\mathscr{P}(a) \in V_\gamma$, and so comprehension holds. Clearly pairing and union hold. Power set is clear. Infinity holds by Theorem 14.8. Clearly choice holds. □

Theorem 14.15. (II.2.3) *If κ is strongly inaccessible, then $H(\kappa) = V_\kappa$.*

Proof. Clearly $\kappa = \beth_\kappa$, so this follows from Lemma 12.54. □

Theorem 14.16. (II.2.3) *If κ is strongly inaccessible, then V_κ is a model of ZFC.*

Proof. By Theorem 14.14 it suffices to prove that replacement holds in V_κ. Using the notation of Theorem 14.6, suppose that $A, w_1, \ldots, w_n \in V_\kappa$ and $\forall x \in A\exists! y \in V_\kappa\varphi^{V_\kappa}(x, y, A, w_1, \ldots, w_n)$. For each $x \in A$ let $f(x) \in V_\kappa$ be such that

$$\varphi^{V_\kappa}(x, f(x), A, w_1, \ldots, w_n).$$

Since $A \in V_\kappa$, we have $|A| < \kappa$ by Theorem 14.15. Hence $|f[A]| < \kappa$. Since κ is regular, $\bigcup_{a \in A} \operatorname{rank}(f(a)) < \kappa$, and so there is an $\alpha < \kappa$ such that $f[A] \subseteq V_\alpha$, so $f[A] \in V_{\alpha+1} \subseteq V_\kappa$. Hence replacement holds. $\square$

Corollary 14.17. *If κ is uncountable, regular, but not strongly inaccessible, then the power set axiom is false in $H(\kappa)$.*

Proof. Say $\lambda < \kappa \leq 2^\lambda$. Clearly $\lambda \in H(\kappa)$. If $X \subseteq \lambda$, then $\operatorname{trcl}(X) \subseteq \lambda$, so$|\operatorname{trcl}(X)| \leq \lambda < \kappa$. Thus $\mathscr{P}(\lambda) \cap H(\kappa) = \mathscr{P}(\lambda)$. By Theorem 14.5, if power set holds in $H(\kappa)$, then there is a $y \in H(\kappa)$ such that $\mathscr{P}(\lambda) \subseteq y$. Then $2^\lambda = |\mathscr{P}(\lambda)| \leq |y| < \kappa$, contradiction. $\square$

15. Reflection theorems

A *set theory structure* is an ordered pair $\overline{A} = (A, R)$ such that A is a non-empty set and R is a binary relation contained in $A \times A$. The model-theoretic notions introduced in Chapter 2 can be applied here.

A notion similar to that of a model is *relativization*. Suppose that $\mathbf{M}$ is a class. We associate with each formula φ its *relativization to* $\mathbf{M}$, denoted by $\varphi^{\mathbf{M}}$. The definition goes by recursion on formulas:

$(x = y)^{\mathbf{M}}$ is $x = y$
$(x \in y)^{\mathbf{M}}$ is $x \in y$
$(\varphi \to \psi)^{\mathbf{M}}$ is $\varphi^{\mathbf{M}} \to \psi^{\mathbf{M}}$.
$(\neg\varphi)^{\mathbf{M}}$ is $\neg\varphi^{\mathbf{M}}$.
$(\forall x\varphi)^{\mathbf{M}}$ is $\forall x[x \in \mathbf{M} \to \varphi^{\mathbf{M}}]$.

The more rigorous version of this definition associates with each pair ψ, φ of formulas a third formula which is called the relativization of φ to ψ.

We say that φ *holds in* $\mathbf{M}$ or *is true in* $\mathbf{M}$, iff $\varphi^{\mathbf{M}}$ holds, i.e., iff ZFC $\vdash \varphi^{\mathbf{M}}$.

Theorem 15.1. *Let Γ be a set of sentences, φ a sentence, and $\mathbf{M}$ a class. Let $\Gamma^{\mathbf{M}} = \{\chi^{\mathbf{M}} : \chi \in \Gamma\}$. Suppose that $\Gamma \models \varphi$. Then*

$$\Gamma^{\mathbf{M}} \models \mathbf{M} \neq \emptyset \to \varphi^{\mathbf{M}}.$$

Proof. Assume the hypothesis of the theorem, let $\overline{A} = (A, E)$ be any set theory structure, assume that $\overline{A}$ is a model of $\Gamma^{\mathbf{M}}$, and suppose that $A \cap \mathbf{M} \neq \emptyset$. We want to show that $\overline{A}$ is a model of $\varphi^{\mathbf{M}}$. To do this, we define another structure $\overline{B} = (B, F)$ for our language. Let $B = A \cap \mathbf{M}$, and let $F = E \cap (B \times B)$. Now we claim:

(*) For any formula χ and any $c \in {}^{\omega}B$, $\overline{A} \models \chi^{\mathbf{M}}[c]$ iff $\overline{B} \models \chi[c]$.

We prove (*) by induction on χ:

$$\begin{aligned}
\overline{A} \models (v_i = v_j)^{\mathbf{M}}[c] \quad &\text{iff} \quad c_i = c_j \\
&\text{iff} \quad \overline{B} \models (v_i = v_j)[c]; \\
\overline{A} \models (v_i \in v_j)^{\mathbf{M}}[c] \quad &\text{iff} \quad c_i E c_j \\
&\text{iff} \quad c_i F c_j \\
&\text{iff} \quad \overline{B} \models (v_i \in v_j)[c]; \\
\overline{A} \models (\neg\chi)^{\mathbf{M}}[c] \quad &\text{iff} \quad \text{not}[\overline{A} \models \chi^{\mathbf{M}}[c]] \\
&\text{iff} \quad \text{not}[\overline{B} \models \chi[c]] \quad \text{(induction hypothesis)} \\
&\text{iff} \quad \overline{B} \models \neg\chi[c]; \\
\overline{A} \models (\chi \to \theta)^{\mathbf{M}}[c] \quad &\text{iff} \quad [\overline{A} \models \chi^{\mathbf{M}}[c] \text{ implies that } \overline{A} \models \theta^{\mathbf{M}}[c]] \\
&\text{iff} \quad [\overline{B} \models \chi[c] \text{ implies that } \overline{B} \models \theta[c] \\
&\qquad\qquad \text{(induction hypothesis)} \\
&\text{iff} \quad \overline{B} \models (\chi \to \theta)[c].
\end{aligned}$$

We do the quantifier step in each direction separately. First suppose that $\overline{A} \models (\forall v_i\chi)^{\mathbf{M}}[c]$. Thus $\overline{A} \models [\forall v_i[v_i \in \mathbf{M} \to \chi^{\mathbf{M}}][c]$. Take any $b \in B$. Then $b \in \mathbf{M}$, so $\overline{A} \models \chi^{\mathbf{M}}[c^i_b]$. By the inductive hypothesis, $\overline{B} \models \chi[c^i_b]$. This proves that $\overline{B} \models \forall v_i\chi[c]$.

Conversely, suppose that $\overline{B} \models \forall v_i\chi[c]$. Suppose that $a \in A$ and $\overline{A} \models (v_i \in \mathbf{M})[c^i_a]$. Then $a \in B$, so $\overline{B} \models \chi[c^i_a]$. By the inductive hypothesis, $\overline{A} \models \chi^{\mathbf{M}}[c^i_a]$. So we have shown that $\overline{A} \models \forall v_i[v_i \in \mathbf{M} \to \chi^M][c]$. That is, $\overline{A} \models (\forall v_i\chi)^{\mathbf{M}}[c]$.

This finishes the proof of (*).

Now $\overline{A}$ is a model of $\Gamma^{\mathbf{M}}$, so by (*), $\overline{B}$ is a model of Γ. Hence by assumption, $\overline{B}$ is a model of φ. So by (*) again, $\overline{A}$ is a model of $\varphi^{\mathbf{M}}$. $\square$

The following theorem gives the basic idea of consistency proofs in set theory; we express this as follows. Remember by the completeness theorem that a set Γ of sentences is consistent iff it has a model.

Corollary 15.2. *Suppose that Γ and Δ are collections of sentences in our language of set theory. Suppose that $\mathbf{M}$ is a class, and $\Gamma \models [\mathbf{M} \neq \emptyset$ and $\varphi^{\mathbf{M}}]$ for each $\varphi \in \Delta$. Then Γ consistent implies that Δ is consistent.*

Proof. Suppose to the contrary that Δ does not have a model. Then trivially $\Delta \models \neg(x = x)$. By Theorem 15.1, $\Delta^{\mathbf{M}} \models \mathbf{M} \neq \emptyset \to \neg(x = x)$. Hence by hypothesis we get $\Gamma \models \neg(x = x)$, contradiction. $\square$

We now want to consider to what extent sentences can reflect to proper subclasses of $\mathbf{V}$; this is a natural extension of our considerations for absoluteness.

Actually we are dealing here with a set-theoretic version of the model theoretic notion of elementary substructure. The model theoretic notion will be important later on, so we describe the basic definition and give an important lemma about the notion.

If $\overline{A} = (A, R)$ and $\overline{B} = (B, S)$ are set theory structures, then we say that $\overline{A}$ is an *elementary substructure* of $\overline{B}$, in symbols $\overline{A} \preceq \overline{B}$, iff $A \subseteq B$, $R = S \cap (A \times A)$, and for every formula $\varphi(x_0, \ldots, x_{n-1})$ and all $a_0, \ldots, a_{n-1} \in A$, $\overline{A} \models \varphi[a_0, \ldots, a_n]$ iff $\overline{B} \models \varphi[a_0, \ldots, a_n]$. (See Chapter 2.)

Lemma 15.3. (Tarski) *Let $\overline{A} = (A, R)$ and $\overline{B} = (B, S)$ be set theory structures, and suppose that $A \subseteq B$ and $R = S \cap (A \times A)$. Then the following conditions are equivalent:*

(i) $\overline{A} \preceq \overline{B}$.

(ii) For every formula $\forall x\varphi(x, y_0, \ldots, y_{n-1})$ and all $a_0, \ldots, a_{n-1} \in A$, if $\forall b \in A[\overline{B} \models \varphi(b, a_0, \ldots, a_{n-1})]$ then $\forall b \in B[\overline{B} \models \varphi(b, a_0, \ldots, a_{n-1})]$.

Proof. (i)$\Rightarrow$(ii): Assume (i), and suppose that $\forall x\varphi(x, y_0, \ldots, y_{n-1})$ is a formula, $a_0, \ldots, a_{n-1} \in A$, and $\forall b \in A[\overline{B} \models \varphi(b, a_0, \ldots, a_{n-1})]$. Since $\overline{A} \preceq \overline{B}$, it follows that $\forall b \in A[\overline{A} \models \varphi(b, a_0, \ldots, a_{n-1})]$. Thus $\overline{A} \models \forall x\varphi(x, a_0, \ldots, a_{n-1})$. Then again by $\overline{A} \preceq \overline{B}$, $\overline{B} \models \forall x\varphi(x, a_0, \ldots, a_{n-1})$. So $\forall b \in B[\overline{B} \models \varphi(b, a_0, \ldots, a_{n-1})]$.

(ii)$\Rightarrow$(i): Assume (ii). We prove for $a_0, \ldots, a_{n-1} \in A$

$$\overline{A} \models \varphi[a_0, \ldots, a_n] \quad \text{iff } \overline{B} \models \varphi[a_0, \ldots, a_n]$$

by induction on φ. The atomic cases are clear, as are the induction steps involving $\neg$ and $\to$. Now suppose that $\overline{A} \models \forall x\varphi(x, a_0, \ldots, a_n)$. Thus $\forall b \in A[\overline{A} \models \varphi(b, a_0, \ldots, a_{n-1})]$.

Hence by the inductive hypothesis, $\forall b \in A[\overline{B} \models \varphi(b, a_0, \ldots, a_{n-1})]$. Hence by (ii), $\forall b \in B[\overline{B} \models \varphi(b, a_0, \ldots, a_{n-1})$, i.e., $\overline{B} \models \forall x\varphi(x, a_0, \ldots, a_n)$.

Conversely, suppose that $\overline{B} \models \forall x\varphi(x, a_0, \ldots, a_n)$. Thus $\forall b \in B[\overline{B} \models \varphi(b, a_0, \ldots, a_n)]$. Hence $\forall b \in A[\overline{B} \models \varphi(b, a_0, \ldots, a_n)]$; then $\forall b \in A[\overline{A} \models \varphi(b, a_0, \ldots, a_n)]$ by the inductive hypothesis, that is, $\overline{A} \models \forall x\varphi(x, a_0, \ldots, a_n)$. $\square$

Lemma 15.3′. (Tarski) *Let $\overline{A} = (A, R)$ and $\overline{B} = (B, S)$ be set theory structures, and suppose that $A \subseteq B$ and $R = S \cap (A \times A)$. Then the following conditions are equivalent:*

(i) $\overline{A} \preceq \overline{B}$.

(ii′) For every formula $\exists x\varphi(x, y_0, \ldots, y_{n-1})$ and all $a_0, \ldots, a_{n-1} \in A$, if $\exists b \in B[\overline{B} \models \varphi(b, a_0, \ldots, a_{n-1})]$ then $\exists b \in A[\overline{B} \models \varphi(b, a_0, \ldots, a_{n-1})]$.

Proof. Assume Lemma 15.3(ii), and suppose we have a formula $\exists x\varphi(x, y_0, \ldots, y_{n-1})$ and elements $a_0, \ldots, a_{n-1} \in A$ such that $\exists b \in B[\overline{B} \models \varphi(b, a_0, \ldots, a_{n-1})]$. Then not$(\forall b \in B[\overline{B} \models \neg\varphi(b, a_0, \ldots, a_{n-1})]$, so by Lemma 15.3(ii) $\exists b \in A[\overline{B} \models \varphi(b, a_0, \ldots, a_{n-1})]$.

The other direction is similar. $\square$

Lemma 15.4. *Suppose that $\mathbf{M}$ and $\mathbf{N}$ are classes with $\mathbf{M} \subseteq \mathbf{N}$. Let $\varphi_0, \ldots, \varphi_n$ be a list of formulas such that if $i \leq n$ and ψ is a subformula of φ_i, then there is a $j \leq n$ such that φ_j is ψ. Then the following conditions are equivalent:*

(i) Each φ_i is absolute for $\mathbf{M}, \mathbf{N}$.

(ii) If $i \leq n$ and φ_i has the form $\forall x\varphi_j(x, y_1, \ldots, y_t)$ with $x, y_1, \ldots, y_t$ exactly all the free variables of φ_j, then

$$\forall y_1, \ldots, y_t \in \mathbf{M}[\forall x \in \mathbf{M}\varphi_j^{\mathbf{N}}(x, y_1, \ldots, y_t) \to \forall x \in \mathbf{N}\varphi_j^{\mathbf{N}}(x, y_1, \ldots, y_t)].$$

Proof. (i)⇒(ii): Assume (i) and the hypothesis of (ii). Suppose that $y_1, \ldots, y_t \in \mathbf{M}$ and $\forall x \in \mathbf{M}\varphi_j^{\mathbf{N}}(x, y_1, \ldots, y_t)$. Thus by absoluteness $\forall x \in \mathbf{M}\varphi_j^{\mathbf{M}}(x, y_1, \ldots, y_t)$. Hence by absoluteness again, $\forall x \in \mathbf{N}\varphi_j^{\mathbf{N}}(x, y_1, \ldots, y_t))$.

(ii)⇒(i): Assume (ii). We prove that φ_i is absolute for $\mathbf{M}, \mathbf{N}$ by induction on the length of φ_i. This is clear if φ_i is atomic, and it easily follows inductively if φ_i has the form $\neg\varphi_j$ or $\varphi_j \to \varphi_k$. Now suppose that φ_i is $\forall x\varphi_j(x, y_1, \ldots, y_t)$, and $y_1, \ldots, y_t \in \mathbf{M}$. then

$$\begin{aligned}
\varphi_i^{\mathbf{M}}(y_1, \ldots, y_t) &\leftrightarrow \forall x \in \mathbf{M}\varphi_j^{\mathbf{M}}(x, y_1, \ldots, y_t) \quad \text{(definition of relativization)}\\
&\leftrightarrow \forall x \in \mathbf{M}\varphi_j^{\mathbf{N}}(x, y_1, \ldots, y_t) \quad \text{(induction hypothesis)}\\
&\leftrightarrow \forall x \in \mathbf{N}\varphi_j^{\mathbf{N}}(x, y_1, \ldots, y_t) \quad \text{(by (ii)}\\
&\leftrightarrow \varphi_i^{\mathbf{N}}(y_1, \ldots, y_t) \quad \text{(definition of relativization)}
\end{aligned}$$

$\square$

Theorem 15.5. *Suppose that $Z(\alpha)$ is a set for every ordinal α, and the following conditions hold:*

(i) If $\alpha < \beta$, then $Z(\alpha) \subseteq Z(\beta)$.

(ii) If γ is a limit ordinal, then $Z(\gamma) = \bigcup_{\alpha<\gamma} Z(\alpha)$.

Let $\mathbf{Z} = \bigcup_{\alpha \in \mathbf{On}} Z(\alpha)$. *Then for any formulas* $\varphi_0, \ldots, \varphi_{n-1}$,

$$\forall \alpha \exists \beta > \alpha[\varphi_0, \ldots, \varphi_{n-1} \text{ are absolute for } Z(\beta), \mathbf{Z}].$$

Proof. Assume the hypothesis, and let an ordinal α be given. We are going to apply Lemma 15.4 with $\mathbf{N} = \mathbf{Z}$, and we need to find an appropriate $\beta > \alpha$ so that we can take $\mathbf{M} = Z(\beta)$ in 15.4.

We may assume that $\varphi_0, \ldots, \varphi_{n-1}$ is subformula-closed; i.e., if $i < n$, then every subformula of φ_i is in the list. Let A be the set of all $i < n$ such that φ_i begins with a universal quantifier. Suppose that $i \in A$ and φ_i is the formula $\forall x \varphi_j(x, y_1, \ldots, y_t)$, where $x, y_1, \ldots, y_t$ are exactly all the free variables of φ_j. We now define a class function $\mathbf{G}_i$ as follows. For any sets $y_1, \ldots, y_t$,

$$\mathbf{G}_i(y_1, \ldots, y_t) = \begin{cases} \text{the least } \eta \text{ such that } \exists x \in Z(\eta) \neg \varphi_j^{\mathbf{Z}}(x, y_1, \ldots, y_t) & \text{if there is such,} \\ 0 & \text{otherwise.} \end{cases}$$

Then for each ordinal ξ we define

$$\mathbf{F}_i(\xi) = \sup\{\mathbf{G}_i(y_1, \ldots, y_t) : y_1, \ldots, y_t \in Z(\xi)\};$$

note that this supremum exists by the replacement axiom.

Now we define a sequence $\gamma_0, \ldots, \gamma_p, \ldots$ of ordinals by induction on $n \in \omega$. Let $\gamma_0 = \alpha + 1$. Having defined γ_p, let

$$\gamma_{p+1} = \max(\gamma_{p+1}, \sup\{\mathbf{F}_i(\xi) : i \in A, \xi \leq \gamma_p\} + 1).$$

Finally, let $\beta = \sup_{p \in \omega} \gamma_p$. Clearly $\alpha < \beta$ and β is a limit ordinal.

(1) If $i \in A$, $y_1, \ldots, y_t \in Z(\beta)$, and $\exists x \in \mathbf{Z} \neg \varphi_i^{\mathbf{Z}}(x, y_1, \ldots, y_t)$, then there is an $x \in Z(\beta)$ such that $\neg \varphi_i^{\mathbf{Z}}(x, y_1, \ldots, y_t)$.

In fact, choose p such that $y_1, \ldots, y_t \in Z(\gamma_p)$. Then $\mathbf{G}_i(y_1, \ldots, y_t) \leq \mathbf{F}_i(\gamma_p) < \gamma_{p+1}$. Hence an x as in (1) exists, with $x \in Z(\gamma_{p+1})$.

(1) clearly gives the desired conclusion. $\square$

Corollary 15.6. (The reflection theorem) *For any formulas* $\varphi_1, \ldots, \varphi_n$,

$$\mathrm{ZF} \models \forall \alpha \exists \beta > \alpha[\varphi_1, \ldots, \varphi_n \text{ are absolute for } V_\beta]. \qquad \square$$

Theorem 15.7. *Suppose that* $\mathbf{Z}$ *is a class and* $\varphi_1, \ldots, \varphi_n$ *are formulas. Then*

$$\forall X \subseteq \mathbf{Z} \exists A[X \subseteq A \subseteq \mathbf{Z} \text{ and } \varphi_1, \ldots, \varphi_n \text{ are absolute for } A, \mathbf{Z} \text{ and } |A| \leq \max(\omega, |X|)].$$

Proof. We may assume that $\varphi_1, \ldots, \varphi_n$ is subformula closed. For each ordinal α let $Z(\alpha) = \mathbf{Z} \cap V_\alpha$. Clearly there is an ordinal α such that $X \subseteq V_\alpha$, and hence $X \subseteq Z(\alpha)$. Now we apply Theorem 15.5 to obtain an ordinal $\beta > \alpha$ such that

(1) $$\varphi_1, \ldots, \varphi_n \text{ are absolute for } Z(\beta), \mathbf{Z}.$$

Let $\prec$ be a well-order of $Z(\beta)$. Let B be the set of all $i < n$ such that φ_i begins with a universal quantifier. Suppose that $i \in B$ and φ_i is the formula $\forall x \varphi_j(x, y_1, \ldots, y_t)$, where $x, y_1, \ldots, y_t$ are exactly all the free variables of φ_j. We now define a function H_i for each $i \in B$ as follows. For any sets $y_1, \ldots, y_t \in Z(\beta)$,

$$H_i(y_1, \ldots, y_t) = \begin{cases} \text{the } \prec\text{-least } x \in Z(\beta) \text{ such that } \neg\varphi_i^{Z(\beta)}(x, y_1, \ldots, y_t) & \text{if there is such,} \\ \text{the } \prec\text{-least element of } Z(\beta) & \text{otherwise.} \end{cases}$$

Let $A \subseteq Z(\beta)$ be closed under each function H_i, with $X \subseteq A$. We claim that A is as desired. To prove the absoluteness, it suffices by Lemma 15.38 to take any formula φ_i with $i \in A$, with notation as above, assume that $y_1, \ldots, y_t \in A$ and $\exists x \in \mathbf{Z} \neg\varphi_j^{\mathbf{Z}}(x, y_1, \ldots, y_t)$, and find $x \in A$ such that $\neg\varphi_j^{\mathbf{Z}}(x, y_1, \ldots, y_t)$. By (1) in the proof of Lemma 4.37, there is an $x \in Z(\beta)$ such that $\neg\varphi_j^{\mathbf{Z}}(x, y_1, \ldots, y_t)$. Hence $H_i(y_1, \ldots, y_t)$ is an element of A such that $\neg\varphi_j^{\mathbf{Z}}(H_i(y_1, \ldots, y_t), y_1, \ldots, y_t)$, as desired.

It remains only to check the cardinality estimate. This is elementary. $\square$

Lemma 15.8. *Suppose that* $\mathbf{F}$ *is a bijection from* A *onto* $\mathbf{M}$*, and for any* $a, b \in A$ *we have* $a \in b$ *iff* $\mathbf{F}(a) \in \mathbf{F}(b)$*. Then for any formula* $\varphi(x_1, \ldots, x_n)$ *and any* $x_1, \ldots, x_n \in A$*,*

$$\varphi^A(x_1, \ldots, x_n) \leftrightarrow \varphi^{\mathbf{M}}(\mathbf{F}(x_1), \ldots, \mathbf{F}(x_n)).$$

Proof. An easy induction on φ. $\square$

Theorem 15.9. *Suppose that* $\mathbf{Z}$ *is a transitive class and* $\varphi_0, \ldots, \varphi_{m-1}$ *are sentences. Suppose that* X *is a transitive subset of* $\mathbf{Z}$*. Then there is a transitive set* M *such that* $X \subseteq M$*,* $|M| \leq \max(\omega, |X|)$*, and for every* $i < m$*,* $\varphi_i^M \leftrightarrow \varphi_i^{\mathbf{Z}}$*.*

Proof. We may assume that the extensionality axiom is one of the φ_i's. Now we apply Theorem 15.7 to get a set A as indicated there. By Proposition 12.31, there is a transitive set M and a bijection mos from A onto M such that for any $a, b \in A$, $a \in b$ iff $\mathrm{mos}_{AR}(a) \in \mathrm{mos}_{AR}(b)$. Hence all of the desired conditions are clear, except possibly $X \subseteq M$. By Lemma 12.33 we have $\mathrm{mos}_{AR}(x) = x$ for all $x \in X$. Hence $X \subseteq M$. $\square$

Corollary 15.10. *Suppose that* S *is a set of sentences containing ZFC. Suppose also that* $\varphi_0, \ldots, \varphi_{n-1} \in S$*. Then*

$$S \models \exists M \left(M \text{ is transitive, } |M| = \omega, \text{ and } \bigwedge_{i<n} \varphi_i^M \right).$$

Proof. Take $\mathbf{Z} = \mathbf{V}$ and $X = \omega$ in Theorem 15.9. $\square$

The following corollary can be taken as a basis for working with countable transitive models of ZFC.

Theorem 15.11. *Suppose that S is a consistent set of sentences containing ZFC. Expand the basic set-theoretic language by adding an individual constant $\mathbf{M}$. Then the following set of sentences is consistent:*

$$S \cup \{\mathbf{M} \text{ is transitive}\} \cup \{|\mathbf{M}| = \omega\} \cup \{\varphi^{\mathbf{M}} : \varphi \in S\}.$$

Proof. Suppose that the indicated set is not consistent. Then there are $\varphi_0, \ldots, \varphi_{m-1}$ in S such that

$$S \models \mathbf{M} \text{ is transitive and } |\mathbf{M}| = \omega \to \neg \bigwedge_{i<n} \varphi_i^{\mathbf{M}};$$

it follows that

$$S \models \neg \exists \mathbf{M} \left(\mathbf{M} \text{ is transitive, } |\mathbf{M}| = \omega, \text{ and } \bigwedge_{i<n} \varphi_i^{\mathbf{M}} \right),$$

contradicting Corollary 15.10. □

Theorem 15.12. (II.5.10) *Suppose that κ is an uncountable regular cardinal and $\langle A(\xi) : \xi \leq \kappa \rangle$ satisfies the following conditions:*

(i) $\forall \xi < \eta \leq \kappa[A(\xi) \subseteq A(\eta)]$.

(ii) $\forall$ limit $\eta \leq \kappa \left[A(\eta) = \bigcup_{\xi<\eta} A(\xi)\right]$.

(iii) $\forall \xi < \kappa[|A(\xi)| < \kappa$.

(iv) $|A(\kappa)| = \kappa$.

Then $\forall \xi < \kappa \exists \eta < \kappa[\xi < \eta$ and $A(\eta) \preceq A(\kappa)$ and η is a limit ordinal].

Proof. Let A be the set of all formulas in the $\in$-language of set theory which begin with a universal quantifier. For each $\varphi \in A$ we define a function G_φ as follows. Say φ is $\forall x \psi_\varphi(x, y_1, \ldots, y_n)$. Then G_φ is a function with domain ${}^n(A(\kappa))$ such that for any $a_1, \ldots, a_n \in A(\kappa)$,

$$G_\varphi(a_1, \ldots, a_n) = \begin{cases} \text{least } \alpha < \kappa : \exists x \in A(\alpha) \neg \psi_\varphi(x, a_1, \ldots, a_n) & \text{if there is such an } \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Then for each ordinal $\alpha < \kappa$ we define

$$F_\varphi(\alpha) = \sup\{G_\varphi(a_1, \ldots, a_n) : a_1, \ldots, a_n \in A(\alpha)\}.$$

Note that $F_\varphi(\alpha) < \kappa$ since $|A(\alpha)| = |\alpha| < \kappa$.

Now we define a sequence γ_i of ordinals less than κ by recursion on $i < \omega$. Now by (iv) we can let γ_0 be greater than ξ such that $A(\gamma_0) \neq \emptyset$. Having defined γ_i, let

$$\gamma_{i+1} = \max(\gamma_i + 1, \sup\{F_\varphi(\eta) : \varphi \text{ a formula, } \eta \leq \gamma_i\}).$$

Let $\eta = \bigcup_{i \in \omega} \gamma_i$. Note that $\eta < \kappa$. Clearly $\xi < \eta$ and $A(\eta) \neq \emptyset$. Now we claim

(*) If $\varphi \in A$, say $\varphi = \forall x \psi_\varphi(x, y_1, \ldots, y_n)$, then

$$\forall a_1, \ldots, a_n \in A(\eta)[\forall x \in A(\eta) \psi_\varphi^{A(\kappa)}(x, a_1, \ldots, a_n) \rightarrow \forall x \in A(\kappa) \psi_\varphi^{A(\kappa)}(x, a_1, \ldots, a_n)].$$

In fact, suppose that $\varphi \in A$, $\varphi = \forall x \psi_\varphi(x, y_1, \ldots, y_n)$, $a_1, \ldots, a_n \in A(\eta)$, and $\exists x \in A(\kappa) \neg \psi_\varphi^{A(\kappa)}(x, a_1, \ldots, a_n)$. Say $a_1, \ldots, a_n \in A(\gamma_i)$. Then $G_\varphi(a_1, \ldots, a_n) < F_\varphi(\gamma_i) < \gamma_{i+1} < \eta$, so $\exists x \in A(\eta) \neg \psi_\varphi(x, a_1, \ldots, a_n)$. This proves (*).

Now we prove by induction on φ that for any $a_1, \ldots, a_n \in A(\eta)$, $\varphi^{A(\eta)}(a_1, \ldots, a_n)$ iff $\varphi^{A(\kappa)}(a_1, \ldots, a_n)$. This is clear for atomic formulas, and the inductive steps for $\neg$ and $\rightarrow$ are clear. Now suppose inductively that φ is $\forall x \psi_\varphi(x, y_1, \ldots, y_n)$. First suppose that $\varphi^{A(\eta)}(a_1, \ldots, a_n)$. Thus $\forall x \in A(\eta) \psi_\varphi^{A(\eta)}(x, a_1, \ldots, a_n)$. Hence by the induction hypothesis, $\forall x \in A(\eta) \psi_\varphi^{A(\kappa)}(x, a_1, \ldots, a_n)$, so by (*), $\forall x \in A(\kappa) \psi_\varphi^{A(\kappa)}(x, a_1, \ldots, a_n)$, i.e., $\varphi^{A(\kappa)}(a_1, \ldots, a_n)$.

Second, suppose that $\varphi^{A(\kappa)}(a_1, \ldots, a_n)$, i.e., $\forall x \in A(\kappa) \psi_\varphi^{A(\kappa)}(x, a_1, \ldots, a_n)$. So $\forall x \in A(\eta) \psi_\varphi^{A(\kappa)}(x, a_1, \ldots, a_n)$, hence by the inductive hypothesis, $\forall x \in A(\eta) \psi_\varphi^{A(\eta)}(x, a_1, \ldots, a_n)$. This finishes the induction. $\square$

16. Consistency of no inaccessibles

Theorem 16.1. *If* ZFC *is consistent, then so is* ZFC+ *"there do not exist uncountable inaccessible cardinals".*

Proof. For brevity we interpret "inaccessible" to mean "uncountable and inaccessible". Let

$$\mathbf{M} = \{x : \forall\alpha[\alpha \text{ inaccessible } \rightarrow x \in V_\alpha]\}$$

Thus $\mathbf{M}$ is a class, and $\mathbf{M} \subseteq V_\alpha$ for every inaccessible α (if there are such). We claim that $\mathbf{M}$ is a model of ZFC+"there do not exist uncountable inaccessible cardinals". To prove this, we consider two possibilities.

Case 1. $\mathbf{M} = \mathbf{V}$. Then of course $\mathbf{M}$ is a model of ZFC. Suppose that α is inaccessible. Then since $\mathbf{M} = \mathbf{V}$ we have $\mathbf{V} \subseteq V_\alpha$, which is not possible, since V_α is a set. Thus $\mathbf{M}$ is a model of ZFC + "there do not exist uncountable inaccessible cardinals".

Case 2. $\mathbf{M} \neq \mathbf{V}$. Let x be a set which is not in $\mathbf{M}$. Then there is an ordinal α such that α is inaccessible and $x \notin V_\alpha$. In particular, there is an inaccessible α, and we let κ be the least such.

(1) $\mathbf{M} = V_\kappa$.

In fact, if $x \in \mathbf{M}$, then $x \in V_\alpha$ for every inaccessible α, so in particular $x \in V_\kappa$. On the other hand, if $x \in V_\kappa$, then $x \in V_\alpha$ for every $\alpha \geq \kappa$, so $x \in V_\alpha$ for every inaccessible α, and so $x \in \mathbf{M}$. So (1) holds.

Now we show that V_κ is as desired. By Theorem 14.16, V_κ is a model of ZFC. Suppose that $x \in V_\kappa$ and (x is an inaccessible cardinal)$^{V_\kappa}$; we want to get a contradiction. In particular, (x is an ordinal)$^{V_\kappa}$, so by absoluteness, x is an ordinal. Absoluteness clearly implies that x is infinite. We claim that x is a cardinal. For, if $f : y \to x$ is a bijection with $y < x$, then clearly $f \in V_\kappa$, and hence by absoluteness ($f : y \to x$ is a bijection and $y < x)^{V_\alpha}$, contradiction. Similarly, x is regular; otherwise there is an injection $f : y \to x$ with $\text{rng}(f)$ unbounded in x, so clearly $f \in V_\kappa$, and absolutenss again yields a contradiction. Thus x is a regular cardinal. Hence, since κ is the smallest inaccessible, there is a $y \in x$ such that there is a one-one function g from x into $\mathscr{P}(y)$. Again, $g \in V_\kappa$, and easy absoluteness results contradicts (x is an inaccessible cardinal)$^{V_\kappa}$. $\square$

(I.5.23) Let $\mathfrak{A}$ be an $\mathscr{L}$-structure, and $P \subseteq A$. Suppose that $k \in \omega\backslash 1$. A set $S \subseteq {}^kA$ is *definable over* $\mathfrak{A}$ *with parameters in* P iff for some $n \geq 0$ there is a formula $\varphi(x_1, \ldots, x_k, y_1, \ldots, y_n)$ of $\mathscr{L}$ such that for some $\overline{b} = \langle b_1, \ldots, b_n\rangle \in P$, $S = \{\overline{a} : \mathfrak{A} \models \varphi(\overline{a}, \overline{b})\}$.

For $k \in \omega\backslash 1$, a set $S \subseteq {}^kA$ is *definable over* $\mathfrak{A}$ *with parameters* iff it is definable over $\mathfrak{A}$ with parameters in A.

For $k \in \omega\backslash 1$, a set $S \subseteq {}^kA$ is *definable over* $\mathfrak{A}$ *without parameters* iff it is definable over $\mathfrak{A}$ with parameters in $\emptyset$.

For $a \in A$ and $P \subseteq A$, we say that *a is definable over $\mathfrak{A}$ with parameters in P* iff $\{a\}$ is definable over $\mathfrak{A}$ with parameters in P.

For $a \in A$, we say that *a is definable over $\mathfrak{A}$ without parameters* iff $\{a\}$ is definable over $\mathfrak{A}$ without parameters.

Proposition 17.1. (I.15.24) *If every element of A is definable without parameters, then for every $k \in \omega\backslash 1$, if $S \subseteq {}^kA$ is definable with parameters, then it is definable without parameters.*

Proof. Assume that every element of A is definable without parameters, $k \in \omega\backslash 1$, and $S \subseteq {}^kA$ is definable with parameters; say $S = \{\overline{a} : \mathfrak{A} \models \varphi(\overline{a}, \overline{b})\}$, where $\overline{a}$ has length k, $\overline{b}$ has length n, and $\overline{b}$ is a sequence of elements of A. For each $i = 1, \ldots, n$ the set $\{b_i\}$ is definable without parameters; say $\{b_i\} = \{x \in A : \mathfrak{A} \models \psi_i(x)\}$. Then

$$S = \{\overline{a} : \mathfrak{A} \models \exists\overline{v}\left[\varphi(\overline{a}, \overline{v}) \wedge \bigwedge_{i=1}^{n} \psi_i(v_i)\right] \qquad \square$$

(I.15.25) If A is a nonempty set and $P \subseteq A$, then $\mathcal{D}(A, P)$ is the set of all subsets of A which are definable over $(A, \in)$ with parameters from P.

$\mathcal{D}^+(A) = \mathcal{D}(A, A)$.

$\mathcal{D}^-(A) = \mathcal{D}(A, \emptyset)$.

$\mathcal{D}^+(\emptyset) = \mathcal{D}^-(\emptyset) = \{\emptyset\}$.

Proposition 17.2. (I.15.26) *Every finite subset of A is in $\mathcal{D}^+(A)$.*

Proof. Let $\{b_1, \ldots, b_n\}$ be a finite subset of A. Then

$$\{b_1, \ldots, b_n\} = \left\{a \in A : \mathfrak{A} \models \bigvee_{i=1}^{n} [a = b_i]\right\} \qquad \square$$

Proposition 17.3. (I.15.28) *Assume that $P \subseteq A$ and R is a relation on A which well-orders A and is definable over $\mathfrak{A}$ with parameters in P. Let $\mathscr{H}(\mathfrak{A}, P)$ be the set of all elements of A that are definable with parameters in P.*

Then $\mathscr{H}(\mathfrak{A}, P)$ is an elementary submodel of $\mathfrak{A}$.

Proof. We apply Lemma 15.3. Assume that $\mathfrak{A} \models \exists y\psi(\overline{a}, y)$ with each $a_i \in \mathscr{H}(\mathfrak{A}, P)$. Say $\mathfrak{A} \models \varphi_i(a_i, \overline{b_i})$ with each $\overline{b_i} \subseteq P$. Let c be the R-least element of A such that

$$\mathfrak{A} \models \exists \overline{x} \left[\bigwedge_{i<m} \varphi_i(x_i, \overline{b_i}) \wedge \psi(\overline{x}, c)) \right]$$

Then $c \in \mathscr{H}(\mathfrak{A}, P)$ and $\mathfrak{A} \models \psi(\overline{a}, c)$. $\square$

Proposition 17.4. (I.15.27) *If $\varphi(x, y_1, \ldots, y_n)$ is a formula in the language of set theory, then*

$$ZFC \vdash \forall A \forall b_1, \ldots, b_n \in A[\{x \in A : \varphi^A(x, b_1, \ldots, b_n)\} \in \mathcal{D}^+(A)]. \qquad \square$$

(II.6.1) We define

$$\begin{aligned} L(0) &= \emptyset; \\ L(\beta + 1) &= \mathcal{D}^+(L(\beta)); \\ L(\gamma) &= \bigcup_{\beta<\gamma} L(\beta) \quad \text{for limit } \gamma; \\ L &= \bigcup_{\alpha \in \mathrm{ON}} L(\alpha). \end{aligned}$$

Lemma 17.5. (II.6.2) $L(\alpha) \subseteq V_\alpha$.

Proof. Induction on α. It is obvious for $\alpha = 0$. Assume it for α, and suppose that $x \in L(\alpha + 1)$. Thus $x \subseteq L(\alpha) \subseteq V_\alpha$, so $x \in V_{\alpha+1}$. The limit case is clear. $\square$

Lemma 17.6. (II.6.2) *$L(\beta)$ is transitive.*

Proof. Induction on β. $\beta = 0$ is clear. Now assume that $L(\beta)$ is transitive. Now $L(\beta) \subseteq L(\beta+1)$, since if $a \in L(\beta)$ then $a = \{x \in L(\beta) : x \in a\}$ because $L(\beta)$ is transitive, and this shows that $a \in \mathcal{D}^+(L_\beta) = L(\beta + 1)$. Now let $a \in L(\beta + 1)$. Then $a \in \mathscr{P}(L(\beta))$, so $a \subseteq L(\beta) \subseteq L(\beta + 1)$.

The case of limit β is clear. $\square$

Lemma 17.7. (II.6.2) *If $\alpha \le \beta$, then $L(\alpha) \subseteq L(\beta)$.*

Proof. By induction, with α fixed. The cases $\beta = \alpha$ and β limit are easy. Now suppose that $a \in L(\beta)$. Then $a = \{x \in L(\beta) : x \in a\}$ because $L(\beta)$ is transitive, and this shows that $a \in \mathcal{D}^+(L_\beta) = L(\beta + 1)$. $\square$

Lemma 17.8. (II.6.2) *$L(\beta) \cap \mathrm{ON} = \beta$.*

Proof. Induction on β. The cases $\beta = 0$ and β limit are clear. Now suppose that $L(\beta) \cap \mathrm{ON} = \beta$. Then $\beta \subseteq L(\beta) \subseteq L(\beta+1)$, so $\beta \subseteq L(\beta+1) \cap \mathrm{ON} \subseteq V_{\beta+1} \cap \mathrm{ON} = \beta+1 = \beta \cup \{\beta\}$. So it suffices to show that $\beta \in L(\beta + 1)$. Now

$$\beta = L(\beta) \cap \mathrm{ON} = \{a \in L(\beta) : (a \text{ is an ordinal})^{L(\beta)}\} \in \mathcal{D}^+(L(\beta)) = L(\beta + 1). \qquad \square$$

(II.6.3) For $x \in L$, the *L-rank* $\rho(x)$ of x is the least α such that $x \in L(\alpha+1)$.

Lemma 17.9. (II.6.4) $L(\alpha) = \{x \in L : \rho(x) < \alpha\}$.

Proof. Suppose that $x \in L(\alpha)$. If $\alpha = \beta+1$, then $\rho(x) \le \beta < \alpha$. If α is limit, then $x \in L(\beta)$ for some $\beta < \alpha$, and $\beta+1 < \alpha$ with $x \in L(\beta+1)$, hence $\rho(x) \le \beta < \alpha$. So $\subseteq$ holds.

If $x \in L$ and $\rho(x) < \alpha$, then $x \in L(\rho(x)+1) \subseteq L(\alpha)$. □

Lemma 17.10. (II.6.4) $L(\alpha+1)\backslash L(\alpha) = \{x \in L : \rho(x) = \alpha\}$. □

Lemma 17.11. (II.6.4) *If $x \in y \in L$, then $\rho(x) < \rho(y)$.*

Proof. Let $\alpha = \rho(y)$. So $x \in y \in L(\alpha+1) = \mathcal{D}^+(L(\alpha)) \subseteq \mathscr{P}(L(\alpha))$, so $x \in L(\alpha)$, hence $\rho(x) < \alpha$ by Lemma 17.9. □

Lemma 17.12. (II.6.5) $L(\alpha) \in L(\alpha+1)$.

Proof. $L_\alpha = \{x \in L(\alpha) : x = x\} \in \mathcal{D}^+(L_\alpha) = L_{\alpha+1}$. □

Lemma 17.13. (II.6.5) $\rho(L(\alpha)) = \rho(\alpha) = \alpha$.

Proof. $\rho(L(\alpha)) \le \alpha$ by Lemma 17.12. If $\rho(L(\alpha)) < \alpha$, then $L(\alpha) \in L(\rho(L(\alpha)+1)) \subseteq L(\alpha)$, contradiction. So $\rho(L(\alpha)) = \alpha$.

By Lemma 17.8 we have $\alpha = \{x \in L(\alpha) : x \text{ is an ordinal}\} \in \mathcal{D}^+(L(\alpha)) = L(\alpha+1)$, so $\rho(\alpha) \le \alpha$. If $\rho(\alpha) < \alpha$, then $\alpha \in L(\rho(\alpha)+1) \subseteq L(\alpha)$, so $\alpha \in L(\alpha) \cap \mathrm{ON} = \alpha$ by Lemma 17.8, contradiction. □

Lemma 17.14. (II.6.6) *Every finite subset of $L(\alpha)$ is in $L(\alpha+1)$.*

Proof. If $F \in [L(\alpha)]^{<\omega}$, then $F = \left\{x \in L(\alpha) : \bigvee_{y \in F}(x = y)\right\} \in \mathcal{D}^+(L(\alpha) = L(\alpha+1)$. □

Lemma 17.15. (II.6.7) $L(\alpha) = V_\alpha$ *for all* $\alpha \le \omega$.

Proof. $L(n) = V_n$ for all $n \in \omega$ by induction, using Lemma 17.14. $L(\omega) = V_\omega$ by taking unions. □

Lemma 17.16. (II.6.9) $|\mathcal{D}^+(A)| = |A|$ *for all infinite A.* □

Lemma 17.17. (II.6.10) $|L(\alpha)| = |\alpha|$ *for all infinite α.*

Proof. Since $\alpha \subseteq L(\alpha)$ by Lemma 17.8, we have $|\alpha| \le |L(\alpha)|$. Now we prove $|L(\alpha)| = |\alpha|$ for all infinite α by induction on α. It is true for $\alpha = \omega$ by Lemma 17.15. Now assume that $|L(\alpha)| = |\alpha|$. Then $|L(\alpha+1)| = |\mathcal{D}^+(L(\alpha))| = |L(\alpha)| = |\alpha|$. using Lemma 17.16. For α limit $> \omega$,

$$|L(\alpha)| = \left|\bigcup_{\beta<\alpha} L(\beta)\right| \le \sum_{\beta<\alpha} |L(\beta)| = \sum_{\omega\le\beta<\alpha} |L(\beta)| = \sum_{\omega\le\beta<\alpha} |\beta| = |\alpha|. \qquad \square$$

Theorem 17.18. (II.6.11) *L is a model of ZF.*

Proof. We will apply Lemma 14.12. Given $x \subseteq L$, let $\alpha = \sup\{\rho(z)+1 : z \in x\}$. Then $x \subseteq L(\alpha) \in L$, using Lemma 17.12.

A typical instance of the comprehension axioms, relativized to L, is as follows:

$$\forall z \in L \forall w_1 \in L \ldots \forall w_n \in L \exists y \in L \forall x \in L[x \in y \leftrightarrow x \in z \text{ and } \varphi^L].$$

So, let $z, w_1, \ldots, w_n \in L$. Let $y = \{x \in z : \varphi^L\}$. Now there is an α such that $z, w_1, \ldots, w_n \in L(\alpha)$. By the reflection theorem (Theorem 15.5) there is a $\beta > \alpha$ such that φ is absolute for $L(\beta), L$. Then

$$\begin{aligned} y &= \{x \in z : \varphi^L\} = \{x \in z : \varphi^{L(\beta)}\} \\ &= \{x \in L(\beta) : x \in z \text{ and } \varphi^{L(\beta)}\} \in \mathcal{D}^+(L(\beta) = L(\beta+1) \subseteq L \end{aligned}$$ □

Lemma 17.19. (II.6.12) *If $x, y \in L$, then $\{x, y\} \in L$ and $\rho(\{x, y\}) = \max(\rho(x), \rho(y)) + 1$.*

Proof. Let $\rho(x) = \alpha$ and $\rho(y) = \beta$. Say $\alpha \leq \beta$. Then $x, y \in L(\beta+1)$. So $\{x, y\} = \{z \in L(\beta+1) : z = x \text{ or } z = y\} \in \mathcal{D}^+(L(\beta+1)) = L(\beta+2)$. This shows that $\{x, y\} \in L$ and $\rho(\{x, y\}) \leq \beta + 1$. Now $y \in \{x, y\}$, so $\beta = \rho(y) < \rho(\{x, y\})$ by Lemma 17.11, so $\rho(\{x, y\}) = \beta + 1$. □

Lemma 17.20. (II.6.12) *If $x, y \in L$, then $(x, y) \in L$ and $\rho((x, y)) = \max(\rho(x), \rho(y)) + 2$.* □

Lemma 17.21. (II.6.12) *If $x \in L$, then $\bigcup x \in L$ and $\rho\left(\bigcup x\right) \leq \rho(x)$.*

Proof. Let $\alpha = \rho(x)$. Then $x \in L(\alpha+1) = \mathcal{D}^+(L(\alpha))$, so there is a formula φ with constants from $L(\alpha)$ and one free variable such that $x = \{y \in L(\alpha) : \varphi^{L(\alpha)}(y)\}$. If $y \in \bigcup x$, say $y \in z \in x$, so $z \in L(\alpha)$ and so $y \in L(\alpha)$ since $L(\alpha)$ is transitive. Thus $\bigcup x = \{y \in L(\alpha) : \exists z \in L(\alpha) \varphi^{L(\alpha)}(z)\}$. Hence $\bigcup x \in \mathcal{D}^+(L(\alpha)) = L(\alpha+1)$; so $\bigcup x \in L$ and $\rho\left(\bigcup x\right) \leq \rho(x)$. □

Lemma 17.22. (II.6.12) *If $x, y \in L$, then $x \cup y \in L$ and $\rho(x \cup y) \leq \max(\rho(x), \rho(y))$.*

Proof. Let $\alpha = \rho(x)$ and $\beta = \rho(y)$. Say $\alpha \leq \beta$. Now $y \in L(\beta+1) = \mathcal{D}^+(L(\beta))$, so there is a formula φ with constants from $L(\beta)$ and with one free variable such that $y = \{z \in L(\beta) : \varphi^{L(\beta)}\}$. Also there is a formula ψ with constants from $L(\beta)$ and with one free variable such that $x = \{z \in L(\beta) : \psi^{L(\beta)}\}$; if $x \in L(\beta)$ we can take ψ to be $v = x$. Now $x \cup y = \{z \in L(\beta) : \varphi^{L(\beta)} \text{ or } \psi^{L(\beta)}\} \in \mathcal{D}^+(L(\beta)$, so $x \cup y \in L$ and $\rho(x \cup y) \leq \max(\rho(x), \rho(y))$. □

Lemma 17.23. *For the language $\mathscr{L}$ of set theory, the formula "φ is a formula of $\mathscr{L}$" is absolute for transitive models of ZF.*

Proof. Let F be the collection of all formula construction sequences; see page 20. Thus for any f, $f \in F$ iff there is an $m \in \omega \backslash 1$ such that f is a function with domain m, and for each $i < m$ one of the following conditions holds:

(1) There exist $j, k < \omega$ such that $f_i = \langle 3, 5(j+1), 5(k+1)\rangle$. ($f_i$ is $v_j = v_k$.)

(2) There exist $j, k < \omega$ such that $f_i = \langle 6, 5(j+1), 5(k+1)\rangle$. ($f_i$ is $v_j \in v_k$, with 6 the "symbol" for $\in$.)

(3) There is a $j < i$ such that $f_i = \langle 1\rangle^\frown f_j$. ($f_i$ is $\neg f_j$.)

(4) There exist $j, k < i$ such that $f_i = \langle 2\rangle^\frown f_j^\frown f_k$. ($f_i$ is $f_j \to f_k$.)

(5) There exist $j < i$ and $k \in \omega$ such that $f_i = \langle 4, 5(k+1)\rangle^\frown f_j$. ($f_i$ is $\forall v_k f_j$.)

Now φ is a formula iff there is an $f \in F$ and an $i \in \mathrm{dmn}(f)$ such that $\varphi = f_i$. □

Lemma 17.24. *In the language for set theory, the relation $(A, \in) \models \varphi[x]$ is absolute for transitive models of ZF, where A is a nonempty set, φ is a formula, and $x \in {}^\omega A$.*

Proof. Let T be the set of all f such that there is an $m \in \omega\backslash 1$ such that f is a function with domain m and for each $i < m$ one of the following conditions holds:

(1) There exist $j, k \in \omega$ such that $f_i = (v_j = v_k, S)$ with $S = \{x \in {}^\omega A : x_j = x_k\}$.

(2) There exist $j, k \in \omega$ such that $f_i = (v_j \in v_k, S)$ with $S = \{x \in {}^\omega A : x_j \in x_k\}$.

(3) There is a $j < i$ such that $f_i = (\neg 1^{st}(f_j), S)$, with $S = {}^\omega A\backslash 2^{nd}(f_j)$.

(4) There exist $j, k < i$ such that $f_i = (1^{st}(f_j) \to 1^{st}(f_k), S)$, with $S = ({}^\omega A\backslash 2^{nd}(f_j)) \cup 2^{nd}(f_k)$.

(5) There exist $j < i$ and $k \in \omega$ such that $f_i = (\forall v_k 1^{st}(f_j), S)$, with $S = \{x \in {}^\omega A : \forall y \in {}^\omega A[y \restriction (\omega\backslash\{k\}) = x \restriction (\omega\backslash\{k\}) \to y \in 2^{nd}(f_j)\}$.

Then $(A, \in) \models \varphi[x]$ iff there exist an $f \in T$ and an $i \in \mathrm{dmn}(f)$ such that $\varphi = 1^{st}(f_i)$ and $x \in 2^{nd}(f_i)$. □

Lemma 17.25. *$\mathcal{D}^+(A)$ is absolute for transitive models of ZF.*

Proof.

$$\begin{aligned} X \in \mathcal{D}^+(A) \quad \text{iff} \quad & \exists\varphi(x, y_1, \ldots, y_m)\exists b_1, \ldots, b_m \in A\forall a \in A \\ & [a \in X \text{ iff } (A, \in) \models \varphi[a, b_1, \ldots, b_m]]. \end{aligned}$$ □

Lemma 17.26. (II.6.13) *The function $\langle L(\alpha) : \alpha \in \mathbf{ON}\rangle$ is absolute for transitive models of ZF.*

Proof. Let $\mathbf{M}$ be a transitive model of ZF. We will apply Theorem 13.11. Let $\mathbf{A} = \mathbf{ON}$ and $\mathbf{R} = \{(\alpha, \beta) : \alpha < \beta\}$. Thus $\mathbf{R}$ is well-founded and set-like on $\mathbf{A}$. Define $\mathbf{G} : \mathbf{A} \times \mathbf{V} \to \mathbf{V}$ as follows:

$$\mathbf{G}(\alpha, x) = \begin{cases} \emptyset & \text{if } x = \emptyset, \\ \mathcal{D}^+(x(\beta)) & \text{if } x \text{ is a function with domain } \alpha = \beta + 1, \\ \bigcup_{\beta<\alpha} x(\beta) & \text{if } x \text{ is a function with domain } \alpha, \text{ and } \alpha \text{ is a limit ordinal,} \\ \emptyset & \text{otherwise.} \end{cases}$$

Thus L is the function obtained by Theorem 9.7. Clearly $\mathbf{G}, \mathbf{R}, \mathbf{A}$ are absolute for $\mathbf{M}$. The statement "$\mathbf{R}$ is set-like on $\mathbf{A}$" is "$\forall x \in \mathbf{ON}[\{y : y < x\}$ is a set]", i.e. "$\forall x \in \mathbf{ON}\exists z \forall y[y \in z \leftrightarrow y \in x]$", and the relativization to $\mathbf{M}$ clearly holds. Since $\mathbf{M}$ is transitive, $\forall x \in \mathbf{ON} \cap \mathbf{M}[x \subseteq \mathbf{M}]$. $\square$

Corollary 17.27. (II.6.14) $(V = L)^L$, *i.e.,* $(\forall x \exists \alpha[x \in L(\alpha)])^L$.

Proof. Since $ON \subseteq L$ and "ordinal" and $\langle L(\alpha) : \alpha \in \mathrm{ON}\rangle$ are absolute, it suffices to show that $\forall x \in L \exists \alpha \in ON[x \in L(\alpha)]$, and this is obvious. $\square$

(II.6.15) For any transitive set M let $o(M) = M \cap ON$.

Proposition 17.28. *If M is a transitive set, then $o(M)$ is the first ordinal not in M.*

Proof. If $x \in M \cap ON$, then $x \subseteq M \cap ON$. So $o(M)$ is a transitive set of ordinals, and hence is an ordinal. If $o(M) \in M$, then $o(M) \in M \cap ON = o(M)$, contradiction. If α is an ordinal not in M, then $\alpha \notin M \cap ON$, so $M \cap ON \leq \alpha$. $\square$

Lemma 17.29. (II.6.16) *If M is transitive and $M \models ZF - P$, then there is no largest ordinal in M.*

Proof. $M \models \forall x \exists y \forall z[z \in y \leftrightarrow z = x$ or $z \in x]$. $\square$

Lemma 17.30. (II.6.16) *If M is transitive and $M \models ZF - P$, then $M \models V = L$ iff $M = L(o(M))$.*

Proof. Let $\gamma = o(M)$. By absoluteness, $L(\alpha) \in M$ for all $\alpha < \gamma$. By Lemma 17.29, γ is a limit ordinal. Since $L(\gamma) = \bigcup_{\alpha<\gamma} L(\alpha)$, it follows that $L(\gamma) \subseteq M$. Now

$$\begin{aligned} M \models V = L \quad & \text{iff} \quad \forall x \in M \exists \alpha \in M \cap ON[x \in L(\alpha)] \\ & \text{iff} \quad \forall x \in M \exists \alpha \in \gamma[x \in L(\alpha) \\ & \text{iff} \quad M \subseteq L(\gamma) \\ & \text{iff} \quad M = L(\gamma). \end{aligned}$$ $\square$

(II.6.17) A formula φ in the language for set theory is *good* iff its free variables are exactly $v_0, \ldots, v_n$ for some $n \in \omega$. Let $\langle \varphi_i : i \in \omega \rangle$ be a one-one list of all the good formulas. Say $\varphi_i = \varphi_i(v_0, \ldots, v_{n_i})$ for each $i \in \omega$. For $A \neq \emptyset$, $i \in \omega$, and $\overline{b} \in {}^{n_i}A$, let $D(A, i, \overline{b}) = \{a \in A : (A, \in) \models \varphi_i(a, \overline{b})\}$. For each $S \in \mathcal{D}^+(A)$ let $i(S, A)$ be the least $i \in \omega$ such that $S = D(A, i, \overline{b})$ for some $\overline{b} \in {}^{n_i}A$.

(II.6.18) If R is a well-order of A, then $R^{(n)}$ is the lexicographic order of ${}^nA$. For $S \in \mathcal{D}^+(A)$, $p(S, R)$ is the $R^{(n_{i(S,A)})}$-first $\overline{b} \in {}^{n_{i(S,A)}}A$ such that $S = D(A, i(S, A), \overline{b})$. Then we define $W = W(A, R)$ by setting

$$S_1 W S_2 \text{ iff } S_1, S_2 \in \mathcal{D}^+(A) \text{ and } \begin{cases} i(S_1, A) < i(S_2, A) \text{ or} \\ i(S_1, A) = i(S_2, A) \text{ and } p(S_1, R) R^{(n_{i(S_1,A)})} p(S_2, R). \end{cases}$$

Clearly $W(A, R)$ is a well-order of $\mathcal{D}^+(A)$.

(II.6.19) We define $\lhd_\delta \subseteq L(\delta) \times L(\delta)$ by recursion as follows:

$$x \lhd_\delta y \text{ iff } x, y \in L(\delta) \text{ and } \begin{cases} \rho(x) < \rho(y) \text{ or} \\ \rho(x) = \rho(y) \text{ and } xW(L(\rho(x)), \lhd_{\rho(x)})y. \end{cases}$$

Then we define

$$x <_L y \text{ iff } x, y \in L \text{ and } \begin{cases} \rho(x) < \rho(y) \text{ or} \\ \rho(x) = \rho(y) \text{ and } x \lhd_{\rho(x)+1} y. \end{cases}$$

Theorem 17.31. (II.6.20) *$<_L$ well-orders L. Each $L(\delta)$ is an initial segment, with $\lhd_\delta = <_L \cap (L(\delta) \times L(\delta))$*

Assuming $V = L$, $<_L$ well-orders V, and AC holds. □

Lemma 17.32. (II.6.22) *If κ is an uncountable regular cardinal, then $L(\kappa) \models ZF - P + V = L$.*

Proof. Let $M = L(\kappa)$. Foundation and extensionality hold since $L(\kappa)$ is transitive. Pairing holds by Lemma 17.19 and Theorem 14.3. Union holds by Lemma 17.21 and Theorem 14.4. By Theorem 14.8, infinity holds in M.

We turn to replacement, where we apply Theorem 14.6. Assume that φ is a formula with free variables among $x, y, A, w_1, \ldots, w_n$ and $A, w_1, \ldots, w_n \in L(\kappa)$ and

$$\forall x \in A \exists! y[y \in L(\kappa) \wedge \varphi^{L(\kappa)}(x, y, A, w_1, \ldots, w_n)].$$

For each $a \in A$ let $f(a) \in L(\kappa)$ be such that $\varphi^{L(\kappa)}(x, f(a), A, w_1, \ldots, w_n)$. Then choose α so that $A, w_1, \ldots, w_n \in L(\alpha)$, with $\omega \leq \alpha < \kappa$. Note that $|A| \leq |L(\alpha)| = |\alpha| < \kappa$ by Lemma 17.17.. Define $f : A \to L(\kappa)$ by setting $f(x) =$ the $y \in L(\kappa)$ such that $\varphi^{L(\kappa)}(x, y, A, w_1, \ldots, w_n)$. Then $\forall x \in A[\rho(f(x)) < \kappa]$. Hence $\beta \stackrel{\text{def}}{=} \sup_{x \in A} \rho(f(x)) < \kappa$. Let $Y = L(\beta)$. So $Y \in L(\kappa)$ by Lemma 17.12. Clearly Y is as desired.

A typical instance of the comprehension axioms, relativized to $L(\kappa)$, is as follows:

$$\forall z \in L(\kappa) \forall w_1 \in L(\kappa) \ldots \forall w_n \in L(\kappa) \exists y \in L(\kappa) \forall x \in L(\kappa)[x \in y \leftrightarrow x \in z \text{ and } \varphi^{L(\kappa)}].$$

So, let $z, w_1, \ldots, w_n \in L(\kappa)$. Let $y = \{x \in z : \varphi^{L(\kappa)}\}$. Now there is an $\alpha < \kappa$ such that $z, w_1, \ldots, w_n \in L(\alpha)$. Now by Theorem 15.11 there is an $\eta \in (\alpha, \kappa)$ such that $L(\eta) \preceq L(\kappa)$. Hence $y = \{x \in L(\eta) : x \in z \text{ and } \varphi^{L(\eta)}\} \in \mathcal{D}^+(L(\eta)) = L(\eta + 1) \subseteq L(\kappa)$.

Thus $L(\kappa)$ is a model of $ZF - P$. Now $o(L(\kappa)) = L(\kappa) \cap \mathbf{ON}$ by Lemma 17.8, so $L(\kappa) = L(o(L(k)))$ and hence by Lemma 17.30, $L(\kappa) \models V = L$. □

Theorem 17.33. (II.6.23) *If $V = L$, then $L(\kappa) = H(\kappa)$ for every infinite cardinal κ.*

Proof. First note that $L(\kappa) \subseteq H(\kappa)$. In fact, suppose that $x \in L(\kappa)$. Choose $\alpha < \kappa$ such that $x \in L(\alpha)$. Then $\text{trcl}(x) \subseteq L(\alpha)$, so $|\text{trcl}(x)| \leq |L(\alpha)| < \kappa$ and hence $x \in H(\kappa)$.

Now assume that $\kappa = \lambda^+$ and $b \in H(\lambda^+)$. Let $T = \text{trcl}(\{b\})$. Then $b \in T$ and $|T| \leq \lambda$. Take an uncountable regular cardinal θ which is greater than the least β such that

$T \in L(\beta)$. By the downward Löwenheim-Skolem theorem there is an A such that $A \preceq L(\theta)$, $T \subseteq A$, and $|A| \leq \lambda$. Let $\text{mos}_{A\in}$ be the Mostowski isomorphism of A onto a transitive set B. By Lemma 12.33, $\text{mos}_{A\in}(x) = x$ for all $x \in T$. In particular, $b = \text{mos}_{A\in}(b) \in B$. By Lemma 17.32, $L(\theta) \models ZF - P + V = L$, so also $B \models ZF - P + V = L$. Hence by Lemma 17.30, there is a β such that $B = L(\beta)$, so $|\beta| = |B| = |A| \leq \lambda$. So $\beta < \lambda^+$ and $b \in B = L(\beta) \subseteq L(\lambda^+)$. This shows that $H(\lambda^+) \subseteq L(\lambda^+)$. It then follows that for limit γ, $H(\gamma) \subseteq L(\gamma)$. □

Theorem 17.34. (II.6.23) *$V = L$ implies GCH.*

Proof. Let λ be an infinite cardinal. Then using Theorem 17.33, $\mathscr{P}(\lambda) \subseteq H(\lambda^+) = L(\lambda^+)$, so $2^\lambda = |\mathscr{P}(\lambda)| \leq |L(\lambda^+| = \lambda^+$. □

Theorem 17.35. **ON** $\subseteq L$.

Proof. By Lemma 17.8. □

Theorem 17.36. (II.6.25) *If κ is any cardinal, then $L \models [\kappa$ is a cardinal$]$.*

Proof. $\neg\exists\alpha < \kappa\exists f : \alpha \to \kappa[f$ is a surjection$]$. This holds in L by downwards absoluteness. □

Theorem 17.37. (II.6.25) *If κ is weakly inaccessible, then $L \models [\kappa$ is strongly inaccessible$]$.*

Proof. $\forall\alpha \in \kappa\exists\lambda \in \kappa[\alpha \in \lambda$ and λ is a cardinal$]$, so this holds in L by absoluteness. Hence $L \models [\kappa$ is a limit cardinal$]$. Also, $\forall\alpha < \kappa\forall f : \alpha \to \kappa\exists\beta < \kappa\forall\xi < \alpha[f(\xi) < \beta]$; this holds in L by absoluteness, so $L \models [\kappa$ is regular limit$]$. Since $L \models [V = L]$, the theorem follows. □

Theorem 17.38. (II.6.25) *Assume that $V = L$ and κ is strongly inaccessible. Then $L(\kappa) = H(\kappa) = V_\kappa \models [ZFC + V = L]$.*

Proof. $L(\kappa) = H(\kappa)$ by Theorem 17.33. $H(\kappa) = V_\kappa$ by Lemma 12.54. $V_\kappa \models ZFC$ by Theorem 14.16. $V_\kappa \models V = L$ by Theorem 17.32. □

Corollary 17.39. (II.6.26) *If there is a weakly inaccessible cardinal, then there is a countable transitive M such that $M \models ZFC + V = L$.*

Proof. Let κ be weakly inaccessible. By Theorem 17.37 we have $L \models [\kappa$ is strongly inaccessible$]$. Now $L \models [V = L]$ by Corollary 17.27. Hence by Theorem 17.38, in L we have $V_\kappa \models [ZFC + V = L]$. Now apply Lemma 14.9. □

An *inner model of ZF* is a transitive class which is a model of ZF and contains all the ordinals.

Theorem 17.40. (minimality) *If M is an inner model of ZF, then $L \subseteq M$.*

Proof. By Lemma 17.26. $L_\alpha^M = L_\alpha$ for every ordinal α. □

Theorem 17.41. (condensation) *For every limit ordinal α, if $M \equiv_{ee} L_\alpha$ and M is transitive, then there is a limit ordinal $\beta \le \alpha$ such that $M = L_\beta$.*

Proof. First we claim that M is extensional. For,

$$L_\alpha \models \forall x, y[\forall z[z \in x \leftrightarrow z \in y] \to x = y], \quad \text{so}$$
$$M \models \forall x, y[\forall z[z \in x \leftrightarrow z \in y] \to x = y].$$

So, suppose that $x, y \in M$ and $\forall z \in M[z \in x \leftrightarrow z \in y]$. Since M is transitive, $\forall z[z \in x \leftrightarrow z \in y]$, hence $x = y$.

Now let $\beta = M \cap \mathbf{ON}$. Since M is transitive, β is an ordinal, and $\beta \subseteq M$.

(1) $0 < \beta$.

For, $L_\alpha \models \exists x \forall y \in x[y \neq y]$, so $M \models \exists x \forall y \in x[y \neq y]$. Choose $x \in M$ so that $\forall y \in M \cap x[y \neq y]$. Thus $M \cap x = \emptyset$. Since M is transitive, $x = \emptyset$. So (1) holds.

(2) β is a limit ordinal.

For,

$$L_\alpha \models \forall \gamma[\gamma \text{ is an ordinal} \to \exists \delta[\delta \text{ is an ordinal} \wedge [\gamma < \delta]]], \quad \text{so}$$
$$M \models \forall \gamma[\gamma \text{ is an ordinal} \to \exists \delta[\delta \text{ is an ordinal} \wedge [\gamma < \delta]]].$$

Now let $\gamma < \beta$. Then $\gamma \in M$ and by absoluteness $M \models [\gamma \text{ is an ordinal}]$, so $\exists \delta \in M[M \models [\delta \text{ is an ordinal}] \wedge [\gamma < \delta]]$. Thus by absoluteness, $\delta \in M$ and $\gamma < \delta$, so (2) holds

(3) $L_\beta \subseteq M$.

For, $L_\alpha \models \forall \delta \in \mathbf{ON} \exists y[y = L_\delta]$. Hence $M \models \forall \delta \in \mathbf{ON} \exists y[y = L_\delta]$. So for every $\delta < \beta$ there is a $y \in M$ such that $M \models [y = L_\delta]$. By absoluteness, $y = L_\delta$. So (3) holds.

(4) $M \subseteq L_\beta$.

For, $L_\alpha \models \forall x \exists y \exists z[y \text{ is an ordinal and } z = L_y \wedge x \in z]$. Hence $M \models \forall x \exists y \exists z[y \text{ is an ordinal and } z = L_y \wedge x \in z]$. Now take any $a \in M$. Choose an ordinal $\gamma \in M$ and $z \in M$ such that $M \models [z = L_\gamma]$ and $x \in z$. By absoluteness, $z = L_\gamma$. □

(II.6.29) For any sets A, B with $A \neq \emptyset$ we define $\mathcal{D}'(A, B) = \{S \subseteq A : S \text{ is definable over } (A, \in, B \cap A) \text{ with parameters from } A\}$. Then for any sets B, C we define

$$\begin{aligned}
L(B, C, 0) &= \begin{cases} \emptyset & \text{if } C = \emptyset, \\ \{C\} \cup \operatorname{trcl}(C) & \text{otherwise;} \end{cases} \\
L(B, C, \beta + 1) &= \mathcal{D}'(L(B, C, \beta), B); \\
L(B, C, \gamma) &= \bigcup_{\beta < \gamma} L(B, C, \beta) \text{ for } \gamma \text{ limit;} \\
L(B, C) &= \bigcup_{\beta \in \mathbf{ON}} L(B, C, \beta); \\
L_\beta(C) &= L(\emptyset, C, \beta); \\
L(C) &= L(\emptyset, C); \\
L_\beta[B] &= L(B, \emptyset, \beta); \\
L[B] &= L(B, \emptyset).
\end{aligned}$$

Proposition 17.42. $L_\alpha[B] \subseteq V_\alpha$.

Proof. See the proof of Lemma 17.5. $\square$

Proposition 17.43. *Let* $\mathrm{rank}(\{C\} \cup \mathrm{trcl}(C)) = \delta$. *Then* $L_\alpha(C) \subseteq V_{\delta+\alpha}$ *for all* α.

Proof. Induction on α. $L_0(C) = L(\emptyset, C, 0) = \{C\} \cup \mathrm{trcl}(C) \in V_{\delta+1} = \mathscr{P}(V_\delta)$, so $L_0(C) \subseteq V_\delta$. Now assume that $L_\alpha(C) \subseteq V_{\delta+\alpha}$. Then

$$\begin{aligned} L_{\alpha+1}(C) &= L(\emptyset, C, \alpha+1) = \mathcal{D}'(L(\emptyset, C, \alpha), \emptyset) \\ &\subseteq \mathscr{P}(L(\emptyset, C, \alpha)) = \mathscr{P}(L_\alpha(C)) \subseteq \mathscr{P}(V_{\delta+\alpha}) = V_{\delta+\alpha+1}. \end{aligned}$$

The limit case is clear. $\square$

Proposition 17.44. $L(B, C, \alpha)$ *is transitive.*

Proof. Induction on α. It is clear for $\alpha = 0$. Now suppose true for α. Then $L(B, C, \alpha) \subseteq L(B, C, \alpha+1)$, since if $a \in L(B, C, \alpha)$ then $a = \{x \in L(B, C, \alpha) : x \in a\}$ since $L(B, C, \alpha)$ is transitive, and so $a \in \mathcal{D}'(L(B, C, \alpha), B) = L(B, C, \alpha+1)$. Now suppose that $x \in L(B, C, \alpha+1)$. Thus $x \in \mathcal{D}'(L(B, C, \alpha), B) \subseteq L(B, C, \alpha) \subseteq L(B, C, \alpha+1)$.

The limit case is clear. $\square$

Proposition 17.45. *If* $\alpha \leq \beta$, *then* $L(B, C, \alpha) \subseteq L(B, C, \beta)$.

Proof. By induction, with α fixed. The cases $\alpha = \beta$ and β limit are clear. Now suppose that $a \in L(B, C, \beta)$. Then $a = \{x \in L(B, C, \beta) : x \in a\}$ since $L(B, C, \beta)$ is transitive, and this shows that $a \in \mathcal{D}'(L(B, C, \beta), B) = L(B, C, \beta+1)$. $\square$

Proposition 17.46. $L_\beta[B] \cap \mathbf{ON} = \beta$ *for all* β.

Proof. See the proof of Lemma 17.8. $\square$

Proposition 17.47. $(\{C\} \cup \mathrm{trcl}(C)) \cap \mathrm{ON}$ *is an ordinal, and if we let* $\delta = (\{C\} \cup \mathrm{trcl}(C)) \cap \mathrm{ON}$, *then* $L_\beta(C) \cap \mathbf{ON} = \delta + \beta$ *for any* β.

Proof. $(\{C\} \cup \mathrm{trcl}(C)) \cap \mathrm{ON}$ is a transitive set of transitive sets, and so it is an ordinal. Let $\delta = (\{C\} \cup \mathrm{trcl}(C)) \cap \mathrm{ON}$. We prove that $L_\beta(C) \cap \mathbf{ON} = \delta + \beta$ for all β by induction on β. $L_0(C) \cap \mathbf{ON} = \delta$ is given. Now suppose that $L_\beta(C) \cap \mathbf{ON} = \delta + \beta$. Now $\delta + \beta \subseteq L_\beta(C) \subseteq L_{\beta+1}(C) \subseteq V_{\delta+\beta+1}$ by Proposition 7.43. Hence $\delta + \beta \subseteq L_{\beta+1}(C) \subseteq V_{\delta+\beta+1} \cap \mathbf{ON} = \delta + \beta + 1 = \delta + \beta + \{\delta + \beta\}$. Hence it suffices to show that $\delta + \beta \in L_{\beta+1}(C)$. Now

$$\delta + \beta = L_\beta(C) \cap \mathbf{ON} = \{a \in L_\beta(C) : (a \text{ is an ordinal})^{L_\beta(C)}\} \in \mathcal{D}'(L_\beta(C)) = L_{\beta+1}(C).$$

The limit case is clear. $\square$

For $x \in L(C)$ let $\rho_C(x)$ be the least α such that $x \in L_{\alpha+1}(C)$. For $x \in L[B]$ let $\rho_B(x)$ be the least α such that $x \in L_{\alpha+1}[B]$.

Proposition 17.48. *(i) $L_0(C) = \{C\} \cup \mathrm{trcl}(C)$, and $\rho_C(x) = 0$ for all $x \in L_0(C)$.*
(ii) If $\alpha > 0$, then $L_\alpha(C) = \{x \in L(C) : \rho_C(x) < \alpha\}$.

Proof. (i): clear. (ii): Assume that $\alpha > 0$. Suppose that $x \in L_\alpha(C)$. If $\alpha = \beta + 1$, then $\rho_C(x) \le \beta < \alpha$. If α is limit, then $x \in L_\beta(C)$ for some $\beta < \alpha$, and $\beta + 1 < \alpha$ with $x \in L_{\beta+1}(C)$, hence $\rho_C(x) \le \beta < \alpha$. So $\subseteq$ holds.

If $x \in L(C)$ and $\rho_C(x) < \alpha$, then $x \in L_{\rho(x)+1}(C) \subseteq L_\alpha(C)$. □

Proposition 17.49. $L_\alpha[B] = \{x \in L[B] : \rho_B(x) < \alpha\}$.

Proof. See the proof of Lemma 17.9. □

Proposition 17.50. (i) If $\alpha > 0$, then $L_{\alpha+1}(C)\backslash L_\alpha(C) = \{x \in L(C) : \rho_C(x) = \alpha\}$.
(ii) $L_{\alpha+1}[B]\backslash L_\alpha[B] = \{x \in L[B] : \rho_B(x) = \alpha\}$. □

Proposition 17.51. *(i) If $x \in y \in L(C)\backslash L_1(C)$, then $\rho_C(x) < \rho_C(y)$.*
(ii) If $x \in y \in L[B]$, then $\rho_B(x) < \rho_B(y)$.

Proof. (i): Let $\alpha = \rho_C(y)$. Thus $\alpha \ge 1$. So $x \in y \in L_{\alpha+1}(C) = \mathcal{D}'(L(\emptyset, C, \alpha), \emptyset) \subseteq \mathscr{P}(L(\emptyset, C, \alpha)$, so $x \in L(\emptyset, C, \alpha)$, hence $\rho_C(x) < \alpha$ by Proposition 17.48.
(ii): similarly. □

Proposition 17.52. (i) $L_\alpha(C) \in L_{\alpha+1}(C)$.
(ii) $L_\alpha[B] \in L_{\alpha+1}[B]$.

Proof. (i): $L_\alpha(C) = \{x \in L_\alpha(C) : x = x\} \in \mathcal{D}'(\emptyset, C, \alpha) = L_{\alpha+1}(C)$.
(ii): similarly. □

Proposition 17.53. (i) $\rho_C(L_\alpha(C)) = \alpha$;
(ii) $\rho_B(L_\alpha[B]) = \alpha$.

Proof. (i): $\rho_C(L_\alpha(C)) \le \alpha$ by Proposition 17.52. If $\rho_C(L_\alpha(C)) < \alpha$, then $L_\alpha(C) \in L_{\rho_C(L_\alpha(C))+1}(C) \le L_\alpha(C)$, contradiction.
(ii): similarly. □

Proposition 17.54. $\rho_B(\alpha) = \alpha$.

Proof. See the proof of Lemma 17.13. □

Proposition 17.55. *Let $\delta = (\{C\} \cup \mathrm{trcl}(C)) \cap \mathrm{ON}$. Then*
(i) $\rho_C(\alpha) = 0$ if $\alpha \le \delta$.
(ii) $\rho_C(\alpha) = \gamma$ if $\alpha = \delta + \gamma$.

Proof. (i): Suppose that $\alpha \le \delta$. Then $\alpha \in L_1(C)$, so $\rho_C(\alpha) = 0$.
(ii): We have $\delta + \gamma \in \delta + \gamma + 1 = L_{\gamma+1}(C)$ by Proposition 17.47. Hence $\rho_C(\delta + \gamma) \le \gamma$. Suppose that $\rho_C(\delta + \gamma) < \gamma$. Then $\delta + \gamma \in L_{\rho_C(\delta+\gamma)+1}(C) \cap \mathbf{ON} \subseteq L_\gamma(C) \cap \mathbf{ON} = \delta + \gamma$, contradiction. □

Proposition 17.56. *Every finite subset of $L(B, C, \alpha)$ is a member of $L(B, C, \alpha+1)$.* □

Proposition 17.57. *$L_n[B] = V_n$ for all $n \leq \omega$.*

Proof. By induction, $L_n[B] = V_n$ for all $n \in \omega$. Hence also $L_\omega[B] = V_\omega$. □

Proposition 17.58. *$|\mathcal{D}'(A, B)| = |A|$ for every infinite A.* □

Proposition 17.59. *Let $\delta = |\{C\} \cup \mathrm{trcl}(C)|$.*

(i) If δ is finite, let $\varepsilon_0 = \delta$ and $\varepsilon_{n+1} = 2^{\varepsilon_n}$ for all $n \in \omega$. Then $|L(B, C, n)| = \varepsilon_n$ for all $n \in \omega$, and $|L(B, C, \omega)| = \omega$.

(ii) If δ is infinite, then $|L(B, C, n)| = |\delta|$ for all $n \leq \omega$. □

Proposition 17.60. *$|L(B, C, \alpha)| = |\alpha| + |\{C\} \cup \mathrm{trcl}(C)|$ for all infinite α.*

Proof. Induction on α. It is true for $\alpha = \omega$ by Proposition 17.59. Now assume that $|L(B, C, \alpha)| = |\alpha| + |\{C\} \cup \mathrm{trcl}(C)|$. Then $|L(B, C, \alpha + 1)| = |\mathcal{D}'(L(B, C, \alpha), B)| = |\alpha + 1| + |\{C\} \cup \mathrm{trcl}(C)|$. For α limit $> \omega$,

$$\begin{aligned}
|L(B, C, \alpha)| &= \left| \bigcup_{\beta<\alpha} L(B, C, \beta) \right| \leq \sum_{\beta<\alpha} |L(B, C, \beta)| = \sum_{\omega\leq\beta<\alpha} |L(B, C, \beta)| \\
&= \sum_{\omega\leq\beta<\alpha} (|\beta| + |\{C\} \cup \mathrm{trcl}(C)|) = |\alpha| + |\{C\} \cup \mathrm{trcl}(C)|.
\end{aligned}$$

□

Proposition 17.61. *$L(B, C)$ is a model of ZF.*

Proof. The proof of Theorem 17.18 carries over. □

Proposition 17.62. *$\mathcal{D}'(A, B)$ is absolute for transitive models of ZF.*

Proof.

$$\begin{aligned}
X \in \mathcal{D}'(A, B)) \quad \text{iff} \quad &\exists\varphi(x, y_1, \ldots, y_m)\exists b_1, \ldots, b_m \in A\forall a \in A \\
&[a \in X \text{ iff } (A, \in, B \cap A) \models \varphi[a, b_1, \ldots, b_m]].
\end{aligned}$$

□

Proposition 17.63. *The function $\langle L(B, C, \alpha) : \alpha \in \mathbf{ON}\rangle$ is absolute for transitive models of ZF.*

Proof. See the proof of Theorem 17.26. □

Proposition 17.64. *Let $\overline{B} = B \cap L[B]$. Then $L[\overline{B}] = L[B]$ and $\overline{B} \in L[\overline{B}]$.*

Proof. We prove by induction that $\forall\alpha \in \mathbf{ON}[L_\alpha[B] = L_\alpha[\overline{B}]]$. This is clear for $\alpha = 0$, and the case α limit is clear. Now assume that $L_\alpha[B] = L_\alpha[\overline{B}]$. Then

$$\begin{aligned}
L_{\alpha+1}[B] &= D'(L(B.0, \alpha), B) \\
&= \{S \subseteq L(B, 0, \alpha) : S \text{ is definable over } (L(B, 0, \alpha), \in, B \cap L(B, 0, \alpha)) \\
&\qquad \text{with parameters from } A\} \\
&= \{S \subseteq L(\overline{B}, 0, \alpha) : S \text{ is definable over } (L(\overline{B}, 0, \alpha), \in, \overline{B} \cap L(\overline{B}, 0, \alpha)) \\
&\qquad \text{with parameters from } A\} \\
&= D'(L(\overline{B}.0, \alpha), \overline{B}) \\
&= L_{\alpha+1}[\overline{B}].
\end{aligned}$$

Thus $L[\overline{B}] = L[B]$. Now there is an ordinal α such that $B \cap L[B] = B \cap L_\alpha[B]$, and then $\overline{B} = \{x \in L(B,0,\alpha) : x \in B \cap L_\alpha[B]\}$, so that $\overline{B}$ is definable over $((L(B,0,\alpha),0,B \cap L_\alpha[B])$. Thus $\overline{B} \in L(B,0,\alpha+1)$. □

Proposition 17.65. *$(V = L[\overline{B}])^{L[B]}$, i.e., $(\forall x \exists \alpha [x \in L(\overline{B},0,\alpha)])^{L[B]}$.*

Proof. $(\forall x \exists \alpha [x \in L(\overline{B},0,\alpha)])^{L[B]}$ is

$$\forall x \in L[B] \exists \alpha \in L[B] (L(\overline{B},0,\alpha)])^{L[B]}.$$

So, given $x \in L[B]$, choose α with $x \in L_\alpha[B]$ and $\overline{B} \in L_\alpha[B]$. Then $x \in (L(\overline{B},0,\alpha)])^{L[B]}$ by absoluteness. □

Lemma 17.66. *Assume that M is an extensional set and $(M,\in,N) \preceq (L_\alpha[B],\in,B \cap L_\alpha[B])$. Then*

(i) $N = M \cap B$.

(ii) If π is the transitive collapse of M onto P then for any formula φ in the expanded language and any $x \in {}^\omega M$,

$$(P,\in,\pi[B \cap M]) \models \varphi[\pi \circ x] \quad \textit{iff} \quad (L_\alpha[B],\in,B \cap L_\alpha[B]) \models \varphi[x].$$

Proof.

(i): $N = M \cap B \cap L_\alpha[B] = M \cap B$.

(ii) Induction on φ. Let $\mathbf{R}$ be the one-place relation symbol corresponding to $B \cap L_\alpha[B]$.

(1) φ is $v_i \in \mathbf{R}$. Then

$$(L_\alpha[B],\in,B \cap L_\alpha[B]) \models x_i \in B \cap L_\alpha[B] \quad \text{iff} \quad (M,\in,B \cap M) \models x_i \in B \cap M$$
$$\text{iff} \quad (P,\in,\pi[B \cap M]) \models \pi(x_i) \in \pi[B \cap M].$$

(2) Other cases are clear. □

Lemma 17.67. *Assume that M is an extensional set and $(M,\in,N) \preceq (L_\alpha[B],\in,B \cap L_\alpha[B])$. Let π be the transitive collapse of M onto P. Then for any $S \subseteq M$, $\pi[S] \in D'(P,\pi[B \cap M])$ iff $S = S' \cap M$ for some $S' \in D'(L_\alpha[B],B)$.*

Proof. Suppose that $S \subseteq M$. First suppose that $\pi[S] \in D'(P,B \cap M)$. Let $\varphi(v_0)$ be a formula in the expanded language such that $\pi[S] = \{\pi(x) : x \in M, (P,\in,P \cap \pi[B \cap M]) \models \varphi(\pi(x))\}$. Hence by Lemma 17.66, $\pi[S] = \{\pi(x) : x \in M, (L_\alpha[B],\in,B \cap L_\alpha[B]) \models \varphi(x)\}$, so that $S = \{x : x \in M, (L_\alpha[B],\in,B \cap L_\alpha[B]) \models \varphi(x)\}$, as desired.

The converse is obtained by reversing steps. □

Lemma 17.68. *Assume that M is an extensional set and $(M,\in,N) \preceq (L_\alpha[B],\in,B \cap L_\alpha[B])$. Let π be the transitive collapse of M onto P. Then for any $\alpha \in P \cap \mathbf{ON}$ and any $S \subseteq M$, $\pi[S] \in L(\pi[B \cap M],0,\alpha)$ iff $S \in \{S' \cap M : S' \in L(B,0,\alpha)\}$.*

Proof. We prove this by induction on α. It is clear for $\alpha = 0$ and for α limit. Now assume it for α. Then for any $S \subseteq M$,

$$\begin{aligned}
\pi[S] \in L_{\alpha+1}(\pi[B \cap M], 0, \alpha) \quad &\text{iff} \quad \pi[S] \in D'(L(\pi[B \cap M], 0, \alpha), \pi[B \cap M]) \\
\text{iff} \quad &\exists S' \in D'(L[B], B)(S = S' \cap M) \\
\text{iff} \quad &S \in \{S' \cap M : S' \in L(B, 0, \alpha + 1)\}.
\end{aligned}$$

□

Theorem 17.69. *Assume that M is an extensional set and $(M, \in, N) \preceq (L_\alpha[B], \in, B \cap L_\alpha[B])$. Let π be the transitive collapse of M onto P. Let $\gamma = P \cap \mathbf{ON}$. Then $(P, \in, \pi[B \cap M]) = L_\gamma[\pi[B \cap M]$.*

Proof.

$$\begin{aligned}
&(L_\alpha[B], \in, B \cap L_\alpha[B]) \models \forall x \exists \beta [x \in L_\beta[B]], \text{ so by Lemma 3,} \\
&(L_\alpha[B], \in, B \cap L_\alpha[B]) \models \forall x \exists \beta [\pi[x \cap M] \in L_\beta[\pi[B \cap M]]], \text{ so} \\
&(P, \in, \pi[B \cap M]) \models \forall x \exists \beta [x \in L_\beta[\pi[B \cap M]]], \text{ hence} \\
&(P, \in, \pi[B \cap M]) = (L_\gamma[B \cap M], \in, \pi[B \cap M]).
\end{aligned}$$

□

A formula φ in the language for $(A, \in, A \cap G)$ is *good*' iff its free variables are exactly $v_0, \ldots, v_n$ for some $n \in \omega$. Let $\langle \varphi_i : i \in \omega \rangle$ be a one-one list of all the good' formulas. Say $\varphi_i = \varphi_i(v_0, \ldots, v_{n_i})$ for each $i \in \omega$. For $A \neq \emptyset$, $i \in \omega$, and $\overline{b} \in {}^{n_i}A$, let $D'(A, B, i, \overline{b}) = \{a \in A : (A, \in, B) \models \varphi_i(a, \overline{b})\}$. For each $S \in \mathcal{D}'(A, B)$ let $i(S, A)$ be the least $i \in \omega$ such that $S = D'(A, B, i, \overline{b})$ for some $\overline{b} \in {}^{n_i}A$.

If R is a well-order of A. then $R^{(n)}$ is the lexicographic order of ${}^nA$. For $S \in \mathcal{D}'(A, B)$, $p(S, R)$ is the $R^{(n_{i(S,A)})}$-first $\overline{b} \in {}^{n_{i(S,A)}}A$ such that $S = D(A, i(S, A), \overline{b})$. Then we define $W = W(A, B, R)$ by setting

$$S_1 W S_2 \text{ iff } S_1, S_2 \in \mathcal{D}^+(A) \text{ and } \begin{cases} i(S_1, A) < i(S_2, A) \text{ or} \\ i(S_1, A) = i(S_2, A) \text{ and } p(S_1, R) R^{(n_{i(S_1,A)})} p(S_2, R). \end{cases}$$

Clearly $W(A, B, R)$ is a well-order of $\mathcal{D}^+(A)$.

(II.6.19) We define $\lhd_\delta \subseteq L_\delta[B] \times L_\delta[B]$ by recursion as follows:

$$x \lhd_\delta y \text{ iff } x, y \in L(\delta) \text{ and } \begin{cases} \rho(x) < \rho(y) \text{ or} \\ \rho(x) = \rho(y) \text{ and } x W(L(\rho(x)), B, \lhd_{\rho(x)}) y. \end{cases}$$

Then we define

$$x <_{L[B]} y \text{ iff } x, y \in L[B] \text{ and } \begin{cases} \rho_B(x) < \rho_B(y) \text{ or} \\ \rho_B(x) = \rho_B(y) \text{ and } x \lhd_{\rho_B(x)+1} y. \end{cases}$$

Theorem 17.70. *$<_{L[B]}$ well-orders $L[B]$, and each $L_\delta[B]$ is an initial segment, with $\lhd_\delta = <_{L[B]} \cap (L_\delta[B] \times L_\delta[B]$*

Assuming $V = L[B]$, $<_{L[B]}$ well-orders $V[B]$, and AC holds. □

Proposition 17.71. *If $C \subseteq \alpha$ for some ordinal α, then there is a well-order of $L(C)$.*

Proof. As in the proof of Theorem 17.31, starting with the given well-order of $L_0(C)$. $\square$

Proposition 17.72. *Choose α so that $B \cap L[B] \in L_\alpha[B]$, using Proposition 17.64. Then in $L[B]$, $2^\kappa = \kappa^+$ for all $\kappa > \alpha$.*

Proof. First we claim

(1) $H(\kappa^+) \subseteq L_\alpha[B]$.

In fact, we repeat the proof of Theorem 17.33 to the application of the Löwenheim-Skolem theorem, obtaining $(A, X) \preceq L(B, 0, \alpha)$. Now let $Y = B \cap L[B]$; so $Y \in L_\alpha[B]$. Thus Y is an allowable parameter in defining $\mathscr{D}'(L_\alpha[B], B)$, and so $\{Y\} \in L_{\alpha+1}[B]$. Hence $(L_{\alpha+1}[B], \in, L_{\alpha+1}[B] \cap B) \models \exists x \forall y[y \in x \leftrightarrow y \in L_{\alpha+1}[B] \cap B]$, so $A \models \exists x \forall y[y \in x \leftrightarrow y \in X]$. Now the rest of the proof gives (1).

Now $\mathscr{P}(\kappa) \subseteq H(\kappa^+) = L_{\kappa^+}[B]$, so $2^\kappa = |\mathscr{P}(\kappa)| \leq |L_{\kappa^+}[B]| = \kappa^+$. $\square$

Theorem 17.73. *If $A \in L[X]$, then $L[A] \subseteq L[X]$.*

Proof. Assume that $A \in L[X]$. Since $L[X]$ is transitive, $A \cap L[X] = A \in L[X]$. Hence $L[A] \subseteq L[X]$ by Theorem 17.65. $\square$

Theorem 17.74. *For every set X there is a set A of ordinals such that $L[X] = L[A]$.*

Proof. Choose α so that $\overline{X} \in L_\alpha[X]$, with α limit. In $L_\alpha[X]$ let f be a bijection from an ordinal θ onto $trcl(\{\overline{X}\})$. Define $\alpha E \beta$ iff $f(\alpha) \in f(\beta)$. Let Γ be the natural bijection of $\mathbf{ON} \times \mathbf{ON}$ onto $\mathbf{ON}$. Let $A = \Gamma(E)$. Then $A \in L[X]$, so $L[A] \subseteq L[X]$ by Theorem 17.65.

Now $A \in L_{\sup(A)+1}[A]$. Hence $E = \Gamma^{-1}[A] \in L[A]$. Hence $(\theta, E) \in L[A]$. Let M be the transitive collapse of (θ, E) in $L[A]$. Then $\overline{X} \in M$ and hence $\overline{X} \in L[A]$. So by Theorem 17.64, $L[A] = L[X]$. $\square$

Theorem 17.75. *If $M \preceq (L_{\omega_1}, \in)$, then $M = L_\alpha$ for some limit ordinal $\alpha \leq \omega_1$.*

Proof. We claim that M is transitive. For, suppose that $X \in M$. Then $X \in L_{\omega_1}$. In L_{ω_1} there is a function g mapping ω onto X. Let f be the $<_{\omega_1}$-least such. Thus

$$(L_{\omega_1}, \in) \models \exists! f[f \text{ maps } \omega \text{ onto } X \wedge \forall g[g \text{ maps } \omega \text{ onto } X \rightarrow f <_{\omega_1} g]], \quad \text{so}$$
$$(M, \in) \models \exists! f[f \text{ maps } \omega \text{ onto } X \wedge \forall g[g \text{ maps } \omega \text{ onto } X \rightarrow f <_{\omega_1} g]].$$

Choose $f \in M$ so that

$$(M, \in) \models [f \text{ maps } \omega \text{ onto } X \wedge \forall g[g \text{ maps } \omega \text{ onto } X \rightarrow f <_{\omega_1} g]]; \quad \text{hence}$$
$$(L_{\omega_1} \models [f \text{ maps } \omega \text{ onto } X \wedge \forall g[g \text{ maps } \omega \text{ onto } X \rightarrow f <_{\omega_1} g]].$$

By absoluteness, f really is a function mapping ω onto X. If $x \in X$, choose n such that $f(n) = x$. Now $(L_{\omega_1}, \in) \models \exists n \in \omega \exists y[f(n) = y]$, so $(M, \in) \models \exists n \in \omega \exists y[f(n) = y]$. Choose

$n \in \omega^M$ and $y \in M$ so that $(M, \in) \models [f(n) = y]$. Then $(L_{\omega_1}, \in) \models [n \in \omega \wedge [f(n) = y]]$. By absoluteness, $n \in \omega$ and $f(n) = y$. So $x = y \in M$.

This shows that M is transitive. By Theorem 17.41, $M = L_\alpha$ for some limit ordinal $\alpha \leq \omega_1$. $\square$

Theorem 17.76. *If $M \preceq (L_{\omega_2}, \in)$, then there is an $\alpha \leq \omega_1$ such that $\omega_1 \cap M = \alpha$.*

Proof. First we claim that if $\gamma < \omega_1$ and $\gamma \in M$, then $\gamma \subseteq M$. This is true by the argument in the proof of Theorem 17.75. Now let $\alpha = \omega_1 \cap M$. So α is a collection of cardinals. If $\gamma \in \alpha$, then $\gamma \subseteq M$ by the claim. So α is transitive. Hence it is an ordinal. $\square$

Theorem 17.77. *If $\alpha \geq \omega$ and X is a constructible subset of α, then $X \in L_\beta$, where β is the least cardinal greater than α.*

Proof. Assume that $\alpha \geq \omega$ and X is a constructible subset of α. Say $X \in L_\kappa$. Let M be an elementary substructure of L_κ such that $X \in M$, $\alpha \subseteq M$, and $|M| = |\alpha|$. Let N be the transitive collapse of M via a function π. Then π is the identity on X, so $X \in N$. Then $N = L_\gamma$ with $|L_\gamma| = |M| = |\alpha|$. Hence $X \in L_\gamma \subseteq L_\beta$, where β is the least cardinal greater than α. $\square$

Theorem 17.78. *Assume $V = L[A]$ with $A \subseteq \omega_1$. Then GCH holds.*

Proof. First we show that $2^\omega = \omega_1$. Take any $X \subseteq \omega$. Say $X \in L_{\beta_X}[A]$ with $\omega_1 \leq \beta_X$. Let $(M_X, \emptyset, Q) \preceq (L_{\beta_X}[A], \emptyset, A)$ be such that $X \in M_X$, $\omega \subseteq M_X$, $M_X \cap \omega_1 \in \omega_1$, and $|M_X| = \omega$. Thus $Q = A \cap M_X$. Let $\delta_X = M \cap \omega_1$. Then $A \cap \delta_X = A \cap M_X \cap \omega_1 = A \cap M_X$. Let $(N_X, \emptyset, \pi[A \cap \delta_X])$ be the transitive collapse of $(M_X, \emptyset, A \cap \delta_X)$ via the function π. Since $\delta_X \subseteq M$, π is the identity on δ_X. Hence $\pi[A \cap \delta_X] = A \cap \delta_X$ and $X \in N_X$. Then there is an ordinal γ_X such that $(N_X, \emptyset, \pi[A \cap \delta_X]) = L_{\gamma_X}[A \cap \delta_X]$. Now $\gamma_X < \omega_1$ since $|\gamma_X| = |L_{\gamma_X}[A \cap \delta_X]| = |N| = |M| = \omega$. Also, $\delta_X < \omega_1$.

Thus $\mathscr{P}(\omega) \subseteq \bigcup_{\mu, \nu < \omega_1} L_\mu[A \cap \nu]$, and this set has size ω_1. So $2^\omega = \omega_1$.

Now suppose that λ is uncountable. We want to show that $2^\lambda = \lambda^+$ in V. Let $Y \subseteq \lambda$. Set $T = \operatorname{trcl}(\{Y\})$. Choose θ so that $T \in L_\theta[A]$. Let $(M, \in, N) \preceq (L_\theta[A], \in, A \cap L_\theta[A])$, with $\omega_1 \cup T \subseteq M$ and $|M| = \lambda$. Hence we get $\gamma < \lambda^+$ such that $Y \in L_\gamma[\pi[A \cap M]]$. Now $A \cap M = A$, so $\pi[A \cap M] = \pi[A] = A$. Thus $Y \in L_\gamma[A] \subseteq L_{\lambda^+}[A]$. This is true for each $Y \subseteq \lambda$, so $\mathscr{P}(\lambda) \subseteq L_{\lambda^+}[A]$. Hence $2^\lambda = \lambda^+$. $\square$

Theorem 17.79. *If $\alpha \geq \omega$ is a countable ordinal, then there is $A \subseteq \omega$ such that $L[A] \models [\alpha$ is countable$]$.*

Proof. Let $W \subseteq \omega \times \omega$ be a well-order of ω of order type α. Let $f : \omega \to \omega \times \omega$ be a bijection, and let $A = f[W]$. Since $A \subseteq \omega$, we have $A = A \cap L[A] \in L[A]$. Hence $W \in L[A]$. In $L[A]$, using W we can define a bijection from α onto ω. $\square$

Theorem 17.80. *If ω_1 of V is not a limit cardinal in L, then there is an $A \subseteq \omega$ such that $\omega_1 = \omega_1^{L[A]}$.*

Proof. Say that α is a cardinal in L and ω_1^L is the successor of α in L. Then $\alpha < \omega_1^L \le \omega_1$, so α is a countable ordinal. Let $A \subseteq \omega$ be such that $L[A] \models [\alpha \text{ is countable}]$. Thus $\alpha < \omega_1^{L[A]} \le \omega_1$. If $\omega_1^{L[A]} < \omega_1$, then in L, $\alpha = |\omega_1^{L[A]}|$; hence this also holds in $L[A]$ since $L \subseteq L[A]$. But $|\alpha| = \omega$ in $L[A]$, contradiction. Thus $\omega_1^{L[A]} = \omega_1$. □

Theorem 17.81. *There is an $A \subseteq \omega_1$ such that $\omega_1 = \omega_1^{L[A]}$.*

Proof. For each α with $\omega \le \alpha < \omega_1$ choose A_α so that $L[A_\alpha] \models [\alpha \text{ is countable}]$. Let $A \subseteq \omega_1 \times \omega_1$ be such that $\forall \alpha < \omega_1 [A_\alpha = \{\xi < \omega_1 : (\alpha, \xi) \in A\}]$.

(1) $\forall \alpha < \omega_1 [A_\alpha \in L[A]]$.

In fact, choose β so that $\alpha, \omega \in L_\beta[A]$. Then $A_\alpha = \{\xi \in L_\beta[A] : (L_\beta[A], \in, A \cap L_\beta[A]) \models (\xi \in \omega \wedge (\alpha, \xi) \in A \cap L_\beta[A])\}$.

From (1), $L[A_\alpha] \subseteq L[A]$. Hence α is countable in $L[A]$. So $\omega_1^{L[A]} = \omega_1$. □

Theorem 17.82. *If ω_2 is not inaccessible in L, then there is an $A \subseteq \omega_1$ such that $\omega_1^{L[A]} = \omega_1$ and $\omega_2^{L[A]} = \omega_2$.*

First note:

(1) $\omega_1^L \le \omega_1$.

In fact, for any $\alpha < \omega_1^L$, α is countable in L, and hence is really countable.

Now for the theorem, since ω_2 is regular in L but not inaccessible, and since GCH holds in L, there is a cardinal κ of L such that $\omega_2 = (\kappa^+)^L$. Now $\omega_1 \le \kappa$. In fact, if $\kappa < \omega_1$, then since ω_1 is a cardinal in L, we get $(\kappa^+)^L \le \omega_1 < \omega_2 = (\kappa^+)^L$, contradiction. So $\omega_1 \le \kappa < \omega_2$. Let W be a well-order of ω_1 of order type κ. For each countable ordinal α, let $A_\alpha \subseteq \omega$ be such that α is countable in $L[A_\alpha]$. Let

$$B = \{(\alpha, \xi, 0) : \alpha < \omega_1, \xi \in A_\alpha\} \cup \{(\alpha, \beta, 1) : (\alpha, \beta) \in W\}.$$

Let $f : \omega_1 \times \omega_1 \times 2 \to \omega_1$ be a bijection, and set $C = f[B]$. Now for each countable ordinal α, $A_\alpha = \{\xi < \omega_1 : (\alpha, \xi, 0) \in B\}$, so $A_\alpha \in L[C]$. Hence $L[A_\alpha] \subseteq L[C]$. It follows that α is countable in $L[C]$. Hence $\omega_1 = \omega_1^{L[C]}$. Also, $W \in L[C]$, so $|\kappa|^{L[C]} = \omega_1$, Hence $\omega_2 = \omega_2^{L[C]}$. □

(II.8.2, II.8.3) If M is a transitive set, then $\mathbf{OD}_M$ is the set of all elements of M that are definable in $(M, \in)$ with parameters in $\mathbf{On} \cap M$. Then we define

$$\mathbf{OD} = \bigcup_{\alpha \in \mathbf{On}} \mathbf{OD}_{V_\alpha}.$$

Theorem 17.83. (II.8.4) *For any formula $\varphi(x_0, \ldots, x_{m-1}, y)$ the following is provable in ZF:*

$$\forall \alpha_0, \ldots, \alpha_{m-1} \forall w [[\varphi(\overline{\alpha}, w) \wedge \exists! y \varphi(\overline{\alpha}, y) \to w \in \mathbf{OD}].$$

Proof. Assume that $\alpha_0, \ldots, \alpha_{m-1}$ are ordinals. Choose β so that $\alpha_0, \ldots, \alpha_{m-1} < \beta$. By Theorem 15.4 let $\gamma > \beta$ be such that $\exists! w\varphi(\overline{v}, w)$ and $\varphi(\overline{v}, w)$ are absolute for V_γ, V. Suppose that $\varphi(\overline{\alpha}, w)$ and $\exists! w\varphi(\overline{\alpha}, w)$. By absoluteness of $\exists! w\varphi(\overline{v}, w)$ we have $\exists! w\varphi^{V_\gamma}(\overline{\alpha}, w)$. So choose $a \in V_\gamma$ such that $\varphi^{V_\gamma}(\overline{\alpha}, a)$. Then by absoluteness of $\varphi(\overline{v}, w)$ we have $\varphi(\overline{\alpha}, a)$. Since $\varphi(\overline{\alpha}, w)$ and $\exists! w\varphi(\overline{\alpha}, w)$, it follows that $a = w$. Hence $\{w\} = \{s \in V_\gamma : V_\gamma \models \varphi(\overline{\alpha}, s\}$. □

Theorem 17.84. *$V_\alpha \in \mathbf{OD}$ for all α.*

Proof. We apply Theorem 17.83. Let $\varphi(x, y)$ be the following formula:

$$\begin{aligned}
&x \text{ is an ordinal and } \exists f[f \text{ is a function and } \mathrm{dmn}(f) = x \cup \{x\}\\
&\text{and } f(0) = \emptyset \text{ and } \forall \beta \in x[f(\beta \cup \{\beta\}) = \mathscr{P}(f(\beta))]\\
&\text{and } \forall \text{limit} \gamma \leq \alpha[f(\gamma) = \bigcup_{\beta \in \gamma} f(\beta)] \text{ and } f(x) = y]
\end{aligned}$$

Then $\varphi(\alpha, V_\alpha)$ and $\exists! w\varphi(\alpha, w)$. So by Theorem 17.83, $V_\alpha \in \mathbf{OD}$. □

(II.8.6) $\mathbf{HOD} = \{x \in \mathbf{OD} : \mathrm{trcl}(x) \subseteq \mathbf{OD}\}$.

Proposition 17.85. (II.8.7) *$\mathbf{On} \subseteq \mathbf{HOD} \subseteq \mathbf{OD}$.*

Proof. For any ordinal α, $\alpha = \{x \in V_\alpha : x \text{ is an ordinal}\}$, so $\alpha \in \mathbf{OD}_{V_\alpha}$, and so $\alpha \in \mathbf{OD}$. Clearly then $\mathbf{On} \subseteq \mathbf{HOD}$. Obviously $\mathbf{HOD} \subseteq \mathbf{OD}$. □

Proposition 17.86. (II.8.7) *$\mathbf{HOD}$ is transitive.*

Proof. Suppose that $x \in y \in \mathbf{HOD}$. Then $y \in \mathbf{OD}$ and $\mathrm{trcl}(y) \subseteq \mathbf{OD}$. Since $x \in \mathrm{trcl}(y)$, we have $x \in \mathbf{OD}$. Also, $\mathrm{trcl}(x) \subseteq \mathrm{trcl}(y)$, so $x \in \mathbf{HOD}$. □

Proposition 17.87. (II.8.8) *For any set a the following are equivalent:*
(i) $a \in \mathbf{HOD}$.
(ii) $a \in \mathbf{OD}$ and $a \subseteq \mathbf{HOD}$.

Proof. (i)⇒(ii): obvious.

(ii)⇒(i): Assume (ii). Since $a \in \mathbf{HOD}$ and $\mathbf{HOD}$ is transitive, we have $\mathrm{trcl}(a) \subseteq \mathbf{HOD}$. So $a \in \mathbf{HOD}$. □

Proposition 17.88. *For any nonzero ordinal α, $\mathbf{OD}_{V_\alpha} \in \mathbf{OD}$.*

Proof. Let $\varphi(\alpha, y)$ be the following formula:

$$\begin{aligned}
&\forall z[z \in y \leftrightarrow \exists \text{ a formula } \psi(x_1, \ldots, x_n, w)\exists \beta_1, \ldots, \beta_n < \alpha\\
&\quad [\{z\} = \{a : V_\alpha \models \psi(\beta_1, \ldots, \beta_n, a)\}]].
\end{aligned}$$

Thus $\varphi(\alpha, \mathbf{OD}_{V_\alpha})$. So by Theorem 17.83, $\mathbf{OD}_{V_\alpha} \in \mathbf{OD}$. □

Now we can define a one-one function F with domain **On** and range all sequences

$$\langle \varphi(x_0, \ldots, x_{n-1}, y), \beta, \alpha_0, \ldots, \alpha_{n-1} \rangle$$

such that φ is a formula with the indicated free variables, and $\alpha_0, \ldots, \alpha_{n-1} < \beta$. Now let $\psi(\gamma, y)$ say that, with

$$F(\gamma) = \langle \varphi(x_0, \ldots, x_{n-1}, w), \beta, \alpha_0, \ldots, \alpha_{n-1} \rangle,$$

$(\exists! w \varphi(\alpha_0, \ldots, \alpha_{n-1}, w))^{V_\beta}$ and $(\varphi(\alpha_0, \ldots, \alpha_{n-1}, y))^{V_\beta}$, or $y = \emptyset$ if there does not exist a unique such w.

Proposition 17.89. (II.8.5) $\forall \gamma \exists! y \psi(\gamma, y)$ *and* $\forall w \neq \emptyset [w \in \mathbf{OD} \leftrightarrow \exists \gamma \psi(\gamma, w)]$.

Proof. Clearly $\forall \gamma \exists! y \psi(\gamma, y)$. Now suppose that $w \neq \emptyset$ and $w \in \mathbf{OD}$. Say $w \in \mathbf{OD}_{V_\beta}$. Then there exist a formula $\varphi(x_0, \ldots, x_{n-1}, y)$ and ordinals $\alpha_0, \ldots, \alpha_{n-1} < \beta$ such that $\{y\} = \{a \in V_\beta : V_\beta \models \varphi(\alpha_0, \ldots, \alpha_{n-1}, a)\}$. Say

$$F(\gamma) = \langle \varphi(x_0, \ldots, x_{n-1}, w), \beta, \alpha_0, \ldots, \alpha_{n-1} \rangle.$$

Then $\psi(\gamma, w)$.

Next suppose that $w \neq \emptyset$ and $\psi(\gamma, w)$. Say

$$F(\gamma) = \langle \varphi(x_0, \ldots, x_{n-1}, w), \beta, \alpha_0, \ldots, \alpha_{n-1} \rangle.$$

Then $(\varphi(\alpha_0, \ldots, \alpha_{n-1}, w))^{V_\beta}$, so $w \in \mathbf{OD}$. □

Proposition 17.90. (II.8.9) *For any ordinal* α, $(V_\alpha \cap \mathbf{HOD}) \in \mathbf{OD}$.

Proof. Let $\varphi(\alpha, w)$ be the following formula:

$$\forall z[z \in w \leftrightarrow z \in V_\alpha \wedge \forall n \backslash 1 \forall y \in {}^n\mathrm{trcl}(z)[\forall i < n-1[y_i \in y_{i+1}] \wedge y_{n-1} \in z \rightarrow \exists \gamma \psi(\gamma, y_0)]]$$

Now $(V_\alpha \cap \mathbf{HOD}) \in \mathbf{OD}$ by Theorem 17.83. □

Proposition 17.91. (II.8.9) *For any ordinal* α, $(V_\alpha \cap \mathbf{HOD}) \in \mathbf{HOD}$.

Proof. By Theorem 17.83. □

Theorem 17.92. (II.8.10) **HOD** *is a model of ZFC.*

Proof. First we consider comprehension. By Theorem 14.2 it suffices to assume that $z, w_1, \ldots, w_n \in \mathbf{HOD}$ and show that $\{x \in z : \varphi^{\mathbf{HOD}}(x, z, w_1, \ldots, w_n)\} \in \mathbf{HOD}$. Choose ordinals $\gamma, \delta_1, \ldots, \delta_n$ such that $\psi(\gamma, z), \psi(\delta_1, w_1), \ldots, \psi(\delta_n, w_n)$. Let $\alpha > \gamma, \delta_1, \ldots, \delta_n$, and also such that $z, w_1, \ldots, w_n \in V_\alpha$. Then by Theorem 15.4 choose $\beta > \alpha$ such that ψ is absolute for V_β, V. Choose ε so that $\psi(\varepsilon, V_\beta \cap \mathbf{HOD})$. Let $y = \{x \in z : \varphi^{\mathbf{HOD}}(x, z, w_1, \ldots, w_n)\}$. Let $\chi(\gamma, \delta_1, \ldots, \delta_n, \varepsilon, x)$ be the following formula:

$$\exists z[\psi(\gamma, z) \wedge x \in z] \wedge \exists w_1 \ldots \exists w_n[\psi(\delta_1, w_1) \wedge \ldots \wedge \psi(\delta_n, w_n) \wedge \varphi'],$$

where φ' is obtained from $\varphi^{V_\beta \cap \mathbf{HOD}}(x, z, w_1, \ldots, w_n)$ by replacing each quantifier $\forall u \in V_\beta \cap \mathbf{HOD}$ by

$$\forall u \in V_\beta[\exists v[\psi(\varepsilon, v) \wedge u \in v] \rightarrow$$

Now for all x,

$$\begin{aligned} x \in y \quad &\text{iff} \quad x \in z \wedge \varphi^{\mathbf{HOD}}(x, z, w_1, \ldots, w_n) \\ &\text{iff} \quad x \in z \wedge \varphi^{V_\beta \cap \mathbf{HOD}}(x, z, w_1, \ldots, w_n) \\ &\text{iff} \quad x \in z \wedge \chi(\gamma, \delta_1, \ldots, \delta_n, \varepsilon, x). \end{aligned}$$

Thus $y = \{x \in z : \chi(\gamma, \delta_1, \ldots, \delta_n, \varepsilon, x)\}$. By Theorem 17.71, $y \in \mathbf{OD}$. Since $y \subseteq z \subseteq \mathbf{HOD}$, we have $y \subseteq \mathbf{HOD}$. Hence $y \in \mathbf{HOD}$ by Proposition 17.75.

For the rest of the axioms of ZF we apply Lemma 14.12. So let $x \subseteq \mathbf{HOD}$. Choose α so that $x \subseteq V_\alpha$. Thus $x \subseteq (V_\alpha \cap \mathbf{HOD}) \in \mathbf{HOD}$.

Finally, for the axiom of choice, suppose that $\emptyset \neq x \in \mathbf{HOD}$. Now for each $u \in x$ there is a smallest ordinal γ_u such that $\psi(\gamma_u, u)$. We define $R = \{(u, v) : u, v \in x$ and $\gamma_x < \gamma_y\}$. Clearly R well-orders x and $R \in \mathbf{OD}$. $\square$

Theorem 17.93. *$V = L$ implies $V = \mathbf{OD} = \mathbf{HOD}$.*

Proof. It suffices to show that an arbitrary set x is ordinal definable. Since there is a definable well-order of V, let $f : \alpha \rightarrow x$ be an onto map, and let $\varphi(\alpha, x)$ say that $f : \alpha \rightarrow x$ is onto. Hence $x \in \mathbf{OD}$. $\square$

18. Real numbers in set theory

We give several ways of thinking about the real numbers, and illustrate them with four invariants, concerning certain ideals.

Let $\mathrm{Fn}(I, J, \kappa) = \{f \in [I \times J]^{<\kappa} : f \text{ is a function}\}$. Let I be an ideal on a set A such that $[A]^{<\omega} \subseteq I$ and $A \notin I$.

$$\begin{aligned}
\mathrm{add}(I) &= \min\left\{\kappa : \exists E \in [I]^\kappa \left[\bigcup E \notin I\right]\right\};\\
\mathrm{cov}(I) &= \min\left\{\kappa : \exists E \in [I]^\kappa \left[A = \bigcup E\right]\right\};\\
\mathrm{non}(I) &= \min\{\kappa : \exists X \in [A]^\kappa [X \notin I]\}\\
\mathrm{cof}(I) &= \min\{\kappa : \exists X \in [I]^\kappa \forall C \in I \exists B \in X[C \subseteq B]\}.
\end{aligned}$$

Lemma 18.1. *For A infinite, the cardinals $\mathrm{add}(I)$, $\mathrm{cov}(I)$, $\mathrm{non}(I)$, $\mathrm{cof}(I)$ are well-defined and infinite.*

Proof.
$\mathrm{add}(I)$: Let $E = \{\{a\} : a \in A\}$. Then $\bigcup E = A \notin I$. If $F \subseteq I$ is finite, then $\bigcup F \in I$.

$\mathrm{cov}(I)$: Let $E = \{\{a\} : a \in A\}$. Then $\bigcup E = A$. If $F \subseteq I$ is finite, then $\bigcup F \in I$, hence $\bigcup F \neq A$.

$\mathrm{non}(I)$: $A \in [A]^{|A|}$ and $A \notin I$. $X \in I$ for all $X \in [A]^{<\omega}$.

$\mathrm{cof}(I)$: $I \in [I]^{|I|}$ and $\forall C \in I \exists B \in I[C \subseteq B]$. If $X \subseteq I$ is finite, then $\bigcup X \in I$, and if $a \in A \subseteq \bigcup X$ then there is no $B \in X$ such that $\{a\} \subseteq B$. □

Lemma 18.2. *For all $X \in [I]^\kappa$ with $\kappa < \mathrm{add}(I)$ we have $\bigcup X \in I$.* □

Lemma 18.3. *Let*

$$\mathrm{add}'(I) = \sup\left\{\kappa : \forall X \in [I]^\kappa \left[\bigcup X \in I\right]\right\}$$

Then $\mathrm{add}(I) = (\mathrm{add}'(I))^+$ if $\mathrm{add}'(I)$ is a successor cardinal, or is a limit cardinal and the supremum is attained; $\mathrm{add}(I) = \mathrm{add}'(I)$ if $\mathrm{add}'(I)$ is a limit cardinal and the supremum is not attained. □

Lemma 18.4. *For all $X \in [I]^\kappa$ with $\kappa < \mathrm{cov}(I)$ we have $\bigcup X \neq A$.* □

Lemma 18.5. *Let*

$$\mathrm{cov}'(I) = \sup\left\{\kappa : \forall X \in [I]^\kappa \left[\bigcup X \neq A\right]\right\}$$

Then $\mathrm{cov}(I) = (\mathrm{cov}'(I))^+$ if $\mathrm{cov}'(I)$ is a successor cardinal, or is a limit cardinal and the supremum is attained; $\mathrm{cov}(I) = \mathrm{cov}'(I)$ if $\mathrm{cov}'(I)$ is a limit cardinal and the supremum is not attained. □

Lemma 18.6. *For all $X \in [A]^\kappa$ with $\kappa < \mathrm{non}(I)$ we have $X \in I$.* □

Lemma 18.7. *Let*

$$\mathrm{non}'(I) = \sup\{\kappa : \forall X \in [A]^\kappa [X \in I]\}$$

Then $\mathrm{non}(I) = (\mathrm{non}'(I))^+$ *if* $\mathrm{non}'(I)$ *is a successor cardinal, or is a limit cardinal and the supremum is attained;* $\mathrm{non}(I) = \mathrm{non}'(I)$ *if* $\mathrm{non}'(I)$ *is a limit cardinal and the supremum is not attained.* □

Lemma 18.8. (III.1.7.1) $\mathrm{add}(I) \le \mathrm{cf}(\mathrm{non}(I)) \le \mathrm{non}(I) \le |A|$.

Proof. Let $\kappa = \mathrm{non}(I)$. Choose $X \in [A]^\kappa$ such that $X \notin I$. Let $X = \bigcup_{\alpha<\mathrm{cf}(\kappa)} Y_\alpha$ with each $|Y_\alpha| < \kappa$. Hence $Y \in {}^{\mathrm{cf}(\kappa)}I$. Then $X = \bigcup_{\alpha<\mathrm{cf}(\kappa)} Y_\alpha \notin I$, so $\mathrm{add}(I) \le \mathrm{cf}(\kappa)$.

We have $A \notin I$, so $\mathrm{non}(I) \le |A|$. □

Lemma 18.9. (III.1.7.2) $\mathrm{add}(I) \le \mathrm{cov}(I) \le |A|$.

Proof. Let $\kappa = \mathrm{cov}(I)$, and choose $X \in [A]^\kappa$ such that $A = \bigcup X$. Then $\bigcup X \notin I$, so $\mathrm{add}(I) \le \kappa$. Since $A \notin I$, we have $\mathrm{cov}(I) \le |A|$. □

Lemma 18.10. (III.1.7.3) $\mathrm{add}(I)$ *is regular.*

Proof. Suppose that $\lambda \overset{\mathrm{def}}{=} \mathrm{add}(I)$ is singular, and let $\kappa \in {}^{\mathrm{cf}(\lambda)}\lambda$ be such that $\sup_{\mu<\mathrm{cf}(\lambda)} \kappa_\mu = \lambda$. By the definition of add, let $X \in {}^\lambda I$ be such that $\bigcup_{\mu<\lambda} X_\mu \notin I$. For each $\mu < \mathrm{cf}(\lambda)$ we have $\bigcup_{\xi<\kappa_\mu} X_\xi \in I$. Since $\mathrm{cf}(\lambda) < \lambda$, it follows that

$$\bigcup_{\mu<\mathrm{cf}(\lambda)} \left(\bigcup_{\xi<\kappa_\mu} X_\xi \right) = \bigcup_{\xi<\lambda} X_\xi \in I,$$

contradiction. □

Proposition 18.11. $\mathrm{add}(I) \le \mathrm{cf}(\mathrm{cof}(I))$.

Proof. Let $\mathrm{cof}(I) = \kappa$, and let $X \in [I]^\kappa$ be such that $\forall C \in I \exists D \in X[C \subseteq D]$. Write $X = \bigcup_{\alpha<\mathrm{cf}(\kappa)} Y_\alpha$ with each $|Y_\alpha| < \kappa$. Then for each $\alpha < \mathrm{cf}(\kappa)$ there is a $C_\alpha \in I$ such that for all $D \in Y_\alpha[C_\alpha \not\subseteq D]$. Let $E = \bigcup_{\alpha<\mathrm{cf}(\kappa)} C_\alpha$. Then $E \notin I$. In fact, otherwise there is a $D \in X$ such that $E \subseteq D$. Say $D \in Y_\alpha$. Then $C_\alpha \subseteq D$, contradiction.

Thus $\mathrm{add}(I) \le \mathrm{cf}(\mathrm{cof}(I))$. □

Proposition 18.12. $\mathrm{cov}(I) \le \mathrm{cof}(I)$.

Proof. Let $\kappa = \mathrm{cof}(I)$, and let $X \in [I]^\kappa$ be such that $\forall C \in I \exists B \in X[C \subseteq B]$. For each $a \in A$ choose $B_a \in X$ such that $\{a\} \subseteq B_a$. Then $A = \bigcup_{a\in A} B_a = \bigcup X$. So $\mathrm{cov}(I) \le \mathrm{cof}(I)$. □

Proposition 18.13. $\mathrm{non}(I) \le \mathrm{cof}(I)$.

Proof. Let $\kappa = \mathrm{cof}(I)$, and let $X \in [I]^\kappa$ be such that $\forall C \in I \exists B \in X[C \subseteq B]$. For each $B \in X$ we have $B \neq A$; let $x_B \in A \backslash B$. Let $C = \{x_B : B \in X\}$. If $C \in I$, choose

$B \in X$ such that $C \subseteq B$. Then $x_B \in C$, so $x_B \in B$, contradiction. Thus $C \notin I$. So $\text{non}(I) \le \text{cof}(I)$. □

Proposition 18.14. *Let* $A = \omega_1$, $I = [\omega_1]^{<\omega}$. *Then* $\text{add}(I) = \omega\chi$, $\text{cov}(I) = \omega_1$, $\text{non}(I) = \omega$, *and* $\text{cof}(I) = \omega_1$.

Proof. These statements are clear, except possibly for $\text{cof}(I) = \omega_1$. Clearly $\text{cof}(I) \le \omega_1$. Suppose that $X \in [I]^{\le\omega}$ and $\forall C \in I \exists B \in X[B \subseteq C]$. Choose $\alpha \in \omega_1 \backslash \bigcup X$. Choose $B \in X$ such that $\{\alpha\} \subseteq B$; this is impossible. □

Proposition 18.15. *Let* $A = \lambda$ *be singular,* $I = [\lambda]^{<\lambda}$. *Then* $\text{add}(I) = \text{cf}(\lambda)$. □

Proposition 18.16. *Let* $A = \lambda$ *be singular,* $I = [\lambda]^{<\lambda}$. *Then* $\text{cov}(I) = \text{cf}(\lambda)$. □

Proposition 18.17. *Let* $A = \lambda$ *be singular,* $I = [\lambda]^{<\lambda}$. *Then* $\text{non}(I) = \lambda$. □

Proposition 18.18. *Let* $A = \lambda$ *be singular,* $I = [\lambda]^{<\omega}$. *Then* $\text{add}(I) = \omega$. □

Proposition 18.19. *Let* $A = \lambda$ *be singular,* $I = [\lambda]^{<\omega}$. *Then* $\text{non}(I) = \omega$. □

Proposition 18.20. *Let* $A = \lambda$ *be singular,* $I = [\lambda]^{<\omega}$. *Then* $\text{cov}(I) = \lambda$. □

A *relational triple* is a triple $\mathbf{A} = (A_0, A_1, A)$ such that A_0 and A_1 are sets and $A \subseteq A_0 \times A_1$. The norm of a relational triple $\mathbf{A} = (A_0, A_1, A)$ is $\min\{|Y| : Y \subseteq A_1 \text{ and } \forall x \in A_0 \exists y \in Y[xAy]\}$; the norm is denoted by $||\mathbf{A}||$. The dual of a relational triple $\mathbf{A} = (A_0, A_1, A)$ is the relational triple $(A_1, A_0, \{(x, y) : (y, x) \notin A\})$; it is denoted $\mathbf{A}^\perp$. We also let $A_0^\perp = A_1$, $A_1^\perp = A_0$, and $A^\perp = \{(x, y) : (y, x) \notin A\}$, A *morphism* from a relational triple (A_0, A_1, A) to a relational triple $(B_0, B_1.B)$ is a pair $\varphi \overset{\text{def}}{=} (\varphi_0, \varphi_1)$ such that:

$\varphi_0 : B_0 \to A_0$;
$\varphi_1 : A_1 \to B_1$;
$\forall a \in A_1 \forall b \in B_0[\varphi_0(b)Aa \to bB\varphi_1(a)]$.

Given such a morphism φ, we define $\varphi^\perp = (\varphi_1, \varphi_0)$.

Proposition 18.21. *If* $\varphi : \mathbf{A} \to \mathbf{B}$, *then* $\varphi^\perp : \mathbf{B}^\perp \to \mathbf{A}^\perp$.

Proof. We have $\varphi_1 : A_1 \to B_1$, so $\varphi_1 : A_0^\perp \to B_0^\perp$. Similarly, $\varphi_0 : B_0 \to A_0$, so $\varphi_0 : B_1^\perp \to A_1^\perp$. Finally, if $b \in B_1^\perp$ and $a \in A_0^\perp$, then $b \in B_0$ and $a \in A_1$, so $\varphi_0(b)Aa \to bB\varphi_1(a)]$, hence $\text{not}(bB\varphi_1(a)) \to \text{not}(\varphi_0(b)Aa)$, hence $\varphi_1(a)B^\perp b \to aA^\perp\varphi_0(a)$. □

Proposition 18.22. *If there is a morphism* $\varphi : \mathbf{A} \to \mathbf{B}$, *then* $||\mathbf{B}|| \le ||\mathbf{A}||$ *and* $||\mathbf{A}^\perp|| \le ||\mathbf{B}^\perp||$.

Proof. Let $Y \subseteq A_1$ be such that $\forall x \in A_0 \exists y \in Y[xAy]\}$, and $|Y| = ||\mathbf{A}||$. Then $\varphi_1[Y] \subseteq B_1$ and for all $x \in B_0$ there is a $y \in Y$ such that $\varphi_0(x)Ay$, hence $xB\varphi_1(y)$. So $||\mathbf{B}|| \le ||\varphi_1[Y]|| \le ||Y|| = ||\mathbf{A}||$.

Applying this to $\varphi^\perp : \mathbf{B}^\perp \to \mathbf{A}^\perp$, we get $||\mathbf{A}^\perp|| \le ||\mathbf{B}^\perp||$. □

For any ideal I on $\mathscr{P}({}^\omega 2)$ let $\mathrm{Cov}(I) = ({}^\omega 2, I. \in)$.

Proposition 18.23. $||\mathrm{Cov}(I)|| = \mathrm{cov}(I)$.

Proof. $||\mathrm{Cov}(I) = \min\{|Y| : Y \subseteq I \text{ and } \forall x \in {}^\omega 2 \exists y \in Y[x \in y]\} = \mathrm{cov}(I)$. □

Proposition 18.24. $||\mathrm{Cov}^\perp(I)|| = \mathrm{non}(I)$.

Proof. We have $\mathrm{Cov}^\perp(I) = (I, {}^\omega 2, \{(a, f) : f \notin a\})$. Hence $||\mathrm{Cov}^\perp(I)|| = \min\{|X| : X \subseteq {}^\omega 2 \text{ and } \forall a \in I \exists f \in X[f \notin a]\} = \mathrm{non}(I)$. □

Proposition 18.25. *Suppose that f is a homeomorphism of a space X onto a space Y. Then*

(i) $\mathrm{add}(\mathrm{meager}_X) = \mathrm{add}(\mathrm{meager}_Y)$*;*
(ii) $\mathrm{cov}(\mathrm{meager}_X) = \mathrm{cov}(\mathrm{meager}_Y)$*;*
(iii) $\mathrm{non}(\mathrm{meager}_X) = \mathrm{non}(\mathrm{meager}_Y)$*;*
(iv) $\mathrm{cof}(\mathrm{meager}_X) = \mathrm{cof}(\mathrm{meager}_Y)$. □

Now we show that add, non, cov, and cof have the same values for meager for each of the following notions of reals:

irrat	$\mathbb{R}$	${}^\omega 2$	$\mathscr{P}(\omega)$	${}^\omega\omega$	C
$(0, 1)$	Ω	$[\omega]^\omega$	$[0, 1]$	Θ	

Here Ω, Θ, and C are defined below.

The irrationals and ${}^\omega\omega$

Theorem 18.26. *${}^\omega\omega$ under the product topology is homeomorphic to the irrationals.*

Proof. Let $a = \langle a_0, a_1, \ldots\rangle$ be an infinite sequence of integers such that $a_i > 0$ for all $i > 0$. We want to give a precise definition of the continued fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots}}}}$$

To start with, we assume that a is a sequence of positive real numbers with domain either ω or some positive integer. We define $[a_0, \ldots, a_l]$ for each $l < \mathrm{dmn}(a)$ by recursion:

$$[a_0] = a_0;$$
$$[a_0, \ldots, a_{k+1}] = a_0 + \frac{1}{[a_1, \ldots, a_{k+1}]}$$

We want to be very explicit as to how these approximations can be written as certain fractions. To this end we make the following recursive definitions:

$$p(a, 0) = a_0; \quad q(a, 0) = 1;$$
$$p(a, 1) = a_0 a_1 + 1; \quad q(a, 1) = a_1.$$

For $k \geq 2$:

$$p(a,k) = a_k p(a,k-1) + p(a,k-2);$$
$$q(a,k) = a_k q(a,k-1) + q(a,k-2). \tag{1}$$

Note that $p(a,k) > 0$ and $q(a,k) > 0$ for all $k \geq 0$. Also, let $a' = \langle a_1, a_2, \ldots \rangle$. Now we claim that for all $i \in \omega$,

$$\begin{aligned} p(a,i+1) &= a_0 p(a',i) + q(a',i); \\ q(a,i+1) &= p(a',i). \end{aligned}$$

We prove these equations by induction on i. For $i = 0$ we have

$$\begin{aligned} p(a,1) &= a_0 a_1 + 1 = a_0 p(a',0) + q(a',0); \\ q(a,1) &= a_1 = p(a',0), \end{aligned}$$

as desired. For $i = 1$,

$$\begin{aligned} p(a,2) &= a_2 p(a,1) + p(a,0) \\ &= a_0 a_1 a_2 + a_2 + a_0 \\ &= a_0(a_1 a_2 + 1) + a_2 \\ &= a_0 p(a',1) + q(a',1); \\ q(a,2) &= a_2 q(a,1) + q(a,0) \\ &= a_1 a_2 + 1 \\ &= p(a',1), \end{aligned}$$

as desired. Now we do the inductive step for $i \geq 2$:

$$\begin{aligned} p(a,i+1) &= a_{i+1} p(a,i) + p(a,i-1) \\ &= a_{i+1}(a_0 p(a',i-1) + q(a',i-1)) + a_0 p(a',i-2) + q(a',i-2) \\ &= a_0(a_{i+1} p(a',i-1) + p(a',i-2)) + a_{i+1} q(a',i-1) + q(a',i-2) \\ &= a_0 p(a',i) + q(a',i); \\ q(a,i+1) &= a_{i+1} q(a,i) + q(a,i-1) \\ &= a_{i+1} p(a',i-1) + p(a',i-2) \\ &= p(a',i), \end{aligned}$$

as desired. So the above equations hold.

Note by an easy induction that $p(a,k), q(a,k) > 0$ for all k. Now we claim:

$$[a_0, \ldots, a_k] = \frac{p(a,k)}{q(a,k)} \tag{2}$$

for every $k \in \omega$. We prove (2) by induction on k. For $k = 0$, we have

$$[a_0] = a_0 = \frac{p(a,0)}{q(a,0)},$$

as desired. For $k = 1$, we have

$$[a_0, a_1] = a_0 + \frac{1}{a_1} = \frac{a_0 a_1 + 1}{a_1} = \frac{p(a,1)}{q(a,1)},$$

as desired. Inductively, for $k \geq 2$,

$$\begin{aligned}
[a_0, \ldots, a_k] &= a_0 + \frac{1}{[a_1, \ldots, a_k]} \\
&= a_0 + \frac{q(a', k-1)}{p(a', k-1)} \\
&= \frac{a_0 p(a', k-1) + q(a', k-1)}{p(a', k-1)} \\
&= \frac{p(a,k)}{q(a,k)},
\end{aligned}$$

as desired.

From now on we shall write p_k, q_k in place of $p(a,k), q(a,k)$ if a is understood. We also define $p_{-1} = 1$ and $q_{-1} = 0$. Then the equations (1) also hold for $k = 1$, since

$$\begin{aligned}
a_1 p_0 + p_{-1} &= a_0 a_1 + 1 = p_1 \quad \text{and} \\
a_1 q_0 + q_{-1} &= a_1 = q_1.
\end{aligned}$$

Next we claim that for $k \geq 1$,

(3) $$q_k p_{k-1} - p_k q_{k-1} = -(q_{k-1} p_{k-2} - p_{k-1} q_{k-2}).$$

In fact, multiply the equations (1) by q_{k-1} and p_{k-1} respectively:

$$\begin{aligned}
p_k q_{k-1} &= a_k p_{k-1} q_{k-1} + p_{k-2} q_{k-1}; \\
q_k p_{k-1} &= a_k q_{k-1} p_{k-1} + q_{k-2} p_{k-1}.
\end{aligned}$$

Subtracting the first of these equations from the second gives (3).

Now $q_0 p_{-1} - p_0 q_{-1} = 1$, so by (3) and induction we get, for $k \geq 0$,

(4) $$q_k p_{k-1} - p_k q_{k-1} = (-1)^k.$$

Hence for $k \geq 1$ we have

(5) $$\frac{p_{k-1}}{q_{k-1}} - \frac{p_k}{q_k} = \frac{(-1)^k}{q_k q_{k-1}}.$$

Next, for any $k \geq 1$,

(6) $$q_k p_{k-2} - p_k q_{k-2} = (-1)^{k-1} a_k.$$

To see this, multiply the equations (1) by q_{k-2} and p_{k-2} respectively:

$$\begin{aligned} p_k q_{k-2} &= a_k p_{k-1} q_{k-2} + p_{k-2} q_{k-2}; \\ q_k p_{k-2} &= a_k q_{k-1} p_{k-2} + q_{k-2} p_{k-2}. \end{aligned}$$

Now subtract the first from the second and use (4): (6) follows.

From (6):

$$\frac{p_{k-2}}{q_{k-2}} - \frac{p_k}{q_k} = \frac{(-1)^{k-1} a_k}{q_k q_{k-2}}. \tag{7}$$

Hence:

$$\left\langle \frac{p_{2k}}{q_{2k}} : k \in \omega \right\rangle \text{ is an increasing sequence;} \tag{8}$$

$$\left\langle \frac{p_{2k+1}}{q_{2k+1}} : k \in \omega \right\rangle \text{ is an decreasing sequence;} \tag{9}$$

Next we claim

$$\frac{p_{2k}}{q_{2k}} < \frac{p_{2l+1}}{q_{2l+1}} \text{ for all } k, l \in \omega \tag{10}$$

In fact, let $m = \max(k, l)$. Then

$$\begin{aligned} \frac{p_{2k}}{q_{2k}} &\le \frac{p_{2m}}{q_{2m}} \quad \text{by (8)} \\ &< \frac{p_{2m+1}}{q_{2m+1}} \quad \text{by (5)} \\ &\le \frac{p_{2l+1}}{q_{2l+1}} \quad \text{by (9)} \end{aligned}$$

So (10) holds. Next we claim:

$$p_k < p_{k+1} \text{ and } q_{k+1} < q_{k+2} \quad \text{for all } k \in \omega. \tag{11}$$

In fact, this is clear from the recursive definitions.

Now we assume that our sequence a is infinite, and all a_i are positive integers. It follows from (8), (9), (10), (11), and (5) that the approximations $\frac{p_k}{q_k}$ converge, and by definition the limit is the value of the infinite continued fraction described at the beginning. For a_0 a negative integer but all a_i positive integers for $i > 0$, we define $a' = \langle 1, a_1, a_2, \ldots \rangle$ and define the continued fraction to be

$$a_0 - 1 + \lim_{k\to\infty} \frac{p(a', k)}{q(a', k)}$$

Now we want to see how to represent any real number as a finite or infinite continued fraction. We make a recursive definition for any real number $\alpha > 1$. Let $r(\alpha, 0) = \alpha$.

Suppose that we have defined $r(\alpha, i) > 1$. Write $r(\alpha, i) = a(\alpha, i) + s(\alpha, i+1)$ with $a(\alpha, i)$ a positive integer and $s(\alpha, i+1)$ a nonnegative real < 1. If $s(\alpha, i+1) = 0$, the construction stops. Otherwise we define $r(\alpha, i+1) = \frac{1}{s(\alpha,i+1)}$. This finishes the construction. Let $l(\alpha)$ be the index i such that $s(\alpha, i+1) = 0$, or $l(\alpha) = \omega$ if there is no such index. We need the following technical fact.

(12) If $\alpha > 1$ and $l(\alpha) > 1$, then $l(r(\alpha, 1)) = l(\alpha) - 1$, and for each $j \leq l(\alpha) - 1$ we have $r(r(\alpha, 1), j) = r(\alpha, j+1)$ and $a(r(\alpha, 1), j) = a(\alpha, j+1)$.

By induction on j we prove that $r(r(\alpha, 1), j)$ is defined and equals $r(\alpha, j+1)$ for each $j \leq l(\alpha) - 1$. For $j = 0$ we have $r(r(\alpha, 1), 0)$ defined and it equals $r(\alpha, 1)$, as desired. Now assume our result for j, with $j + 1 \leq l(\alpha) - 1$. Then

$$r(r(\alpha, 1), j) = r(\alpha, j+1) = a(\alpha, j+1) + s(\alpha, j+2).$$

Now $j + 2 \leq l(\alpha)$, so $s(\alpha, j+2) > 0$, and hence by definition, $r(\alpha, j+2) = \frac{1}{s(\alpha,j+2)} = r(r(\alpha, 1), j+1)$. This completes the inductive proof.

Now if $j \leq l(\alpha) - 1$, then

$$\begin{aligned} r(r(\alpha, 1), j) &= a(r(\alpha), 1), j) + s(r(\alpha, 1), j+1); \\ r(\alpha, j+1) &= a(\alpha, j+1) + s(\alpha, j+2); \end{aligned}$$

so $a(r(\alpha, 1), j) = a(\alpha, j+1)$. Finally, if $j = l(\alpha)$, then $r(\alpha, j) = a(\alpha, j)$, and hence $r(r(\alpha, 1), j-1) = r(\alpha, j) = a(\alpha, j)$ and so $l(r(\alpha, 1)) = j - 1$, as desired in (12).

(13) If $\alpha > 1$ and $n \leq l(\alpha)$, then $\alpha = [a(\alpha, 0), a(\alpha, 1), \ldots, a(\alpha, n-1), r(\alpha, n)]$.

We prove this by induction on n. For $n = 0$, $[r(\alpha, 0)] = \alpha$. Assume that our condition is true for n, and $n + 1 \leq l(\alpha)$. Then

$$\begin{aligned} &[a(\alpha, 0), a(\alpha, 1), \ldots, a(\alpha, n), r(\alpha, n+1)] \\ &\quad = a(\alpha, 0) + \frac{1}{[a(\alpha, 1), a(\alpha, 2), \ldots, a(\alpha, n), r(\alpha, n+1)]} \\ &\quad = a(\alpha, 0) + \frac{1}{[a(r(\alpha, 1), 0) a(r(\alpha, 1), 1), \ldots, a(r(\alpha, 1), n-1), r(r(\alpha, 1), n)]} \\ &\quad = a(\alpha, 0) + \frac{1}{r(\alpha, 1)} \\ &\quad = a(\alpha, 0) + s(\alpha, 1) \\ &\quad = \alpha, \end{aligned}$$

completing the inductive proof.

(14) If $\alpha > 1$ is rational, then the above definition of $r(\alpha, i)$'s terminates after finitely many steps.

In fact, it suffices to show that if $r(\alpha, i) = \frac{b}{c}$ with b, c positive integers and g.c.d$(b, c) = 1$, and $r(\alpha, i+1)$ is defined, then $r(\alpha, i+1)$ has the form $\frac{d}{e}$, with d and e positive integers with $e < c$. To prove this, recall that $r(\alpha, i) = a(\alpha, i) + s(\alpha, i+1)$, with $s(\alpha, i+1)$ a nonnegative real < 1, and $r(\alpha, i+1) = \frac{1}{s(\alpha,i+1)}$. Thus

$$\frac{b}{c} = r(\alpha, i) = a(\alpha, i) + s(\alpha, i+1) \quad \text{and hence}$$

$$b = ca(\alpha, i) + cs(\alpha, i+1); \tag{15}$$

Hence

$$\begin{aligned} r(\alpha, i+1) &= \frac{1}{s(\alpha, i+1)} \\ &= \frac{1}{r(\alpha, i) - a(\alpha, i)} \\ &= \frac{1}{\frac{b}{c} - a(\alpha, i)} \\ &= \frac{c}{b - ca(\alpha, i)} \\ &= \frac{c}{cs(\alpha, i+1)} \quad \text{by (15),} \end{aligned}$$

and $cs(\alpha, i+1)$ is a positive integer $< c$, as desired.

(16) If α is rational, then there exist integers $a_0, a_1, \ldots, a_n$ with $a_i > 0$ for all $i > 0$ such that $\alpha = [a_0, a_1, \ldots, a_n]$.

In fact, let m be an integer such that $\alpha + m > 1$; if $\alpha > 1$, let $m = 0$. By (14), $n \stackrel{\text{def}}{=} l(\alpha + m)$ is finite. We then have $r(\alpha + m, n) = a(\alpha + m, n)$. Hence by (13) we have $\alpha + m = [a(\alpha + m, 0), \ldots, a(\alpha + m, n)]$, and the desired conclusion follows.

(17) If $\langle a_0, a_1, \ldots \rangle$ is a sequence of rational numbers each greater than 0, then also $[a_0, a_1, \ldots, a_n]$ is rational for each n.

This is clear from the basic definition, by induction.

(18) Let $\alpha > 1$ be irrational. Then by (17), the sequence

$$b \stackrel{\text{def}}{=} \langle a(\alpha, 0), a(\alpha, 1), \ldots \rangle$$

never terminates. We claim that for each positive integer n,

$$\alpha = \frac{p(b, n-1)r(\alpha, n) + p(b, n-2)}{q(b, n-1)r(\alpha, n) + q(b, n-2)}.$$

We prove by induction that for every positive integer n, this holds for all irrationals $\alpha > 1$. First, the case $n = 1$:

$$\begin{aligned} \frac{p(b, 0)r(\alpha, 1) + p(b, -1)}{q(b, 0)r(\alpha, 1) + q(b, -1)} &= \frac{a(\alpha, 0)r(\alpha, 1) + 1}{r(\alpha, 1)} \\ &= a(\alpha, 0) + \frac{1}{r(\alpha, 1)} \\ &= a(\alpha, 0) + s(\alpha, 1) \\ &= r(\alpha, 0) \\ &= \alpha, \end{aligned}$$

as desired. Now we assume our statement for n. In fact, we apply it to $r(\alpha,1)$ rather than α. Note that $r(\alpha,1)>1$, and it is irrational by (17) and (13). Let

$$\begin{aligned} c &= \langle a(\alpha,1), a(\alpha,2), \ldots\rangle \\ &= \langle a(r(\alpha,1),0), a(r(\alpha,1),1), \ldots\rangle, \end{aligned}$$

by (12). Hence, starting with the inductive hypothesis,

$$\begin{aligned} r(\alpha,1) &= \frac{p(c,n-1)r(r(\alpha,1),n)+p(c,n-2)}{q(c,n-1)r(r(\alpha,1),n)+q(c,n-2)} \\ &= \frac{p(c,n-1)r(\alpha,n+1)+p(c,n-2)}{q(c,n-1)r(\alpha,n+1)+q(c,n-2)}. \end{aligned}$$

Hence, using the equations following (1),

$$\begin{aligned} \alpha &= r(\alpha,0) \\ &= a(\alpha,0)+s(\alpha,1) \\ &= a(\alpha,0)+\frac{1}{r(\alpha,1)} \\ &= a(\alpha,0)+\frac{q(c,n-1)r(\alpha,n+1)+q(c,n-2)}{p(c,n-1)r(\alpha,n+1)+p(c,n-2)} \\ &= \frac{a(\alpha,0)p(c,n-1)r(\alpha,n+1)+a(\alpha,0)p(c,n-2)+q(c,n-1)r(\alpha,n+1)+q(c,n-2)}{p(c,n-1)r(\alpha,n+1)+p(c,n-2)} \\ &= \frac{p(b,n)r(\alpha,n+1)+p(b,n-1)}{q(b,n)r(\alpha,n+1)+q(b,n-1)}, \end{aligned}$$

which finishes the inductive proof of (18).

We now omit the parameter b, as it is understood in what follows.

(19) Let $\alpha>1$ be irrational. Then for every positive integer n,

$$\alpha-\frac{p_n}{q_n}=\frac{(p_{n-1}q_{n-2}-q_{n-1}p_{n-2})(r_n-a_n)}{(q_{n-1}r_n+q_{n-2})(q_{n-1}a_n+q_{n-2})}.$$

To prove this, first note by (18) and (1) that

$$\alpha-\frac{p_n}{q_n}=\frac{p_{n-1}r_n-p_{n-2}}{q_{n-1}r_n+q_{n-2}}-\frac{p_{n-1}a_n+p_{n-2}}{q_{n-1}a_n+q_{n-2}}. \tag{20}$$

Now we have

$$\begin{aligned} &(p_{n-1}r_n-p_{n-2})(q_{n-1}a_n+q_{n-2})-(p_{n-1}a_n+p_{n-2})(q_{n-1}r_n+q_{n-2}) \\ &\quad= p_{n-1}q_{n-1}a_nr_n+p_{n-1}q_{n-2}r_n+p_{n-2}q_{n-1}a_n+p_{n-2}q_{n-2} \\ &\qquad - p_{n-1}q_{n-1}a_nr_n-p_{n-1}q_{n-2}a_n-p_{n-2}q_{n-1}r_n-p_{n-2}q_{n-2} \\ &\quad= (p_{n-1}q_{n-2}-q_{n-1}p_{n-2})(r_n-a_n). \end{aligned}$$

Hence from (20) we get (19).

(21) For irrational $\alpha > 1$ we have

$$\alpha = [a(\alpha, 0), a(\alpha, 1), \ldots].$$

In fact, note from (4) that $p_{n-1}q_{n-2} - q_{n-1}p_{n-2} = (-1)^{n-1}$, while by definition we have $r(\alpha, n) - a(\alpha, n) = s(\alpha, n+1) < 1$. Hence by (19),

$$\left|\alpha - \frac{p_n}{q_n}\right| \leq \frac{1}{(q_{n-1}r_n + q_{n-2})(q_{n-1}a_n + q_{n-2})} < \frac{1}{q_{n-2}^2},$$

and hence (21) follows from (11).

Now for any irrational $\alpha > 1$, define

$$f(\alpha) = \langle a(\alpha, 0), a(\alpha, 1), \ldots\rangle.$$

Then by the above results, f is a one-to-one function mapping the set $\mathcal{N}$ of irrationals $>$ 1 onto the set ${}^\omega(\omega\backslash 1)$. The latter set is clearly homeomorphic to ${}^\omega\omega$.

(22) The set of irrationals > 1 is homeomorphic to the entire set of irrationals.

To see this, define g by setting, for each irrational $x > 1$,

$$g(x) = \begin{cases} x + m & \text{if } 0 < m < x < m + 1 \text{ with } m \in \omega, \\ x + 3m + 1 & \text{if } -m < x < -m + 1 \text{ with } m \in \omega. \end{cases}$$

Then g maps $(m, m+1)_{\text{irr}}$ one-one onto $(2m, 2m+1)_{\text{irr}}$ for each positive integer m, and $(-m, -m+1)_{\text{irr}}$ one-one onto $(2m+1, 2m+2)_{\text{irr}}$ for each $m \in \omega$. Clearly g is the desired homeomorphism.

Thus to finish the proof of Theorem 18.26 it suffices to show that f, defined above, is a homeomorphism. To do this, we need the following fact.

(23) Suppose that $a_0, \ldots, a_n, b_0, \ldots, b_{n-1}$ are positive integers and r is a real number > 1. Assume that

$$\begin{array}{ll} [a_0, \ldots, a_{n-1}] < [b_0, \ldots, b_{n-1}, r] < [a_0, \ldots, a_n] & \text{if } n \text{ is odd} \\ [a_0, \ldots, a_{n-1}] > [b_0, \ldots, b_{n-1}, r] > [a_0, \ldots, a_n] & \text{if } n \text{ is even} \end{array}$$

Then $a_i = b_i$ for all $i < n$. Cf here (2), (8), (9), (10).

We prove (23) by induction on n. For $n = 1$ the assumption is that $a_0 < b_0 + \frac{1}{r} < a_0 + \frac{1}{a_1}$. So clearly $a_0 = b_0$. Now assume (23) for an odd n; we prove it for $n + 1$ and $n + 2$. So, first suppose that

$$[a_0, \ldots, a_n] > [b_0, \ldots, b_n, r] > [a_0, \ldots, a_{n+1}].$$

Thus

$$a_0 + \frac{1}{[a_1, \ldots, a_n]} > b_0 + \frac{1}{[b_1, \ldots, b_n, r]} > a_0 + \frac{1}{[a_1, \ldots, a_{n+1}]},$$

and it follows that $a_0 = b_0$ and

$$[a_1, \ldots, a_n] < [b_1, \ldots, b_n, r] < [a_1, \ldots, a_{n+1}];$$

then the inductive hypothesis yields $a_i = b_i$ for all $i = 1, \ldots, n$, which proves our statement for $n + 1$.

The inductive step to $n + 2$ is clearly similar. So (23) holds.

Now to show that f is continuous, suppose that $s \in {}^n(\omega\backslash 1)$; we want to show that $f^{-1}[O(s)]$ is open. We may assume that $n = 2m + 1$ for some natural number m. Let $\alpha \in f^{-1}[O_s]$. Define $a_i = a(\alpha, i)$ for all i. Thus $a_0 = s_0, \ldots, a_{2m} = s_{2m}$. By (2) and (8)–(10) we have $[a_0, \ldots, a_{2m}] < \alpha < [a_0, \ldots, a_{2m+1}]$. Choose ε so that $[a_0, \ldots, a_{2m}] + \varepsilon < \alpha < \alpha + \varepsilon < [a_0, \ldots, a_{2m+1}]$. We claim:

(24) For every irrational $\beta > 1$, if $|\alpha - \beta| < \varepsilon$, then $\beta \in f^{-1}[O(s)]$.

This will prove continuity of f. To prove (24), assume its hypothesis, and let $b_i = b(\beta, i)$ for all i.

Case 1. $\beta < \alpha$. Thus $\alpha - \beta < \varepsilon$. Hence $[a_0, \ldots, a_{2m}] < [a_0, \ldots, a_{2m}] + \varepsilon < \alpha < \beta + \varepsilon$, so $[a_0, \ldots, a_{2m}] < \beta$. If $[a_0, \ldots, a_{2m+1}] \le \beta$, then by (8)–(10), $\alpha < \beta$, contradiction. So $\beta < [a_0, \ldots, a_{2m+1}]$. Now $\beta = [b_0, \ldots, b_{2m}, r_{2m+1}]$ by (13), so by (23), $a_i = b_i$ for all $i \le 2m$, as desired.

Case 2. $\alpha < \beta$. Thus $\beta - \alpha < \varepsilon$, so $\beta < \alpha + \varepsilon$. Hence

$$[a_0, \ldots, a_{2m}] < \alpha < \beta < \alpha + \varepsilon < [a_0, \ldots, a_{2m+1}],$$

and the argument is finished as in Case 1.

So (24) holds, and f is continuous.

(25) f is an open mapping.

For, suppose that $\alpha > 1$ is irrational, and ε is a positive real number; we want to show that $f[S_\varepsilon(\alpha)]$ is open. Let $b \in f[S_\varepsilon(\alpha)]$; we want to find a finite sequence s such that $b \in O(s) \subseteq f[S_\varepsilon(\alpha)]$. Say $b = f(\beta)$ with $\beta \in S_\varepsilon(\alpha)$. So $|\alpha - \beta| < \varepsilon$. Choose m such that

$$\frac{1}{q(b, 2m)q(b, 2m+1)} < \varepsilon - |\alpha - \beta|.$$

This is possible by (11). Let $s = \langle b_0, \ldots, b_{2m+1}\rangle$. So $b \in O(s)$. Now suppose that $c \in O(s)$. Then

$$[b_0, \ldots, b_{2m}] = [c_0, \ldots, c_{2m}] < [c] < [c_0, \ldots, c_{2m+1}] = [b_0, \ldots, b_{2m+1}]$$

by (8)–(10). Also,

$$[b_0, \ldots, b_{2m}] = [c_0, \ldots, c_{2m}] < \beta < [c_0, \ldots, c_{2m+1}] = [b_0, \ldots, b_{2m+1}]$$

by (8)–(10). Now

$$\begin{aligned}
[b_0, \ldots, b_{2m+1}] - [b_0, \ldots, b_{2m}] &= \frac{p(b, 2m+1)}{q(b, 2m+1)} - \frac{p(b, 2m)}{q(b, 2m)} \quad \text{by (2)} \\
&= \frac{1}{q(b, 2m)q(b, 2m+1)} \\
&< \varepsilon - |\alpha - \beta|.
\end{aligned}$$

Hence

$$|[c] - \alpha| \le |[c] - \beta| + |\beta - \alpha| < \varepsilon,$$

and so $c = f([c]) \in f[S_\varepsilon(\alpha)]$, as desired. $\square$

Meager for $\mathbb{R}$ and $(0, 1)$.

Proposition 18.27. *$\mathbb{R}$ is homeomorphic to $(0, 1)$.*

Proof. For each $x \in (0, 1)$ let $f(x) = -\frac{1}{x} + \frac{1}{1-x}$. Then if $x < y$ we have

$$\begin{aligned}
&1 < \frac{y}{x}; \quad \frac{1}{y} < \frac{1}{x} \quad -\frac{1}{x} < -\frac{1}{y};\\
&-y < -x; \quad 1 - y < 1 - x; \quad 1 < \frac{1-x}{1-y}; \quad \frac{1}{1-x} < \frac{1}{1-y};\\
&f(x) < f(y).
\end{aligned}$$

In particular, f is one-one. Also, $\lim_{x\to 0} f(x) = -\infty$ and $\lim_{x\to 1} f(x) = \infty$. So the proposition follows. $\square$

Proposition 18.28. *If $A \subseteq (0, 1)$ is nowhere dense in $(0, 1)$, then A is nowhere dense in $[0, 1]$.*

Proof. Suppose that $A \subseteq (0, 1)$ is nowhere dense in $(0, 1)$. Take any $a < b$ with $(a, b) \cap [0, 1] \neq \emptyset$; we want to show that $(a, b) \cap [0, 1]\backslash A \neq \emptyset$. Clearly $(a, b) \cap (0, 1) \neq \emptyset$, so $(a, b) \cap (0, 1)\backslash A \neq \emptyset$. So $(a, b) \cap [0, 1]\backslash A \neq \emptyset$. $\square$

Corollary 18.29. *If $A \subseteq (0, 1)$ is meager in $(0, 1)$, then A is meager in $[0, 1]$.* $\square$

Proposition 18.30. *If $A \subseteq [0, 1]$ is nowhere dense in $[0, 1]$, then $A \cap (0, 1)$ is nowhere dense in $(0, 1)$.*

Proof. Suppose that $A \subseteq [0, 1]$ is nowhere dense in $[0, 1]$. Take any $a < b$ with $(a, b)\cap (0, 1) \neq \emptyset$; we want to show that $(a, b)\cap(0, 1)\backslash(A\cap(0, 1)) \neq \emptyset$. Now $(\max(a, 0), \min(b, 1)) = (a, b)\cap(0, 1) \neq \emptyset$ and $(\max(a, 0), \min(b, 1) \subseteq [0, 1]$, so $(\max(a, 0), \min(b, 1))\backslash A \neq \emptyset$. Clearly $(\max(a, 0), \min(b, 1))\backslash A \subseteq (a, b) \cap (0, 1)\backslash(A \cap (0, 1))$.qed

Corollary 18.31. *If $A \subseteq [0, 1]$ is meager in $[0, 1]$, then $A\cap(0, 1)$ is meager in $(0, 1)$.* $\square$

Proposition 18.32. *(i)* $\mathrm{add}(\mathrm{meager}_{[0,1]}) = \mathrm{add}(\mathrm{meager}_{(0,1)})$*;*
(ii) $\mathrm{cov}(\mathrm{meager}_{[0,1]}) = \mathrm{cov}(\mathrm{meager}_{(0,1)})$*;*
(iii) $\mathrm{non}(\mathrm{meager}_{[0,1]}) = \mathrm{non}(\mathrm{meager}_{(0,1)})$*;*
(iv) $\mathrm{cof}(\mathrm{meager}_{[0,1]}) = \mathrm{cof}(\mathrm{meager}_{(0,1)})$*.*

Proof. (i): First let $\kappa = \mathrm{add}(\mathrm{meager}_{[0,1]})$ and suppose that $E \in [\mathrm{add}(\mathrm{meager}_{[0,1]})]^\kappa$ with $\bigcup E \notin \mathrm{add}(\mathrm{meager}_{[0,1]})$. Then by Corollary 18.31, $E' = \{A \cap (0, 1) : A \in E\} \subseteq \mathscr{P}(\mathrm{meager}_{(0,1)})$. If $\bigcup E' \in \mathrm{add}(\mathrm{meager}_{(0,1)})$, then clearly

$$\bigcup E \subseteq \bigcup E' \cup \{0, 1\} \in \mathrm{add}(\mathrm{meager}_{[0,1]}),$$

contradiction.

Second let $\kappa = \text{add}(\text{meager}_{(0,1)})$ and suppose that $E \in [\text{add}(\text{meager}_{(0,1)})]^\kappa$ with $\bigcup E \notin \text{add}(\text{meager}_{(0,1)})$. Then by Corollary 18.29, $E \subseteq \mathscr{P}(\text{meager}_{[0,1]})$. If $\bigcup E \in \text{add}(\text{meager}_{[0,1]})$, then by Corollary 18.31, $\bigcup E = (\bigcup E) \cap (0,1) \in \text{add}(\text{meager}_{(0,1)})$, contradiction.

(ii): First let $\kappa = \text{cov}(\text{meager}_{[0,1]})$ and suppose that $E \in [\text{add}(\text{meager}_{[0,1]})]^\kappa$ with $[0,1] = \bigcup E$. Then by Corollary 18.31, $E' = \{A \cap (0,1) : A \in E\} \subseteq \mathscr{P}(\text{meager}_{(0,1)})$. Hence $(0,1) = \bigcup E'$.

Second let $\kappa = \text{cov}(\text{meager}_{(0,1)})$ and suppose that $E \in [\text{add}(\text{meager}_{(0,1)})]^\kappa$ with $(0,1) = \bigcup E$. Then by Corollary 18.29, $E \subseteq \mathscr{P}(\text{meager}_{[0,1]})$. Now $[0,1] = \bigcup E \cup \{0,1\}$.

(iii): First let $\kappa = \text{non}(\text{meager}_{[0,1]})$ and $X \in [[0,1]]^\kappa$ with $X \notin \text{non}(\text{meager}_{[0,1]})$. Then $X\backslash\{0,1\} \in [(0,1)]^\kappa$ and $X\backslash\{0,1\} \notin \text{non}(\text{meager}_{(0,1)})$ by Corollary 18.29.

Second let $\kappa = \text{non}(\text{meager}_{(0,1)})$ and $X \in [(0,1)]^\kappa$ with $X \notin \text{non}(\text{meager}_{(0,1)})$. Then $X \in [(0,1)]^\kappa$ with $X \notin \text{non}(\text{meager}_{[0,1]})$ by Corollary 18.31.

(iv): First let $\kappa = \text{cof}(\text{meager}_{[0,1]})$ and $X \in [\text{cof}(\text{meager}_{[0,1]})]^\kappa$ such that $\forall A \in \text{cof}(\text{meager}_{[0,1]})\exists B \in X[A \subseteq B]$. Let $X' = \{B \cap (0,1) : B \in X\}$. Then

$$X' \in \mathscr{P}(\text{cof}(\text{meager}_{(0,1)}))$$

by Corollary 18.31. Suppose that $A \in \text{cof}(\text{meager}_{(0,1)})$. Then by Corollary 18.29,

$$A \in \text{cof}(\text{meager}_{[0,1]}).$$

So there is a $B \in X$ such that $A \subseteq B$. Hence $A \subseteq B \cap (0,1)$.

Second let $\kappa = \text{non}(\text{meager}_{(0,1)})$ and $X \in [\text{cof}(\text{meager}_{(0,1)})]^\kappa$ such that

$$\forall A \in \text{cof}(\text{meager}_{(0,1)})\exists B \in X[A \subseteq B].$$

Let $X' = \{B \cup \{0,1\} : B \in X\}$. Clearly $X' \in [\text{cof}(\text{meager}_{[0,1]})]^\kappa$. Suppose that $A \in \text{cof}(\text{meager}_{[0,1]})$. Then $A \cap (0,1) \in \text{cof}(\text{meager}_{(0,1)})$ by Corollary 18.31. Choose $B \in X$ such that $A \cap (0,1) \subseteq B$. Then $A \subseteq B \cup \{0,1\}$. $\square$

Meager for irrat and $\mathbb{R}$

Lemma 18.33. *If $A \subseteq \mathbb{R}$ is nowhere dense in $\mathbb{R}$, then $A \cap \text{irrat}$ is nowhere dense in* irrat.

Proof. Assume that $A \subseteq \mathbb{R}$ is nowhere dense in $\mathbb{R}$. Then $\forall a,b \in \mathbb{R}[a < b$ implies that $(a,b)\backslash\overline{A} \neq \emptyset]$, so $\forall a,b \in \mathbb{R}[a < b$ implies that $\exists c,d \in \mathbb{R}[c < d$ and $(c,d) \subseteq (a,b)\backslash\overline{A}]]$. Now the closure of $A \cap \text{irrat}$ in irrat is $\overline{A} \cap \text{irrat}$. Given $a,b \in \mathbb{R}$ with $a < b$, choose $c,d \in \mathbb{R}$ such that $c < d$ and $(c,d) \subseteq (a,b)\backslash\overline{A}$. Then

$$(c,d) \cap \text{irrat} \subseteq ((a,b) \cap \text{irrat})\backslash(\overline{A} \cap \text{irrat}).$$

Thus $A \cap \text{irrat}$ is nowhere dense in irrat. $\square$

Lemma 18.34. *If A is* $\text{meager}_\mathbb{R}$*, then $A \cap \text{irrat}$ is* $\text{meager}_{\text{irrat}}$. $\square$

Lemma 18.35. *If $A \subseteq$ irrat is nowhere dense in irrat, then it is nowhere dense in $\mathbb{R}$.*

Proof. Assume that $A \subseteq$ irrat is nowhere dense in irrat. Then $\forall a, b \in \mathbb{R}[a < b$ implies that $(a, b) \cap \text{irrat} \backslash (\overline{A} \cap \text{irrat}) \neq \emptyset$. Given $a, b \in \mathbb{R}$ with $a < b$, choose $c, d \in \mathbb{R}$ with $c < d$ and $(c, d) \cap \text{irrat} \subseteq (a, b) \cap \text{irrat} \backslash (\overline{A} \cap \text{irrat})$. We claim that $(c, d) \subseteq (a, b) \backslash \overline{A}$. Now $(c, d) \cap (a, b) = (\max(c, a), \min(d, b))$. Suppose that $(\max(c, a), \min(d, b)) \cap \overline{A} \neq \emptyset$. Then there is an $x \in (\max(c, a), \min(d, b)) \cap A$. So x is irrational, contradiction. $\square$

Lemma 18.36. *If A is* $\text{meager}_{\text{irrat}}$*, then A is* $\text{meager}_{\mathbb{R}}$*.* $\square$

Lemma 18.37. $\text{add}(\text{meager}_{\mathbb{R}}) = \text{add}(\text{meager}_{\text{irrat}})$.

Proof. First let $\kappa = \text{add}(\text{meager}_{\text{irrat}})$, and let $E \subseteq \text{meager}_{\text{irrat}}$ be such that $|E| = \kappa$ and $\bigcup E \notin \text{meager}_{\text{irrat}}$. By Lemma 18.35, $E \subseteq \text{meager}_{\mathbb{R}}$. We have $\bigcup E \notin \text{meager}_{\mathbb{R}}$ by Lemma 18.34.

Second, let $\kappa = \text{add}(\text{meager}_{\mathbb{R}})$, and let $E \subseteq \text{meager}_{\mathbb{R}}$ be such that $|E| = \kappa$ and $\bigcup E \notin \text{meager}_{\mathbb{R}}$. Then $E' = \{X \cap \text{irrat} : X \in E\} \subseteq \text{meager}_{\text{irrat}}$ by Lemma 18.34. Hence $E' \cup \{\mathbb{Q}\} \subseteq \text{meager}_{\text{irrat}}$. If $\bigcup(E' \cup \{\mathbb{Q}\}) \in \text{meager}_{\text{irrat}}$, then $\bigcup E \subseteq \bigcup(E' \cup \{\mathbb{Q}\}) \in \text{meager}_{\text{irrat}} \subseteq \text{meager}_{\mathbb{R}}$ by Lemma 18.36, contradiction. $\square$

Lemma 18.38. $\text{cov}(\text{meager}_{\mathbb{R}}) = \text{cov}(\text{meager}_{\text{irrat}})$.

Proof. First let $\kappa = \text{cov}(\text{meager}_{\mathbb{R}})$, and let $E \in [\text{meager}_{\mathbb{R}}]^\kappa$ be such that $\mathbb{R} = \bigcup E$. For each $A \in E$ we have $A \cap \text{irrat} \in \text{meager}_{\text{irrat}}$ by Corollary 18.34. Moreover, $\bigcup_{A \in E}(A \cap \text{irrat}) = \text{irrat}$. So $\text{cov}(\text{meager}_{\text{irrat}}) \leq \kappa$.

Second let $\kappa = \text{cov}(\text{meager}_{\text{irrat}})$, and let $E \in [\text{meager}_{\text{irrat}}]^\kappa$ be such that $\text{irrat} = \bigcup E$. Let $E' = E \cup \{\{q\} : q \in \mathbb{Q}\}$. Then by Lemma 18.36, $E' \in \mathscr{P}(\text{meager}_{\mathbb{R}})$, and $\bigcup E' = \mathbb{R}$. So $\text{cov}(\text{meager}_{\mathbb{R}}) \leq \kappa$. $\square$

Lemma 18.39. $\text{non}(\text{meager}_{\mathbb{R}}) = \text{non}(\text{meager}_{\text{irrat}})$.

Proof. First let $\kappa = \text{non}(\text{meager}_{\mathbb{R}})$, and let $X \in [\mathbb{R}]^\kappa$ such that $X \notin \text{meager}_{\mathbb{R}}$. Then $X \cap \text{irrat} \notin \text{meager}_{\text{irrat}}$, as otherwise, with $Y = (X \cap \text{irrat}) \cup \{\{q\} : q \in \mathbb{Q}\}$ we would have $X \subseteq Y \in \text{meager}_{\mathbb{R}}$, using Lemma 18.36.

Second let $\kappa = \text{non}(\text{meager}_{\text{irrat}})$, and let $X \in [\text{irrat}]^\kappa$ such that $X \notin \text{meager}_{\text{irrat}}$. By Lemma 18.34, $X \notin \text{meager}_{\mathbb{R}}$. Hence $\text{non}(\text{meager}_{\mathbb{R}}) \leq \kappa$. $\square$

Lemma 18.40. $\text{cof}(\text{meager}_{\mathbb{R}}) = \text{cof}(\text{meager}_{\text{irrat}})$.

Proof. First let $\kappa = \text{cof}(\text{meager}_{\mathbb{R}})$, and let $X \in [\text{meager}_{\mathbb{R}}]^\kappa$ be such that $\forall A \in \text{meager}_{\mathbb{R}} \exists B \in X[A \subseteq B]$. Let $Y = \{A \cap \text{irrat} : A \in X\}$. Then by Lemma 18.34, $Y \in \mathscr{P}(\text{meager}_{\text{irrat}})$, and $\forall A \in \text{meager}_{\text{irrat}} \exists B \in Y[A \subseteq B]$, using also Lemma 18.36. Hence $\text{cof}(\text{meager}_{\text{irrat}}) \leq \kappa$.

Second let $\kappa = \text{cof}(\text{meager}_{\text{irrat}})$, with $X \in [\text{meager}_{\text{irrat}}]^\kappa$ so that $\forall A \in \text{meager}_{\text{irrat}} \exists B \in X[A \subseteq B]$. Let $Y = \{A \cup \mathbb{Q} : A \in X$. Then by Lemma 18.36, $Y \in \mathscr{P}(\text{meager}_{\mathbb{R}})$. If $A \in \text{meager}_{\mathbb{R}}$, then $A \cap \text{irrat} \in \text{meager}_{\text{irrat}}$ by Lemma 18.34, and so there is a $B \in X$ such that $A \cap \text{irrat} \subseteq B$. Then $A \subseteq B \cup \mathbb{Q} \in Y$. Hence $\text{cof}(\text{meager}_{\mathbb{R}}) \leq \kappa$. $\square$

The Cantor set and ${}^{\omega}2$.

Let

$$C = \left\{ x \in [0,1] : \exists t \in {}^{\omega\setminus 1}\{0,2\} \left[x = \sum_{i=1}^{\infty} \frac{t_i}{3^i} \right] \right\}.$$

C is the *Cantor set*. For $a < b$ let

$$f([a,b]) = \left\{ \left[a, a + \frac{1}{3}(b-a) \right], \left[a + \frac{2}{3}(b-a), b \right] \right\}.$$

Define A with domain ω recursively by

$$A_0 = \{[0,1]\};$$
$$A_{n+1} = \bigcup_{X \in A_n} f(X).$$

Lemma 18.41. *For every positive integer n and every set Y, $Y \in A_n$ iff there is a $t : (n+1)\setminus 1 \to \{0,2\}$ such that*

$$Y = \left[\sum_{i=1}^{n} \frac{t_i}{3^i}, \sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{1}{3^n} \right].$$

Proof. For $n = 1$ we have $A_1 = f([0,1]) = \{[0, \frac{1}{3}], [\frac{2}{3}, 1]\}$. With $t_1 = 0$ we have $[\sum_{i=1}^{n} \frac{t_i}{3^i}, \sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{1}{3^n}] = [0, \frac{1}{3}]$, and with $t_1 = 2$ we have $[\sum_{i=1}^{n} \frac{t_i}{3^i}, \sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{1}{3^n}] = [\frac{2}{3}, 1]$, as desired.

Now assume the equality for $n \geq 1$. First suppose that $Y \in A_{n+1}$. Then there is an $X \in A_n$ such that $Y \in f(X)$. By the inductive hypothesis choose $t : (n+1)\setminus 1 \to \{0,2\}$ such that

$$X = \left[\sum_{i=1}^{n} \frac{t_i}{3^i}, \sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{1}{3^n} \right].$$

Note that X has size $\frac{1}{3^n}$.

Case 1.

$$Y = \left[\sum_{i=1}^{n} \frac{t_i}{3^i}, \sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{1}{3^{n+1}} \right].$$

Let $s \restriction ((n+1)\setminus 1) = t \restriction ((n+1)\setminus 1)$ and $s(n+1) = 0$. Then

$$(*) \qquad Y = \left[\sum_{i=1}^{n+1} \frac{s_i}{3^i}, \sum_{i=1}^{n+1} \frac{s_i}{3^i} + \frac{1}{3^{n+1}} \right].$$

Case 2.

$$Y = \left[\sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{2}{3^{n+1}}, \sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{1}{3^n} \right].$$

Let $s \restriction ((n+1)\backslash 1) = t \restriction ((n+1)\backslash 1)$ and $s(n+1) = 2$. Then $(*)$ holds.

Second, suppose that $(*)$ holds. Let $t = s \restriction ((n+1)\backslash 1)$. If $s(n+1) = 0$, then

$$Y = \left[\sum_{i=1}^{n} \frac{t_i}{3^i}, \sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{1}{3^{n+1}}\right];$$

If $s(n+1) = 2$, then

$$Y = \left[\sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{2}{3^{n+1}}, \sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{1}{3^n}\right].$$

Hence in either case,

$$Y \in f\left(\left[\sum_{i=1}^{n} \frac{t_i}{3^i}, \sum_{i=1}^{m} \frac{t_i}{3^i} + \frac{1}{3^n}\right]\right),$$

and

$$\left[\sum_{i=1}^{n} \frac{t_i}{3^i}, \sum_{i=1}^{m} \frac{t_i}{3^i} + \frac{1}{3^n}\right] \in A_n$$

by the inductive hypothesis. Hence $Y \in A_{n+1}$. □

Theorem 18.42. $C = \bigcap_{n\in\omega} \bigcup A_n$.

Proof. Suppose that $x \in C$ and $n \in \omega$. Choose $s \in {}^{\omega\backslash 1}\{0,2\}$ such that $x = \sum_{i=1}^{\infty} \frac{s_i}{3^i}$. Let $t = s \restriction ((n+1)\backslash 1)$. Then

$$x \in \left[\sum_{i=1}^{n} \frac{t_i}{3^i}, \sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{1}{3^n}\right]$$

since $\sum_{i=n+1}^{\infty} \frac{t_i}{3^i} \leq \frac{1}{3^n}$. Thus by Lemma 18.41, $x \in \bigcup A_n$.

Now suppose that $x \notin C$. If $x \notin [0,1]$, clearly $x \notin \bigcap_{n\in\omega} (\bigcup A_n)$. So suppose that $x \in [0,1]$, and choose $t \in {}^{\omega\backslash 1}3$ such that $x = \sum_{i=1}^{\infty} \frac{t_i}{3^i}$. Choose n minimal such that $t_n = 1$. Suppose $x \in \bigcup A_n$. By Lemma 18.41, choose $s \in {}^{(n+1)\backslash 1}\{0,2\}$ such that

$$\sum_{i=1}^{n} \frac{s_i}{3^i} \leq x \leq \sum_{i=1}^{n} \frac{s_i}{3^i} + \frac{1}{3^n}.$$

We claim that $t \restriction n = s \restriction n$. Otherwise there is a least $m < n$ such that $t_m \neq s_m$. If $t_m < s_m$, then $t_m = 0$ since $t_m \in \{0,2\}$ because $m < n$. Hence

$$x = \sum_{i=1}^{\infty} \frac{t_i}{3^i} = \sum_{i=1}^{m} \frac{t_i}{3^i} + \sum_{i=m+1}^{\infty} \frac{t_i}{3^i} \leq \sum_{i=1}^{m} \frac{t_i}{3^i} + \frac{1}{3^m} < \sum_{i=1}^{m} \frac{s_i}{3^i} \leq \sum_{i=1}^{n} \frac{s_i}{3^i} \leq x,$$

contradiction. If $s_m < t_m$, then $s_m = 0$ and $t_m = 2$, and

$$x \leq \sum_{i=1}^{n} \frac{s_i}{3^i} + \frac{1}{3^n} \leq \sum_{i=1}^{m} \frac{s_i}{3^i} + \sum_{i=m+1}^{\infty} \frac{2}{3^i} = \sum_{i=1}^{m} \frac{s_i}{3^i} + \frac{1}{3^m} < \sum_{i=1}^{m} \frac{t_i}{3^i} \leq x,$$

contradiction.

So $s \upharpoonright n = t \upharpoonright n$.

Case 1. $s_n = 2$. Then

$$x = \sum_{i=1}^{\infty} \frac{t_i}{3^i} = \sum_{i=1}^{n} \frac{t_i}{3^i} + \sum_{i=n+1}^{\infty} \frac{t_i}{3^i} \le \sum_{i=1}^{n} \frac{t_i}{3^i} + \frac{1}{3^n} = \sum_{i=1}^{n} \frac{s_i}{3^i} \le x$$

It follows that $t_i = 2$ for all $i \ge n+1$, and hence $x = (t \upharpoonright n)^\frown\langle 2, 0, 0, 0, \ldots\rangle \in C$, contradiction.

Case 2. $s_n = 0$. Then

$$x = \sum_{i=1}^{\infty} \frac{t_i}{3^i} \le \sum_{i=1}^{n} \frac{s_i}{3^i} + \frac{1}{3^n} = \sum_{i=1}^{n} \frac{t_i}{3^i} \le x;$$

it follows that $t_i = 0$ for all $i > n$; hence $x = (t \upharpoonright n)^\frown\langle 0, 2, 2, 2, \ldots\rangle \in C$, contradiction.

$\square$

Theorem 18.43. *C is homeomorphic to ${}^\omega 2$.*

Proof. For each $t \in {}^\omega 2$ let

$$f(t) = \sum_{i=1}^{\infty} \frac{2t_{i-1}}{3^i}.$$

Clearly f maps onto C. It is one-one; for suppose that $s, t \in {}^\omega 2$ with $s \ne t$. Let n be minimum such that $s_n \ne t_n$. Say $s_n = 0$ and $t_n = 1$. Then

$$\begin{aligned} f(s) = \sum_{i=1}^{\infty} \frac{2s_{i-1}}{3^i} &= \sum_{i=1}^{n} \frac{2s_{i-1}}{3^i} + \sum_{i=n+1}^{\infty} \frac{2s_{i-1}}{3^i} \\ &\le \sum_{i=1}^{n} \frac{2s_{i-1}}{3^i} + \sum_{i=n+1}^{\infty} \frac{2}{3^i} = \sum_{i=1}^{n} \frac{2s_{i-1}}{3^i} + \frac{1}{3^n} < \sum_{i=1}^{n} \frac{2s_{i-1}}{3^i} + \frac{2}{3^n} \le \sum_{i=1}^{\infty} \frac{2t_{i-1}}{3^i} = f(t). \end{aligned}$$

So f is one-one. Now f is continuous. For, suppose that U is open in $\mathbb{R}$ and $t \in f^{-1}[U]$. Say $(f(t) - \varepsilon, f(t) + \varepsilon) \subseteq U$. Choose n so that $3^{-n} < \varepsilon$. Let $V = \{s \in {}^\omega 2 : s \upharpoonright n = t \upharpoonright n\}$. For any $s \in V$ we have

$$|f(s) - f(t)| = \left| \sum_{n+1 \le i}^{\infty} \frac{2(s(i-1) - t(i-1))}{3^i} \right| \le \sum_{n+1 \le i}^{\infty} \frac{2}{3^i} = \frac{1}{3^n} < \varepsilon.$$

Thus $V \subseteq f^{-1}[U]$. So f is continuous. By Engelking Theorem 3.1.13, f is a homeomorphism. $\square$

Ω and ${}^\omega 2$.

If $z \in {}^{\omega}2$ let $z^\circ = \sum_{i=1}^{\infty}(z_{i-1}2^{-i})$. Note that clearly $0 \le z^\circ$. Also, if $z \ne \langle 1, 1, \ldots\rangle$ then $z^\circ < 1$. In fact, choose j such that $z_j = 0$. then

$$\begin{aligned} z^\circ &= \sum_{i=1}^{\infty}(z_{i-1}2^{-i}) = \sum_{i=1}^{j}(z_{i-1}2^{-i}) + \sum_{i=j+2}^{\infty}(z_{i-1}2^{-i}) \\ &\le \sum_{i=1}^{j}(z_{i-1}2^{-i}) + \sum_{i=j+2}^{\infty} 2^{-i} = \sum_{i=1}^{j}(z_{i-1}2^{-i}) + 2^{-j-1} < 1. \end{aligned}$$

For $z \in [0, 1)$ let $z' \in {}^{\omega}2$ be such that $z = \sum_{i=1}^{\infty}(z'_{i-1}2^{-i})$, with z' not eventually 1. Let $\Omega = \{x \in {}^{\omega}2 : x \text{ is not eventually 1 and } x \ne \langle 0, 0, 0, \ldots\rangle\}$. For each $m \in \omega$ and each $f \in {}^{m}2$ let $W_f = \{x \in {}^{\omega}2 : f \subseteq x\}$ and $W'_f = \{x \in \Omega : f \subseteq x\}$. Let $M = \{x \in {}^{\omega}2 : x$ is eventually 1 or $x = \langle 0, 0, 0, \ldots\rangle\}$.

Lemma 18.44. *If $x, y \in {}^{\omega}2$ and neither x nor y is eventually 1, and if $x \ne y$, then $\sum_{i=1}^{\infty}(x_{i-1}2^{-i}) \ne \sum_{i=1}^{\infty}(y_{i-1}2^{-i})$.*

Proof. Suppose that $x, y \in {}^{\omega}2$ and neither is eventually 1, and $x \ne y$. Let j be minimum such that $x_j \ne y_j$. Wlog $x_j = 0$ and $y_j = 1$. Choose $k > j$ such that $x_k = 0$. Then

$$\begin{aligned} x &= \sum_{i=1}^{k}(x_{i-1}2^{-i}) + \sum_{i=k+2}^{\infty} x_{i-1}2^{-i} \le \sum_{i=1}^{k}(x_{i-1}2^{-i}) + \sum_{i=k+2}^{\infty} 2^{-i} \\ &= \sum_{i=1}^{k}(x_{i-1}2^{-i}) + 2^{-k-1} < \sum_{i=1}^{k}(x_{i-1}2^{-i}) + \sum_{i=k+1}^{\infty} 2^{-i} \\ &= \sum_{i=1}^{j}(x_{i-1}2^{-i}) + 2^{-k} \le \sum_{i=1}^{\infty}(y_i 2^{-i}), \end{aligned}$$

so $\sum_{i=1}^{\infty}(x_{i-1}2^{-i}) \ne \sum_{i=1}^{\infty}(y_{i-1}2^{-i})$. $\square$

Lemma 18.45. *(i) If $x \in {}^{\omega}2$ is not eventually 1, then $x^{\circ\prime} = x$.*

(ii) If $x \in [0, 1)$, then $x'^{\circ} = x$.

Proof. (i): Suppose that $x \in {}^{\omega}2$ is not eventually 1. Now $x^{\circ\prime} = z'$, where $x^\circ = \sum_{i=1}^{\infty}(z'_{i-1}2^{-i})$ with z' not eventualy 1; but also $x^\circ = \sum_{i=1}^{\infty}(x_{i-1}2^{-i})$. So $x = z'$ by Lemma 18.44.

(ii): obvious. $\square$

Lemma 18.46. *Ω is dense in ${}^{\omega}2$.*

Proof. Given $m \in \omega$ and $f \in {}^{m}2$, let $x \in W'_f$ be such that x is not eventually 1. $\square$

Lemma 18.47. *If $X \subseteq {}^{\omega}2$ is nowhere dense in ${}^{\omega}2$, then $X \cap \Omega$ is nowhere dense in Ω.*

Proof. Suppose that $X \subseteq {}^{\omega}2$ is nowhere dense in ${}^{\omega}2$. Suppose that W'_f is given. Now ${}^{\omega}2 \backslash \overline{X}$ is dense in ${}^{\omega}2$, so $W_f \backslash \overline{X} \neq \emptyset$. Choose g with $W_g \subseteq W_f \backslash \overline{X}$. Take $x \in W_G \cap \Omega$. Then $x \in W'_f \backslash (\overline{X} \cap \Omega)$. This shows that $X \cap \Omega$ is nowhere dense in Ω. □

Corollary 18.48. *If $X \subseteq {}^{\omega}2$ is meager in ${}^{\omega}2$, then $X \cap \Omega$ is meager in Ω.* □

Lemma 18.49. *If $X \subseteq \Omega$ is nowhere dense in Ω, then X is nowhere dense in ${}^{\omega}2$.*

Proof. Suppose that $X \subseteq \Omega$ is nowhere dense in Ω, We want to show that for any $f \in {}^{<\omega}2$, $W_f \backslash \overline{X} \neq \emptyset$. We have $W'_f \backslash (\overline{X} \cap \Omega) \neq \emptyset$, so $W_f \backslash \overline{X} \neq \emptyset$. □

Corollary 18.50. *If $X \subseteq \Omega$ is meager in Ω, then X is meager in ${}^{\omega}2$.* □

Lemma 18.51. $\mathrm{add}(\mathrm{meager}_{{}^{\omega}2}) = \mathrm{add}(\mathrm{meager}_{\Omega})$.

Proof. First let $\kappa = \mathrm{add}(\mathrm{meager}_{{}^{\omega}2})$ and suppose that $E \in [\mathrm{meager}_{{}^{\omega}2}]^{\kappa}$ with $\bigcup E \notin \mathrm{meager}_{{}^{\omega}2}$. Let $E' = \{A \cap \Omega : A \in E\}$. Then by Corollary 18.48, $E' \subseteq \mathrm{meager}_{\Omega}$, and $|E'| \leq \kappa$. Suppose that $\bigcup E' \in \mathrm{meager}_{\Omega}$. Then by Corollary 18.50, $\bigcup E' \in \mathrm{meager}_{{}^{\omega}2}$. Now M is countable, and $\bigcup E \subseteq \bigcup E' \cup M$, so $\bigcup E \in \mathrm{meager}_{{}^{\omega}2}$, contradiction. Thus $\mathrm{add}(\mathrm{meager}_{\Omega}) \leq \mathrm{add}(\mathrm{meager}_{{}^{\omega}2})$.

Second let $\kappa = \mathrm{add}(\mathrm{meager}_{\Omega})$ and suppose that $E \in [\mathrm{meager}_{\Omega}]^{\kappa}$ with $\bigcup E \notin \mathrm{meager}_{\Omega}$. By Corollary 18.50, $E \in [\mathrm{meager}_{{}^{\omega}2}]^{\kappa}$. Suppose that $\bigcup E \in \mathrm{meager}_{{}^{\omega}2}$. By Corollary 18.48, $\bigcup E = (\bigcup E) \cap \Omega \in \mathrm{meager}_{\Omega}$, contradiction. □

Lemma 18.52. $\mathrm{cov}(\mathrm{meager}_{{}^{\omega}2}) = \mathrm{cov}(\mathrm{meager}_{\Omega})$.

Proof. First let $\kappa = \mathrm{cov}(\mathrm{meager}_{{}^{\omega}2})$ and suppose that $E \in [\mathrm{meager}_{{}^{\omega}2}]^{\kappa}$ with $\bigcup E = {}^{\omega}2$. Let $E' = \{A \cap \Omega : A \in E\}$. Then by Corollary 18.48, $E' \subseteq \mathscr{P}(\mathrm{meager}_{\Omega})$, and $|E'| \leq \kappa$. Clearly $\bigcup E' = \Omega$. So $\mathrm{cov}(\mathrm{meager}_{\Omega}) \leq \kappa$.

Second let $\kappa = \mathrm{cov}(\mathrm{meager}_{\Omega})$ and suppose that $E \in [\mathrm{meager}_{\Omega}]^{\kappa}$ with $\bigcup E = \Omega$. By Corollary 18.50, $E \in [\mathrm{meager}_{{}^{\omega}2}]^{\kappa}$. Then $M \cup \bigcup E = {}^{\omega}2$. Hence $\mathrm{cov}(\mathrm{meager}_{{}^{\omega}2}) = \mathrm{cov}(\mathrm{meager}_{\Omega})$. □

Lemma 18.53. $\mathrm{non}(\mathrm{meager}_{{}^{\omega}2}) = \mathrm{non}(\mathrm{meager}_{\Omega})$.

Proof. First let $\kappa = \mathrm{non}(\mathrm{meager}_{{}^{\omega}2})$ and suppose that $X \in [{}^{\omega}2]^{\kappa}$ with $X \notin \mathrm{meager}_{{}^{\omega}2}$. Then $|X \cap \Omega| \leq \kappa$. Suppose that $X \cap \Omega \in \mathrm{meager}_{\Omega}$. Then by Corollary 18.50, $X \cap \Omega \in \mathrm{meager}_{{}^{\omega}2}$) so $X \subseteq (X \cap \Omega) \cup M \in \mathrm{meager}_{{}^{\omega}2}$, contradiction.

Second let $\kappa = \mathrm{non}(\mathrm{meager}_{\Omega})$ and suppose that $X \in [\Omega]^{\kappa}$ with $X \notin \mathrm{meager}_{\Omega}$. Then $X \in [{}^{\omega}2]^{\kappa}$ and by Corollary 18.48 $X \notin \mathrm{meager}_{{}^{\omega}2}$. □

Lemma 18.54. $\mathrm{cof}(\mathrm{meager}_{{}^{\omega}2}) = \mathrm{cof}(\mathrm{meager}_{\Omega})$.

Proof. First let $\kappa = \mathrm{cof}(\mathrm{meager}_{{}^{\omega}2})$ and suppose that $X \in [\mathrm{meager}_{{}^{\omega}2}]^{\kappa}$ is such that $\forall A \in \mathrm{meager}_{{}^{\omega}2} \exists B \in X[A \subseteq B]$. Let $X' = \{A \cap \Omega : A \in X\}$. So $|X| \leq \kappa$, and by Corollary 18.48, $X' \subseteq \mathscr{P}(\mathrm{meager}_{\Omega})$. Suppose that $A \in \mathrm{meager}_{\Omega}$. Then by Corollary 18.50, $A \in \mathrm{meager}_{{}^{\omega}2}$, so there is a $B \in X$ such that $A \subseteq B$. Then $A \subseteq B \cap \Omega \in X'$. Hence $\mathrm{cof}(\mathrm{meager}_{\Omega}) \leq \kappa$.

Second let $\kappa = \mathrm{cof}(\mathrm{meager}_\Omega)$ and suppose that $X \in [\mathrm{meager}_\Omega]^\kappa$ is such that $\forall A \in \mathrm{meager}_\Omega \exists B \in X[A \subseteq B]$. Let $X' = \{A \cup M : A \in X\}$. So $|X'| \leq \kappa$. Suppose that $A \in \mathrm{meager}_{^\omega 2}$. Then by Corollary 18.48, $A \cap \Omega \in \mathrm{meager}_\Omega$, so there is a $B \in X$ such that $A \cap \Omega \subseteq B$. Then $A \subseteq B \cup M \in X'$. Hence $\mathrm{cof}(\mathrm{meager}_{^\omega 2}) = \mathrm{cof}(\mathrm{meager}_\Omega)$. □

$(0,1)$ and Ω

Lemma 18.55. *If $h \in {}^\omega 2$ is not eventually 1 and $h_m = 0$, then*

$$\sum_{i=1}^{\infty}(h_{i-1}2^{-i}) < \sum_{i=1}^{m}(h_{i-1}2^{-i}) + 2^{-m-1}.$$

Proof. Assume that $h \in {}^\omega 2$ is not eventually 1 and $h_m = 0$. Choose $n > m$ so that $h_n = 0$. Then

$$\begin{aligned}\sum_{i=1}^{\infty}(h_{i-1}2^{-i}) &\leq \sum_{i=1}^{n}(h_{i-1}2^{-i}) + \sum_{i=n+2}^{\infty} 2^{-i}\\ &= \sum_{i=1}^{n}(h_{i-1}2^{-i}) + 2^{-n-1}\\ &< \sum_{i=1}^{n}(h_{i-1}2^{-i}) + 2^{-n-1} + 2^{-n-2}\\ &\leq \sum_{i=1}^{m}(h_{i-1}2^{-i}) + \sum_{i=m+2}^{\infty} 2^{-i}\\ &= \sum_{i=1}^{m}(h_{i-1}2^{-i}) + 2^{-m-1}.\end{aligned}$$

□

If $A \subseteq {}^\omega 2$, we let $A^\circ = \{x^\circ : x \in A\}$; and for $X \subseteq [0,1)$ we let $X^p = \{x' : x \in X\}$.

Lemma 18.56. *If $0 < a < b < 1$, then there is an f such that $(W'_f)^\circ \subseteq (a,b)$.*

Proof. Assume that $0 < a < b < 1$. Let m be minimum such that $a'_m \neq b'_m$. If $a'_m = 1$ and $b'_m = 0$, then

$$\begin{aligned}b = \sum_{i=1}^{\infty}(b'_{i-1}2^{-i}) &\leq \sum_{i=1}^{m}(b'_{i-1}2^{-i}) + \sum_{i=m+2}^{\infty} 2^{-i}\\ &= \sum_{i=1}^{m}(b'_{i-1}2^{-i}) + 2^{-m-1} = \sum_{i=1}^{m+1}(a'_{i-1}2^{-i}) \leq a,\end{aligned}$$

contradiction. Hence $a'_m = 0$ and $b'_m = 1$.

Choose $p > n > m$ such that $a'_n = a'_p = 0$. Let $f = \langle a'_i : i < n\rangle^\frown\langle 1\rangle^\frown\langle a'_i : n+1 \leq i \leq p\rangle$. We claim that $(W'_f)^\circ \subseteq (a,b)$. Take any $g \in W'_f$; we want to show that $a < g^\circ < b$, i.e.,

$$\sum_{i=1}^{\infty}(a'_{i-1}2^{-i}) < \sum_{i=1}^{\infty}(g_{i-1}2^{-i}) < \sum_{i=1}^{\infty}(b'_{i-1}2^{-i}) \tag{1}$$

We have

$$\begin{aligned}
\sum_{i=1}^{\infty}(a'_{i-1}2^{-i}) &< \sum_{i=1}^{n}(a'_{i-1}2^{-i}) + 2^{-n-1} \quad \text{by Lemma 18.55}\\
&= \sum_{i=1}^{n+1}(f_{i-1}2^{-i}) = \sum_{i=1}^{n+1}(g_{i-1}2^{-i}) \leq \sum_{i=1}^{\infty}(g_{i-1}2^{-i})\\
&\leq \sum_{i=1}^{p+1}(g_{i-1}2^{-i}) + \sum_{i=p+2}^{\infty}2^{-i} = \sum_{i=1}^{p+1}(g_{i-1}2^{-i}) + 2^{-p-1}\\
&< \sum_{i=1}^{n+1}(g_{i-1}2^{-i}) + \sum_{i=n+2}^{\infty}2^{-i}\\
&= \sum_{i=1}^{n+1}(g_{i-1}2^{-i}) + 2^{-n-1} = \sum_{i=1}^{n}(a'_{i-1}2^{-i}) + 2^{-n} + 2^{-n-1}\\
&\leq \sum_{i=1}^{m}(b'_{i-1}2^{-i}) + \sum_{i=m+1}^{\infty}2^{-i} = \sum_{i=1}^{m+1}(b'_{i-1}2^{-i}) \leq \sum_{i=1}^{\infty}(b'_{i-1}2^{-i}). \qquad \square
\end{aligned}$$

Lemma 18.57. *For any $f \in {}^{<\omega}2$ there exist a, b with $0 < a < b < 1$ and $(a,b) \subseteq (W'_f)^\circ$.*

Proof. Say $f \in {}^m2$. Let $a = f^\frown\langle 1,0,0,0,\ldots\rangle$ and $b = f^\frown\langle 1,1,0,0,\ldots\rangle$. Clearly $0 < a^\circ < b^\circ < 1$. We claim that $(a^\circ, b^\circ) \subseteq {}^\circ W'_f$. Suppose that $a^\circ < z < b^\circ$. In particular, $z \in (0,1)$ and $z' \in \Omega$. If $a = z'$, then $a^\circ = z'^\circ = z$, contradiction. So $a \neq z'$. Similarly $b \neq z'$. Let n be minimum such that $a_n \neq z'_n$. Suppose that $n < m$.

Subcase 1. $a_n = 0$, $z'_n = 1$. Then

$$\begin{aligned}
b^\circ = \sum_{i=1}^{\infty}(b_{i-1}2^{-i}) &= \sum_{i=1}^{n+1}(a_{i-1}2^{-i}) + \sum_{i=n+2}^{\infty}(b_{i-1}2^{-i})\\
&\leq \sum_{i=1}^{n}(a_{i-1}2^{-i}) + \sum_{i=n+2}^{\infty}2^{-i} = \sum_{i=1}^{n}(a_{i-1}2^{-i}) + 2^{-n-1}\\
&= \sum_{i=1}^{n}(z'_{i-1}2^{-i}) + 2^{-n-1} \leq \sum_{i=1}^{\infty}(z'_{i-1}2^{-i}) = z,
\end{aligned}$$

contradiction.

Subcase 2. $a_n = 1$, $z'_n = 0$. Then

$$z = \sum_{i=1}^{\infty}(z'_{i-1}2^{-i}) \leq \sum_{i=1}^{n}(z'_{i-1}2^{-i}) + \sum_{i=n+2}^{\infty} 2^{-i}$$
$$= \sum_{i=1}^{n}(a_{i-1}2^{-i}) + 2^{-n-1} \leq \sum_{i=1}^{\infty}(a_{i-1}2^{-i}) = a^\circ,$$

contradiction.

It follows that $m \leq n$, hence $f \subseteq z'$ and so $z' \in W'_f$. Thus $z = z'^\circ \in (W'_f)^\circ$. $\square$

Lemma 18.58. *If $A \subseteq \Omega$ is nowhere dense, then A° is nowhere dense in $(0,1)$.*

Proof. Suppose that $A \subseteq \Omega$ is nowhere dense, U is open, $U \cap (0,1) \neq \emptyset$. We want to show that $U\backslash\overline{A^\circ} \neq \emptyset$. Wlog $U = (a,b)$ with $0 \leq a < b \leq 1$. We want to find $0 \leq c < d \leq 1$ such that $(c,d) \subseteq (a,b)\backslash\overline{A^\circ}$. By Lemma 18.56 choose f such that $(W'_f)^\circ \subseteq (a,b)$. Now $W'_f\backslash\overline{A} \neq \emptyset$, so there is a g such that $W'_g \subseteq W'_f\backslash\overline{A}$. By Lemma 18.57 choose $0 < c < d < 1$ so that $(c,d) \subseteq (W'_g)^\circ$. Thus $(c,d) \subseteq (W'_g)^\circ \subseteq (W'_f)^\circ \subseteq (a,b)$. Suppose that $x \in (c,d) \cap \overline{A^\circ}$; we want to get a contradiction. Say $y \in (c,d) \cap A^\circ$. Say $y = z^\circ$ with $z \in A$. But also $y \in (c,d) \subseteq (W'_g)^\circ$, so there is a $w \in W'_g$ such that $y = w^\circ$. Since $W'_g \subseteq W'_f\backslash\overline{A}$, we have $w \notin A$. But $z, w \in \Omega$, so $z = z^{\circ\prime} = y' = w^{\circ\prime} = w$, contradiction. $\square$

Corollary 18.59. *If $A \subseteq \Omega$ is meager, then A° is meager in $(0,1)$.* $\square$

Lemma 18.60. *If $A \subseteq (0,1)$ is nowhere dense in $(0,1)$, then A^p is nowhere dense in Ω.*

Proof. Assume that $A \subseteq (0,1)$ is nowhere dense in $(0,1)$ Let W_f be given; we want to show that $W_f \cap \Omega\backslash\overline{A^p} \neq \emptyset$. By Lemma 18.57 choose $0 < a < b < 1$ so that $(a,b) \subseteq (W'_f)^\circ$. Now $(a,b)\backslash\overline{A}$ is dense, hence nonempty and open. Choose $0 < c < d < 1$ with $(c,d) \subseteq (a,b)\backslash\overline{A}$. By Lemma 18.56 choose W'_g such that $(W'_g)^\circ \subseteq (c,d)$. So $(W'_g)^\circ \subseteq (W'_f)^\circ$, and hence $W'_g \subseteq W'_f$. Suppose that $W'_g \subseteq \overline{A^p}$; we want to get a contradiction. Then $W'_g \cap A^p \neq \emptyset$. Say $x \in W'_g \cap A_p$. Choose $y \in A$ with $x = y'$. Then $y = x^\circ \in (W'_g)^\circ \subseteq (c,d) \subseteq (a,b)\backslash\overline{A}$, contradiction. $\square$

Corollary 18.61. *If $A \subseteq (0,1)$ is meager in $(0,1)$. then A^p is meager in Ω.*

Lemma 18.62. $\text{add}(\text{meager}_\Omega) = \text{add}(\text{meager}_{(0,1)})$.

Proof. First let $\kappa = \text{add}(\text{meager}_\Omega)$ and suppose that $E \in [\text{meager}_\Omega]^\kappa$ with $\bigcup E \notin \text{meager}_\Omega$. Let $E' = \{A^\circ : A \in E\}$. So by Corollary 18.59, $E' \in \mathscr{P}(\text{meager}_{(0,1)})$. Clearly $|E'| \leq \kappa$. Suppose that $\bigcup E' \in \text{meager}_{(0,1)}$. By Corollary 18.61, $(\bigcup E')^p \in \text{meager}_\Omega$. Take any $A \in E$. Then $A^\circ \in E'$, so $A^\circ \subseteq \bigcup E'$. Hence $A = A^{\circ p} \subseteq (\bigcup E')^p$. Thus $\bigcup E \subseteq (\bigcup E')^p \in \text{meager}_\Omega$, so $\bigcup E \in \text{meager}_\Omega$, contradiction.

Second let $\kappa = \text{add}(\text{meager}_{(0,1)})$ and suppose that $E \in [\text{meager}_{(0,1)}]^\kappa$ with $\bigcup E \notin \text{meager}_{(0,1)}$. Let $E' = \{A^p : A \in E\}$. So by Corollary 18.61, $E' \subseteq \mathscr{P}(\text{meager}_\Omega)$. Suppose that $\bigcup E' \in \text{meager}_\Omega$. By Corollary 18.59, $(\bigcup E')^\circ \in \text{meager}_{(0,1)}$. If $A \in E$, then $A^p \in E'$, so $A^p \subseteq \bigcup E'$, hence $A = A^{p\circ} \subseteq (\bigcup E')^\circ$, so $\bigcup E \subseteq (\bigcup E')^\circ$, contradiction. $\square$

Lemma 18.63. $\text{cov}(\text{meager}_\Omega) = \text{cov}(\text{meager}_{(0,1)})$.

Proof. First let $\kappa = \text{cov}(\text{meager}_\Omega)$ and suppose that $E \in [\text{meager}_\Omega]^\kappa$ with $\Omega = \bigcup E$. Let $E' = \{A^\circ : A \in E\}$. Then by Corollary 18.59, $E' \subseteq \mathscr{P}(\text{meager}_{(0,1)})$. If $a \in (0,1)$, then $a' \in \Omega$, hence there is an $A \in E$ such that $a' \in A$. So $a = a^{p\circ} \in {}^\circ[A] \in E'$. Thus $(0,1) \subseteq \bigcup E'$.

Seecond let $\kappa = \text{cov}(\text{meager}_{(0,1)})$ and suppose that $E \in [\text{meager}_{(0,1)}]^\kappa$ with $(0,1) = \bigcup E$. Let $E' = \{A^p : A \in E\}$. Then $E' \subseteq \mathscr{P}(\text{meager}_\Omega)$ by Corollary 18.61. Suppose that $x \in \Omega$. Then $x^\circ \in (0,1)$, so there is an $A \in E$ such that $x^\circ \in A$. Hence $x = x^{\circ p} \in A^p \in E'$. This shows that $\bigcup E' = \Omega$. $\square$

Lemma 18.64. $\text{non}(\text{meager}_\Omega) = \text{non}(\text{meager}_{(0,1)})$.

Proof. First let $\kappa = \text{non}(\text{meager}_\Omega)$ and suppose that $X \in [\Omega]^\kappa$ such that $X \notin \text{meager}_\Omega$. Suppose that X° is meager in $(0,1)$. Then $X = X^{\circ p}$ is meager in Ω by Corollary 18.61, contradiction. It follows that $\text{non}(\text{meager}_{(0,1)}) \leq \kappa$.

Second let $\kappa = \text{non}(\text{meager}_{(0,1)})$ and suppose that $X \in [(0,1)]^\kappa$ such that $X \notin \text{meager}_{(0,1)}$. Suppose that $X^p \in \text{meager}_\Omega$. Then $X = X^{p\circ} \in \text{meager}_{(0,1)}$ by Lemma 18.59, contradiction. Hence $\text{non}(\text{meager}_\Omega) = \text{non}(\text{meager}_{(0,1)})$. $\square$

Lemma 18.65. $\text{cof}(\text{meager}_\Omega) = \text{cof}(\text{meager}_{(0,1)})$.

Proof. First let $\kappa = \text{cof}(\text{meager}_\Omega)$ and suppose that $X \in [\text{meager}_\Omega]^\kappa$ is such that $\forall A \in \text{meager}_\Omega \exists B \in X[A \subseteq B]$. Let $X' = \{A^\circ : A \in X\}$. Then by Corollary 18.59, $X' \subseteq \mathscr{P}(\text{meager}_{(0,1)})$, and clearly $|X'| \leq \kappa$. Suppose that $A \in \text{meager}_{(0,1)}$. Then by Corollary 18.61, $A^p \in \text{meager}_\Omega$. Hence there is a $B \in X$ such that $A^p \subseteq B$. Then $A = A^p \circ \subseteq {}^\circ[B] \in X'$. It follows that $\text{cof}(\text{meager}_{(0,1)}) \leq \kappa$.

Second let $\kappa = \text{cof}(\text{meager}_{(0,1)})$ and suppose that $X \in [\text{meager}_{(0,1)}]^\kappa$ is such that $\forall A \in \text{meager}_{(0,1)} \exists B \in X[A \subseteq B]$. Let $X' = \{A^p : A \in X\}$. Then by Corollary 18.61, $X' \subseteq \mathscr{P}(\text{meager}_\Omega)$, and clearly $|X'| \leq \kappa$. Suppose that $A \in \text{meager}_\Omega$. Then $A^\circ \in \text{meager}_{(0,1)}$ by Corollary 18.59, so there is a $B \in X$ such that $A^\circ \subseteq B$. Then $A = A^{\circ p} \subseteq {}'[B] \in X'$. It follows that $\text{cof}(\text{meager}_\Omega) = \text{cof}(\text{meager}_{(0,1)})$. $\square$

For finite disjoint $F, G \subseteq \omega$ we define $U_{FG} = \{X \subseteq \omega : F \subseteq X \text{ and } X \cap G = \emptyset\}$.

Lemma 18.66. *$\{U_{FG} : F, G \in [\omega]^{<\omega}$ and $F \cap G = \emptyset\}$ forms a basis for a topology on $\mathscr{P}(\omega)$, and χ is a homeomorphism from ${}^\omega 2$ onto $\mathscr{P}(\omega)$.* $\square$

Let $\Theta = \{x \in {}^\omega 2 : \{i \in \omega : x(i) = 1\} \text{ is infinite}\}$.

Lemma 18.67. *Θ is dense in ${}^\omega 2$, and if $X \subseteq {}^\omega 2$ is nowhere dense, then $X \cap \Theta$ is nowhere dense in Θ.*

Proof. Clearly Θ is dense in ${}^\omega 2$. Now ${}^\omega 2 \backslash \overline{X}$ is nonempty and open, so there is a $W_f \subseteq {}^\omega 2 \backslash \overline{X}$. Then $W_f \cap \Theta \subseteq \Theta \backslash (\overline{X} \cap \Theta)$. $\square$

Lemma 18.68. *If $X \subseteq {}^{\omega}2$ is meager$_{^\omega 2}$, then $X \cap \Theta$ is meager in Θ.* $\square$

Lemma 18.69. *If $X \subseteq \Theta$ is nowhere dense in Θ, then X is nowhere dense in ${}^{\omega}2$.*

Proof. Let U be a nonempty open set in ${}^{\omega}2$. We want to show that $U \cap ({}^{\omega}2 \backslash \overline{X}) \neq \emptyset$. Now $U \cap \Theta \neq \emptyset$, and $\overline{X} \cap \Theta$ is the closure of X in Θ. Hence

$$\emptyset \neq U \cap \Theta \cap (\Theta \backslash (\overline{X} \cap \Theta)) = U \cap \Theta \backslash \overline{X} \subseteq U \cap ({}^{\omega}2 \backslash \overline{X}),$$

as desired. $\square$

Lemma 18.70. *If $X \subseteq \Theta$ is meager in Θ, then X is meager in ${}^{\omega}2$.* $\square$

Lemma 18.71. $\mathrm{add}(\mathrm{meager}_{^\omega 2}) = \mathrm{add}(\mathrm{meager}_\Theta)$.

Proof. Suppose that $E \in [\mathrm{meager}_{^\omega 2}]^\kappa$ and $\bigcup E \notin \mathrm{meager}_{^\omega 2}$, with $\kappa = \mathrm{add}(\mathrm{meager}_{^\omega 2})$. Let $E' = \{X \cap \Theta : X \in E\}$. Then $E' \subseteq \mathrm{meager}_\Theta$ by Lemma 18.68. Suppose that $\bigcup E' \in \mathrm{meager}_\Theta$. By Lemma 18.70, $\bigcup E' \in \mathrm{meager}_{^\omega 2}$. Let $M = \{x \in {}^{\omega}2 : \{i \in I : x(i) = 1\}$ is finite$\}$. Then M is countable, and $\bigcup E \subseteq \bigcup E' \cup M$, so $\bigcup E$ is meager, contradiction.

Second suppose that $E \in [\mathrm{meager}_\Theta]^\kappa$ and $\bigcup E \notin \mathrm{meager}_\Theta$, with $\kappa = \mathrm{add}(\mathrm{meager}_\Theta)$. By Lemma 18.70, $E \subseteq \mathrm{meager}_{^\omega 2}$. Suppose that $\bigcup E \in \mathrm{meager}_{^\omega 2}$. By Lemma 18.68, $\bigcup E \in \mathrm{meager}_\Theta$, contradiction. $\square$

Lemma 18.72. $\mathrm{cov}(\mathrm{meager}_{^\omega 2}) = \mathrm{cov}(\mathrm{meager}_\Theta)$.

Proof. First suppose that $E \in [\mathrm{meager}_{^\omega 2}]^\kappa$ and ${}^{\omega}2 = \bigcup E$, with $\kappa = \mathrm{cov}(\mathrm{meager}_{^\omega 2})$. Let $E' = \{X \cap \Theta : X \in E\}$. Then $E' \subseteq \mathrm{meager}_\Theta$ by Lemma 18.68. We have $\bigcup E' = (\bigcup E) \cap \Theta = {}^{\omega}2 \cap \Theta = \Theta$.

Second suppose that $E \in [\mathrm{meager}_\Theta]^\kappa$ and $\bigcup E = \Theta$, $\kappa = \mathrm{cov}(\mathrm{meager}_\Theta)$. Let $M = \{x \in {}^{\omega}2 : \{i \in I : x(i) = 1\}$ is finite$\}$. Then M is countable, hence meager in ${}^{\omega}2$, and $\bigcup E \cup M = {}^{\omega}2$. $\square$

Lemma 18.73. $\mathrm{non}(\mathrm{meager}_{^\omega 2}) = \mathrm{non}(\mathrm{meager}_\Theta)$.

Proof. First suppose that $X \in [{}^{\omega}2]^\kappa$ with $X \notin \mathrm{meager}_{^\omega 2}$ and $|X| = \mathrm{non}(\mathrm{meager}_{^\omega 2})$. If $X \cap \Theta \in \mathrm{meager}_\Theta$, then $X \subseteq (X \cap \Theta) \cup M \in \mathrm{meager}_{^\omega 2}$ by Lemma 18.70, with M as above, contradiction.

Second suppose that $X \in [\Theta]^\kappa$ with $X \notin \mathrm{meager}_\Theta$ and $|X| = \mathrm{non}(\mathrm{meager}_\Theta)$. Then by Lemma 18.68, X is not meager in ${}^{\omega}2$. $\square$

Lemma 18.74. $\mathrm{cof}(\mathrm{meager}_{^\omega 2}) = \mathrm{cof}(\mathrm{meager}_\Theta)$.

Proof. First suppose that $X \in [\mathrm{meager}_{^\omega 2}]^\kappa$ such that $\forall A \in \mathrm{meager}_{^\omega 2} \exists B \in X[A \subseteq B]$, with $\kappa = \mathrm{cof}(\mathrm{meager}_{^\omega 2})$. Let $Y = \{B \cap \Theta : B \in X\}$. Then $Y \subseteq \mathrm{meager}_\Theta$ by Lemma 18.68. Suppose that $A \in \mathrm{meager}_\Theta$. Then $A \in \mathrm{meager}_{^\omega 2}$ by Lemma 18.70. Hence there is a $B \in X$ such that $A \subseteq B$. So $B \cap \Theta \in Y$ and $A \subseteq B \cap \Theta$.

Second suppose that $X \in [\mathrm{meager}_\Theta]^\kappa$ such that $\forall A \in \mathrm{meager}_\Theta \exists B \in X[A \subseteq B]$, with $\kappa = \mathrm{cof}(\mathrm{meager}_\Theta)$. Let M be as above. Let $Y = \{B \cup M : B \in X\}$. Then $Y \subseteq \mathrm{meager}_{^\omega 2}$

by Lemma 18.70. Suppose that $A \in \text{meager}_{^\omega 2}$. Then $A \cap \Theta \in \text{meager}_\Theta$ by Lemma 18.68. Choose $B \in X$ such that $A \cap \Theta \subseteq B$. Then $A \subseteq B \cup M \in Y$. □

Lemma 18.75. *With the relative topology on $[\omega]^\omega$, Θ is homeomorphic to $[\omega]^\omega$.* □

measures

We give background on measures, and prove that add, non, cov, and cof are the same applied to null sets in the sense of $[0,1]$, ${}^\omega 2$, Θ, or $[\omega]^\omega$.

If A is a σ-algebra of subsets of X, then a *measure* on A is a function $\mu : A \to [0,\infty]$ such that $\mu(\emptyset) = 0$ and $\mu(\bigcup_{i\in\omega} a_i) = \sum_{i\in\omega} \mu(a_i)$ if $a \in {}^\omega A$ and $a_i \cap a_j = \emptyset$ for all $i \neq j$. Note that $a_i = \emptyset$ is possible for some $i \in \omega$.

We give some important properties of measures:

Proposition 18.76. *Suppose that μ is a measure on a σ-algebra A of subsets of X. Then:*

(i) If $Y, Z \in A$ and $Y \subseteq Z$, then $\mu(Y) \leq \mu(Z)$.

(ii) If $Y \in {}^\omega A$, then $\mu(\bigcup_{n\in\omega} Y_n) \leq \sum_{n\in\omega} \mu(Y_n)$.

(iii) If $Y \in {}^\omega A$ and $Y_n \subseteq Y_{n+1}$ for all $n \in \omega$, then $\mu(\bigcup_{n\in\omega} Y_n) = \sup_{n\in\omega} \mu(Y_n)$.

(iv) If $Y \in {}^\omega A$ and $\mu(Y_0) < \infty$ and $Y_n \supseteq Y_{n+1}$ for all $n \in \omega$, then $\mu(\bigcap_{n\in\omega} Y_n) = \inf_{n\in\omega} \mu(Y_n)$.

Proof. (i): We have $\mu(Z) = \mu(Y) + \mu(Z\backslash Y) \geq \mu(Y)$.

(ii): Let $Z_n = Y_n \backslash \bigcup_{m<n} Y_m$. By induction, $\bigcup_{m\leq n} Z_m = \bigcup_{m\leq n} Y_m$, and hence $\bigcup_{m\in\omega} Z_m = \bigcup_{m\in\omega} Y_m$. Now

$$\mu\left(\bigcup_{m\in\omega} Y_m\right) = \mu\left(\bigcup_{m\in\omega} Z_m\right) = \sum_{m\in\omega} \mu(Z_m) \leq \sum_{m\in\omega} \mu(Y_m).$$

(iii): Again let $Z_n = Y_n \backslash \bigcup_{m<n} Y_m$. By induction, $Y_n = \bigcup_{m\leq n} Z_m$. Hence

$$\begin{aligned}
\mu\left(\bigcup_{n\in\omega} Y_n\right) &= \mu\left(\bigcup_{n\in\omega} Z_n\right) \\
&= \sum_{n\in\omega} \mu(Z_n) \\
&= \lim_{n\to\infty} \sum_{m\leq n} \mu(Z_m) \\
&= \lim_{n\to\infty} \mu\left(\bigcup_{m\leq n} Z_m\right) \\
&= \lim_{n\to\infty} \mu(Y_n) \\
&= \sup_{n\in\omega} \mu(Y_n).
\end{aligned}$$

(iv): $Y_0\backslash Y_n \subseteq Y_0\backslash Y_{n+1}$ for all n, so, by (iii),

$$\mu\left(\bigcup_{n\in\omega}(Y_0\backslash Y_n)\right) = \sup_{n\in\omega}\mu(Y_0\backslash Y_n).$$

Hence

$$\mu(Y_0) = \mu\left(Y_0\backslash\bigcap_{n\in\omega}Y_n\right) + \mu\left(\bigcap_{n\in\omega}Y_n\right),$$

so

$$\mu\left(\bigcap_{n\in\omega}Y_n\right) = \mu(Y_0) - \mu\left(Y_0\backslash\bigcap_{n\in\omega}Y_n\right) = \mu(Y_0) - \sup_{n\in\omega}\mu(Y_0\backslash Y_n).$$

Now for any $n\in\omega$, $\mu(Y_0) = \mu(Y_0\backslash Y_n) + \mu(Y_n)$, and hence

$$\mu(Y_0) - \sup_{n\in\omega}\mu(Y_0\backslash Y_n) \le \mu(Y_0) - \mu(Y_0\backslash Y_n) = \mu(Y_n).$$

Also, if $x \le \mu(Y_n)$ for all n, then $x \le \mu(Y_0) - \mu(Y_0\backslash Y_n)$, hence $\mu(Y_0\backslash Y_n) \le \mu(Y_0) - x$ for all n, so $\sup_{n\in\omega}\mu(Y_0\backslash Y_n) \le \mu(Y_0) - x$, and so $x \le \mu(Y_0) - \sup_{n\in\omega}\mu(Y_0\backslash Y_n)$. This proves (iv). □

measure spaces and outer measures

A *measure space* is a triple (X, Σ, μ) such that:

(1) X is a set

(2) Σ is a σ-algebra of subsets of X.

(3) μ is a measure on Σ.

Given a measure space as above, a subset A of X is a μ-*null* set iff there is an $E\in\Sigma$ such that $A\subseteq E$ and $\mu(E) = 0$.

Theorem 18.77. *If (X, Σ, μ) is a measure space, then the collection of μ-null sets is a σ-ideal of subsets of X.*

Proof. Let I be the collection of all μ-null sets. Clearly $\emptyset\in I$, and $B\subseteq A\in I$ implies that $B\in I$. Now suppose that $\langle A_i : i\in\omega\rangle$ is a system of members of I. For each $i\in\omega$ choose $E_i\in\Sigma$ such that $A_i\subseteq E_i$ and $\mu(E_i) = 0$. Then $\bigcup_{i\in I}A_i\subseteq\bigcup_{i\in I}E_i$, and

$$\mu\left(\bigcup_{i\in\omega}E_i\right) \le \sum_{i\in\omega}\mu(E_i) = 0.$$ □

An *outer measure* on a set X is a function $\mu : \mathscr{P}(X)\to[0,\infty]$ satisfying the following conditions:

(1) $\mu(\emptyset) = 0$.

(2) If $A \subseteq B \subseteq X$, then $\mu(A) \le \mu(B)$.

(3) For every $A \in {}^{\omega}\mathscr{P}(X)$, $\mu(\bigcup_{n\in\omega} A_n) \le \sum_{n\in\omega} \mu(A_n)$.

If θ is an outer measure on a set X, then a subset E of X is *θ-measurable* iff for every $A \subseteq X$,

$$\theta(A) = \theta(A \cap E) + \theta(A \backslash E).$$

Note that every subset $E \subseteq X$ such that $\theta(E) = 0$ is automatically θ-measurable.

Theorem 18.78. *Let θ be an outer measure on a set X. Let Σ be the collection of all θ-measurable subsets of X. Then $(X, \Sigma, \theta \restriction \Sigma)$ is a measure space. Moreover, if $E \subseteq X$ and $\theta(E) = 0$, then $E \in \Sigma$.*

Proof. Note that Σ is obviously closed under complementation. Obviously

(1) If $A, E \subseteq X$, then $\theta(A) \le \theta(A \cap E) + \theta(A \backslash E)$.

Clearly $\emptyset \in \Sigma$ and Σ is closed under complements. Next we show that Σ is closed under $\cup$. Suppose that $E, F \in \Sigma$ and $A \subseteq X$. Then

$$\begin{aligned}
\theta(A \cap (E \cup F)) + \theta(A \backslash (E \cup F)) &\le \theta((A \cap (E \cup F) \cap E)) + \theta(A \cap (E \cup F) \backslash E))) \\
&\quad + \theta(A \backslash (E \cup F)) \\
&= \theta(A \cap E) + \theta((A \backslash E) \cap F) + \theta((A \backslash E) \backslash F) \\
&= \theta(A \cap E) + \theta(A \backslash E) \\
&= \theta(A) \\
&\le \theta(A \cap (E \cup F)) + \theta(A \backslash (E \cup F)) \quad \text{by (1).}
\end{aligned}$$

This proves that $E \cup F \in \Sigma$. Thus we have shown that Σ is a field of subsets of X.

Next we show that Σ is closed under countable unions. So, suppose that $E \in {}^{\omega}\Sigma$, and let $K = \bigcup_{n\in\omega} E_n$. For every $m \in \omega$ let

$$G_m = \bigcup_{n \le m} E_n.$$

Then clearly each G_m is in Σ. Now we define $F_0 = G_0$, and for $m > 0$, $F_m = G_m \backslash G_{m-1}$. Then also each F_m is in Σ. By induction, $\bigcup_{n\le m} F_n = G_m$. Hence $\bigcup_{n\in\omega} F_n = \bigcup_{n\in\omega} E_n$. Now temporarily fix a positive integer n and an $A \subseteq X$. Then

$$\theta(A \cap G_n) = \theta(A \cap G_n \cap G_{n-1}) + \theta(A \cap G_n \backslash G_{n-1}) = \theta(A \cap G_{n-1}) + \theta(A \cap F_n);$$

hence by induction $\theta(A \cap G_n) = \sum_{m\le n} \theta(A \cap F_m)$.

Now we unfix n. Now $A \cap K = \bigcup_{n\in\omega}(A \cap F_n)$, so

$$\theta(A \cap K) \le \sum_{n\in\omega} \theta(A \cap F_n) = \lim_{n\to\infty} \sum_{m\le n} \theta(A \cap F_m) = \lim_{n\to\infty} \theta(A \cap G_m).$$

Also, note that if $m<n$ then $G_m\subseteq G_n$, hence $X\backslash G_n\subseteq X\backslash G_m$, and so

$$\theta(A\backslash K)=\theta\left(A\backslash\bigcup_{n\in\omega}G_n\right)=\theta\left(\bigcap_{n\in\omega}(A\backslash G_n)\right)\le\inf_{n\in\omega}\theta(A\backslash G_n)=\lim_{n\to\infty}\theta(A\backslash G_n).$$

Hence

$$\begin{aligned}\theta(A\cap K)+\theta(A\backslash K)&\le\lim_{n\to\infty}\theta(A\cap G_n)+\lim_{n\to\infty}\theta(A\backslash G_n)\\&=\lim_{n\to\infty}(\theta(A\cap G_n)+\theta(A\backslash G_n))\\&=\theta(A)\\&\le\theta(A\cap K)+\theta(A\backslash K).\end{aligned}$$

This proves that $K\in\Sigma$, so that Σ is closed under countable unions.

Finally, suppose that $\langle E_n:n\in\omega\rangle$ is a system of pairwise disjoint members of Σ. Let $K=\bigcup_{n\in\omega}E_n$. Hence $\theta(K)\le\sum_{n\in\omega}\theta(E_n)$. Conversely, for each $n\in\omega$ let $G_n=\bigcup_{m\le n}E_m$. Then

$$\theta(G_{n+1})=\theta(G_{n+1}\cap E_{n+1})+\theta(G_{n+1}\backslash E_{n+1})=\theta(E_{n+1})+\theta(G_n).$$

Hence by induction, $\theta(G_n)=\sum_{m\le n}\theta(E_m)$ for every n, and hence

$$\theta(K)\ge\theta(G_n)=\sum_{m\le n}\theta(E_m),$$

and so $\theta(K)\ge\sum_{n\in\omega}\theta(E_n)$.

For the "moreover" statement, suppose that $E\subseteq X$ and $\theta(E)=0$, Then for any $A\subseteq X$, $\theta(A)\le\theta(A\cap E)+\theta(A\backslash E)=\theta(A\backslash E)\le\theta(A)$. □

measure on $^\kappa 2$

Let κ be an infinite cardinal. For each $f\in\mathrm{Fn}(\kappa,2,\omega)$ let $U_f=\{g\in{}^\kappa 2:f\subseteq g\}$. Hence $U_\emptyset={}^\kappa 2$. Note that the function taking f to U_f is one-one. For each $f\in\mathrm{Fn}(\kappa,2,\omega)$ let $\theta_0(U_f)=1/2^{|\mathrm{dmn}(f)|}$. Thus $\theta_0(U_\emptyset)=1$. Let $\mathcal{C}=\{U_f:f\in\mathrm{Fn}(\kappa,2,\omega)\}$. Note that ${}^\kappa 2\in\mathcal{C}$. For any $A\subseteq{}^\kappa 2$ let

$$\theta(A)=\inf\left\{\sum_{n\in\omega}\theta_0(C_n):C\in{}^\omega\mathcal{C}\text{ and }A\subseteq\bigcup_{n\in\omega}C_n\right\}.$$

Proposition 18.79. *θ is an outer measure on $^\kappa 2$.*

Proof. For (1), for any $m\in\omega$ let $f\in\mathrm{Fn}(\kappa,2,\omega)$ have domain of size m. Then $\emptyset\subseteq U_f$ and $\theta_0(U_f)=\frac{1}{2^m}$. Hence $\theta(\emptyset)=0$.

For (2), if $A \subseteq B \subseteq {}^{\kappa}2$, then

$$\left\{C \in {}^{\omega}\mathcal{C} : B \subseteq \bigcup_{n\in\omega} C_n\right\} \subseteq \left\{C \in {}^{\omega}\mathcal{C} : A \subseteq \bigcup_{n\in\omega} C_n\right\},$$

and hence $\mu(A) \leq \mu(B)$.

For (3), assume that $A \in {}^{\omega}\mathscr{P}({}^{\kappa}2)$. We may assume that $\sum_{n\in\omega} \theta(A_n) < \infty$. Let $\varepsilon > 0$; we show that $\theta(\bigcup_{n\in\omega} A_n) \leq \sum_{n\in\omega} \theta(A_n) + \varepsilon$, and the arbitrariness of ε then gives the desired result. For each $n \in \omega$ choose $C^n \in {}^{\omega}\mathcal{C}$ such that $A_n \subseteq \bigcup_{m\in\omega} C^n_m$ and $\sum_{m\in\omega} \theta_0(C^n_m) \leq \theta(A_n) + \frac{\varepsilon}{2^n}$. Then $\bigcup_{n\in\omega} A_n \subseteq \bigcup_{n\in\omega} \bigcup_{m\in\omega} C^n_m$ and

$$\theta\left(\bigcup_{n\in\omega} A_n\right) \leq \sum_{n\in\omega}\sum_{n\in\omega} \theta_0(C^n_m) \leq \sum_{n\in\omega} \theta(A_n) + \varepsilon,$$

as desired. $\square$

Let Σ_0 be the set of all θ-measurable subsets of ${}^{\omega}2$.

Proposition 18.80. *If $\varepsilon \in 2$ and $\alpha < \kappa$, then $\{f \in {}^{\kappa}2 : f(\alpha) = \varepsilon\} \in \Sigma_0$.*

Proof. Let $E = \{f \in {}^{\kappa}2 : f(\alpha) = \varepsilon\}$, and let $X \subseteq {}^{\kappa}2$; we want to show that $\theta(X) = \theta(X \cap E) + \theta(X \backslash E)$. $\leq$ holds by the definition of outer measure. Now suppose that $\delta > 0$. Choose $C \in {}^{\omega}\mathcal{C}$ such that $X \subseteq \bigcup_{n\in\omega} C_n$ and $\sum_{n\in\omega} \theta_0(C_n) < \theta(X) + \delta$. For each $n \in \omega$ let $C_n = U_{f_n}$ with $f_n \in \mathrm{Fn}(\kappa, 2, \omega)$. For each $n \in \omega$, if $\alpha \notin \mathrm{dmn}(f_n)$, replace C_n by U_g and U_h, where $g = f_n \cup \{(\alpha, 0)\}$ and $h = f_n \cup \{(\alpha, 1)\}$; let the new sequence be $C' \in {}^{\omega}\mathcal{C}$. Note that

$$\theta_0(C_n) = \theta_0(U_{f_n}) = \frac{1}{2^{|\mathrm{dmn}(f_n)|}} = \theta_0(U_g) + \theta_0(U_h).$$

Then $\sum_{n\in\omega} \theta(C_n) = \sum_{n\in\omega} \theta(C'_n)$ and $X \subseteq \bigcup_{n\in\omega} C'_n$. Say $C'_n = U_{g_n}$ for each $n \in \omega$. Note that $\alpha \in \mathrm{dmn}(g_n)$ for each $n \in \omega$. Let $M = \{n \in \omega : g_n(\alpha) = \varepsilon\}$ and $N = \{n \in \omega : g_n(\alpha) = 1 - \varepsilon\}$. Then M, N is a partition of ω such that $X \cap E \subseteq \bigcup_{n\in M} C'_n$ and $X \backslash E \subseteq \bigcup_{n\in N} C'_n$. Hence

$$\theta(X \cap E) + \theta(X \backslash E) \leq \sum_{n\in M} \theta(C'_n) + \sum_{n\in N} \theta(C'_n) = \sum_{n\in\omega} \theta(C'_n) < \theta(X) + \delta.$$

Since δ is arbitrary, it follows that $\theta(X) = \theta(X \cap E) + \theta(X \backslash E)$. $\square$

For $f : 2 \to \mathbb{R}$ we define $\int f = \frac{1}{2} f(0) + \frac{1}{2} f(1)$.

Proposition 18.81. *If $f_n : 2 \to [0, \infty)$ for each $n \in \omega$ and $\forall t < 2[\sum_{n\in\omega} f_n(t) < \infty]$, then $\sum_{n\in\omega} \int f_n < \infty$, and $\sum_{n\in\omega} \int f_n = \int \sum_{n\in\omega} f_n$.*

Proof.

$$\int \sum_{n\in\omega} f_n = \frac{1}{2}\sum_{n\in\omega} f_n(0) + \frac{1}{2}\sum_{n\in\omega} f_n(1) = \sum_{n\in\omega}\left(\frac{1}{2} f_n(0) + \frac{1}{2} f_n(1)\right) = \sum_{n\in\omega}\int f_n.$$

Proposition 18.82. $\theta({}^\kappa 2) = 1$.

Proof. It is obvious that ${}^\kappa 2 \in \Sigma_0$, and that $\theta({}^\kappa 2) \le \theta_0({}^\kappa 2) = 1$. Suppose that $\theta({}^\kappa 2) < 1$. Choose $C \in {}^\omega\mathcal{C}$ such that $2^\kappa = \bigcup_{n\in\omega} C_n$ and $\sum_{n\in\omega} \theta_0(C_n) < 1$, with C one-one. For each $n \in \omega$ let $C_n = U_{f_n}$, where $f_n \in \mathrm{Fn}(\kappa, 2, \omega)$.

(1) $\forall g \in \mathrm{Fn}(\kappa, 2, \omega)\exists n \in \omega[f_n \subseteq g \text{ or } g \subseteq f_n]$.

In fact, let $g \in \mathrm{Fn}(\kappa, 2, \omega)$. Let $h \in {}^\kappa 2$ with $g \subseteq h$. Choose n such that $h \in C_n$. Then $f_n \subseteq h$. So $f_n \subseteq g$ or $g \subseteq f_n$.

(2) Let $M = \{n \in \omega : \forall m \ne n[f_m \not\subseteq f_n]\}$. Then ${}^\kappa 2 \subseteq \bigcup_{n\in M} U_{f_n}$.

For, given $g \in {}^\kappa 2$ choose $m \in \omega$ such that $g \in C_m$. Thus $f_m \subseteq g$. Let $n \in \omega$ with $f_n \subseteq f_m$ and $|\mathrm{dmn}(f_n)|$ minimum. Then $f_n \subseteq g$ and $n \in M$, as desired.

(3) $|M| \ge 2$.

In fact, obviously $M \ne \emptyset$. Suppose that $M = \{n\}$. Since $\sum_{n\in M} \theta_0(C_n) < 1$, we have $f_n \ne \emptyset$. Then ${}^\kappa 2 \subseteq U_{f_n}$, contradiction.

(4) M is infinite.

In fact, suppose that M is finite, and let $m = \sup\{|\mathrm{dmn}(f_n)| : n \in M\}$. Let $g \in \mathrm{Fn}(\kappa, 2, \omega)$ be such that $|\mathrm{dmn}(g)| = m+1$. Then by (1), $f_n \subseteq g$ for all $n \in M$. By (3), this contradicts the definition of M.

Let $J = \bigcup_{n\in M} \mathrm{dmn}(f_n)$.

(5) J is infinite.

For, suppose that J is finite. Now $M = \bigcup_{G\subseteq J}\{n \in M : \mathrm{dmn}(f_n) = G\}$, so there is a $G \subseteq J$ such that $\{n \in M : \mathrm{dmn}(f_n) = G\}$ is infinite. But clearly $|\{n \in M : \mathrm{dmn}(f_n) = G\}| \le 2^{|G|}$, contradiction.

Let $i : \omega \to J$ be a bijection. For $n, k \in \omega$ let f'_{nk} be the restriction of f_n to the domain $\{\alpha \in \mathrm{dmn}(f_n) : \forall j < k[\alpha \ne i_j]\}$, and let

$$\alpha_{nk} = \frac{1}{2^{|\mathrm{dmn}(f'_{nk})|}}.$$

Now for $n, k \in \omega$ and $t < 2$ we define

$$\varepsilon_{nk}(t) = \begin{cases} \alpha_{n,k+1} & \text{if } i_k \notin \mathrm{dmn}(f_n), \\ \alpha_{n,k+1} & \text{if } i_k \in \mathrm{dmn}(f_n) \text{ and } f_n(i_k) = t, \\ 0 & \text{otherwise.} \end{cases}$$

(6) $\int \varepsilon_{nk} = \alpha_{nk}$ for all $n, k \in \omega$.

In fact,

$$\begin{aligned} \int \varepsilon_{nk} &= \frac{1}{2}\varepsilon_{nk}(0) + \frac{1}{2}\varepsilon_{nk}(1) \\ &= \begin{cases} \alpha_{n,k+1} & \text{if } i_k \notin \mathrm{dmn}(f_n), \\ \frac{1}{2}\alpha_{n,k+1} & \text{if } i_k \in \mathrm{dmn}(f_n) \end{cases} \\ &= \alpha_{nk}. \end{aligned}$$

Now we define by induction elements $t_k \in 2$ and subsets M_k of M. Let $M_0 = M$. Note that

$$\alpha_{n0} = \frac{1}{2^{|\text{dmn}(f_n)|}}; \quad \sum_{n\in M} \alpha_{n0} = \sum_{n\in M} \frac{1}{2^{|\text{dmn}(f_n)|}} = \sum_{n\in M} \theta_0(C_n) < 1.$$

Now suppose that M_k and t_i have been defined for all $i < k$, so that $\sum_{n\in M_k} \alpha_{nk} < 1$. Note that this holds for $k = 0$. Now

$$\begin{aligned} 1 > \sum_{n\in M_k} \alpha_{nk} &= \sum_{n\in M_k} \int \varepsilon_{nk} \quad \text{by (6)} \\ &= \int \sum_{n\in M_k} \varepsilon_{nk} \quad \text{by Proposition 18.81.} \end{aligned}$$

It follows that there is a $t_k < 2$ such that $\left(\sum_{n\in M_k} \varepsilon_{nk}\right)(t_k) < 1$. Let

$$M_{k+1} = \{n \in M : \forall j < k+1[i_j \notin \text{dmn}(f_n), \text{ or } i_j \in \text{dmn}(f_n) \text{ and } f_n(i_j) = t_j]\}.$$

If $n \in M_{k+1}$, then $\varepsilon_{nk}(t_k) = \alpha_{n,k+1}$. Hence

$$\sum_{n\in M_{k+1}} \alpha_{n,k+1} = \sum_{n\in M_{k+1}} \varepsilon_{nk}(t_k) \le \left(\sum_{n\in M_k} \varepsilon_{nk}\right)(t_k) < 1.$$

Also, $M_{k+1} \neq \emptyset$. For, let $g \in {}^\kappa 2$ such that $g(i_j) = t_j$ for all $j \le k$. Say $g \in C_n$ with $n \in M$. Then $f_n \subseteq g$. Hence $i_j \notin \text{dmn}(f_n)$, or $i_j \in \text{dmn}(f_n)$ and $f_n(i_j) = t_j$. Thus $n \in M_{k+1}$.

This finishes the construction. Now let $g \in {}^\kappa 2$ be such that $g(i_j) = t_j$ for all $j \in \omega$. Say $g \in C_n$ with $n \in M$. Then $f_n \subseteq g$. The domain of f_n is a finite subset of J. Choose $k \in \omega$ so that $\text{dmn}(f_n) \subseteq \{i_j : j < k\}$. Then $n \in M_k$. Hence $f'_{nk} = \emptyset$ and so $\alpha_{nk} = 1$. This contradicts $\sum_{m\in M_k} \alpha_{mk} < 1$. □

Let ν be the tiny function with domain 2 which interchanges 0 and 1. For any $f \in {}^\kappa 2$ let $F(f) = \nu \circ f$.

Proposition 18.83.

(i) F is a permutation of ${}^\kappa 2$.
(ii) For any $f \in \text{Fn}(\kappa, 2, \omega)$ we have $F[U_f] = U_{\nu\circ f}$.
(iii) For any $X \subseteq {}^\kappa 2$ we have $\theta(X) = \theta(F[X])$.
(iv) $\forall E \in \Sigma_0[F[E] \in \Sigma_0]$.

Proof. (i): Clearly F is one-one, and $F(F(f)) = f$ for any $f \in {}^\kappa 2$. So (i) holds.
(ii): For any $g \in {}^\kappa 2$,

$$\begin{aligned} g \in F[U_f] \quad &\text{iff} \quad \exists h \in U_f[g = F(h)] \\ &\text{iff} \quad \exists h \in {}^\kappa 2[f \subseteq h \text{ and } g = \nu \circ h] \\ &\text{iff} \quad \exists h \in {}^\kappa 2[\nu \circ f \subseteq \nu \circ h \text{ and } g = \nu \circ h] \\ &\text{iff} \quad \nu \circ f \subseteq g \\ &\text{iff} \quad g \in U_{\nu\circ f} \end{aligned}$$

(iii): Clearly $\theta_0(U_f) = \theta_0(F[U_f])$ for any $f \in \mathrm{Fn}(\kappa, 2, \omega)$. Also, $A \subseteq \bigcup_{n\in\omega} C_n$ iff $F[A] \subseteq \bigcup_{n\in\omega} F[C_n]$. So (iii) holds.

(iv): Suppose that $E \in \Sigma_0$. Let $X \subseteq {}^\kappa 2$. Then

$$\begin{aligned}\theta(X \cap F[E]) + \theta(X \backslash F[E]) &= \theta(F[F[X]] \cap F[E]) + \theta(F[F[X]] \backslash F[E]) \\ &= \theta(F[F[X] \cap E]) + \theta(F[F[X] \backslash E]) \\ &= \theta(F[X] \cap E) + \theta(F[X] \backslash E) \\ &= \theta(E) = \theta(F[E]). \end{aligned}$$

□

Proposition 18.84. *If $\alpha < \kappa$ and $\varepsilon < 2$, then $\theta(U_{\{(\alpha,\varepsilon)\}}) = \frac{1}{2}$.*

Proof. By Proposition 18.83 we have $\theta(U_{\{(\alpha,\varepsilon)\}}) = \theta(U_{\{(\alpha,1-\varepsilon)\}})$, so the result follows from Proposition 18.82. □

Proposition 18.85. *For each $f \in \mathrm{Fn}(\kappa, 2, \omega)$ we have $U_f \in \Sigma_0$ and $\theta(U_f) = \frac{1}{2^{|\mathrm{dmn}(f)|}}$.*

Proof. We have $U_f = \bigcap_{\alpha\in\mathrm{dmn}(f)} U_{\{(\alpha,f(\alpha))\}}$. Note that if $\alpha \in \mathrm{dmn}(f)$, then $U_{\{(\alpha,f(\alpha))\}} = \{g \in {}^\kappa 2 : g(\alpha) = f(\alpha)\}$; hence $U_{\{(\alpha,f(\alpha))\}} \in \Sigma_0$ by Proposition 18.80, and so $U_f \in \Sigma_0$. We prove that $\theta(U_f) = \frac{1}{2^{|\mathrm{dmn}(f)|}}$ by induction on $|\mathrm{dmn}(f)|$. For $|\mathrm{dmn}(f)| = 1$, this holds by Proposition 18.84. Now assume that it holds for $|\mathrm{dmn}(f)| = m$. For any f with $|\mathrm{dmn}(f)| = m$ and $\alpha \notin \mathrm{dmn}(f)$ we have $2^{-|\mathrm{dmn}(f)|} = \theta(U_f) = \theta(U_{f\cup\{(\alpha,0)\}}) + \theta(U_{f\cup\{(\alpha,1)\}})$. Since $\theta(U_{f\cup\{(\alpha,\varepsilon)\}}) \leq \theta_0(U_{f\cup\{(\alpha,\varepsilon)\}}) = 2^{-|\mathrm{dmn}(f)|-1}$ for each $\varepsilon \in 2$, it follows that $\theta(U_{f\cup\{(\alpha,\varepsilon)\}}) = 2^{-|\mathrm{dmn}(f)|-1}$ for each $\varepsilon \in 2$. □

Proposition 18.86. *If F is a finite subset of ${}^\kappa 2$, then $F \in \Sigma_0$ and $\theta(F) = 0$.*

Proof. This is obvious if $F = \emptyset$. For $F = \{f\}$ we have $F \subseteq U_{f\restriction n}$ for each $n \in \omega$, and so $\theta(F) = 0$. Then it is clear that $F \in \Sigma_0$. Now the general case follows easily. □

Proposition 18.87. *If $X \subseteq {}^\kappa 2$ is measurable, then $\theta(X) = \inf\{\varphi(U) : X \subseteq U$ and U is open$\}$.*

Proof. By Proposition 18.85, $\theta(U_f) = \theta_0(U_f)$ for each $f \in \mathrm{Fn}(\kappa, 2, \omega)$. Hence by the definition preceding Proposition 18.79,

$$\begin{aligned}\theta(X) &\leq \inf\left\{\theta\left(\bigcup_{n\in\omega} U_{f_n}\right) : f \in {}^\omega\mathrm{Fn}(\kappa, 2, \omega),\ X \subseteq \bigcup_{n\in\omega} U_{f_n}\right\} \\ &\leq \inf\left\{\sum\{\theta(U_{f_n}) : f \in {}^\omega\mathrm{Fn}(\kappa, 2, \omega),\ X \subseteq \bigcup_{n\in\omega} U_{f_n}\right\} \\ &= \inf\left\{\sum\{\theta_0(U_{f_n}) : f \in {}^\omega\mathrm{Fn}(\kappa, 2, \omega),\ X \subseteq \bigcup_{n\in\omega} U_{f_n}\right\} \\ &= \theta(X). \end{aligned}$$

□

Proposition 18.88. *If $X \subseteq {}^\kappa 2$ is measurable, then there is a system $\langle f^n_m : n, m \in \omega\rangle$ with each $f^n_m \in \mathrm{Fn}(\kappa, 2, \omega)$ such that $X \subseteq \bigcap_{n\in\omega} \bigcup_{m\in\omega} U_{f^n_m}$ and $\theta((\bigcap_{n\in\omega} \bigcup_{m\in\omega} U_{f^n_m})\backslash X) = 0$.*

Proof. By the proof of Proposition 18.87, for each $n \in \omega$ let $\langle f^n_m : m \in \omega\rangle$ be such that each $f^n_m \in \mathrm{Fn}(\kappa, 2, \omega)$, $X \subseteq \bigcup_{m\in\omega} U_{f^n_m}$, and $\theta(\bigcup_{m\in\omega} U_{f^n_m}) - \theta(X) \le \frac{1}{n+1}$. Then

$$\forall n \in \omega \left[X \subseteq \bigcap_{p\in\omega} \bigcup_{m\in\omega} U_{f^p_m} \subseteq \bigcup_{m\in\omega} U_{f^n_m}\right];$$

$$\forall n \in \omega \left[\theta\left(\bigcap_{n\in\omega} \bigcup_{m\in\omega} U_{f^n_m}\right) - \theta(X) \le \theta\left(\bigcup_{m\in\omega} U_{f^n_m}\right) - \theta(X) \le \frac{1}{n+1}\right];$$

$$\theta\left(\bigcap_{n\in\omega} \bigcup_{m\in\omega} U_{f^n_m}\right) - \theta(X) = 0;$$

$$\theta\left(\bigcap_{n\in\omega} \bigcup_{m\in\omega} U_{f^n_m}\right) = \theta\left(\left(\bigcap_{n\in\omega} \bigcup_{m\in\omega} U_{f^n_m}\right)\backslash X\right) + \theta(X);$$

$$\theta\left(\left(\bigcap_{n\in\omega} \bigcup_{m\in\omega} U_{f^n_m}\right)\backslash X\right) = 0. \qquad \square$$

measure on $\mathbb{R}$

For any $a, b \in \mathbb{R}$ let $[a, b) = \{x \in \mathbb{R} : a \le x < b\}$. Note that if $a \ge b$, then $[a, b) = \emptyset$. Note that if $[a, b) = [c, d)$, $a < b$, and $c < d$, then $a = c$ and $b = d$. For any $a, b \in \mathbb{R}$ we define

$$\lambda([a,b)) = \begin{cases} 0 & \text{if } a \ge b, \\ b-a & \text{if } a < b. \end{cases}$$

A set of the form $[a, b)$ is called a *half-open interval.*

Lemma 18.89. *Suppose that I is a half-open interval, $\langle J_i : i \in \omega\rangle$ is a system of half-open intervals, and $I \subseteq \bigcup_{i\in\omega} J_i$. Then*

$$\lambda(I) \le \sum_{j\in\omega} \lambda(J_i).$$

Proof. If $I = \emptyset$ this is obvious. So suppose that $I \ne \emptyset$. Then there exist real numbers $a < b$ such that $I = [a, b)$. Let

$$A = \left\{x \in [a,b] : x - a \le \sum_{j\in\omega} \lambda(J_j \cap (-\infty, x))\right\}.$$

Obviously $a \in A$, and A is bounded above by b, so $c \stackrel{\text{def}}{=} \sup(A)$ exists. Now

$$\begin{aligned} c - a &= \sup_{x\in A}(x-a) \\ &\le \sup_{x\in A} \sum_{j\in\omega} \lambda(J_j \cap (-\infty, x)) \\ &\le \sum_{j\in\omega} \lambda(J_j \cap (-\infty, c)). \end{aligned}$$

Hence $c \in A$. Now suppose that $c < b$. Thus $c \in [a, b)$, so there is a $k \in \omega$ such that $c \in J_k$. Say $J_k = [u, v)$. Then $x \stackrel{\text{def}}{=} \min(v, b) > c$. Then $\lambda(J_j \cap (-\infty, c)) \leq \lambda(J_j \cap (-\infty, x))$ for each j, and $\lambda(J_k \cap (-\infty, x)) = \lambda(J_k \cap (-\infty, c)) + x - c$. Hence

$$\begin{aligned}\sum_{j\in\omega} \lambda(J_j \cap (-\infty, x)) &\geq \sum_{j\in\omega} \lambda(J_j \cap (-\infty, c)) + x - c \\ &\geq c - a + x - c = x - a.\end{aligned}$$

Here we used the above inequality on $c - a$. Thus we have shown that $x \in A$. But $x > c = \sup(A)$, contradiction.

Hence $c = b$, so $b \in A$. $\square$

Now for any $A \subseteq \mathbb{R}$ let

$$\theta'(A) = \inf\left\{\sum_{j\in\omega} \lambda(I_j) : \langle I_j : j \in \omega\rangle \text{ is a sequence of half-open intervals} \right. \\ \left. \text{such that } A \subseteq \bigcup_{j\in\omega} I_j\right\}.$$

Lemma 18.90. *(i) θ' is an outer measure on $\mathbb{R}$.*

(ii) $\theta'(I) = \lambda(I)$ for every half-open interval I.

Proof. (i): Clearly (1) and (2) hold. Now for (3), suppose that $\langle A_i : i \in \omega\rangle$ is a sequence of subsets of X. Let $B = \bigcup_{i\in\omega} A_i$. For each $i \in \omega$ let $\langle I_{ij} : j \in \omega\rangle$ be a sequence of half-open intervals such that $A_i \subseteq \bigcup_{j\in\omega} I_{ij}$ and

$$\sum_{j\in\omega} \lambda(I_{ij}) \leq \theta'(A_i) + \frac{\varepsilon}{2^i}.$$

Note that this holds even if $\theta'(A_i) = \infty$. Let $p : \omega \to \omega \times \omega$ be a bijection.

(1) $$B \subseteq \bigcup_{m\in\omega} I_{1^{st}(p(m)),2^{nd}(p(m))}.$$

In fact, if $b \in B$, choose $i \in I$ such that $b \in A_i$, and then choose $j \in \omega$ such that $b \in I_{ij}$. Let $m = p^{-1}(i, j)$. Then

$$b \in I_{1^{st}(p(m)),2^{nd}(p(m))},$$

as desired in (1).

(2) $$\sum_{m\in\omega} \lambda(I_{1^{st}(p(m)),2^{nd}(p(m))}) \leq \sum_{i\in\omega}\sum_{j\in\omega} \lambda(I_{ij}).$$

In fact, let $m \in \omega$, and set

$$n = \max(\{1^{st}(p(i)) : i \leq m\} \cup \{2^{nd}(p(i)) : i \leq m\}).$$

Then

$$\sum_{i=0}^{m} \lambda(I_{1^{st}(p(m)),2^{nd}(p(m))}) \leq \sum_{i=0}^{n}\sum_{j=0}^{n} \lambda(I_{ij}) \leq \sum_{i\in\omega}\sum_{j\in\omega} \lambda(I_{ij}),$$

and (2) follows.

Hence using (1) we have

$$\begin{aligned}
\theta'\left(\bigcup_{i\in\omega} A_i\right) &= \theta'(B) \\
&\leq \sum_{m\in\omega} \lambda(I_{1^{st}(p(m)),2^{nd}(p(m))}) \\
&\leq \sum_{i\in\omega}\sum_{j\in\omega} \lambda(I_{ij}) \\
&\leq \sum_{i\in\omega} \left(\theta'(A_i) + \frac{\varepsilon}{2^i}\right) \\
&= \sum_{i\in\omega} \theta'(A_i) + \sum_{i\in\omega} \frac{\varepsilon}{2^i} \\
&= \sum_{i\in\omega} \theta'(A_i) + 2\varepsilon.
\end{aligned}$$

Hence (3) in the definition of outer measure holds.

Clearly $\theta'(I) \leq \lambda(I)$. The other inequality follows from Lemma 18.89. □

Corollary 18.91. *For θ' the explicit outer measure defined above on $\mathbb{R}$, and with*

$$\begin{aligned}
\Sigma_1 = \{E \subseteq \mathbb{R} : & \textit{ for every } A \subseteq X, \\
& \theta'(A) = \theta'(A \cap E) + \theta'(A \backslash E)\},
\end{aligned}$$

the system $(\mathbb{R}, \Sigma_1, \theta' \restriction \Sigma_1)$ is a measure space. □

Lemma 18.92. *$(-\infty, x)$ is measurable for every $x \in \mathbb{R}$.*

Proof. First we show

(1) $\lambda(I) = \lambda(I \cap (-\infty, x)) + \lambda(I \backslash (-\infty, x))$ for every half-open interval I.

This is obvious if $I \subseteq (-\infty, x)$ or $I \subseteq [x, \infty)$. So assume that neither of these cases hold. Then with $I = [a, b)$ we must have $a < x < b$. Then

$$\begin{aligned}
\lambda(I \cap (-\infty, x)) + \lambda(I \backslash (-\infty, x)) &= \lambda([a, x)) + \lambda([x, b)) \\
&= \lambda([a, x)) + \lambda([x, b)) \\
&= x - a + b - x \\
&= b - a \\
&= \lambda([a, b)) \\
&= \lambda(I).
\end{aligned}$$

So (1) holds.

Now for the proof of the lemma, let $A \subseteq \mathbb{R}$ and let $\varepsilon > 0$. We show that $\theta'(A \cap (-\infty, x)) + \theta'(A \backslash (-\infty, x)) \leq \theta'(A) + \varepsilon$, which will prove the lemma. By the definition of θ', there is a sequence $\langle I_j : j \in \omega \rangle$ of half-open intervals such that $A \subseteq \bigcup_{j \in \omega} I_j$ and $\sum_{j \in \omega} \lambda(I_j) \leq \theta'(A) + \varepsilon$. Now $\langle I_j \cap (-\infty, x) : j \in \omega \rangle$ and $\langle I_j \backslash (-\infty, x) : j \in \omega \rangle$ are sequences of half-open intervals, $A \cap (-\infty, x) \subseteq \bigcup_{j \in \omega} (I_j \cap (-\infty, x))$, and $A \backslash (-\infty, x) \subseteq \bigcup_{j \in \omega} (I_j \backslash (-\infty, x))$, so

$$\begin{aligned} \theta'(A \cap (-\infty, x)) + \theta'(A \backslash (-\infty, x)) &\leq \sum_{j=0}^{\infty} \lambda(I_j \cap (-\infty, x)) + \sum_{j=0}^{\infty} \lambda(I_j \backslash (-\infty, x)) \\ &= \sum_{j=0}^{\infty} \lambda(I_j) \leq \theta'(A) + \varepsilon. \end{aligned}$$ □

Theorem 18.93. *Every Borel subset of $\mathbb{R}$ is Lebesgue measurable.*

Proof. It suffices to show that every open set is Lebesgue measurable. It then suffices to prove the following:

(1) If U is a nonempty open subset of $\mathbb{R}$, then there is a family $\mathscr{A}$ of half-open intervals with rational coefficients such that $U = \bigcup \mathscr{A}$.

To prove (1), let $\mathscr{A}$ be the set of all half-open intervals contained in U. Now take any $x \in U$. Since U is open, there are real numbers $y < z$ such that $x \in (y, z) \subseteq U$. Choose rational numbers r, s such that $y < r < x < s < z$. Then $x \in [r, s) \subseteq U$, as desired. □

Corollary 18.94. *Every Lebesgue null set is Lebesgue measurable. Every singleton is a null set, and every countable set is a null set.* □

Lemma 18.95. *Suppose that μ is a measure and E, F, G are μ-measurable. Then*

$$\mu(E \triangle F) \leq \mu(E \triangle G) + \mu(G \triangle F).$$

Proof.

$$\begin{aligned} \mu(E \triangle F) &= \mu(E \backslash F) + \mu(F \backslash E) \\ &= \mu((E \backslash F) \cap G) + \mu((E \backslash F) \backslash G) + \mu(F \backslash E) \cap G) + \mu((F \backslash E) \backslash G) \\ &\leq \mu(G \backslash F) + \mu(E \backslash G) + \mu(G \backslash E) + \mu(F \backslash G) \\ &= \mu(E \triangle G) + \mu(G \triangle F). \end{aligned}$$ □

Lemma 18.96. *If E is Lebesgue measurable with finite measure, then for any $\varepsilon > 0$ there is an open set $U \supseteq E$ such that $\theta'(E) \leq \theta'(U) \leq \theta'(E) + \varepsilon$. Moreover, there is a system $\langle K_j : j < \omega \rangle$ of open intervals such that $U = \bigcup_{j < \omega} K_j$ and $\theta'(U) \leq \sum_{j < \omega} \theta'(K_j) \leq \theta'(E) + \varepsilon$.*

Proof. By the basic definition of Lebesgue measure,

$$0 = \theta'(E) = \inf\left\{\sum_{j\in\omega}\theta'(I_j) : \langle I_j : j \in \omega\rangle \text{ is a sequence of half-open intervals}\right.$$
$$\left.\text{such that } A \subseteq \bigcup_{j\in\omega} I_j\right\}.$$

Hence we can choose a sequence $\langle I_j : j \in \omega\rangle$ of half-open intervals such that $E \subseteq \bigcup_{j\in\omega} I_j$ and

$$\theta'\left(\bigcup_{j\in\omega} I_j\right) \leq \sum_{j\in\omega}\theta'(I_j) \leq \theta'(E) + \frac{\varepsilon}{2}.$$

Write $I_j = [a_j, b_j)$ with $a_j < b_j$. Define

$$\begin{aligned}
K_j &= \left(a_j - \frac{\varepsilon}{2^{j+2}}, b_j\right); \quad \text{then} \\
E &\subseteq \bigcup_{j\in\omega} K_j \quad \text{and} \\
\theta'\left(\bigcup_{j\in\omega} K_j\right) &\leq \sum_{j\in\omega}\theta'(K_j) \\
&= \sum_{j\in\omega}\left(\frac{\varepsilon}{2^{j+2}} + \theta'(I_j)\right) \\
&= \sum_{j\in\omega}\frac{\varepsilon}{2^{j+2}} + \sum_{j\in\omega}\theta'(I_j) \\
&\leq \frac{\varepsilon}{2} + \theta'(E) + \frac{\varepsilon}{2} = \theta'(E) + \varepsilon.
\end{aligned}$$

□

Corollary 18.97. *(i) If A is Lebesgue measurable and $\theta'(A)$ is finite, then $\theta'(A) = \inf\{\theta'(U) : U$ open, $A \subseteq U\}$.*

(ii) If A is Lebesgue measurable with finite measure, then $\theta'(A) = \sup\{\theta'(C) : C$ closed, $C \subseteq A\}$.

(iii) If A is measurable and $\theta'(A) = \infty$, then $\sup\{\theta'(C) : C$ closed, $C \subseteq A\} = \infty$.

Proof. Only (iii) needs a proof. Let $\varepsilon > 0$. For each $n \in \omega$ let

$$\begin{aligned}
a_{2n} &= n; \\
b_{2n} &= n + 1; \\
a_{2n+1} &= -n - 1; \\
b_{2n+1} &= -n.
\end{aligned}$$

For each $n \in \omega$ let C_n be a closed subset of $[a_n, b_n) \cap A$ such that

$$\theta'([a_n, b_n) \cap A \backslash C_n) < \frac{\varepsilon}{2^n}.$$

Then

$$\begin{aligned}
\theta'(A) &= \sum_{n\in\omega} \theta'([a_n, b_n) \cap A) \\
&= \lim_{n=0}^{\infty} \theta'([[a_0, b_0) \cap A] \cup \ldots \cup [[a_n, b_n) \cap A]) \\
&= \lim_{n=0}^{\infty} \theta'([[a_0, b_0) \cap A\backslash C_0] \cup \ldots \cup [[a_n, b_n) \cap A\backslash C_n]) \\
&\qquad + \theta'(C_0 \cup \ldots \cup C_n) \\
&= \lim_{n=0}^{\infty} \theta'([[a_0, b_0) \cap A\backslash C_0] \cup \ldots \cup [[a_n, b_n) \cap A\backslash C_n]) \\
&\qquad + \lim_{n\to\infty} \theta'(C_0 \cup \ldots \cup C_n) \\
&= \varepsilon + \lim_{n\to\infty} \theta'(C_0 \cup \ldots \cup C_n),
\end{aligned}$$

as desired.

The following is an elementary lemma concerning the topology of the reals.

Lemma 18.98. *Suppose that U is a bounded open set.*

(i) There is a collection $\mathscr{A}$ of pairwise disjoint open intervals such that $U = \bigcup \mathscr{A}$.

(ii) There exist a countable subset C of $\mathbb{R}$ and a collection $\mathscr{B}$ of pairwise disjoint open intervals with rational endpoints such that $U = C \cup \bigcup \mathscr{B}$ and $C \cap \bigcup \mathscr{B} = \emptyset$.

Proof. (i): For $x, y \in \mathbb{R}$, define $x \equiv y$ iff one of the following conditions holds: (1) $x = y$; (2) $x < y$ and $[x, y] \subseteq U$; (3) $y < x$ and $[y, x] \subseteq U$. Clearly $\equiv$ is an equivalence relation on $\mathbb{R}$. If $x < z < y$ and $x \equiv y$, then obviously $x \equiv z$. Thus each equivalence class is convex. If C is an equivalence class with more than one element, then it must be an open interval (a, b), since if for example the left endpoint a is in C then some real to the left of a must be in C, contradiction. It follows now that the collection $\mathscr{A}$ of all equivalence classes with more than one element is as desired in (i).

(ii): First note that the set $\mathscr{A}$ of (i) must be countable. Now take any $(a, b) \in \mathscr{A}$, $a < b$. Let $c_0 < c_1 < \cdots < c_m < \cdots$ be rational numbers in (a, b) which converge to b, and $c_0 = d_0 > d_1 > \cdots > d_m > \cdots$ rational numbers which converge to a. Then let $L^{ab}_{2i} = (c_i, c_{i+1})$ and $L^{ab}_{2i+1} = (d_{i+1}, d_i)$ for all $i \in \omega$. Let $D^{ab} = \{c_i : i < \omega\} \cup \{d_i : i < \omega\}$. Define $\mathscr{B} = \{L^{ab}_i : (a, b) \in \mathscr{A}, i < \omega\}$ and $C = \bigcup_{(a,b)\in\mathscr{A}} D^{ab}$. Clearly this works for (ii). □

Lemma 18.99. *If E is Lebesgue measurable and $\varepsilon > 0$, then there is an $m \in \omega$ and a sequence $\langle I_i : i < m\rangle$ of open intervals with rational endpoints such that $\theta'\left(E \triangle \bigcup_{i<m} I_i\right) \leq \varepsilon$.*

Proof. By Corollary 18.97, let $U \supseteq E$ be open such that $\theta'(E) \leq \theta'(U) \leq \theta'(E) + \frac{\varepsilon}{2}$. Then choose C and $\mathscr{B}$ as above. Let $W = \bigcup \mathscr{B}$. So $\theta'(W) = \sum_{I \in \mathscr{B}} \theta'(I)$. Then choose $m \in \omega$ and $\langle I_i : i < m \rangle$ elements of $\mathscr{B}$ such that $\sum_{I \in \mathscr{B}} \theta'(I) - \sum_{i<m} \theta'(I_i) \leq \frac{\varepsilon}{2}$. Now $\theta'(W) = \sum_{I \in \mathscr{B}} \theta'(I)$ and $\theta'(\bigcup_{i<m} I_i) = \sum_{i<m} \theta'(I_i)$. Let $V = \bigcup_{i<m} I_i$. Thus $\theta'(W) - \theta'(V) \leq \frac{\varepsilon}{2}$. Hence $V \subseteq W \subseteq U$, and

$$\begin{aligned}
\theta'(E \triangle V) &\leq \theta'(E \triangle U) + \theta'(U \triangle W) + \theta'(W \triangle V) \\
&= \theta'(U \backslash E) + \theta'(C) + \theta'(W \backslash V) \\
&= \theta'(U) - \theta'(E) + \theta'(W) - \theta'(V) \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$ □

Lemma 18.100. *(i)* $\theta'([a,b)) = b - a$ *if* $a < b$.
(ii) $\theta'([a,b]) = b - a$ *if* $a < b$.

Proof. (i) holds by Lemma 18.90(ii). Then for $a < b$, $\theta'([a,b]) = \theta'([a,b)) + \theta'(\{b\}) = \varphi'([a,b)) = b - 1$ using Corollary 18.94. □

Connections between different measures

Lemma 18.101. *If* (X, Σ, μ) *is a measure space and* $Y \subseteq X$*, then*

$$(Y, \{A \cap Y : A \in \Sigma\}, \mu \restriction \{A \cap Y : A \in \Sigma\})$$

is a measure space. □

At this point we have two important measure spaces: $({}^\omega 2, \Sigma_0, \theta)$ and $(\mathbb{R}, \Sigma_1, \theta')$. We now define $\Sigma_2 = \{A \cap \Omega : A \in \Sigma_0\}$ and $\theta_2 = \theta \restriction \{A \cap \Omega : A \in \Sigma_0\}$. Thus

Corollary 18.102. $(\Omega, \Sigma_2, \theta_2)$ *is a measure space.* □

Let $\Sigma_3 = \{A \cap [0,1] : A \in \Sigma_1\}$ and $\theta_3 = \theta' \restriction \{A \cap [0,1] : A \in \Sigma_1\}$.

Corollary 18.103. $([0,1], \Sigma_3, \theta_3)$ *is a measure space.* □

If (X, Σ, μ) is a measure space, then $A \in \Sigma$ is an *atom* iff $\mu(A) > 0$. and for all $B \in \Sigma$ with $B \subseteq A$, either B or $A \backslash B$ has measure 0.

Let λ be the usual measure on ${}^\omega 2$ and μ Lebesgue measure on $[0,1]$. Consider the measure spaces $({}^\omega 2, \Sigma_0, \lambda)$ and $([0,1], \Sigma_1, \mu)$. For each $x \in {}^\omega 2$ let $\varphi(x) = \sum_{i=0}^\infty (2^{-i-1} x_i)$.

Theorem 18.104. *There is a bijection* $\tilde{\varphi} : {}^\omega 2 \to [0,1]$ *which is equal to* φ *except at countably many points, and any such bijection is an isomorphism from* $({}^\omega 2, \Sigma_0, \lambda)$ *to* $([0,1], \Sigma_1, \mu)$*. That is:*
(a) $\forall X \subseteq {}^\omega 2 [X \in \Sigma_0$ *iff* $\tilde{\varphi}[X] \in \Sigma_1]$;
(b) $\forall X \subseteq [0,1][X \in \Sigma_1$ *iff* $\tilde{\varphi}^{-1}[X] \in \Sigma_0]$;
(c) $\forall X \in \Sigma_0[\lambda(X) = \mu(\tilde{\varphi}[X])]$;
(d) $\forall X \in \Sigma_1[\mu(X) = \lambda(\tilde{\varphi}^{-1}[X])]$.

Proof. Let $H = \{x \in {}^\omega 2 : \exists m \in \omega \forall i \geq m[x_i = x_m]\}$ and $H' = \{2^{-n}k : n \in \omega, k \leq 2^n\}$. Then H and H' are countable.

(1) $\varphi \restriction ({}^\omega 2 \backslash H)$ is a bijection from ${}^\omega 2 \backslash H$ onto $[0,1] \backslash H'$.

For, first we show that $\varphi \restriction ({}^\omega 2 \backslash H)$ maps into $[0,1] \backslash H'$. Let $x \in ({}^\omega 2 \backslash H)$. Thus

(2) $\forall m \in \omega \exists i > m[x_i \neq x_m]$.

It follows that $\varphi(x) \neq 1$, for by (2) there is a j such that $x_j = 0$, and then

$$\varphi(x) = \sum_{i=0}^{\infty}(2^{-i-1}x_i) \leq \sum_{i=0}^{j-1}(2^{-i-1}x_i) + \sum_{i=j+1}^{\infty} 2^{-i-1} = \sum_{i=0}^{j-1}(2^{-i-1}x_i) + 2^{-j-1} < 1.$$

Suppose that $\varphi(x) \in H'$. Thus there exist $n \in \omega$ and $k < 2^n$ such that

$$\sum_{i=0}^{\infty}(2^{-i-1}x_i) = 2^{-n}k. \tag{3}$$

Since $\varphi(x) \neq 1$, we can write $2^{-n}k = \sum_{i=0}^{n-1}(2^{-i-1}y_i)$ with each $y_i \in 2$. Thus by (3) we have

$$\sum_{i=0}^{\infty}(2^{-i-1}x_i) = \sum_{i=0}^{n-1}(2^{-i-1}y_i). \tag{4}$$

Now we claim that $y \subseteq x$. For, suppose not, and let $j < n$ be minimum such that $x_j \neq y_j$. Hence by (4) we have

$$\sum_{i=j}^{\infty}(2^{-i-1}x_i) = \sum_{i=j}^{n-1}(2^{-i-1}y_i).$$

Case 1. $x_j = 0$ and $y_j = 1$. By (2) choose $k > j$ so that $x_k = 1$ and choose $l > k$ so that $x_l = 0$. Then

$$\begin{aligned}\sum_{i=j}^{\infty}(2^{-i-1}x_i) &\leq \sum_{i=j}^{l-1}(2^{-i-1}x_i) + \sum_{i=l+1}^{\infty} 2^{-i-1} = \sum_{i=j}^{l-1}(2^{-i-1}x_i) + 2^{-l-1} \\ &< \sum_{i=0}^{j-1}(2^{-i-1}x_i) + \sum_{i=j+1}^{\infty} 2^{-i-1} \leq \sum_{i=j}^{n-1}(2^{-i-1}y_i),\end{aligned}$$

contradiction.

Case 2. $x_j = 1$ and $y_j = 0$. Then

$$\sum_{i=j}^{n-1}(2^{-i-1}y_i) \leq \sum_{i=j+1}^{n} 2^{-i-1} < \sum_{i=j+1}^{\infty} 2^{-i-1} = 2^{-j-1} \leq \sum_{i=j}^{\infty}(2^{-i-1}x_i),$$

contradiction.

Thus $y \subseteq x$. Now by (2) there is a $j \geq n$ such that $x_j = 1$. Hence

$$\sum_{i=j}^{\infty}(2^{-i-1}x_i) > \sum_{i=j}^{n-1}(2^{-i-1}y_i),$$

contradiction.

Thus $\varphi(x) \notin H'$.

To show that $\varphi \restriction ({}^{\omega}2 \backslash H)$ maps onto $[0,1]\backslash H'$, let $t \in [0,1]\backslash H'$. Since $1 \in H'$, we can write $t = \sum_{i=0}^{\infty}(2^{-i-1}x_i)$ with x not eventually 1. We claim that $x \notin H$. For, suppose that $x \in H$. Say $m \in \omega$ and $\forall i > m[x_i = x_m]$. Since x is not eventually 1, we have $x_m = 0$. Since $t \notin H'$, we have $t \neq 0$, so x is not the all 0 sequence. Choose n maximum such that $x_n \neq 0$. Thus $t = \sum_{i=0}^{n}(2^{-i-1}x_i)$. Hence

$$\begin{aligned} 2^{n+1}t &= 2^{n+1}2^{-1}x_0 + 2^{n+1}2^{-2}x_1 + \cdots + x_0 \\ &= 2^n x_0 + 2^{n-1}x_1 + \cdots + x_0. \end{aligned}$$

Hence with $k = 2^n x_0 + 2^{n-1}x_1 + \cdots + x_0$ we have $k \leq 2^{n+1}$ and $t = 2^{-n-1}k \in H'$, contradiction. So $x \notin H$. Clearly $\varphi(x) = t$.

For $\varphi \restriction ({}^{\omega}2\backslash H)$ one-one, suppose that $x, y \in ({}^{\omega}2\backslash H)$ and $x \neq y$. Let m be minimum such that $x_m \neq y_m$. By symmetry, say $x_m = 0$ and $y_m = 1$. Choose $n > m$ so that $x_n = 0$; this is possible since $x \notin H$. Then

$$\sum_{i=0}^{\infty}(2^{-i-1}x_i) \leq \sum_{i=0}^{n-1}(2^{-i-1}x_i) + \sum_{i=n+1}^{\infty} 2^{-i-1} = \sum_{i=0}^{n-1}(2^{-i-1}x_i) + 2^{-n} < \sum_{i=0}^{n-1}(2^{-i-1}y_i).$$

This finishes the proof of (1).

Now H and H' are countable and infinite. Hence there is an extension of $\varphi \restriction ({}^{\omega}2\backslash H)$ to a bijection of ${}^{\omega}2$ onto $[0,1]$. Let $\tilde{\varphi}$ be any bijection of ${}^{\omega}2$ onto $[0,1]$ which is equal to φ except for countably many points. Let M be the countable set $\{x \in {}^{\omega}2 : \varphi(x) \neq \tilde{\varphi}(x)\}$ and let N be the countable set $\varphi[M] \cup \tilde{\varphi}[M]$.

(5) $\forall A \subseteq {}^{\omega}2[\varphi[A] \triangle \tilde{\varphi}[A] \subseteq N]$.

In fact, if $b \in \varphi[A]\backslash\tilde{\varphi}[A]$, then there is an $x \in A$ such that $b = \varphi(x)$. Since $b \notin \tilde{\varphi}[A]$, we have $\tilde{\varphi}(x) \neq b$. Hence $x \in M$, so $b \in \varphi[M] \subseteq N$. Now suppose that $b \in \tilde{\varphi}[A]\backslash\varphi[A]$. Say $b = \tilde{\varphi}(x)$ with $x \in A$. Since $b \notin \varphi[A]$, we have $\varphi(x) \neq b$. So $x \in M$ and $b \in \tilde{\varphi}[M] \subseteq N$. So (5) holds.

(6) If $t \in [0,1]$, then $\lambda(\tilde{\varphi}^{-1}[\{t\}]) = 0$ and hence $\lambda(\tilde{\varphi}^{-1}[\{t\}]) = \mu(\{t\})$.

We have $\tilde{\varphi}^{-1}[\{t\}] = \{\tilde{\varphi}^{-1}(t)\}$, so $\lambda(\tilde{\varphi}^{-1}[\{t\}]) = 0$ by Proposition 18.86. $\mu(\{t\}) = 0$ by Corollary 18.94.

(7) If $n \in \omega$, $k < 2^n$, and $E = [2^{-n}k, 2^{-n}(k+1)]$, then $\tilde{\varphi}^{-1}[E] \in \Sigma_0$ and $\lambda(\tilde{\varphi}^{-1}[E]) = \mu(E) = 2^{-n}$.

$\mu(E) = 2^{-n}$ by Lemma 18.100. Further,

$$\varphi^{-1}[E] = \left\{ x \in {}^{\omega}2 : 2^{-n}k \leq \sum_{i=0}^{\infty}(2^{-i-1}x_i) \leq 2^{-n}(k+1) \right\}.$$

Let $k = 2^{n-1}y_0 + 2^{n-2}y_1 + \cdots + y_{n-1}$ with each $y_i \in 2$. Then $2^{-n}k = 2^{-1}y_0 + 2^{-2}y_1 + \cdots + 2^{-n}y_{n-1} = \sum_{i=0}^{n-1}(2^{-i-1}y_i)$.

Case 1. $y_{n-1} = 0$. Then $k+1 = \sum_{i=0}^{n-2}(2^{n-i-1}y_i) + 1$ and so $2^{-n}(k+1) = \sum_{i=0}^{n-2}(2^{-i-1}y_i) + 2^{-n}$.

(8) If $x \in \varphi^{-1}[E]$ and x is not eventually 1, then $\forall i < n[x_i = y_i]$.

For, suppose that $j < n$ is minimum such that $x_i \neq y_i$. Choose $l > k > j$ with $x_l = x_k = 0$.

Subcase 1.1. $x_j = 0$, $y_j = 1$. Then

$$\begin{aligned} \sum_{i=0}^{\infty}(2^{-i-1}x_i) &\leq \sum_{i=0}^{l-1}(2^{-i-1}x_i) + \sum_{i=l+1}^{\infty} 2^{-i-1} = \sum_{i=0}^{l-1}(2^{-i-1}x_i) + 2^{-l-1} \\ &< \sum_{i=0}^{j-1}(2^{-i-1}x_i) + 2^{-j-1} \leq \sum_{i=0}^{n-1}(2^{-i-1}y_i) = 2^{-n}k. \end{aligned}$$

This contradicts $x \in \varphi^{-1}[E]$.

Subcase 1.2. $x_j = 1$, $y_j = 0$. Then

$$2^{-n}(k+1) = \sum_{i=0}^{n-2}(2^{-i-1}y_i) + 2^{-n} < \sum_{i=0}^{\infty}(2^{-i-1}x_i),$$

contradiction.

Thus (8) holds.

(9) If $x \in {}^{\omega}2$ and $\forall i < n[x_i = y_i]$, then $x \in \varphi^{-1}[E]$.

In fact, assume that $x \in {}^{\omega}2$ and $\forall i < n[x_i = y_i]$. Then

$$\begin{aligned} 2^{-n}k &= \sum_{i=0}^{n-1}(2^{-i-1}y_i) \leq \sum_{i=0}^{\infty}(2^{-i-1}x_i) \\ &\leq \sum_{i=0}^{n-1}(2^{-i-1}x_i) + \sum_{i=n+1}^{\infty} 2^{-i-1} = \sum_{i=0}^{n-2}(2^{-i-1}x_i) + 2^{-n} = 2^{-n}(k+1). \end{aligned}$$

This proves (9).

Case 2. $y_{n-1} = 1$ and there is a $j < n-1$ such that $y_j = 0$. Take the greatest such j. Then $k+1 = 2^{n-1}y_0 + 2^{n-2}y_1 + \cdots + 2^{n-j}y_{j-1} + 2^{n-j-1}$, and hence $2^{-n}(k+1) = \sum_{i=0}^{j-1}(2^{-i-1}y_i) + 2^{-j-1}$. Now suppose that $x \in \varphi^{-1}[E]$. Again we claim that (8) and (9) hold. For (8), suppose that $x \in \varphi^{-1}[E]$, x is not eventually 1, and $l < n$ is minimum such that $x_l \neq y_l$. Choose $t > s > l$ with $x_t = x_s = 0$.

Subcase 2.1. $x_l = 0$, $y_l = 1$. Then

$$\begin{aligned}\sum_{i=0}^{\infty}(2^{-i-1}x_i) &\leq \sum_{i=0}^{t-1}(2^{-i-1}x_i) + \sum_{i=t+1}^{\infty} 2^{-i-1} = \sum_{i=0}^{t-1}(2^{-i-1}x_i) + 2^{-t-1} \\ &< \sum_{i=0}^{l-1}(2^{-i-1}x_i) + 2^{-l-1} \leq \sum_{i=0}^{n-1}(2^{-i-1}y_i) = 2^{-n}k.\end{aligned}$$

This contradicts $x \in \varphi^{-1}[E]$.

Subcase 2.2. $x_l = 1$, $y_l = 0$. Then $l \leq j$, and

$$\begin{aligned}2^{-n}(k+1) &= \sum_{i=0}^{j-1}(2^{-i-1}y_i) + 2^{-j-1} = \sum_{i=0}^{l-1}(2^{-i-1}y_i) + \sum_{i=l}^{j-1}(2^{-i-1}y_i) + 2^{-j-1} \\ &< \sum_{i=0}^{l-1}(2^{-i-1}y_i) + \sum_{i=l+1}^{\infty} 2^{-i-1} = \sum_{i=0}^{l-1}(2^{-i-1}x_i) + 2^{-i-1} \leq \sum_{i=0}^{\infty}(2^{-i-1}x_i,\end{aligned}$$

contradiction.

Hence (8) holds.

Now for (9), assume that $x \in {}^{\omega}2$ and $\forall i < n[x_i = y_i]$. Then

$$\begin{aligned}2^{-n}k &= \sum_{i=0}^{n-1}(2^{-i-1}y_i) \leq \sum_{i=0}^{\infty}(2^{-i-1}x_i) \\ &\leq \sum_{i=0}^{j-1}(2^{-i-1}x_i) + \sum_{i=j+1}^{\infty} 2^{-i-1} = \sum_{i=0}^{j-1}(2^{-i-1}y_i) + 2^{-j-1} = 2^{-n}(k+1).\end{aligned}$$

So (9) holds. This finishes Case 2.

Case 3. $\forall i < n[y_i = 1]$. Then $k+1 = 2^n$ and $2^{-n}k = 1$. To check (8), suppose that $x \in \varphi^{-1}[E]$, x is not eventually 1, and j is minimum such that $x_j \neq y_j$. Take $s > t > j$ with $x_s = x_t = 0$. Then $x_j = 0$, and

$$\sum_{i=0}^{\infty}(2^{-i-1}x_i) \leq \sum_{i=0}^{s-1}(2^{-i-1}x_i) + \sum_{i=s+1}^{\infty} 2^{-i-1} = \sum_{i=0}^{s-1}(2^{-i-1}x_i) + 2^{-s-1} < \sum_{i=0}^{n-1}(2^{-i-1}y_i),$$

contradiction. Thus (8) holds.

For (9), assume that $x \in {}^{\omega}2$ and $\forall i < n[x_i = y_i]$. Clearly $x \in \varphi^{-1}[E]$.

So (8) and (9) hold in all cases.

Now let $S = \{x \in {}^{\omega}2 : x \text{ is eventually } 1\}$. So S is countable. By (8) we have $\varphi^{-1}[E] \backslash S \subseteq \{x \in {}^{\omega}2 : x \upharpoonright n = y \upharpoonright n\}$, and by (9) we have $\{x \in {}^{\omega}2 : x \upharpoonright n = y \upharpoonright n\} \subseteq \varphi^{-1}[E]$. Let $T = \varphi^{-1}[E] \backslash \{x \in {}^{\omega}2 : x \upharpoonright n = y \upharpoonright n\}$. Now $\{x \in {}^{\omega}2 : x \upharpoonright n = y \upharpoonright n\} \in \Sigma_0$

and $\lambda(\{x \in {}^\omega 2 : x \restriction n = y \restriction n\}) = 2^{-n}$ by Proposition 18.85. Note that $T \subseteq S$, so T is countable. Since $\varphi^{-1}[E] = \{x \in {}^\omega 2 : x \restriction n = y \restriction n\} \cup T$, it follows that $\varphi^{-1}[E] \in \Sigma_0$ and $\lambda(\varphi^{-1}[E]) = 2^{-n}$. Since $\varphi^{-1}[E] = (\varphi^{-1}[E] \cap M) \cup (\varphi^{-1}[E]\backslash M)$ and M is countable, it follows that $(\varphi^{-1}[E]\backslash M) \in \Sigma_0$ and $\lambda(\varphi^{-1}[E]\backslash M) = 2^{-n}$. Clearly $\tilde{\varphi}^{-1}[E]\backslash M = \varphi^{-1}[E]\backslash M$, so $(\tilde{\varphi}^{-1}[E]\backslash M) \in \Sigma_0$ and $\lambda(\tilde{\varphi}^{-1}[E]\backslash M) = 2^{-n}$. Hence $\tilde{\varphi}^{-1}[E] \in \Sigma_0$ and $\lambda(\tilde{\varphi}^{-1}[E]) = 2^{-n}$. This proves (7).

(10) If $n \in \omega$ and $k < l \leq 2^n$, and $E = [2^{-n}k, 2^{-n}l]$, then $E \in \Sigma_0$, and $\lambda(\tilde{\varphi}^{-1}[E]) = 2^{-n}(l-k) = \mu(E)$.

This is true by (7) since $E = \bigcup_{k \leq i < l}[2^{-n}i, 2^{-n}(i+1)]\backslash\{2^{-n}i : 0 < i < l\}$.

(11) Suppose that $0 \leq t < u \leq 1$ and $E = [t, u)$. Then $\tilde{\varphi}^{-1}[E]) \in \Sigma_0$, and $\lambda(\tilde{\varphi}^{-1}[E]) = u - t = \mu(E)$.

In fact, for each $n \in \omega$ let $k_n = \lfloor 2^n t \rfloor$ and $l_n = \lfloor 2^n u \rfloor$. Then $k_n \leq 2^n t < k_n + 1$ and $l_n \leq 2^n u < l_n + 1$; hence $2^{-n}k_n \leq t < 2^{-n}k_n + 2^{-n}$ and $2^{-n}l_n \leq u < 2^{-n}l_n + 2^{-n}$. It follows that $\bigcap_{n\in\omega}[2^{-n}k_n, 2^{-n}(l_n+1)] = [t, u]$. Hence by (10), $\tilde{\varphi}^{-1}[E] = \tilde{\varphi}^{-1}[[t, u]] = \bigcap_{n\in\omega}\tilde{\varphi}^{-1}[[2^{-n}k_n, 2^{-n}(l_n+1)]] \in \Sigma_0$. Also, if $m < n$ then $[2^{-n}k_n, 2^{-n}(l_n+1)] \subseteq [2^{-m}k_m, 2^{-m}(l_m+1)]]$, hence $\tilde{\varphi}^{-1}[[2^{-n}k_n, 2^{-n}(l_n+1)]] \subseteq \tilde{\varphi}^{-1}[[2^{-m}k_m, 2^{-m}(l_m+1)]]]$. Hence by Proposition 18.76(iv) we have $\lambda(\tilde{\varphi}^{-1}[E]) = u - t$. Clearly $\mu(E) = u - t$.

Now for each $X \subseteq {}^\omega 2$ define

$$\lambda^*(X) = \inf\{\lambda(E) : X \subseteq E \in \Sigma_0\}.$$

(12) For every $X \subseteq {}^\omega 2$ there is an $E \in \Sigma_0$ such that $X \subseteq E$ and $\lambda^*(X) = \lambda(E)$.

In fact, suppose that $X \subseteq {}^\omega 2$. For each $n \in \omega$ choose $E_n \in \Sigma_0$ such that $X \subseteq E_n$ and $\lambda(E_n) \leq \lambda^*(X) + \frac{1}{2^n}$. Then $E \stackrel{\text{def}}{=} \bigcap_{n\in\omega} E_n \in \Sigma_0$, $X \subseteq E$, and

$$\lambda^*(X) \leq \lambda(E) \leq \inf_{n\in\omega}\lambda(E_n) \leq \lambda^*(X),$$

proving (12).

(13) If $E \in \Sigma_1$, then $\lambda^*(\tilde{\varphi}^{-1}[E]) \leq \mu(E)$ and there is a $V \in \Sigma_0$ such that $\tilde{\varphi}^{-1}[E] \subseteq V$ and $\lambda(V) \leq \mu(E)$.

To prove (13), assume that $E \in \Sigma_1$. By the basic definition of Lebesgue measure,

$$\mu(E\backslash\{1\}) = \inf\Bigg\{\sum_{n\in\omega}\mu(I_n) : \langle I_n : n \in \omega\rangle \text{ is a sequence of half-open subintervals of } [0,1] \text{ such that } E \subseteq \bigcup_{n\in\omega} I_n\Bigg\}$$

Hence for every $\varepsilon > 0$ there is a system $\langle I_n : n \in \omega\rangle$ of half-open subintervals of $[0,1]$ such that $E \subseteq \bigcup_{n\in\omega} I_n$ and $\sum_{n\in\omega}\mu(I_n) \leq \mu(E\backslash\{1\}) + \varepsilon$. Hence

$$\tilde{\varphi}^{-1}[E] \subseteq \{\tilde{\varphi}^{-1}[\{1\}] \cup \bigcup_{n\in\omega}\tilde{\varphi}^{-1}[I_n],$$

and hence

$$\lambda^*(\tilde\varphi^{-1}[E]) \le \lambda\left(\bigcup_{n\in\omega}\tilde\varphi^{-1}[I_n]\right) \le \sum_{n\in\omega}\lambda(\tilde\varphi^{-1}[I_n]) = \sum_{n\in\omega}\mu(I_n) \le \mu(E)+\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $\lambda^*(\tilde\varphi^{-1}[E]) \le \mu(E)$. By (12) there is a $V \in \Sigma_0$ such that $\tilde\varphi^{-1}[E] \subseteq V$ and $\lambda^*(\tilde\varphi^{-1}[E]) = \lambda(V)$. So (13) holds.

(14) If $E \in \Sigma_1$, then $\tilde\varphi^{-1}[E] \in \Sigma_0$ and $\lambda(\tilde\varphi^{-1}[E]) = \mu(E)$.

For, by symmetry with (13) there is a $V' \in \Sigma_0$ such that $\tilde\varphi^{-1}[[0,1]\backslash E] \subseteq V'$ and $\lambda(V') \le \mu([0,1]\backslash E)$. Then ${}^\omega2\backslash\tilde\varphi^{-1}[E] = \tilde\varphi^{-1}[[0,1]\backslash E] \subseteq V'$ and $\tilde\varphi^{-1}[E] \subseteq V$, so $V \cup V' = {}^\omega2$. Now

$$\lambda(V)+\lambda(V') \le \mu(E)+\mu([0,1]\backslash E) = 1 \le \lambda(V\cup V') \le \lambda(V)+\lambda(V').$$

So $\lambda(V)+\lambda(V') = \lambda(V\cup V')$. Hence

$$\begin{aligned}\lambda(V)+\lambda(V') &= \lambda(V\backslash V')+\lambda(V\cap V')+\lambda(V'\backslash V)+\lambda(V\cap V')\\ &= \lambda(V\cup V')+\lambda(V\cap V') = \lambda(V)+\lambda(V')+\lambda(V\cap V').\end{aligned}$$

It follows that $\lambda(V\cap V') = 0$. In particular, $V\cap V'\cap\tilde\varphi^{-1}[E] \in \Sigma_0$. Now $\tilde\varphi^{-1}[[0,1]\backslash E] = \tilde\varphi^{-1}[[0,1]]\backslash\tilde\varphi^{-1}[E] = ({}^\omega2\backslash\tilde\varphi^{-1}[E]) \subseteq V'$, so $({}^\omega2\backslash V') \subseteq \tilde\varphi^{-1}[E]$. Also, $\tilde\varphi^{-1}[E] \subseteq V$. Hence $\tilde\varphi^{-1}[E] = ({}^\omega2\backslash V')\cup(V'\cap\tilde\varphi^{-1}[E] = ({}^\omega2)\backslash V')\cup(V'\cap V\cap\tilde\varphi^{-1}[E]) \in \Sigma_0$.

Now

$$\lambda(\tilde\varphi^{-1}[E]) \le \lambda(V) \le \mu(E) \quad\text{and}\quad 1-\lambda(\tilde\varphi^{-1}[E]) \le \lambda(V') \le 1-\mu(E),$$

so $\lambda(\tilde\varphi^{-1}[E]) = \mu(E)$. Thus (14) holds.

(15) If $n \in \omega$, $\varepsilon \in {}^{n+1}2$, $t = \sum_{i=0}^n(2^{-i-1}\varepsilon_i)$, and $C = \{x \in {}^\omega2 : x \upharpoonright (n+1) = \varepsilon\}$, then $\varphi[C] = [t, t+2^{-n-1}]$.

For, first let $x \in C$. Then

$$t = \sum_{i=0}^n(2^{-i-1}\varepsilon_i) \le \sum_{i=0}^\infty(2^{-i-1}x_i) \le \sum_{i=0}^n(2^{-i-1}\varepsilon_i) + \sum_{i=n+1}^\infty 2^{-i-1} = t+2^{-n-1}.$$

Thus $\varphi(x) \in [t, t+2^{-n-1}]$.

Second, suppose that $u \in [t, t+2^{-n-1}]$.

Case 1. $\varepsilon_n = 0$. Let $u = \sum_{i=0}^\infty(2^{-i-1}x_i)$, with x not eventually 1.

(16) $x \upharpoonright (n+1) = \varepsilon$.

For, suppose that j is minimum such that $x_j \ne \varepsilon_j$. Choose $s > t > j$ such that $x_s = x_t = 0$.

Subcase 2.1. $x_j = 0$ and $\varepsilon_j = 1$. Then

$$u = \sum_{i=0}^\infty(2^{-i-1}x_i) \le \sum_{i=0}^{s-1}(2^{-i-1}x_i) + \sum_{i=s+1}^\infty 2^{-i-1} < \sum_{i=0}^n(2^{-i-1}\varepsilon_i) = t,$$

contradiction.

Subcase 2.2. $x_j = 1$ and $\varepsilon_j = 0$. Then

$$t + 2^{n-1} < \sum_{i=0}^{\infty}(2^{-i-1}x_i) = u,$$

contradiction.

Thus (16) holds, as desired in Case 1.

Case 2. ε is the all 1 sequence, and $u = t + 2^{-n-1}$. Then $u = 1$. Let $x = \langle 1 : i \in \omega \rangle$. Then $\sum_{i=0}^{\infty}(2^{-i-1}x_i) = 1$. Again (16) holds.

Case 3. ε is the all 1 sequence, and $u < t + 2^{-n-1}$. Let $u = \sum_{i=0}^{\infty}(2^{-i-1}x_i)$, with x not eventually 1. We claim that (16) holds again. Otherwise there is a $j \le n$ such that $x_j = 0$. Then $u = \sum_{i=0}^{\infty}(2^{-i-1}x_i) < \sum_{i=0}^{n} 2^{-i-1} = t$, contradiction.

Case 4. $\varepsilon_n = 1$, ε not the all 1 sequence, $u < t + 2^{-n-1}$. Let $u = \sum_{i=0}^{\infty}(2^{-i-1}x_i)$, with x not eventually 1. We claim that (16) holds. Otherwise let j be minimum such that $x_j \neq \varepsilon_j$.

Subcase 4.1. $x_j = 0$ and $\varepsilon_j = 1$. Then

$$u = \sum_{i=0}^{\infty}(2^{-i-1}x_i) < \sum_{i=0}^{n}(2^{-i-1}\varepsilon_i) = t,$$

contradiction.

Subcase 4.2. $x_j = 1$ and $\varepsilon_j = 0$. Then

$$\begin{aligned}
t + 2^{-n-1} &= \sum_{i=0}^{n}(2^{-i-1}\varepsilon_i) + 2^{-n-1} \\
&= \sum_{i=0}^{j-1}(2^{-i-1}\varepsilon_i) + \sum_{i=j+1}^{n}(2^{-i-1}\varepsilon_i) + 2^{n-1} \\
&\le \sum_{i=0}^{j-1}(2^{-i-1}\varepsilon_i) + \sum_{i=j+1}^{n} 2^{-i-1} + 2^{n-1} \\
&= \sum_{i=0}^{j-1}(2^{-i-1}\varepsilon_i) + 2^{j-1} \\
&\le \sum_{i=0}^{\infty}(2^{-i-1}x_i) = u,
\end{aligned}$$

contradiction.

Case 5. $\varepsilon_n = 1$, ε not the all 1 sequence, $u = t + 2^{-n-1}$. Let $x = \varepsilon^\frown\langle 1 : i \in \omega \rangle$. Then with j maximum such that $\varepsilon_j = 0$ we have

$$u = t + 2^{-n-1} = \sum_{i=0}^{j-1}(2^{-i-1}\varepsilon_i) + 2^{-j-1} = \sum_{i=0}^{\infty}(2^{-i-1}x_i).$$

This finishes the proof of (15).

(17) If $n \in \omega$, $\varepsilon \in {}^{n+1}2$, $t = \sum_{i=0}^{n}(2^{-i-1}\varepsilon_i)$, and $C = \{x \in {}^{\omega}2 : x \restriction (n+1) = \varepsilon\}$, then $\mu(\varphi[C]) = \lambda(C) = 2^{-n-1}$.

This is clear from (15).

(18) If $n \in \omega$, $\varepsilon \in {}^{n+1}2$, $t = \sum_{i=0}^{n}(2^{-i-1}\varepsilon_i)$, and $C = \{x \in {}^{\omega}2 : x \restriction (n+1) = \varepsilon\}$, then $\mu(\tilde{\varphi}[C]) = \lambda(C) = 2^{-n-1}$.

Recall that M is countable, and $N = \varphi[M] \cup \tilde{\varphi}[M]$ is countable. Clearly $\varphi[C] \backslash N = \tilde{\varphi}[C] \backslash N$. Hence

$$\begin{aligned}\lambda(C) = \mu(\varphi[C]) &= \mu(\varphi[C] \cap N) + \mu(\varphi[C] \backslash N)\\ &= \mu(\varphi[C] \backslash N) = \mu(\tilde{\varphi}[C] \backslash N)\\ &= \mu(\tilde{\varphi}[C] \backslash N) + \mu(\tilde{\varphi}[C] \cap N) = \mu(\tilde{\varphi}[C]).\end{aligned}$$

(19) If $F \in [\omega]^{<\omega}$, $h \in {}^{F}2$, and $C = \{x \in {}^{\omega}2 : h \subseteq x\}$, then $\mu(\tilde{\varphi}[C]) = \lambda(C)$.

In fact, choose $m \in \omega$ such that $F \subseteq m$. Then

$$C = \bigcup\{\{x \in {}^{\omega}2 : k \subseteq x\} : k \in {}^{m}2 \text{ and } h \subseteq k\}.$$

For each $k \in {}^{m}2$ such that $h \subseteq k$ let $D_k = \{x \in {}^{\omega}2 : k \subseteq x\}$. Note that $D_k \cap D_l = \emptyset$ when $k \neq l$. Let $I = \{k \in {}^{m}2 : h \subseteq k\}$ Note that $|\{k \in {}^{m}2 : h \subseteq k\}| = 2^{m-|F|}$. Now $\lambda(C) = 2^{-|F|}$ by Proposition 18.85 and by (18),

$$\mu(\tilde{\varphi}[C]) = \mu\left(\bigcup_{k \in I} \tilde{\varphi}[D_k]\right) = \sum_{k \in I} 2^{-m} = 2^{-m}2^{m-|F|} = 2^{-|F|}.$$

So (19) holds.

Now for each $X \subseteq [0,1]$ define

$$\mu^*(X) = \inf\{\mu(E) : X \subseteq E \in \Sigma_1\}.$$

(20) For every $X \subseteq [0,1]$ there is an $E \in \Sigma_1$ such that $X \subseteq E$ and $\mu^*(X) = \mu(E)$.

In fact, suppose that $X \subseteq [0,1]$. For each $n \in \omega$ choose $E_n \in \Sigma_1$ such that $X \subseteq E_n$ and $\mu(E) \leq \mu^*(X) + \frac{1}{2^n}$. Then $E \stackrel{\text{def}}{=} \bigcap_{n \in \omega} E_n \in \Sigma_1$, $X \subseteq E$, and

$$\mu^*(X) \leq \lambda(E) \leq \inf_{n \in \omega} \mu(E_n) \leq \mu^*(X),$$

proving (20).

(21) If $E \in \Sigma_0$, then $\mu^*(\tilde{\varphi}[E]) \leq \lambda(E)$ and there is a $V \in \Sigma_1$ such that $\tilde{\varphi}[E] \subseteq V$ and $\mu(V) \leq \lambda(E)$.

To prove (21), assume that $E \in \Sigma_0$. By the basic definition of measure on ${}^\omega 2$,

$$\lambda(E) = \inf\left\{\sum_{n\in\omega} \theta_0(U_{f_n}) : E \subseteq \bigcup_{n\in\omega} U_{f_n}\right\}.$$

(For θ_0 see before Proposition 18.29.) Hence for every $\varepsilon > 0$ there is a system $\langle f_n : n \in \omega\rangle$ such that $E \subseteq \bigcup_{n\in\omega} U_{f_n}$ and $\sum_{n\in\omega} \lambda(U_{f_n}) \leq \lambda(E) + \varepsilon$. Hence

$$\tilde{\varphi}[E] \subseteq \bigcup_{n\in\omega} \tilde{\varphi}[U_{f_n}],$$

and hence, using (19),

$$\mu^*(\tilde{\varphi}[E]) \leq \mu\left(\bigcup_{n\in\omega} \tilde{\varphi}[U_{f_n}]\right) \leq \sum_{n\in\omega} \mu(\tilde{\varphi}[U_{f_n}]) = \sum_{n\in\omega} \lambda(U_{f_n}) \leq \lambda(E) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $\mu^*(\tilde{\varphi}[E]) \leq \lambda(E)$. By (20) there is a $V \in \Sigma_1$ such that $\tilde{\varphi}[E] \subseteq V$ and $\mu^*(\tilde{\varphi}[E]) = \mu(V)$. So (21) holds.

(22) If $E \in \Sigma_0$, then $\tilde{\varphi}[E] \in \Sigma_1$ and $\mu(\tilde{\varphi}[E]) = \lambda(E)$.

For, by symmetry with (21) there is a $V' \in \Sigma_1$ such that $\tilde{\varphi}[{}^\omega 2\backslash E] \subseteq V'$ and $\mu(V') \leq \lambda({}^\omega 2\backslash E)$. Then $V \cup V' = [0, 1]$ and

$$\mu(V) + \mu(V') \leq \lambda(E) + \lambda({}^\omega 2\backslash E) = 1 \leq \mu(V \cup V') \leq \mu(V) + \mu(V').$$

So $\mu(V) + \mu(V') = \mu(V \cup V')$. Hence

$$\begin{aligned} \mu(V) + \mu(V') &= \mu(V\backslash V') + \mu(V \cap V') + \mu(V'\backslash V) + \mu(V \cap V') \\ &= \mu(V \cup V') + \mu(V \cap V') = \mu(V) + \mu(V') + \mu(V \cap V'). \end{aligned}$$

It follows that $\mu(V \cap V') = 0$. In particular, $V \cap V' \cap \tilde{\varphi}[E] \in \Sigma_1$. Now $\tilde{\varphi}[{}^\omega 2\backslash E] = \tilde{\varphi}[{}^\omega 2]\backslash\tilde{\varphi}[E] = [0, 1]\backslash\tilde{\varphi}[E] \subseteq V'$, so $[0, 1]\backslash V' \subseteq \tilde{\varphi}[E]$. Hence $\tilde{\varphi}[E] = ([0, 1]\backslash V') \cup (V' \cap \tilde{\varphi}[E]) = [0, 1]\backslash V') \cup (V' \cap V \cap \tilde{\varphi}[E]) \in \Sigma_1$.

Now

$$\mu(\tilde{\varphi}[E]) \leq \mu(V) \leq \mu(E) \quad \text{and} \quad 1 - \mu(\tilde{\varphi}[E]) \leq \mu(V') \leq 1 - \mu(E),$$

so $\mu(\tilde{\varphi}[E]) = \mu(E)$. Thus (22) holds.

Now (14) gives (d) of Theorem 18.104 and $\Rightarrow$ of (b). (22) gives (c) and $\Rightarrow$ in (a). For $\Leftarrow$ of (a), suppose that $X \subseteq {}^\omega 2$ and $\tilde{\varphi}[X] \in \Sigma_1$. By $\Rightarrow$ of (b), $X = \tilde{\varphi}^{-1}[\tilde{\varphi}[X] \in \Sigma_0$. For $\Leftarrow$ of (b), suppose that $X \subseteq [0, 1]$ and $\tilde{\varphi}^{-1}[X] \in \Sigma_0$. Then by (a), $X = \tilde{\varphi}[\tilde{\varphi}^{-1}[X]] \in \Sigma_0 \in \Sigma_1$. $\square$

Lemma 18.105. *If $E \subseteq \mathscr{P}(\Sigma_0)$, then $\tilde{\varphi}\left[\bigcup E\right] = \bigcup_{A\in E} \tilde{\varphi}[A]$.* $\square$

Proposition 18.106. $\text{add}(\text{null}_{^\omega 2}) = \text{add}(\text{null}_{[0,1]})$.

Proof. First let $\kappa = \text{add}(\text{null}_{^\omega 2})$, and let $E \in [\text{null}_{^\omega 2}]^\kappa$ with $\bigcup E \notin \text{null}_{^\omega 2}$. For each $A \in E$ let $A' = \tilde{\varphi}[A]$, and let $E' = \{A' : A \in E\}$. Then by Theorem 18.104(c), $E' \subseteq \mathscr{P}(\text{null}_{[0,1]})$. Suppose that $\bigcup E' \in \text{null}_{[0,1]}$. By Theorem 18.104(d),

$$\bigcup E = \bigcup_{A \in E} \tilde{\varphi}^{-1}[\tilde{\varphi}[A]] = \bigcup_{B \in E'} \tilde{\varphi}^{-1}[B] = \tilde{\varphi}^{-1}\left[\bigcup E'\right] \in \text{null}_{^\omega 2},$$

contradiction.

Second let $\kappa = \text{add}(\text{null}_{[0,1]})$, and let $E \in [\text{null}_{[0,1]}]^\kappa$ with $\bigcup E \notin \text{null}_{[0,1]}$. For each $A \in E$ let $A' = \tilde{\varphi}^{-1}[A]$. Thus $A' \in \text{null}_{^\omega 2}$ by Theorem 18.104(d). Continue as in the first case. $\square$

Proposition 18.107. $\text{cov}(\text{null}_{^\omega 2}) = \text{cov}(\text{null}_{[0,1]})$.

Proof. First let $\kappa = \text{cov}(\text{null}_{^\omega 2})$, and let $E \in [\text{null}_{^\omega 2}]^\kappa$ with ${}^\omega 2 = \bigcup E$.

$$[0,1] = \tilde{\varphi}[{}^\omega 2] = \tilde{\varphi}\left[\bigcup E\right] = \bigcup_{A \in E} \tilde{\varphi}[A],$$

and each $\tilde{\varphi}[A] \in \text{null}_{[0,1]}$.

The other direction is similar. $\square$

Proposition 18.108. $\text{non}(\text{null}_{^\omega 2}) = \text{non}(\text{null}_{[0,1]})$.

Proof. First let $\kappa = \text{non}(\text{null}_{^\omega 2})$, and let $X \in [{}^\omega 2]^\kappa$ such that $X \notin \text{null}_{^\omega 2}$. If $\tilde{\varphi}[X] \in \text{null}_{[0,1]}$, this is a contradiction.

The other direction is similar. $\square$

Proposition 18.109. $\text{cof}(\text{null}_{^\omega 2}) = \text{cof}(\text{null}_{[0,1]})$.

Proof. First let $\kappa = \text{non}(\text{null}_{^\omega 2})$, and let $X \in [{}^\omega 2]^\kappa$ such that $\forall A \in \text{null}_{^\omega 2} \exists B \in X[A \subseteq B]$. Let $X' = \{\tilde{\varphi}[C] : C \in X\}$. Take any $A \in \text{null}_{[0,1]}$. Then $\tilde{\varphi}^{-1}[A] \in \text{null}_{^\omega 2}$, so there is a $B \in X$ such that $\tilde{\varphi}^{-1}[A] \subseteq B$. Then $\tilde{\varphi}[\tilde{\varphi}^{-1}[A]] = A \subseteq \tilde{\varphi}[B]$.

The other direction is similar. $\square$

Let $\mathscr{A} = \{X \cap \Theta : X \subseteq {}^\omega 2 \text{ is measurable}\}$. Then $\mathscr{A}$ is a σ-field of subsets of Θ. For any X measurable in ${}^\omega 2$ the set $X \cap \Theta$ is also measurable.

Proposition 18.110. $\text{add}(\text{null}_{^\omega 2}) = \text{add}(\text{null}_\Theta)$.

Proof. First suppose that $E \in [\text{null}_{^\omega 2}]^\kappa$, $\bigcup E \notin \text{null}_{^\omega 2}$, and $|E| = \text{add}(\text{null}_{^\omega 2})$. Let $E' = \{X \cap \Theta : X \in E\}$. Then $E' \subseteq \text{null}_\Theta$. If $\bigcup E' \in \text{null}_\Theta$, then $\bigcup E \subseteq \bigcup E' \cup N \in \text{null}_{^\omega 2}$, contradiction, where $N = \{x \in {}^\omega 2 : \{i \in \omega : x(i) = 1\} \text{ is finite}\}$.

Second suppose that $E \in [\text{null}_\Theta]^\kappa$, $\bigcup E \notin \text{null}_\Theta$, and $|E| = \text{add}(\text{null}_\Theta)$. Then $E \subseteq \text{null}_{^\omega 2}$ and $\bigcup E \notin \text{null}_{^\omega 2}$. $\square$

Proposition 18.111. $\mathrm{cov}(\mathrm{null}_{^\omega 2}) = \mathrm{cov}(\mathrm{null}_\Theta)$.

Proof. First suppose that $E \in [\mathrm{null}_{^\omega 2}]^\kappa$, ${}^\omega 2 = \bigcup E$, and $|E| = \mathrm{cov}(\mathrm{null}_{^\omega 2})$. Let $E' = \{X \cap \Theta : X \in E\}$. Then $\bigcup E' = \Theta$ and $E' \subseteq \mathrm{null}_\Theta$.

Second suppose that $E \in [\mathrm{null}_\Theta]^\kappa$, $\Theta = \bigcup E$, and $|E| = \mathrm{cov}(\mathrm{null}_\Theta)$. Then $E \subseteq \mathrm{null}_{^\omega 2}$ and ${}^\omega 2 = \bigcup E \cup N$, with N as above. $\square$

Proposition 18.112. $\mathrm{non}(\mathrm{null}_{^\omega 2}) = \mathrm{non}(\mathrm{null}_\Theta)$.

Proof. First let $X \in [{}^\omega 2]^\kappa$ such that $X \notin \mathrm{null}_{^\omega 2}$ and $\kappa = \mathrm{non}(\mathrm{null}_{^\omega 2})$. Then $X \cap \Theta \subseteq \Theta$ and $X \cap \Theta \notin \mathrm{null}_\Theta$, as otherwise $X \subseteq (X \cap \Theta) \cup N \in \mathrm{null}_{^\omega 2}$, with N as above.

Second let $X \in [\Theta]^\kappa$ such that $X \notin \mathrm{null}_\Theta$ and $\kappa = \mathrm{non}(\mathrm{null}_\Theta)$. Then $X \notin \mathrm{null}_{^\omega 2}$. $\square$

Proposition 18.113. $\mathrm{cof}(\mathrm{null}_{^\omega 2}) = \mathrm{cof}(\mathrm{null}_\Theta)$.

Proof. First suppose that $X \in [\mathrm{null}_{^\omega 2}]^\kappa$, $\forall A \in \mathrm{null}_{^\omega 2} \exists B \in X[A \subseteq B]$, and $\kappa = \mathrm{cof}(\mathrm{null}_{^\omega 2})$. Let $Y = \{B \cap \Theta : B \in X\}$. Thus $Y \subseteq \mathrm{null}_\Theta$. Suppose that $A \in \mathrm{null}_\Theta$. Then $A \in \mathrm{null}_{^\omega 2}$, so there is a $B \in X$ such that $A \subseteq B$. Hence $A \subseteq B \cap \Theta \in Y$.

Second suppose that $X \in [\mathrm{null}_\Theta]^\kappa$, $\forall A \in \mathrm{null}_\Theta \exists B \in X[A \subseteq B]$, and $\kappa = \mathrm{cof}(\mathrm{null}_\Theta)$. Let $Y = \{B \cup N : B \in X\}$, with N as above. So $Y \subseteq \mathrm{null}_{^\omega 2}$. Suppose that $A \in \mathrm{null}_{^\omega 2}$. Then $A \cap \Theta \in \mathrm{null}_\Theta$, so there is a $B \in X$ such that $A \cap \Theta \subseteq B$, so $A \subseteq B \cup N \in Y$. $\square$

There is a bijection f from Θ onto $[\omega]^\omega$. So the measure on Θ can be carried over to a measure on $[\omega]^\omega$.

19. The Cichoń diagram

To express this diagram we need to introduce two of the continuum cardinals considered more completely later.

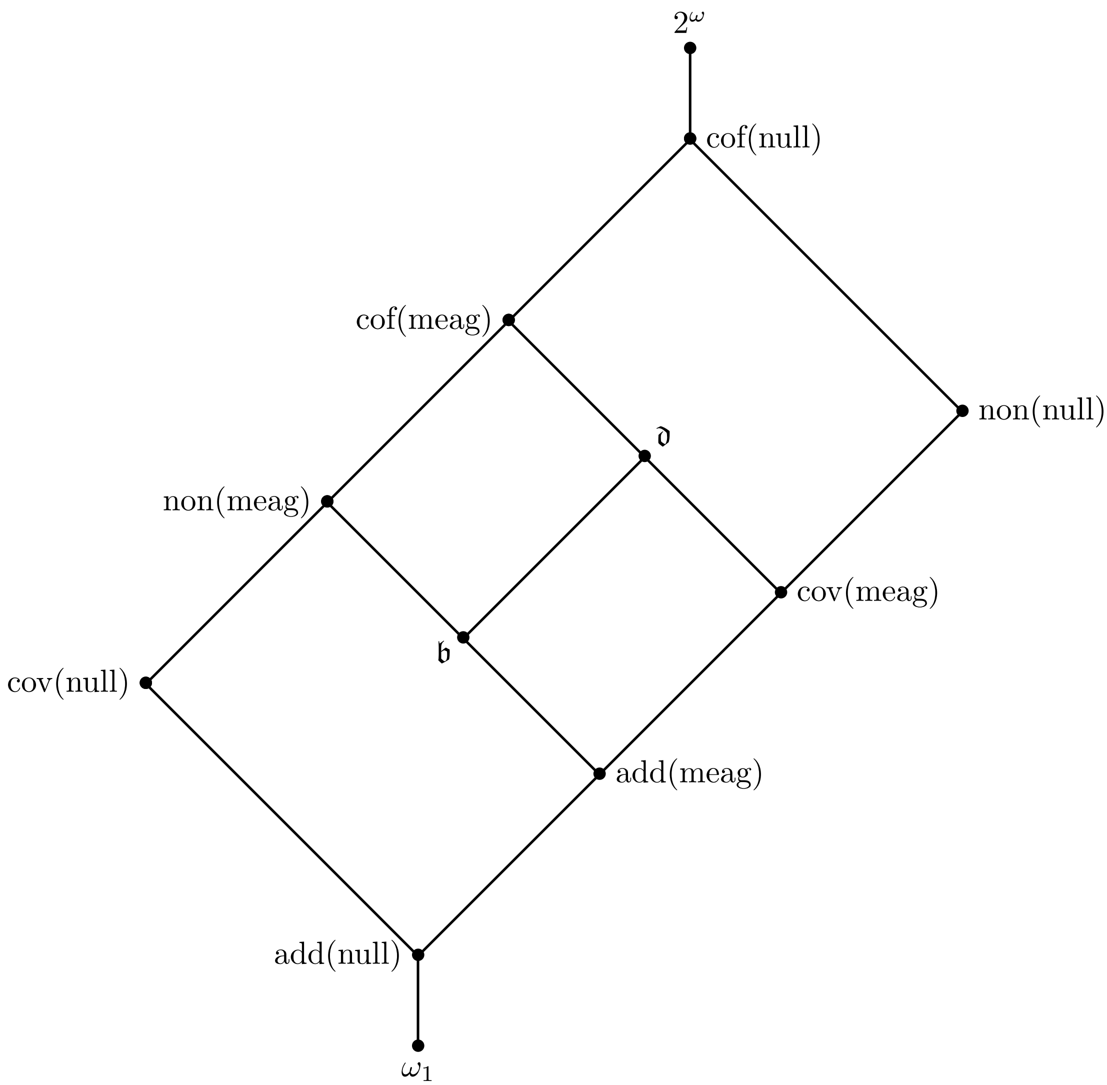
We define $f \leq g$ iff $f, g \in {}^\omega\omega$ and $f(m) \leq g(m)$ for all $m \in \omega$.

We define $f \leq^* g$ iff $f, g \in {}^\omega\omega$ and $\exists m \forall n \geq m[f(n) \leq g(n)]$.

A family $\mathscr{D} \subseteq {}^\omega\omega$ is *almost dominating* iff $\forall f \in {}^\omega\omega \exists g \in \mathscr{D}[f \leq^* g]$. Let $\mathfrak{d}$ be the smallest size of a almost dominating family; this is the *dominating* number.

A family $\mathscr{B} \subseteq {}^\omega\omega$ is *almost unbounded* iff there is no $f \in {}^\omega\omega$ such that $\forall g \in \mathscr{B}[g \leq^* f]$. Let $\mathfrak{b}$ be the smallest size of an almost unbounded family.

We prove that the relations expressed in the following diagram hold; this diagram will be expanded later.



Proposition 19.1. $\omega_1 \leq \text{add(null)}$. $\square$

Proposition 19.2. $\text{add(null)} \leq \text{cov(null)}$.

Proof. By Lemma 18.9. □

Proposition 19.3. add(meag) $\le$ cov(meag).

Proof. By Lemma 18.9. □

Let Inc $= \{i \in {}^\omega\omega : i$ is strictly increasing and $i_0 = 0\}$. For any $i \in$ Inc we define for any $n \in \omega$ $I^i_n = [i_n, i_{n+1})$. A *chopped real* is a pair (x, i) such that $x \in {}^\omega 2$ and $i \in$ Inc. A function $y \in {}^\omega 2$ *matches* a chopped real (x, i) iff $x \restriction I^i_n = y \restriction I^i_n$ for infinitely many $n \in \omega$.

Theorem 19.4. *For any chopped real (x, i) the set $\{y \in {}^\omega 2 : y$ does not match $(x, i)\}$ is a meager subset of ${}^\omega 2$.*

Proof. Let (x, i) be a chopped real. Then

$$\{y \in {}^\omega 2 : y \text{ matches } (x, i)\} = \bigcap_k \bigcup_{n \ge k} \{y \in {}^\omega 2 : x \restriction I^i_n = y \restriction I^i_n\}. \tag{1}$$

Now each set $\{y \in {}^\omega 2 : x \restriction I^i_n = y \restriction I^i_n\}$ is open in ${}^\omega 2$, since it is equal to $U_{x \restriction I^i_n}$. Hence for any $k \in \omega$ the set $\bigcup_{n \ge k} \{y \in {}^\omega 2 : x \restriction I^i_n = y \restriction I^i_n\}$ is open. It is also dense; for suppose that $z \subseteq \omega \times 2$ is a finite function. Choose $n \ge k$ such that $I^i_n \cap \mathrm{dmn}(z) = \emptyset$, and let $y \in {}^\omega 2$ extend z and $x \restriction I^i_n$. Then $y \in U_z \cap \bigcup_{n \ge k} \{y \in {}^\omega 2 : x \restriction I^i_n = y \restriction I^i_n\}$.

Hence (1) says that $\{y \in {}^\omega 2 : y$ matches $(x, i)\}$ is a countable intersection of dense open sets. Hence its complement is a countable union of nowhere dense sets, i.e., it is meager. □

Let Cov(meag) $= ({}^\omega 2, \text{meag}, \in))$. This is a relational triple; see after Proposition 18.20 for the definition.

Proposition 19.5. $||$Cov(meag)$||$ = cov(meag) *in* ${}^\omega 2$.

Proof. $||\mathrm{Cov(meag)}|| = \min\{|Y| : Y \subseteq \text{meag and } \forall x \in {}^\omega 2 \exists y \in Y[x \in y]\} = \mathrm{cov(meag)}$. □

Let Cov(null) $= ({}^\omega 2, \text{null}, \in))$.

Proposition 19.6. $||$Cov(null)$||$ = cov(null) *in* ${}^\omega 2$.

Proof. $||\mathrm{Cov(null)}|| = \min\{|Y| : Y \subseteq \text{null and } \forall x \in {}^\omega 2 \exists y \in Y[x \in y]\} = \mathrm{cov(null)}$. □

Proposition 19.7. *Define $i \in {}^\omega\omega$ recursively by $i_0 = 0$ and $i_{n+1} = i_n + n + 1$. Thus $i \in$ Inc. Define fRg iff $f, g \in {}^\omega 2$ and g matches the chopped real (f, i). Let $\mathbf{R} = ({}^\omega 2, {}^\omega 2, R)$. Thus $\mathbf{R}$ is a relational triple. Let ψ_0 be the identity on ${}^\omega 2$ and let $\psi_1(g) = {}^\omega 2 \backslash \{f : fRg\}$. Then (ψ_0, ψ_1) is a morphism from $\mathbf{R}^\perp$ to* Cov(*meag*)*.*

Proof. Recall that

$$\begin{aligned} \mathbf{R}^\perp &= ({}^\omega 2, {}^\omega 2, \{(g, f) : g, f \in {}^\omega 2 \text{ and } (f, g) \notin \mathbf{R}\}) \\ &= ({}^\omega 2, {}^\omega 2, \{(g, f) : g, f \in {}^\omega 2 \text{ and } g \text{ does not match}(f, i)\}) \quad \text{and} \\ \mathrm{Cov(meag)} &= ({}^\omega 2, \text{meag}, \in). \end{aligned}$$

Thus the first two conditions for a morphism hold, using Theorem 19.4. Now suppose that $f, g \in {}^{\omega}2$ and $(\psi_0(g), f) \in \{(g, f) : g, f \in {}^{\omega}2 \text{ and } g \text{ does not match}(f, i)\}$. Thus $g = \psi_0(g)$ does not match (f, i). Hence $f \in \psi_1(g)$. $\square$

Proposition 19.8. (Continuing Proposition 19.7) *(i) For any $g \in {}^{\omega}2$, $\{f : fRg\}$ has measure zero.*

(ii) Let φ_0 be the identity on ${}^{\omega}2$ and $\varphi_1(g) = \{f : fRg\}$ for any $g \in {}^{\omega}2$. Then (φ_0, φ_1) is a morphism from **R** *to* Cov(null).

Proof. (i): Note that $|I_n^i = |[i_n, i_n + n + 1)| = n + 1$. If $g \in {}^{\omega}2$ and $n \in \omega$, then $\{f \in {}^{\omega}2 : f \text{ agrees with } g \text{ on } I_n\}$ has measure $1/2^{n+1}$. Hence for any $g \in {}^{\omega}2$, $\{f \in {}^{\omega}2 : fRg\}$ has measure 0; see the proof of Theorem 19.4.

(ii): the first two conditons are clear, using (i). Now suppose that $f, g \in {}^{\omega}2$ and $\varphi_0(f)Rg$. Thus fRg, so $f \in \varphi_1(g)$. $\square$

Proposition 19.9. *Suppose that* **A**, **B** *and* **C** *are relational triples, (φ_0, φ_1) is a morphism from* **A** *to* **B**, *and (ψ_0, ψ_1) is a morphism from* **B** *to* **C**. *Then $(\varphi_0 \circ \psi_0, \psi_1 \circ \varphi_1)$ is a morphism from* **A** *to* **C**.

Proof. Since $\varphi_0 : B_0 \to A_0$ and $\psi_0 : C_0 \to B_0$, it follows that $\varphi_0 \circ \psi_0 : C_0 \to A_0$. Since $\varphi_1 : A_1 \to B_1$ and $\psi_1 : B_1 \to C_1$, it follows that $\psi_1 \circ \varphi_1 : A_1 \to C_1$. Now suppose that $a \in A_1$, $c \in C_0$, and $\varphi_0(\psi_0(c))Aa$. Then $\psi_0(c)B\varphi_1(a)$ and hence $cC\psi_1(\varphi_1(a))$. $\square$

Proposition 19.10. $||\mathrm{Cov}^{\perp}(\mathrm{meag})|| = \mathrm{non}(\mathrm{meag})$.

Proof. We have $\mathrm{Cov}^{\perp}(\mathrm{meag}) = (\mathrm{meag}, {}^{\omega}2, \not\ni)$. Hence $||\mathrm{Cov}^{\perp}(\mathrm{meag})|| = \min\{|Y| : Y \subseteq {}^{\omega}2 \text{ and } \forall X \in \mathrm{meag} \exists x \in Y[x \notin X]\} = \mathrm{non}(\mathrm{meag})$. $\square$

Proposition 19.11. $\mathrm{cov}(\mathrm{null}) \leq \mathrm{non}(\mathrm{meag})$.

Proof. Let φ_0, φ_1, ψ_0, ψ_1 be as above. Then by propositions 18.18 and 19.7, (ψ_1, ψ_0) is a morphism from $\mathrm{Cov}^{\perp}(\mathrm{meag})$ to **R**. Now (φ_0, φ_1) is a morphism from **R** to Cov(null). Hence by Proposition 19.9, $(\psi_1 \circ \varphi_0, \varphi_1 \circ \psi_0)$ is a morphism from $\mathrm{Cov}^{\perp}(\mathrm{meag})$ to Cov(null). Hence by Propositions 18.19, 19.6, and 19.10, $\mathrm{cov}(\mathrm{null}) = ||\mathrm{Cov}(\mathrm{null})|| \leq ||\mathrm{Cov}^{\perp}(\mathrm{meag})|| = \mathrm{non}(\mathrm{meag})$. $\square$

Proposition 19.12. $||\mathrm{Cov}^{\perp}(\mathrm{null})|| = \mathrm{non}(\mathrm{null})$.

Proof. We have $\mathrm{Cov}^{\perp}(\mathrm{null}) = (\mathrm{null}, {}^{\omega}2, \not\ni)$. Hence $||\mathrm{Cov}^{\perp}(\mathrm{null})|| = \min\{|Y| : Y \subseteq {}^{\omega}2 \text{ and } \forall X \in \mathrm{null} \exists x \in Y[x \notin X]\} = \mathrm{non}(\mathrm{null})$. $\square$

Proposition 19.13. $\mathrm{cov}(\mathrm{meag}) \leq \mathrm{non}(\mathrm{null})$.

Proof. Let φ_0, φ_1, ψ_0, ψ_1 be as above. Then by propositions 18.18 and 19.7, (ψ_1, ψ_0) is a morphism from $\mathrm{Cov}^{\perp}(\mathrm{null})$ to **R**. Now (φ_0, φ_1) is a morphism from **R** to Cov(meag). Hence by Proposition 19.9, $(\varphi_1 \circ \psi_0, \psi_1 \circ \varphi_0)$ is a morphism from $\mathrm{Cov}^{\perp}(\mathrm{null})$ to Cov(meag). Hence by Propositions 18.19, 19.5, and 19.12, $\mathrm{cov}(\mathrm{meag}) = ||\mathrm{Cov}(\mathrm{meag})|| \leq ||\mathrm{Cov}^{\perp}(\mathrm{null})|| = \mathrm{non}(\mathrm{null})$. $\square$

Theorem 19.14. $\mathrm{cf}(\mathfrak{b}) = \mathfrak{b} \leq \mathrm{cf}(\mathfrak{d}) \leq \mathfrak{d}$.

Proof. Suppose that $\mathrm{cf}(\mathfrak{b}) < \mathfrak{b}$. Let X be almost unbounded with $|X| = \mathfrak{b}$. Then we can write $X = \bigcup_{\alpha<\mathrm{cf}(\mathfrak{b})} Y_\alpha$ with $|Y_\alpha| < \mathfrak{b}$ for all $\alpha < \mathrm{cf}(\mathfrak{b})$. Choose a bound g^α for Y_α for each $\alpha < \mathrm{cf}(\mathfrak{b})$, and then by the above argument choose a bound h for $\{g^\alpha : \alpha < \mathrm{cf}(\mathfrak{b})\}$. Then h is a bound for X, contradiction. Thus $\mathrm{cf}(\mathfrak{b}) = \mathfrak{b}$.

To prove that $\mathfrak{b} \leq \mathrm{cf}(\mathfrak{d})$, let D be a almost dominating family of size $\mathfrak{d}$, and write $D = \bigcup_{\alpha<\mathrm{cf}(\mathfrak{d})} E_\alpha$, with each E_α of size less than $\mathfrak{d}$. Since then E_α is not almost dominating, there is an $f^\alpha \in {}^\omega\omega$ such that for all $g \in E_\alpha$ we have $f^\alpha \not\leq^* g$. Suppose that $\mathrm{cf}(\mathfrak{d}) < \mathfrak{b}$, and accordingly let $h \in {}^\omega\omega$ be such that $f^\alpha \leq^* h$ for all $\alpha < \mathrm{cf}(\mathfrak{d})$. Choose $k \in D$ such that $h \leq^* k$. Say $k \in E_\alpha$. But $f^\alpha \leq^* h \leq^* k$, contradiction. $\square$

For each $i \in \mathrm{Inc}$ we associate $\mathrm{func}^i \in {}^\omega\omega$ as follows. Let $x \in \omega$ choose n so that $i_n \leq x < i_{n+1}$. Let $\mathrm{func}^i(x) = i_{n+2} - 1$.

With each $g \in {}^\omega\omega$ we associate $\nu^g \in \mathrm{Inc}$ as follows; $\nu^g(n)$ is defined by recursion on n. Let $\nu^g_0 = 0$. If ν^g_n has been defined, let ν^g_{n+1} be minimum such that $\nu^g_n < \nu^g_{n+1}$ and $\forall x \leq \nu^g_n[g(x) < \nu^g_{n+1}]$.

Given $i, j \in \mathrm{Inc}$, we say that i *almost dominates* j iff $\exists m\forall n \geq m\exists k[[j_k, j_{k+1}) \subseteq [i_n, i_{n+1})]$.

Proposition 19.15. *If $i \in \mathrm{Inc}$, $g \in {}^\omega\omega$, and i almost dominates ν^g, then $g \leq^* \mathrm{func}^i$.*

Proof. By definition, choose m so that for all $n \geq m$ there is a k such that $[\nu^g_k, \nu^g_{k+1}) \subseteq [i_n, i_{n+1})$. Take any $x \geq i_m$; we claim that $g(x) \leq \mathrm{func}^i(x)$. For, take n such that $x \in [i_n, i_{n+1})$. Then $i_{n+1} > x \geq i_m$, so $n + 1 > m$. Hence there ia a k such that $[\nu^g_k, \nu^g_{k+1}) \subseteq [i_{n+1}, i_{n+2})$. Now $x < i_{n+1} \leq \nu^g_k$, so $g(x) \leq \nu^g_{k+1} - 1 \leq i_{n+2} - 1 = \mathrm{func}^i(x)$. $\square$

Proposition 19.16. *If $i \in \mathrm{Inc}$, $g \in {}^\omega\omega$, and $\mathrm{func}^i \leq^* g$, then ν^g almost dominates i.*

Proof. Assume the hypotheses, and choose m so that $\forall n \geq m[\mathrm{func}^i(n) \leq g(n)]$. We claim that $\forall n \geq m\exists k[[i_{k+1}, i_{k+2}) \subseteq [\nu^g_n, \nu^g_{n+1})$ (as desired). For, take any $n \geq m$. Choose k such that $\nu^g_n \in [i_k, i_{k+1})$. Take any $x \in [i_{k+1}, i_{k+2})$. Then $m \leq n \leq \nu^g_n$, so

$$\nu^g_n < i_{k+1} \leq x \leq i_{k+2} - 1 = \mathrm{func}^i(\nu^g_n) \leq g(\nu^g_n) < \nu^g_{n+1}. \qquad \square$$

Proposition 19.17. $\mathfrak{b} = \min\{|X| : X \subseteq \mathrm{Inc}$ *and* $\neg\exists i \in \mathrm{Inc}\forall j \in X[i$ *almost dominates* $j]\}$.

Proof. First suppose that $X \subseteq {}^\omega\omega$ is almost unbounded, with $|X| = \mathfrak{b}$. Let $Y = \{\nu^g : g \in X\}$. Thus $|Y| \leq \mathfrak{b}$. Suppose that $i \in \mathrm{Inc}$ almost dominates each ν^g for $g \in X$. Then by Proposition 19.15, $g \leq^* \mathrm{func}^i$ for each $g \in X$, contradiction. Thus Y is one of the sets on the right, and hence rhs $\leq \mathfrak{b}$.

Second, suppose that $Y \subseteq \mathrm{Inc}$ is such that $\neg\exists i \in \mathrm{Inc}\forall j \in Y[i$ almost dominates j, with $|Y|$ minimum. Let $X = \{\mathrm{func}^i : i \in Y\}$. Suppose that $f \in {}^\omega\omega$ and $\mathrm{func}^i \leq^* f$ for all $i \in X$. Then by Proposition 19.16, ν^f almost dominates each $i \in X$, contradiction. Hence X is unbouned, and so $\mathfrak{b} \leq |X| \leq |Y|$. $\square$

Lemma 19.18. *$X \subseteq {}^{\omega}2$ is nowhere dense iff for every finite function $z \subseteq \omega \times 2$ there is a finite function $w \subseteq \omega \times 2$ such that $z \subseteq w$ and $U_w \cap X = \emptyset$.*

Proof. Suppose that X is nowhere dense and $z \subseteq \omega \times 2$ is a finite function. Then $U_z \backslash \overline{X} \neq \emptyset$. Take any $f \in U_z \backslash \overline{X}$. Then there is a finite function $t \subseteq \omega \times 2$ such that $f \in U_t$ and $U_t \cap X = \emptyset$. Now $z \cup t \subseteq f$, so $z \cup t$ is a function. Let $w = z \cup t$. Then w is as desired.

Conversely, suppose that for every finite function $z \subseteq \omega \times 2$ there is a finite function $w \subseteq \omega \times 2$ such that $z \subseteq w$ and $U_w \cap X = \emptyset$. Let $z \subseteq \omega \times 2$ be a finite function. Choose w as indicated. Let $f \in U_w$. Then $f \in U_z \backslash \overline{X}$, as desired. $\square$

Proposition 19.19. *If $M \subseteq {}^{\omega}2$ is meager, then there is a chopped real (x, i) such that $M \subseteq \{y \in {}^{\omega}2 : y$ does not match $(x, i)\}$.*

Proof. Say $M = \bigcup_{n\in\omega} F_n$ with each F_n nowhere dense. Since the union of two nowhere dense sets is nowhere dense, we may assume that $F_0 \subseteq F_1 \subseteq \cdots$. We now define $i \in \mathrm{Inc}$ and a sequence of functions $\langle z_n : n \in \omega\rangle$, each $z_n \in {}^{[i_n,i_{n+1})}2$. Let $i_0 = 0$. Let $t \subseteq \omega \times 2$ be a finite function such that $U_t \cap F_0 = \emptyset$. Wlog there is an $i_1 > 0$ such that $t \in {}^{[0,i_1)}2$. Let $z_0 = t$. Now suppose that $n > 0$, and i_n and z_{n-1} have been defined. Let $m = i_n$. Let $\langle u_i : i < 2^m\rangle$ enumerate ${}^m2$. We now define an increasing sequence $\langle j_k : k < 2^m\rangle$ of members of ω and functions $\langle w_k : k < 2^m\rangle$, each $w_k \in {}^{[j_k,j_{k+1})}2$. Let $j_0 = m$. Suppose that $j_0 < \cdots < j_k$ and $w_0, \ldots, w_{k-1}$ have been defined, with $k < 2^m - 1$, such that $\bigcup_{l<k} w_l \subseteq \omega \times 2$ is a finite function. Now $u_k \cup \bigcup_{l<k} w_l \subseteq \omega \times 2$ is a function, so by Lemma 19.18 there is a finite function $t \subseteq \omega \times 2$ such that $u_k \cup \bigcup_{l<k} w_l \subseteq t$ and $U_t \cap F_n = \emptyset$. Increasing t if necessary, we may assume that there is a $j_{k+1} > j_k$ such that $t \in {}^{[0,j_{k+1})}2$. Let $w_k = t \restriction [j_k, j_{k+1})$.

Let $i_{n+1} = j_{2^m-1}$ and $z_n = \bigcup_{k<2^m} w_k$. This completes the construction of i and $\langle z_n : n \in \omega\rangle$. Let $x = \bigcup_{n\in\omega} z_n$.

Thus (x, i) is a chopped real. We claim that no member of M matches (x, i). For, let $y \in M$; say $y \in F_k$ with $k > 0$. We claim that $x \restriction [i_n, i_{n+1}) \neq y \restriction [i_n, i_{n+1})$ for all $n > k$. Let $m = i_n$ and $u_k = y \restriction m$. In the construction we get $j_{k+1} \in [i_n, i_{n+1})$ and $t \in {}^{[0,j_{k+1})}2$ with $u_k \subseteq t$ and $U_t \cap F_n = \emptyset$. Since $y \in F_n$ and $x \restriction [i_n, j_{k+1}) \subseteq t$, it follows that $x \restriction [i_n, j_{k+1}) \neq y \restriction [i_n, j_{k+1})$. $\square$

Proposition 19.20. *If (x, i) and (y, j) are chopped reals and $\{z \in {}^{\omega}2 : z$ matches $(x, i)\} \subseteq \{z \in {}^{\omega}2 : z$ matches $(y, j)\}$, then there is an $m \in \omega$ such that for all $n \geq m$ there is a $k \in \omega$ such that $[j_k, j_{k+1}) \subseteq [i_n, i_{n+1})$ and $x \restriction [j_k, j_{k+1}) = y \restriction [j_k, j_{k+1})$.*

Proof. Assume that (x, i) and (y, j) are chopped reals and $\{z \in {}^{\omega}2 : z$ matches $(x, i)\} \subseteq \{z \in {}^{\omega}2 : z$ matches $(y, j)\}$, but suppose that for every $m \in \omega$ there is an $n \geq m$ such that for all $k \in \omega$, if $[j_k, j_{k+1}) \subseteq [i_n, i_{n+1})$ then $x \restriction [j_k, j_{k+1}) \neq y \restriction [j_k, j_{k+1})$. Then there is an infinite $I \subseteq \omega$ such that for all $n \in I$ and all $k \in \omega$, if $[j_k, j_{k+1}) \subseteq [i_n, i_{n+1})$ then $x \restriction [j_k, j_{k+1}) \neq y \restriction [j_k, j_{k+1})$. Let $\langle n(s) : s \in \omega\rangle$ be the strictly increasing enumeration of I. Let $K = \{n(2s) : s \in \omega\}$, and let $L = \bigcup_{s\in\omega}[i_{n(2s)}, i_{n(2s)+1})$. Let $z \restriction L = x \restriction L$ while $z(t) = 1 - y(t)$ for all $t \in \omega \backslash L$. Thus z matches (x, i), so by hypothesis z matches (y, j). Thus $U \stackrel{\text{def}}{=} \{t : z \restriction [j_t, j_{t+1}) = y \restriction [j_t, j_{j+1})\}$ is infinite. If $z \restriction [j_t, j_{t+1}) = y \restriction [j_t, j_{j+1})$ then by the definition of z, also $z \restriction [j_t, j_{t+1}) = x \restriction [j_t, j_{j+1})$. Now if $t \in U$, then

$[j_t, j_{t+1}) \subseteq L$. Since the components of L are not contiguous, it follows that for all $t \in U$ there is an s such that $[j_t, j_{t+1}) \subseteq [i_{n(2s)}, i_{n(2s)+1})$. Hence by supposition, for all $t \in U$ we have $x \restriction [j_t, j_{t+1}) \neq y \restriction [j_t, j_{t+1})$. But $x \restriction [j_t, j_{t+1}) = z \restriction [j_t, j_{t+1}) = y \restriction [j_t, j_{t+1})$, contradiction. □

Theorem 19.21. add(meag) $\leq \mathfrak{b}$.

Proof. By Proposition 19.17, let $X \subseteq \text{Inc}$ be of size $\mathfrak{b}$ such that there is no $i \in \text{Inc}$ which almost dominates each $j \in X$. Take any $x \in {}^\omega 2$ and let

$$Y = \{\{y \in {}^\omega 2 : y \text{ does not match } (x, i)\} : i \in X\}.$$

Let $Z = \bigcup Y$. We claim that Z is not meager. Since $|Y| \leq |X| = \mathfrak{b}$, this will prove the theorem. In fact, suppose that Z is meager. By Proposition 19.19, let (y, j) be a chopped real such that $Z \subseteq \{z \in {}^\omega 2 : z \text{ does not match } (y, j)\}$. Thus for all $i \in X$ we have $\{z \in {}^\omega 2 : z \text{ matches } (y, j)\} \subseteq \{z \in {}^\omega 2 : z \text{ matches } (x, i)\}$. Hence by Proposition 19.20 we get

$$\forall i \in X \exists m \in \omega \forall n \geq m \exists k \in \omega[[i_k, i_{k+1}) \subseteq [j_n, j_{n+1})].$$

But then j almost dominates each $i \in X$, contradicting Proposition 19.17. □

We consider ${}^\omega\omega$ as a topological space: the ω-power of the discrete space ω. Recall the definition of a base for this topology. For F a finite subset of ω and $G \in {}^F\mathscr{P}(\omega)$, define

$$\mathscr{O}_{F,G} = \{f \in {}^\omega\omega : \forall n \in F[f(n) \in G(n)]\}.$$

Proposition 19.22. *$\{\mathscr{O}_{F,G} : F \in [\omega]^\omega, G \in {}^F\mathscr{P}(\omega)\}$ is a base for a topology on ${}^\omega\omega$.*

Proof. Suppose that $f \in \mathscr{O}_{F,G} \cap \mathscr{O}(F', G')$. Let $F'' = F \cap F'$, $\text{dmn}(G'') = F \cap F'$, and $G''(n) = G(n) \cap G'(n)$ for all $n \in F \cap F'$. Clearly $f \in \mathscr{O}_{F'',G''} \subseteq \mathscr{O}_{F,G} \cap \mathscr{O}(F', G')$.

For any $f \in {}^\omega\omega$, $f \in \mathscr{O}_{\{0\},\{(0,\{f(0)\})\}}$. □

Proposition 19.23. *${}^\omega\omega$ is Hausdorff.*

Proof. Let f, g be distinct members of ${}^\omega\omega$. Say $f(n) \neq g(n)$. Then

$$f \in \mathscr{O}_{\{n\},\{(n,\{f(n)\})\}}, \ g \in \mathscr{O}_{\{n\},\{(n,\{g(n)\})\}}, \ \mathscr{O}_{\{n\},\{(n,\{f(n)\})\}} \cap \mathscr{O}_{\{n\},\{(n,\{g(n)\})\}} = \emptyset. \quad \square$$

Proposition 19.24. *Let $C \subseteq {}^\omega\omega$. Then the following are equivalent:*

(i) C is compact.

(ii) C is closed, and there is an $F \in \prod_{m \in \omega} [\omega]^{<\omega}$ such that $C \subseteq \prod_{m \in \omega} F_m$.

(iii) C is closed, and there is a $g \in {}^\omega\omega$ such that $C \subseteq \{f \in {}^\omega\omega : f \leq g\}$.

Proof. (i)⇒(ii): C is closed, as a compact subset of a Hausdorff space. Now for each $m \in \omega$ let $F_m = \{f(m) : f \in C\}$. Thus $C \subseteq \prod_{m \in \omega} F_m$. Suppose that there is an $m \in \omega$ with F_m infinite. Then

$$C \subseteq \bigcup_{n \in F_m} \mathscr{O}_{\{m\},\{(m,\{n\})\}}$$

with no finite subcover, contradiction.

(ii)⇒(iii): With F as in (ii), choose $g(m) > F_m$ for all $m \in \omega$.

(iii)⇒(i): Suppose that C is covered by a family $\mathscr{F}$ of open sets. Then

$$\prod_{m\in\omega}[0,g(m)] \subseteq ({}^\omega\omega\backslash C)\cup\bigcup\mathscr{F},$$

so there is a finite subset $\mathscr{F}' \subseteq \mathscr{F}$ such that

$$\prod_{m\in\omega}[0,g(m)] \subseteq ({}^\omega\omega\backslash C)\cup\bigcup\mathscr{F}'.$$

Hence $C \subseteq \bigcup\mathscr{F}'$, as desired. □

Let $\mathscr{K}_\sigma$ be the least σ-ideal of subsets of ${}^\omega\omega$ containing all the compact sets.

Proposition 19.25. *For any $g \in {}^\omega\omega$ we have $\{f \in {}^\omega\omega : f \leq^* g\} \in \mathscr{K}_\sigma$.*

Proof. Let $I = \{(m,s) : m \in \omega \text{ and } s \in {}^m\omega\}$. So I is countable. Now

$$\{f \in {}^\omega\omega : f \leq^* g\} \subseteq \bigcup_{(m,s)\in I}\{f \in {}^\omega\omega : f \leq s \cup (g \restriction (\omega\backslash m)\}.$$

Now note that for any $h \in {}^\omega\omega$ the set $\{f \in {}^\omega\omega : f \leq h\}$ is closed. In fact, if f is not in this set, then there is an $n \in \omega$ such that $f(n) > h(n)$, and hence $f \in \mathscr{O}_{\{n\},\{(n,f(n))\}}$ with $\mathscr{O}_{\{n\},\{(n,f(n))\}} \cap \{f \in {}^\omega\omega : f \leq h\} = \emptyset$. Now the proposition follows from Proposition 19.24. □

Proposition 19.26. *For any $F \subseteq {}^\omega\omega$ the following are equivalent:*

(i) $F \in \mathscr{K}_\sigma$.

(ii) F is covered by a countable union of compact subsets.

(iii) There is a $g \in {}^\omega\omega$ such that $F \subseteq \{f \in {}^\omega\omega : f \leq^* g\}$.

Proof. (i)⇒(ii): Let $I = \{X \subseteq {}^\omega\omega : X$ is covered by a countable union of compact subsets of ${}^\omega\omega\}$. Clearly I is a σ-ideal containing the compact sets. So $\mathscr{K}_\sigma \subseteq I$.

(ii)⇒(iii): Assume (ii). By Proposition 19.24 we can write $F \subseteq \bigcup_{g\in M}\{f : f \leq g\}$ for some countable $M \subseteq {}^\omega\omega$. Now M is bounded under $\leq^*$ since $\omega_1 \leq \mathfrak{b}$, so there is an $h \in {}^\omega\omega$ such that $g \leq^* h$ for all $g \in M$. Thus $F \subseteq \{f \in {}^\omega\omega : f \leq^* h\}$.

(iii)⇒(i): by Proposition 19.25. □

Proposition 19.27. $\text{non}(\mathscr{K}_\sigma) = \mathfrak{b}$.

Proof. First suppose that $X \subseteq {}^\omega\omega$ such that $X \notin \mathscr{K}_\sigma$, with $|X| = \text{non}(\mathscr{K}_\sigma)$. Suppose that $f \leq^* g$ for all $f \in X$. Then $X \subseteq \{f \in {}^\omega\omega : f \leq^* g\}$, so $X \in \mathscr{K}_\sigma$ by Proposition 19.26, contradiction. It follows that $\mathfrak{b} \leq \text{non}(\mathscr{K}_\sigma)$.

Second suppose that $X \subseteq {}^\omega\omega$ is almost unbounded, with $|X| = \mathfrak{b}$. Then $X \notin \mathscr{K}_\sigma$ by Proposition 19.26. So $\text{non}(\mathscr{K}_\sigma) \leq \mathfrak{b}$. □

Proposition 19.28. $\mathscr{K}_\sigma \subseteq$ meag.

Proof. For any $g \in {}^\omega\omega$ the set $\{f \in {}^\omega\omega : f \leq g\}$ is nowhere dense. In fact, given $g \in {}^\omega\omega$ and a basic open set $U_{F,G}$, take any $n \in (\omega\backslash F)$ and let $F' = F \cup \{n\}$ and $G' = G \cup \{(n, \{g(n)+1\})\}$. Thus $U_{F,G} \subseteq U_{F',G'}$ and $U_{F',G'} \cap \{f \in {}^\omega\omega : f \leq g\} = \emptyset$. So $\{f \in {}^\omega\omega : f \leq g\}$ is nowhere dense. Then by Proposition 19.24, every compact subset of ${}^\omega\omega$ is nowhere dense. Hence by Proposition 19.26, $\mathscr{K}_\sigma \subseteq$ meag. $\square$

Proposition 19.29. $\mathfrak{b} \leq \mathrm{non}(\mathrm{meag})$.

Proof. By Proposition 19.25, $\mathfrak{b} = \mathrm{non}(\mathscr{K}_\sigma)$. By Proposition 19.26, clearly $\mathrm{non}(\mathscr{K}_\sigma) \leq \mathrm{non}(\mathrm{meag})$. $\square$

Proposition 19.30. $\mathfrak{d} = \min\{|P| : P \subseteq \mathrm{Inc}$ *and* $\forall i \in \mathrm{Inc}\exists j \in P[j$ *almost dominates* $i]\}$.

Proof. First suppose that $P \subseteq \mathrm{Inc}$ and $\forall i \in \mathrm{Inc}\exists j \in P[j$ almost dominates $i]$, with $|P|$ minimum. Then $\{\mathrm{func}^j : j \in P\}$ is a dominating subset of ${}^\omega\omega$. In fact, let $g \in {}^\omega\omega$. Choose $j \in P$ which almost dominates ν^g. By Proposition 19.15, $g \leq^* \mathrm{func}^j$. This shows that $\leq$ holds in the proposition.

Second suppose that $D \subseteq {}^\omega\omega$ is an almost dominating set, with $|D| = \mathfrak{d}$. We claim that $\{\nu^g : g \in D\}$ is a set P as in the proposition. In fact, let $i \in \mathrm{Inc}$. Choose $g \in D$ such that $\mathrm{func}^i \leq^* g$. Then by Proposition 19.16, ν^g almost dominates i. This proves $\geq$ in the proposition. $\square$

Proposition 19.31. $\mathrm{cov}(\mathscr{K}_\sigma) = \mathfrak{d}$.

Proof. First suppose that $E \subseteq \mathscr{K}_\sigma$, $\bigcup E = {}^\omega\omega$, and $|E| = \mathrm{cov}(\mathscr{K}_\sigma)$. For each $F \in E$ choose $g_F \in {}^\omega\omega$ such that $F \subseteq \{f \in {}^\omega\omega : f \leq^* g_F\}$, using Proposition 19.26. We claim that $\{g_F : F \in E\}$ is almost dominating. For, let $f \in {}^\omega\omega$. Choose $F \in E$ such that $f \in F$. Then $f \leq^* g_F$, as desired. This shows that $\mathfrak{d} \leq \mathrm{cov}(\mathscr{K}_\sigma)$.

Second, suppose that $D \subseteq {}^\omega\omega$ is dominating. For each $g \in D$ let $F_g = \{f \in {}^\omega\omega : f \leq^* g\}$. Then $F_g \in \mathscr{K}_\sigma$ by Proposition 19.25. For any $f \in {}^\omega\omega$ there is a $g \in D$ such that $f \leq^* g$. Then $f \in F_g$. Thus $\bigcup_{g\in D} F_g = {}^\omega\omega$. This shows that $\mathrm{cov}(\mathscr{K}_\sigma) \leq \mathfrak{d}$. $\square$

Proposition 19.32. $\mathrm{cov}(\mathrm{meag}) \leq \mathfrak{d}$.

Proof. By Proposition 19.28, $\mathscr{K}_\sigma \subseteq$ meag. Hence by Proposition 19.31, $\mathrm{cov}(\mathrm{meag}) \leq \mathrm{cov}(\mathscr{K}_\sigma) = \mathfrak{d}$. $\square$

Proposition 19.33. *For any proper ideal I,* $\mathrm{non}(I) \leq \mathrm{cof}(I)$.

Proof. Let $\kappa = \mathrm{cof}(I)$, and let $X \in [I]^\kappa$ be as in the definition of $\mathrm{cof}(I)$. For each $B \in X$ choose $x_B \in A\backslash B$. Then $|\{x_B : B \in X\}| \leq \kappa$. Suppose that $\{x_B : B \in X\} \in I$. Choose $C \in X$ such that $\{x_B : B \in X\} \subseteq C$. Then $x_C \in C$, contradiction. $\square$

Proposition 19.34. $\mathfrak{d} \leq \mathrm{cof}(\mathrm{meag})$.

Proof. Let $\mathscr{A}$ be a family of meager sets such that for any meager set M there is an $N \in \mathscr{A}$ such that $M \subseteq N$, with $|\mathscr{A}| = \mathrm{cof}(\mathrm{meag})$. By Proposition 19.19, for each $N \in \mathscr{A}$

there is a chopped real (x_N, i^N) such that $N \subseteq \{y \in {}^\omega 2 : y \text{ does not match } (x_n, i^N)\}$. We claim that $\{i^N : N \in \mathscr{A}\}$ almost dominates every $j \in \text{Inc}$; by Proposition 19.30 this will prove our proposition. To prove the claim, suppose that $j \in \text{Inc}$. Fix $y \in {}^\omega 2$. Then (y, j) is a chopped real, and so by Theorem 19.4 $\{z \in {}^\omega 2 : z \text{ does not match } (y, j)\}$ is meager. say $\{z \in {}^\omega 2 : z \text{ does not match } (y, j)\} \subseteq N \in \mathscr{A}$. Hence $\{z \in {}^\omega 2 : z \text{ does not match } (y, j)\} \subseteq \{z \in {}^\omega 2 : z \text{ not match } (x_n, i^N)\}$. By Proposition 19.20, i^N almost dominates j. □

If P and Q are partial orders, a function $f : P \to Q$ is a *Tukey function* iff $\forall X \subseteq Q[X$ bounded in Q imples that $f^{-1}[X]$ is bounded in $P]$.

Proposition 19.35. *$f : P \to Q$ is a Tukey function iff for any $B \in Q$ there is an $A \in P$ such that*

$$\forall C \in P[f(C) \leq B \to C \leq A]$$

Proof. $\Rightarrow$: Assume that $f : P \to Q$ is a Tukey function and $B \in Q$. Let $X = \{Z \in Q : Z \leq B\}$. Thus X is bounded. Hence $f^{-1}[X]$ is bounded, say by $A \in P$. Now suppose that $C \in P$ and $f(C) \leq B$. Thus $C \in f^{-1}[X]$, so $C \leq A$.

$\Leftarrow$: Assume the indicated condition, and suppose that $X \subseteq Q$ is bounded by B. Choose $A \in P$ such that $\forall C \in P[f(C) \leq B \to C \leq A]$. Thus $f^{-1}[X]$ is bounded by A. □

For each $B \in Q$ we select one $f^*(B)$ satisfying the condition for A in Proposition 19.35.

meag is the poset of meager subsets of ${}^\omega 2$ with inclusion, and null is the poset of null subsets of ${}^\omega 2$ with inclusion.

Proposition 19.36. *If $f : \text{meag} \to \text{null}$ is a Tukey function, then* $\text{add}(\text{null}) \leq \text{add}(\text{meag})$.

Proof. Suppose that $X \subseteq \text{meag}$ and $|X| < \text{add}(\text{null})$; we show that $\bigcup X$ is meager; hence the proposition follows. We have $B \stackrel{\text{def}}{=} \bigcup_{A \in X} f(A) \in \text{null}$. If $A \in X$, then $f(A) \subseteq B$, so $A \subseteq f^*(B)$. Thus $\bigcup X \subseteq f^*(B)$, and so $\bigcup X$ is meager. □

Proposition 19.37. *If $f : \text{meag} \to \text{null}$ is a Tukey function, then* $\text{cof}(\text{meag}) \leq \text{cof}(\text{null})$.

Proof. Suppose that $X \subseteq \text{null}$ is cofinal in null. Then, we claim, $\{f^*(B) : B \in X\}$ is cofinal in meag. For, suppose that $C \in \text{meag}$. Choose $B \in X$ such that $f(C) \subseteq B$. Then $C \subseteq f^*(B)$. □

Lemma 19.38. *For each $n \in \omega$ let $B_n = \{x \in {}^\omega 2 : x(n) = 1\}$. Then for any finite nonempty $I \subseteq \omega$ we have*

$$\mu\left(\bigcap_{n \in I} B_n\right) = \prod_{n \in I} \mu(B_n) = \frac{1}{2^{|I|}}. \qquad \square$$

Lemma 19.39. *For each $a \in (0, 1)$ there is a $k \in {}^\omega\omega$ such that*

$$\prod_{m \in \omega} \left(1 - \frac{1}{2^{k_m}}\right) = a.$$

Note that $\exists s \forall t \geq s \left[1 - \frac{1}{2^t} > a\right]$. Let k_0 be minimum such that $1 - \frac{1}{2^{k_0}} > a$. Suppose that k_i has been defined for all $i \leq m$ so that

$$\prod_{i \leq m} \left(1 - \frac{1}{2^{k_i}}\right) > a.$$

Then

$$\exists s \forall t > s \left[\left(1 - \frac{1}{2^t}\right) \prod_{i \leq m} \left(1 - \frac{1}{2^{k_i}}\right) > a\right].$$

Let k_{m+1} be minimum such that

$$\left(1 - \frac{1}{2^{k_{m+1}}}\right) \prod_{i \leq m} \left(1 - \frac{1}{2^{k_i}}\right) > a.$$

Now we claim that for all $m \in \omega$,

$$(*) \qquad \prod_{i \leq m} \left(1 - \frac{1}{2^{k_i}}\right) - a \leq \frac{1}{2^{k_m}}.$$

For $m = 0$ we clearly have $k_0 > 0$. Then $a \geq 1 - \frac{1}{2^{k_0 - 1}} = 1 - \frac{1}{2^{k_0}} - \frac{1}{2^{k_0}}$ and hence $1 - \frac{1}{2^{k_0}} - a \leq \frac{1}{2^{k_0}}$. Now assume that $(*)$ holds for m. Clearly $k_{m+1} > 0$. Hence, with $P = \prod_{i \leq m} \left(1 - \frac{1}{2^{k_i}}\right)$,

$$\begin{aligned}
\left(1 - \frac{1}{2^{k_{m+1}-1}}\right) P &\leq a; \\
\left(1 - \frac{2}{2^{k_{m+1}}}\right) P &\leq a; \\
\left(1 - \frac{1}{2^{k_{m+1}}}\right) P - \frac{1}{2^{k_{m+1}}} P &\leq a \\
\left(1 - \frac{1}{2^{k_{m+1}}}\right) P - a &\leq \frac{1}{2^{k_{m+1}}} P \leq \frac{1}{2^{k_{m+1}}}.
\end{aligned}$$

Thus $(*)$ holds. Now there is no n such that $k_m \leq n$ for all m. Otherwise,

$$a < \prod_{i \leq m} \left(1 - \frac{1}{2^{k_i}}\right) \leq \left(1 - \frac{1}{2^n}\right)^{m+1}$$

for all m, and so $a = 0$, contradiction.

Now $(*)$ implies that

$$\prod_{m \in \omega} \left(1 - \frac{1}{2^{k_m}}\right) = a. \qquad \square$$

A system $\langle A_n : m \in \omega\rangle$ of measurable subsets of ${}^{\omega}2$ is *μ-independent* iff

$$\forall I \in [\omega]^{<\omega}\left[\mu\left(\bigcap_{n\in I} A_n\right) = \prod_{n\in I}\mu(A_n)\right]$$

Proposition 19.40. *Suppose that $\langle A_m : m \in \omega\rangle$ is a system of measurable μ-independent subsets of ${}^{\omega}2$. Then for any finite $I \subseteq \omega$ and any $\varepsilon \in {}^{I}2$ we have*

$$\mu\left(\bigcap_{i\in I} A_i^{\varepsilon(i)}\right) = \prod_{i\in I}\mu(A_i^{\varepsilon(i)}),$$

where $X^1 = X$ and $X^0 = {}^{\omega}2\backslash X$.

Proof. We prove this by induction on $|I|$, starting with $|I| = 1$, where this is obvious. Now suppose true for $|I| = m$. We do the case $|I| = m+1$ by induction on the number n of $i \in I$ such that $\varepsilon(i) = n$. The case $n = 0$ is clear. Now assume true for n, where $n < m+1$. Take any $i \in I$ such that $\varepsilon(i) = 0$. Then

$$\mu\left(\bigcap_{j\in I} A_j^{\varepsilon(j)}\right) + \mu\left(\bigcap_{\substack{j\in I\\ j\neq i}} A_j^{\varepsilon(j)} \cap A_i\right) = \mu\left(\bigcap_{\substack{j\in I\\ j\neq i}} A_j^{\varepsilon(j)}\right).$$

By the induction hypothesis

$$\mu\left(\bigcap_{\substack{j\in I\\ j\neq i}} A_j^{\varepsilon(j)} \cap A_i\right) = \prod_{\substack{j\in I\\ j\neq i}}\mu(A_j^{\varepsilon(j)})\mu(A_i)$$

and

$$\mu\left(\bigcap_{\substack{j\in I\\ j\neq i}} A_j^{\varepsilon(j)}\right) = \prod_{\substack{j\in I\\ j\neq i}}\mu(A_j^{\varepsilon(j)}).$$

Hence

$$\begin{aligned}\mu\left(\bigcap_{j\in I} A_j^{\varepsilon(j)}\right) &= \prod_{\substack{j\in I\\ j\neq i}}\mu(A_j^{\varepsilon(j)}) - \prod_{\substack{j\in I\\ j\neq i}}\mu(A_j^{\varepsilon(j)})\mu(A_i)\\ &= \prod_{\substack{j\in I\\ j\neq i}}\mu(A_j^{\varepsilon(j)})(1-\mu(A_i)),\end{aligned}$$

finishing the inductive proof. $\square$

Theorem 19.41. *If $a \in {}^{\omega}(0,1)$, then there exists a system $\langle A_n : n \in \omega\rangle$ of μ-independent open subsets of ${}^{\omega}2$ such that $\forall n \in \omega[\mu(A_n) = a_n]$.*

Proof. By Lemma 19.39 let $\langle k^n_m : m, n \in \omega\rangle$ be a system of natural numbers such that

$$\forall n \in \omega \left[\prod_{m\in\omega}\left(1 - \frac{1}{2^{k^n_m}}\right) = 1 - a_n\right].$$

Note that for any $n \in \omega$,

$$(*) \qquad \sum_{m\in\omega} \frac{1}{2^{k^n_m}} = a_n.$$

In fact, for any $m \in \omega \left[1 - a_n \leq 1 - \frac{1}{2^{k^n_m}}\right]$, and so $\forall m \in \omega \left[\frac{1}{2^{k^n_m}} \leq a_n\right]$. Thus a_n is an upper bound for $\left\{\frac{1}{2^{k^n_m}} : m \in \omega\right\}$. Now suppose that x is any upper bound for $\left\{\frac{1}{2^{k^n_m}} : m \in \omega\right\}$. Then $1 - x$ is a lower bound for $\left\{1 - \frac{1}{2^{k^n_m}} : m \in \omega\right\}$, hence $1 - x \leq 1 - a_n$ and so $a_n \leq x$. This proves $(*)$.

Now let $\langle I^n_m : n, m \in \omega\rangle$ be a system of pairwise disjoint set such that $|I^n_m| = k^n_m$ for all $m, n \in \omega$. For each $n \in \omega$ let

$$A_n = {}^{\omega}2 \backslash \bigcap_{m\in\omega}\left({}^{\omega}2\backslash \bigcap_{j\in I^n_m} B_j\right),$$

where B_n is as in Lemma 19.38. Note that each B_n is clopen. Hence also for any $m, n \in \omega$ the set $\bigcap_{j\in I^n_m} B_j$ is clopen, and so is ${}^{\omega}2\backslash \bigcap_{j\in I^n_m} B_j$. Hence $\bigcap_{m\in\omega}\left({}^{\omega}2\backslash \bigcap_{j\in I^n_m} B_j\right)$ is closed, and so A_n is open. Now for any $n \in \omega$,

$$\begin{aligned}
A_n &= \left\{x \in {}^{\omega}2 : x \notin \bigcap_{m\in\omega}\left({}^{\omega}2\backslash \bigcap_{j\in I^n_m} B_j\right)\right\}\\
&= \left\{x \in {}^{\omega}2 : \exists m \in \omega \left[x \in \bigcap_{j\in I^n_m} B_j\right]\right.\\
&= \{x \in {}^{\omega}2 : \exists m \in \omega \forall j \in I^n_m [x \in B_j]\}\\
&= \{x \in {}^{\omega}2 : \exists m \in \omega \forall j \in I^n_m [x(j) = 1]\}.
\end{aligned}$$

Now for any $m, n \in \omega$ let $C_{mn} = \{x \in {}^{\omega}2 : \forall j \in I^n_m[x(j) = 1]\}$. Then $\mu(C_{mn}) = \frac{1}{2^{k^n_m}}$. A_n is the union of the pairwise disjoint sets C_{mn}, so $\mu(A_n) = \sum_{m\in\omega} \frac{1}{2^{k^n_m}} = a_n$ by $(*)$. Finally, $\langle A_n : n \in \omega\rangle$ is a μ-independent system, since the C_{mn} are pairwise disjoint. $\square$

If $H \in {}^{\omega\backslash 1}\omega$ is an increasing function with nonzero entries such that $\sum_{m\in\omega\backslash 1} \frac{1}{H(m)} < \infty$, then we define

$$\mathcal{C}_H = \left\{S \in {}^{\omega\backslash 1}([\omega\backslash 1]^{<\omega}) : \sum_{m\in\omega\backslash 1} \frac{|S(m)|}{H(m)} < \infty\right\}.$$

With $H(n) = n^2$ for all positive n, H is an increasing function with nonzero entries such that $\sum_{m\in\omega\backslash 1}\frac{1}{H(m)} < \infty$; this is a p-series. See Heinbockel Calculus 1, example 4-9 on page 291. For this H we write $\mathcal{C}$ in place of $\mathcal{C}_H$.

If $S_1, S_2 \in \mathcal{C}$, we define $S_1 \leq S_2$ iff $\exists m\forall n \geq m[S_1(n) \subseteq S_2(n)]$. Clearly this is a partial order of $\mathcal{C}$.

Lemma 19.42. *If $G \subseteq {}^\omega 2$ is a null set, then there is a closed $K \subseteq {}^\omega 2$ such that $K\cap G = \emptyset$, $\mu(K) > 0$, and for every open $U \subseteq {}^\omega 2$, if $K \cap U \neq \emptyset$ then $\mu(K \cap U) > 0$.*

Proof. By Theorem 18.78 and Proposition 18.87, there is an open set $U \supseteq G$ such that $\mu(U) \leq \frac{1}{2}$. Let $K' = {}^\omega 2\backslash U$, and

$$K = K'\backslash\bigcup\{K' \cap U_f : K' \cap U_f \neq \emptyset \text{ and } \mu(K' \cap U_f) = 0\}.$$

Note that there are only countably many sets U_f. Hence $\mu(K) = \mu(K') \geq \frac{1}{2}$. If $K\cap U_f \neq \emptyset$, then also $K'\cap U_f \neq \emptyset$. If $\mu(K\cap U_f) = 0$, then also $\mu(K'\cap U_f) = 0$ and so $K\cap K'\cap U_f = \emptyset$, contradiction. If V is any open set and $K \cap V \neq \emptyset$, then there is an f with $U_f \subseteq V$ such that $K \cap U_f \neq \emptyset$. Hence $\mu(K \cap V) \geq \mu(K \cap U_v) > 0$. $\square$

Lemma 19.43. $\forall x \in [-1, 0][1 + x \leq e^x]$.

Proof. Let $f(x) = e^x - x - 1$ for all $x \in [-1, 0]$. Then $f'(x) = e^x - 1$ and $f''(x) = e^x$. So $f(x)$ has a minimum 0 at $x = 0$. $\square$

Lemma 19.44. *If $-1 < a_n < 0$ for all $n \in \omega\backslash 1$ and $\prod_{n\in\omega\backslash 1}(1 + a_n)$ exists and is greater than 0, then $\sum_{n\in\omega\backslash 1} a_n$ exists and is greater than $-\infty$.*

Proof. For each $n \in \omega\backslash 1$ let $S_n = \sum_{r=1}^n a_r$ and $P_n = \prod_{r=1}^n(1 + a_r)$. S is strictly decreasing, so $\sum_{n\in\omega\backslash 1} a_n$ exists but is possibly $-\infty$. By Lemma 19.43,

$$0 < P_n \leq \prod_{r=1}^n e^{a_r} = e^{\sum_{r=1}^n a_r}.$$

If $\sum_{n\in\omega\backslash 1} a_n = -\infty$, then $\prod_{n\in\omega\backslash 1}(1 + a_n) = 0$, contradiction. $\square$

Lemma 19.45. *If $\langle S_n : n \in \omega\rangle$ is a sequence of members of $\mathcal{C}$, then there is a $T \in \mathcal{C}$ such that $\{i \in \omega\backslash 1 : S_n(i) \not\subseteq T(i)\}$ is finite, for all n.*

Proof. We begin with the following elementary fact.

(1) If $\langle a_{ik} : i \in \omega, k \leq n\rangle$ is a system of positive real numbers, where $n \in \omega\backslash 1$, and if $\sum_{i\in\omega} a_{ik} < \infty$ for every $k \leq n$, then $\sum_{i\in\omega}\sum_{k\leq n} a_{ik} \leq \sum_{k\leq n}\sum_{i\in\omega} a_{ik}$.

It follows that for any $k \in \omega\backslash 1$,

$$\sum_{j\in\omega\backslash 1}\sum_{i\leq k}\frac{|S_i(j)|}{j^2} < \infty.$$

Hence

$$(*)\qquad \forall k\in\omega\backslash 1\exists n\left[\sum_{n<j}\sum_{i\leq k}\frac{|S_i(j)|}{j^2}<2^{-k}\right].$$

Now we define $\langle n_k : k\in\omega\rangle$ by recursion. Let $n_0=0$. If n_k has been defined, by $(*)$ let $n_{k+1}>n_k$ be such that

$$\sum_{n_{k+1}<j}\sum_{i\leq k+1}\frac{|S_i(j)|}{j^2}<2^{-k-1}.$$

Now for any $y\in\omega\backslash 1$ choose k so that $n_k<j\leq n_{k+1}$ and define $T(j)=\bigcup_{i\leq k}S_i(j)$. Then

$$\begin{aligned}\sum_{j\in\omega\backslash 1}\frac{|T(j)|}{j^2}&=\sum_{k\in\omega}\sum_{n_k<j\leq n_{k+1}}\frac{|T(j)|}{j^2}\\ &\leq\sum_{k\in\omega}\sum_{n_k<j\leq n_{k+1}}\sum_{i\leq k}\frac{|S_i(j)|}{j^2}\\ &\leq\sum_{k\in\omega}2^{-k}.\end{aligned}$$

Thus $T\in\mathcal{C}$. Clearly $\{j\in\omega\backslash 1 : S_i(j)\not\subseteq T(j)\}$ is finite, for all i. $\square$

Lemma 19.46. *Suppose that $K\subseteq{}^{\omega}2$ is nonempty and closed, and $\langle V_n : n\in\omega\rangle$ is a decreasing sequence of open sets such that $K\cap\bigcap_{n\in\omega}V_n=\emptyset$. Then there is an open set U and a $k\in\omega$ such that $K\cap U\neq\emptyset$ and $\forall n\geq k[K\cap U\cap V_n=\emptyset]$.*

Proof. Assume the hypotheses. Then $K=\bigcup_{n\in\omega}(K\backslash V_n)$, so by the Baire category theorem there is a $k\in\omega$ and an open U such that $\emptyset\neq K\cap U\subseteq K\backslash V_k$. Since $\langle K\backslash V_s : s\in\omega\rangle$ is increasing, it follows that $\forall n\geq k[K\cap U\subseteq K\backslash V_n]$. $\square$

Theorem 19.47. *There is a Tukey function $\varphi_2:\mathcal{C}\to$ null.*

Proof. By Theorem 19.41 let $\langle G_{ij} : i,j\in\omega\backslash 1\rangle$ be a system of μ-independent open subset of ${}^{\omega}2$ such that $\mu(G_{ij})=i^{-2}$ for all $i,j\in\omega\backslash 1$. For each $S\in\mathcal{C}$ define

$$\varphi_2(S)=G_S\stackrel{\text{def}}{=}\bigcap_{n\in\omega\backslash 1}\bigcup_{m>n}\bigcup_{k\in S(m)}G_{mk}.$$

We show that $\varphi_2(S)\in$ null; for any $n>0$,

$$\mu(\varphi_2(S))=\mu\left(\bigcap_{n\in\omega\backslash 1}\bigcup_{m>n}\bigcup_{k\in S(m)}G_{mk}\right)$$

$$\begin{aligned}
&\le \mu\left(\bigcup_{m>n}\bigcup_{k\in S(m)} G_{mk}\right)\\
&\le \sum_{m>n}\mu\left(\bigcup_{k\in S(m)} G_{mk}\right)\\
&= \sum_{m>n}\frac{|S(m)|}{m^2},
\end{aligned}$$

and the last sum approaches 0 as $n \to \infty$.

Now we will construct a function φ_2^* which will be an explicit function showing that φ_2 is a Tukey function. Let $H \in \text{null}$. By Lemma 19.42 let $K^H \subseteq {}^\omega 2$ be a closed set such that $K^H \cap H = \emptyset$, $\mu(K^H) > 0$ and for every open $U \subseteq {}^\omega 2$, if $K^H \cap U \neq \emptyset$ then $\mu(K^H \cap U) > 0$.

Let $\langle U_n : n \in \omega\rangle$ be an enumeration of all the basic open subsets of ${}^\omega 2$ which intersect K^H. For all $n \in \omega$ and $i \in \omega\backslash 1$ let

$$A^H_{ni} = \{j \in \omega\backslash 1 : K^H \cap U_n \cap G_{ij} = \emptyset\}.$$

Now for all $n \in \omega$ and $i, j \in \omega\backslash 1$, if $j \in A^H_{ni}$ then $K^H \cap U_n \subseteq {}^\omega 2\backslash G_{ij}$. Hence if F is a finite subset of A^H_{ni}, then $K^H \cap U_n \subseteq \bigcap_{j\in F}({}^\omega 2\backslash G_{ij})$, and hence $0 < \mu(K^H \cap U_n) \le \mu(\bigcap_{j\in F}({}^\omega 2\backslash G_{ij})) = (1 - \frac{1}{i^2})^{|F|}$, using Proposition 19.40. Since $\left(1-\frac{1}{i^2}\right)^s \to 0$ as $s \to \infty$, it follows that A^H_{ni} is finite. Now

$$\prod_{j\in A^H_{ni}}\left(1-\frac{1}{i^2}\right) = \left(1-\frac{1}{i^2}\right)^{|A^H_{ni}|};$$

$$\begin{aligned}
\prod_{i\le m+1}\left(1-\frac{1}{i^2}\right)^{|A^H_{ni}|} &= \left(1-\frac{1}{(m+1)^2}\right)^{|A^H_{ni}|}\prod_{i\le m}\left(1-\frac{1}{i^2}\right)^{|A^H_{ni}|}\\
&\le \prod_{i\le m}\left(1-\frac{1}{i^2}\right)^{|A^H_{ni}|}
\end{aligned}$$

It follows that $\prod_{i\in\omega}\left(1-\frac{1}{i^2}\right)^{|A^H_{ni}|}$ exists. Now we have

$$\prod_{i\in\omega}\left(1-\frac{1}{i^2}\right)^{|A^G_{ni}|} = \prod_{i\in\omega}\prod_{j\in A^G_{ni}}\left(1-\frac{1}{i^2}\right) = \prod_{i\in\omega}\prod_{j\in A^G_{ni}}\mu({}^\omega 2\backslash G_{ij}) \ge \mu(K^H \cap U_n) > 0,$$

Applying Lemma 19.44 with $a_n = -\frac{1}{n^2}$ for all $n > 0$, it follows that

$$\sum_{i=1}^{\infty} -\frac{|A^G_{ni}|}{i^2} \text{ exists and is } > -\infty,$$

so
$$\sum_{i=1}^{\infty} \frac{|A^G_{ni}|}{i^2} \text{ exists and is } < \infty,$$
Hence $\langle A^G_{ni} : i \in \omega\rangle \in \mathcal{C}$ for all n.

We now define, for any $H \in$ null, $\varphi_2^*(H)$ is some element of $\mathcal{C}$ such that

$$\forall n \in \omega\backslash 1 \exists k \in \omega \forall i \geq k[A^H_{ni} \subseteq (\varphi_2^*(H))_n];$$

this exists by Lemma 19.45. (Replace S_n there by $\langle A^H_{ni} : i \in \omega\rangle$.)

Now to show that $\varphi_2 : \mathcal{C} \to$ null is a Tukey function, with associated map φ_2^*, suppose that $X \subseteq$ null and $\forall A \in X[A \subseteq H]$. with $H \in$ null. We want to show that $\varphi_2^{-1}[X]$ is bounded by $\varphi_2^*(H)$. That is, we want to show that for all $S \in \mathcal{C}$, if $\varphi_2(S) \in X$ then $S \subseteq \varphi_2^*(H)$. Now by the definition of φ_2 we have

$$\varphi_2(S) = G_S = \bigcap_{n\in\omega\backslash 1} \bigcup_{m>n} \bigcup_{k\in S(m)} G_{mk} \subseteq H.$$

Now $K^H \cap H = \emptyset$, so
$$K^H \cap \bigcap_{n\in\omega\backslash 1} \bigcup_{m>n} \bigcup_{k\in S(m)} G_{mk} = \emptyset.$$

Now note that if $n < n' \in \omega$ then

$$\bigcup_{m>n'} \bigcup_{k\in S(m)} G_{mk} \subseteq \bigcup_{m>n} \bigcup_{k\in S(m)} G_{mk}.$$

Hence by Lemma 19.46 there exist an open set U and a $t \in \omega$ such that $U \cap K^H \neq \emptyset$ and for all $s \geq t$,
$$U \cap K^H \cap \bigcup_{m>s} \bigcup_{k\in S(m)} G_{mk} = \emptyset.$$

We may assume that U is a basic open set U_n; see the beginning of the construction of φ_2^*. Now for all $m > t$, $U_n \cap K^H \cap \bigcup_{u\in S(m)} G_{mu} = \emptyset$. So $S(m) \subseteq A^H_{nm}$. Also, there is a v such that $\forall m > v[A^H_{nm} \subseteq (\varphi_2^*(H))_m]$. Hence for all $m > \max(t, v)$ we have $S(m) \subseteq A^H_{nm} \subseteq (\varphi_2^*(H))_m$. So $S \leq \varphi_2^*(H)$, as desired. $\square$

Lemma 19.48. *If $U \subseteq {}^\omega 2$ is a nonempty open set and $n \in \omega$, then there is a countable family $\mathscr{A}$ of open subsets of U such that:*

(i) Every dense open subset of ${}^\omega 2$ contains a member of $\mathscr{A}$.

(ii) The intersection of any n elements of $\mathscr{A}$ is nonempty.

Proof. Let $\langle V_m : m \in \omega\rangle$ enumerate all nonempty clopen subsets of U. For each $k \in \omega$ let
$$A_k = \left\{ m > k : \forall I \subseteq k+1 \left[\bigcap_{i\in I} V_i \neq \emptyset \to V_m \cap \bigcap_{i\in I} V_i \neq \emptyset \right] \right\}.$$

Define

$$\mathscr{A} = \left\{ \bigcup_{i \le n} V_{m_i} : m_0 \in \omega \text{ and } \forall i \le n[m_{i+1} \in A_{m_i}] \right\}.$$

Now to prove (i), let W be a dense open subset of ${}^\omega 2$.

(1) $\forall k \in \omega[A_k \cap \{m \in \omega : V_m \subseteq W\} \neq \emptyset]$.

In fact, let $k \in \omega$. Now for each $I \subseteq k+1$ such that $\bigcap_{i \in I} V_i \neq \emptyset$ choose a nonempty clopen $C_I \subseteq ((\bigcap_{i \in I} V_i) \cap W)$. Then

$$T \stackrel{\text{def}}{=} \bigcup \left\{ C_I : I \subseteq k+1 \text{ and } \bigcap_{i \in I} V_i \neq \emptyset \right\}$$

is a nonempty clopen subset of U. Let V_l be any nonempty clopen subset of T with $k < l$. Note that $V_l \subseteq U \cap W$. Thus $l \in A_k \cap \{m \in \omega : V_m \subseteq W\} \neq \emptyset]$. This proves (1).

Now we define $\langle m_i : i \le n \rangle$ by recursion. By (1), let m_0 be such that $m_0 \in A_0$ and $V_{m_0} \subseteq W$. Having defined m_i, by (1) let m_{i+1} be such that $m_i < m_{i+1}$ and $m_{i+1} \in A_{m_i}$ and $V_{m_{i+1}} \subseteq W$. Then $\bigcup_{i \le n} V_{m_i} \subseteq W$ and $\bigcup_{i \le n} V_{m_i} \in \mathscr{A}$. This proves (i).

To prove (ii), suppose that $W_1, \ldots, W_n \in \mathscr{A}$. Say $W_j = \bigcup_{i \le n} V_{m_i^j}$ with $m_0^j \in \omega$ and $\forall i \le n[m_{i+1}^j \in A_{m_i^j}]$. Let $s_0 = \min\{m_0^j : 1 \le j \le n\}$. If s_i has been defined, with $0 \le i < n$, if for all $j = 1, \ldots, n$ we have $m_k^j = s_k$ for some $k \le i$, the construction stops. Otherwise there is a $j \in \{1, \ldots, n\}$ such that $\forall k \le i[m_k^j \neq s_k]$, and we let $s_{i+1} = \min\{m_{i+1}^j : 1 \le j \le n \text{ and } m_k^j \neq s_k \text{ for all } k \le i\}$. So this constructs $s_0, \ldots, s_l$ with $l \le n$. Now for each $t \le l$ let $C_t = \{j : \forall u < t[m_u^j \neq s_u]\}$. Thus $s_t = \min\{m_t^j : j \in C_t\}$.

(2) If $t+1 \le l$, then $s_t < s_{t+1}$.

For, let $s_{t+1} = m_{t+1}^{j(1)}$, with $j(1) \in C_{t+1}$, and $s_t = m_t^{j(0)}$ with $j(0) \in C_t$. Now $C_{t+1} \subseteq C_t$, so $j(1) \in C_t$. Hence $s_t = m_t^{j(0)} \le m_t^{j(1)} < m_{t+1}^{j(1)} = s_{t+1}$.

Now for each $t \in \{0, \ldots, l\}$ choose $j(t) \in C_t$ such that $m_t^{j(t)} = s_t$.

(3) For all $t \in 0, \ldots, l$ we have $V_{m_0^{j(0)}} \cap \ldots \cap V_{m_t^{j(t)}} \neq \emptyset$.

We prove this by induction. It is clear for $t = 0$. Assume it for t, with $t+1 \le l$. Then by (2) we get $V_{m_0^{j(0)}} \cap \ldots \cap V_{m_{t+1}^{j(t+1)}} \neq \emptyset$.

Now if $l = n$, then (3) gives (ii). If $l < n$, then for all v with $l < v \le n$ there is a $k \le l$ such that $m_k^v = s_k = m_k^{j(k)}$, and (ii) again follows. $\square$

Theorem 19.49. *There are functions* $\varphi_1 : \text{meag} \to \mathcal{C}$ *and* $\varphi_1^* : \mathcal{C} \to \text{meag}$ *such that for any* $F \in \text{meag}$ *and any* $S \in \mathcal{C}$,

$$\varphi_1(F) \le S \text{ implies that } F \subseteq \varphi_1^*(S).$$

Proof. Let $\langle U_n : n \in \omega\rangle$ enumerate all the nonempty clopen subsets of ${}^\omega 2$. For each $n \in \omega$ let $\langle V^n_m : m, n \in \omega\rangle$ be obtained from Lemma 19.48 using U_n for U and n^2 for n.

Suppose that F is a meager set. Say $F = \bigcup_{n\in\omega} F'_n$, each F'_n nowhere dense, and $F'_n \subseteq F'_m$ for $n < m$. We define $\varphi_1(F) \stackrel{\text{def}}{=} S_F$ as follows. S_F is a function with domain ω, and for each $n \in \omega$,

$$S_F(n) = \{\min\{k \in \omega : \overline{F}'_n \cap V^n_k = \emptyset\}\}.$$

Note that since each set ${}^\omega 2\backslash\overline{F}'_n$ is dense open, this definition makes sense. Clearly $S_F \in \mathcal{C}$.

Now for each $S \in \mathcal{C}$ let

$$\varphi^*_1(S) \stackrel{\text{def}}{=} F^S = {}^\omega 2\backslash \bigcap_{n\in\omega} \bigcup_{m>n} \bigcap_{i\in S(m)} V^m_i.$$

Now $|S(m)| \le m^2$ for all $m \in \omega$, so by Lemma 19.48, $\emptyset \ne \bigcap_{i\in S(m)} V^m_i \subseteq U_m$. We claim that each set $\bigcup_{m>n}\bigcap_{i\in S(m)} V^m_i$ is dense, hence open dense. For, if W is any open set, then there is a U_m with $m > n$ such that $U_m \subseteq W$. Hence

$$W \cap \bigcup_{m>n} \bigcap_{i\in S(m)} V^m_i \ge U_m \cap \bigcap_{i\in S(m)} V^m_i = \bigcap_{i\in S(m)} V^m_i \ne \emptyset.$$

It follows that F^S is meager.

Now suppose that $\varphi_1(F) \le S$. Choose n so that for all $m > n$ $S_F(m) \subseteq S(m)$. For each $m > n$ let k_m be such that $S_F(m) = \{k_m\}$ and k_m is minimum such that $F'_m \cap V^m_{k_m} = \emptyset$. Now for any $m > n$ we have $\bigcap_{i\in S(m)} V^m_i \subseteq V^m_{k_m}$, and so $\bigcap_{i\in S(m)} V^m_i \cap F'_m = \emptyset$. Hence also $\bigcap_{i\in S(m)} V^m_i \cap F'_n = \emptyset$. It follows that $\bigcup_{m>n}\bigcap_{i\in S(m)} V^m_i \cap F'_n = \emptyset$. Hence $\bigcap_{n\in\omega}\bigcup_{m>n}\bigcap_{i\in S(m)} V^m_i \cap F = \emptyset$. So $F \subseteq F^S = \varphi^*_1(S)$. ☐

Theorem 19.50. $\varphi_2 \circ \varphi_1 : \text{meag} \to \text{null}$ *is a Tukey function.*

Proof. Suppose that $C \in \text{meag}$, $B \in \text{null}$, and $\varphi_2(\varphi_1(C) \subseteq B$. Then

$$\begin{aligned} \varphi_1(C) &\le \varphi^*_2(B) \quad \text{by Theorem 19.47 and its proof} \\ C &\subseteq \varphi^*_1(\varphi^*_2(B)). \quad \text{by Theorem 19.49} \end{aligned}$$

☐

Corollary 19.51. add(null) $\le$ add(meag).

Proof. By Proposition 19.36 and Theorem 19.50. ☐

Corollary 19.52. cof(meag) $\le$ cof(null).

Proof. By Proposition 19.37 and Theorem 19.50. ☐

Lemma 19.53. *There are* 2^ω *Borel sets.*

Proof. Every open set is the union of a family of open intervals with rational endpoints, so there are at most 2^ω open sets. There are exactly 2^ω, since one can take the set

$\{(m, m+\frac{1}{2}) : m \in \mathbb{Z}\}$ and form all open sets which are unions of subsets of it. Now the Borel sets can be obtained as follows:

$$\begin{aligned}
B_0 &= \{X : X \text{ open}\};\\
B_{2\alpha+1} &= B_{2\alpha} \cup \{\mathbb{R}\backslash a : a \in B_\alpha\};\\
B_{2\alpha+2} &= B_{2\alpha+1} \cup \left\{\bigcup_{n\in\omega} a_n : a \in {}^{\omega}B_{2\alpha+1}\right\}.
\end{aligned}$$

Now the collection of Borel sets is $\bigcup_{\alpha<\omega_1} B_\alpha$. By induction, $|B_\alpha| = 2^\omega$ for all $\alpha < \omega_1$. $\square$

Lemma 19.54. *Every null set is contained in a Borel null set.*

Proof. Let C be a null set. Then there is an open set U_n such that $C \subseteq U_n$ and $\mu(U_n) \le 2^{-n}$. Then $C \subset \bigcap_{n\in\omega} U_n$ and $\mu(\bigcap_{n\in\omega} U_n) = 0$. $\square$

Theorem 19.55. $\mathrm{cof}(\mathrm{null}) \le 2^\omega$. $\square$

20. Continuum cardinals

We consider several cardinal numbers involving the continuum; most of them are similar to $\mathfrak{b}$ and $\mathfrak{d}$ introduced in Chapter 19. First some definitions.

For $a, b \in [\omega]^\omega$ we write $a \subseteq^* b$ iff $a \backslash b$ is finite.

A set $S \subseteq [\omega]^\omega$ is *splitting* iff $\forall a \in [\omega]^\omega \exists s \in S[a \cap s$ and $a \backslash s$ are infinite$]$.

A *tower* is a system $\langle a_\xi : \xi < \alpha \rangle$ of members of $[\omega]^\omega$ such that:
(i) $a_\xi \subseteq^* a_\eta$ if $\xi < \eta$.
(ii) $a_\eta \backslash a_\xi$ is infinite if $\xi < \eta$.
(iii) α is a limit ordinal.
(iv) If $b \in [\omega]^\omega$ and $a_\xi \subseteq^* b$ for all $\xi < \alpha$, then $\omega \backslash b$ is finite.

$X \subseteq [\omega]^\omega$ is *weakly dense* iff $\forall a \in [\omega]^\omega \exists x \in X[x \subseteq^* a$ or $x \cap a$ is finite$]$.

$X \subseteq [\omega]^\omega$ is *ideal independent* iff $\forall x \in X \forall F \in [X \backslash \{x\}]^{<\omega} \left[x \not\subseteq^* \bigcup_{y \in F} y\right]$.

A *free sequence* is a sequence $\langle a_\xi : \xi < \alpha \rangle$ of members of $[\omega]^\omega$ such that for all $F, G \in [\alpha]^{<\omega}$ with $F < G$ we have $\bigcap_{\xi \in F} a_\xi \cap \bigcap_{\xi \in G} (\omega \backslash a_\xi)$ is infinite. Here we write $F < G$ to mean that $\xi < \eta$ for all $\xi \in F$ and $\eta \in G$. We allow $F = \emptyset$ or $G = \emptyset$. A free sequence is *maximal* iff there is no $b \in [\omega]^\omega$ such that $\langle a_\xi : \xi < \alpha \rangle^\frown \langle b \rangle$ is a free sequence, where $\langle a_\xi : \xi < \alpha \rangle^\frown \langle b \rangle$ is the result of adjoining b at the end of the sequence $\langle a_\xi : \xi < \alpha \rangle$.

A *nonprincipal ultrafilter* on $\mathscr{P}(\omega)$ is a subset D of $\mathscr{P}(\omega)$ such that:
(i) $\omega \in D$.
(ii) If $x \in D$ and $x \subseteq y \subseteq \omega$, then $y \in D$.
(iii) If $x, y \in D$ then $x \cap y \in D$.
(iv) For all $x \subseteq \omega$, $x \in D$ or $(\omega \backslash x) \in D$.
(v) Every member of D is infinite.

A set $X \subseteq [\omega]^\omega$ *generates* a nonprincipal ultrafilter D iff $X \subseteq D$ and for all $x \in D$ there is a finite $F \subseteq X$ such that $\bigcap F \subseteq x$.

Sets $a, b \in [\omega]^\omega$ are *almost disjoint* iff $a \cap b$ is finite. A MAD family is a set $X \subseteq [\omega]^\omega$ such that $\forall x, y \in X[x \neq y \to x \cap y$ is finite$]$, and for all $a \subseteq \omega$, if $\forall x \in X[x \subseteq^* a]$ then $\omega \backslash a$ is finite.

A MAD family X *refines* a MAD family Y iff $\forall x \in X \exists y \in Y[x \subseteq^* y]$.

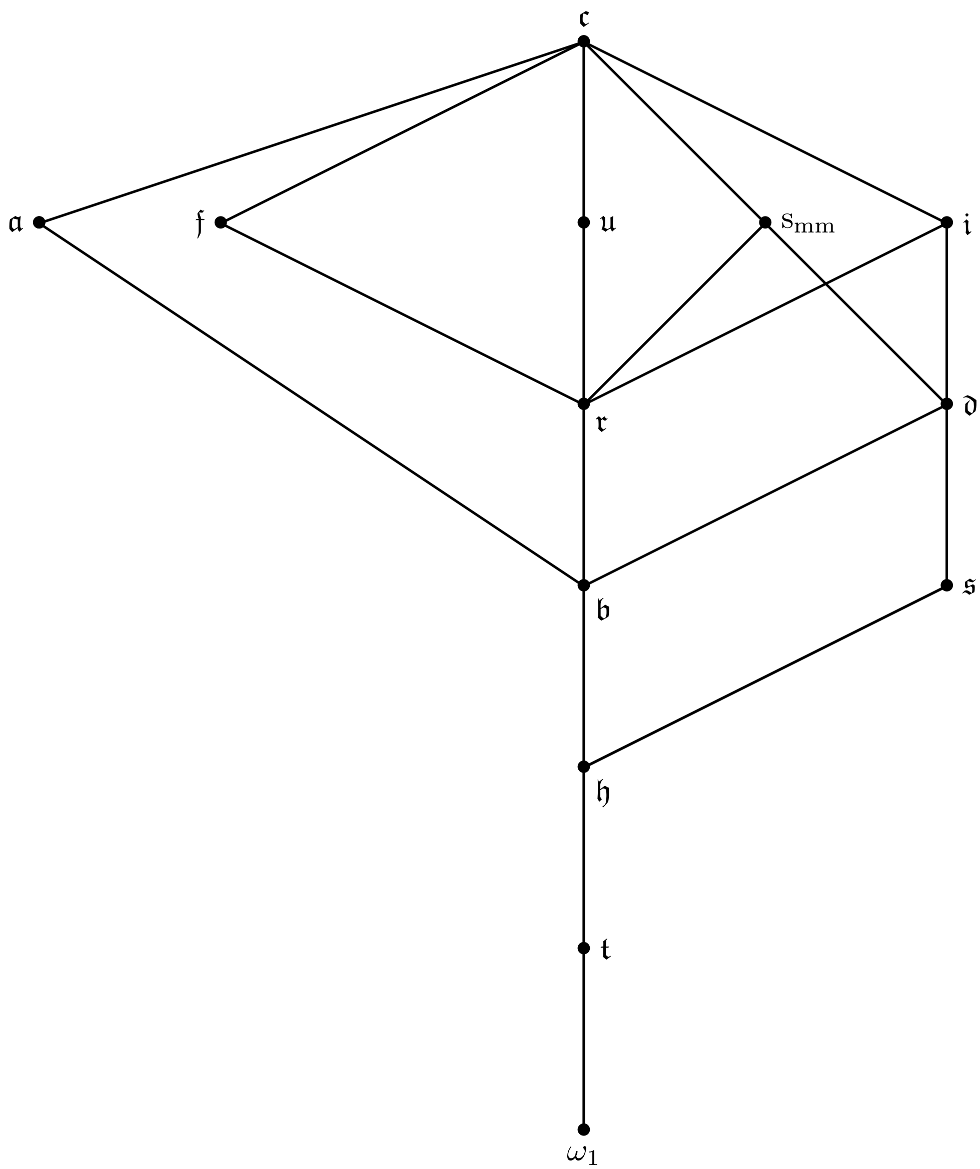
A set $X \subseteq [\omega]^\omega$ is *independent* iff

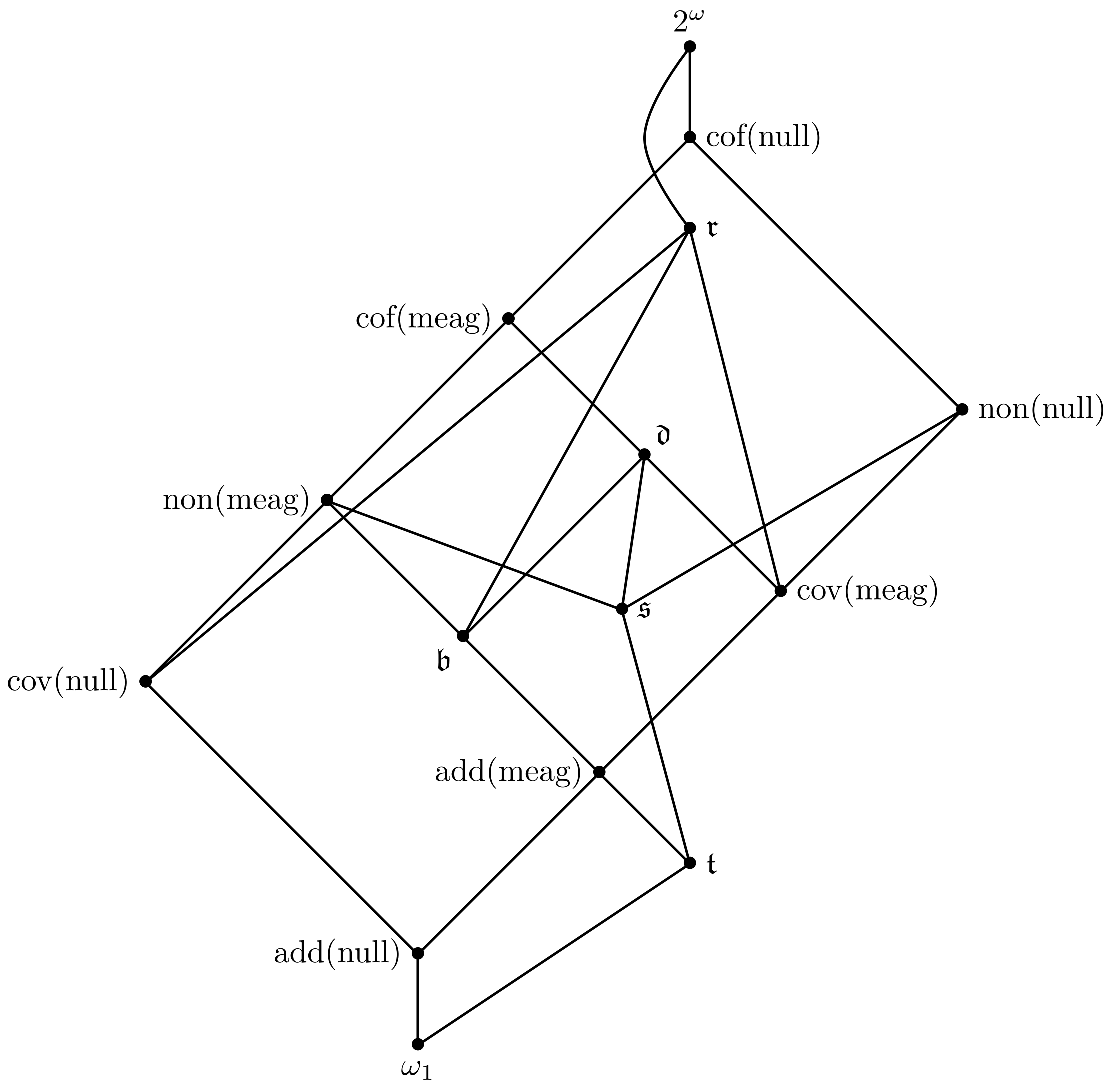
$$\forall F, G \in [X]^{<\omega} \left[F \cap G = \emptyset \to \bigcap_{a \in F} a \cap \bigcap_{a \in G} (\omega \backslash a) \text{ is infinite}\right]$$

Now we can define the cardinals to be studied in this chapter.

$$\mathfrak{s} = \min\{|X| : X \text{ is splitting}\};$$
$$\mathfrak{t} = \min\{|\alpha| : \text{there is a tower of length } \alpha\}$$

$$\begin{aligned}
\mathfrak{h} &= \min\{|X| : X \text{ is a set of MAD families with no common refinement}\}\\
\mathfrak{r} &= \min\{|X| : X \text{ is weakly dense}\}\\
\mathfrak{a} &= \min\{|X| : X \text{ is an infinite MAD family}\}\\
\mathfrak{f} &= \min\{|\alpha| : \text{ there is a maximal free sequence of length } \alpha\}\\
\mathrm{s_{mm}} &= \min\{|X| : X \text{ is infinite and maximal ideal independent}\}\\
\mathfrak{i} &= \min\{|X| : X \text{ is maximal independent}\};\\
\mathfrak{u} &= \min\{|X| : X \text{ generates a nonprincipal ultrafilter}\}.
\end{aligned}$$





We will prove the indicated relationships in these two diagrams.

Recall the definition of Inc from page 270. If $i \in \mathrm{Inc}$ then we let $\varphi(i) = \bigcup_{n\in\omega}[i_{2n}, i_{2n+1})$. If $X \in [\omega]^\omega$ define a member $\psi(X) = i^X$ of Inc as follows:

$$\begin{aligned} i_0^X &= 0; \\ i_{n+1}^X &= \text{ least } j > i_n^X \text{ such that } [i_n^X, j) \cap X \neq \emptyset. \end{aligned}$$

Lemma 20.1. *If $X \in [\omega]^\omega$ and i almost dominates $\psi(X)$, then $\varphi(i)$ splits X.*

Proof. Choose m such that $\forall n \geq m \exists k[[i_k^X, i_{k+1}^X) \subseteq [i_n, i_{n+1})]$; hence $\forall n \geq m[X \cap [i_n, i_{n+1}) \neq \emptyset]$. So the lemma follows. $\square$

Theorem 20.2. $\mathfrak{s} \leq \mathfrak{d}$.

Proof. Let D be an almost dominating family of members of Inc, with $|D| = \mathfrak{d}$; see Proposition 19.30. Then $\{\varphi(i) : i \in D\}$ is a splitting family by Lemma 20.1. $\square$

Proposition 20.3. $\mathfrak{b} \leq \mathfrak{r}$.

Proof. Let $R \subseteq [\omega]^\omega$ be $*$-dense with $|R| = \mathfrak{r}$. We claim that no $i \in \mathrm{Inc}$ almost dominates each member of $\{\psi(X) : X \in R\}$. In fact, otherwise by Lemma 20.1, $\varphi(i)$ splits each member of R, contradiction. So the proposition follows by Proposition 19.17. $\square$

If $\mathscr{F}$ is a family of sets, a *pseudo-intersection* of $\mathscr{F}$ is an infinite set A such that $A \subseteq^* B$ for all $B \in \mathscr{F}$.

A set $\mathscr{D} \subseteq [\omega]^\omega$ is *open* iff $\forall X, Y \in [\omega]^\omega[X \subseteq^* Y \in \mathscr{D} \to X \in \mathscr{D}]$. $\mathscr{D}$ is *dense* iff $\forall Y \in [\omega]^\omega \exists X \in \mathscr{D}[X \subseteq Y]$. Obviously $[\omega]^\omega$ itself is dense open. We say that $\mathscr{D}$ is $*$-*dense* iff $\forall Y \in [\omega]^\omega \exists X \in \mathscr{D}[X \subseteq^* Y]$.

Proposition 20.4. *If $\mathscr{D}$ is $*$-dense, then $\mathscr{D}$ is infinite.*

Proof. Suppose not; say $\mathscr{D}$ is $*$-dense and finite. For each set $X \subseteq \omega$ let $X^1 = X$ and $X^0 = \omega \backslash X$. Then $\omega = \bigcup_{\varepsilon \in {}^{\mathscr{D}}2} \bigcap_{X \in \mathscr{D}} X^{\varepsilon(X)}$, so there is an $\varepsilon \in {}^{\mathscr{D}}2$ such that $\bigcap_{X \in \mathscr{D}} X^{\varepsilon(X)}$ is infinite. Take an infinite $Y \subseteq \bigcap_{X \in \mathscr{D}} X^{\varepsilon(X)}$ with $(\bigcap_{X \in \mathscr{D}} X^{\varepsilon(X)}) \backslash Y$ infinite. Choose $Z \in \mathscr{D}$ such that $Z \subseteq^* Y$, Then $Z \backslash Z^{\varepsilon(Z)} \subseteq Z \backslash Y$, so $\varepsilon(Z) = 1$. (Otherwise $Z \subseteq Z \backslash Y$ with Z infinite and $Z \backslash Y$ finite, contradiction.) But then $\bigcap_{X \in \mathscr{D}} X^{\varepsilon(X)}) \backslash Y \subseteq Z \backslash Y$ with $\bigcap_{X \in \mathscr{D}} X^{\varepsilon(X)}) \backslash Y$ infinite and $Z \backslash Y$ finite, contradiction. $\square$

Proposition 20.5. *If $\mathscr{D}$ is $*$-dense, then there is a $\mathscr{D}'$ such that $\mathscr{D} \subseteq \mathscr{D}' \subseteq [\omega]^\omega$, $|\mathscr{D}| = |\mathscr{D}'|$, and $\mathscr{D}'$ is dense.*

Proof. Let $\mathscr{D}' = \{X \in [\omega]^\omega :$ there is a finite $F \subseteq \omega$ such that $X \cup F \in \mathscr{D}\}$. Then $|\mathscr{D}| = |\mathscr{D}'|$ by Proposition 20.4. Clearly $\mathscr{D} \subseteq \mathscr{D}'$. $\mathscr{D}'$ is dense, since if $Y \in [\omega]^\omega$, choose $X \in \mathscr{D}$ such that $X \subseteq^* Y$. Then $X = (X \cap Y) \cup (X \backslash Y)$, so $X \cap Y \subseteq Y$ and $X \backslash Y$ is finite, so $X \cap Y \in \mathscr{D}'$. $\square$

Proposition 20.6. *For every $X \in [\omega]^\omega$ there is a $*$-dense open family $\mathscr{D}$ such that $X \notin \mathscr{D}$.*

Proof. Let $X = Y \cup Z$ with $Y, Z \in [\omega]^\omega$ and $Y \cap Z = \emptyset$. Define

$$\mathscr{D} = \{W \in [\omega]^\omega : W \subseteq^* Y \text{ or } W \subseteq^* Z \text{ or } W \cap X \text{ is finite}\}.$$

Clearly $\mathscr{D}$ is as desired. $\square$

Lemma 20.7. *If $X \subseteq [\omega]^\omega$ and $\forall x, y \in X[x \neq y \to |x \cap y| < \omega]$, then the following are equivalent:*

(i) X is a MAD family.

(ii) $\forall a \in [\omega]^\omega \exists x \in X[|a \cap x| = \omega]$.

Proof. (i)$\Rightarrow$(ii): Assume that X is a MAD family and $a \in [\omega]^\omega$. If $\forall x \in X[|a \cap x| < \omega]$, then $\forall x \in X[x \subseteq^* (\omega \backslash a)]$, so $a = \omega \backslash (\omega \backslash a)$ is finite, contradiction.

(ii)$\Rightarrow$(i): Assume (ii), and suppose that $a \subseteq \omega$ and $\omega \backslash a$ is infinite. Choose $x \in X$ such that $x \cap (\omega \backslash a)$ is infinite. Thus $x \not\subseteq^* a$. $\square$

Proposition 20.8. $\mathfrak{h} = \min\{|\mathscr{A}| :$ *$\mathscr{A}$ is a family of open $*$-dense sets and $\bigcap \mathscr{A} = \emptyset$.*

Proof. Let $\mathfrak{h}' = \min\{|\mathscr{A}| : \mathscr{A}$ is a family of open $*$-dense sets and $\bigcap \mathscr{A} = \emptyset\}$.

$\mathfrak{h}' \le \mathfrak{h}$: Suppose that $\mathscr{P}$ is a family of MAD families such that $|\mathscr{P}| = \mathfrak{h}$ and $\mathscr{P}$ does not have a common refinement. Let Q be maximal subject to the following conditions:

(i) Q is a family of members of $[\omega]^\omega$;

(ii) $\forall a, b \in Q[a \ne b \to |a \cap b| < \omega]$.

(iii) $\forall a \in Q \forall P \in \mathscr{P} \exists b \in P[a \subseteq^* b]$.

Then Q is not a MAD family, as otherwise it would refine $\mathscr{P}$. Thus there is an $a \subseteq \omega$ such that $\forall x \in Q[x \subseteq^* a]$ but $X \overset{\text{def}}{=} \omega \backslash a$ is infinite. Then $x \cap X$ is finite for all $x \in Q$. Let $f : \omega \to X$ be a bijection. For each $P \in \mathscr{P}$ let

$$\mathscr{D}_P = \{Y \in [\omega]^\omega : \exists Z \in [\omega]^\omega[Z \in P \text{ and } [f[Y] \subseteq^* Z]]\}.$$

We claim that $\mathscr{D}_P$ is $*$-dense open. It is clearly open. Now suppose that $W \in [\omega]^\omega$. Since $f[W]$ is infinite, there is a $Z \in [\omega]^\omega$ such that $Z \in P$ and $f[W] \cap Z$ is infinite. Let $Y = W \cap f^{-1}[Z]$. Then $Y \in [\omega]^\omega$ and $f[Y] \subseteq^* Z$. So $Y \in \mathscr{D}_P$. So the claim is established.

Now suppose that $\bigcap_{P \in \mathscr{P}} \mathscr{D}_P \ne \emptyset$; we will get a contradiction, and this will prove $\mathfrak{h} \le \mathfrak{h}'$. Take $Y \in \bigcap_{P \in \mathscr{P}} \mathscr{D}_P$. For any $P \in \mathscr{P}$ we have $Y \in \mathscr{D}_P$, and so we can choose $Z \in [\omega]^\omega$ such that $Z \in P$ and $f[Y] \subseteq^* Z$. Then $f[Y] \subseteq^* X$ and $Q \cup \{f[Y]\}$ satisfies the conditions defining Q, contradiction.

$\mathfrak{h} \le \mathfrak{h}'$: Suppose that $\mathscr{A}$ is a family of open $*$-dense sets with empty intersection, with $|\mathscr{A}| = \mathfrak{h}'$. Let $\kappa = |\mathscr{A}|$. If $\mathscr{D} \in \mathscr{A}$, let $P_{\mathscr{D}}$ be a maximal subset $[\omega]^\omega$ satisfying the following conditions:

(i) $\forall a, b \in P_{\mathscr{D}}[a \ne b \to |a \cap b| < \omega]$.

(ii) $P_{\mathscr{D}} \subseteq \mathscr{D}$.

Now $P_{\mathscr{D}}$ is a MAD family. For suppose that $a \in [\omega]^\omega$ and $\forall x \in P_{\mathscr{D}}[|a \cap x| < \omega]$. Choose $X \in \mathscr{D}$ such that $X \subseteq^* a$. Then $P_{\mathscr{D}} \cup \{X\}$ satisfies the conditions defining $P_{\mathscr{D}}$, contradiction. Suppose that Q is a common refinement of $\{\mathscr{D}_P : P \in \mathscr{A}\}$; we will get a contradiction, which will finish the proof. Take any $X \in Q$. If $\mathscr{D} \in \mathscr{A}$, then there is a $Y \in P_{\mathscr{D}}$ such that $X \subseteq^* Y$. By the definition of $P_{\mathscr{D}}$, we have $Y \in \mathscr{D}$, hence $X \in \mathscr{D}$ since $\mathscr{D}$ is open. So $X \in \bigcap \mathscr{A}$, contradiction. $\square$

Proposition 20.9. $\mathfrak{t} \le \mathfrak{h}$.

Proof. Suppose that $\mathscr{A}$ is a family of $*$-dense open sets with $|\mathscr{A}| < \mathfrak{t}$; we want to find a member of $\bigcap \mathscr{A}$. Write $\mathscr{A} = \{\mathscr{D}_\alpha : \alpha < \kappa\}$ with $\kappa < \mathfrak{t}$. We now define a sequence $\langle T_\alpha : \alpha \le \kappa\}$ by recursion. Let $T_0 = \omega$. If $T_\alpha \in [\omega]^\omega$ has been chosen, let $T_{\alpha+1} \in \mathscr{D}_\alpha$ be a $*$-subset of T_α; this is possible because $\mathscr{D}_\alpha$ is $*$-dense. For $\alpha \le \kappa$ limit, let T_α be a pseudo-intersection of $\{T_\beta : \beta < \alpha\}$; this is possible because $\alpha < \mathfrak{t}$. This finishes the construction.

By openness we have $T_\kappa \in \bigcap \mathscr{A}$. $\square$

Theorem 20.10. $\omega_1 \le \mathfrak{t}$.

Proof. It suffices to take any sequence $\langle x_n : n \in \omega\rangle$ of elements of $[\omega]^\omega$ such that $\forall m, n \in \omega[m < n \to x_n \subseteq^* x_m]$ and find a pseudo-intersection of $\{x_n : n \in \omega\}$. Now $\bigcap_{m \le n} x_m$ is infinite for each n, since $x_{n+1} \backslash \bigcap_{m \le n} x_m = \bigcup_{m \le n}(x_{n+1} \backslash x_m)$ is finite. For each $n \in \omega$ choose $k_n \in \bigcap_{m \le n} x_m \backslash \{k_m : m < n\}$. Then $y \stackrel{\text{def}}{=} \{k_n : n \in \omega\}$ is infinite, and for each n, $y \backslash x_n \subseteq \{k_m : m < n\}$. $\square$

Let $n \in \omega \backslash 1$, $k \in \omega \backslash 2$, $f : [\omega]^n \to k$, and $H \subseteq \omega$. Then H is *homogeneous for* f iff $f \restriction [H]^n$ is constant; it is *almost homogeneous for* f iff there is a finite set $F \subseteq H$ such that $H \backslash F$ is homogeneous for f. Note that if H is homogeneous for F, then it is almost homogeneous for F.

Proposition 20.11. *If $n \in \omega \backslash 1$ and $k \in \omega \backslash 2$, then there is no $H \in [\omega]^\omega$ such that H is almost homogeneous for all $f \in {}^{[\omega]^n}k$.*

Proof. Suppose that such an H exists. Let $H = \{m_0, m_1, \ldots\}$ with $m_0 < m_1 < \cdots$. Define $f : [\omega]^n \to k$ by setting, for each $Y \in [\omega]^n$,

$$f(Y) = \begin{cases} 0 & \text{if } Y \notin [H]^n, \\ 0 & \text{if } Y \in [H]^n \text{ and } \min(Y) \text{ has the form } m_{2i} \text{ for some } i, \\ 1 & \text{otherwise.} \end{cases}$$

Clearly H is not almost homogeneous for f, contradiction. $\square$

For the next results we need Ramsey's theorem. So we prove it here. Further results along this line will be given later. These are results concerning the *partition calculus*; see also the Dushnik-Miller theorem, Theorem 12.58. The basic definition is as follows:

• Suppose that ρ is a nonzero cardinal number, $\langle \lambda_\alpha : \alpha < \rho \rangle$ is a sequence of cardinals, and σ, κ are cardinals. We also assume that $1 \le \sigma \le \lambda_\alpha \le \kappa$ for all $\alpha < \rho$. Then we write

$$\kappa \to (\langle \lambda_\alpha : \alpha < \rho \rangle)^\sigma$$

provided that the following holds:

For every $f : [\kappa]^\sigma \to \rho$ there exist $\alpha < \rho$ and $\Gamma \in [\kappa]^{\lambda_\alpha}$ such that $f[[\Gamma]^\sigma]] \subseteq \{\alpha\}$.

In this case we say that Γ is *homogeneous for* f. This generalizes the notion above. The following colorful terminology is standard. We imagine that α is a color for each $\alpha < \rho$, and we color all of the σ-element subsets of κ. To say that Γ is homogeneous for f is to say that all of the σ-element subsets of Γ get the same color. Usually we will take σ and ρ to be a positive integers. If $\rho = 2$, we have only two colors, which are conventionally taken to be red (for 0) and blue (for 1). If $\sigma = 2$ we are dealing with ordinary graphs.

Note that if $\rho = 1$ then we are using only one color, and so the arrow relation obviously holds by taking $\Gamma = \kappa$. If κ is infinite and $\sigma = 1$ and ρ is a positive integer, then the relation holds no matter what σ is, since

$$\kappa = \bigcup_{i<\rho} \{\alpha < \kappa : f(\{\alpha\}) = i\},$$

and so there is some $i < \rho$ such that $|\{\alpha < \kappa : f(\{\alpha\}) = i\}| = \kappa \ge \lambda_i$, as desired.

The general infinite Ramsey theorem is as follows.

Theorem 20.12. (Ramsey) *If n and r are positive integers, then*

$$\omega \to (\underbrace{\omega, \ldots, \omega}_{r \text{ times}})^n.$$

Proof. We proceed by induction on n. The case $n = 1$ is trivial, as observed above. So assume that the theorem holds for $n \geq 1$, and now suppose that $f : [\omega]^{n+1} \to r$. For each $m \in \omega$ define $g_m : [\omega \backslash \{m\}]^n \to r$ by:

$$g_m(X) = f(X \cup \{m\}).$$

Then by the inductive hypothesis, for each $m \in \omega$ and each infinite $S \subseteq \omega$ there is an infinite $H_m^S \subseteq S \backslash \{m\}$ such that g_m is constant on $[H_m^S]^n$. We now construct by recursion two sequences $\langle S_i : i \in \omega \rangle$ and $\langle m_i : i \in \omega \rangle$. Each m_i will be in ω, and we will have $S_0 \supseteq S_1 \supseteq \cdots$. Let $S_0 = \omega$ and $m_0 = 0$. Suppose that S_i and m_i have been defined, with S_i an infinite subset of ω. We define

$$\begin{aligned} S_{i+1} &= H_{m_i}^{S_i} \quad \text{and} \\ m_{i+1} &= \text{the least element of } S_{i+1} \text{ greater than } m_i. \end{aligned}$$

Clearly $S_0 \supseteq S_1 \supseteq \cdots$ and $m_0 < m_1 < \cdots$. Moreover, $m_i \in S_i$ for all $i \in \omega$.

(1) For each $i \in \omega$, the function g_{m_i} is constant on $[\{m_j : j > i\}]^n$.

In fact, $\{m_j : j > i\} \subseteq S_{i+1}$ by the above, and so (1) is clear by the definition.

Let $p_i < r$ be the constant value of $g_{m_i} \restriction [\{m_j : j > i\}]^n$, for each $i \in \omega$. Hence

$$\omega = \bigcup_{j<r} \{i \in \omega : p_i = j\};$$

so there is a $j < r$ such that $K \overset{\text{def}}{=} \{i \in \omega : p_i = j\}$ is infinite. Let $L = \{m_i : i \in K\}$. We claim that $f[[L]^{n+1}] \subseteq \{j\}$, completing the inductive proof. For, take any $X \in [L]^{n+1}$; say $X = \{m_{i_0}, \ldots, m_{i_n}\}$ with $i_0 < \cdots < i_n$. Then

$$f(X) = g_{m_{i_0}}(\{m_{i_1}, \ldots, m_{i_n}\}) = p_{i_0} = j. \qquad \square$$

As a digression, we also prove the finite version of Ramsey's theorem:

Theorem 20.13. (Ramsey) *Suppose that $n, r, l_0, \ldots, l_{r-1}$ are positive integers, with $n \leq l_i$ for each $i < r$. Then there is a $k \geq l_i$ for each $i < r$ and $k \geq n$ such that*

$$k \to (l_0, \ldots, l_{r-1})^n.$$

Proof. Assume the hypothesis, but suppose that the conclusion fails. Thus for every k such that $k \geq l_i$ for each $i < r$ with $k \geq n$ also, we have $k \not\to (l_0, \ldots, l_{r-1})^n$, which means that there is a function $f_k : [k]^n \to r$ such that for each $i < r$, there is no set $S \in [k]^{l_i}$

such that $f_k[[S]^n] \subseteq \{i\}$. We use these functions to define a certain $g : [\omega]^n \to r$ which will contradict the infinite version of Ramsey's theorem. Let $M = \{k \in \omega : k \geq l_i$ for each $i < r$ and $k \geq n\}$.

To define g, we define functions $h_i : [i]^n \to r$ by recursion. h_0 has to be the empty function. Now suppose that we have defined h_i so that $S_i \stackrel{\text{def}}{=} \{s \in M : f_s \upharpoonright [i]^n = h_i\}$ is infinite. This is obviously true for $i = 0$. Then

$$S_i = \bigcup_{s:[i+1]^n \to r} \{k \in S_i : f_k \upharpoonright [i+1]^n = s\},$$

and so there is a $h_{i+1} : [i+1]^n \to r$ such that $S_{i+1} \stackrel{\text{def}}{=} \{k \in S_i : f_k \upharpoonright [i+1]^n = h_{i+1}\}$ is infinite, finishing the construction.

Clearly $h_i \subseteq h_{i+1}$ for all $i \in \omega$. Hence $g = \bigcup_{i\in\omega} h_i$ is a function mapping $[\omega]^n$ into r. By the infinite version of Ramsey's theorem choose $v < r$ and $Y \in [\omega]^\omega$ such that $g[[Y]^n] \subseteq \{v\}$. Take any $Z \in [Y]^{l_v}$. Choose i so that $Z \subseteq i$, and choose $k \in S_i$. Then for any $X \in [Z]^n$ we have

$$f_k(X) = h_i(X) = g(X) = v,$$

so Z is homogeneous for f_k, contradiction. $\square$

Proposition 20.14. *If $n \in \omega\backslash 1$, $k \in \omega\backslash 2$, and $F \subseteq {}^{[\omega]^n}k$ is finite, then there is a $H \in [\omega]^\omega$ which is homogeneous for each $f \in F$.*

Proof. Let $F = \{f_0, \ldots, f_m\}$. We define infinite sets $K_0, \ldots K_{m+1} \subseteq \omega$ by recursion. Let $K_0 = \omega$. If $i \leq m$ and K_i has been defined, then $f_{i+1} \upharpoonright [K_i]^n : [K_i]^n \to k$, and so by Ramsey's theorem there is a $K_{i+1} \in [K_i]^\omega$ such that K_{i+1} is homogeneous for $f_{i+1} \upharpoonright [K_i]^n$, hence also for f. Clearly K_{m+1} is as desired in the proposition. $\square$

We now define, for every positive integer n,

$$\mathrm{par}_n = \min\{|F| : F \subseteq {}^{[\omega]^n}2 : \neg\exists X \in [\omega]^\omega \forall f \in F[X \text{ is almost homogeneous for } f\}.$$

By Proposition 20.11 this definition makes sense. Also, by Proposition 20.14, $\omega \leq \mathrm{par}_n$ for every positive integer n.

Proposition 20.15. $\mathfrak{s} = \mathrm{par}_1$.

Proof. First suppose that F satisfies the condition in the definiton of par_1, with $|F| = \mathrm{par}_1$. For each $f \in F$ let $P_f = \{m \in \omega : f(m) = 1\}$, and let $M = \{P_f : f \in F\}$. We claim that M is a splitting family; this will prove $\mathfrak{s} \leq \mathrm{par}_1$. So, suppose that $Y \in [\omega]^\omega$. Choose $f \in F$ such that Y is not almost homogeneous for f. Then $Y \cap P_f$ is infinite, as otherwise, since f has the constant value 0 on $Y\backslash P_f$, $Y\backslash P_f$ would an infinite set homogeneous for f. Similarly $Y\backslash P_f$ is infinite.

Second, suppose that S is a splitting family. Let F be the collection of all characteristic functions of members of S. So if we show that F satisfies the conditions in the definition of par_1, this will prove that $\mathfrak{s} \geq \mathrm{par}_1$. Suppose that $Y \in [\omega]^\omega$, and choose $M \in S$ which splits Y. Let f be the characteristic function of M. If N is any finite subset of Y, then $(Y\backslash N)\cap M$ and $(Y\backslash N)\backslash M$ are both infinite, and so f is not constant on $Y\backslash N$. $\square$

Proposition 20.16. *Suppose that $2 \le k \in \omega$ and n is a positive integer. Then*

$$\mathrm{par}_n = \min\{|F| : F \subseteq {}^{[\omega]^n}k : \neg\exists X \in [\omega]^\omega \forall f \in F[X \text{ is almost homogeneous for } f]\}$$

Proof. If F is as in the definition of par_n, clearly F works as in the right side. So $\ge$ holds. Now suppose that F is as in the right side. For each $f \in F$ and $i < k$ define $g_{fi} : [\omega]^n \to 2$ by setting, for any $x \in [\omega]^n$,

$$g_{fi}(x) = \begin{cases} 0 & \text{if } f(x) = i, \\ 1 & \text{otherwise.} \end{cases}$$

Now $G \overset{\text{def}}{=} \{g_{fi} : f \in F, i < k\}$ has the same size as F by Proposition 20.14, so it suffices, in order to prove $\le$, to show that G satisfies the condition in the definition of par_n. So suppose that $X \in [\omega]^\omega$ and X is almost homogeneous for each g_{fi}. We claim that X is almost homogeneous for each $f \in F$ (contradiction). For, take any $f \in F$. For each $i < k$ let M_i be a finite subset of X such that g_{fi} is constant on $[X\backslash M_i]^n$. We claim that f is constant on $\left[X\backslash \bigcup_{i<k} M_i\right]$, as desired. For, take any two $x, y \in X\backslash \bigcup_{i<k} M_i$. Say $f(x) = i$. Then since $x, y \in X\backslash M_i$, we get $g_{fi}(y) = g_{fi}(x) = 0$, and hence $f(y) = i$, as desired. $\square$

Example 20.17. *If n is a positive integer and $k \in \omega\backslash 2$, then there is a countable $F \subseteq {}^{[\omega]^n}k$ such that there is no $M \in [\omega]^\omega$ such that M is homogeneous for each $f \in F$.*

Proof. Let $[\omega]^n = \{a_\alpha : \alpha < \omega\}$, with $a_\alpha \ne a_\beta$ if $\alpha \ne \beta$. For each $\alpha < \omega$ we define $g_\alpha : [\omega]^n \to k$ by setting, for each $x \in [\omega]^n$,

$$g_\alpha(x) = \begin{cases} 1 & \text{if } x = a_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Let $F = \{g_\alpha : \alpha < \omega\}$. Suppose that $M \in [\omega]^\omega$. Choose $\alpha < \omega$ so that $a_\alpha \in [M]^n$, and choose $x \in [M]^n$ with $x \ne a_\alpha$. Then $g_\alpha(a_\alpha) = 1$ and $g_\alpha(x) = 0$, so M is not homogeneous for g_α. $\square$

Lemma 20.18. *If $m \le n$, then* $\mathrm{par}_n \le \mathrm{par}_m$.

Proof. Assume that $m \le n$. Choose F with $|F|$ minimum such that $F \subseteq {}^{[\omega]^m}2$ and there is no $X \in [\omega]^\omega$ such that $\forall f \in F[X$ is almost homogeneous for $f\}$. For each $f \in F$ define $g_f \in {}^{[\omega]^n}2$ as follows. Let $x = \{a_0, \ldots, a_{n-1}\} \in [\omega]^n$, with $a_0 < \cdots < a_{n-1}$. We define $g_f(x) = f(\{x_0, \ldots, x_{m-1}\})$. We claim that there is no $X \in [\omega]^\omega$ such that $\forall f \in F[X$ is almost homogeneous for $g_f\}$; this will prove the lemma. Suppose that there is such an X. We claim that X is almost homogeneous for each $f \in F$. (Contradiction). For, let $G \in [X]^{<\omega}$ be such that $X\backslash G$ is homogeneous for g_f. Say $g_f \restriction [X\backslash G]^n$ has constant value ε. Then for all $a_0 < \cdots < a_{m-1}$ in ω, choose $a_m \ldots, a_{n-1}$ so that $a_{m-1} < a_m < \cdots < a_{n-1}$. Then $f(\{a_0, \ldots, a_{m-1}\}) = g_f(\{a_0, \ldots, a_{n-1}\} = \varepsilon$, as desired. $\square$

Corollary 20.19. $\mathrm{par}_n \leq \mathfrak{s}$ *for every positive integer* n. □

Theorem 20.20. $\omega < \mathfrak{s}$.

Proof. First suppose that $\{F_0, \ldots, F_{n-1}\}$ is splitting, with $n \in \omega$. For each $\varepsilon \in {}^n2$ let $G_\varepsilon = \bigcap_{i<n} F_i^{\varepsilon(i)}$, where $F_i^1 = F_i$ and $F_i^0 = (\omega \backslash F_i)$. Then $\omega = \bigcup_{\varepsilon \in {}^n2} G_\varepsilon$, so there is an $\varepsilon \in {}^n2$ such that G_ε is infinite. Now G_ε is not split by $\{F_0, \ldots, F_{n-1}\}$, contradiction.

Suppose that $\{Y_i : i < \omega\}$ is a splitting family. It is clear how to construct by recursion an $\varepsilon \in {}^\omega 2$ such that $\bigcap_{j<i} Y_j^{\varepsilon(j)}$ is infinite for every $i < \omega$. Now construct $\langle m_i : i < \omega$ by letting $m_i \in \bigcap_{j<i} Y_j^{\varepsilon(j)} \backslash \{m_j : j < i\}$ for every $i < \omega$. Clearly $Z \stackrel{\text{def}}{=} \{m_i : i < \omega\}$ is not split by any Y_i. □

Theorem 20.21. *For every integer* $n \geq 2$, $\mathrm{par}_n = \min(\mathfrak{b}, \mathfrak{s})$.

Proof. By Corollary 20.19, $\mathrm{par}_n \leq \mathfrak{s}$. Next we show that $\mathrm{par}_n \leq \mathfrak{b}$. By Lemma 20.18 it suffices to take the case $n = 2$. Let B be an almost unbounded subset of ${}^\omega\omega$ with $|B| = \mathfrak{b}$. We may assume that each member of B is strictly increasing. For each $g \in B$ define $f_g : [\omega]^2 \to 2$ by setting for any $\{x, y\} \in [\omega]^2$ with $x < y$,

$$f_g(\{x, y\}) = \begin{cases} 1 & \text{if } g(x) < y, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that there is no set $H \in [\omega]^\omega$ which is almost homogeneous for all f_g's; this will prove $\mathrm{par}_n \leq \mathfrak{b}$. Suppose that there is such an H.

(1) If $K \subseteq \omega$ and $f_g[[K]^2] \subseteq \{0\}$, then K is finite.

For, assume that $K \neq \emptyset$, and let x be its first element. If $z \in K \backslash \{x\}$, then $f_g(\{x, z\}) = 0$, and hence $z \leq g(x)$. So (1) holds.

Now we define $h, k : \omega \to \omega$ as follows. For any $x \in \omega$, $h(x)$ and $k(x)$ are the first and second elements of H which are greater than x. Now take any $g \in B$; we will show that $g <^* k$ (contradiction). Let F be a finite subset of H such that $f_g \restriction [H \backslash F]^2$ is constant. By (1), this constant value is 1. Thus if $x > F$, we have $h(x), k(x) \in H \backslash F$ and $h(x) < k(x)$, so $f_g(\{h(x), k(x)\}) = 1$, and hence $g(h(x)) < k(x)$. So $g(x) < g(h(x)) < k(x)$. Thus $g <^* k$, as desired.

So we have shown $\leq$ in the theorem.

For $\geq$, we prove the following statement by induction on n:

(2) If n is a positive integer, $\langle f_\xi : \xi < \kappa \rangle$ is a system of members of ${}^{[\omega]^n}2$, and $\kappa < \min(\mathfrak{s}, \mathfrak{b})$, then there is a set almost homogeneous for all of the f_ξ's.

This holds for $n = 1$ by Proposition 20.15. Now suppose that $n > 1$ and we know the result for $n - 1$. Suppose that $\langle f_\xi : \xi < \kappa \rangle$ is a sequence of members of ${}^{[\omega]^n}2$ with $\kappa < \min(\mathfrak{b}, \mathfrak{s})$. We want to find a set almost homogeneous for all of them. Let $c : \omega \to [\omega]^{n-1}$ be a bijection. For each $\xi < \kappa$ and $p \in \omega$ let

$$f_{\xi,p}(m) = \begin{cases} f_\xi(c(p) \cup \{m\}) & \text{if } m \notin c(p), \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\{f_{\xi,p} : \xi < \kappa, p \in \omega\}$ is a family of less than $\mathfrak{s}$ functions mapping ω into 2 (using Theorem 20.20). Hence by Proposition 20.15 there is an infinite set A almost homogeneous for all of them. So for each $\xi < \kappa$ and $p \in \omega$ we can choose $g_\xi(p) \in \omega$ and $j_\xi(p) \in 2$ such that $f_{\xi,p}(x) = j_\xi(p)$ for all $x \in A$ such that $x \geq g_\xi(p)$. Write $A = \{m_i : i < \omega\}$, m strictly increasing. For each $a \in [\omega]^{n-1}$ let $c_\xi(a) = j_\xi(c^{-1}(\{m_i : i \in a\}))$. By the inductive hypothesis, let M be an infinite set almost homogeneous for each j_ξ. Choose b_ξ and k_ξ such that c_ξ takes on the constant value k_ξ on $[M\backslash b_\xi]^{n-1}$. Let $B = \{m_i : i \in M\}$.

(3) If $a \in [B\backslash m_{b_\xi}]^{n-1}$ then $j_\xi(c^{-1}(a)) = k_\xi$.

In fact, write $a = \{m_i : i \in s\}$. Then $s \subseteq M$, and $m_i \geq m_{b_\xi}$ and hence $i \geq b_\xi$, for each $i \in s$. So $s \in [M\backslash b_\xi]^{n-1}$, so $k_\xi = c_\xi(s) = j_\xi(c^{-1}(\{m_i : i \in s\})) = j_\xi(c^{-1}(a))$. So (3) holds.

Since $\kappa < \mathfrak{b}$, choose h such that $g_\xi \leq^* h$ for all $\xi < \kappa$. Choose a_ξ so that $g_\xi(p) \leq h(p)$ for all $p \geq a_\xi$.

Now we define $x_0 < x_1 < \cdots$ in B by recursion. Suppose that x_s has been defined for all $s < t$. Choose $x_t \in B$ so that $x_s < x_t$ for all $s < t$, and also $h(p) < x_t$ for all p such that $c(p) \in [\{x_0, \ldots, x_{t-1}\}]^{n-1}$. Let $H = \{x_i : i < \omega\}$. We claim that H is almost homogeneous for each f_ξ. Let $\xi < \kappa$. Choose t such that $t > c(p)$ for each $p < a_\xi$, and also $t \geq m_{b_\xi}$. Suppose that $a \in [H\backslash t]^n$. Let m be the largest element of a, and let $p = c^{-1}(a\backslash\{m\})$. Then $c(p)$ consists of members of H which are $\geq t$, so $a_\xi \leq p$, as otherwise $t > c(p)$, contradiction. Thus $g_\xi(p) \leq h(p) < m$. Also note that $a\backslash\{m\} \in [B\backslash m_{b_\xi}]^{n-1}$. So

$$f_\xi(a) = f_{\xi,p}(m) = j_\xi(p) = j_\xi(c^{-1}(a\backslash\{m\})) = k_\xi. \qquad \square$$

Proposition 20.22. $\mathfrak{h} \leq \mathfrak{b}, \mathfrak{s}$.

Proof. By Proposition 20.21 it suffices to show that $\mathfrak{h} \leq \mathrm{par}_2$. So, suppose that $F \subseteq {}^{[\omega]^2}2$ and $|F| < \mathfrak{h}$; we want to find $X \in [\omega]^\omega$ which is almost homogeneous for all $f \in F$. For each $f \in F$ let $\mathscr{D}_f = \{X \in [\omega]^\omega : X$ is almost homogeneous for $f\}$. We claim that each $\mathscr{D}_f$ is $*$-dense open. For openness, suppose that $X \subseteq^* Y \in \mathscr{D}_f$. Choose G, H finite such that $f \restriction [Y\backslash G]^2$ is constant and $X\backslash Y = H$. Then $f \restriction [X\backslash(G \cup H)]^2 \subseteq f \restriction [Y\backslash G]^2$ is constant; so $X \in \mathscr{D}_f$. For denseness, take any $Y \in [\omega]^\omega$. Then $f \restriction [Y]^2 : [Y]^2 \to 2$, so by Ramsey's theorem there is an infinite $X \subseteq Y$ such that $f \restriction [X]^2$ is constant; so $X \in \mathscr{D}_f$, as desired.

By Proposition 20.8, take $H \in \bigcap_{f\in F} \mathscr{D}_f$. Clearly H is almost homogeneous for all $f \in F$, as desired. $\square$

Proposition 20.23. $\mathfrak{b} \leq \mathfrak{a}$.

Proof. Suppose that $\mathscr{A}$ is an infinite MAD family, with $|\mathscr{A}| = \mathfrak{a}$. Let $\langle C_n : n \in \omega\rangle$ be a one-one enumeration of some of the members of $\mathscr{A}$. We define

$$D_0 = C_0 \cup \left(\omega\backslash \bigcup_{n\in\omega} C_n\right);$$

$$D_{n+1} = C_{n+1}\backslash \bigcup_{m\leq n} C_m.$$

Clearly $\langle D_n : n \in \omega\rangle$ is a partition of ω into infinite subsets. For each $n \in \omega$, let $f_n : D_n \to \omega$ be a bijection. Let $\mathscr{A}' = \mathscr{A}\backslash\{C_n : n \in \omega\}$. Then

$$\text{(1)} \qquad \mathscr{A}'' \stackrel{\text{def}}{=} \mathscr{A}' \cup \{D_n : n \in \omega\} \text{ is a MAD family}$$

In fact, $\mathscr{A}''$ is clearly an almost disjoint family. If $Y \subseteq \omega$ is infinite, choose $E \in \mathscr{A}$ such that $Y \cap E$ is infinite, using Lemma 20.7. If $E = C_n$ for some n, then also $Y \cap D_n$ is infinite, since $Y \cap D_n = Y \cap C_n \backslash \bigcup_{p<n} C_p = Y \cap C_n \backslash \bigcup_{p<n}(C_n \cap C_p)$ and $\bigcup_{p<n}(C_n \cap C_p)$ is finite. If $E \in \mathscr{A}'$, then also $E \in \mathscr{A}''$. So (1) holds.

For each $A \in \mathscr{A}'$ and each $n \in \omega$ the set $A \cap D_n$ is finite. Define $g_A : \omega \to \omega$ by letting $g_A(n)$ be the least natural number such that $\forall m \in A \cap D_n[f_n(m) < g_A(n)]$, for any $n \in \omega$. We claim

$$\text{(2)} \qquad \{g_A : A \in \mathscr{A}'\} \text{ is almost unbounded}$$

This will complete the proof

To show (2), suppose that $g_A \leq^* h$ for all $A \in \mathscr{A}'$. Define $X = \{f_n^{-1}(h(n)) : n \in \omega\}$. Thus

$$\text{(3)} \qquad \forall n \in \omega[|D_n \cap X| = 1]$$

In fact, $f_n^{-1}(h(n)) \in D_n \cap X$. If $m \in D_n \cap X$, then there is a $p \in \omega$ such that $m = f_p^{-1}(h(p))$, so $m \in D_p$, hence $p = n$ and so $m = f_n^{-1}(h(n)$. This proves (3).

It follows from (3) that X is infinite. Now take any $A \in \mathscr{A}'$; we show that $X \cap A$ is finite. Since also $A \cap D_n$ is finite for all $n \in \omega$, this will contradict (1), and hence will complete the proof.

Choose $p \in \omega$ so that $g_A(n) \leq h(n)$ for all $n \geq p$. Then, we claim,

$$\text{(4)} \qquad A \cap X \subseteq \bigcup_{n<p}(A \cap D_n).$$

(Hence $A \cap X$ is finite, as desired.) To prove (4), suppose that $m \in A \cap X$. Choose $n \in \omega$ so that $m = f_n^{-1}(h(n))$. Then $m \in A \cap D_n$, so $f_n(m) < g_A(n)$. But $f_n(m) = h(n)$, so it follows that $n < p$. □

Proposition 20.24. $\mathfrak{r} \leq \mathfrak{i}$.

Proof. Let $\mathscr{I} \subseteq [\omega]^\omega$ be maximal independent, with size $\mathfrak{i}$. Let

$$R = \left\{\prod_{i\in F} a_i^{\varepsilon(i)} : F \in [I]^{<\omega} \text{ and } \varepsilon \in {}^F2\right\},$$

where $b^1 = b$ and $b^0 = (\omega\backslash b)$. By maximality, R satisfies the conditions defining $\mathfrak{r}$. □

Lemma 20.25. *Suppose that $\langle C_n : n \in \omega\rangle$ is a sequence of infinite subsets of ω such that $C_n \subseteq^* C_m$ if $m < n$. Suppose that $\mathscr{A}$ is a family of size less than $\mathfrak{d}$ of infinite subsets* *of ω, each of which has infinite intersection with each C_n. Then $\{C_n : n \in \omega\}$ has a pseudo-intersection B that has infinite intersection with each member of $\mathscr{A}$.*

Proof. Let $C'_n = \bigcap_{m \leq n} C_m$ for all $n \in \omega$. If $A \in \mathscr{A}$, then

$$A \cap C'_n = (A \cap C_n) \backslash \bigcup_{m<n} (C_n \backslash C_m),$$

so $A \cap C'_n$ is still infinite. So it suffices to work with the C'_n's rather than the C_n's.

For each $h \in {}^\omega\omega$ let $B_h = \bigcup_{n \in \omega}(C'_n \cap h(n))$. Then $B_h \backslash C'_n \subseteq \bigcup_{m<n} h(m)$. In fact, if $x \in B_h \backslash C'_n$, say $x \in C'_p \cap h(p)$. Since $x \notin C'_n$, we have $p < n$. Thus $x \in \bigcup_{m<n} h(m)$. It follows that $B_h \subseteq^* C'_n$. Hence it suffices to find $h \in {}^\omega\omega$ so that B_h has infinite intersection with each member of $\mathscr{A}$.

For each $A \in \mathscr{A}$ and each $n \in \omega$, let $f_A(n)$ be the n-th element of the infinite set $A \cap C'_n$ (starting the numbering at 0). Since $|\mathscr{A}| < \mathfrak{d}$, the set $\{f_A : A \in \mathscr{A}\}$ is not almost dominating, and so we can choose $h \in {}^\omega\omega$ such that $h \not\leq^* f_A$ for all $A \in \mathscr{A}$. Thus for each $A \in \mathscr{A}$, the set $\{n \in \omega : h(n) > f_A(n)\}$ is infinite, so that $h(n) \cap A \cap C'_n$ has at least n elements for infinitely many n, and so $B_h \cap A$ is infinite, as desired. □

Proposition 20.26. $\mathfrak{d} \leq \mathfrak{i}$.

Proof. Suppose that $\mathscr{I} \subseteq [\omega]^\omega$ is independent and $|\mathscr{I}| < \mathfrak{d}$; we show that it is not maximal.

Let $\langle D_n : n \in \omega \rangle$ be a one-one enumeration of some of the elements of $\mathscr{I}$, and let $\mathscr{I}' = \mathscr{I} \backslash \{D_n : n \in \omega\}$. For each $\varepsilon \in {}^\omega 2$ and each $n \in \omega$ define

$$C_n^\varepsilon = \bigcap_{k<n} D_k^{\varepsilon(k)}.$$

Let

$$\mathscr{A} = \left\{ \bigcap_{X \in F} X \cap \bigcap_{X \in G} (\omega \backslash X) : F, G \text{ are finite disjoint subsets of } \mathscr{I} \right\}.$$

We apply Lemma 20.25 to $\langle C_n^\varepsilon : n \in \omega \rangle$ and $\mathscr{A}$ to get a pseudo-intersection B^ε of $\{C_n^\varepsilon : n \in \omega\}$ which has infinite intersection with each element of $\mathscr{A}$. Thus

(1) $B^\varepsilon \subseteq^* \bigcap_{k<n} D_k^{\varepsilon(k)}$ for all $n \in \omega$.

(2) B^ε has infinite intersection with each element of $\mathscr{A}$.

(3) $B^\varepsilon \cap B^\delta$ is finite for distinct $\varepsilon, \delta \in {}^\omega 2$.

This is clear from (1).

(4) There are countable disjoint $Q, Q' \subseteq {}^\omega 2$ such that for every $p \in {}^{<\omega}2$ there are $f \in Q$ and $g \in Q'$ such that $p \subseteq f$ and $p \subseteq g$.

In fact, enumerate ${}^{<\omega}2$ as $\langle p_n : n \in \omega \}$. Now we define functions $f_n, g_n \in {}^\omega 2$ by induction as follows: they are distinct elements of the set

$$\{h \in {}^\omega 2 : p_n \subseteq h\} \backslash \{f_m, g_m : m < n\}.$$

Then we let $Q = \{f_n : n \in \omega\}$ and $Q' = \{g_n : n \in \omega\}$. Clearly (4) holds.

(5) There exists $\langle E^\varepsilon : \varepsilon \in Q \cup Q' \rangle$ such that the E^ε's are pairwise disjoint, $E^\varepsilon \subseteq B^\varepsilon$, and $B^\varepsilon \backslash E^\varepsilon$ is finite.

To prove this, enumerate $Q \cup Q'$ as $\langle \varepsilon_n : n \in \omega \rangle$ without repetitions, and let $E^{\varepsilon_n} = B^{\varepsilon_n} \backslash \bigcup_{m<n} B^{\varepsilon_m}$ for all m; clearly (5) then holds.

Now we define

$$Z = \bigcup_{\varepsilon \in Q} E^\varepsilon, \quad \text{and} \quad Z' = \bigcup_{\varepsilon \in Q'} E^\varepsilon.$$

(6) Z has infinite intersection with each set $\left(\bigcap_{X \in F} X\right) \cap \left(\bigcap_{X \in G} (\omega \backslash X)\right)$ with F, G finite disjoint subsets of $\mathscr{I}$.

In fact, take such F, G. Let $F' = F \cap \mathscr{I}'$ and $G' = G \cap \mathscr{I}'$. Choose $n \in \omega$ such that for all $k \in \omega$, if $D_k \in F \cup G$ then $k < n$. Define $p \in {}^n 2$ by setting, for each $k < n$,

$$p(k) = \begin{cases} 1 & \text{if } D_k \in F, \\ 0 & \text{otherwise.} \end{cases}$$

Choose $\varepsilon \in Q$ such that $p \subseteq \varepsilon$. Then

$$\begin{aligned}
\left(\bigcap_{X \in F} X\right) \cap \left(\bigcap_{X \in G} (\omega \backslash X)\right) &= \left(\bigcap_{X \in F'} X\right) \cap \left(\bigcap_{X \in G'} (\omega \backslash X)\right) \cap \left(\bigcap_{D_k \in F \cup G} D_k^{\varepsilon(k)}\right) \\
&\supseteq \left(\bigcap_{X \in F'} X\right) \cap \left(\bigcap_{X \in G'} (\omega \backslash X)\right) \cap \left(\bigcap_{k<n} D_k^{\varepsilon(k)}\right) \\
&\supseteq^* \left(\bigcap_{X \in F'} X\right) \cap \left(\bigcap_{X \in G'} (\omega \backslash X)\right) \cap B^\varepsilon. \\
&\supseteq^* \left(\bigcap_{X \in F'} X\right) \cap \left(\bigcap_{X \in G'} (\omega \backslash X)\right) \cap E^\varepsilon.
\end{aligned}$$

The last intersection is infinite, and is a subset of Z since $\varepsilon \in Q$, as desired; (6) holds.

Similarly,

(7) Z' has infinite intersection with each set $\bigcap_{X \in F} X \cap \bigcap_{X \in G} (\omega \backslash X)$, with F, G finite disjoint subsets of $\mathscr{I}$.

Since $\omega \backslash Z \supseteq Z'$, this finishes the proof. □

Proposition 20.27. $\mathfrak{r} \leq \mathfrak{u}$.

Proof. Let X filter-generate a nonprincipal ultrafilter U. We may assume that X is closed under $\cap$. For any $a \subseteq \omega$, either $a \in U$ or $(\omega \backslash a) \in U$; so there is a $b \in X$ such that $b \subseteq a$ or $b \subseteq (\omega \backslash a)$. □

Proposition 20.28. $\mathfrak{r} \leq \mathrm{s}_{\mathrm{mm}}$.

Proof. Suppose that $X \subseteq [\omega]^\omega$ is maximal ideal independent. Let

$$Y = X \cup \left\{\omega \backslash \bigcup F : F \in [X]^{<\omega}\right\} \cup \left\{b \backslash \bigcup F : b \notin F,\ F \cup \{b\} \in [X]^{<\omega}\right\}.$$

If $F \cup \{b\} \in [X]^{<\omega}$ and $b \notin F$, then $b \backslash \bigcup F$ is infinite by ideal independence. If $F \in [X]^{<\omega}$, then there is a $b \in X \backslash F$ since X is infinite. So $\omega \backslash \bigcup F \supseteq (b \backslash \bigcup F)$, so $\omega \backslash \bigcup F$ is infinite. Thus all members of Y are infinite. We claim that Y is weakly dense in A. For, suppose that $a \in A \backslash X$. Then $X \cup \{a\}$ is no longer ideal independent, so we have two cases.

Case 1. $a \subseteq^* \bigcup F$ for some $F \in [X]^{<\omega}$. Then $\omega \backslash \bigcup F \subseteq^* (\omega \backslash a)$, as desired.

Case 2. There exist a finite subset F of X and a $b \in X \backslash F$ such that $b \subseteq^* a \cup \bigcup F$. Then $b \backslash \bigcup F \subseteq^* a$, as desired. □

Proposition 20.29. *Suppose that X is maximal ideal independent, $f : X \to [\omega]^\omega$, and $x \triangle f(x)$ is finite for all $x \in X$. Then*

(i) $\forall F \in [X]^{<\omega}[(\bigcup F) \triangle \left(\bigcup_{x \in F} f(x)\right)$ is finite].

(ii) $f[X]$ is ideal independent.

(iii) $f[X]$ is maximal ideal independent.

Proof. (i): Suppose that $F \in [X]^{<\omega}$. Then

$$\begin{aligned}
\left(\bigcup F\right) \triangle \left(\bigcup_{x \in F} f(x)\right) &= \left(\left(\bigcup F\right) \backslash \left(\bigcup_{x \in F} f(x)\right)\right) \cup \left(\left(\bigcup_{x \in F} f(x)\right) \backslash (\left(\bigcup F\right)\right) \\
&= \left(\bigcup_{x \in F} \left(x \backslash \bigcup_{y \in F} f(y)\right)\right) \cup \bigcup_{x \in F} \left(f(x) \backslash \bigcup_{y \in F} y\right) \\
&\subseteq \left(\bigcup_{x \in F} (x \backslash f(x))\right) \cup \bigcup_{x \in F} (f(x) \backslash x),
\end{aligned}$$

and the last set here is obviously finite.

(ii): Suppose that $y \in f[X]$, $F \in [f[X] \backslash \{y\}]^{<\omega}$, and $y \subseteq^* \bigcup F$. Say $y = f(x)$ and $z = f(u_z)$ for each $z \in F$, with each $x, u_z \in X$. Clearly $x \neq u_z$ for all $z \in F$. Hence $x \backslash \bigcup_{z \in F} u_z$ is infinite. Now $x \backslash \bigcup_{z \in F} u_z \subseteq (x \backslash \bigcup F) \cup (\bigcup F \backslash \bigcup_{z \in F} u_z)$, and $\bigcup F \backslash \bigcup_{z \in F} u_z$ is finite by (i). Hence $x \backslash \bigcup F$ is infinite. Also, $x \backslash \bigcup F \subseteq (y \backslash \bigcup F) \cup (x \backslash y)$, and $x \backslash y$ is finite, so $y \backslash \bigcup F$ is infinite.

(iii): Suppose that $y \in [\omega]^\omega \backslash f[X]$; we want to show that $f[X] \cup \{y\}$ is ideal dependent.

Case 1. $y \triangle x$ is finite for some $x \in X$. Then $y \backslash f(x) \subseteq (y \backslash x) \cup (x \backslash f(x))$, so $y \backslash f(x)$ is finite, and consequently $y \subseteq^* f(x)$, as desired.

Case 2. $y \triangle x$ is infinite for all $x \in X$. In particular, $y \notin X$.

Subcase 2.1. $y \subseteq^* \bigcup F$ for some $F \in [X]^{<\omega}$. Then $y \backslash \bigcup_{x \in F} f(x) \subseteq (y \backslash \bigcup F) \cup (\bigcup F \backslash \bigcup_{x \in F} f(x))$. Now $y \backslash \bigcup F$ is finite by the case assumption, and $\bigcup F \backslash \bigcup_{x \in F} f(x)$ is finite by (i), so $y \backslash \bigcup_{x \in F} f(x)$ is finite, hence $y \subseteq^* \bigcup_{x \in F} f(x)$, as desired.

Subcase 2.2. There exist $x \in X$ and $F \in [X \backslash \{x\}]^{<\omega}$ such that $x \subseteq^* y \cup \bigcup F$. Thus

(1) $(x \backslash \bigcup F) \backslash y$ is finite.

Now if $f(x) \in f[F]$, say $f(x) = f(z)$ with $z \in F$. Then $x \backslash z \subseteq (x \backslash f(x)) \cup (f(z) \backslash z)$, so $x \backslash z$ is finite. So $x \subseteq^* z$, contradiction. Thus $f(x) \notin f[F]$. Now

$$\left(x \backslash \bigcup_{z \in F} f(z) \right) \backslash y \subseteq \left(\left(x \backslash \bigcup F \right) \backslash y \right) \cup \left(\left(\bigcup F \right) \backslash \bigcup_{z \in F} f(z) \right)$$

By (1) and (i) it follows that $(x \backslash \bigcup_{z \in F} f(z)) \backslash y$ is finite. Now

$$(f(x) \backslash \bigcup_{z \in F} f(z)) \backslash y \subseteq ((x \backslash \bigcup_{z \in F} f(z)) \backslash y) \cup (f(x) \backslash x),$$

which is finite. Thus $f(x) \subseteq^* y \cup \bigcup_{z \in F} f(z)$, as desired. $\square$

Theorem 20.30. $\mathfrak{d} \leq \mathrm{s_{mm}}$.

Proof. Suppose, in order to get a contradiction, that $\mathrm{s_{mm}} < \mathfrak{d}$. Let $X \subseteq [\omega]^\omega$ be maximal ideal independent with $\omega \leq |X| < \mathfrak{d}$. Let $\langle A_i : i \in \omega \rangle$ be a sequence of distinct elements of X. Define $A'_i = A_i \cup \{i\}$. Let

$$X' = (X \backslash \{A_i : i \in \omega\}) \cup \{A'_i : i \in \omega\}$$

By Proposition 20.29, X' is maximal ideal independent. Define $C_i = A'_i \backslash \bigcup_{j<i} A'_j$ for each $i \in \omega$. Then each C_i is infinite, by the ideal independence of X'. Also note that $\bigcup_{j<n} C_j = \bigcup_{j<n} A'_j$, by induction on n. Hence $\bigcup_{j \in \omega} C_j = \bigcup_{j \in \omega} A'_j = \omega$.

(1) If $F \in [X']^{<\omega}$, $B \in X' \backslash (F \cup \{A'_i : i \in \omega\})$, and $n \in \omega$, then there is a $j \geq n$ such that $C_j \cap (B \backslash \bigcup F) \neq \emptyset$.

In fact, otherwise we have $\left(\bigcup_{j \geq n} C_j \right) \cap (B \backslash \bigcup F) = \emptyset$, hence $B \subseteq (\bigcup F) \cup \bigcup_{j<n} C_j = (\bigcup F) \cup \bigcup_{j<n} A'_j$, contradicting the ideal independence of X'.

By (1) we have

(2) If $F \in [X']^{<\omega}$, $B \in X' \backslash (F \cup \{A'_i : i \in \omega\})$, and $n \in \omega$, then there exist k and $j \geq n$ such that $(C_j \cap B \cap k) \backslash \bigcup F \neq \emptyset$.

Now we define for $F \in [X']^{<\omega}$, $B \in X' \backslash (F \cup \{A'_i : i \in \omega\})$ and $n \in \omega$

$$\varphi_{FB}(n) = \min\{k \in \omega : \exists j \geq n[(C_j \cap B \cap k) \backslash \bigcup F \neq \emptyset]\}.$$

The number of pairs (F, B) as above is less than $\mathfrak{d}$, so the set of all such functions φ_{FB} is not dominating. Hence there is a function $h_0 \in {}^\omega\omega$ not dominated by any of them. We may assume that h_0 is strictly increasing. For each $n \in \omega$ let $D_n = C_n \backslash h_0(n)$.

(3) If $F \in [X']^{<\omega}$ and $n \in \omega$, then there is a $j \geq n$ such that $D_j \backslash \bigcup F \neq \emptyset$.

In fact, otherwise we have $\left(\bigcup_{j \geq n} D_j \right) \backslash \bigcup F = \emptyset$, i.e., $\left(\bigcup_{j \geq n} (C_j \backslash h_0(j)) \right) \backslash \bigcup F = \emptyset$. Hence $\left(\bigcup_{j \geq n} (C_j \backslash h_0(j)) \right) \subseteq \bigcup F$. Choose $j \geq n$ so that $A'_j \notin F$. Then $C_j \backslash h_0(j) \subseteq \bigcup F$, i.e.,

$(A'_j\backslash\bigcup_{k<j}A'_k)\backslash h_0(j)\subseteq\bigcup F$. Hence $A'_j\subseteq^*\bigcup_{k<j}A'_k\cup\bigcup F$, contradicting ideal independence.

From (3) we obtain

(4) If $F\in[X']^{<\omega}$ and $n\in\omega$, then there exist k and $j\geq n$ such that $(D_j\cap k)\backslash\bigcup F\neq\emptyset$.

Now for $F\in[X']^{<\omega}$ and $n\in\omega$ let

$$\varphi'_F(n)=\min\{k:\exists j\geq n[(D_j\cap k)\backslash\bigcup F\neq\emptyset]\}.$$

Again, there are fewer than $\mathfrak{d}$ of these functions φ'_F, so there is a function $k\in{}^\omega\omega$ not dominated by any of them. For any $n\in\omega$ let

$$h_1(n)=\max\{h_1(n-1)+1,k(n)+1,\min(C_n\backslash h_0(n))+1\},$$

with $h_1(n-1)+1$ omitted if $n=0$.

Now let

$$Y=\bigcup_{n\in\omega}[D_n\cap h_1(n)].$$

Note that for all $n\in\omega$, $D_n\cap h_1(n)\neq\emptyset$, since $\min(D_n)=\min(C_n\backslash h_0(n))<h_1(n)$. Hence Y is infinite. We claim that $Y\notin X'$ and $X'\cup\{Y\}$ is ideal independent. (Contradiction.)

(5) $\forall F\in[X']^{<\omega}[Y\not\subseteq^*\bigcup F]$; in particular, $Y\notin X'$.

In fact, let $n\in\omega$; we will find $j\geq n$ such that $(D_j\cap h_1(j))\backslash\bigcup F\neq\emptyset$. We have $k\not\leq^*\varphi'_F$, so choose $m\geq n$ so that $\varphi'_F(m)<k(m)$. By the definition of $\varphi'_F(m)$, there is a $j\geq m$ such that $D_j\cap\varphi'_F(m)\backslash\bigcup F\neq\emptyset$. We have $\varphi'_F(m)<k(m)<h_1(m)\leq h_1(j)$, so $D_j\cap h_1(j)\backslash\bigcup F\neq\emptyset$. This proves (5).

(6) For all $F\in[X']^{<\omega}$ and all $n\in\omega$ with $A'_n\notin F$ we have $A'_n\not\subseteq^* Y\cup\bigcup F$.

In fact, assume otherwise. Now for $m>n$ we have $A_n\cap C_m=\emptyset$, and hence $A_n\cap D_m=\emptyset$. Hence $A_n\subseteq^*\bigcup_{m\leq n}[D_n\cap h_1(n)]\cup\bigcup F$. Since $\bigcup_{m\leq n}[D_n\cap h_1(n)]$ is finite, it follows that $A_n\subseteq^*\bigcup F$, contradiction. So (6) holds.

(7) If $F\in[X']^{<\omega}$ and $B\in X'\backslash(F\cup\{A'_n:n\in\omega\})$, then $B\not\subseteq^* Y\cup\bigcup F$.

To prove (7) we first prove

(8) If $F\in[X']^{<\omega}$ and $B\in X'\backslash(F\cup\{A'_n:n\in\omega\})$, then for all $n\in\omega$ there is a $j\geq n$ such that $(C_j\cap B\cap h_0(j))\backslash\bigcup F\neq\emptyset$.

To prove (8), since h_0 is not dominated by φ_{FB}, choose $m\geq n$ such that $\varphi_{FB}(m)<h_0(m)$. Then by definition of φ_{FB} there is a $j\geq m$ such that $C_j\cap B\cap\varphi_{FB}(m)\backslash\bigcup F\neq\emptyset$. So $C_j\cap B\cap h_0(m)\backslash\bigcup F\neq\emptyset$, hence $C_j\cap B\cap h_0(j)\backslash\bigcup F\neq\emptyset$. This proves (8)

Now for (7), note that $C_j\cap h_0(j)\cap Y=\emptyset$. Thus $(C_j\cap B\cap h_0(j))\backslash\bigcup F\subseteq B\backslash(Y\cup\bigcup F)$. Since for all $n\in\omega$ there is a $j\geq n$ with this property, (7) follows. $\square$

Proposition 20.31. $\omega\leq\mathfrak{f}$.

Proof. We need to show that any free sequence $\langle a_\xi : \xi < n\rangle$ with n a positive integer is not maximal. For each $i \le n$ let

$$c_i = \left(\bigcap_{\xi<i} a_\xi\right) \cap \left(\bigcap_{i\le\xi<n} (\omega\backslash a_\zeta)\right)$$

These sets are pairwise disjoint and infinite. Let b be such that $|c_i \cap b| = \omega = |c_i\backslash b|$ for all $i \le n$. Clearly $\langle a_\xi : \xi < n\rangle^\frown\langle b\rangle$ is a free sequence. $\square$

Theorem 20.32. $\mathfrak{r} \le \mathfrak{f}$.

Proof. Suppose that $\langle a_\xi : \xi < \alpha\rangle$ is a maximal free sequence with $|\alpha| = \mathfrak{f}$. By Proposition 20.31, α is infinite. We claim that

$$\left\{\bigcap_{\xi\in F} a_\xi \cap \bigcap_{\xi\in G} (\omega\backslash a_\xi) : F, G \in [\alpha]^{<\omega}, F < G\right\}$$

is weakly dense. To see this, let $b \in [\omega]^\omega$. If $b = a_\xi$ for some $\xi < \alpha$, the conclusion is clear. Otherwise $\langle a_\xi : \xi < \alpha\rangle^\frown\langle b\rangle$ is not a free sequence. Define $a_\alpha = b$. Then there exist finite $F, G \subseteq \alpha + 1$ such that $F < G$ and $\bigcap_{\xi\in F} a_\xi \cap \bigcap_{\xi\in G}(\omega\backslash a_\xi)$ is finite.

Case 1. $G = \emptyset$. Then clearly $\alpha \in F$. Since b is infinite, $F \ne \{\alpha\}$. So $\bigcap_{\xi\in F\backslash\{\alpha\}} a_\xi \cap b$ is finite, as desired.

Case 2. $G \ne \emptyset$. Clearly then α is the greatest element of G, and so $\bigcap_{\xi\in F} a_\xi \cap \bigcap_{\xi\in G\backslash\{\alpha\}}(\omega\backslash a_\xi) \subseteq^* b$. $\square$

Theorem 20.33. $\mathfrak{a}, \mathfrak{f}, \mathfrak{u}, s_{\mathrm{mm}}, \mathfrak{i} \le \mathfrak{c}$.

Proof. $\mathfrak{a}, \mathfrak{u}, s_{\mathrm{mm}}, \mathfrak{i}$ are all defined as the minimum of $|X|$ for some $X \subseteq [\omega]^\omega$, so they are all at most $|[\omega]^\omega| = \mathfrak{c}$. A free sequence is a one-one sequence of members of $[\omega]^\omega$, so the sequence has length of size at most $\mathfrak{c}$. $\square$

This finishes checking the first diagram at the beginning of this chapter.

Lemma 20.34. *Suppose that $\lambda < \mathfrak{t}$ and $\langle T_\alpha : \alpha < \lambda\rangle$ is a sequence of dense subsets of $\mathbb{Q}$ such that $\forall\alpha, \beta < \lambda[\alpha < \beta \to T_\beta \subseteq^* T_\alpha]$. Then there is a dense $X \subseteq \mathbb{Q}$ which is almost contained in every T_α.*

Proof. Let $h : \mathbb{Q} \to \omega$ be a bijection. For each interval I with rational endpoints the sequence $\langle h[T_\alpha \cap I] : \alpha < \lambda\rangle$ consists of infinite subsets of ω and it is almost decreasing; hence it has an infinite pseudo-intersection Y_I. Let $\langle y_{In} : n \in \omega\rangle$ be a one-one enumeration of Y_I. Now for each $\alpha < \lambda$ we have $Y_I \subseteq^* h[T_\alpha \cap I]$, so we can let $f_\alpha(I)$ be greater than each member of $\{n \in \omega : y_{In} \notin h[T_\alpha \cap I]\}$. Let M be the set of all rational intervals, and let $k : \omega \to M$ be a bijection. Then each $f_\alpha \circ k$ maps ω into ω. Since $\lambda < \mathfrak{t} \le \mathfrak{b}$, let $g \in {}^\omega\omega$ be such that $f_\alpha \circ k \le^* g$ for all $\alpha < \lambda$. Let

$$X = \bigcup_{I\in M} \{y_{In} : n > g(k^{-1}(I))\}.$$

Now $h^{-1}[X]$ is dense in $\mathbb{Q}$. For, given $I \in M$, we have $Y_I \backslash X \subseteq \{y_{In} : n \leq g(k^{-1}(I))\}$, and so $Y_I \subseteq^* X$. Hence $Y_I \cap X$ is infinite. Now $Y_I \subseteq^* h[I]$, so there is an $n \in X \cap h[I]$, so $h^{-1}(n) \in I$, as desired.

Finally, let $\alpha < \lambda$. Choose m such that $\forall n \geq m[f_\alpha(k(n)) \leq g(n)]$. Then $\forall I \in M[g(k^{-1}(I)) < f_\alpha(I) \rightarrow k^{-1}(I) < m]$. So $Z \stackrel{\text{def}}{=} \{I \in M : g(k^{-1}(I) < f_\alpha(I)\}$ is finite. We claim that

$$X \backslash T_\alpha \subseteq \{h^{-1}(y_{In}) : I \in Z \text{ and } g(k^{-1}(I)) < n\};$$

hence $X \backslash T_\alpha$ is finite. For, suppose that $I \in M$, $g(k^{-1}(I)) < n$, and $h^{-1}(y_{In}) \notin T_\alpha$; thus $h^{-1}(y_{In})$ is a typical member of $X \backslash T_\alpha$. Then $y_{In} \notin h[T_\alpha]$, hence $n < f_\alpha(I)$, so $g(k^{-1}(I)) < f_\alpha(I)$ and so $I \in Z$. $\square$

Lemma 20.35.

$$\begin{aligned}\text{add(meager)} = \min\{|X| : X \textit{ is a collection of open dense subsets of } \mathbb{R}\\ \textit{and } \bigcap X \textit{ does not contain a countable intersection of open dense subsets of } \mathbb{R}\}.\end{aligned}$$

Proof. Let X be a collection of meager sets such that $\bigcup X$ is not meager, and $|X| = \text{add(meager)}$. Then $|X| \geq \omega_1$ by the Cichón diagram. For each $x \in X$ let B_x be a countable set of nowhere dense sets such that $x = \bigcup B_x$. For each $x \in X$ and $y \in B_x$ the set $\mathbb{R} \backslash \overline{y}$ is open dense. Let $Y = \{\mathbb{R} \backslash \overline{y} : x \in X, y \in B_x\}$. Then $|Y| = |X|$ and each member of Y is open dense. Suppose that $\bigcap C \subseteq \bigcap Y$, with C a countable set of open dense subsets of $\mathbb{R}$. Then

$$\begin{aligned}\bigcup X &= \bigcup_{x \in X} \bigcup B_x = \bigcup \{y : \exists x \in X[y \in B_x]\}\\ &\subseteq \bigcup \{\overline{y} : \exists x \in X[y \in B_x]\} = \mathbb{R} \backslash \bigcap Y \subseteq \mathbb{R} \backslash \bigcap C = \bigcup_{c \in C} (\mathbb{R} \backslash c).\end{aligned}$$

Thus $\bigcup X$ is meager, contradiction.

Conversely, suppose that X is a collection of open dense subsets of $\mathbb{R}$ and $\bigcap X$ does not contain a countable intersection of open dense subsets of $\mathbb{R}$. Let $Y = \{\mathbb{R} \backslash x : x \in X\}$. Then Y is a set of nowhere dense subsets of $\mathbb{R}$. Suppose that $\bigcup Y$ is meager. Then there is a countable set Z of nowhere dense subsets of $\mathbb{R}$ such that $\bigcup Y \subseteq \bigcup Z$. Now each set $\mathbb{R} \backslash \overline{z}$ for $z \in Z$ is open dense. We have

$$\bigcap_{z \in Z} (\mathbb{R} \backslash \overline{z}) \subseteq \bigcap_{z \in Z} (\mathbb{R} \backslash z) \subseteq \bigcap_{y \in Y} (\mathbb{R} \backslash y) = \bigcap X,$$

contradiction. $\square$

Theorem 20.36. $\mathfrak{t} \leq \text{add(meag)}$.

Proof. By Lemma 20.35 it suffices to take any $\kappa < \mathfrak{t}$, assume that $\langle G_\alpha : \alpha < \kappa \rangle$ is a family of open dense subsets of $\mathbb{R}$, and show that $\bigcap_{\alpha < \kappa} G_\alpha$ contains a countable intersection of open dense subsets of $\mathbb{R}$.

First we construct an almost descending sequence $\langle T_\alpha : \alpha \leq \kappa\rangle$ of dense subsets of $\mathbb{Q}$. Let $T_0 = \mathbb{Q}$. At limit stages, apply Lemma 20.34. At successor stages define $T_{\alpha+1} = T_\alpha \cap G_\alpha$. Clearly $T_{\alpha+1}$ is dense in $\mathbb{Q}$ since both T_α and G_α are, and G_α is open.

Note that $T_\kappa \subseteq^* G_\alpha$ for each $\alpha < \kappa$.

Let $h : \mathbb{Q} \to \omega$ be a bijection.

For each $t \in T_\kappa$ and $\alpha < \kappa$, let $f_\alpha(t)$ be some $n \in \omega\backslash 1$ such that $(t - \frac{1}{n}, t + \frac{1}{n}) \subseteq G_\alpha$ if $t \in G_\alpha$, and $f_\alpha(t) = 0$ otherwise. Now T_κ is countable and $\kappa < \mathfrak{t} \leq \mathfrak{b}$, so there is a $g : \omega \to \omega$ such that $f_\alpha \circ h^{-1} \leq^* g$ for all $\alpha < \kappa$.

Now for each finite $F \subseteq T_\kappa$ let

$$U_F = \bigcup_{t\in T_\kappa\backslash F} \left(t - \frac{1}{g(h(t))+1}, t + \frac{1}{g(h(t))+1}\right).$$

Obviously U_F is open. Clearly $T_\kappa\backslash F \subseteq U_F$, so U_F is dense in $\mathbb{Q}$. Furthermore,

$$\bigcap_{F\in[T_\kappa]^{<\omega}} U_F \subseteq \bigcap_{\alpha<\kappa} G_\alpha.$$

For, let $\alpha < \kappa$. Then

$$F \stackrel{\text{def}}{=} (T_\kappa\backslash G_\alpha) \cup \{t \in \mathbb{Q} : g(h(t)) < f_\alpha(t)\}$$

is finite, since $f_\alpha(t) = f(h^{-1}(h(t))$. For any $t \in T_\kappa\backslash F$ we have $f_\alpha(t) \leq g(h(t))$ and $t \in G_\alpha$, and so

$$\left(t - \frac{1}{g(h(t))+1}, t + \frac{1}{g(h(t))+1}\right) \subseteq \left(t - \frac{1}{f_\alpha(t)}, t + \frac{1}{f_\alpha(t)}\right) \subseteq G_\alpha. \qquad \square$$

Lemma 20.37. *If $A \in [\omega]^\omega$, let $U_A = \{X \in [\omega]^\omega : A \cap X$ is finite, or $A\backslash X$ is finite$\}$. Then U_A is meager.*

Proof. For each $F \in [\omega]^{<\omega}$ let $V_F = \{X \in [\omega]^\omega : A \cap X = F\}$. Then V_F is closed. For, suppose that $X \in [\omega]^\omega$ and $X \notin V_F$. Thus $A \cap X \neq F$.

Case 1. There is an $x \in A \cap X\backslash F$. Then $X \in \{Y \in [\omega]^\omega : x \in Y\}$ and $\{Y \in [\omega]^\omega : x \in Y\}$ is open and disjoint from V_F.

Case 2. There is an $x \in F\backslash(A \cap X)$.

Subcase 2.1. $x \in F\backslash A$. Then $V_F = \emptyset$.

Subcase 2.2. $F \subseteq A$ and $x \in F\backslash X$. Then $X \in \{Y \in [\omega]^\omega : x \notin Y\}$, and $\{Y \in [\omega]^\omega : x \notin Y\}$ is open and disjoint from V_F.

Now let U_{HG} be any open set with $U_{HG} \cap [\omega]^\omega \neq \emptyset$. Choose $X \in [\omega]^\omega$ with $H \subseteq X$, $G \cap X = \emptyset$, and $A \cap X\backslash F \neq \emptyset$. This shows that $U_{HG} \cap [\omega]^\omega\backslash V_F \neq \emptyset$. So V_F is nowhere dense.

Also $W_F \stackrel{\text{def}}{=} \{X \in [\omega]^\omega : A\backslash X = F\}$ is closed and nowhere dense. In fact, suppose that $X \in [\omega]^\omega$ and $X \notin W_F$. Thus $A\backslash X \neq F$.

Case 1. There is an $x \in (A\backslash X)\backslash F$. Then $X \in \{Y \in [\omega]^\omega : x \notin Y\}$ and $\{Y \in [\omega]^\omega : x \notin Y\}$ is open and disjoint from W_F.

Case 2. There is an $x \in F \backslash (A \backslash X)$. Thus $x \in F$, and either $x \notin A$, or $x \in A \cap X$.

Subcase 2.1. $x \notin A$. Then $W_F = \emptyset$.

Subcase 2.2. $F \subseteq A$ and $x \in A \cap X$. Then $X \in \{Y \in [\omega]^\omega : x \in Y\}$, and $\{Y \in [\omega]^\omega : x \in Y\}$ is open and disjoint from W_F.

Now let U_{HG} be any open set with $U_{HG} \cap [\omega]^\omega \neq \emptyset$. Choose $X \in [\omega]^\omega$ so that $H \subseteq X$, $G \cap X = \emptyset$, and $(A \backslash X) \backslash F \neq \emptyset$. So W_F is nowhere dense.

Now

$$U_A = \bigcup_{F \in [\omega]^{<\omega}} V_F \cup \bigcup_{F \in [\omega]^{<\omega}} W_F,$$

so U_A is meager. □

Let $\mathscr{R} = (\mathscr{P}(\omega), [\omega]^\omega, R)$, where $R = \{(A, B) : A \subseteq \omega$ and $B \in [\omega]^\omega$, and ($A \cap B$ is finite or $B \backslash A$ is finite)$\}$.

Proposition 20.38. $||\mathscr{R}|| = \mathfrak{r}$.

Proof.

$$||\mathscr{R}|| = \min\{|X| : X \subseteq [\omega]^\omega \text{ and } \forall A \in \mathscr{P}(\omega) \exists B \in X [A \cap B \text{ is finite or } B \backslash A \text{ is finite}]\} = \mathfrak{r}.$$

□

Proposition 20.39. $||\mathscr{R}^\perp|| = \mathfrak{s}$.

Proof. $\mathscr{R}^\perp = ([\omega]^\omega, \mathscr{P}(\omega), S)$, where $S = \{(B, A) : B \in [\omega]^\omega, A \in \mathscr{P}(\omega)$, and $A \cap B$ is infinite and $B \backslash A$ is infinite$\}$. Hence

$$\begin{aligned} ||\mathscr{R}^\perp|| = \min\{|X| : X \subseteq \mathscr{P}(\omega) \text{ and } \forall B \in [\omega]^\omega \exists A \subseteq \omega \\ [A \cap B \text{ is infinite and } B \backslash A \text{ is infinite}]\} = \mathfrak{s} \end{aligned}$$

□

Let $\mathrm{Cov}'(\mathrm{meag}) = ([\omega]^\omega, \mathrm{meag}, \in)$.

Proposition 20.40. $||\mathrm{Cov}'(\mathrm{meag})|| = \mathrm{cov}(\mathrm{meager}_{[\omega]^\omega})$.

Proof. See the proof of Proposition 19.5 □

Proposition 20.41. *For any $B \in [\omega]^\omega$ let $f(B) = B$. For any $A \in [\omega]^\omega$ let $g(A) = U_A$ from Lemma 20.37. Then (f, g) is a morphism from $\mathscr{R}$ to* $\mathrm{Cov}'(\mathrm{meag})$.

Proof. Recall that $\mathscr{R} = (\mathscr{P}(\omega), [\omega]^\omega, R)$, where $R = \{(A, B) : A \subseteq \omega$ and $B \in [\omega]^\omega$, and ($A \cap B$ is finite or $B \backslash A$ is finite)$\}$, and $\mathrm{Cov}'(\mathrm{meag}) = ([\omega]^\omega, \mathrm{meag}, \in)$. Thus by Lemma 20.37 the functions f and g are of the correct form. Now suppose that $A \in [\omega]^\omega$, $B \in [\omega]^\omega$, and $(f(A), B) \in R$. Thus $A \cap B$ is finite or $B \backslash A$ is finite. Thus $A \in C_B = g(B)$, as desired. □

Proposition 20.42. $\mathrm{cov}(\mathrm{meager}_{[\omega]^\omega}) \leq \mathfrak{r}$.

Proof. By Proposition 18.22, Proposition 20.38, and Proposition 20.40. □

Proposition 20.43. $\mathfrak{s} \le \text{non}(\text{meager}_{[\omega]^\omega})$.

Proof. By Proposition 18.21, Proposition 19.10, and Proposition 20.39. $\square$

Lemma 20.44. *If $A \in [\omega]^\omega$, let $U_A = \{X \in [\omega]^\omega : A \cap X$ is finite, or $A \backslash X$ is finite$\}$. Then U_A is null.*

Proof. Let $A \in [\omega]^\omega$. For $F \subseteq A$, F finite, and $G \subseteq A \backslash F$, G finite, let f_{FG} be the function with domain $F \cup G$ such that

$$f_{FG}(m) = \begin{cases} 1 & \text{if } m \in F, \\ 0 & \text{if } m \in G. \end{cases}$$

Let $B_{FG} = \{X \in [\omega]^\omega : f_{FG} \subseteq \chi_X\}$. Then B_{FG} is measurable, and $\mu(B_{FG}) = 2^{-|F \cup G|}$. Hence $D_F \stackrel{\text{def}}{=} \bigcap\{B_{FG} : G \in [A \backslash F]^{<\omega}\}$ is a null set.

(1) $D_F = \{X \in [\omega]^\omega : A \cap X = F\}$.

In fact, let $X \in D_F$. Since $X \subseteq B_{F\emptyset}$, we have $f_{F\emptyset} \subseteq \chi_X$, and so $F \subseteq X$. Hence $F \subseteq A \cap X$. Now suppose that $m \in A \cap X \backslash F$. Then $\{m\} \subseteq A \backslash F$, so $f_{F\{m\}} \subseteq \chi_X$, since $X \subseteq B_{F\{m\}}$. Hence $m \notin X$, contradiction. Thus $\subseteq$ holds in (1).

Now suppose that $X \in [\omega]^\omega$ and $A \cap X = F$. Suppose that $G \in [A \backslash F]^{<\omega}$. If $m \in F$, then $m \in X$, and so $f_{FG}(m) = \chi_X(m)$. If $m \in G$, then $m \in A \backslash F$ and so $m \notin X$ and hence again $f_{FG}(m) = 0 = \chi_X(m)$. Thus $f_{FG} \subseteq \chi_X$ and so $X \in B_{FG}$. This proves (1).

Now suppose that $F \in [A]^{<\omega}$ and $G \in [A \backslash F]^{<\omega}$. Define g_{FG} with domain $F \cup G$ by

$$g_{FG}(m) = \begin{cases} 0 & \text{if } m \in F, \\ 1 & \text{if } m \in G. \end{cases}$$

Let $E_{FG} = \{X \in [\omega]^\omega : g_{FG} \subseteq \chi_X\}$. Then E_{FG} is measurable, and $\mu(E_{FG}) = 2^{-|F \cup G|}$. Hence $H_F \stackrel{\text{def}}{=} \bigcap\{E_{FG} : G \in [A \backslash F]^{<\omega}\}$ is a null set.

(2) $H_F = \{X \in [\omega]^\omega : A \backslash X = F\}$.

For, let $X \in H_F$. Since $X \subseteq E_{F\emptyset}$, we have $g_{F\emptyset} \subseteq \chi_X$, and hence $F \cap X = \emptyset$. So $F \subseteq A \backslash X$. Suppose that $m \in (A \backslash X) \backslash F$. Then $\{m\} \subseteq A \backslash F$, so $g_{F\{m\}} \subseteq \chi_X$. Hence $m \in X$, contradiction. Thus $A \backslash X = F$. This proves $\subseteq$ in (2).

Now suppose that $X \in [\omega]^\omega$ and $A \backslash X = F$. Suppose that $G \in [A \backslash F]^{<\omega}$. If $m \in F$, then $g_{FG}(m) = 0 = \chi_X(m)$. If $m \in G$, then $m \in X$, and $g_{FG}(m) = 1 = \chi_X(m)$. Thus $g_{FG} \subseteq \xi_X$ and so $X \in E_{FG}$. This proves (2).

Now

$$U_A = \bigcup\{D_F : F \in [\omega]^{<\omega}\} \cup \bigcup\{H_F : F \in [\omega]^{<\omega}\},$$

so U_A is a null set. $\square$

Proposition 20.45. *For any $B \in [\omega]^\omega$ let $f(B) = B$. For any $A \in [\omega]^\omega$ let $g(A) = U_A$ from Lemma 20.37. Then (f, g) is a morphism from $\mathscr{R}$ to* Cov(null).

Proof. Recall that $\mathscr{R} = (\mathscr{P}(\omega), [\omega]^\omega, R)$, where $R = \{(A, B) : A \subseteq \omega$ and $B \in [\omega]^\omega$, and $(A \cap B$ is finite or $B \backslash A$ is finite$)\}$, and $\text{Cov(null)} = ([\omega]^\omega, \text{null}, \in)$. Thus by Lemma

20.44 the functions f and g are of the correct form. Now suppose that $A \in [\omega]^\omega$, $B \in [\omega]^\omega$, and $(f(A), B) \in R$. Thus $A \cap B$ is finite or $B \backslash A$ is finite. Thus $A \in U_B = g(B)$, as desired. $\square$

Proposition 20.46. $\mathrm{cov}(\mathrm{null}_{[\omega]^\omega}) \leq \mathfrak{r}$.

Proof. By Proposition 18.22, Proposition 19.6, and Proposition 20.38. $\square$

Proposition 20.47. $\mathfrak{s} \leq \mathrm{non}(\mathrm{null}_{[\omega]^\omega})$.

Proof. By Proposition 18.22, Proposition 19.12, and Proposition 20.39. $\square$

We give some additional results about these functions.

Proposition 20.48. *If I is an ideal on a set A, then* $\mathrm{add}(I)$ *is regular.*

Proof. Suppose that $\mathrm{add}(I) = \lambda$ is singular. Let $\langle \kappa_\xi : \xi < \mathrm{cf}(\lambda) \rangle$ be strictly increasing with supremum λ. Let $\langle a_\xi : \xi < \lambda \rangle$ be members of I such that $\bigcup_{\xi<\lambda} a_\xi \notin I$. For each $\eta < \mathrm{cf}(\lambda)$ we have $\bigcup_{\xi<\kappa_\eta} a_\xi \in I$, and so

$$\bigcup_{\xi<\lambda} a_\xi = \bigcup_{\eta<\mathrm{cf}(\lambda)} \bigcup_{\xi<\kappa_\eta} a_\xi \in I,$$

contradiction. $\square$

A family of sets $\mathscr{E}$ has the *finite intersection property*, FIP, iff $\bigcap F \neq \emptyset$ for all finite $F \subseteq \mathscr{E}$; it has the *strong finite intersection property*, SFIP, iff $\bigcap F$ is infinite for all finite $F \subseteq \mathscr{E}$

Proposition 20.49. *Let $Z_n = \{x \in \omega : n|x\}$, $\mathscr{E} = \{Z_n : n \in \omega\}$.*
(i) $\mathscr{E}$ has SFIP.
(ii) $\bigcap \mathscr{E} = \emptyset$.
(iii) $K \stackrel{\text{def}}{=} \{n! : n \in \omega\}$ is a pseudo-intersection of $\mathscr{E}$.
(iv) $L \stackrel{\text{def}}{=} \{7n! : n \in \omega\}$ is a pseudo-intersection of $\mathscr{E}$.
(v) $M \stackrel{\text{def}}{=} \{n!/7 : n \in \omega\}$ is a pseudo-intersection of $\mathscr{E}$.

Proof. (i): If F is a finite subset of ω, let $X = \{x \in \omega : n|x \text{ for all } n \in F\}$. Then X is infinite and $X \subseteq Z_n$ for all $n \in F$.
(ii): obvious.
(iii): For any $n \in \omega$, $K \backslash Z_n \subseteq \{m! : m < n\}$.
(iv): For any $n \in \omega$, $L \backslash Z_n \subseteq \{7m! : m < n\}$.
(v) For any $n \in \omega$, $M \backslash Z_n \subseteq \{m!/7 : m \leq 7, n\}$. $\square$

Theorem 20.50. *If $\omega \leq \kappa < \mathfrak{t}$, then $2^\kappa = 2^\omega$.*

Proof. By recursion we define $t_\eta \in [\omega]^\omega$ for every $\eta \in {}^{<\mathfrak{t}}2$. Let $t_\emptyset = \omega$. If t_η has been defined, let $t_{\eta 0}$ and $t_{\eta 1}$ be disjoint infinite subsets of t_η. If η has limit length $\alpha < \mathfrak{t}$ and $t_{\eta \restriction \beta}$ has been defined for all $\beta < \alpha$, in such a way that $\beta < \gamma < \alpha$ implies that $t_{\eta \restriction \gamma} \subseteq^* t_{\eta \restriction \beta}$, let t_η be an infinite set $\subseteq^*$ each $t_{\eta \restriction \beta}$ for $\beta < \alpha$; this is possible because $\alpha < \mathfrak{t}$. Now $\langle t_\eta : \eta$ has length $\kappa\rangle$ is a system of infinite almost disjoint subsets of ω, and the desired conclusion follows. $\square$

Proposition 20.51. $\mathfrak{t} \le \mathrm{cf}(2^\omega)$.

Proof. If $\kappa < \mathfrak{t}$, then $\kappa < \mathrm{cf}(2^\kappa) = \mathrm{cf}(2^\omega)$ by Theorem 20.50; hence $\mathrm{cf}(2^\omega) < \mathfrak{t}$ is impossible. $\square$

We define

$$\mathfrak{p} = \min\{|\mathscr{F}| : \mathscr{F} \subseteq [\omega]^\omega,\ \mathscr{F} \text{ has SFIP, but has no pseudo-intersection}\}.$$

This is well defined, since a tower is such a family $\mathscr{F}$.

Proposition 20.52. $\omega_1 \le \mathfrak{p} \le \mathfrak{t}$.

Proof. Obviously $\mathfrak{p} \le \mathfrak{t}$. Now suppose that $\mathscr{F} = \{A_m : m \in \omega\}$ has SFIP; we show that it has a pseudo-intersection, thus proving the first inequality in the proposition. For each $n \in \omega$ choose $k_n \in \bigcap_{m \le n} A_m \backslash \{k_m : m < n\}$. Then $y \stackrel{\text{def}}{=} \{k_n : n \in \omega\}$ is infinite, and for each n, $y \backslash A_n \subseteq \{k_m : m < n\}$. $\square$

Actually $\mathfrak{p} = \mathfrak{t}$, as we will prove later.

Proposition 20.53. *Suppose that $\mathscr{A}, \mathscr{C} \subseteq [\omega]^\omega$, $|\mathscr{A}| + |\mathscr{C}| < \mathfrak{p}$, and $\forall \mathscr{F} \in [\mathscr{C}]^{<\omega}$ and all $A \in \mathscr{A}$, the set $\bigcap \mathscr{F} \cap A$ is infinite. Then $\mathscr{C}$ has a pseudo-intersection X such that $X \cap A$ is infinite for all $A \in \mathscr{A}$.*

Proof. Define

$$\mathscr{H}_1 = \{[C]^{<\omega} : C \in \mathscr{C}\},$$
$$\mathscr{H}_2 = \{\{F \in [\omega \backslash n]^{<\omega} : F \cap A \ne \emptyset\} : A \in \mathscr{A},\ n \in \omega\}.$$

Note that $\mathscr{H}_1, \mathscr{H}_2 \subseteq \mathscr{P}([\omega]^{<\omega})$.

(1) $\mathscr{H}_1 \cup \mathscr{H}_2$ has the strong finite intersection property.

To prove this, suppose that $\mathscr{F}$ is a finite subset of $\mathscr{C}$ and $\mathscr{G}$ is a finite set of pairs (A, n) such that $A \in \mathscr{A}$ and $n \in \omega$; we want to show that

$$(*) \qquad \bigcap_{C \in \mathscr{F}} [C]^{<\omega} \cap \bigcap_{(A,n) \in \mathscr{G}} \{F \in [\omega \backslash n]^{<\omega} : F \cap A \ne \emptyset\}$$

is infinite. Choose $m > n$ for each n such that $(A, n) \in \mathscr{G}$ for some A. Note that if $(A, n) \in \mathscr{G}$ for some n, then $A \in \mathscr{A}$, and so by hypothesis the set $\bigcap \mathscr{F} \cap A$ is infinite; choose $p_A \in (\bigcap \mathscr{F} \cap A) \backslash m$. Then if F is any finite subset of $\omega \backslash m$ such that $p_A \in F$ for all A for which $(A, n) \in G$ for some n, it follows that F is in the set $(*)$.

Thus (1) holds. Note that $|\mathscr{H}_1| + |\mathscr{H}_2| < \mathfrak{p}$. Now $|[\omega]^{<\omega}| = \omega$, so we can apply the definition of $\mathfrak{p}$ to $\mathscr{P}([\omega]^{<\omega})/\mathrm{fin}$ rather than $\mathscr{P}(\omega)/\mathrm{fin}$. It follows that there is an $\mathscr{I} \subseteq [\omega]^{<\omega}$ which is a pseudo-intersection of $\mathscr{H}_1 \cup \mathscr{H}_2$.

Since $\mathscr{I}$ is infinite, clearly also $\bigcup \mathscr{I}$ is infinite.

(2) $\bigcup \mathscr{I}$ is a pseudo-intersection of $\mathscr{C}$.

For, suppose that $C \in \mathscr{C}$. Now $\mathscr{I} \subseteq^* [C]^{<\omega}$. Choose $\mathscr{K} \in [[\omega]^{<\omega}]^{<\omega}$ such that $\mathscr{I}\backslash\mathscr{K} \subseteq [C]^{<\omega}$. Let $n \in \omega$ be greater than each member of $\bigcup \mathscr{K}$. We claim that $\bigcup \mathscr{I}\backslash n \subseteq C$. For, suppose that $m \in \bigcup \mathscr{I}\backslash n$. Choose $I \in \mathscr{I}$ such that $m \in I$. Since $m \geq n$, we have $I \notin \mathscr{K}$. So $I \in [C]^{<\omega}$, and hence $m \in C$, as desired. This proves (2).

Finally, suppose that $A \in \mathscr{A}$. Take any $n < \omega$; we show that $\bigcup \mathscr{I} \cap A$ has a member $\geq n$; this will finish the proof. Now $\mathscr{I} \subseteq^* \{F \in [\omega\backslash n]^{<\omega} : F \cap A \neq \emptyset\}$, so we can find a finite $\mathscr{G} \subseteq [\omega]^{<\omega}$ such that $\mathscr{I}\backslash\mathscr{G} \subseteq \{F \in [\omega\backslash n]^{<\omega} : F \cap A \neq \emptyset\}$. Take any $F \in \mathscr{I}\backslash\mathscr{G}$. Then $F \in [\omega\backslash n]^{<\omega}$ and $F \cap A \neq \emptyset$, so $\bigcup \mathscr{I} \cap A$ has a member $\geq n$. □

Proposition 20.54. $\mathfrak{p}$ *is regular.*

Proof. Suppose not. Let $\mathscr{A} \subseteq [\omega]^\omega$ with $|\mathscr{A}| = \mathfrak{p}$, $\mathscr{A}$ closed under finite intersections, and write $\mathscr{A} = \bigcup_{\alpha<\mathrm{cf}(\mathfrak{p})} \mathscr{B}_\alpha$, where $|\mathscr{B}_0| = \omega$, $\mathscr{B}_\alpha \subseteq \mathscr{B}_\beta$ for $\alpha < \beta < \mathrm{cf}(\mathfrak{p})$, and $|\mathscr{B}_\alpha| < \mathfrak{p}$ for each $\alpha < \mathrm{cf}(\mathfrak{p})$. Moreover, let each $\mathscr{B}_\alpha$ be closed under finite intersection.

(1) Suppose that $\mathscr{C} \subseteq [\omega]^\omega$, $|\mathscr{C}| < \mathfrak{p}$, and $\forall F \in [\mathscr{C}]^{<\omega}$ and all $Y \in \mathscr{A}$, the set $\bigcap F \cap Y$ is infinite. Then $\mathscr{C}$ has a pseudo-intersection X such that $X \cap Y$ is infinite for all $Y \in \mathscr{A}$.

To prove (1), note that for any $\alpha < \mathrm{cf}(\mathfrak{p})$, the set $\mathscr{C} \cup \mathscr{B}_\alpha$ has the SFIP and size less than $\mathfrak{p}$, so it has a pseudo-intersection Z_α. Now we apply Proposition 20.51 to $\mathscr{C}$ and $\{Z_\alpha : \alpha < \mathrm{cf}(\mathfrak{p})\}$: we get a pseudo-intersection X of $\mathscr{C}$ such that $X \cap Z_\alpha$ is infinite for all $\alpha < \mathrm{cf}(\mathfrak{p})$. For any $Y \in \mathscr{A}$, choose $\alpha < \mathrm{cf}(\mathfrak{p})$ with $Y \in \mathscr{B}_\alpha$. Then $Z_\alpha\backslash Y$ is finite, hence $X \cap Z_\alpha\backslash Y$ is finite, hence $X \cap Y$ is infinite, as desired.

Now we are going to define by recursion $\langle C_\alpha : \alpha \leq \mathrm{cf}(\mathfrak{p})\rangle$ so that the following conditions hold:

(2) $C_\alpha \in [\omega]^\omega$.

(3) C_α is a pseudo-intersection of $\mathscr{A}_\alpha$.

(4) If $\beta < \alpha$, then $C_\alpha\backslash C_\beta$ is finite.

(5) $\forall Y \in \mathscr{A}(C_\alpha \cap Y$ is infinite).

As soon as we have done this, a contradiction is reached because $C_{\mathrm{cf}(\mathfrak{p})}$ is a pseudo-intersection of $\mathscr{A}$ by (3) and (4).

So, suppose that C_α has been defined for all $\alpha < \gamma$ so that (2)–(5) hold, where $\gamma \leq \mathrm{cf}(\mathfrak{p})$. We want to apply (1) with $\mathscr{C}$ replaced by $\{C_\alpha : \alpha < \gamma\} \cup \mathscr{A}_\gamma$. To check the hypotheses, suppose that $F \in [\gamma]^{<\omega}$, $G \in [\mathscr{A}_\gamma]^{<\omega}$, and $Y \in \mathscr{A}$; we want to show that $\bigcap_{\alpha\in F} C_\alpha \cap \bigcap G \cap Y$ is infinite. Wlog $F \neq \emptyset \neq G$. Let β be the largest element of F. Since $\mathscr{A}$ is closed under $\cap$, we have $\bigcap G \cap Y \in \mathscr{A}$. By (5), $C_\beta \cap \bigcap G \cap Y$ is infinite. Now $\bigcup_{\alpha\in F\backslash\{\beta\}}(C_\beta\backslash C_\alpha) = C_\beta\backslash \bigcap_{\alpha\in F\backslash\{\beta\}} C_\alpha$ is finite, so $\bigcap_{\alpha\in F} C_\alpha \cap \bigcap G \cap Y$ is infinite, as desired.

So we apply (1) and get C_γ such that C_γ is a pseudo-intersection of $\{C_\alpha : \alpha < \gamma\} \cup \mathscr{A}_\gamma$ and $C_\gamma \cap Y$ is infinite for all $Y \in \mathscr{A}$. So (2)–(5) hold, and the construction is complete. □

21. Linear orders

In this chapter we prove some results about linear orders which form a useful background in much of set theory. Among these facts are: any two denumerable densely ordered sets are isomorphic, the existence of η_α sets, the existence of completions, a discussion of Suslin lines, and a proof of a very useful theorem of Hausdorff.

A linear order $(A,<)$ is *densely ordered* iff $|A|>1$, and for any $a<b$ in A there is a $c\in A$ such that $a<c<b$. A subset X of a linearly ordered set L is *dense in* L iff for any two elements $a<b$ in L there is an $x\in X$ such that $a<x<b$. Note that if X is dense in L and L has at least two elements, then L itself is dense.

Theorem 21.1. *If L is a dense linear order, then L is the disjoint union of two dense subsets.*

Proof. Let $\langle a_\alpha : \alpha<\kappa\rangle$ be a well-order of L, with $\kappa=|L|$. We put each a_α in A or B by recursion, as follows. Suppose that we have already done this for all $\beta<\alpha$. Let $C=\{a_\beta : \beta<\alpha \text{ and } a_\beta<a_\alpha\}$, and let $D=\{a_\beta : \beta<\alpha \text{ and } a_\beta>a_\alpha\}$. We take two possibilities.

Case 1. C has a largest element a_β, D has a smallest element a_γ, and $a_\beta,a_\gamma\in A$. Then we put a_α in B.

Case 2. Otherwise, we put a_α in A.

Now we want to see that this works. So, suppose that elements $a_\xi<a_\eta$ of L are given. Let $a_\beta<a_\gamma$ be the elements of L with smallest indices which are in the interval (a_ξ,a_η). If one of these is in A and the other in B, this gives elements of A and B in (a_ξ,a_η). So, suppose that they are both in A, or both in B. Let a_ν be the member of L with smallest index that is in (a_β,a_γ). Thus $a_\xi<a_\beta<a_\nu<a_\gamma<a_\eta$, so by the minimality of β and γ we have $\beta,\gamma<\nu$. Thus $\beta<\nu$ and $a_\beta<a_\nu$.

(1) a_β is the largest element of $\{a_\rho : \rho<\nu,\ a_\rho<a_\nu\}$.

In fact, a_β is in this set, as just observed. If $a_\beta<a_\rho$, $\rho<\nu$, and $a_\rho<a_\nu$, then also $a_\rho<a_\gamma$ since $a_\nu<a_\gamma$, so the definition of ν is contradicted. Hence (1) holds.

(2) a_γ is the smallest element of $\{a_\rho : \rho<\nu,\ a_\rho>a_\nu\}$.

In fact, $\gamma<\nu$ as observed just before (1), and $a_\gamma>a_\nu$ by the definition of a_ν. If $a_\rho<a_\gamma$, $\rho<\nu$, and $a_\rho>a_\nu$, then also $a_\rho>a_\beta$ since $a_\nu>a_\beta$, so the definition of ν is contradicted. Hence (2) holds.

So by construction, if $a_\beta,a_\gamma\in A$ then $a_\nu\in B$, while if $a_\beta,a_\gamma\in B$, then $a_\nu\in A$. So again we have found elements of both A and B which are in (a_ξ,a_η). □

The proof of the following result uses the important *back-and-forth* argument.

Theorem 21.2. *Any two denumerable densely ordered sets without first and last elements are order-isomorphic.*

Proof. Let $(A,<)$ and $(B,\prec)$ be denumerable densely ordered sets without first and last elements. Write $A=\{a_i : i\in\omega\}$ and $B=\{b_i : i\in\omega\}$. We now define by recursion sequences $\langle c_i : i \in \omega\rangle$ of elements of A and $\langle d_i : i \in \omega\rangle$ of elements of B. Let $c_0 = a_0$ and $d_0 = b_0$.

Now suppose that c_{2m} and d_{2m} have been defined so that the following condition hold:

(*) For all $i, j \le 2m$, $c_i < c_j$ iff $d_i < d_j$.

(Note that then a similar equivalence holds for $=$ and for $>$.) We let $c_{2m+1} = a_{m+1}$. Now we consider several cases.

Case 1. $a_{m+1} = c_i$ for some $i \le 2m$. Take the least such i, and let $d_{2m+1} = d_i$.

Case 2. $a_{m+1} < c_i$ for all $i \le 2m$. Let d_{2m+1} be any element of B less than each d_i, $i \le 2m$.

Case 3. $c_i < a_{m+1}$ for all $i \le 2m$. Let d_{2m+1} be any element of B greater than each d_i, $i \le 2m$.

Case 4. Case 1 fails, and there exist $i, j \le 2m$ such that $c_i < a_{m+1} < c_j$. Let d_{2m+1} be any element b of B such that $d_i < b < d_j$ whenever $c_i < a_{m+1} < c_j$; such an element b exists by (*).

This finishes the definition of d_{2m+1}. d_{2m+2} and c_{2m+2} are defined similarly. Namely, we let $d_{2m+2} = b_{m+1}$ and then define c_{2m+2} similarly to the above, with a, b interchanged and c, d interchanged.

Note that each a_i appears in the sequence of c_i's, namely $c_0 = a_0$ and $c_{2i+1} = a_{i+1}$, and similarly each b_i appears in the sequence of d_i's. Hence it is clear that $\{(c_i, d_i) : i \in \omega\}$ is the desired order-isomorphism. □

Proposition 21.3. *If L is countable and M is dense with no first or last element, then L can be isomorphically embedded in M.*

Proof. Wlog $L \neq \emptyset$. Let $\langle a_i : i \in \omega\rangle$ be such that $L = \{a_i : i \in \omega\}$. L could be finite, so there could be repetitions in the sequence $\langle a_i : i \in \omega\rangle$. Now for each $m \in \omega$ we define $g_m : \{a_i : i < m\} \to M$ by recursion. Let $g_0 = \emptyset$. If g_m has been defined, and it is an isomorphic embedding, let $g_m \subseteq g_{m+1}$; we consider several cases for a_m.

Case 1. $a_m = a_i$ for some $i < m$. Let $g_{m+1} = g_m$.

Case 2. $a_i < a_m$ for all $i < m$. Let $g_{m+1}(a_m)$ be some element of M greater than each $g(a_i)$ for $i < m$.

Case 3. $a_i > a_m$ for all $i < m$. Let $g_{m+1}(a_m)$ be some element of M less than each $g(a_i)$ for $i < m$.

Case 4. $a_i < a_m$ for some $i < m$ and $a_j > a_m$ for some $j < m$. Let $A = \{i < m : a_i < a_m\}$ and $B = \{j < m : a_j > a_m\}$. Thus $\forall i \in A \forall j \in B[a_i < a_j]$, so $\forall i \in A \forall j \in B[g_m(a_i) < g_m(a_j)]$. Let $g_{m+1}(a_m)$ be an element of M such that $\forall i \in A \forall j \in B[g_m(a_i) < g_{m+1}(a_m) < g_m(a_j)]$. □

Theorem 21.4. *If L is an infinite linear order, then there is a subset M of L which is order isomorphic to $(\omega, <)$, or to $(\omega, >)$.*

Proof. Suppose that L does not have a subset order isomorphic to $(\omega, >)$. We claim then that L is well-ordered, and therefore is isomorphic to an infinite ordinal and hence has a subset isomorphic to $(\omega, <)$. To prove this claim, suppose it is not true. So L has some nonempty subset P with no least element. We now define a sequence $\langle a_i : i \in \omega\rangle$ of elements of P by recursion. Let a_0 be any element of P. If $a_i \in P$ has been defined, then it is not the least element of P and so there is an $a_{i+1} \in P$ with $a_{i+1} < a_i$. This finishes the construction. Thus we have essentially produced a subset of L order isomorphic to $(\omega, >)$, contradiction. $\square$

It would be natural to conjecture that Theorem 21.4 generalizes in the following way: for any infinite cardinal κ and any linear order L of size κ, there is a subset M of L order isomorphic to $(\kappa, <)$ or to $(\kappa, >)$. This is clearly false, as the real numbers under their usual order form a counterexample. (Given a set of real numbers order isomorphic to 2^ω, one could choose rationals between successive members of the set, and produce 2^ω rationals, contradiction.) We want to give an example that works for many cardinals. The construction we use is very important for later purposes too.

The following definitions apply to any infinite ordinal γ.

- If f and g are distinct elements of ${}^\gamma 2$, we define

$$\chi(f,g) = \min\{\alpha < \gamma : f(\alpha) \neq g(\alpha)\}.$$

- Let $f < g$ iff f and g are distinct elements of ${}^\gamma 2$ and $f(\chi(f,g)) < g(\chi(f,g))$. (Thus $f(\chi(f,g)) = 0$ and $g(\chi(f,g)) = 1$.) Clearly $({}^\gamma 2, <)$ is a linear order; this is called the *lexicographic order.*

We also need some general set-theoretic notation. If A is any set and κ any cardinal, then

$$\begin{aligned}
[A]^\kappa &= \{X \subseteq A : |X| = \kappa\};\\
[A]^{<\kappa} &= \{X \subseteq A : |X| < \kappa\};\\
[A]^{\leq\kappa} &= \{X \subseteq A : |X| \leq \kappa\}.
\end{aligned}$$

Theorem 21.5. *For any infinite cardinal κ, the linear order ${}^\kappa 2$ does not contain a subset order isomorphic to κ^+ or to $(\kappa^+, >)$.*

Proof. The two assertions are proved in a very similar way, so we give details only for the first assertion. In fact, we assume that $\langle f_\alpha : \alpha < \kappa^+\rangle$ is a strictly increasing sequence of members of ${}^\kappa 2$, and try to get a contradiction. The contradiction will follow rather easily from the following statement:

(1) If $\gamma \leq \kappa$, $\Gamma \in [\kappa^+]^{\kappa^+}$, and $f_\alpha \restriction \gamma < f_\beta \restriction \gamma$ for any $\alpha, \beta \in \Gamma$ such that $\alpha < \beta$, then there exist $\delta < \gamma$ and $\Delta \in [\Gamma]^{\kappa^+}$ such that $f_\alpha \restriction \delta < f_\beta \restriction \delta$ for any $\alpha, \beta \in \Delta$ such that $\alpha < \beta$.

To prove this, assume the hypothesis. For each $\alpha \in \Gamma$ let $f'_\alpha = f_\alpha \restriction \gamma$. Clearly Γ does not have a largest element. For each $\alpha \in \Gamma$ let α' be the least member of Γ which is greater than α. Then

$$\Gamma = \bigcup_{\xi<\gamma} \{\alpha \in \Gamma : \chi(f'_\alpha, f'_{\alpha'}) = \xi\}.$$

Since $|\Gamma| = \kappa^+$, it follows that there are $\delta < \gamma$ and $\Delta \in [\Gamma]^{\kappa^+}$ such that $\chi(f'_\alpha, f'_{\alpha'}) = \delta$ for all $\alpha \in \Delta$. We claim now that $f'_\alpha \restriction \delta < f'_\beta \restriction \delta$ for any two $\alpha, \beta \in \Delta$ such that $\alpha < \beta$, as desired in (1). For, take any such α, β. Suppose that $f'_\alpha \restriction \delta = f'_\beta \restriction \delta$. (Note that we must have $f'_\alpha \restriction \delta \le f'_\beta \restriction \delta$.) Now from $\chi(f'_\alpha, f'_{\alpha'}) = \delta$ we get $f'_{\alpha'}(\delta) = 1$, and from $\chi(f'_\beta, f'_{\beta'}) = \delta$ we get $f'_\beta(\delta) = 0$. Now $f'_{\alpha'} \restriction \delta = f'_\alpha \restriction \delta = f'_\beta \restriction \delta$, so we get $f'_\beta < f'_{\alpha'} \le f'_\beta$, contradiction. This proves (1).

Clearly from (1) we can construct an infinite decreasing sequence $\kappa > \gamma_1 > \gamma_2 > \cdots$ of ordinals, contradiction. $\square$

Now we give some more definitions, leading to a kind of generalization of Theorem 21.2.

- If $(L, <)$ is a linear order and $A, B \subseteq L$, we write $A < B$ iff $\forall x \in A \forall y \in B[x < y]$. If $A = \{a\}$ here, we write $a < B$; similarly for $A < b$.

- *Intervals* in linear orders are defined in the usual way. For example, $[a, b) = \{c : a \le c < b\}$.

- An η_α*-set* is a linear order $(L, <)$ such that if $A, B \subseteq L$, $A < B$, and $|A|, |B| < \aleph_\alpha$, then there is a $c \in L$ such that $A < c < B$. Taking $A = \emptyset$ and $B = \{a\}$ for some $a \in L$, we see that η_α-sets do not have first elements; similarly they do not have last elements. Note that an η_0-set is just a densely ordered set without first or last elements.

- For any ordinal α, we define

$$H_\alpha = \{f \in {}^{\aleph_\alpha}2 : \text{there is a } \xi < \aleph_\alpha \text{ such that } f(\xi) = 1 \text{ and } f(\eta) = 0 \text{ for all } \eta \in (\xi, \aleph_\alpha)\}.$$

We take the order on H_α induced by that on ${}^{\aleph_\alpha}2$: $f < g$ in H_α iff $f < g$ as members of ${}^{\aleph_\alpha}2$.

Theorem 21.6. *Let α be an ordinal, and let* $\mathrm{cf}(\aleph_\alpha) = \aleph_\gamma$. *Then the following conditions hold:*

(i) H_α *is an* η_γ*-set.*
(ii) $\mathrm{cf}(H_\alpha, <) = \aleph_\gamma$.
(iii) $\mathrm{cf}(H_\alpha, >) = \aleph_\gamma$.
(iv) $|H_0| = \aleph_0$*, and for* $\alpha > 0$, $|H_\alpha| = \sum_{\beta<\alpha} 2^{\aleph_\beta}$.

Proof. For each $f \in H_\alpha$ let $\zeta_f < \aleph_\alpha$ be such that $f(\zeta_f) = 1$ and $f(\eta) = 0$ for all $\eta \in (\zeta_f, \aleph_\alpha)$.

For (i), suppose that $A, B \subseteq H_\alpha$ with $A < B$ and $|A|, |B| < \aleph_\gamma$. Obviously we may assume that one of A, B is nonempty. Then there are three possibilities:

Case 1. $A \ne \emptyset \ne B$. Let

$$\begin{aligned}\xi &= \sup\{\zeta_f : f \in A\};\\ \rho &= \max(\xi + 1, \sup\{\zeta_f : f \in B\}).\end{aligned}$$

Thus $\xi, \rho < \aleph_\alpha$ since $|A|, |B| < \aleph_\gamma = \mathrm{cf}(\aleph_\alpha)$. We now define $g \in {}^{\aleph_\alpha}2$ by setting, for each $\eta < \aleph_\alpha$,

$$g(\eta) = \begin{cases} 1 & \text{if } \eta \le \xi \text{ and } \exists f \in A(f \restriction \eta = g \restriction \eta \text{ and } f(\eta) = 1);\\ 0 & \text{if } \eta \le \xi \text{ and there is no such } f;\\ 0 & \text{if } \xi < \eta \le \rho;\\ 1 & \text{if } \eta = \rho + 1;\\ 0 & \text{if } \rho + 1 < \eta < \aleph_\alpha.\end{cases}$$

Clearly $g \in H_\alpha$. We claim that $A < g < B$. Note that $g \notin A \cup B$ since $g(\rho+1) = 1$ while $f(\rho+1) = 0$ for any $f \in A \cup B$.

To prove the claim first suppose that $f \in A$. Assume that $g < f$; we will get a contradiction. Let $\eta = \chi(g, f)$. Then $g(\eta) = 0$ and $f(\eta) = 1$. It follows that $\eta \leq \xi$ and $g \restriction \eta = f \restriction \eta$, contradicting the definition of $g(\eta)$.

Second, suppose that $f \in B$. Assume that $f < g$; we will get a contradiction. Let $\eta = \chi(f, g)$. Thus $f(\eta) = 0$ and $g(\eta) = 1$. We claim that $\eta = \rho + 1$. For, otherwise since $g(\eta) = 1$ we must have $\eta \leq \xi$, and then there is an $h \in A$ such that $h \restriction \eta = g \restriction \eta$ and $h(\eta) = 1$. So $f \restriction \eta = g \restriction \eta = h \restriction \eta$, $f(\eta) = 0$, and $h(\eta) = 1$, so $f < h$. But $f \in B$ and $h \in A$, contradiction. This proves our claim that $\eta = \rho + 1$.

Now clearly $\zeta_f \leq \rho$. Since

$$g \restriction (\rho+1) = g \restriction \eta = f \restriction \eta = f \restriction (\rho+1),$$

it follows that $g(\zeta_f) = 1$. So from $\zeta_f \leq \rho$ we infer that $\zeta_f \leq \xi$. Thus since $g(\zeta_f) = 1$, it follows that there is a $k \in A$ such that $k \restriction \zeta_f = g \restriction \zeta_f$ and $k(\zeta_f) = 1$. But now we have $k \restriction (\zeta_f + 1) = g \restriction (\zeta_f + 1) = f \restriction (\zeta_f + 1)$ and $f(\sigma) = 0$ for all $\sigma \in (\zeta_f, \aleph_\alpha)$. Hence $f \leq k$, which contradicts $f \in B$ and $k \in A$.

This finishes the proof of (i) in Case 1.

Case 2. $A = \emptyset \neq B$. Let

$$\rho = \sup\{\zeta_f : f \in B\}.$$

Define $g \in H_\alpha$ by setting, for each $\xi < \aleph_\alpha$,

$$g(\xi) = \begin{cases} 0 & \text{if } \xi \leq \rho, \\ 1 & \text{if } \xi = \rho + 1, \\ 0 & \text{if } \rho + 1 < \xi. \end{cases}$$

Clearly $g < B$, as desired.

Case 3. $A \neq \emptyset = B$. Let ξ be as in Case 1. Define $g \in H_\alpha$ by setting, for each $\eta < \aleph_\alpha$,

$$g(\eta) = \begin{cases} 1 & \text{if } \eta \leq \xi + 1, \\ 0 & \text{if } \xi + 1 < \eta. \end{cases}$$

Clearly $A < g$, as desired.

This finishes the proof of (i).

For (ii), let $\langle \delta_\xi : \xi < \aleph_\gamma \rangle$ be a strictly increasing sequence of ordinals with supremum $\aleph_\alpha$. For each $\xi < \aleph_\gamma$ define $f_\xi \in H_\alpha$ by setting, for each $\eta < \aleph_\alpha$,

$$f_\xi(\eta) = \begin{cases} 1 & \text{if } \eta \leq \delta_\xi, \\ 0 & \text{if } \delta_\xi < \eta. \end{cases}$$

Clearly $\langle f_\xi : \xi < \aleph_\gamma \rangle$ is a strictly increasing sequence of members of H_α and $\{f_\xi : \xi < \aleph_\gamma\}$ is cofinal in H_α. So (ii) holds.

For (iii), take $\langle \delta_\xi : \xi < \aleph_\gamma \rangle$ as in the proof of (ii). For each $\xi < \alpha_\gamma$ define $f_\xi \in H_\alpha$ by setting, for each $\eta < \aleph_\alpha$,

$$f_\xi(\eta) = \begin{cases} 0 & \text{if } \eta < \delta_\xi, \\ 1 & \text{if } \eta = \delta_\xi, \\ 0 & \text{if } \delta_\xi < \eta. \end{cases}$$

Clearly $\langle f_\xi : \xi < \aleph_\gamma$ is a strictly decreasing sequence of members of H_α and $\{f_\xi : \xi < \aleph_\gamma\}$ is cofinal in $(H_\alpha, >)$. So (iii) holds.

Finally, for (iv), for each $\delta < \aleph_\alpha$ let

$$L_\delta = \{f \in H_\alpha : f(\delta) = 1 \text{ and } f(\varepsilon) = 0 \text{ for all } \varepsilon \in (\delta, \aleph_\alpha)\}.$$

Clearly these sets are pairwise disjoint, and their union is H_α. For $\alpha = 0$,

$$|H_\alpha| = \sum_{\delta<\omega} |L_\delta| = \sum_{\delta<\omega} 2^{|\delta|} = \aleph_0.$$

For $\alpha > 0$,

$$\begin{aligned}
|H_\alpha| &= \sum_{\delta<\aleph_\alpha} |L_\delta| \\
&= \sum_{\delta<\omega} |L_\delta| + \sum_{\omega\leq\delta<\aleph_\alpha} |L_\delta| \\
&= \aleph_0 + \sum_{\omega\leq\delta<\aleph_\alpha} 2^{|\delta|} \\
&= \aleph_0 + \sum_{\beta<\alpha} \left(2^{\aleph_\beta} \cdot |\{\delta < \aleph_\alpha : |\delta| = \aleph_\beta\}|\right) \\
&= \aleph_0 + \sum_{\beta<\alpha} \left(2^{\aleph_\beta} \cdot \aleph_{\beta+1}\right) \\
&= \sum_{\beta<\aleph_\alpha} 2^{\aleph_\beta}.
\end{aligned}$$

□

Corollary 21.7. *If $\aleph_\alpha$ is regular, then*

(i) H_α is an η_α-set.

(ii) $\mathrm{cf}(H_\alpha, <) = \aleph_\alpha$.

(iii) $\mathrm{cf}(H_\alpha, >) = \aleph_\alpha$.

(iv) $|H_0| = \aleph_0$, and for $\alpha > 0$, $|H_\alpha| = \sum_{\beta<\alpha} 2^{\aleph_\beta}$. □

Corollary 21.8. *For each regular cardinal $\aleph_\alpha$ there is an η_α-set.* □

Corollary 21.9. *For each ordinal α there is an $\eta_{\alpha+1}$-set of size $2^{\aleph_\alpha}$.* □

Corollary 21.10. (GCH) *For each regular cardinal $\aleph_\alpha$ there is an η_α-set of size $\aleph_\alpha$.* □

One of the most useful facts about η_α-sets is their *universality*, expressed in the following theorem.

Theorem 21.11. *Suppose that $\aleph_\alpha$ is regular. If K is an η_α-set, then any linearly ordered set of size $\leq \aleph_\alpha$ can be isomorphically embedded in K.*

Proof. Let L be a linearly ordered set of size at most $\aleph_\alpha$, and write $L = \{a_\xi : \xi < \aleph_\alpha\}$. We define a sequence $\langle f_\xi : \xi < \aleph_\alpha \rangle$ of functions by recursion. Suppose that f_η has been defined for all $\eta < \xi$ so that it is a strictly increasing function mapping a subset of L of size less than $\aleph_\alpha$ into K, and such that $f_\rho \subseteq f_\eta$ whenever $\rho < \eta < \xi$. Let $g = \bigcup_{\eta<\xi} f_\eta$. Then g is still a strictly increasing function mapping a subset of L of size less than $\aleph_\alpha$ into K. If $a_\xi \in \mathrm{dmn}(g)$, we set $f_\xi = g$. Suppose that $a_\xi \notin \mathrm{dmn}(g)$. Let

$$A = \{g(b) : b \in \mathrm{dmn}(g) \text{ and } b < a_\xi\};$$
$$B = \{g(b) : b \in \mathrm{dmn}(g) \text{ and } a_\xi < b\}.$$

Then $A < B$, and $|A|, |B| < \aleph_\alpha$. So by the η_α-property, there is an element c of K such that $A < c < B$. We let $f_\xi = g \cup \{(a_\xi, c)\}$ for such an element c. (AC is used.)

This finishes the construction, and clearly $\bigcup_{\xi<\aleph_\alpha} f_\xi$ is as desired. □

Given a linearly ordered set L, a subset X of L, and an element a of L, we call a an *upper bound* for X iff $x \le a$ for all $x \in X$. Thus every element of L is an upper bound of the empty set. We say that a is a *least upper bound* for X iff a is an upper bound for X and is $\le$ any upper bound for X. Clearly a least upper bound for X is unique if it exists. If a is the least upper bound of the empty set, then a is the smallest element of L. We use lub or sub to abbreviate least upper bound. Also "supremum" is synonymous with "least upper bound".

Similarly one defines *lower bound* and *greatest lower bound.* Any element is a lower bound of the empty set, and if a is the greatest lower bound of the empty set, then a is the largest element of L. We use glb or inf to abbreviate greatest lower bound. "infimum" is synonymous with "greatest lower bound".

A linear order L is *complete* iff every subset of L has a greatest lower bound and a least upper bound.

Proposition 21.12. *For any linear order L the following conditions are equivalent:*

(i) L is complete.

(ii) Every subset of L has a least upper bound.

(iii) Every subset of L has a greatest lower bound.

Proof. (i)⇒(ii): obvious. (ii)⇒(iii): Assume that every subset of L has a least upper bound, and let $X \subseteq L$; we want to show that X has a greatest lower bound. Let Y be the set of all lower bounds of X. Then let a be a least upper bound for Y. Take any $x \in X$. Then $\forall y \in Y[y \le x]$, so $a \le x$ since a is the lub of Y. This shows that a is a lower bound for X. Suppose that y is any lower bound for X. Then $y \in Y$, and hence $y \le a$ since a is an upper bound for Y.

(iii)⇒(i) is treated similarly. □

Let $(L, <)$ be a linear order. We say that a linear order $(M, \prec)$ is a *completion* of L iff the following conditions hold:

(C1) $L \subseteq M$, and for any $a, b \in L$, $a < b$ iff $a \prec b$.

(C2) M is complete.

(C3) Every element of M is the lub of a set of elements of L.

(C4) If $a \in L$ is the lub in L of a subset X of L, then a is the lub of X in M.

Theorem 21.13. *Any linear order has a completion.*

Proof. Let $(L, <)$ be a linear order. We let M' be the collection of all $X \subseteq L$ such that the following conditions hold:

(1) For all $a, b \in L$, if $a < b \in X$ then $a \in X$.

(2) If X has a lub a in L, then $a \in X$.

We consider the structure $(M', \subset)$. It is clearly a partial order; we claim that it is a linear order. (Up to isomorphism it is the completion that we are after.) Suppose that $X, Y \in M'$ and $X \neq Y$; we want to show that $X \subset Y$ or $Y \subset X$. By symmetry take $a \in X \backslash Y$. Then we claim that $Y \subseteq X$ (hence $Y \subset X$). For, take any $b \in Y$. If $a < b$, then $a \in Y$ by (1), contradiction. Hence $b \leq a$, and so $b \in X$ by (1), as desired. This proves the claim.

Next we claim that $(M', \subset)$ is complete. For, suppose that $\mathscr{X} \subseteq M'$. Then $\bigcup \mathscr{X}$ satisfies (1). In fact, suppose that $c < d \in \bigcup \mathscr{X}$. Choose $X \in \mathscr{X}$ such that $d \in X$. Then $c \in X$ by (1) for X, and so $c \in \bigcup \mathscr{X}$. Now we consider two cases.

Case 1. $\bigcup \mathscr{X}$ does not have a lub in L. Then $\bigcup \mathscr{X} \in M'$, and it is clearly the lub of $\mathscr{X}$.

Case 2. $\bigcup \mathscr{X}$ has a lub in L; say a is its lub. Then

(3) $\bigcup \mathscr{X} \cup \{a\} = (-\infty, a]$.

In fact, $\subseteq$ is clear. Suppose that $b < a$. Then b is not an upper bound for $\bigcup \mathscr{X}$, so we can choose $c \in \bigcup \mathscr{X}$ such that $b < c$. Then $b \in \bigcup \mathscr{X}$ since $\bigcup \mathscr{X}$ satisfies (1). This proves (3).

Clearly $(-\infty, a] \in M'$. We claim that it is the lub of $\mathscr{X}$. Clearly it is an upper bound. Now suppose that Z is any upper bound. Then $\bigcup \mathscr{X} \subseteq Z$. If $a \notin Z$, then $\bigcup \mathscr{X} = Z$, contradicting (2) for Z. So $a \in Z$ and hence $(-\infty, a] \subseteq Z$.

Hence we have shown that $(M', \subset)$ is complete.

Now for each $a \in L$ let $f(a) = \{b \in L : b \leq a\}$. Clearly $f(a) \in M'$.

(4) For any $a, b \in L$ we have $a < b$ iff $f(a) \subset f(b)$.

For, suppose that $a, b \in L$. If $a < b$, clearly $f(a) \subseteq f(b)$, and even $f(a) \subset f(b)$ since $b \in f(b) \backslash f(a)$. The other implication in (4) follows easily from this implication by assuming that $b \leq a$.

(5) Every element of M' is a lub of elements of $f[L]$.

For, suppose that $X \in M'$, and let $\mathscr{X} = \{f(a) : a \in X\}$; we claim that X is the lub of $\mathscr{X}$. Clearly $f(a) \subseteq X$ for all $a \in X$, so X is an upper bound of $\mathscr{X}$. Suppose that $Y \in M'$ is any upper bound for $\mathscr{X}$. If $a \in X$, then $a \in f(a) \subseteq Y$, so $a \in Y$. Thus $X \subseteq Y$, as desired. So (5) holds.

(6) If $a \in L$ is the lub in L of $X \subseteq L$, then $f(a)$ is the lub in M' of $f[X]$.

For, assume that $a \in L$ is the lub in L of $X \subseteq L$. If $x \in X$, then $x \leq a$, so $f(x) \subseteq f(a)$. Thus $f(a)$ is an upper bound for $f[X]$ in M'. Now suppose that $Y \in M'$ and Y is an upper bound for $f[X]$. If $b \in L$ and $b < a$, then since a is the lub of X, there is a $d \in X$ such that $b < d \leq a$. So $f(d) \subseteq Y$, and hence $d \in Y$. Since $b < d$, we also have $b \in Y$. This shows that $f(a)\backslash\{a\} \subseteq Y$. If $a \in X$, then $f(a) \in f[X]$ and so $f(a) \subseteq Y$, as desired. Assume that $a \notin X$. Since a is the lub of X in L, there is no largest member of L which is less than a. Now suppose that $a \notin Y$. If $u \in Y$, then $u < a$, as otherwise $a \leq u$ and so $a \in Y$, contradiction. It follows that $Y = \{u \in L : u < a\}$. Clearly then a is the lub of Y. This contradicts (2). It follows that $a \in Y$. Hence $f(a) \subseteq Y$. So (6) holds.

Thus M' is as desired, up to isomorphism.

Finally, we need to take care of the "up to isomorphism" business. Non-rigorously, we just identify a with $f(a)$ for each $a \in L$. This is the way things are done in similar contexts in mathematics. Rigorously we proceed as follows; and a similar method can be used in other contexts. Let A be a set disjoint from L such that $|A| = |M'\backslash f[L]|$. For example, we could take $A = \{(L, X) : X \in M'\backslash f[L]\}$; this set is clearly of the same size as $M'\backslash f[L]$, and it is disjoint from L by the foundation axiom. Let g be a bijection from A onto $M'\backslash f[L]$. Now let $N = L \cup A$, and define $h : N \to M'$ by setting, for any $x \in N$,

$$h(x) = \begin{cases} f(x) & \text{if } x \in L, \\ g(x) & \text{if } x \in A. \end{cases}$$

Thus h is a bijection from N to M', and it extends f. We now define $x \ll y$ iff $x, y \in N$ and $h(x) \subset h(y)$. We claim that $(N, \ll)$ really is a completion of L. (Not just up to isomorphism.) We check the conditions for this. Obviously $L \subseteq N$. Suppose that $a, b \in L$. Then $a < b$ iff $f(a) \subset f(b)$ iff $h(a) \subset h(b)$ iff $a \ll b$. Now h is obviously an order-isomorphism from $(N \subset)$ onto $(M' \subset)$, so N is complete. Now take any element a of N. Then by (5), $h(a)$ is the lub of a set $f[X]$ with $X \subseteq L$. By the isomorphism property, a is the lub of X. Finally, suppose that $a \in L$ is the lub of $X \subseteq L$. Then by (6), $f(a)$ is the lub of $f[X]$ in M', i.e., $h(a)$ is the lub of $h[X]$ in M'. By the isomorphism property, a is the lub of X in N. □

Theorem 21.14. *If L is a linear order and M, N are completions of L, then there is an isomorphism f of M onto N such that $f \restriction L$ is the identity.*

Proof. It suffices to show that if P is a completion of L and M', f, g, h, N are as in the proof of Theorem 21.13, then there is an isomorphism g from P onto N such that $g \restriction L$ is the identity.

For any $x \in P$ let $g'(x) = \{a \in L : a \leq_P x\}$. We claim that $g'(x) \in M'$. Clearly condition (1) holds. Now suppose that $g'(x)$ has a lub b in L. By (C4) for P, b is the lub of $g'(x)$ in P. But obviously x is the lub of $g'(x)$ in P, so $b = x \in g'(x)$. So (2) holds for $g'(x)$, and hence $g'(x) \in M'$.

Now we let $g(x) = h^{-1}(g'(x))$ for any $x \in P$. If $x \in L$, then $g'(x) = f(x) = h(x)$, and hence $g(x) = x$.

If $x <_P y$, clearly $g'(x) \subseteq g'(y)$, and hence $g(x) \leq_N g(y)$. By (C3) for P and y, there is an $a \in L$ such that $x <_P a \leq_P y$. So $a \in g'(y)\backslash g'(x)$. Hence $g'(x) \subset g'(y)$ and so $g(x) <_N g(y)$. Thus $\forall x, y \in P[x <_P y \to g(x) <_N g(y)]$. Hence $x \not<_P y$ iff $y \leq_P x$ iff $g(y) \leq_N g(x)$ iff $g(x) \not<_N g(y)$. So $\forall x, y \in P[x <_P y \leftrightarrow g(x) <_N g(y)]$.

It remains only to show that g is a surjection. Let $x \in N$. Set $y = \sup_P h(x)$. If $a \in h(x)$, then $a \le_P y$ and so $a \in g'(y)$. Thus $h(x) \subseteq g'(y)$. Now suppose that $a \in g'(y)$. So $a \le_P y$. If $a <_P y$, then there is a $z \in h(y)$ such that $a <_P z \le_P y$. It follows that $a \in h(y)$. If $a = y$, then $a \in h(x)$ by (2). So $g'(y) \subseteq h(y)$, showing that $g'(y) = h(x)$. Hence $g(y) = h^{-1}(g'(y)) = x$. □

Corollary 21.15. *Suppose that L is a dense linear order, and M is a linear order. Then the following conditions are equivalent:*

(i) M is a completion of L.

(ii) (a) $L \subseteq M$

(b) M is complete.

(c) For any $a, b \in L$, $a <_L b$ iff $a <_M b$.

(d) For any $x, y \in M$, if $x <_M y$ then there is an $a \in L$ such that $x <_M a <_M y$.

Proof. (i)⇒(ii): Assume that M is a completion of L. then (a)–(c) are clear. Suppose that $x, y \in M$ and $x <_M y$. By (C3), choose $b \in L$ such that $x <_M b \le_M y$. If $x \in L$, then choose $a \in L$ such that $x <_L a <_L b$; so $x <_M a <_M y$, as desired. Assume that $x \notin L$. Then by (C4), b is not the lub in L of $\{u \in L : u <_M x\}$, so there is some $a \in L$ such that $a <_L b$ and a is an upper bound of $\{u \in L : u <_M x\}$. Since by (C3) x is the lub of $\{u \in L : u <_M x\}$, it follows that $x <_M a <_M b \le_M y$, as desired.

(ii)⇒(i): Assume (ii). Then (C1) and (C2) are clear. For (C3), let $x \in M$, and let $X = \{a \in L : a < x\}$. Then x is an upper bound for X, and (ii)(d) clearly implies that it is the lub of X. For (C4), suppose that $a \in L$ is the lub in L of a set X of elements of L. Suppose that $x \in M$ is an upper bound for X and $x < a$. Then by (ii)(d) there is an element $b \in L$ such that $x < b < a$. Then there is an element $c \in X$ such that $b < c \le a$. It follows that $c \le x$, contradiction. □

Note from this corollary that the completion of a dense linear order is also dense.

We now take up a special topic, Suslin lines.

• A subset U of a linear order L is *open* iff U is a union of open intervals (a, b) or $(-\infty, a)$ or (a, ∞). Here $(-\infty, a) = \{b \in L : b < a\}$ and $(a, \infty) = \{b \in L : a < b\}$. L itself is also counted as open. (If L has at least two elements, this follows from the other parts of this definition.) Note that if L has a largest element a, then $(a, \infty) = \emptyset$; similarly for smallest elements.

• An *antichain* in a linear order L is a collection of pairwise disjoint nonempty open sets.

• A linear order L has the *countable chain condition*, abbreviated ccc, iff every antichain in L is countable.

• A subset D of a linear order L is *topologically dense* in L iff $D \cap U \neq \emptyset$ for every nonempty open subset U of L. Then dense in the sense at the beginning of the chapter implies topologically dense. In fact, if D is dense in the original sense and U is a nonempty open set, take some non-empty open interval (a, b) contained in U. There is a $d \in D$ with $a < d < b$, so $D \cap U \neq \emptyset$. If $\emptyset \neq (a, \infty) \subseteq U$ for some a, choose $b \in (a, \infty)$, and then choose $d \in D$ such that $a < d < b$. Then again $D \cap U \neq \emptyset$. Similarly if $(-\infty, a) \subseteq U$ for some a.

Conversely, if L itself is dense, then topological denseness implies dense in the order sense; this is clear. On the other hand, take for example the ordered set ω; ω itself is topologically dense in ω, but ω is not dense in ω in the order sense.

• A linear order L is *separable* iff there is a countable subset C of L which is topologically dense in L. Note that if L is separable and (a, b) is a nonempty open interval of L, then (a, b), with the order induced by L ($x < y$ for $x, y \in (a, b)$ iff $x < y$ in L) is separable. In fact, if C is countable and topologically dense in L clearly $C \cap (a, b)$ is countable and topologically dense in (a, b). Similarly, $[a, b]$ is separable, taking $(C \cap [a, b]) \cup \{a, b\}$. This remark will be used shortly.

• A *Suslin line* is a linear ordered set $(S, <)$ satisfying the following conditions:
(i) S has ccc.
(ii) S is not separable.

• *Suslin's Hypothesis* () is the statement that there do not exist Suslin lines.

Later in these notes we will prove that MA + ¬CH implies SH. Here MA is *Martin's axiom*, which we will define and discuss later. The consistency of MA + ¬CH requires iterated forcing, and will be proven much later in these notes. Also later in these notes we will prove that $\diamondsuit$ implies ¬SH, and still later we will prove that $\diamondsuit$ is consistent with ZFC, namely it follows from $V = L$. Both $\diamondsuit$ and L are defined later.

For now we want to connect our notion of Suslin line with more familiar mathematics, and with the original conjecture of Suslin. The following is a theorem of elementary set theory.

Theorem 21.16. *For any linear order $(L, \prec)$ the following conditions are equivalent:*
(i) $(L, \prec)$ is isomorphic to $(\mathbb{R}, <)$.
(ii) The following conditions hold:
(a) L has no first or last elements.
(b) L is dense.
(c) Every nonempty subset of L which is bounded above has a least upper bound.
(d) L is separable.

Proof. (i)⇒(ii): standard facts about real numbers.

(ii)⇒(i): By (d), let C be a countable subset of L such that $(a, b) \cap C \neq \emptyset$ whenever $a < b$ in L. Clearly C is infinite, is dense, and has no first or last element. By Theorem 21.2, let f be an isomorphism from $(C, <)$ onto $(\mathbb{Q}, <)$. We now apply the procedure used at the end of the proof of 21.13. Let P be a set disjoint from $\mathbb{Q}$ such that $|L \backslash C| = |P|$, and let $R = \mathbb{Q} \cup P$. Let g be a bijection from $L \backslash C$ onto P, and define $h = f \cup g$. Define $x \prec y$ iff $x, y \in R$ and $h^{-1}(x) <_L h^{-1}(y)$. This makes R into a linearly ordered set with h an isomorphism from L onto R. Now we adjoin first and last elements a_R, b_R to R and similarly $a_{\mathbb{R}}, b_{\mathbb{R}}$ for $\mathbb{R}$; call the resulting linearly ordered sets R' and $\mathbb{R}'$. Then R' and $\mathbb{R}'$ are both completions of $\mathbb{Q}$ according to Corollary 21.15. Hence (i) holds by Theorem 21.14. $\square$

Originally, Suslin made the conjecture that separability in Theorem 21.16 can be replaced by the condition that every family of pairwise disjoint open intervals is countable. The following theorem shows that this conjecture and our statement of Suslin's hypothesis are equivalent.

Theorem 21.17. *The following conditions are equivalent:*

(i) There is a Suslin line.

(ii) There is a linearly ordered set $(L, <)$ *satisfying the following conditions:*

(a) L *has no first or last elements.*

(b) L *is dense.*

(c) Every nonempty subset of L *which is bounded above has a least upper bound.*

(d) No nonempty open subset of L *is separable.*

(e) L *is ccc.*

Proof. Obviously (ii) implies (i). Now suppose that (i) holds, and let S be a Suslin line. We obtain (ii) in two steps: first taking care of denseness, and then taking the completion to finish up.

We define a relation $\sim$ on S as follows: for any $a, b \in S$,

$$\begin{aligned} a \sim b \quad \text{iff} \quad & a = b, \\ & \text{or } a < b \text{ and } [a,b] \text{ is separable,} \\ & \text{or } b < a \text{ and } [b,a] \text{ is separable.} \end{aligned}$$

Clearly $\sim$ is an equivalence relation on S. Let L be the collection of all equivalence classes under $\sim$.

(1) If $I \in L$, then I is convex, i.e., if $a < c < b$ with $a, b \in I$, then also $c \in I$.

For, $[a,b]$ is separable, so $[a,c]$ is separable too, and hence $a \sim c$; so $c \in I$.

(2) If $I \in L$, then I is separable.

For, this is clear if I has only one or two elements. Suppose that I has at least three elements. Then there exist $a, b \in I$ with $a < b$ and $(a,b) \neq \emptyset$. Let $\mathscr{M}$ be a maximal pairwise disjoint set of such intervals. Then $\mathscr{M}$ is countable. Say $\mathscr{M} = \{(x_n, y_n) : n \in \omega\}$. Since $x_n \sim y_n$, the interval $[x_n, y_n]$ is separable, so we can let D_n be a countable dense subset of it. We claim that the following countable set E is dense in I:

$$\begin{aligned} E = \bigcup_{n\in\omega} D_n & \cup \{e : e \text{ is the largest element of } I\} \\ & \cup \{a : a \text{ is the smallest element of } I\}. \end{aligned}$$

Thus e and a are added only if they exist. To show that E is dense in I, first suppose that $a, b \in I$, $a < b$, and $(a,b) \neq \emptyset$. Then by the maximality of $\mathscr{M}$, there is an $n \in \omega$ such that $(a,b) \cap (x_n, y_n) \neq \emptyset$. Choose $c \in (a,b) \cap (x_n, y_n)$. Then $\max(a, x_n) < c < \min(b, y_n)$, so there is a $d \in D_n \cap (\max(a, x_n), \min(b, y_n)) \subseteq (a,b)$, as desired. Second, suppose that $a \in I$ and $(a, \infty) \neq \emptyset$; here $(a, \infty) = \{x \in I : a < x\}$. We want to find $d \in E$ with $a < d$. If I has a largest element e, then e is as desired. Otherwise, there are $b, c \in I$ with $a < b < c$, and then an element of $(a, c) \cap E$, already shown to exist, is as desired. Similarly one deals with $-\infty$. Thus we have proved (2).

Now we define a relation $<$ on L by setting $I < J$ iff $I \neq J$ and $a < b$ for some $a \in I$ and $b \in J$. By (1) this is equivalent to saying that $I < J$ iff $I \neq J$ and $a < b$ for all $a \in I$ and $b \in J$. In fact, suppose that $a \in I$ and $b \in J$ and $a < b$, and also $c \in I$ and $d \in J$, while $d \leq c$. If $d \leq a$, then $d \leq a < b$ with $d, b \in J$ implies that $a \in J$, contradiction. Hence $a < d$. Since also $d \leq c$ this gives $d \in I$, contradiction.

Clearly $<$ makes L into a simply ordered set. Except for not being complete in the sense of (c), L is close to the linear order we want.

To see that L is dense, suppose that $I < J$ but $(I, J) = \emptyset$. Take any $a \in I$ and $b \in J$. Then $(a, b) \subseteq I \cup J$, and $I \cup J$ is separable by (2), so $a \sim b$, contradiction.

For (d), by a remark in the definition of separable it suffices to show that no open interval (I, J) is separable. Suppose to the contrary that (I, J) is separable. Let $\mathscr{A}$ be a countable dense subset of (I, J). Also, let $\mathscr{B} = \{K \in L : I < K < J \text{ and } |K| > 2\}$. Any two distinct members of $\mathscr{B}$ are disjoint, and hence by ccc $\mathscr{B}$ is countable. In fact, each $K \in \mathscr{B}$ has the form (a, b), $[a, b)$, $(a, b]$, or $[a, b]$. since $|K| > 2$, and in each case the open interval (a, b) is nonempty. So ccc applies.

Define $\mathscr{C} = \mathscr{A} \cup \mathscr{B} \cup \{I, J\}$. By (2), each member of $\mathscr{C}$ is separable, so for each $K \in \mathscr{C}$ we can let D_K be a countable dense subset of K. Let $E = \bigcup_{K \in \mathscr{C}} D_K$. So E is a countable set. Fix $a \in I$ and $b \in J$. We claim that $E \cap (a, b)$ is dense in (a, b). (Hence $a \sim b$ and so $I = J$, contradiction.) For, suppose that $a \leq c < d \leq b$ with $(c, d) \neq \emptyset$.

Case 1. $[c]_\sim = [d]_\sim = I$. Then $D_I \cap (c, d) \neq \emptyset$, so $E \cap (c, d) \neq \emptyset$, as desired.

Case 2. $[c]_\sim = [d]_\sim = J$. Similarly.

Case 3. $I < [c]_\sim = [d]_\sim < J$. Then $[c]_\sim \in \mathscr{B} \subseteq \mathscr{C}$, so the desired result follows again.

Case 4. $[c]_\sim < [d]_\sim$. Choose $K \in \mathscr{A}$ such that $[c]_\sim < K < [d]_\sim$. Hence $c < e < d$ for any $e \in D_K$, as desired.

Thus we have obtained a contradiction, which proves that (I, J) is not separable.

Next, we claim that L has ccc. In fact, suppose that $\mathscr{A}$ is an uncountable family of pairwise disjoint open intervals. Let $\mathscr{B}$ be the collection of all endpoints of members of $\mathscr{A}$, and for each $I \in \mathscr{B}$ choose $a_I \in I$. Then

$$\{(a_I, a_J) : (I, J) \in \mathscr{A}\}$$

is an uncountable collection of pairwise disjoint nonempty open intervals in S, contradiction. In fact, given $(I, J) \in \mathscr{A}$, choose K with $I < K < J$. then $a_K \in (a_I, a_J)$. So $(a_I, a_J) \neq \emptyset$. Suppose that $(I, J), (I', J')$ are distinct members of $\mathscr{A}$. Wlog $J \leq I'$. Then $a_J \leq a_{I'}$, and it follows that $(a_I, a_J) \cap (a_{I'}, a_{J'}) = \emptyset$.

This finishes the first part of the proof. We have verified that L satisfies (b), (d), and (e). Now let M be the completion of L, and let N be M without its first and last elements. We claim that N finally satisfies all of the conditions in (ii). Clearly N is dense, it has no first or last elements, and every nonempty subset of it bounded above has a least upper bound. Next, suppose that $a < b$ in N and C is a countable subset of (a, b) which is dense in (a, b). Choose $c, d \in L$ such that $a < c < d < b$. For any $u, v \in C$ with $c < u < v < d$ choose $e_{uv} \in L$ such that $u < e_{uv} < v$; such an element exists by Corollary 21.15. We claim that $\{e_{uv} : u, v \in C,\ u < v\}$ is dense in (c, d) in L, which is a contradiction. For, given x, y such that $c < x < y < d$ in L, by the definition of denseness we can find $u, v \in C$ such that $x < u < v < y$; and then $x < e_{uv} < y$, as desired.

It remains only to prove that N has ccc. Suppose that $\mathscr{A}$ is an uncountable collection of nonempty open intervals of N. By Corollary 21.15, for each $(a, b) \in \mathscr{A}$ we can find $c, d \in L$ such that $a < c < d < b$. So this gives an uncountable collection of nonempty open intervals in L, contradiction. □

In the next part of this chapter we prove a very useful theorem on characters of points and gaps in linearly ordered sets due to Hausdorff.

For any cardinal κ, the order type which is the reverse of κ is denoted by κ^*. Reg is the class of all regular cardinals. We define *regular* so that every regular cardinal is infinite. If $\kappa < \lambda$ are cardinals, then $[\kappa, \lambda]_{\text{reg}}$ is the collection of all regular cardinals in the interval $[\kappa, \lambda]$; similarly for half-open and open intervals.

Let $R \subseteq \text{Reg} \times \text{Reg}$. We define

$$\chi_{\text{left}}(R) = \text{the least cardinal greater than each member of dmn}(R);$$
$$\chi_{\text{right}}(R) = \text{the least cardinal greater than each member of rng}(R).$$

Let L be a linear order, and let $x \in L$. If x is the first element of L, then its *left character* is 0. If x has an immediate predecessor, then its *left character* is 1. Finally, suppose that x is not the first element of L and does not have an immediate predecessor. Then the *left character* of x is the smallest cardinal κ such that there is a strictly increasing sequence of elements of L with supremum x. This cardinal κ is then regular. Similarly, if x is the last element of L, then its *right character* is 0. If x has an immediate successor, then its *right character* is 1. Finally, suppose that x is not the last element of L and it does not have an immediate successor. Then the *right character* of x is the smallest cardinal λ such that there is a strictly decreasing sequence of elements of L with infimum x. The *character* of x is the pair (κ, λ) where κ is the left character and λ^* is the right character. The *point-character set* of L is the collection of all characters of points of L; we denote it by $\text{Pchar}(L)$. Note that $\text{Pchar}(L) \neq \emptyset$.

A *gap* of L is an ordered pair (M, N) such that $M \neq \emptyset \neq N$, $L = M \cup N$, M has no largest element, N has no smallest element, and $\forall x \in M \forall y \in N(x < y)$. The definitions of left and right characters of a gap are similar to the above definitions for points; but they are always infinite regular cardinals. Again, the *character* of (M, N) is the pair (κ, λ) where κ is the left character and λ^* is the right character. The *gap-character set* of L is the collection of all characters of gaps of L; we denote it by $\text{Gchar}(L)$. We say that L is *Dedekind complete* iff every nonempty subset of L which is bounded above has a least upper bound. For L dense this is equivalent to saying that $\text{Gchar}(L) = \emptyset$.

The *full character set* of L is the pair $(\text{Pchar}(L), \text{Gchar}(L))$.

If L does not have a first element, then the *coinitiality* of L is the least cardinal κ such that there is a strictly decreasing sequence $\langle a_\alpha : \alpha < \kappa \rangle$ of elements of L such that $\forall x \in L \exists \alpha < \kappa [a_\alpha < x]$; we denote this cardinal by $\text{ci}(L)$. Similarly for the right end, if L does not have a greatest element then we define the *cofinality* of L, denoted by .

L is *irreducible* iff it has no first or last elements, and the full character set of (x,y) is the same as the full character set of L for any two elements $x,y\in L$ with $x<y$.

Now a *complete character system* is a set $R\subseteq \mathrm{Reg}\times\mathrm{Reg}$ with the following properties:

(C1) $\mathrm{dmn}(R)=[\omega,\chi_{\mathrm{left}}(R))_{\mathrm{reg}}$.

(C2) $\mathrm{rng}(R)=[\omega,\chi_{\mathrm{right}}(R))_{\mathrm{reg}}$.

(C3) There is a κ such that $(\kappa,\kappa)\in R$.

Note these conditions do not mention orderings.

Proposition 21.18. *If L is an irreducible infinite Dedekind complete dense linear order, then* $\mathrm{Pchar}(L)$ *is a complete character system. Moreover,* $\mathrm{ci}(L)\le\chi_{\mathrm{right}}(\mathrm{Pchar}(L))$ *and* $\mathrm{cf}(L)\le\chi_{\mathrm{left}}(\mathrm{Pchar}(L))$.

Proof. Let $R=\mathrm{Pchar}(L)$. (C1): the inclusion $\subseteq$ is obvious. Now suppose that $\kappa\in[\omega,\chi_{\mathrm{left}}(R))_{\mathrm{reg}}$. Then there is a point x of L with character (μ,ν) such that $\kappa\le\mu$. Let $\langle a_\xi:\xi<\mu\rangle$ be a strictly increasing sequence of elements of L with supremum x. Let $y=\sup_{\xi<\kappa}a_\xi$. Clearly y has left character κ, as desired.

(C2): symmetric to (C1).

(C3): By a straightforward transfinite construction one gets (for some ordinal α) a strictly increasing sequence $\langle x_\xi:\xi<\alpha\rangle$ and a strictly decreasing sequence $\langle y_\xi:\xi<\alpha\rangle$ such that $x_\xi<y_\eta$ for all $\xi,\eta<\alpha$, and such that there is exactly one point z with $x_\xi<z<y_\eta$ for all $\xi,\eta<\alpha$. Then α is a limit ordinal, and z has character $(\mathrm{cf}(\alpha),\mathrm{cf}(\alpha))$, as desired.

Finally, suppose that $\mathrm{cf}(L)>\chi_{\mathrm{left}}(R)$. Then by the argument for (C1), L has a point with left character $\chi_{\mathrm{left}}(R)$, contradiction. A similar argument works for ci. $\square$

We shall use the sum construction for linear orders. If $\langle L_i:i\in I\rangle$ is a system of linear orders, and I itself is an ordered set, then by $\sum_{i\in I}L_i$ we mean the set

$$\{(i,a):i\in I,\ a\in L_i\}$$

ordered lexicographically.

The following lemma is probably well-known.

Lemma 21.19. *If $\langle L_i:i\in I\rangle$ is a system of complete linear orders, and I is a complete linear order, then $\sum_{i\in I}L_i$ is also complete.*

Proof. Suppose that C is a nonempty subset of $\sum_{i\in I}L_i$. Let $i_0=\sup\{i\in I:(i,a)\in C$ for some $a\in L_i\}$. We consider two cases.

Case 1. There is an $a\in L_{i_0}$ such that $(i_0,a)\in C$. Then we let $a_0=\sup\{a\in L_{i_0}:(i_0,a)\in C\}$. Clearly (i_0,a_0) is the supremum of C.

Case 2. There is no $a\in L_{i_0}$ such that $(i_0,a)\in C$. Then the supremum of C is (i_0,a), where a is the first element of L_{i_0}. $\square$

Another construction we shall use is the infinite product. Suppose that I is a well-ordered set and $\langle L_i:i\in I\rangle$ is a system of linear orders. Then we make $\prod_{i\in I}L_i$ into a linear order by defining, for $f,g\in\prod_{i\in I}L_i$,

$$f<g \quad\text{iff}\quad f\ne g\text{ and }f(i)<g(i),$$

where $i = \text{f.d.}(f,g)$, and $\text{f.d.}(f,g)$ is the first $i \in I$ such that $f(i) \neq g(i)$.

Given such an infinite product, and given a strictly increasing sequence $x = \langle x_\alpha : \alpha < \lambda\rangle$ of members of it, with λ a limit ordinal, we call x of *argument type* if the following two conditions hold:

(A1) $\langle \text{f.d.}(x_\alpha, x_{\alpha+1}) : \alpha < \lambda\rangle$ is strictly increasing

(A2) For each $\alpha < \lambda$, the sequence $\langle \text{f.d.}(x_\alpha, x_\beta) : \alpha < \beta < \kappa\rangle$ is a constant sequence.

On the other hand, x is *of basis type* iff there is an $i \in I$ such that $\text{f.d.}(x_\alpha, x_\beta) = i$ for all distinct $\alpha, \beta < \kappa$.

Lemma 21.20. *Let $\langle M_i : i \in I\rangle$ be a system of ordered sets, with I well-ordered. If $x < y < z$ in $\prod_{i\in I} M_i$, then $f.d.(x,z) = \min\{f.d.(x,y), f.d.(y,z)\}$.*

Proof. Let $i = \min\{\text{f.d.}(x,y), \text{f.d.}(y,z)\}$.

Case 1. $i = \text{f.d.}(x,y) = \text{f.d.}(y,z)$. Then $x \restriction i = y \restriction i = z \restriction i$ and $x(i) < y(i) < z(i)$, so $\text{f.d.}(x,z) = i$.

Case 2. $i = \text{f.d.}(x,y) < \text{f.d.}(y,z)$. Then $x \restriction i = y \restriction i = z \restriction i$ and $x(i) < y(i) = z(i)$, so $\text{f.d.}(x,z) = i$.

Case 3. $i = \text{f.d.}(y,z) < \text{f.d.}(x,y)$. Then $x \restriction i = y \restriction i = z \restriction i$ and $x(i) = y(i) < z(i)$, so $\text{f.d.}(x,z) = i$. $\square$

The following is Satz XIV in Hausdorff [1908].

Theorem 21.21. *Let $\langle M_i : i \in I\rangle$ be a system of ordered sets, with I well-ordered. Suppose that κ is regular and $\langle x_\alpha : \alpha < \kappa\rangle$ is a strictly increasing sequence of elements of $\prod_{i\in I} M_i$. Then this sequence has a subsequence of length κ which is either of argument type or of basis type.*

Proof. First we claim

(1) For every $\alpha < \kappa$ there is a $\beta > \alpha$ and an $i \in I$ such that $\text{f.d.}(x_\alpha, x_\gamma) = i$ for all $\gamma \geq \beta$.

This is true because, by Lemma 21.20, if $\alpha < \beta < \gamma < \kappa$, then $\text{f.d.}(x_\alpha, x_\beta) \geq \text{f.d.}(x_\alpha, x_\gamma)$; hence

$$\text{f.d.}(x_\alpha, x_{\alpha+1}) \geq \text{f.d.}(x_\alpha, x_{\alpha+2}) \geq \cdots \geq \text{f.d.}(x_\alpha, x_{\alpha+\beta}) \geq \cdots$$

for all $\beta < \kappa$; so this sequence of elements of I has a minimum, and (1) holds.

Now for each $\alpha < \kappa$, let $\varphi(\alpha)$ be the least $\beta > \alpha$ so that an i as in (1) exists, and let $i(\alpha)$ be such an i. Thus

(2) For each $\alpha < \kappa$ we have $\alpha < \varphi(\alpha)$, and for all $\gamma \geq \varphi(\alpha)$ we have $\text{f.d.}(x_\alpha, x_\gamma) = i(\alpha)$.

Now we define a function $\alpha \in {}^\kappa\kappa$ by setting

$$\begin{aligned}
\alpha(0) &= 0;\\
\alpha(\xi+1) &= \varphi(\alpha(\xi));\\
\alpha(\eta) &= \sup_{\xi<\eta} \alpha(\xi) \quad \text{for } \eta \text{ limit}
\end{aligned}$$

Then we clearly have

(3) α is a strictly increasing function, and f.d.$(x_{\alpha(\xi)}, x_{\alpha(\eta)}) = i(\alpha(\xi))$ for all $\xi, \eta < \kappa$ with $\xi < \eta$.

Moreover,

(4) If $\xi < \eta < \theta < \kappa$, then $i(\alpha(\xi)) = \text{f.d.}(x_{\alpha(\xi)}, x_{\alpha(\theta)}) \leq \text{f.d.}(x_{\alpha(\eta)}, x_{\alpha(\theta)}) = i(\alpha(\eta))$.

In fact, this is clear by Lemma 21.20.

Now we consider two cases.

Case 1. $\forall\xi < \kappa\exists\eta < \kappa[\xi < \eta$ and $i(\alpha(\xi)) < i(\alpha(\eta))]$. Then there is a strictly increasing $\beta \in {}^{\kappa}\kappa$ such that for all $\xi, \eta \in \kappa$, if $\xi < \eta$ then $i(\alpha(\beta(\xi))) < i(\alpha(\beta(\eta)))$. Hence for $\xi < \eta < \kappa$ we have

$$\text{f.d.}(x_{\alpha(\beta(\xi))}, x_{\alpha(\beta(\xi+1))}) = i(\alpha(\beta(\xi))) < i(\alpha(\beta(\eta))) = \text{f.d.}(x_{\alpha(\beta(\eta))}, x_{\alpha(\beta(\eta+1))});$$

moreover, if $\xi < \eta < \kappa$, then f.d.$(x_{\alpha(\beta(\xi))}, x_{\alpha(\beta(\eta))}) = i(\alpha(\beta(\xi)))$; so $\langle x_{\alpha(\beta(\xi))} : \xi < \kappa\rangle$ is of argument type.

Case 2. $\exists\xi < \kappa\forall\eta < \kappa[\xi < \eta$ implies that $i(\alpha(\xi)) = i(\alpha(\eta))]$. Hence $\langle x_{\alpha(\xi+\eta)} : \eta < \kappa\rangle$ is of basis type. $\square$

A variant of the product construction will be useful. Let κ be an infinite regular cardinal. A *κ-system* is a pair (T, M) with the following properties:

(V1) For each $\alpha < \kappa$ and each $x \in T_\alpha$, $M_{x\alpha}$ is a linear order.

(V2) $T_0 = \{\emptyset\}$.

(V3) For each $\alpha < \kappa$ we have

$$T_{\alpha+1} = \{f : \text{dmn}(f) = \alpha + 1,\ (f \restriction \alpha) \in T_\alpha,\ f(\alpha) \in M_{(f\restriction\alpha)\alpha}\}$$

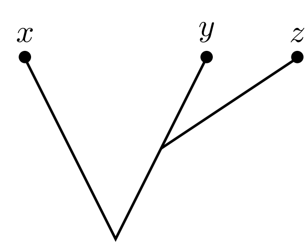
(V4) If $\beta \leq \kappa$ is a limit ordinal $\leq \kappa$, then $T_\beta = \{f : \text{dmn}(f) = \beta \text{ and } \forall\alpha < \beta[f \restriction \alpha \in T_\alpha]\}$.

We define a relation $<$ on T_κ by setting, for any $x, y \in T_\kappa$, $x < y$ iff $x \neq y$, and $x(\xi) < y(\xi)$, where $\xi = \text{f.d.}(x, y)$. Here the second $<$ relation is that of $M_{(x\restriction\xi)\xi}$.

Lemma 21.22. *Under the above assumptions, $<$ is a linear order on T_κ.*

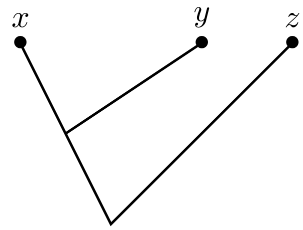
Proof. Clearly $<$ is irreflexive, and $\forall x, y \in T_\kappa[x < y$ or $x = y$ or $y < x]$. For transitivity, suppose that $x < y < z$. Let $\xi =$f.d.(x, y), $\eta =$f.d.(y, z), and $\theta =$f.d.(x, z).

Case 1. $\xi < \eta$. Then $\xi = \theta$ and $x(\theta) = x(\xi) < y(\xi) = z(\xi)$, so $x < z$.



Case 2. $\xi = \eta$. Then $\xi = \theta$ and $x(\theta) = x(\xi) < y(\xi) = y(\eta) < z(\eta) = z(\theta)$.

Case 3. $\eta < \xi$. Then $\theta = \eta$ and $x(\theta) = y(\theta) = y(\eta) < z(\eta) = z(\theta)$. $\square$



$\square$

The idea is that this is a *variable product*: not all functions in a cartesian product are allowed. If $x \in T_\kappa$, then for each $\alpha < \kappa$ the value $x(\alpha)$ lies in an ordered set $M_{(x\restriction\alpha)\alpha}$ which depends on $x \restriction \alpha$. Thus the linear order has a tree-like property.

Theorem 21.23. *Assume the above notation. For each $\gamma < \kappa$ let $M'_\gamma = \{(x, y) : x \in T_\gamma, y \in M_{x\gamma}\}$. Let M'_γ have the lexicographic ordering. Then there is an isomorphism of T_κ into $\prod_{\gamma<\kappa} M'_\gamma$.*

Namely, for each $x \in T_\kappa$ define $f(x)$ by $(f(x))_\gamma = (x \restriction \gamma, x(\gamma))$ for any $\gamma < \kappa$. Then f is the indicated isomorphism. Moreover, for all $x, y \in T_\kappa$ we have f.d.$(x, y) =$ f.d.$(f(x), f(y))$.

Proof. Clearly f maps T_κ into $\prod_{\gamma<\kappa} M'_\gamma$. Suppose that $x, y \in T_\kappa$ and $x < y$. Choose α minimum such that $x(\alpha) \neq y(\alpha)$; so $x(\alpha) < y(\alpha)$. Hence $(x \restriction \alpha, x(\alpha)) < (x \restriction \alpha, y(\alpha)) = (y \restriction \alpha, y(\alpha))$. If $\beta < \alpha$, then $(x \restriction \beta, x(\beta)) = (y \restriction \beta, y(\beta))$. Hence $f(x) < f(y)$ and f.d.$(x, y) =$ f.d.$(f(x), f(y))$. On the other hand, suppose that $f(x) < f(y)$. Let $\alpha =$ f.d.$(f(x), f(y))$. If $\beta < \alpha$, then $(f(x))_\beta = (f(y))_\beta$, i.e., $(x \restriction \beta, x(\beta)) = (y \restriction \beta, y(\beta))$. Hence $x \restriction \alpha = y \restriction \alpha$. Since $(f(x))_\alpha < (f(y))_\alpha$, we have $(x \restriction \alpha, x(\alpha)) < (y \restriction \alpha, y(\alpha))$. Hence $x(\alpha) < y(\alpha)$. It follows that $x < y$. $\square$

Theorem 21.24. *If (T, M) is a κ-system on a regular cardinal κ and each linear order $M_{x\alpha}$ is complete, then T_κ is complete.*

Proof. It suffices to take any regular cardinal ν, suppose that $x = \langle x_\theta : \theta < \nu\rangle$ is a strictly increasing sequence in T_κ, and show that it has a supremum. By Theorems 21.21 and 21.23 let $\langle f(x_{\theta(\xi)}) : \xi < \nu\rangle$ be a subsequence of $\langle f(x_\theta) : \theta < \nu\rangle$ which is of basis type or argument type.

Case 1. $\langle f(x_{\theta(\xi)}) : \xi < \nu\rangle$ is of basis type. Say $\gamma < \kappa$ and f.d.$(f(x_{\theta(\xi)}), f(x_{\theta(\eta)})) = \gamma$ for all distinct $\xi, \eta < \nu$. Thus by Theorem 21.23, f.d.$(x_{\theta(\xi)}, x_{\theta(\eta)}) = \gamma$ for all distinct $\xi, \eta < \nu$. Let $a = \sup\{x_{\theta(\xi)}(\gamma) : \xi < \nu\}$. Now define $y \in T_\kappa$ by setting

$$
\begin{aligned}
y \restriction \gamma &= x_{\theta(0)} \restriction \gamma;\\
y(\gamma) &= (y \restriction \gamma) \cup \{(\gamma, a)\};\\
y(\delta + 1) &= (y \restriction \delta) \cup \{(\delta, b)\} \quad \text{with } b \text{ the least element of } M_{(y\restriction\delta)\delta)} \text{ for } \gamma \leq \delta;\\
y(\psi) &= \bigcup_{\delta<\psi} (y \restriction \delta) \quad \text{for } \psi \text{ limit } > \gamma.
\end{aligned}
$$

Clearly y is an upper bound for $\langle x_\theta : \theta < \nu\rangle$. Now suppose that $z \in T_\kappa$ is any upper bound. If $\xi < \nu$, $\varphi < \gamma$, and $x_{\theta(\xi)}(\varphi) \neq z(\varphi)$, let $\rho =$ f.d.$(x_{\theta(\xi)}, z)$. Then $\rho < \gamma$ and

$x_{\theta(\xi)}(\rho) < z(\rho)$. Clearly then f.d.$(x_{\theta(\eta)}, z) = \rho$ for all $\eta < \nu$, and $y < z$. Thus we may assume that $x_{\theta(\xi)}(\varphi) = z(\varphi)$ for all $\xi < \nu$ and $\varphi < \gamma$. It follows that $a \leq z(\gamma)$. If $a < z(\gamma)$, clearly $y < z$. Suppose that $a = z(\gamma)$. Then again clearly $y \leq z$. So y is the least upper bound for $\langle x_\theta : \theta < \nu \rangle$.

Case 2. $\langle f(x_{\theta(\xi)}) : \xi < \nu \rangle$ is of argument type. By (A1), $\langle$f.d.$(x_{\theta(\xi)}, x_{\theta(\xi+1)} : \xi < \nu\rangle$ is strictly increasing. Thus $\nu \leq \kappa$. Let $\beta = \sup\{\text{f.d.}(x_{\theta(\xi)}, x_{\theta(\xi+1)}) : \xi < \nu\}$. Thus $\beta \leq \kappa$. Let $y_\xi = x_{\theta(\xi)} \restriction \text{f.d.}(x_{\theta(\xi)}, x_{\theta(\xi+1)})$ for each $\xi < \nu$. Hence $y_\xi \in T_{\text{f.d.}(x_{\theta(\xi)}, x_{\theta(\xi+1)})}$. Now

(1) $y_\xi \subseteq y_\eta$ if $\xi < \eta < \nu$.

In fact, suppose that this is not true; say that $\xi < \eta < \nu$ but $y_\xi \not\subseteq y_\eta$. So there is an $\alpha <$ f.d.$(x_{\theta(\xi)}, x_{\theta(\xi+1)})$ such that $y_\xi(\alpha) \neq y_\eta(\alpha)$. Thus $x_{\theta(\xi)}(\alpha) \neq x_{\theta(\eta)}(\alpha)$; so f.d.$(x_{\theta(\xi)} x_{\theta(\eta)}) \leq \alpha <$ f.d.$(x_{\theta(\xi)}, x_{\theta(\xi+1)})$, contradicting (A2).

From (1), clearly

(2) $y_\xi \subset y_\eta$ if $\xi < \eta < \nu$.

Now consider the function $z \stackrel{\text{def}}{=} \bigcup_{\xi<\nu} y_\xi$. We consider two cases.

Case 1. $\nu = \kappa$. Then $z \in T_\kappa$ and $\beta = \kappa$. We claim that z is the supremum of x in this case. If $\xi < \nu$, let $\alpha =$ f.d.$(x_{\theta(\xi)}, x_{\theta(\xi+1)})$. Then $z \restriction \alpha = y_\xi = x_{\theta(\xi)} \restriction \alpha$. Now $\alpha =$ f.d.$(x_{\theta(\xi)}, x_{\theta(\xi+1)}) <$ f.d.$(x_{\theta(\xi+1)}, x_{\theta(\xi+2)})$ by (A1) in the definition of argument type. So $x_{(\theta(\xi)}(\alpha) < x_{\theta(\xi+1)}(\alpha) = y_{\xi+1}(\alpha) \leq z(\alpha)$. Thus $x_{\theta(\xi)} < z$. Now suppose that $w < z$. Let $\xi =$ f.d.(w, z). Since $\beta = \kappa$, choose $\eta < \nu$ such that $\xi <$ f.d.$(x_{\theta(\eta)}, x_{\theta(\eta+1)})$. Then $w \restriction \xi = z \restriction \xi = y_\eta \restriction \xi$, and $w(\xi) < z(\xi) = y_\eta(\xi)$. So $w < y_\eta$, as desired.

Case 2. $\nu < \kappa$. Then also $\beta < \kappa$. Also, $z \in T_\beta$. We define an extension $v \in T_\kappa$ of z by recursion. Let $w_0 = z$. If w_α has been defined as a member of $T_{\beta+\alpha}$, with $\beta + \alpha < \kappa$, let $a(\alpha)$ be the least member of $M_{w_\alpha \alpha}$, and set $w_{\alpha+1} = w_\alpha \cup \{(\alpha, a(\alpha))\}$. So $w_{\alpha+1} \in T_{\beta+\alpha+1}$. If γ is limit and w_α has been defined as a member of $T_{\beta+\alpha}$ for all $\alpha < \gamma$, and if $\beta + \gamma < \kappa$, let $w_\gamma = \bigcup_{\alpha<\gamma} w_\alpha$. Finally, let $v = \bigcup_{\alpha<\kappa} w_\alpha$. So $v \in T_\kappa$ and it is an extension of z. We claim that it is the l.u.b. of $\langle x_\alpha : \alpha < \nu \rangle$. First suppose that $\xi < \nu$. Then $x_{\theta(\xi)} \restriction \text{f.d.}(x_{\theta(\xi)}, x_{\theta(\xi+1)}) = y_\xi = z \restriction \text{f.d.}(x_{\theta(\xi)}, x_{\theta(\xi+1)})$, and

$$\begin{aligned} x_{\theta(\xi)}(\text{f.d.}(x_{\theta(\xi)}, x_{\theta(\xi+1)})) &< x_{\theta(\xi+1)}(\text{f.d.}(x_{\theta(\xi)}, x_{\theta(\xi+1)})) \\ &= y_{\xi+1}(\text{f.d.}(x_{\theta(\xi)}, x_{\theta(\xi+1)})) \\ &= z(\text{f.d.}(x_{\theta(\xi)}, x_{\theta(\xi+1)})). \end{aligned}$$

Thus $x_{\theta(\xi)} < v$. Now suppose that $t < v$. Then $\alpha \stackrel{\text{def}}{=}$ f.d.(t, v) is less than β by construction, so $t \restriction \alpha = z \restriction \alpha$ and $t(\alpha) < z(\alpha)$. By the definition of z this gives a $\xi < \nu$ such that $t \restriction \alpha = x_{\theta(\xi)} \restriction \alpha$ and $t(\alpha) < x_{\theta(\xi)}(\alpha)$. So $t < x_{\theta(\xi)}$, as desired. □

Our main theorem is as follows; it is Satz XVII of Hausdorff [1908].

Theorem 21.25. *Suppose that R is a complete character system, and κ, λ are regular cardinals with $\kappa \leq \chi_{\text{right}}(R)$ and $\lambda \leq \chi_{\text{left}}(R)$, and $\chi_{\text{left}}(R)$ and $\chi_{\text{right}}(R)$ are successor cardinals. Then there is an irreducible Dedekind complete dense order L such that* Pchar$(L) = R$, *with* ci$(L) = \lambda$ *and* cf$(L) = \kappa$.

Proof. We may assume that $\chi_{\text{left}}(R) \geq \chi_{\text{right}}(R)$; otherwise we replace R by $R^{-1} \stackrel{\text{def}}{=} \{(\kappa, \lambda) : (\lambda, \kappa) \in R\}$, and replace the final order by its reverse. Let R be ordered lexicographically. Note by hypothesis that R has a largest element. We define some important orders which are components of the final order L. Let α and β be regular cardinals. Now we define

$$\varphi_{\alpha\beta} = \alpha + 1 + \beta^*;$$
$$\Phi = \sum_{(\alpha,\beta)\in R} \varphi_{\alpha\beta};$$
$$\mu(\alpha, \beta) = 1 + \alpha^* + \Phi + \beta + 1.$$

The symmetry of this definition will enable us to shorten several proofs below. Since we are using the standard notation for sums of order types, and some order types are repeated, it is good to have an exact notation for the indicated orders. m, f, l are new elements standing for "middle", "first", and "last" respectively. We suppose that with each ordinal ξ we associate a new element ξ', used in forming things like β^*. Thus more precisely,

$$\varphi_{\alpha\beta} = \alpha \cup \{m\} \cup \{\xi' : \xi < \beta\};$$

the ordering here is: α has its natural order; for $\xi, \eta < \beta$, we define $\xi' < \eta'$ iff $\eta < \xi$; $\xi < m$ for each $\xi < \alpha$; $m < \xi'$ for each $\xi < \beta$; and transitivity gives the rest.

$$\Phi = \{((\alpha, \beta), a) : (\alpha, \beta) \in R,\ a \in \varphi_{\alpha\beta}\} \quad \text{with lexicographic order;}$$

$$\mu(\alpha, \beta) = \{f\} \cup \{\xi' : \xi < \alpha\} \cup \Phi \cup \beta \cup \{l\};$$

the ordering should be obvious on the basis of the above remarks. We implicitly assume the distinctness of the various objects making up $\mu(\alpha, \beta)$.

(1) $\mu(\alpha, \beta)$ is a complete linear order, for any regular cardinals α, β.

This is clear on the basis of Lemma 21.19.

(2) If $(\alpha, \beta) \in R$, $(\alpha, \beta) \neq (\omega, \omega)$, and a is the smallest element of $\varphi_{\alpha\beta}$, then the right character of $((\alpha, \beta), a)$ is 1, and the left character is:

(a) 1, if β is a successor cardinal;
(b) β if β is a (regular) limit cardinal $> \omega$;
(c) 1, if $\beta = \omega$ and α is a successor cardinal;
(d) α if $\beta = \omega$ and α is a (regular) limit cardinal $> \omega$.

We prove this by cases:

(2)(a): Say $\beta = \gamma^+$. Then (α, β) is the lexicographic successor of (α, γ), and the left character of $((\alpha, \beta), a)$ is 1, since $((\alpha, \gamma), 0')$ is the immediate predecessor of $((\alpha, \beta), a)$.

(2)(b): $\langle((\alpha, \gamma), 0) : \gamma < \beta\rangle$ is strictly increasing with supremum $((\alpha, \beta), a)$.

(2)(c): Say $\alpha = \gamma^+$. Let $\chi_{\text{right}}(R) = \delta^+$. Then $((\gamma, \delta), 0')$ is the immediate predecessor of $((\alpha, \omega), a)$.

(2)(d): $\langle((\gamma, \omega), 0') : \gamma < \alpha\rangle$ is strictly increasing with supremum $((\alpha, \omega), a)$

(3) If a is the smallest element of $\varphi_{\omega\omega}$, then the character of $((\omega,\omega),a)$ in $\mu(\alpha,\beta)$ is $(1,1)$.

(4) Let $\chi_{\rm left}(R)=\delta^+$ and $\chi_{\rm right}(R)=\varepsilon^+$. If $(\alpha,\beta)\in R$, $(\alpha,\beta)\neq(\delta,\varepsilon)$, and a is the largest element of $\varphi_{\alpha\beta}$, then the left character of $((\alpha,\beta),a)$ is 1, and the right character is:

(a) 1, if $\beta<\varepsilon$;

(b) 1, if $\beta=\varepsilon$;

We prove this by cases:

(4)(a): $((\alpha,\beta^+),0)$ is the immediate successor of $((\alpha,\beta),a)$.

(4)(b): $((\alpha^+,\omega),0)$ is the immediate successor of $((\alpha,\beta),a)$.

(5) If a is the greatest element of $\varphi_{\delta\varepsilon}$, then the character of $((\delta,\varepsilon),a)$ in $\mu(\alpha,\beta)$ is $(1,1)$.

(6) If $\alpha<\chi_{\rm right}(R)$ is regular and $\beta<\chi_{\rm left}(R)$ is regular, then:

(a) the right character of the left end point of $\mu(\alpha,\beta)$ is α;

(b) the left character of the right end point of $\mu(\alpha,\beta)$ is β;

(c) if $a\in\mu(\alpha,\beta)$ is not an end point, and its character is (γ,δ^*), then $\gamma<\chi_{\rm left}(R)$ and $\delta<\chi_{\rm right}(R)$.

In fact, (a) and (b) are clear. Now suppose that $a\in\mu(\alpha,\beta)$ is not an end point. If a is in the α^* portion but not equal to $0'$, or it is in the β portion but is not the first element of β, the conclusion of (c) is clear. The character of $0'$ is $(1,1)$. The character of the first element of β is $(1,1)$ since R has a largest element. If a is within some $\varphi_{\alpha\beta}$ but is not the first or last element of $\varphi_{\alpha\beta}$, then the conclusion of (c) holds by (2) and (4). If a is the first or last element of $\varphi_{\alpha\beta}$ then the conclusion of (2) holds by (3) and (5).

Let p be a new element, not appearing in any of the above orders. Let σ be the least regular cardinal such that $(\sigma,\sigma)\in R$; it exists by condition (C3) in the definition of a complete character system. For each regular $\alpha<\chi_{\rm left}(R)$, let ξ_α be the least cardinal such that $(\alpha,\xi_\alpha)\in R$; it exists by (C1) in the definition of complete character set. Similarly, for each regular $\alpha\in\chi_{\rm right}(R)$ let η_α be the least cardinal such that $(\eta_\alpha,\alpha)\in R$.

Now we define by recursion a σ-system (T,M). Let $T_0=\{\emptyset\}$ and $M_{\emptyset 0}=\mu(\lambda,\kappa)$. Now except for $M_{\emptyset 0}$, $M_{x\alpha}$ will have the form $\{p\}$ or $\mu(\rho,\sigma)$ with $\rho<\chi_{\rm left}(R)$ and $\sigma<\chi_{\rm right}(R)$.

Suppose that $\gamma\le\sigma$ is a limit ordinal. We let T_γ be the set of all x with domain γ such that $x\restriction\alpha\in T_\alpha$ for all $\alpha<\gamma$. Now suppose that $\gamma<\sigma$, still with γ a limit ordinal. Now if $x\in T_\gamma$ and $|M_{(x\restriction\alpha)\alpha}|>1$ for all $\alpha<\gamma$, we set

$$M_{x\gamma}=\mu(\xi_{{\rm cf}(\gamma)},\eta_{{\rm cf}(\gamma)}).$$

On the other hand, if $|M_{(x\restriction\alpha)\alpha}|=1$ for some $\alpha<\gamma$, we set $M_{x\gamma}=\{p\}$.

Now suppose that $\gamma=\beta+1$. Then we set

$$T_\gamma=\{x^\frown\langle b\rangle : x\in T_\beta \text{ and } b\in M_{x\beta}\}.$$

Now we define $M_{x\gamma}$ for each $x\in T_\gamma$.

(7) If $x(\beta)=p$, then $M_{x\gamma}=\{p\}$.

(8) If $x(\beta)$ is an endpoint of $M_{(x\restriction\beta)\beta}$ or has no immediate neighbors, then $M_{x\gamma}=\{p\}$.

(9) If $x(\beta)$ has a right neighbor but no left neighbor, and the left character of $x(\beta)$ in $M_{(x\upharpoonright\beta)\beta}$ is α, then $M_{x\gamma} = \mu(\xi_\alpha, \sigma)$. Note that $\alpha < \chi_{\text{left}}(R)$.

(10) If $x(\beta)$ has a left neighbor but no right neighbor, and the right character of $x(\beta)$ in $M_{(x\upharpoonright\beta)\beta}$ is α, then $M_{x\gamma} = \mu(\sigma, \eta_\alpha)$. Note that $\alpha < \chi_{\text{right}}(R)$.

(11) If $x(\beta)$ has both a left and a right neighbor, then $M_{x\gamma} = \mu(\sigma, \sigma)$.

This finishes the definition of (T, M). The linear order T_σ is close to the order we are after.

Note the following two facts:

(12) If $x \in T_\sigma$, $\alpha < \beta < \sigma$, and $x(\alpha) = p$, then $x(\beta) = p$.

In fact, we prove by induction on $\beta \in [\alpha, \sigma)$ that $x(\beta) = p$. This is given for $\beta = \alpha$. Now assume that $\beta \in [\alpha, \sigma)$ and $x(\beta) = p$. By the above definition with $\gamma = \beta + 2$, $x(\beta + 1) \in M_{(x\upharpoonright(\beta+1))(\beta+1)}$. By (7), $M_{(x\upharpoonright(\beta+1))(\beta+1)} = \{p\}$. Hence $x(\beta + 1) = p$. Assume that $\beta \in [\alpha, \sigma)$, β is limit, and $x(\gamma) = p$ for all $\gamma \in [\alpha, \beta)$. By definition we then have $M_{(x\upharpoonright\beta)\beta} = \{p\}$. Then by the above definition with $\gamma = \beta + 1$, $x(\beta) = p$. So (12) holds.

(13) If $x \in T_\sigma$ and $\alpha < \sigma$, then either $M_{(x\upharpoonright\alpha)\alpha} = \{p\}$ or $M_{(x\upharpoonright\alpha)\alpha} = \mu(\theta, \varphi)$ for some θ, φ; except for $M_{\emptyset 0}$ we have $\theta < \chi_{\text{left}}(R)$ and $\varphi < \chi_{\text{right}}(R)$.

From Theorem 21.24 we know that T_σ is complete. Now we find the characters of the elements of T_σ.

(14) The smallest element of T_σ has character $(0, \lambda^*)$.

To prove this, note that the smallest element of T_σ is $x \stackrel{\text{def}}{=} \langle f, p, p, \ldots\rangle$, where f is the first element of $M_{\emptyset 0} = \mu(\lambda, \kappa)$. For each $\alpha < \lambda$ let $y_\alpha = \langle(\alpha + 1)', f, p, p, \ldots\rangle$. Note here that $(\alpha = 1)'$ has both a left and a right neighbor, so $M_{\langle(\alpha+1)'\rangle 1} = \mu(\sigma, \sigma)$ by (11). f here is the first element of $\mu(\sigma, \sigma)$. By (8), $M_{\langle(\alpha+1)', f\rangle 2} = \{p\}$. Thus y_α is an element of T_σ. Clearly $x < y_\alpha$ for each $\alpha < \lambda$, and $y_\beta < y_\alpha$ if $\alpha < \beta < \lambda$. Now suppose that $x < w$. Hence $w(0) \neq f$. If $w(0)$ is not in the λ^* part, clearly $y_\alpha < w$ for every $\alpha < \lambda$. If $w(0) = \beta'$ for some $\beta < \lambda$, then $y_{\beta+1} < w$. This proves (14).

(15) The largest element of T_σ has character $(\kappa, 0)$.

In fact, the largest element of T_σ is $x \stackrel{\text{def}}{=} \langle l, p, p, \ldots\rangle$, where l is thee last element of $M_{\emptyset 0} = \mu(\lambda, \kappa)$. For each $\alpha < \kappa$ let $y_\alpha = \langle\alpha + 1, l, p, p, \ldots\rangle$. We see that $y_\alpha \in T_\sigma$ as in the proof of (14). Clearly $y_\alpha < x$ for each $\alpha < \kappa$. Now suppose that $w < x$. Clearly then $w(0) < l$. If $w(0)$ is not in the κ part, clearly $w < y_\alpha$ for all $\alpha < \kappa$. If $w(0) = \alpha < \kappa$, then $w < y_\alpha$. This proves (15).

Let a be the first element of T_σ, and b the last element.

(16) If $a < x < b$ and $|M_{x\upharpoonright\alpha}| > 1$ for every $\alpha < \sigma$, then x has character (σ, σ).

For, first we show that x has left character σ. For each $\alpha < \sigma$ let

$$y_\alpha = (x \upharpoonright \alpha)^\frown\langle f, p, p, \ldots\rangle.$$

Clearly $y_\alpha < x$ for all $\alpha < \sigma$, and $\langle y_\alpha : \alpha < \sigma\rangle$ is strictly increasing. Now suppose that $z \in T_\sigma$ and $z < x$. Let $\alpha = \text{f.d.}(z, x)$. Then clearly $z < y_{\alpha+1}$, as desired.

Now for right character σ, for each $\alpha < \sigma$ let

$$z_\alpha = (x \upharpoonright \alpha)^\frown\langle l, p, p, \ldots\rangle.$$

Clearly $x < z_\alpha$ for all $\alpha < \sigma$, and $\langle z_\alpha : \alpha < \sigma\rangle$ is strictly decreasing. Now suppose that $w \in T_\sigma$ and $x < w$. Let $\alpha = \text{f.d.}(w, x)$. Then clearly $z_{\alpha+1} < w$, as desired. Thus (16) holds.

Now suppose that $a < x < b$ and $x(\alpha) = p$ for some $\alpha < \sigma$, and let α be minimum with this property. Then by construction, α is a successor ordinal $\gamma+1$, and $x(\gamma)$ is an endpoint of $M_{(x\upharpoonright\gamma)\gamma}$ or is an element of $M_{(x\upharpoonright\gamma)\gamma}$ with no neighbor.

Case 1. $x(\gamma)$ is an element of $M_{(x\upharpoonright\gamma)\gamma}$ with no neighbor. Then by definition, there is a $(\rho, \xi) \in R$ such that $x(\gamma) = ((\rho, \xi), m)$, i.e., $x(\gamma)$ is the middle element of $\varphi_{\rho\xi}$. We claim that x has character (ρ, ξ^*). To see this, for each $\alpha < \rho$ let

$$y_\alpha = (x \upharpoonright \gamma)^\frown\langle((\rho, \xi), \alpha + 1), f, p, p, \ldots\rangle.$$

Then $y_\alpha < x$, the sequence $\langle y_\alpha : \alpha < \rho\rangle$ is strictly increasing, and x is its supremum. So the left character of x is ρ, and similarly the right character of x is ξ.

Case 2. $x(\gamma)$ is the endpoint f of $M_{(x\upharpoonright\gamma)\gamma}$. We now consider three subcases.

Subcase 2.1. $\gamma = 0$. This would imply that $x = a$, contradiction.

Subcase 2.2. γ is a limit ordinal. Then $\text{cf}(\gamma) < \sigma$. So by construction, $M_{(x\upharpoonright\gamma)\gamma}$ is $\mu(\xi_{\text{cf}(\gamma)}, \eta_{\text{cf}(\gamma)})$. For each $\delta < \gamma$ let $y_\delta = (x \upharpoonright \delta)^\frown\langle f, p, p, \ldots\rangle$. Clearly $\langle y_\delta : \delta < \gamma\rangle$ is strictly increasing with limit x. Hence x has character $(\text{cf}(\gamma), \xi_{\text{cf}(\gamma)}) \in R$.

Subcase 2.3. $\gamma = \beta + 1$ for some β. Then

$$x = (x \upharpoonright \beta)^\frown\langle x(\beta), f, p, p, \ldots\rangle.$$

Clearly then one of (9)–(11) holds for $x(\beta)$.

Subsubcase 2.3.1. (9) holds for $x(\beta)$. So $x(\beta)$ has a right neighbor, but no left neighbor. Say the left character of $x(\beta)$ is α. Then by (9), $M_{(x\upharpoonright\gamma)\gamma}$ is $\mu(\xi_\alpha, \sigma)$. We claim that x has character (α, ξ_α). To see this, let $\langle \alpha_\theta : \theta < \alpha\rangle$ be strictly increasing with supremum $x(\beta)$. Then for each $\theta < \alpha$ let y_θ be any element of N such that $x \upharpoonright \beta = y_\theta \upharpoonright \beta$ and $y_\theta(\beta) = \alpha_\theta$. So clearly $y_\theta < x$ and $\langle y_\theta : \theta < \alpha\rangle$ is strictly increasing. Now suppose that $z \in N$ and $z < x$. If $\text{f.d.}(z, x) < \beta$, then $z < y_0$. Suppose that $\text{f.d.}(z, x) = \beta$. Then $z(\beta) < x(\beta)$, so $z(\beta) < \alpha_\theta$ for some $\theta < \alpha$, and hence $z < y_\theta$. Clearly by the form of x, one of these possibilities for z must hold. Hence the left character of x is α. Clearly its right character is ξ_α.

Subsubcase 2.3.2. (10) holds for $x(\beta)$. So $x(\beta)$ has a left neighbor ε, but no right neighbor. Hence ε has a right neighbor, and hence (9) or (11) holds for $(x \upharpoonright \beta)^\frown\langle\varepsilon\rangle$ in place of x and β in place of γ. Hence $M_{(x\upharpoonright\beta)^\frown\langle\varepsilon\rangle,\beta+1}$ is $\mu(\xi_\alpha, \sigma)$ for some α, or $\mu(\sigma, \sigma)$. Now

(17) $y \stackrel{\text{def}}{=} (x \upharpoonright \beta)^\frown\langle\varepsilon, l, p, p, \ldots\rangle$ is the immediate predecessor of x.

In fact, clearly $y < x$. Suppose that $z < x$. Clearly f.d.$(z, x) \leq \beta$. If f.d.$(z, x) < \beta$, obviously $z < y$. If f.d.$(z, x) = \beta$, then $z(\beta) \leq \varepsilon$. If $z(\beta) < \varepsilon$, then $z < y$. If $z(\beta) = \varepsilon$, then $z(\gamma) \leq l$, and so $z \leq y$. So (17) holds.

Clearly the left character of y is σ. Now $M_{x\restriction\gamma,\gamma}$ is $\mu(\sigma, \eta_\alpha)$ for some α, and so it is clear that the right character of x is also σ.

Subsubcase 2.3.3. (11) holds for $x(\beta)$. So $x(\beta)$ has a left neighbor ε and a right neighbor ρ. Then y as above is the immediate predecessor of x, and it has left character σ. Since $M_{(x\restriction\gamma)\gamma}$ is $\mu(\sigma, \sigma)$, it is clear that the right character of x is σ.

Case 3. $x(\gamma)$ is the endpoint l of $M_{(x\restriction\gamma)\gamma}$. We now consider three subcases.

Subcase 3.1. $\gamma = 0$. This would imply that $x = b$, contradiction.

Subcase 3.2. γ is a limit ordinal. Then $\mathrm{cf}(\gamma) < \sigma$. So by construction, $M_{(x\restriction\gamma)\gamma}$ is $\mu(\xi_{\mathrm{cf}(\gamma)}, \eta_{\mathrm{cf}(\gamma)})$. Clearly the left character of x is $\eta_{\mathrm{cf}(\gamma)}$. Now for each $\delta < \gamma$ let $y_\delta = (x \restriction \delta)^\frown\langle l, p, p, \ldots\rangle$. Thus $x < y_\delta$ and $\langle y_\delta : \delta < \gamma$ is strictly decreasing. Suppose that $x < z$. Then there is a $\delta < \gamma$ such that $x \restriction \delta = z \restriction \delta$ and $x(\delta) < z(\delta)$. Then $y_{\delta+1} < z$. This shows that the right character of x is $\mathrm{cf}(\gamma)$. So the character of x is $(\eta_{\mathrm{cf}(\gamma)}, \mathrm{cf}(\gamma))$.

Subcase 3.3. $\gamma = \beta + 1$ for some β. Then

$$x = (x \restriction \beta)^\frown\langle x(\beta), l, p, p, \ldots\rangle.$$

Clearly then one of (9)–(11) holds for $x(\beta)$.

Subsubcase 3.3.1. (9) holds for $x(\beta)$. So $x(\beta)$ has a right neighbor, but no left neighbor. $M_{(x\restriction\gamma)\gamma}$ is $\mu(\xi_\alpha, \sigma)$ for some α, so the left character of x is σ. Let ε be the right neighbor of $x(\beta)$, and set $y = (x \restriction \beta)^\frown\langle \varepsilon, f, p, p, \ldots\rangle$. Then y is the right neighbor of x. y has a left neighbor, so one of (10),(11) holds, and hence the right character of y is σ.

Subsubcase 3.3.2. (10) holds for $x(\beta)$. So $x(\beta)$ has a left neighbor ε, but no right neighbor. Let the right character of $x(\beta)$ be α, and let $\langle \delta_\xi : \xi < \alpha\rangle$ be strictly decreasing with limit $x(\beta)$. For each $\xi < \alpha$ let

$$y_\xi = (x \restriction \beta)^\frown\langle \delta_\xi, l, p, p, \ldots\rangle.$$

It is clear that $\langle y_\xi : \xi < \alpha\rangle$ is strictly decreasing with limit x. So the right character of x is α. Now $x_{x\restriction\gamma,\gamma}$ is $\mu(\sigma, \eta_\alpha)$, so the left character of x is η_α. Thus x has character (η_α, α).

Subsubcase 3.3.3. (11) holds for $x(\beta)$. So $x(\beta)$ has a right neighbor ρ. Moreover, $M_{x\restriction\gamma,\gamma}$ is $\mu(\sigma, \sigma)$, so the left character of x is σ. Now let

$$y = (x \restriction \beta)^\frown\langle \rho, f, p, p, \ldots\rangle.$$

Then y is the right neighbor of x. Since ρ has a left neighbor, (10) or (11) holds for $(x \restriction \beta)^\langle\rho\rangle$. Hence $M_{(x\restriction\beta)^\frown\langle\rho\rangle,\gamma}$ is $\mu(\sigma, \eta_\alpha)$ for some α, or it is $\mu(\sigma, \sigma)$. Hence the right character of y is σ.

Summarizing our investigation of characters of elements of T_σ, we have:

(18) If $a < x < b$, then one of the following holds:

(a) x has no neighbors, and its character is in R.

(b) x has an immediate predecessor y, and the characters of x, y are $(1, \sigma)$ and $(\sigma, 1)$ respectively.

(c) x has an immediate successor y, and the characters of x, y are $(\sigma, 1)$ and $(1, \sigma)$ respectively.

Now we show that if $x < y$ in L_σ and y is not the immediate successor of x, then for every $(\xi, \eta) \in R$ there is a $z \in (x, y)$ with character (ξ, η). Let α be minimum such that $x(\alpha) \neq y(\alpha)$. Then $x(\alpha) \neq p$, as otherwise $M_{x\restriction\alpha,\alpha} = \{p\}$ and so also $y(\alpha) = p$. Now we consider two cases.

Case I. $y(\alpha)$ is not the immediate successor of $x(\alpha)$. Choose z with $x(\alpha) < z < y(\alpha)$. Say $M_{(x\restriction\alpha)^\frown\langle z\rangle} = \mu(\tau, \rho)$. In the Φ portion of $\mu(\tau, \rho)$ take the middle element $((\xi, \eta), m)$ of $\varphi(\xi, \eta)$. Let

$$w = (x \restriction \alpha)^\frown\langle z\rangle^\frown\langle((\xi, \eta), m)\rangle, p, p, \ldots\rangle.$$

Then $x < w < y$ and w has character (ξ, η).

Case II. $y(\alpha)$ is the immediate successor of $x(\alpha)$. Then

$$\begin{aligned} x &\leq (x \restriction \alpha)^\frown\langle x(\alpha)\rangle^\frown\langle l, p, p, \ldots\rangle \\ &< (x \restriction \alpha)^\frown\langle y(\alpha)\rangle^\frown\langle f, p, p, \ldots\rangle \\ &\leq y. \end{aligned}$$

Since y is not the immediate successor of x, one of the $\leq$s here is really $<$.

Case IIa. $x < (x \restriction \alpha)^\frown\langle x(\alpha)\rangle^\frown\langle l, p, p, \ldots\rangle$. Then $\alpha + 1$ is the first argument where these two sequences differ. Let $M_{(x\restriction(\alpha+1)),\alpha+1} = \mu(\tau, \rho)$. Then $x(\alpha + 1)$ is not the immediate predecessor of l, and so the argument in Case I gives an element w with character (ξ, η) such that

$$x < w < (x \restriction \alpha)^\frown\langle x(\alpha)\rangle^\frown\langle l, p, p, \ldots\rangle \leq y.$$

Case IIb. $(x \restriction \alpha)^\frown\langle y(\alpha)\rangle^\frown\langle f, p, p, \ldots\rangle^y$. This is similar to Case IIa.

Now let L be obtained from T_σ by deleting the second element of any pair (x, y) of elements of T_σ such that y is the immediate successor of x. Clearly L is as desired in the theorem. □

Theorem 21.26. *Suppose that R is a complete character system and κ and λ are regular cardinals with $\kappa \leq \chi_{\text{right}}(R)$ and $\lambda \leq \chi_{\text{left}}(R)$, and $\chi_{\text{right}}(R)$ and $\chi_{\text{left}}(R)$ are successor cardinals. Also suppose that $R = R_0 \cup R_1$ with $R_0 \neq \emptyset$. Then there is an irreducible dense linear order M such that* $\mathrm{Pchar}(M) = R_0$, $\mathrm{Gchar}(M) = R_1$, $\mathrm{ci}(M) = \lambda$, *and* $\mathrm{cf}(M) = \kappa$.

Proof. Let L be given by Theorem 21.25. For each $(\alpha, \beta) \in R$ let $M_{\alpha\beta} = \{x \in L :$ the character of x is $(\alpha, \beta)\}$. Note that $M_{\alpha\beta}$ is dense in L. For each $(\alpha, \beta) \in R_0 \cap R_1$ write $M_{\alpha\beta} = P_{\alpha\beta} \cup Q_{\alpha\beta}$ with $P_{\alpha\beta} \cap Q_{\alpha\beta} = \emptyset$ and both dense in $M_{\alpha\beta}$. Now we define

$$N = \bigcup_{(\alpha,\beta)\in R_0\cap R_1} P_{\alpha\beta} \cup \bigcup_{(\alpha,\beta)\in R_0\setminus R_1} M_{\alpha\beta}.$$

We claim that N is as desired. Take any $x < y$ in N. If $z \in (x, y) \cap N$, then $z \in M_{\alpha\beta}$ for some $(\alpha, \beta) \in R_0$, and so z has character (α, β). If $(\alpha, \beta) \in R_0 \cap R_1$, take any $z \in P_{\alpha\beta}$ such that $x < z < y$. Then $z \in N$ and it has character (α, β). Similarly, there is an element of $(x, y) \cap N$ whose character is a given member of $R_0 \backslash R_1$. Thus $\text{Pchar}(N) = R_0$. The elements of L that are omitted in N are all the members of

$$\bigcup_{(\alpha,\beta)\in R_0\cap R_1} Q_{\alpha\beta} \cup \bigcup_{(\alpha,\beta)\in R_1\backslash R_0} M_{\alpha\beta}.$$

It follows that $\text{Gchar}(N) = R_1$. □

We now give some further results on ccc.

A topological space is ccc iff every family of pairwise disjoint open sets is countable.

Lemma 21.27. (III.2.2) *Every separable space is ccc.*

Proof. Suppose that $\mathscr{A}$ is an uncountable family of pairwise disjoint open sets, while D is a countable dense set. For each $U \in \mathscr{A}$ choose $d_U \in D \cap U$. Then $\langle d_U : U \in \mathscr{A} \rangle$ is injective, contradiction. □

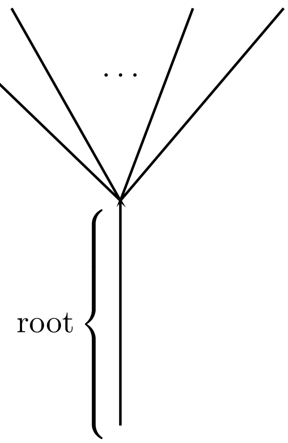
Proposition 21.28. (III.2.3) *Every ccc metric space is separable.*

Proof. For x in the space and $\varepsilon > 0$ let $B(x, \varepsilon) = \{y : \rho(x, y) < \varepsilon\}$. For each positive integer n, let Y_n be a maximal pairwise disjoint set of elements of the form $B(x, \frac{1}{n})$. Let D have at least one element in common with each member of $\bigcup_{n\geq 1} Y_n$, with D countable. We claim that D is dense. For, let $B(x, \frac{1}{n})$ be given. Then there is a $B(y, \frac{1}{3n}) \in Y_{3n}$ such that $B(x, \frac{1}{3n}) \cap B(y, \frac{1}{3n}) \neq \emptyset$. Choose $w \in B(x, \frac{1}{3n}) \cap B(y, \frac{1}{3n})$ and $z \in D \cap B(y, \frac{1}{3n})$. Then

$$d(x, z) \leq d(x, w) + d(w, y) + d(y, z) < \frac{1}{3n} + \frac{1}{3n} + \frac{1}{3n} = \frac{1}{n},$$

and so $z \in B(x, \frac{1}{n})$, as desired.

To proceed we need an important theorem from infinite combinatorics. A collection $\mathscr{A}$ of sets forms a Δ-*system* iff there is a set r (called the *root* or *kernel* of the Δ-system) such that $A \cap B = r$ for any two distinct $A, B \in \mathscr{A}$. This is illustrated as follows:



The existence theorem for Δ-systems that is most often used is as follows.

Theorem 21.29. (III.2.6) Δ-system theorem) *If κ is an uncountable regular cardinal and $\mathscr{A}$ is a collection of finite sets with $|\mathscr{A}| \geq \kappa$, then there is a $\mathscr{B} \in [\mathscr{A}]^\kappa$ such that $\mathscr{B}$ is a Δ-system.*

Proof. First we prove the following special case of the theorem.

$(*)$ If $\mathscr{A}$ is a collection of finite sets each of size $m \in \omega$, with $|\mathscr{A}| = \kappa$, then there is a $\mathscr{B} \in [\mathscr{A}]^\kappa$ such that $\mathscr{B}$ is a Δ-system.

We prove this by induction on m. The hypothesis implies that $m > 0$. If $m = 1$, then each member of $\mathscr{A}$ is a singleton, and so $\mathscr{A}$ is a collection of pairwise disjoint sets; hence it is a Δ-system with root $\emptyset$. Now assume that $(*)$ holds for m, and suppose that $\mathscr{A}$ is a collection of finite sets each of size $m+1$, with $|\mathscr{A}| = \kappa$, and with $m > 0$. We consider two cases.

Case 1. There is an element x such that $\mathscr{C} \stackrel{\text{def}}{=} \{A \in \mathscr{A} : x \in A\}$ has size κ. Let $\mathscr{D} = \{A \backslash \{x\} : A \in \mathscr{C}\}$. Then $\mathscr{D}$ is a collection of finite sets each of size m, and $|\mathscr{D}| = \kappa$. Hence by the inductive assumption there is an $\mathscr{E} \in [\mathscr{D}]^\kappa$ which is a Δ-system, say with kernel r. Then $\{A \cup \{x\} : A \in E\} \in [\mathscr{A}]^\kappa$ and it is a Δ-system with kernel $r \cup \{x\}$.

Case 2. Case 1 does not hold. Let $\langle A_\alpha : \alpha < \kappa\rangle$ be a one-one enumeration of $\mathscr{A}$. Then from the assumption that Case 1 does not hold we get:

$(**)$ For every x, the set $\{\alpha < \kappa : x \in A_\alpha\}$ has size less than κ.

We now define a sequence $\langle \alpha(\beta) : \beta < \kappa\rangle$ of ordinals less than κ by recursion. Suppose that $\alpha(\beta)$ has been defined for all $\beta < \gamma$, where $\gamma < \kappa$. Then $\Gamma \stackrel{\text{def}}{=} \bigcup_{\beta<\gamma} A_{\alpha(\beta)}$ has size less than κ, and so by $(**)$, so does the set

$$\bigcup_{x \in \Gamma} \{\delta < \kappa : x \in A_\delta\}.$$

Thus we can choose $\alpha(\gamma) < \kappa$ such that for all $x \in \Gamma$ we have $x \notin A_{\alpha(\gamma)}$. This implies that $A_{\alpha(\gamma)} \cap A_{\alpha(\beta)} = \emptyset$ for all $\beta < \gamma$. Thus we have produced a pairwise disjoint system $\langle A_{\alpha(\beta)} : \beta < \kappa \rangle$, as desired. (The root is $\emptyset$ again.)

This finishes the inductive proof of $(*)$

Now the theorem itself is proved as follows. Let $\mathscr{A}'$ be a subset of $\mathscr{A}$ of size κ. Then

$$\mathscr{A}' = \bigcup_{m \in \omega} \{A \in \mathscr{A}' : |A| = m\}.$$

Hence there is an $m \in \omega$ such that $\{A \in \mathscr{A}' : |A| = m\}$ has size κ. So $(*)$ applies to give the desired conclusion. □

Proposition 21.30. (III.2.7) *If κ is singular, then there is a family $\mathscr{A}$ of κ two-element subsets of κ such that no $\mathscr{B} \in [\mathscr{A}]^\kappa$ forms a Δ-system.*

Proof. Suppose that κ is uncountable and singular. Let $\langle \lambda_\xi : \xi < \mathrm{cf}(\kappa) \rangle$ be a strictly increasing continuous sequence of cardinals with supremum κ. Let $\mathscr{A} = \{\{\lambda_\xi, \alpha\} : \xi < \mathrm{cf}(\kappa), \lambda_\xi < \alpha < \lambda_{\xi+1}\}$. Clearly $|\mathscr{A}| = \kappa$. Suppose that $\mathscr{B} \in [\mathscr{A}]^\kappa$ is a Δ-system, say with kernel G. If $G = \emptyset$, then for each $\xi < \mathrm{cf}(\kappa)$, $\mathscr{B}$ has at most one member in $[\lambda_\xi, \lambda_{\xi+1})$, and so $|\mathscr{B}| \le \mathrm{cf}(\kappa) < \kappa$, contradiction. Let $\alpha \in G$. Say $\lambda_\xi \le \alpha < \lambda_{\xi+1}$. Now B has some member F not in $[\lambda_\xi, \lambda_{\xi+1})$, as otherwise $|\mathscr{B}| \le \lambda_{\xi+1}$. Then $G \not\subseteq F$, contradiction. □

Theorem 21.31. (III.2.8) *If $\prod_{i \in I} X_i$ is not ccc, then there is a finite $F \subseteq I$ such that $\prod_{i \in F} X_i$ is not ccc.*

Proof. Suppose that $\langle V^\alpha : \alpha < \omega_1 \rangle$ is a system of pairwise disjoint open sets in $\prod_{i \in I} X_i$. We may assume that each V^α is basic open; say $V^\alpha = \prod_{i \in I} U_i^\alpha$ where there is a finite $F^\alpha \subseteq I$ such that $\forall i \in I \backslash F^\alpha [U_i^\alpha = X_i]$. By the Δ-system lemma, let $\Gamma \in [\omega_1]^{\omega_1}$ and G be such that $F^\alpha \cap F^\beta = G$ for all distinct $\alpha, \beta \in \Gamma$. For distinct $\alpha, \beta \in \Gamma$ we have $V^\alpha \cap V^\beta = \emptyset$, and hence $\left(\prod_{i \in G} U_i^\alpha\right) \cap \left(\prod_{i \in G} U_i^\beta\right) = \emptyset$. □

Theorem 21.32. *A finite product of separable spaces is separable.*

Proof. Let $\prod_{i \in F} X_i$ be given with F finite and each X_i separable. Say $D_i \subseteq X_i$, D_i dense, D_i countable. Suppose that $U \subseteq \prod_{i \in F} X_i$ is open. Wlog U has the form $\prod_{i \in F} V_i$, each V_i open in X_i. For each $i \in F$ choose $x_i \in D_i \cap V_i$. Then $x \in \left(\prod_{i \in F} D_i\right) \cap \left(\prod_{i \in F} V_i\right)$. □

Corollary 21.33. (III.2.10) *A product of separable spaces is ccc.*

Proof. By Lemma 21.27 and Theorems 21.31, 21.32. □

Lemma 21.34. (III.2.11) *Let X_i be Hausdorff with at least two points, for each $i \in I$, with $|I| > \mathfrak{c}$. Then $\prod_{i \in I} X_i$ is not separable.*

Proof. Suppose that $x_n \in \prod_{i \in I} X_i$ for all $n \in \omega$; we show that $\{x_n : n \in \omega\}$ is not dense. For each $i \in I$ choose disjoint open $U_i, V_i \subseteq X_i$. For each $n \in \omega$ and $i \in I$ define

$$g_n(i) = \begin{cases} 1 & \text{if } x_n(i) \in V_i, \\ 0 & \text{otherwise.} \end{cases}$$

For each $i \in I$ we have $\langle g_n(i) : n \in \omega\rangle \in {}^\omega 2$, while $|I| > \mathfrak{c}$, so there are distinct $j, k \in I$ such that $\langle g_n(j) : n \in \omega\rangle = \langle g_n(k) : n \in \omega\rangle$. Let $W = \{y \in \prod_{i\in I} X_i : y(j) \in U_j$ and $y(k) \in V_k\}$. Then W is open. If $x_n \in W$, then $g_n(k) = 1$ and $g_n(j) = 0$, contradiction. $\square$

Theorem 21.35. (III.2.12) *$\mathscr{F} \subseteq {}^\omega\omega$ is an* **independent family** *iff for all $m \in \omega$, all $j_1, \ldots, j_m \in \omega$, and all distinct $f_1, \ldots, f_m \in \mathscr{F}$,*

$$|\{c \in \omega : f_1(c) = j_1 \wedge \ldots \wedge f_m(c) = j_m\}| = \omega.$$

There is an independent family $\mathscr{F}$ of size 2^ω.

Proof. Let $E = \{(s, p) : s \in [\omega]^{<\omega},\ p : \mathscr{P}(s) \to \omega\}$. For each $A \subseteq \omega$ define $f_A(s, p) = p(A \cap s)$. Suppose that $j_1, \ldots, j_m \in \omega$ are given, and $A_1, \ldots, A_m \subseteq \omega$ are distinct. For distinct $i, k \in \{1, \ldots, m\}$ choose $a_{ik} \in A_i \triangle A_k$. Suppose that $s \in [\omega]^{<\omega}$ and $\{a_{ik} : i, k \in \{1, \ldots, m\}, i \neq k\} \subseteq s$. Note that $A_i \cap s \neq A_k \cap s$ for $i \neq k$. Define $p : \mathscr{P}(s) \to \omega$ by setting

$$p(X) = \begin{cases} j_i & \text{if } X = A_i \cap s, \\ 0 & \text{otherwise} \end{cases}$$

If $i \in 1, \ldots, m$, then $f_{A_i}(s, p) = p(A_i \cap s) = j_i$.

Let F be a bijection from ω onto E. For each $A \subseteq \omega$ let $g_A = f_A \circ F$. We claim that $\{g_A : A \subseteq \omega\}$ is independent. For, let $j_1, \ldots, j_m \in \omega$ and let $A_1, \ldots, A_m \subseteq \omega$ be distinct. Let $\langle (s_k, p_k) : k \in \omega\rangle$ be distinct members of E such that $f_{A_i}((s_k, p_k)) = j_i$ for each $i = 1, \ldots, m$. Then for each $k \in \omega$ and $i = 1, \ldots, m$ we have

$$g_{A_i}(F^{-1}(s_k, p_k)) = f_{A_i}(s_k, p_k) = j_i,$$

as desired. $\square$

Theorem 21.36. (III.2.13) *If X_i is a separable space for each $i \in I$, and $|I| \leq 2^\omega$, then $\prod_{i\in I} X_i$ is separable.*

Proof. Let $\langle f_i : i \in I\rangle$ be an independent family of functions, each $f_i \in {}^\omega\omega$. For each $i \in I$ let $\{d^i_j : j \in \omega\}$ be dense in X_i. For any $e \in \omega$ and $i \in I$ let $x_e(i) = d^i_{f_i(e)}$. We claim that $\{x_e : e \in \omega\}$ is dense in $\prod_{i\in I} X_i$. Let U be basic open in $\prod_{i\in I} X_i$. Say $F \in [I]^{<\omega}$ and $\emptyset \neq U = \prod_{i\in I} V_i$ with $V_i = X_i$ for all $i \in I\backslash F$ and V_i open in X_i for all $i \in F$. Say $d^i_{j_i} \in V_i$ for all $i \in F$. Choose $e \in \omega$ such that $f_i(e) = j_i$ for all $i \in F$. Then for any $i \in F$ we have $x_e(i) = d^i_{f_i(e)} = d^i_{j_i} \in V_i$. So $x_e \in U$. $\square$

Theorem 21.37. (III.2.17) *If X is a dense linear order without endpoints which is separable and connected in the order topology, then X is isomorphic to $\mathbb{R}$.*

Proof. By Theorem 21.16 it suffices to show that every nonempty subset Y of X bounded above has a least upper bound. Suppose not. Let $Z = \{z \in X : z < x$ for some $x \in Y\}$, and let $W = \{x \in X : z < x$ for all $z \in Y\}$. Then Z and W are disjoint clopen sets with union X, contradicting connectedness. $\square$

Lemma 21.38. (III.2.18) *If X is a Suslin line, then $X \times X$ is not ccc.*

Proof. We will choose $a_\alpha, b_\alpha, c_\alpha \in X$ for $\alpha < \omega_1$ so that

(1) $a_\alpha < b_\alpha < c_\alpha$.
(2) $(a_\alpha, b_\alpha) \neq \emptyset \neq (b_\alpha, c_\alpha)$.
(3) $\forall \xi < \alpha [b_\xi \notin (a_\alpha, c_\alpha)$.

After doing this, the open sets $(a_\alpha, b_\alpha) \times (b_\alpha, c_\alpha) \subseteq X \times X$ are nonempty by (2); they are pairwise disjoint since if $\xi < \alpha$ then by (3) either $b_\xi \leq a_\alpha$ and so $(a_\xi, b_\xi) \cap (a_\alpha, b_\alpha) = \emptyset$, or $c_\alpha \leq b_\xi$ and so $(b_\alpha, c_\alpha) \cap (b_\xi, c_\xi) = \emptyset$.

To do the construction, first let I be the set of all isolated points. Then I is countable. We construct $a_\alpha, b_\alpha, c_\alpha \in X$ by recursion. Suppose that they have been constructed for all $\xi < \alpha$, where $\alpha < \omega_1$. Choose $a_\alpha < c_\alpha$ so that $(a_\alpha, c_\alpha) \neq \emptyset$ and $(a_\alpha, c_\alpha) \cap (I \cup \{b_\xi : \xi < \alpha\}) = \emptyset$. This is possible because the countable set $I \cup \{b_\xi : \xi < \alpha\}$ is not dense in X. Then $(a_\alpha, c_\alpha) \cap I = \emptyset$, so (a_α, c_α) is infinite. Choose $b_\alpha \in (a_\alpha, c_\alpha)$. Then $(a_\alpha, b_\alpha) \neq \emptyset \neq (b_\alpha, c_\alpha)$. □

Proposition 21.39. (III.2.19) *If X is a Suslin line, then the completion of X is a compact Suslin line.*

Proof. Let Y be the completion of X; see the definition following Proposition 21.12. First we show that Y has ccc. Clearly it suffices to show that if (a, b) is a nonempty interval in Y, then there exist $c, d \in X$ with $a \leq c < d \leq b$ and $(c, d) \neq \emptyset$ in X.

Case 1. $b \notin X$. Then by (C3) there are $c, e, d \in X$ such that $a < c < e < d < b$, as desired.

Case 2. $b \in X$ but $a \notin X$. Say $a < e < b$. By the argument in Case 1 we may assume that $e \in X$. If there are no members of X in (a, e), then e is the least upper bound of $\{x \in X : x < a\}$ in X, so by (C4) it is the least upper bound of this set in Y, contradiction. Thus there is a $d \in X$ such that $a < d < e$, as desired.

Thus ccc holds in Y.

Next we claim:

($*$) If $y \in Y \backslash X$ then there is a strictly increasing sequence $\langle x_n^y : n \in \omega \rangle$ of elements of X such that $\sup_{n \in \omega} x_n^y = y$.

This is clear by (C3) and ccc.

Now suppose that $D \subseteq Y$ is countable and dense. Let

$$D' = (D \cap X) \cup \bigcup_{y \in D \backslash X} \{x_n^y : n \in \omega\}.$$

Thus D' is countable. We claim that it is dense in X. (Contradiction) For suppose that $a, b \in X$, $a < b$, $(a, b) \cap X \neq \emptyset$. Choose $d \in D$ such that $a < d < b$. If $d \in X$, then $d \in D'$. If $d \notin X$, choose n so that $a < x_n^d < d$. so $a < x_n^d < b$ and $x_n^d \in D'$, as desired.

Thus Y is not separable.

To show that Y is compact, let $\mathscr{U}$ be an open cover of Y by open intervals. Then there is an open interval in $\mathscr{U}$ with $-\infty$ as a member; this interval must have the form $[-\infty, a)$. Let

$$y_0 = \sup\{y \in Y : [-\infty, y] \subseteq \bigcup F \text{ for some } F \in [\mathscr{U}]^{<\omega}\}.$$

Now there is a member of $\mathscr{U}$ with y_0 as a member. This is only possible if $y_0 = \infty$ and $[-\infty, \infty] \subseteq \bigcup F$ for some $F \in [\mathscr{U}]^{<\omega}$.

Proposition 21.40. (III.2.20) *A space X has ccc iff there is no family $\langle U_\alpha : \alpha < \omega_1 \rangle$ such that $\forall \alpha, \beta[\alpha < \beta < \omega_1 \to [U_\alpha \subseteq U_\beta$ and $U_\beta \backslash \overline{U_\alpha} \neq \emptyset]]$.*

Proof. $\Rightarrow$: Suppose that there is a family of the sort indicated. We claim that $(U_{\alpha+1} \backslash \overline{U_\alpha}) \cap (U_{\beta+1} \backslash \overline{U_\beta}) = \emptyset$ for $\alpha < \beta$. For,

$$(U_{\alpha+1} \backslash \overline{U_\alpha}) \cap (U_{\beta+1} \backslash \overline{U_\beta}) = U_{\alpha+1} \backslash \overline{U_\beta} \subseteq U_{\alpha+1} \backslash U_\beta = \emptyset$$

$\Leftarrow$: Suppose that $\langle U_\alpha : \alpha < \omega_1 \rangle$ are pairwise disjoint nonempty open sets. Let $V_\alpha = \bigcup_{\beta \leq \alpha} U_\beta$ for each $\alpha < \omega_1$. Thus $V_\alpha \subseteq V_\beta$ for $\alpha < \beta$. For $\alpha < \beta$ we have $V_\beta \backslash \overline{V_\alpha} \neq \emptyset$. For, assume otherwise. Thus $V_\beta \subseteq \overline{V_\alpha}$. Hence $U_\beta \subseteq \overline{V_\alpha}$, so

$$\emptyset \neq U_\beta \cap V_\alpha = U_\beta \cap \bigcup_{\gamma \leq \alpha} U_\gamma = \emptyset,$$

contradiction. $\square$

Proposition 21.41. (III.2.20) *There is a family $\langle U_\alpha : \alpha < \omega_1$ of open sets in ${}^{\omega_1}2$ such that $\forall \alpha, \beta[\alpha < \beta \to U_\alpha \subset U_\beta]$.*

Proof. For each $\alpha < \omega_1$ let $U_\alpha = \{f \in {}^{\omega_1}2 : f(\xi) = 1$ for some $\xi < \alpha\}$. Then $U_\alpha \subseteq U_\beta$ when $\alpha < \beta$. Also, for $\alpha < \beta$ let f equal 0 below α and 1 for α and above. Then $f \in U_\beta \backslash U_\alpha$. $\square$

Proposition 21.42. (III.2.22) *We say that κ is a* **caliber** *for a space X iff for every system $\langle U_\alpha : \alpha < \kappa \rangle$ of nonempty open subsets of X there is a $B \in [\kappa]^\kappa$ such that $\bigcap_{\alpha \in B} U_\alpha \neq \emptyset$.*

Suppose that $\langle X_i : i \in I \rangle$ is a system of separable spaces. Then ω_1 is a caliber for $\prod_{i \in I} X_i$.

Proof. Let $Y = \prod_{i \in I} X_i$. Let $\langle U_\alpha : \alpha < \omega_1 \rangle$ be a system of nonempty open sets in Y. We may assume that each U_α is basic open; say $U_\alpha = \prod_{i \in I} Y_i^\alpha$ with $Y_i^\alpha = X_i$ for all $i \notin F_\alpha$, $F_\alpha \subseteq I$ finite, and Y_i^α nonempty open for all α and i. Let $\langle F_\alpha : \alpha \in M \rangle$ be a Δ-system with kernel G, $|M| = \omega_1$. Now $\prod_{i \in G} X_i$ is separable by Theorem 21.36, so let $D \subseteq \prod_{i \in G} X_i$ be countable and dense. For each $\alpha \in M$ choose $d^\alpha \in D \cap \prod_{i \in G} Y_i^\alpha$. Then choose $e \in D$ such that $N \stackrel{\text{def}}{=} \{\alpha \in M : d^\alpha = e\}$ has size ω_1. Fix $y \in \prod_{i \in I} X_i$. For each $\alpha \in N$ and $i \in I$ let $x(\alpha, i) \in Y_i^\alpha$. We now define $f \in \prod_{i \in I} X_i$. Let $f \restriction G = e$. and define for $i \in I \backslash G$

$$f_i = \begin{cases} x(\alpha, i) & \text{if } i \in F_\alpha \backslash G, \\ y_i & \text{if } i \notin \bigcup_{\alpha \in N} F_\alpha. \end{cases}$$

Since $(F_\alpha \backslash G) \cap (F_\beta \backslash G) = \emptyset$ for $\alpha \neq \beta$, this definition is unambiguous. We claim that $f \in \bigcap_{\alpha \in N} U_\alpha$. For, let $\alpha \in N$. If $i \in G$, then $f_i = e_i = d_i^\alpha \in Y_i^\alpha$. If $i \in F_\alpha \backslash G$, then $f_i = x(\alpha, i) \in Y_i^\alpha$. If $i \notin F_\alpha$, then $y_i \in X_i = Y_i^\alpha$. This proves the claim. $\square$

Proposition 21.43. (III.2.23) *Let X be the set of all $x \in {}^{\omega_1}2$ such that $x(\alpha) = 0$ for all but finitely many α. Then X is ccc, but ω_1 is not a caliber for X.*

Proof. Suppose that $\langle U^\alpha \cap X : \alpha < \omega_1 \rangle$ is a system of nonempty pairwise disjoint open sets, each U^α open in ${}^{\omega_1}2$. Wlog for each $\alpha < \omega_1$ $U^\alpha = \{f \in {}^{\omega_1}2 : k^\alpha \subseteq f\}$, where k^α is a finite function $\subseteq \omega_1 \times 2$. Let $M \in [\omega_1]^{\omega_1}$ and G be such that $\langle \mathrm{dmn}(k^\alpha) : \alpha \in M \rangle$ is a Δ-system with kernel G. Then $M = \bigcup_{g \in {}^G 2} \{\alpha \in M : k^\alpha \restriction G = f\}$. So there exist an $N \in [M]^{\omega_1}$ and $g \in {}^G 2$ such that $\forall \alpha \in N[k^\alpha \restriction G = g]$. Take any two distinct $\alpha, \beta \in N$. For any $i < \omega_1$ let

$$f(i) = \begin{cases} g(i) & \text{if } i \in G, \\ k^\alpha(i) & \text{if } i \in \mathrm{dmn}(k^\alpha) \backslash G, \\ k^\beta(i) & \text{if } i \in \mathrm{dmn}(k^\beta) \backslash G, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f \in U^\alpha \cap U^\beta \cap X$, contradiction.

For the caliber, for each $\alpha < \omega_1$ let $U^\alpha = \{f \in {}^{\omega_1}2 : f(\alpha) = 1\}$. Then $U^\alpha \cap X$ is nonempty and open, but there is no $M \in [\omega_1]^{\omega_1}$ such that $\bigcap_{\alpha \in M}(U^\alpha \cap X) \neq \emptyset$. $\square$

Proposition 21.44. *Theorem 21.2 does not extend to ω_1. In fact, consider $\omega_1 \times \mathbb{Q}$ and $\omega_1^* \times \mathbb{Q}$, both with the lexicographic order, where ω_1^* is ω_1 under the reverse order ($\alpha <^* \beta$ iff $\beta < \alpha$). These two orders are dense with no endpoints, both of size ω_1, but are not isomorphic.*

Clearly each of these linear orders is dense with no first or last element. In fact, concerning the first linear order, suppose that $(\xi, r) < (\eta, s)$. If $\xi < \eta$, then $(\xi, r) < (\xi, r+1) < (\eta, s)$. If $\xi = \eta$, then $r < s$, and so

$$(\xi, r) < \left(\xi, \frac{r+s}{2}\right) < (\xi, s).$$

Thus $\omega_1 \times \mathbb{Q}$ is dense. It does not have a first or last element, since if (ξ, r) is given, then $(0, r-1) < (\xi, r) < (\xi, r+1)$. The other order is treated similarly.

We claim that these two linear orders are not isomorphic. Suppose to the contrary that f is an isomorphism of $\omega_1 \times \mathbb{Q}$ onto $\omega_1^* \times \mathbb{Q}$. For all $\xi < \omega_1$ let $f(\xi, 0) = (\alpha(\xi), q(\xi))$. Now if $\xi < \eta < \omega_1$, then $\alpha(\xi) \geq \alpha(\eta)$. Hence

(1) There is a $\rho < \omega_1$ such that for all $\xi \in [\rho, \omega_1)$ we have $\alpha(\xi) = \alpha(\rho)$.

In fact, suppose not. Thus for every $\rho < \omega_1$ there is a $\xi \in (\rho, \omega_1)$ such that $\alpha(\xi) < \alpha(\rho)$. Define $\langle \rho_n : n \in \omega \rangle$ by recursion as follows. Let $\rho_0 = 0$. If ρ_n has been defined, choose $\rho_{n+1} > \rho_n$ such that $\alpha(\rho_{n+1}) < \alpha(\rho_n)$. Then $\langle \alpha(\rho_n) : n \in \omega \rangle$ is a strictly decreasing sequence of ordinals, contradiction.

Thus (1) holds. Now $\langle q(\xi) : \rho \leq \xi < \omega_1 \rangle$ is a strictly increasing sequence of rationals, contradicting the fact that $|\mathbb{Q}| = \omega$. $\square$

Proposition 21.45. *For any infinite cardinal κ, the set ${}^\kappa 2$ under the lexicographic order is complete.*

Proof. Let $A \subseteq {}^\kappa 2$; we want to find a lub for A. We define $f \in {}^\kappa 2$ by recursion, as follows: for any $\alpha < \kappa$,

$$f(\alpha) = \begin{cases} 1 & \text{if there is a } g \in A \text{ such that } g \upharpoonright \alpha = f \upharpoonright \alpha \text{ and } g(\alpha) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $g \in A$ and $f < g$. Then $f \upharpoonright \chi(f,g) = g \upharpoonright \chi(f,g)$, $f(\chi(f,g)) = 0$, and $g(\chi(f,g)) = 1$. This contradicts the definition of f. (We are using $\chi(f,g)$ in the same way as for H_α.) Thus $g \leq f$ for all $g \in A$.

Suppose that h is an upper bound for A and $h < f$. Thus $h \upharpoonright \chi(h,f) = f \upharpoonright \chi(h,f)$, $h(\chi(h,f)) = 0$, and $f(\chi(h,f)) = 1$. By the definition of f, there is a $g \in A$ such that $\chi(g,f) = \chi(h,f)$ and $g(\chi(g,f) = 1$. But then $\chi(g,h) = \chi(h,f)$ too, and it follows that $h < g$. This contradicts h being an upper bound for A.

Thus f is the desired lub of A. $\square$

Proposition 21.46. *Suppose that κ and λ are cardinals, with $\omega \leq \lambda \leq \kappa$. Let μ be minimum such that $\kappa < \lambda^\mu$. Take the lexicographic order on ${}^\mu\lambda$, as for H_α. Then this gives a dense linear order of size λ^μ with a dense subset of size κ.*

Proof. Note that obviously $\mu \leq \kappa$. Clearly ${}^\mu\lambda$ is a linear order. Let

$$D = \{f \in {}^\mu\lambda : \text{ there is a } \xi < \mu \text{ such that } f(\xi) = 1 \text{ and } f(\eta) = 0 \text{ for all } \eta \in (\xi, \mu)\}.$$

Clearly $|D| \leq \kappa$. We can show that ${}^\mu\lambda$ is dense, and D is dense in it, by finding an element of D between any two elements $f < g$ of ${}^\mu\lambda$. For brevity let $\alpha = \chi(f,g)$. Now define h by:

$$h(\beta) = \begin{cases} f(\beta) & \text{if } \beta \leq \alpha; \\ f(\alpha+1)+1 & \text{if } \beta = \alpha+1; \\ 1 & \text{if } \beta = \alpha+2; \\ 0 & \text{if } \beta \in (\alpha+2, \mu). \end{cases}$$

Clearly h is as desired.

If $|D| < \kappa$, we can simply add κ elements to it to satisfy the requirement that $|D| = \kappa$. $\square$

Proposition 21.47. *$\mathscr{P}(\omega)$ under $\subseteq$ contains a chain of size 2^ω.*

Proof. Let $f : \omega \to \mathbb{Q}$ be a bijection. For each real number r, let $a_r = f^{-1}[\{s \in \mathbb{Q} : s < r\}]$. Clearly $r < t$ implies that $a_r \subseteq a_t$; and in fact $a_r \subset a_t$ since there is a rational number u such that $r < u < t$, and so $f^{-1}(u) \in a_t \backslash a_r$. Now a is an isomorphic embedding. $\square$

A subset S of a linear order L is *weakly dense* iff for all $a, b \in L$, if $a < b$ then there is an $s \in S$ such that $a \leq s \leq b$.

Proposition 21.48. *For any linear order L the following are equivalent:*

(i) L has a countable subset weakly dense in it.

(ii) L can be isomorphically embedded in $\mathbb{R}$.

Proof. (i)$\Rightarrow$(ii): Assume that $D \subseteq L$ is countable and weakly dense in L. Let $P = \{(L, a, b, q) : a, b \in L,\ a < b$, and there is no c such that $a < c < b$, and $q \in \mathbb{Q}\}$. Note that $L \cap P = \emptyset$. We order $L \cup P$ as follows: for $c, d \in L \cup P$, $c < d$ iff one of the following holds

(1) $c, d \in L$ and $c <_L d$.
(2) $c \in L$, $d = (L, a, b, q) \in P$, and $c \leq a$.
(3) $c = (L, a, b, q) \in P$, $d \in L$, and $b \leq c$.
(4) $c = (L, a, b, q) \in P$, $d = (L, a', b', q') \in P$, and $b \leq a'$.
(5) $c = (L, a, b, q) \in P$, $d = (L, a, b, r) \in P$, and $q <_{\mathbb{Q}} r$.

Clearly $L \cup P$ is dense, $D \cup P$ is countable, and $D \cup P$ is dense in $L \cup P$. We may assume that D contains the endpoints of L, if any. Moreover, $L \subseteq L \cup P$ as ordered sets. Hence it suffices to find an embedding of $L \cup P$ into $\mathbb{R}$.

By Proposition 21.4 let g be a strictly increasing map from $D \cup P$ into the set of rationals in $(0, 1)$. Now we define $f : L \cup P \to \mathbb{R}$: for any $a \in L \cup P$,

$$f(a) = \begin{cases} g(a) & \text{if } a \in D \cup P, \\ \sup\{g(b) : b \in D \cup P,\ b < a\} & \text{otherwise.} \end{cases}$$

We check that f is order-preserving: suppose that $a, c \in L \cup P$ and $a < c$.

Case 1. $a, c \in D \cup P$. Then $f(a) = g(a) < g(c) = f(c)$.

Case 2. $a \in D \cup P$, $c \notin D \cup P$. Then $f(a) = g(a) < \sup\{g(b) : b \in D \cup P,\ b < c\} = f(c)$.

Case 3. $a \notin D \cup P$, $c \in D \cup P$. Then $f(a) = \sup\{g(b) : b \in D \cup P,\ b < a\} < g(c) = f(c)$.

Case 4. $a \notin D \cup P$, $c \notin D \cup P$. Choose $d \in D \cup P$ with $a < d < c$. Then $f(a) = \sup\{g(b) : b \in D \cup P,\ b < a\} < g(d) < \sup\{g(b) : b \in D \cup P,\ b < c\} = f(c)$.

(ii)$\Rightarrow$(i): Assume that f is an order-isomorphism from L into $\mathbb{R}$. For rationals $r < s$ such that $(r, s) \cap \mathrm{rng}(f) \neq \emptyset$, choose $x_{rs} \in (r, s) \cap \mathrm{rng}(f)$ and choose d_{rs} such that $f(d_{rs} = x_{rs}$. Let $M = \{(a, b) : a, b \in L,\ a < b$, and there is no $c \in L$ such that $a < c < b\}$. Then M is countable, since $\{(f(a), f(b)) : (a, b) \in M\}$ is a collection of pairiwise disjoint open intervals in $\mathbb{R}$. Let

$$D = \{d_{rs} : r, s \in \mathbb{Q} \text{ and } (r, s) \cap \mathrm{rng}(f) \neq \emptyset\} \cup \{a : (a, b) \in M\}.$$

Then D is countable. Suppose that $a, b \in L$ and $a < b$. If there is a c such that $a < c < b$, then there exist rationals r, s such that $r < s$ and $(r, s) \subseteq (f(a), f(b))$. $f(c) \in (r, s)$, and so $d_{rs} \in (a, b)$. If there is no such c, then $a \in D$ by definition. $\square$

Proposition 21.49. *The following conditions are equivalent for any cardinals κ, λ such that $\omega \leq \kappa \leq \lambda$:*

(i) There is a linear order of size λ with a weakly dense subset of size κ.

(ii) $\mathscr{P}(\kappa)$ has a chain of size λ.

$\Rightarrow$: Clearly we may assume that $\kappa < \lambda$. Assume that L is a linear order with a weakly dense subset D of size κ. Let f be a bijection from D to κ. For each $r \in L \backslash D$ let

$a_r = f[\{d \in D : d < r\}]$. thus $a_r \in \mathscr{P}(\kappa)$, and $a_r \subseteq a_s$ if $r, s \in L\backslash D$ with $r < s$. Also, by the weak denseness of D, there is a $d \in D$ such that $r \le d \le s$. Since $r, s \in L\backslash D$, we have $r < d < s$, and so $d \in a_s\backslash a_r$. Thus $r < s$ implies that $a_r \subset a_s$. The other implication follows from this one. Note that $|L\backslash D| = \lambda$.

$\Leftarrow$: Again we may assume that $\kappa < \lambda$. Let L be a chain in $\mathscr{P}(\kappa)$ of size λ. For each $\alpha < \kappa$ let

$$x_\alpha = \bigcup \{a : a \in L,\ \alpha \notin a\}.$$

(1) If $\alpha, \beta < \kappa$, then $x_\alpha \subseteq x_\beta$ or $x_\beta \subseteq x_\alpha$.

For, suppose that $\gamma \in x_\alpha\backslash x_\beta$ and $\delta \in x_\beta\backslash x_\alpha$. Say $\gamma \in a \in L$ with $\alpha \notin a$ and $\delta \in b \in L$ with $\beta \notin b$. Say by symmetry $a \subseteq b$. Since $\beta \notin b$, it follows that $\beta \notin a$. Since $\gamma \in a$, we have $\gamma \in x_\beta$, contradiction.

(2) If $\alpha \in \kappa$ and $a \in L$, then $a \subseteq x_\alpha$ or $x_\alpha \subseteq a$.

For, suppose that $\alpha \in \kappa$ and $a \in L$. If $\alpha \notin a$, then $a \subseteq x_\alpha$. Suppose that $\alpha \in a$. If $\alpha \notin b \in L$, we then must have $b \subset a$, since a and b are comparable. Thus $x_\alpha \subseteq a$ in this case. So (2) holds.

By (1) and (2), the set $M \stackrel{\text{def}}{=} L \cup \{x_\alpha : \alpha \in \kappa\}$ is a chain. Its size is clearly λ. We claim that $\{x_\alpha : \alpha < \kappa\}$ is weakly dense in it, which will finish the proof (expanding $\{x_\alpha : \alpha < \kappa\}$ to a set of size κ if necessary). For, suppose that $a, b \in L$ and $a \subset b$. Choose $\alpha \in b\backslash a$. Then clearly $a \subseteq x_\alpha$. Also, $x_\alpha \subseteq b$ by the proof of (2). □

Proposition 21.50. *Suppose that L_i is a linear order with at least two elements, for each $i \in \omega$. Let $\prod_{i\in\omega} L_i$ have the lexicographic order. Then it is not a well-order.*

Proof. For each $i \in \omega$ let $a_i < b_i$ be elements of L_i. For each $j \in \omega$ define f^j by setting, for each $i \in \omega$,

$$f^j(i) = \begin{cases} a_i & \text{if } i \le j \text{ or } j+1 < i; \\ b_i & \text{if } i = j+1. \end{cases}$$

Then $f^0 > f^1 > \cdots$. □

Proposition 21.51. *Let $\langle L_i : i \in I\rangle$ be a system of linear orders, with I itself an ordered set. Then if each L_i is dense without first or last elements, then also $\sum_{i\in I} L_i$ is dense without first or last elements.*

Suppose that $(i, a) < (j, b)$. If $i < j$, let c be an element of L_i such that $a < c$. Then $(i, a) < (i, c) < (j, b)$. If $i = j$, let c be an element of L_i such that $a < c < b$. Then $(i, a) < (i, c) < (i, b)$. Thus $\sum_{i\in I} L_i$ is dense. Given (i, a), choose $c, d \in L_i$ with $c < a < d$. Then $(i, c) < (i, a) < (i, d)$. Thus $\sum_{i\in I} L_i$ is does not have a first or last element. □

Proposition 21.52. *Let κ be any infinite cardinal number. Let L_0 be a linear order similar to $\omega^* + \omega + 1$; specifically, let it consist of a copy of $\mathbb{Z}$ followed by one element a greater than every integer, and let L_1 be a linear order similar to $\omega^* + \omega + 2$; say it consists* *of a copy of $\mathbb{Z}$ followed by two elements $a < b$ greater than every integer. For any $f \in {}^{\kappa}2$ let*

$$M_f = \sum_{\alpha<\kappa} L_{f(\alpha)}.$$

If $f, g \in {}^{\kappa}2$ then M_f and M_g are not isomorphic.

Conclusion: there are exactly 2^{κ} linear orders of size κ up to isomorphism.

Proof. Let $f, g \in {}^{\kappa}2$. We assume that M_f is isomorphic to M_g and show that $f = g$. Let F be an isomorphism of M_f onto M_g. Clearly

(1) For all $x \in M_f \cup M_g$ the following conditions are equivalent:

(a) x does not have an immediate predecessor.

(b) $x = (\xi, a)$ for some $\xi < \kappa$.

Now for any $x \in M_f$, x has an immediate predecessor iff $F(x)$ has an immediate predecessor, as is easily seen. We claim then that $F(\xi, a) = (\xi, a)$ for all $\xi < \kappa$. We prove this by transfinite induction. Suppose that $F(\eta, a) = (\eta, a)$ for all $\eta < \xi$. Now $F(\xi, a)$ does not have an immediate predecessor, so by (1) it has the form (ρ, a) for some ρ. We cannot have $\rho < \xi$, since this would contradict F being one-one, by the supposition. If $\xi < \rho$, then we would have $F^{-1}(\xi, a) < F^{-1}(\rho, a) = (\xi, a)$; hence $F^{-1}(\xi, a) = (\sigma, a)$ for some $\sigma < \xi$; but $F(\sigma, a) = (\sigma, a)$ by the inductive hypothesis. contradiction. Thus $F(\xi, a) = (\xi, a)$, finishing the inductive proof.

Next we claim

(2) For any $x \in M_f$ the following conditions are equivalent:

(a) x does not have an immediate predecessor, but it has an immediate successor y which in turn does not have an immediate successor.

(b) $x = (\xi, a)$ for some ξ such that $f(\xi) = 1$.

This is obvious, and a similar condition for M_g holds.

Now the property given in (2)(a) is preserved under isomorphisms, so by the above, for any $\xi < \kappa$,

$$\begin{aligned} f(\xi) = 1 \quad &\text{iff} \quad (\xi, a) \text{ satisfies (2)(a)} \\ &\text{iff} \quad F(\xi, a) \text{ satisfies (2)(a)} \\ &\text{iff} \quad g(\xi) = 1. \end{aligned}$$

Thus $f = g$, as desired.

The required conclusion in the proposition is clear. □

Proposition 21.53. *Let κ be an uncountable cardinal. Let L_0 be a linear order similar to $\eta + 1 + \eta \cdot \omega_1^*$; specifically consisting of a copy of the rational numbers in the interval $(0, 1]$ followed by $\mathbb{Q} \times \omega_1$, where $\mathbb{Q} \times \omega_1$ is ordered as follows: $(r, \alpha) < (s, \beta)$ iff $\alpha > \beta$, or $\alpha = \beta$ and $r < s$. Let L_1 be a linear order similar to $\eta \cdot \omega_1 + 1 + \eta \cdot \omega_1^*$; specifically, we take L_1 to be the set*

$$\{(q, \alpha, 0) : q \in \mathbb{Q}, \alpha < \omega_1\} \cup \{(0, 0, 1)\} \cup \{(q, \alpha, 2) : q \in \mathbb{Q}, \alpha < \omega_1\},$$

with the following ordering:

$$\begin{aligned}
&(q,\alpha,0)<(r,\beta,0) \quad \textit{iff} \quad \alpha<\beta, \textit{ or } \alpha=\beta \textit{ and } q<r;\\
&(q,\alpha,0)<(0,0,1)<(r,\beta,2) \textit{ for all relevant } q,r,\alpha,\beta;\\
&(q,\alpha,2)<(r,\beta,2) \quad \textit{iff} \quad \alpha>\beta, \textit{ or } \alpha=\beta \textit{ and } q<r.
\end{aligned}$$

For each $f \in {}^{\kappa}2$ let

$$M_f = \sum_{\alpha<\kappa} L_{f(\alpha)}.$$

Then each M_f is a dense linear order without first or last elements, and if $f, g \in {}^{\kappa}2$ and $f \neq g$, then M_f and M_g are not isomorphic.

Conclusion: for κ uncountable there are exactly 2^κ dense linear orders without first or last elements, of size κ, up to isomorphism.

Proof. For L_0, if q is an element in the initial $(0,1]$ part, then $q-1<q$; so L_0 has no least element. If (q,α) is an element in the second part, then $(q,\alpha)<(q+1,\alpha)$; so L_0 has no greatest element. For denseness, suppose that $x<y$ in L_0. If both are in the first part, clearly there is a z such that $x<z<y$. Suppose that x is in the first part and $y=(q,\alpha)$ is in the second part. Then $x<(q-1,\alpha)<y$. Finally, suppose that $x=(r,\beta)$ and $y=(s,\gamma)$ are both in the second part. If $\beta>\gamma$, then $x<(r+1,\beta)<y$. If $\beta=\gamma$, then $r<s$ and the desired element is clear.

For L_1, given any element $(q,\alpha,0)$ in the first part, we have $(q-1,\alpha,0)<(q,\alpha,0)$, so L_1 does not have a least element. If $(q,\alpha,2)$ is any element in the third part, then $(q,\alpha,2)<(q+1,\alpha,2)$, so L_2 does not have a greatest element. To prove denseness, suppose that $x<y$ in L_1. We can consider several cases.

Case 1. $x=(q,\alpha,0)$, $y=(r,\beta,0)$, and $\alpha<\beta$. Then $x<(q+1,\alpha,0)<y$.

Case 2. $x=(q,\alpha,0)$, $y=(r,\alpha,0)$. Then $q<r$; so with $q<t<r$ we have $x<(t,\alpha,0)<y$.

Case 3. $x=(q,\alpha,0)$, $y=(0,0,1)$. Then $x<(q+1,\alpha,0)<y$.

Case 4. $x=(q,\alpha,0)$, $y=(r,\beta,2)$. Then $x<(0,0,1)<y$.

Case 5. $x=(0,0,1)$, $y=(q,\alpha,2)$. Then $x<(q-1,\alpha,2)<y$.

Case 6. $x=(q,\alpha,2)$, $y=(r,\beta,2)$, and $\alpha>\beta$. Then $x<(q+1,\alpha,2)<y$.

Case 7. $x=(q,\alpha,2)$, $y=(r,\alpha,2)$. Thus $q<r$. Then with $q<t<r$ we have $x<(t,\alpha,2)<y$.

Thus L_1 is dense.

Before beginning the main part of the proposition, we note the following facts.

(1) Each q in the first part of L_0, except for the final 1, has character (ω,ω).

(2) The final 1 has character (ω,ω_1). A strictly decreasing sequence with limit 1 is

$$\langle (0,\alpha) : \alpha<\omega_1\rangle.$$

(3) Every element in the second part of L_0 has character (ω,ω).

(4) Every element in the first part of L_1 has character (ω,ω).

(5) $(0,0,1)$ has character (ω_1,ω_1).

In fact, a strictly increasing sequence with limit $(0,0,1)$ is

$$\langle (0,\alpha,0) : \alpha < \omega_1 \rangle.$$

A strictly decreasing sequence with limit $(0,0,1)$ is

$$\langle (0,\alpha,2) : \alpha < \omega_1 \rangle.$$

(6) Every element in the third part of L_1 has character (ω,ω).

Now we suppose that $f \in {}^{\kappa}2$. We give some properties of M_f:

(7) The character of an element (x,ξ) of M_f is equal to the character of x in $L_{f(\xi)}$.

This is true because $L_{f(\xi)}$ is a convex set in M_f, i.e., if $u < v < w$ with $u, w \in L_{f(\xi)}$, then $v \in L_{f(\xi)}$. Also, the fact that $L_{f(\xi))}$ does not have a least or greatest element is needed to see (7).

(8) For each $\xi < \kappa$ there is a unique element of M_f of the form (ξ, x_ξ) which has character (ω,ω_1) if $f(\xi) = 0$ and has character (ω_1,ω_1) if $f(\xi) = 1$; all other elements of the form (ξ, y) have character (ω,ω).

Now we can treat the main part of the proposition. Suppose that $f, g \in {}^{\kappa}2$ and M_f is isomorphic to M_g; say F is an isomorphism. The sequence

$$\langle F(x_\xi,\xi) : \xi < \kappa \rangle,$$

with x_ξ given in (8), is an increasing sequence of elements of M_g such that all other elements of M_g have character (ω,ω). In M_g there is only one sequence of order type κ consisting of elements which do not have character (ω,ω), by (8) for M_g. Hence $F(x_\xi,\xi) = (y_\xi,\xi)$, where y_ξ is defined for M_g like x_ξ was for M_f in (8). But x_ξ and y_ξ then have the same characters, and so $f(\xi) = g(\xi)$. Thus $f = g$.

The final statement of the proposition is clear. $\square$

λ is the order type of $\mathbb{R}$. ζ is the order type of Z.

Theorem 21.54. *If A is isomorphic to an initial segment of B and B is isomorphic to a terminal segement of A, then $A \cong B$.*

Proof. Let f be an isomorphism of A onto an initial segment of B, say with $B = f[A] + E$, and let g be an isomorphism of B onto a terminal segment of A, say with $A = F + g[B]$. Then $B = f[F] + f[g[B]] + E$. Let $h = f \circ g$. Thus $B = f[F] + h[B] + E$. Hence

(1) $h^n[B] = h^n[f[F]] + h^{n+1}[B] + h^n[E]$.

(2) $B = f[F] + h[f[F]] + h^2[f[F]] + \cdots + G$ for some G.

Let $\alpha = \text{o.t.}(A)$, $\beta = \text{o.t.}(B)$, $\sigma = \text{o.t.}(F)$, $\tau = \text{o.t.}(G)$. Then $\alpha = \sigma + \beta$ and $\beta = \sigma \cdot \omega + \tau$, so $\alpha = \beta$. $\square$

Theorem 21.55. *There are exactly continuum many countable order types of linear orders.*

Proof. For each $n \in \omega$ and each $X \subseteq \omega$ let $A_n(X)$ be the order type of $\eta+2$ if $n \in X$, and the order type of of $\eta + 3$ if $n \notin X$. Then let $L(X) = \sum_{n\in\omega} A_n(X)$. We claim that L is one-one. Suppose not. So there are distinct $X, Y \subseteq \omega$ such that $L(X) = L(Y)$. Let $n \in X\triangle Y$ be minimum. Say $n \in X\backslash Y$. Let f be an isomorphism of $L(X)$ onto $L(Y)$.

(1) $\forall m < n[f[\sum_{p<m} A_p(X)] = \sum_{p<m} A_p(Y)]$.

We prove this by induction. Suppose that $m < n$ and $f[A_p(X)] = A_p(Y)$ for all $p < m$. Then $f[A_m(X)]$ must equal $A_(Y)$. Thus (1) holds. Then $f[A_n]$ must equal the $\eta + 2$ portion of A_Y; so the last element of $\eta+3$ is outside of the range of f, contradiction. $\square$

Theorem 21.56. *Every countable linear order is isomorphic to a suborder of* $\mathbb{Q}$.

Proof. Let L be any countable linear order. We may assume that L is infinite. Say $L = \{a_i : i < \omega\}$, We define $(b_i, c_i) : i < \omega\}$ by recursion. Let $b_0 = a_0$ and let c_0 be any rational number. Now suppose that b_i and c_i have been defined for all $i < j$ so that

(1) $\forall i < j[b_i \in L$ and $c_i \in \mathbb{Q}]$.

(2) $\forall i, k < j[b_i < b_k$ iff $c_i < c_k]$.

(3) $\langle b_i : i < j\rangle$ is one-one and $\langle c_i : i < j\rangle$ is one-one.

Then we let $b_j = a_k$ with k minimum such that $a_k \notin \{b_l : l < j\}$. To define c_j we consider several cases. Suppose that $b_{s(0)} < \cdots < b_{s(j-1)}$ with each $s(l) < j$.

Case 1. $b_j < b_{s(0)}$. Let $c_j < c_{s(0)}$,

Case 2. $b_{s(j-1)} < b_j$. Let c_j be such that $c_{s(j-1)} < c_j$.

Case 3, $\exists k[0 < k < j - 1$ and $b_{s(k)} < b_j < c_{b(k+1)})$. Let c_j be such that $c_{s(k)} < c_j < c_{s(k+1)}$.

In fact, suppose that $i < \omega$ and $\forall j < \omega[a_i \neq b_j]$. Let k be the least such i. For each $s < k$ choose t_s such that $a_s = b_{t_s}$. Let w be greater than each t_s. Then $b_w = a_k$, contradiction. $\square$

Lemma 21.57. *If L and M are non-isomorphic countable subsets of $\mathbb{R}$, then $\mathbb{R}\backslash L$ and $\mathbb{R}\backslash M$ are dense and non-isomorphic.*

Proof. To show that $\mathbb{R}\backslash L$ is dense, suppose that $a, b \in \mathbb{R}\backslash L$ and $a < b$. Since (a, b) is uncountable, there is a c such that $a < c < b$. Thus $\mathbb{R}\backslash L$ is dense. Similarly, $\mathbb{R}\backslash M$ is dense.

Now suppose that $f : \mathbb{R}\backslash L \to \mathbb{R}\backslash M$ is an isomorphism. For each $a \in L$ let $\downarrow a = \{x \in \mathbb{R}\backslash L : x < a\}$.

(1) If $a, b \in \mathbb{R}$ and $a < b$, then $(a, b) \cap (\mathbb{R}\backslash L)$ is uncountable.

(2) If $a, b \in \mathbb{R}$ and $a < b$, then $\downarrow a \subset \downarrow b$.

Now $\downarrow a$ is bounded by a. Hence $f[\downarrow a]$ is bounded.

(3) The least upper bound of $f[\downarrow a]$ is in M.

In fact, $\downarrow a$ does not have a lub in $\mathbb{R}\backslash L$, so $f[\mathbb{R}\backslash L]$ does not have a lub in $\mathbb{R}\backslash M$. Hence (3) holds.

Define $g : L \to M$ by $g(a) = lub(f[\downarrow a])$. Clear $a < b$ implies that $g(a) < g(b)$. If $b \in M$, choose $a \in L$ such that $f(a) = b$. If $x < a$, then $f(x) \in f[\downarrow a]$, and so $f(x) \leq g(a)$. Thus g is an isomorphism of L onto M, contradicion. $\square$

Corollary 21.58. *There are exactly continuum many isomorphism types of dense linear orders of subsets of* $\mathbb{R}$.

Proposition 21.59. *If L is a doubly transitive linear order, then L is dense.*

Proof. Assume not. Then there are elements $a, b \in L$ such that b is the immediate successor of a. Since L is transitive, let f be an automorphism of L such that $f(a) = b$. Then let $c = f(b)$. Since $\neg\exists d[a < d < b]$, it follows that $\neg\exists d[f(a) < d < f(b)]$, i.e., $\neg\exists d[b < d < c]$. Suppose that g is an automorphism of L such that $g(a) = a$ and $g(b) = c$. Then $\neg\exists d[a < d < b]$ but $a = g(a) < b < g(b)$, contradiction. $\square$

Theorem 21.60. *The following are equivalent:*

(i) L is doubly transitive;

(ii) $\forall k \geq 2[L$ is k-tuply transitive$]$;

(iii) $\exists k \geq 2[L$ is k-tuply transitive$]$.

Proof. (i)⇒(ii): Assume (i) and assume that $a_0 < a_1 < \cdots a_{k-1}$ and $b_0 < b_1 < \cdots b_{k-1}$ with $k \geq 2$, By double transitivity let $f_0, \ldots, f_{k-2}$ be such that $f_0(a_0) = b_0$, $f_0(a_1) = b_1$, ..., $f_{k-2}(a_{k-2}) = b_{k-2}$, $f_{k-2}(a_{k-1}) = b_{k-1}$. Then define $g : L \to L$ by

$$g(x) = \begin{cases} f_0(x) & \text{if } x < a_1; \\ f_1(x) & \text{if } a_1 \leq x < a_2; \\ . \\ . \\ . \\ f_{k-3}(x) & \text{if } a_{k-3} \leq x < a_{k-2}; \\ f_{k-2}(x) & \text{if } a_{k-2} \leq x. \end{cases}$$

Clearly g is as desired.

(ii)⇒(iii): clear.

(iii)⇒(i): clear. $\square$

Proposition 21.61. *If, $(I, \leq)$ is scattered and $\forall i \in I[L_i$ is scattered$]$ then $\sum_{i\in I} L_i$ is scattered.*

Proof. Assume that $(I, \leq)$ is scattered and $\forall i \in I[L_i$ is scattered. Suppose that $D \subseteq \sum_{i\in I} L_i$ is dense. For each $i \in I$ let $D_i = D \cap A_i$. Since D_i is an interval of D, D_i is dense. But it is also an interval of A_i, so $D_i \leq 1$. Let $I_0 = \{i \in I : D_i = 1\}$. Then I_0 is isomorphic to D and is a subordering of I. Since I is scattered, $|D| = 1$. Hence $\sum_{i \in I} L_i$ is scattered. □

Corollary 21.62. *If L and M are scattered, then so are $L + M$ and $L \cdot M$.* □

Proposition 21.63. *The following are equivalent:*

(i) L is complete;

(ii) There do not exist nonempty $X, Y \subseteq L$ such that $L = X \cup Y$, $X < Y$, X has no greatest element, Y has no least element and there is no a such that $X < a < Y$;

(iii) Every non empty convex subset of L has one of the following forms:

(a) L;
(b) (a, b);
(c) $(a, b]$;
(d) $[a, b)$;
(e) $[a, b]$;
(f) $(-\infty, b)$;
(g) $(-\infty, b]$;
(h) (a, ∞);
(i) $[a, \infty)$.

Proof. (i)⇒(ii): Assume (i) but suppose that X, Y are as in (ii). Since $X < \sum X < Y$, this is a contradiction.

(ii)⇒(iii): Assume (ii). Let X be a nonempty convex subset of L.

Case 1. X has a smallest element a and a largest element b. Suppose that there are $\emptyset \neq Y, Z \subseteq X$ with $X = Y \cup Z$ and $Y < Z$. Let $Y' = Y \cup (-\infty, a)$ and $Z' = Z \cup (b, \infty)$. By (ii), there is a c such that $Y' < c < Z'$. Since X is convex, it follows that $c \in X$. Hence $X = [a, b]$. Thus (e) holds.

Case 2. X has a smallest element a, does not have a largest element, and there is a y with $X < y$. Similarly to Case 1, $\{y : X < y\}$ has a smallest element b, and $X = [a, b)$. So (d) holds.

Case 3. X has a smallest element a, does not have a largest element, and there is no y with $X < y$. Then $X = [a, \infty)$ and (i) holds

Case 4. X has no smallest element, has a largest element b, and there is an $x < X$. By (ii), $\{x : x < X\}$ has a largest element a. Then by an argument similar to that in Case 1, we have $X = (a, b]$. So (c) holds.

Case 5. X has no smallest element, has a largest element b, and there is no $x < X$. Then by an argument similar to that in Case 1, we have $X = (-\infty, b]$. So (g) holds.

Case 6. X has no smallest element and no largest element, and there exist x, y with $x < X < y$. Then $\{x : x < X\}$ has a largest element a and $\{y : X < y\}$ has a smallest element b. Then $X = (a, b)$, and (b) holds.

Case 7. X has no smallest element and no largest element, there exists $x < X$, but there is no y such that $X < y$. Then $\{x : x < X\}$ has a largest element a, so $X = (a, \infty)$.

Case 8. X has no smallest element and no largest element, there is no $x < X$, but there is a y such that $X < y$. Then $\{y : X < y\}$ has a smallest b, and $X = (-\infty, b)$.

Case 9. There is no $x < X$, and there is no y with $X < y$. Then $X = L$.

(iii)⇒(i): Assume (iii), and suppose that $X \subseteq L$, X nonempty, X bounded above by b. Let $X' = \{x \in L : \exists y \in X[x \le y]\}$. Clearly X' is convex. Clearly in each of the cases (a)-(i) a lub of X' exists and this is a lub of X. □

If L is a linear order and L' is a partition of L consisting of non-empty convex sets, define $\ll$ on L' by

$$I_1 \ll I_2 \quad \text{iff} \quad \forall x \in I_1 \forall y \in I_2 [x < y].$$

Then $(I', \ll)$ is a *condensation* of I.

A *homomorphism* from a linear order L to a linear order M is a mapping $f : L \to M$ such that $\forall a, b \in L[a < b \to f(a) \le f(b)]$.

If L is a linear order, then we define $\forall x \in L[c_F^L(x) = \{y \in L : x < y$ and $[x, y]$ is finite, or $y < x$ and $[y, x]$ is finite$\}]$, Clearly each $c^L(x)$ is nonempty and convex.

Proposition 21.64. *$\{c_F^L(x) : x \in L\}$ is a partition of L.* □

We denote the partition $\{c_F^L(x) : x \in L\}$ by L_F.

Theorem 21.65. *If L is a countably infinite linear ordering, then there is an order-preserving map of L onto a proper subset of L.*

Proof. *Case 1.* There is an $x \in L$ such that $c_F^L(x)$ has order type ω. We may assume that x is the smallest element of $c_F^L(x)$. We define

$$f(y) = \begin{cases} y & \text{if } y \notin c_F^L(x); \\ \text{the immediate successor of } x & \text{otherwise.} \end{cases}$$

Clearly f is as desired.

Case 2. There is an $x \in L$ such that $c_F^L(x)$ has order type ω^*. This is similar to Case 1.

Case 3. There is an $x \in L$ such that $c_F^L(x)$ has order type $\omega^* + \omega$. Write $c_F^L(x) = \{x_i : i \in \omega^* + \omega\}$ with $\cdots x_{-2} < x_{-1} < x_0 < x_1 \cdots$. Define

$$f(y) = \begin{cases} y & \text{if } y \notin c_F^L(x); \\ x_{-i} & \text{if } y = x_{-i}; \\ x_{i+1} & \text{if } y = x_i. \end{cases}$$

Again it is clear that f works.

Case 4. $\forall x \in L[c_F^L(x)$ is finite$]$.

(1) $(L', \ll)$ is dense.

For, suppose that $L_F^L(x) \ll L_F^L(y)$ and there is no z such that $L_F^L(x) \ll L_F^L(z) \ll L_F^L(y)$. Then $x < y$ and $[x, y]$ is finite, or $y < x$ and $[y, x]$ is finite. Hence $L_F^L(x) = L_F^L(y)$, contradiction.

Now clearly there is no x such that $L_F^L(x) = L$. Now let L'' have exactly one element in each $L_F^L(x)$. Then L'' has order type η. Take any $y \in L''$. Then $L''\backslash\{y\}$ has order type η. There is an isomorphism of L onto $L''\backslash\{y\}$, as desired. □

We let $c^L_W(x) = \{y : x \le y$ and $[x, y]$ is well-ordered, or $y < x$ and $[y, x]$ is well-ordered$\}$. Clearly each $c^L_W(x)$ is convex, and $\{c^L_W(x) : x \in L\}$ is a partition of L.

Proposition 21.66. *If L^* is not well-ordered but $\forall x \in L^*[\{y : x \le y\}$ is well-ordered$]$, then there exist well-orderings $L_0, L_1, \ldots$ such that*

$$L^* = \cdots + L_3 + L_2 + L_1 + L_0.$$

Proof. Assume the hypothesis. Say $\cdots < x_3 < x_2 < x_1 < x_0$.

(1) $\forall x \in L^* \exists i \in \omega[x_i \le x]$

For, otherwise there is an $x \in L^*$ such that $\forall i \in \omega[x < x_i]$ and so $\{y : x \le y\}$ is not well-ordered, contradiction.

Hence it follows that

$$L^* = \cdots [x_{n+1}, x_n] + \cdots [x_1, x_0] + [x_0, \infty),$$

as desired. □

Proposition 21.67. *If $L_0, L_1, \ldots$ are well-orderings and*

$$L^* = \cdots + L_3 + L_2 + L_1 + L_0,$$

then for every $x \in L^*[\{y : x \le y\}$ is a well-ordering$]$. □

Theorem 21.68. *Suppose that L is a countable linear ordering, and for every $f : L \to L$, if $\forall x, y \in L[x < y \to f(x) < f(y)]$, then $\forall x \in L[x \le f(x)]$. Then L is a well-ordering.*

Proof. Assume the hypotheses.

(1) There is no $L' \subseteq L$ which is a dense ordering.

In fact, assume otherwise. Take $x < y$ in L' with $x < y$. By the proof of Theorem 3, there is an isomorphic embedding $f : L \to L'$ such that $f(y) = x$. Since $x < y$, we have $f(y) < y$, contradicting the hypothesis. Thus (1) holds.

(2) $\forall x \in L[c_W(x)$ is well-ordered$]$.

For, suppose that $x \in L$ and $c_W(x)$ is not well-ordered. Then by Proposition 13 there are well-orderings $L_0, L_1, \ldots$ such that

$$c_W(x) = \cdots + L_3 + L_2 + L_1 + L_0.$$

For each $i < \omega$ let α_i be the order type of L_i; so α_i is an ordinal. For each $i \in \omega$ let $A_i = \{j \ge i : \alpha_i \le \alpha_j\}$.

If A_i is finite, then there is a $j(i)$ such that $\forall j > j(i)[j \notin A_i]$ hence $\forall j > j(i)[\alpha_j < \alpha_i]$.

(3) $\exists i_0 \forall i \ge i_0[A_i$ is infinite$]$.

In fact, otherwise there is a strictly increasing sequence $i_0, i_1, \ldots$ such that $\forall k \in \omega[A_{i_k}$ is finite and $j(k) < i_{k+1}$, i.e. $\forall k \in \omega \forall j > j(k)[\alpha_j < \alpha_{i_k}]$ and $i_{k+1} > j(k)]$. Thus $\alpha_{i_0} > \alpha_{i_1} > \alpha_{i_2} \cdots$, contradiction. So (3) holds.

Thus for all $i \geq i_0$ there are infinitely many $j \geq i$ such that $\alpha_i \leq \alpha_j$. We now define $k : \{i \in \omega : i \geq i_0\} \to \{i \in \omega : i \geq i_0\}$ by

$$k(i) = \begin{cases} \text{least } j > i_0[\alpha_j \geq \alpha_{i_0}] & \text{if } i = i_0; \\ \text{least } j > k(n)[\alpha_j \geq \alpha_{n+1}] & \text{if } i = n+1 \text{ with } n \geq i_0 \end{cases}.$$

Now we define $f : L \to L$ by

$$f(y) = \begin{cases} y & \text{if } y \notin c_W(x); \\ y & \text{if } y \in L_{i_0-1} + L_{i_0-2} + \cdots + L_0; \\ \text{the } \gamma\text{-th member of } L_{k(n)} & \text{if } i_0 \leq n \text{ and } \gamma < \alpha_n \\ & \text{and } y \text{ is the } \gamma\text{-th member of } L_n. \end{cases}$$

Clearly $y < z \to f(y) < f(z)$. Now $k(i_0) > i_0$ and $\alpha_{k(i_0)} \geq \alpha_{i_0}$. Hence $k(k(i_0)+1) > k(i_0)$ and $\alpha_{k(k(i_0)+1)} \geq \alpha_{k(i_0)+1}$. If y is the 0-th member of L_{i_0}, then $f(y)$ is the 0-th member of $L_{k(i_0)}$. Thus $f(y) < y$, contradiction. This proves (2).

(4) If $c_W(x) \ll c_W(y)$, then there is a z such that $c_W(x) \ll c_W(z) \ll c_W(y)$.

In fact, suppose not. Suppose that $x \leq u \leq y$. If $u \in c_W(x)$ then $[x, u]$ is well-ordered, hence $u \in c_W(x)$. If $u \in c_W(y)$, then $[u, y]$ is well-ordered, hence $u \in c_W(y)$. So (4) holds.

By (4), choosing one element from each $c_W(x)$ we get a dense order $L \subseteq L$. This contradicts (1). It follows that L is a well-order. $\square$

$$\begin{aligned} aE_0^L b &\text{ iff } a, b \in L \text{ and } a = b; \\ aE_{\alpha+1}^L b &\text{ iff } ([a]_\alpha E_\alpha^L [b]_\alpha \text{ and } ([a]_\alpha, [b]_\alpha)_{E_\alpha^L} \text{ is finite}) \\ &\quad \text{ or } ([b]_\alpha E_\alpha^L [a]_\alpha \text{ and } ([b]_\alpha, [a]_\alpha)_{E_\alpha^L} \text{ is finite}) \\ aE_\lambda^L b &\text{ iff } \exists \alpha < \lambda[aE_\alpha^L b] \text{ for } \lambda \text{ limit.} \end{aligned}$$

Proposition 21.69. *For L countable there is an $\alpha < \omega_1$ such that $E_\alpha^L = E_{\alpha+1}^L$.*

Proof. Clearly $\forall \alpha, \beta < \omega_1[\alpha < \beta \to E_\alpha^L \subseteq E_\beta^L]$ and $\forall \alpha < \omega_1[E_\alpha^L \subseteq L \times L]$. The proposition follows. $\square$

For L a countable linear order, $\lambda(L)$ is the least ordinal $\alpha < \omega_1$ such that $E_\alpha^L = E_{\alpha+1}^L$.

Proposition 21.70. *If L is a countable scattered linear order, then $|E_{\lambda(L)}^L| = 1$.*

Proof. Let $\alpha = \lambda(L)$, and suppose that $|E_\alpha^L| > 1$. Suppose that $aE_\alpha b$ with $a \neq b$. Then $aE_{\alpha+1}^L b$; say $[a]_\alpha E_\alpha^L [b]_\alpha$ with $([a]_\alpha, [b]_\alpha)_{E_\alpha^L}$ finite. Thus E_α^L is dense, contradiction. $\square$

Proposition 21.71. *If $M \subseteq L$ is convex, then $\forall a, b \in M \forall \alpha < \omega_1[aE_\alpha^M b \leftrightarrow aE_\alpha^L b]$.*

Proof. Induction on α. $\square$

Proposition 21.72. *If M is an equivalence class of E^L_α, then $\lambda(M) \le \alpha$.*

Proof. Assume that M is an equivalence class of E^L_α. Then M is convex, so Proposition 21.71 applies, and it follows that $|M| = 1$. Hence $\lambda(M) \le \alpha$. $\square$

Now let $\mathscr{L}$ be the intersection of all K such that each countable well-ordering is in K, K is closed under isomorphisms and sums over countable well-orderings and inverse well orderings, and $\forall L \in K \forall A[\{x \in L : x \in A\} \in K]$,

Proposition 21.73. *Let L be a linear ordering, and define*

$$\begin{aligned} E = \{(a,b) \in L \times L : & a = b \text{ or } (a < b \text{ and } (a,b) \text{ is finite}) \\ & \text{or } (b < a \text{ and } (b,a) \text{ is finite})\} \end{aligned}$$

Then E is an equivalence relation on L, and if L is a single equivalence class under E^L, then $L \in \mathscr{L}$.

Proof. *Case 1.* L does not have a first element. Then we can write

$$L = \cdots \cup [a_{n+1}, a_n) \cup [a_n, a_{n-1}) \cup \cdots \cup [a_1, a_0)$$

or

$$L = \cdots \cup [a_{n+1}, a_n) \cup [a_n, a_{n-1}) \cup \cdots \cup [a_1, a_0) \cup [a_0, a_{-1}) \cup \ldots$$

with each $[a_{i+1}, a_i)$ finite. Clearly $L \in \mathscr{L}$.

Case 2. L has a first element. Then we can write

$$L = [a_0, a_1) \cup [a_1, a_2) \cup \cdots$$

with each $[a_i, a_{i+1})$ finite. Clearly $L \in \mathscr{L}$. $\square$

Proposition 21.74. *$\mathscr{L}$ is closed under lexicographic sums with index set in $\mathscr{L}$.*

Proof. Let $\mathscr{B}L$ consist of all finite linear orders along with all countable well-orders and inverses of countable well-orders. Define L^α for $\alpha < \omega_1$ by recursion:

$$\begin{aligned} \mathscr{L}^0 &= \mathscr{B}L; \\ \mathscr{L}^{\alpha+1} &= \left\{ L : L = \sum_{a \in J} I_a : I_a \in \mathscr{L}^\alpha, J \in \mathscr{B}L \right\} \\ &\quad \cup \{L : \exists M \in \mathscr{L}^\alpha : L \cong M\}; \\ \mathscr{L}^\lambda &= \bigcup_{\alpha < \lambda} \mathscr{L}^\alpha \text{ for } \lambda \text{ limit.} \end{aligned}$$

Now we show by induction on α that $\mathscr{L}$ is closed under lexicographic products with index set in $\mathscr{L}^{\alpha+1}$; clearly this will prove the proposition. This is obvious for $\alpha = 0$. Now suppose that $M_a \in \mathscr{L}$ for all $a \in L$, with $L \in \mathscr{L}^{\alpha+1}$. Say $L = \sum_{b\in L'} L_b$ with each L_b in $\mathscr{L}^\alpha$ and $L' \in \mathscr{B}L$. For each $b \in L'$ let $I_b = \{(b, x) : x \in L_b\}$. Then I_b is a convex subset of L which is isomorphic to L_b. Hence $I_b \in \mathscr{L}^\alpha$.

Now for each $b \in L'$ let $N_b = \sum_{a\in I_b} M_a$. Then $N_b \in \mathscr{L}$ by the inductive hypothesis. Clearly $\sum_{a\in L} M_a$ is isomorphic to $\sum_{b\in L'} N_b$, so $\sum_{a\in L} M_a \in \mathscr{L}$. □

The following is a theorem of Hausdorff; we follow a paper of Abraham, Bonnet, Cummings, Džamonja, Thompson.

Theorem 21.75. *For any countable ordered set L the following are equivalent:*
(i) L is scattered.
(ii) $L \in \mathscr{L}$.
(iii) $|E_{\lambda(L)}| = 1$.

Proof. Clearly (ii) implies (i), and (i) implies (iii) by Proposition 21.70. Now we prove by induction on $\lambda(L)$ that (iii) implies (ii). If $\lambda(L) = 0$, then $|L| = 1$ and so $L \in \mathscr{L}$. Next, suppose that $\lambda(L) = \alpha + 1$. If M is an equivalence class of E^L_α, then $\lambda(M) = 1$ by Proposition 21.72, so $M \in \mathscr{L}$ by the inductive hypothesis. Since E_α is a single equivalence class under $E^L_{\alpha+1}$, we have $E_\alpha \in \mathscr{L}$. Then by Proposition 21.74, $L \in \mathscr{L}$.

Now we take the case $\lambda(L)$ limit. Let $\lambda = \lambda(L)$. Fix $a \in L$, and for each $\gamma < \omega_1$ let A_γ be the E^L_γ-class of a. Note that A_γ is convex, A_γ increases with γ, and $L = \bigcup_{\gamma<\omega_1} A_\gamma$. By Proposition 21.72, $\lambda(A_\gamma) \le \gamma$, so by the inductive hypothesis, $A_\gamma \in \mathscr{L}$. We now define

$$L_\gamma = \{b < a : b \in A_{\gamma+1}\backslash A_\gamma\};$$
$$R_\gamma = \{b > a : b \in A_{\gamma+1}\backslash A_\gamma\}.$$

Then by definition of $\mathscr{L}$, $L_\gamma, R_\gamma \in \mathscr{L}$. Now if $\gamma < \delta < \omega_1$. then $L_\delta < L_\gamma < \{a\} < R_\gamma < R_\delta$. Hence L is the lexicographic sum with index $\omega_1^* + 1 + \omega_1$ of members of $\mathscr{L}$, so $L \in \mathscr{L}$. □

Proposition 21.76. *Let L be a linear order. Then there exist ordinals γ and δ such that L is the sum of points and bounded intervals indexed by $\gamma^* + \delta$.*

Proof. Let L be a linear order.

Case 1. L does not have a first or last element. Let γ be the coinitiality of L and δ the cofinality of L. Say that $a \in {}^\gamma L$ is strictly decreasing and coinitial in L, and $b \in {}^\delta L$ is strictly increasing and cofinal in L with $a_0 < b_0$ and $(a_0, b_0) \neq \emptyset$, and define, for $\xi < \gamma$ and $\eta < \delta$,

$$c_\xi = (a_{\xi+1}, a_\xi]$$
$$d_\eta = [b_\eta, b_{\eta+1}).$$

Then L is the sum of the cs and $\{(a_0, b_0)\}$ and ds.

Case 2. L has a first, but no last, element. This is similar to Case 1, with $\gamma = 0$.

Case 3. L has a last, but no first, element. This is similar to Case 1, with $\delta = 0$.

Case 4. L has a first and last element. We can take $\gamma = 0$ and $\delta = 1$. □

22. Trees

In this chapter we study infinite trees. The main things we look at are König's tree theorem, Aronszajn trees, Suslin trees, and Kurepa trees.

A *tree* is a partially ordered set $(T, <)$ such that for each $t \in T$, the set $\{s \in T : s < t\}$ is well-ordered by the relation $<$. Thus every ordinal is a tree, but that is not so interesting in the present context. We introduce some standard terminology concerning trees.

• $(t\downarrow) = \{s \in T : s < t\}$; $(t\uparrow) = \{s \in T : t < s\}$.

• For each $t \in T$, the order type of $\{s \in T : s < t\}$ is called the *height* of t, and is denoted by $\mathrm{ht}(t, T)$ or simply $\mathrm{ht}(t)$ if T is understood.

• A *root* of a tree T is an element of T of height 0, i.e., it is an element of T with no elements of T below it. Frequently we will assume that there is only one root.

• For each ordinal α, the α-th *level* of T, denoted by ,$\mathrm{Lev}_\alpha(T)$ is the set of all elements of T of height α.

• The *height* of T itself is the least ordinal greater than the height of each element of T; it is denoted by $\mathrm{ht}(T)$.

• A *chain* in T is a subset of T linearly ordered by $<$.

• A *branch* of T is a maximal chain of T.

• For each $\alpha \leq \mathrm{ht}(T)$ let $T_\alpha = \bigcup_{\beta<\alpha} \mathrm{Lev}_\beta(T)$.

Note that chains and branches of T are actually well-ordered, and so we may talk about their *lengths.*

Some further terminology concerning trees will be introduced later. A typical tree is ${}^{<\omega}2$, which is by definition the set of all finite sequences of 0s and 1s, with $\subset$ as the partial order. More generally, one can consider ${}^{<\alpha}2$ for any ordinal α.

Theorem 22.1. (König) *Every tree of height ω in which every level is finite has an infinite branch.*

Proof. Let T be a tree of height ω in which every level is finite. We define a sequence $\langle t_m : m \in \omega\rangle$ of elements of T by recursion. Clearly $T = \bigcup_{r \text{ a root}}\{s \in T : r \leq s\}$, and the index set is finite, so we can choose a root t_0 such that $\{s \in T : t_0 \leq s\}$ is infinite. Suppose now that we have defined an element t_m of height m such that $\{s \in T : t_m \leq s\}$ is infinite. Let $S = \{u \in T : t_m < u$ and u has height $\mathrm{ht}(t_m) + 1\}$. Clearly

$$\{s \in T : t_m \leq s\} = \{t_m\} \cup \bigcup_{u \in S}\{s \in T : u \leq s\}$$

and the index set of the big union is finite, so we can choose t_{m+1} of height $\mathrm{ht}(t_m) + 1$ such that $\{s \in T : t_{m+1} \leq s\}$ is infinite.

This finishes the construction. Clearly $\{t_m : m \in \omega\}$ is an infinite branch of T. $\square$

In attempting to generalize König's theorem, one is naturally led to Aronszajn trees and Suslin trees. For the following definitions, let κ be any infinite cardinal.

• A tree $(T, <)$ is a *κ-tree* iff it has height κ and every level has size less than κ.

• A *κ-Aronszajn tree* is a κ-tree which has no chain of size κ.

• A subset X of a tree T is an *antichain* iff any two distinct members of X are incomparable. Note that each set $\mathrm{Lev}_\alpha(T)$ is an antichain. This notion is different from antichains as introduced in Chapters 21.

• A *κ-Suslin tree* is a tree of height κ which has no chains or antichains of size κ.

• An *Aronszajn tree* is an ω_1-Aronszajn tree, and a *Suslin tree* is an ω_1-Suslin tree.

It is natural to guess that Aronszajn trees and Suslin trees are the same thing, since the definition of κ-tree implies that all levels have size less than κ, and a guess is that this implies that all antichains are of size less than κ. This guess is not right though. Even our simplest example of a tree, ${}^{<\omega}2$, forms a counterexample. This tree has all levels finite, but it has infinite antichains, for example

$$\{\langle 0\rangle, \langle 1,0\rangle, \langle 1,1,0\rangle, \langle 1,1,1,0\rangle, \ldots\}.$$

In the rest of this chapter we investigate these notions, and state some consistency results, some of which will be proved later. There is also one difficult natural open problem which we will formulate.

First we consider Aronszajn trees. Note that Theorem 22.1 can be rephrased as saying that there does not exist an ω-Aronszajn tree. As far as existence of Aronszajn trees is concerned, the following theorem takes care of the case of singular κ:

Theorem 22.2. *If κ is singular, then there is a κ-Aronszajn tree.*

Proof. Let $\langle \lambda_\alpha : \alpha < \mathrm{cf}(\kappa)\rangle$ be a strictly increasing sequence of infinite cardinals with supremum κ. Consider the tree which has a single root, and above the root has disjoint chains which are copies of the λ_α's. Clearly this tree is a κ-Aronszajn tree. We picture this tree on the next page. Very rigorously, we could define T to be the set $\{0\} \cup \{(\alpha, \beta) : \alpha < \mathrm{cf}(\kappa) \text{ and } \beta < \lambda_\alpha\}$, with the ordering $0 < (\alpha, \beta)$ for all $\alpha < \mathrm{cf}(\kappa)$ and $\beta < \lambda_\alpha$, and $(\alpha, \beta) < (\alpha', \beta')$ iff $\alpha = \alpha'$ and $\beta < \beta'$. □

Turning to regular κ, we first prove

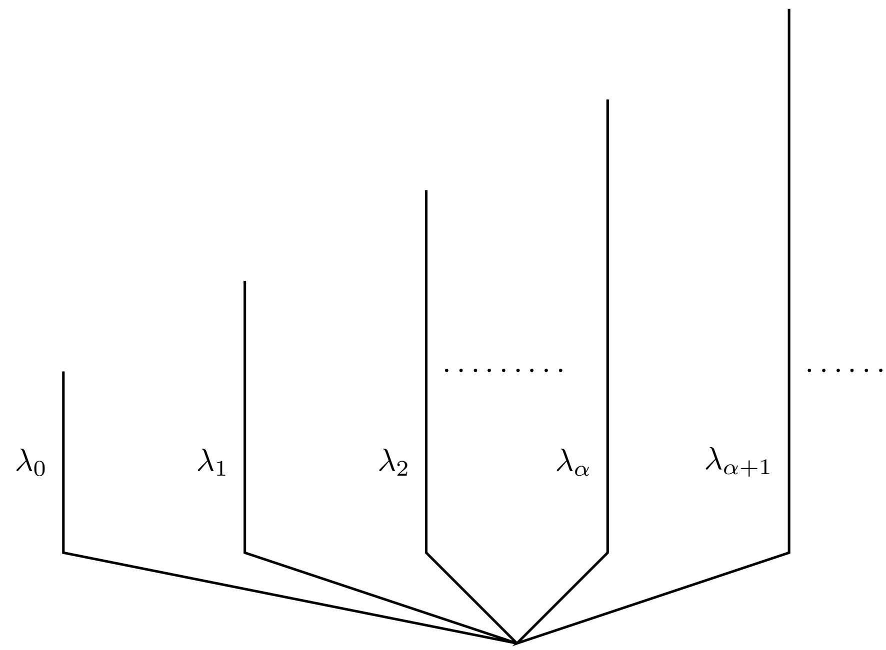
Theorem 22.3. *There is an Aronszajn tree.*

Proof. We start with the tree

$$T = \{s \in {}^{<\omega_1}\omega : s \text{ is one-one}\}.$$

under $\subset$. This tree clearly does not have a chain of size ω_1. But all of its infinite levels are uncountable, so it is not an ω_1-Aronszajn tree. We will define a subset of it that is the desired tree. We define a system $\langle S_\alpha : \alpha < \omega_1\rangle$ of subsets of T by recursion; these will be the levels in the new tree.

Let $S_0 = \{\emptyset\}$. Now suppose that $\alpha > 0$ and S_β has been constructed for all $\beta < \alpha$ so that the following conditions hold for all $\beta < \alpha$:



(1_β) $S_\beta \subseteq {}^\beta\omega \cap T$.

(2_β) $\omega\backslash\mathrm{rng}(s)$ is infinite, for every $s \in S_\beta$.

(3_β) For all $\gamma < \beta$, if $s \in S_\gamma$, then there is a $t \in S_\beta$ such that $s \subset t$.

(4_β) $|S_\beta| \leq \omega$.

(5_β) If $s \in S_\beta$, $t \in T$, and $\{\gamma < \beta : s(\gamma) \neq t(\gamma)\}$ is finite, then $t \in S_\beta$.

(6_β) If $s \in S_\beta$ and $\gamma < \beta$, then $s \restriction \gamma \in S_\gamma$.

(Vacuously these conditions hold for all $\beta < 0$.) If α is a successor ordinal $\varepsilon + 1$, we simply take

$$S_\alpha = \{s \cup \{(\varepsilon, n)\} : s \in S_\varepsilon \text{ and } n \notin \mathrm{rng}(s)\}.$$

Clearly (1_β)–(6_β) hold for all $\beta < \alpha + 1$.

Now suppose that α is a limit ordinal less than ω_1 and (1_β)–(6_β) hold for all $\beta < \alpha$. Since α is a countable limit ordinal, it follows that $\mathrm{cf}(\alpha) = \omega$. Let $\langle\delta_n : n \in \omega\rangle$ be a strictly increasing sequence of ordinals with supremum α. Now let $U = \bigcup_{\beta<\alpha} S_\beta$. Take any $s \in U$; we want to define an element $t_s \in {}^\alpha\omega \cap T$ which extends s. Let $\beta = \mathrm{dmn}(s)$.

Choose n minimum such that $\beta \leq \delta_n$. Now we define a sequence $\langle u_i : i \in \omega\rangle$ of members of U; u_i will be a member of $S_{\delta_{n+i}}$. By (3_{δ_n}), let u_0 be a member of S_{δ_n} such that $s \subseteq u_0$. Having defined a member u_i of $S_{\delta_{n+i}}$, use $(3_{\delta_{n+i+1}})$ to get a member u_{i+1} of $S_{\delta_{n+i+1}}$ such that $u_i \subseteq u_{i+1}$. This finishes the construction. Let $v = \bigcup_{i\in\omega} u_i$. Thus

$s \subseteq v \in {}^\alpha\omega \cap T$. Unfortunately, condition (2) may not hold for v, so this is not quite the element t_s that we are after. We define $t_s \in {}^\alpha\omega$ as follows. Let $\gamma < \alpha$. Then

$$t_s(\gamma) = \begin{cases} v(\delta_{2n+2i}) & \text{if } \gamma = \delta_{n+i} \text{ for some } i \in \omega, \\ v(\gamma) & \text{if } \gamma \notin \{\delta_{n+i} : i \in \omega\}. \end{cases}$$

Clearly $t_s \in {}^\alpha\omega \cap T$. Since $v(\delta_{2n+2i+1}) \notin \text{rng}(t_s)$ for all $i \in \omega$, it follows that $\omega\backslash\text{rng}(t_s)$ is infinite.

We now define

$$S_\alpha = \bigcup_{s\in U} \{w \in {}^\alpha\omega \cap T : \{\varepsilon < \alpha : w(\varepsilon) \neq t_s(\varepsilon)\} \text{ is finite}\}.$$

Now we want to check that (1_α)–(6_α) hold. Conditions (1_α) and (3_α) are very clear. For (2_α), suppose that $w \in S_\alpha$. Then $w \in {}^\alpha\omega \cap T$ and there is an $s \in U$ such that $\{\varepsilon < \alpha : w(\varepsilon) \neq t_s(\varepsilon)\}$ is finite. Since $\omega\backslash\text{rng}(t_s)$ is infinite, clearly $\omega\backslash\text{rng}(w)$ is infinite. For (4_α), note that U is countable by the assumption that (4_β) holds for every $\beta < \alpha$, while for each $s \in U$ the set

$$\{w \in {}^\alpha\omega \cap T : \{\varepsilon < \alpha : w(\varepsilon) \neq t_s(\varepsilon)\} \text{ is finite}\}$$

is also countable. So (4_α) holds. For (5_α), suppose that $w \in S_\alpha$, $x \in T$, and $\{\gamma < \alpha : w(\gamma) \neq x(\gamma)\}$ is finite. Choose $s \in U$ such that $\{\varepsilon < \alpha : w(\varepsilon) \neq t_s(\varepsilon)\}$ is finite. Then of course also $\{\varepsilon < \alpha : x(\varepsilon) \neq t_s(\varepsilon)\}$ is finite. So $x \in S_\alpha$, and (5_α) holds. Finally, for (6_α), suppose that $w \in S_\alpha$ and $\gamma < \alpha$; we want to show that $w \restriction \gamma \in S_\gamma$. Choose $s \in U$ such that $\{\varepsilon < \alpha : w(\varepsilon) \neq t_s(\varepsilon)\}$ is finite. Assume the notation introduced above when defining t_s. Choose $i \in \omega$ such that $\gamma \leq \delta_{n+i}$. Then

$$\begin{aligned}\{\varepsilon < \delta_{n+i} : w(\varepsilon) \neq u_i(\varepsilon)\} &= \{\varepsilon < \delta_{n+i} : w(\varepsilon) \neq v(\varepsilon)\} \\ &\subseteq \{\varepsilon < \delta_{n+i} : w(\varepsilon) \neq t_s(\varepsilon)\} \cup \{\delta_{n+j} : j < i\},\end{aligned}$$

and the last union is clearly finite. It follows from $(5_{\delta_{n+1}})$ that $w \in S_\gamma$. So (6_α) holds.

This finishes the construction. Clearly $\bigcup_{\alpha<\omega_1} S_\alpha$ is the desired Aronszajn tree. $\square$

We defer a discussion of possible generalizations of Theorem 22.3 until we discuss the closely related notion of a Suslin tree.

The proof of Theorem 22.2 gives

Theorem 22.4. *If κ is singular, then there is a κ-Suslin tree.* $\square$

Note also that Theorem 22.1 implies that there are no ω-Suslin trees. There do not exist ZFC results about existence or non-existence of κ-Suslin trees for κ uncountable and regular. We limit ourselves at this point to some simple facts about Suslin trees.

Proposition 22.5. *If T is a κ-Suslin tree with κ uncountable and regular, then T is a κ-tree.* $\square$

Proposition 22.6. *For any infinite cardinal κ, every κ-Suslin tree is a κ-Aronszajn tree.* $\square$

This is a good place to notice that the construction of an ω_1-Aronszajn tree given in the proof of Theorem 22.3 does not give an ω_1-Suslin tree. In fact, assume the notation of that proof, and for each $n \in \omega$ let

$$A_n = \bigcup_{\alpha<\omega_1} \{s \in S_{\alpha+1} : s(\alpha) = n\}.$$

Clearly A_n is an antichain in $\bigcup_{\alpha<\omega_1} S_\alpha$, and $\bigcup_{n\in\omega} A_n = \bigcup_{\alpha<\omega_1} S_{\alpha+1}$. Hence $\left|\bigcup_{n\in\omega} A_n\right| = \omega_1$. It follows that some A_n is uncountable, so that $\bigcup_{\alpha<\omega_1} S_\alpha$ is not a Suslin tree.

We now introduce some notions that are useful in talking about κ-trees; these conditions were implicit in part of the proof of Theorem 22.3.

- A *well-pruned* κ-tree is a κ-tree T with exactly one root such that for all $\alpha < \beta < \mathrm{ht}(T)$ and for all $x \in \mathrm{Lev}_\alpha(T)$ there is a $y \in \mathrm{Lev}_\beta(T)$ such that $x < y$.

- A *normal subtree* of a tree $(T, <)$ is a tree $(S, \prec)$ satisfying the following conditions:
  (i) $S \subseteq T$.
  (ii) For any $s_1, s_2 \in S$, $s_1 \prec s_2$ iff $s_1 < s_2$.
  (iii) For any $s, t \in T$, if $s < t$ and $t \in S$, then $s \in S$.

Note that each level of a normal subtree is a subset of the corresponding level of T. Clearly a normal subtree of height κ of a κ-Aronszajn tree is a κ-Aronszajn tree; similarly for κ-Suslin trees.

- A tree T is *eventually branching* iff for all $t \in T$, the set $\{s \in T : t \leq s\}$ is not a chain.

Clearly a well-pruned κ-Aronszajn tree is eventually branching; similarly for κ-Suslin trees.

Theorem 22.7. *If κ is regular, then any κ-tree T has a normal subtree T' which is a well-pruned κ-tree. Moreover, if $x \in T$ and $|\{y \in T : x \leq y\}| = \kappa$ then we may assume that $x \in T'$.*

Proof. Let κ be regular, and let T be a κ-tree. We define

$$S = \{t \in T : |\{s \in T : t \leq s\}| = \kappa\}.$$

Clearly S is a normal subtree of T, although it may contain more than one root of T. Now we claim

(1) Some root of T is in S.

In fact, $\mathrm{Lev}_0(T)$ has size less than κ, and

$$T = \bigcup_{s\in\mathrm{Lev}_0(T)} \{t \in T : s \leq t\},$$

so there is some $s \in \mathrm{Lev}_0(T)$ such that $|\{t \in T : s \leq t\}| = \kappa$. This element s is in S, as desired in (1).

We now take an s as indicated. To satisfy the second condition in the Theorem, we can take s below the element x of that condition.

Now we let $S' = \{t \in S : s \leq t\}$. We claim that S' is as desired. Clearly it is a normal subtree of T, and it has exactly one root, namely s. To show that it has height κ and is well-pruned, it suffices now to prove

(2) If $u \in S'$, $\alpha < \beta < \kappa$, and $\mathrm{ht}(u, S') = \alpha$, then there is a $v \in S' \cap \mathrm{Lev}_\beta(T)$ such that $u < v$.

In fact,

$$\{t \in T : u \leq t\} = \bigcup_{\alpha \leq \gamma < \beta} \{t \in \mathrm{Lev}_\gamma(T) : u \leq t\} \cup \bigcup_{\substack{v \in \mathrm{Lev}_\beta(T) \\ u < v}} \{t \in T : v \leq t\},$$

and the first big union here is the union of fewer than κ sets, each of size less than κ. Hence there is a $v \in \mathrm{Lev}_\beta(T)$ such that $u < v$ and $|\{t \in T : v \leq t\}| = \kappa$. So $v \in S'$ and $u < v$, as desired. □

Proposition 22.8. *Let κ be an uncountable regular cardinal. If T is an eventually branching κ-tree in which every antichain has size less than κ, then T is a Suslin tree.*

Proof. Suppose to the contrary that C is a chain of length κ. We may assume that C is maximal, so that it has elements of each level less than κ. For each $t \in T$ choose $f(t) \in T$ such that $t < f(t) \notin C$; this is possible by the eventually branching hypothesis. Now we define $\langle s_\alpha : \alpha < \kappa\rangle$ by recursion, choosing

$$s_\alpha \in \left\{ t \in C : \sup_{\beta < \alpha} \mathrm{ht}(f(s_\beta), T) < \mathrm{ht}(t, T) \right\};$$

this is possible since κ is regular. Now $\langle f(s_\alpha) : \alpha < \kappa\rangle$ is an antichain. In fact, if $\beta < \alpha$ and $f(s_\beta)$ and $f(s_\alpha)$ are comparable, then by construction $\mathrm{ht}(f(s_\beta), T) < \mathrm{ht}(s_\alpha, T) < \mathrm{ht}(f(s_\alpha), T)$, and so $f(s_\beta) < f(s_\alpha)$. But then the tree property yields that $f(s_\beta) < s_\alpha$ and so $f(s_\beta) \in C$, contradiction.

Thus we have an antichain of size κ, contradiction. □

One of the main motivations for the notion of a Suslin tree comes from a correspondence between linear orders and trees. Under this correspondence, Suslin trees correspond to Suslin lines, and the existence of Suslin trees is equivalent to the existence of Suslin lines.

First we show how to go from a tree to a line, in a fairly general setting. Suppose that T is a well-pruned κ-tree, and let $\prec$ be a linear order of T. Here $\prec$ may have nothing to do with the order of the tree. Note that every branch of T has limit ordinal length. For each branch B of T, let $\mathrm{len}(B)$ be its length, and let $\langle b^B(\alpha) : \alpha < \mathrm{len}(B)\rangle$ be an enumeration of B in increasing order. For distinct branches B_1, B_2, neither is included in the other, and so we can let $d(B_1, B_2)$ be the smallest ordinal $\alpha < \min(\mathrm{len}(B_1), \mathrm{len}(B_2))$ such that

$b^{B_1}(\alpha) \neq b^{B_2}(\alpha)$. We define the *$\prec$-branch linear order of T*, denoted by $\mathscr{B}(T, \prec)$, to be the collection of all branches of T, where the order $<$ on $\mathscr{B}(T, \prec)$ is defined as follows: for any two distinct branches B_1, B_2,

$$B_1 < B_2 \quad \text{iff} \quad b^{B_1}(d(B_1, B_2)) \prec b^{B_2}(d(B_1, B_2)).$$

This is a kind of lexicographic ordering of the branches. Clearly this is an irreflexive relation, and clearly any two branches are comparable. The following lemma gives that it is transitive.

Lemma 22.9. *Assume that $B_1 < B_2 < B_3$. Then exactly one of the following holds:*

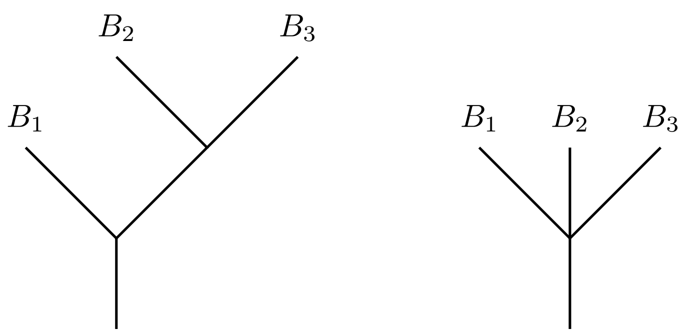
(i) $d(B_1, B_3) = d(B_1, B_2) < d(B_2, B_3)$.

(ii) $d(B_1, B_3) = d(B_1, B_2) = d(B_2, B_3)$.

(iii) $d(B_1, B_3) = d(B_2, B_3) < d(B_1, B_2)$.

In any case $B_1 < B_3$.

Clearly at most one of (i)–(iii) holds. These three conditions are illustrated as follows:



Case 1. $d(B_1, B_2) < d(B_2, B_3)$. Then, we claim, $d(B_1, B_3) = d(B_1, B_2)$. In fact, if $\alpha < d(B_1, B_2)$, then

$$b^{B_1}(\alpha) = b^{B_2}(\alpha) = b^{B_3}(\alpha),$$

while

$$b^{B_1}(d(B_1, B_2) \prec b^{B_2}(d(B_1, B_2)) = b^{B_3}(d(B_1, B_2)).$$

Hence the claim holds, and $B_1 < B_3$.

Case 2. $d(B_1, B_2) = d(B_2, B_3)$. Then, we claim, $d(B_1, B_3) = d(B_1, B_2)$. In fact, if $\alpha < d(B_1, B_2)$, then

$$b^{B_1}(\alpha) = b^{B_2}(\alpha) = b^{B_3}(\alpha),$$

while

$$b^{B_1}(d(B_1, B_2) \prec b^{B_2}(d(B_1, B_2)) \prec b^{B_3}(d(B_1, B_2)).$$

This proves the claim, and $B_1 < B_3$.

Case 3. $d(B_1, B_2) > d(B_2, B_3)$. Then, we claim, $d(B_1, B_3) = d(B_2, B_3)$. In fact, if $\alpha < d(B_2, B_3)$, then

$$b^{B_1}(\alpha) = b^{B_2}(\alpha) = b^{B_3}(\alpha),$$

while

$$b^{B_1}(d(B_2, B_3) = b^{B_2}(d(B_2, B_3)) \prec b^{B_3}(d(B_2, B_3)).$$

This proves the claim, and $B_1 < B_3$. $\square$

Thus the construction gives a linear order.

We describe another way of going from a tree to a linear order. Let T be a tree. We define $s \sim t$ iff $s, t \in T$ and $(s\downarrow) = (t\downarrow)$. Clearly $\sim$ is an equivalence relation on T.

Proposition 22.10. *Let T be a tree, and let $X \subseteq T$. Then the following are equivalent:*
(i) X is a $\sim$-class.
(ii) One of the following holds:
(a) There is an $s \in T$ such that $X = \{t \in T : t$ is an immediate successor of $s\}$.
(b) X is the set of all roots of T.
(c) There is a limit ordinal α such that all elements of X have height α, and there is a chain C having elements of each height less than α such that $X = \{t \in T : (t\downarrow) = C\}$. $\square$

For any tree T, a *full chain* is a chain C such that for some ordinal $\alpha < \mathrm{ht}(T)$, C has elements of each height less than α, and all elements of C have height less than α. Note that $\emptyset$ is a full chain. If X is a $\sim$-class and $s \in X$, then $s\downarrow$ is a full chain, which we denote by C_X. If C is a full chain, then $X \overset{\text{def}}{=} \{s : s$ is an immediate successor of $C\}$ is either empty or is an $\sim$-class. If C is a full chain with an upper bound, then we denote this set X by $\sim_C$.

If $s \in T$ and $\alpha < \mathrm{ht}(s)$, then we denote by s_α the element of T of height α which is below s. If $s, t \in T$ are incomparable, then $f.d.(s, t)$ is the smallest $\alpha < \mathrm{ht}(s), \mathrm{ht}(t)$ such that $s_\alpha \neq t_\alpha$. Note that $(s_{f.d.(s,t)}\downarrow) = (t_{f.d.(s,t)}\downarrow)$.

Now for each $\sim$-class X associate a linear order $\leq_X$ of X. We now define a relation $\leq_\sim$ on T by setting $s \leq_\sim t$ iff

(1) $s \leq_T t$, or
(2) s, t are incomparable, and $s_{f.d.(s,t)} \leq_X t_{f.d.(s,t)}$, where X is the $\sim$-class of $(s_{f.d.(s,t)}\downarrow)$.

Proposition 22.11. *$\leq_\sim$ is a linear order on T, and $\forall s, t \in T[s \leq_T t$ implies that $s \leq_\sim t]$.*

Proof. The second statement is obvious. Clearly $\leq_\sim$ is reflexive and antisymmetric. Now suppose that $s \leq_\sim t \leq_\sim u$.

Case 1. $s \leq_T t \leq_T u$. Then $s \leq_T u$, so $s \leq_\sim u$.

Case 2. $s \leq_T t$, t and u are incomparable, and $s \leq_T u$. Then $s \leq_\sim u$.

Case 3. $s \leq_T t$, t and u are incomparable, and s and u are incomparable. Say $s = t_\alpha$. Then $f.d.(t, u) \leq \alpha$, for if $\alpha < f.d.(t, u)$, then $s = t_\alpha = u_\alpha \leq u$, contradiction. Now if $\beta < f.d.(t, u)$, then $\beta < \alpha$, and $s_\beta = t_\beta = u_\beta$. Also, $s_{f.d.(t,u)} = t_{f.d.(t,u)} \leq_X u_{f.d.(t,u)}$, where X is the $\sim$-class of $(t_{f.d.(t,u)}\downarrow)$. Thus $f.d.(s, u) = f.d.(t, u)$ and $s_{f.d.(s,u)} \leq_X u_{f.d.(t,u)}$, where X is the $\sim$-class of $(t_{f.d.(s,u)}\downarrow)$. So $s \leq_\sim u$.

Case 4. s and t are incomparable, and $t \leq_T u$. Clearly then s and u are incomparable. Say $t = u_\alpha$. Then $f.d.(s, t) \leq \alpha$, for if $\alpha < f.d.(s, t)$ then $t = u_\alpha = s_\alpha \leq s$, contradiction.

Now if $\beta < f.d.(s,t)$, then $\beta < \alpha$, and $u_\beta = t_\beta = s_\beta$. Also, $s_{f.d.(s,t)} \leq_X t_{f.d.(s,t)} = u_{f.d.(s,t)}$, where X is the $\sim$-class of $(s_{f.d.(s,t)}\downarrow)$. Thus $f.d.(s,u) = f.d.(s,t)$ and so $s \leq_\sim u$.

Case 5. s and t are incomparable, and t and u are incomparable, and $u \leq_T s$. Say $u = s_\alpha$. Then $f.d.(s,t) \leq \alpha$, for if $\alpha < f.d.(s,t)$, then $u = s_\alpha = t_\alpha \leq t$, contradiction. Now if $\beta < f.d.(s,t)$, then $u_\beta = s_\beta = t_\beta$. Moreover, $u_{f.d.(s,t)} = s_{f.d.(s,t)} \leq_X t_{f.d.(s,t)}$, where X is the $\sim$-class of $(s_{f.d.(s,t)}\downarrow)$. Hence $f.d.(u,t) = f.d.(s,t)$ and so $u \leq_\sim t$, contradiction.

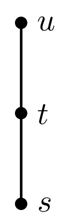
Case 6. s and t are incomparable, and t and u are incomparable, and $s \leq_T u$. Then $s \leq_\sim u$.

Case 7. s and t are incomparable, and t and u are incomparable, s and u are incomparable, and $f.d.(s,t) < f.d.(t,u)$. If $\beta < f.d.(s,t)$, then $s_\beta = t_\beta = u_\beta$ and $s_{f.d.(s,t)} \leq_X t_{f.d.(s,t)} = u_{f.d.(s,t)}$, where X is the $\sim$-class of $(s_{f.d.(s,t)}\downarrow)$. Hence $f.d.(s,t) = f.d.(s,u)$ and so $s \leq_\sim u$.

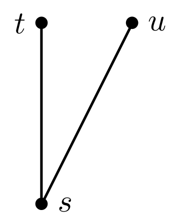
Case 8. s and t are incomparable, and t and u are incomparable, s and u are incomparable, and $f.d.(s,t) = f.s.(t,u)$. If $\beta < f.d.(s,t)$, then $s_\beta = t_\beta = u_\beta$ and $s_{f.d.(s,t)} \leq_X t_{f.d.(s,t)} \leq_X u_{f.d.(s,t)}$, and so $f.d.(s,t) = f.d.(s,u)$ and $s \leq_\sim u$.

Case 9. s and t are incomparable, and t and u are incomparable, s and u are incomparable, and $f.d.(t,u) < f.d.(s,t)$. If $\beta < f.d.(t,u)$, then $u_\beta = t_\beta = s_\beta$ and $s_{f.d.(t,u)} = t_{f.d.(t,u)} \leq_X u_{f.d.(t,u)}$. Hence $f.d.(s,u) = f.d.(t,u)$ and $s \leq_\sim u$. $\square$

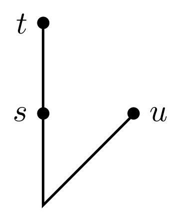
Here are some diagrams illustrating the above proof.



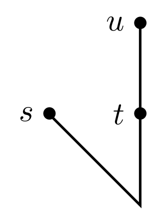
Case 1



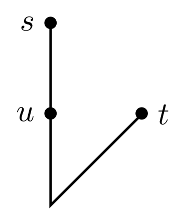
Case 2



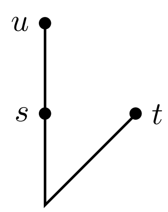
Case 3



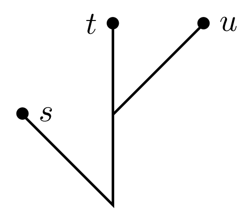
Case 4



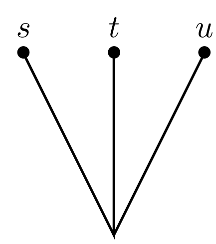
Case 5



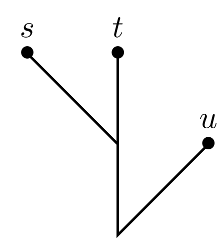
Case 6



Case 7



Case 8



Case 9

An *Aronszajn line* is an ordered set L such that:

(i) ω_1 cannot be embedded in L;

(ii) ω_1^* cannot be embedded in L;

(iii) For any $X \in [\mathbb{R}]^{\omega_1}$, X cannot be embedded in L.

(iv) $|L| = \omega_1$.

Lemma 22.12. *If T is an ω_1-tree and $\langle a_\xi : \xi < \omega_1 \rangle$ is a sequence of distinct elements* *of T, then for all $\alpha < \omega_1$ there is a $t^\alpha \in \mathrm{Lev}_\alpha(T)$ such that $\{\xi < \omega_1 : t^\alpha \le_T a_\xi\}$ is uncountable.*

Proof. Let $\alpha < \omega_1$. Then

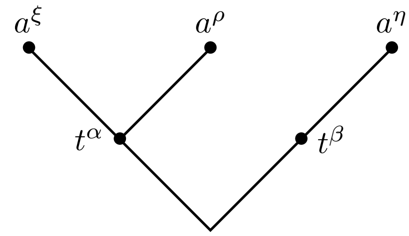
$$\omega_1 = \{\xi < \omega_1 : \mathrm{ht}(a^\xi) < \alpha\} \cup \bigcup_{t \in \mathrm{Lev}_\alpha(T).} \{\xi < \omega_1 : t \le_T a^\xi\}$$

The first set here is countable, so there is a $t \in \mathrm{Lev}_\alpha(T)$ such that $\{\xi < \omega_1 : t \le_T a^\xi\}$ is uncountable, as desired. $\square$

Theorem 22.13. *If T is an Aronszajn tree, then $(T, \le_\sim)$ is an Aronszajn line.*

Proof. First suppose that $\langle a^\xi : \xi < \omega_1\rangle$ is $<_\sim$-increasing; we want to get a contradiction. Choose t^α as in Lemma 22.12, for each $\alpha < \omega_1$. Now since T does not have a chain of length ω_1, there exist $\alpha < \beta$ such that $t^\alpha \not<_T t^\beta$. Choose $\xi < \eta < \rho$ such that $t^\alpha \le_T a^\xi$, $t^\alpha \le_T a^\rho$, and $t^\beta \le_T a^\eta$. Then $f.d.(a^\xi, a^\eta) = f.d.(a^\rho, a^\eta)$. In fact, if $\gamma < f.d.(a^\xi, a^\eta)$, then $\gamma < \alpha$, since $\alpha \le \gamma$ would imply that $t^\alpha = a^\xi_\alpha = a^\eta_\alpha = t^\beta_\alpha \le t^\beta$, contradiction. Then $a^\xi_\gamma = t^\alpha_\gamma = a^\rho_\gamma = a^\eta_\gamma$. Thus $f.d.(a^\xi, a^\eta) \le f.d.(a^\rho, a^\eta)$. By symmetry, $f.d.(a^\xi, a^\eta) = f.d.(a^\rho, a^\eta)$. Thus $a^\xi_{f.d.(a^\xi,a^\eta)} \sim a^\eta_{f.d.(a^\xi,a^\eta)} \sim a^\rho_{f.d.(a^\xi,a^\eta)}$. With X the $\sim$-class of $a^\xi_{f.d.(a^\xi,a^\eta)}$, $a^\xi <_\sim a^\eta$ implies that $a^\xi_{f.d.(a^\xi,a^\eta)} <_X a^\eta_{f.d.(a^\xi,a^\eta)}$, while $a^\eta < a^\rho$ implies that $a^\eta_{f.d.(a^\xi,a^\eta)} <_X a^\rho_{f.d.(a^\xi,a^\eta)}$ Now $f.d.(a^\xi, a^\eta) \le \alpha$, so $a^\xi_{f.d.(a^\xi,a^\eta)} = t^\alpha_{f.d.(a^\xi,a^\eta)} = a^\rho_{f.d.(a^\xi,a^\eta)}$, so this is a contradiction.

Illustration:



The case $\langle a^\xi : \xi < \omega_1\rangle$ $<_\sim$-decreasing is similar, but we give the details. Again choose t^α as in Lemma 22.12, for each $\alpha < \omega_1$. Now since T does not have a chain of length ω_1, there exist $\alpha < \beta$ such that $t^\alpha \not<_T t^\beta$. Choose $\xi < \eta < \rho$ such that $t^\alpha \le_T a^\xi$, $t^\alpha \le_T a^\rho$, and $t^\beta \le_T a^\eta$. Then $f.d.(a^\xi, a^\eta) = f.d.(a^\rho, a^\eta)$. In fact, if $\gamma < f.d.(a^\xi, a^\eta)$, then $\gamma < \alpha$, since $\alpha \le \gamma$ would imply that $t^\alpha = a^\xi_\alpha = a^\eta_\alpha = t^\beta_\alpha < t^\beta$, contradiction. Then $a^\xi_\gamma = t^\alpha_\gamma = a^\rho_\gamma = a^\eta_\gamma$. Thus $f.d.(a^\xi, a^\eta) \le f.d.(a^\rho, a^\eta)$. By symmetry, $f.d.(a^\xi, a^\eta) = f.d.(a^\rho, a^\eta)$. Thus $a^\xi_{f.d.(a^\xi,a^\eta)} \sim a^\eta_{f.d.(a^\xi,a^\eta)} \sim a^\rho_{f.d.(a^\xi,a^\eta)}$. With X the $\sim$-class of $a^\xi_{f.d.(a^\xi,a^\eta)}$, $a^\eta <_\sim a^\xi$ implies that $a^\eta_{f.d.(a^\xi,a^\eta)} <_X a^\xi_{f.d.(a^\xi,a^\eta)}$, while $a^\rho < a^\eta$ implies that $a^\rho_{f.d.(a^\xi,a^\eta)} <_X a^\eta_{f.d.(a^\xi,a^\eta)}$ Now $f.d.(a^\xi, a^\eta) \le \alpha$, so $a^\xi_{f.d.(a^\xi,a^\eta)} = t^\alpha_{f.d.(a^\xi,a^\eta)} = a^\rho_{f.d.(a^\xi,a^\eta)}$, so this is a contradiction.

Finally, suppose that $\langle a^\xi : \xi < \omega_1\rangle$ is a sequence of distinct elements of T and $f : \{a_\xi : \xi < \omega_1\} \to \mathbb{R}$ is such that $a_\xi <_\sim a_\eta$ implies that $f(a_\xi) < f(a_\eta)$; we want to get a contradiction. Again choose t^α as in Lemma 22.12, for each $\alpha < \omega_1$. Let D be a countable subset of ω_1 such that $\{f(a_\xi) : \xi \in D\}$ is dense in $\text{rng}(f)$. Let α be such that all elements a_ξ for $\xi \in D$ have height less than α. Let $a^\xi <_\sim a^\eta <_\sim a^\rho$ be above t. Thus the interval $(a^\xi, a^\rho) \neq \emptyset$ in $\{a^\mu : \mu < \omega_1\}$. So there is a $\mu \in D$ such that $a^\xi <_\sim a^\mu <_\sim a^\rho$.

Case 1. $a^\xi <_T a^\rho$. Then $a^\mu <_T a^\xi$ is not possible, as then $a^\mu <_\sim a^\xi <_\sim a^\rho$. So a^μ and a^ξ are incomparable. Also, a^μ and a^ρ are incomparable. If $\gamma < f.d.(a^\xi, a^\mu)$, then $\gamma < f.d.(a^\rho, a^\mu)$; so $f.d.(a^\xi, a^\mu) \leq f.d.(a^\rho, a^\mu)$. The other inequality is also clear, so $f.d.(a^\xi, a^\mu) = f.d.(a^\rho, a^\mu)$. Clearly $a^\xi_{f.d.(a^\xi,a^\mu)} = a^\rho_{f.d.(a^\xi,a^\mu)}$. Let X be the $\sim$-class of $a^\xi_{f.d.(a^\xi,a^\mu)}$. Then $a^\xi <_\sim a^\mu$ yields $a^\xi_{f.d.(a^\xi,a^\mu)} <_X a^\mu_{f.d.(a^\xi,a^\mu)}$, and $a^\mu <_\sim a^\rho$ yields $a^\mu_{f.d.(a^\xi,a^\mu)} <_X a^\xi_{f.d.(a^\xi,a^\mu)}$, contradiction.

Case 2. a^ξ and a^ρ are incomparable. Now $f.d.(a^\xi, a^\mu) < \alpha$, so if $\gamma < f.d.(a^\xi, a^\mu)$ then $a^\mu_\gamma = t^\alpha_\gamma = a^\rho_\gamma$. Hence $f.d.(a^\xi, a^\mu) \leq f.d.(a^\rho, a^\mu)$. The other inequality holds by symmetry, so $f.d.(a^\xi, a^\mu) = f.d.(a^\rho, a^\mu)$. Also, $a^\xi_{f.d.(a^\xi,a^\mu)} = t^\alpha_{f.d.(a^\xi,a^\mu)} = a^\rho_{f.d.(a^\xi,a^\mu)}$. Let X be the $\sim$-class of $a^\xi_{f.d.(a^\xi,a^\mu)}$. Then $a^\xi <_\sim a^\mu$ yields $a^\xi_{f.d.(a^\xi,a^\mu)} <_X a^\mu_{f.d.(a^\xi,a^\mu)}$, and $a^\mu <_\sim a^\rho$ yields $a^\mu_{f.d.(a^\xi,a^\mu)} <_X a^\xi_{f.d.(a^\xi,a^\mu)}$, contradiction. $\square$

Theorem 22.14. *If there is a Suslin tree then there is a Suslin line.*

Proof. By Theorem 22.7 we may assume that T is well-pruned. Take any linear order $\prec$ of T. To show that $\mathscr{B}(T, \prec)$ is ccc, suppose that $\mathscr{A}$ is an uncountable collection of nonempty pairwise disjoint open intervals in $\mathscr{B}(T, \prec)$. For each $(B, C) \in \mathscr{A}$ choose $E_{(B,C)} \in (B, C)$. Remembering that each branch has limit length, we can also select an ordinal $\alpha_{(B,C)}$ such that

$$d(B, E_{(B,C)}), d(E_{(B,C)}, C) < \alpha_{(B,C)} < \text{len}(E_{(B,C)})$$

We claim that $\langle b^{E(B,C)}(\alpha_{(B,C)}) : (B, C) \in \mathscr{A}\rangle$ is a system of pairwise incomparable elements of T, which contradicts the definition of a Suslin tree. In fact, suppose that (B, C) and (B', C') are distinct elements of $\mathscr{A}$ and $b^{E(B,C)}(\alpha_{(B,C)}) \leq b^{E(B',C')}(\alpha_{(B',C')})$. It follows that $\alpha_{(B,C)} \leq \alpha_{(B',C')}$ and

(1) $b^{E(B,C)}(\beta) = b^{E(B',C')}(\beta)$ for all $\beta \leq \alpha_{(B,C)}$.

Hence

(2) If $\beta < d(B, E_{(B,C)})$, then $\beta < \alpha_{(B,C)}$, and so $b^B(\beta) = b^{E_{(B,C)}}(\beta) = b^{E_{(B',C')}}(\beta)$.

Now recall that $d(B, E_{(B,C)}) < \alpha_{(B,C)}$. Hence

$$b^B(d(B, E_{(B,C)})) \prec b^{E_{(B,C)}}(d(B, E_{(B,C)})) = b^{E_{(B',C')}}(d(B, E_{(B,C)})),$$

and so $B < E_{(B',C')}$. Similarly, $E_{(B',C')} < C$, as follows:

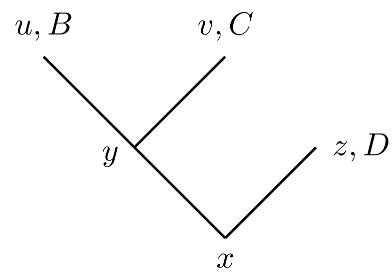
(3) If $\beta < d(C, E_{(B,C)})$, then $\beta < \alpha_{(B,C)}$, and so $b^C(\beta) = b^{E_{(B,C)}}(\beta) = b^{E_{(B',C')}}(\beta)$.

Now recall that $d(C, E_{(B,C)}) < \alpha_{(B,C)}$. Hence

$$b^C(d(C, E_{(B,C)})) \succ b^{E_{(B,C)}}(d(C, E_{(B,C)})) = b^{E_{(B',C')}}(d(C, E_{(B,C)})),$$

and so $C > E_{(B',C')}$. Hence $E_{(B',C')} \in (B, C)$. But also $E_{(B',C')} \in (B', C')$, contradiction.

To show that $\mathscr{B}(T, \prec)$ is not separable, it suffices to show that for each $\delta < \omega_1$ the set $\{B \in \mathscr{B}(T, \prec) : \mathrm{len}(B) < \delta\}$ is not dense in $\mathscr{B}(T, \prec)$. Take any $x \in T$ of height δ. Since $\{y : y > x\}$ has elements of every level greater than δ, it cannot be a chain, as this would give a chain of size ω_1. So there exist incomparable $y, z > x$. Similarly, there exist incomparable $u, v > y$. Let B, C, D be branches containing u, v, z respectively. By symmetry say $B < C$. Illustration:



(4) $\mathrm{ht}(y) < d(B, C)$

This holds since $y \in B \cap C$.

(5) $d(B, D) \leq \mathrm{ht}(y)$ and $d(C, D) \leq \mathrm{ht}(y)$; hence $d(B, D) < d(B, C)$ and $d(C, D) < d(B, C)$.

In fact, $y \in B \backslash D$, so $d(B, D) \leq \mathrm{ht}(y)$ follows. Similarly $d(C, D) \leq \mathrm{ht}(y)$. Now the rest follows by (4).

(6) $d(B, D) = d(C, D)$.

For, if $d(B, D) < d(C, D)$, then $b^C(d(B, D)) = b^D(d(B, D)) \neq b^B(d(B, D))$, contradicting $d(B, D) < d(B, C)$, part of (5). If $d(C, D) < d(B, D)$, then $b^B(d(C, D)) = b^D(d(C, D)) \neq b^C(d(C, D))$, contradicting $d(C, D) < d(B, C)$, part of (5).

By (6) we have $B, C < D$, or $D < B, C$. Since we are assuming that $B < C$, it follows that

(7) $B < C < D$ or $D < B < C$.

Case 1. $B < C < D$. Thus (B, D) is a nonempty open interval. Suppose that there is some branch E with $\mathrm{len}(E) < \delta$ and $B < E < D$. Then $d(B, E), d(E, D) < \delta$. By Lemma 22.9 one of the following holds: $d(B, D) = d(B, E) < d(E, D)$; $d(B, D) = d(B, E) = d(E, D)$; $d(B, D) = d(E, D) < d(B, E)$. Hence $d(B, D) < \delta$. Since $x \in B \cap D$ and x has height δ, this is a contradiction.

Case 2. $D < B < C$. Thus (D, C) is a nonempty open interval. Suppose that there is some branch E with $\mathrm{len}(E) < \delta$ and $D < E < C$. Then $d(D, E), d(E, C) < \delta$. By Lemma 22.9 one of the following holds: $d(D, C) = d(D, E) < d(E, C)$; $d(D, C) = d(D, E) = d(E, C)$; $d(D, C) = d(E, D) < d(D, E)$; hence $d(D, C) < \delta$. Since $x \in C \cap D$ and x is of height δ, this is a contradiction. $\square$

Another important kind of tree of height ω_1 is the Kurepa trees. A tree T is a *Kurepa tree* iff T is an ω_1-tree which has more than ω_1 branches of length ω_1. A linear order L is a

Kurepa line iff $|L| > \omega_1$, L has a weakly dense subset of size ω_1, and L does not contain an uncountable subset isomorphic to a set of real numbers.

Theorem 22.15. *If T is a Kurepa tree, $\mathscr{B}'$ is the set of all branches of T of length ω_1, and $\prec$ is a linear order on T, then $\mathscr{B}'$ with the order of $\mathscr{B}(T,\prec)$ is a Kurepa line.*

Proof. Obviously $|\mathscr{B}'| > \omega_1$. Fix $C \in \mathscr{B}'$. For each $t \in T$ let $M_t = \{B \in \mathscr{B}' : t \in B\}$, fix $E \in M_t$ if $M_t \neq \emptyset$, and define

$$B_t = \begin{cases} C & \text{if } M_t = \emptyset, \\ D & \text{if } D \text{ is the smallest element of } M_t, \\ E & \text{if } M_t \neq \emptyset \text{ but } M_t \text{ does not have a smallest element.} \end{cases}$$

Let $\mathscr{E} = \{B_t : t \in T\}$. Thus $|\mathscr{E}| \le \omega_1$. We claim that $\mathscr{E}$ is dense in $\mathscr{B}'$. For, suppose that $U, V \in \mathscr{B}'$, $U < V$, and $(U,V) \neq \emptyset$. Say $U < W < V$. Then by Lemma 22.9 we have three possibilities.

Case 1. $d(U,V) = d(U,W) < d(W,V)$. Let $t = W_{d(U,W)}$. Then $d(U,B_t) = d(U,W) = d(U,V) < d(B_t,V)$ and so $U < B_t < V$.

Case 2. $d(U,V) = d(U,W) = d(W,V)$. Let $t = W_{d(U,W)}$. Then $d(U,B_t) = d(U,W) = d(U,V) = d(B_t,V)$ and so $U < B_t < V$.

Case 3. $d(U,V) = d(W,V) < d(U,W)$. Let $t = W_{d(U,W)}$. Then $d(U,B_t) = d(U,W) = d(U,V) < d(U,B_t)$ and so $U < B_t < V$.

Now suppose that M is an uncountable subset of $\mathscr{B}'$ and $f : M \to \mathbb{R}$ is order preserving. Let N be a countable subset of M such that $f[N]$ is dense in $\mathrm{rng}(f)$. Then N is dense in M.

(1) $\forall B \in M \exists t_B \in B \forall C \in N\backslash\{B\}[t_B \notin C]$.

In fact, choose $\alpha > d(B,C)$ for all $C \in N\backslash\{B\}$, and let $t_B = B_\alpha$.

(2) $\exists s \in T[\forall C \in N[\mathrm{ht}(t_C) < \mathrm{ht}(s)]$ and $B_s \cap M$ is infinite$]$.

For, let $\alpha = (\sup_{C\in N} \mathrm{ht}(t_C)) + 1$. Applying Lemma 22.12 to an enumeration of M we get (2).

Now we take s as in (2). Take $C < D < E < F$ in $B_s \cap M$. Choose $U, V \in N$ with $C < U < E$ and $D < V < F$. Since $s \in C \cap D \cap E \cap F$, we have $\mathrm{ht}(s) < d(C,E)$, $d(D,F)$. By Lemma 22.9, $d(C,E) \le d(C,U)$. So $\mathrm{ht}(s) < d(C,U)$. So $s \in U$. Since $t_U \in U$ and $\mathrm{ht}(t_U) < \mathrm{ht}(s)$ we have $t_U < s$. Similarly $s \in V$, so $t_U \in V$, contradiction. □

We now go the other direction, from a linear order to a tree. The basic construction goes as follows. Suppose that L is an infinite linear order. A subset I of L is *convex* iff $\forall a, b \in I \forall c \in L[a < c < b \to c \in I]$. Let $\mathbb{I}$ be the collection of all nonempty convex subsets of L. We are now going to define a sequence $\langle \mathbb{J}_\alpha : \alpha < \omega_1 \rangle$ of convex subsets of $\mathbb{I}$. Let $\mathbb{J}_0 = \{L\}$. Now suppose that $0 < \beta < \omega_1$ and we have defined $\mathbb{J}_\alpha$ for all $\alpha < \beta$ so that the following conditions hold:

(4_α) The elements of $\mathbb{J}_\alpha$ are nonempty and pairwise disjoint.

(5_α) If $\gamma < \alpha$, $I \in \mathbb{J}_\gamma$, and $J \in \mathbb{J}_\alpha$, then either $I \cap J = \emptyset$, or else $J \subseteq I$.

Note that (4_0) and (5_0) hold. Now suppose that β is a successor ordinal $\delta+1$. For each $M \in \mathbb{J}_\delta$ such that $|M| \geq 2$ choose nonempty convex $I_1^M, I_2^M \in \mathbb{I}$ such that $\forall a \in I_1^M \forall b \in I_2^M [a, b]$ and $I_1^M \cup I_2^M \subseteq M$. Let $\mathbb{J}_\beta = \bigcup_{M \in \mathbb{J}_\delta} \{I_1^M, I_2^M\}$. Clearly (4_β) and (5_β) hold.

Next, suppose that β is a limit ordinal. Let $\mathbb{J}_\beta$ consist of some sets P such that there is an $N \in \prod_{\alpha<\beta} \mathbb{J}_\beta$ such that $\emptyset \neq P \subseteq \bigcap_{\alpha<\beta} N_\alpha$. This finishes the construction. There is a least ordinal γ such that $\mathbb{J}_\gamma$ is not defined. We also define $T_I = \bigcup_{\beta<\gamma} \mathbb{J}_\beta$.

This is called the *partition construction.* Several parts of it can be varied.

Proposition 22.16. *If L is a linear ordering and $\langle \mathbb{J}_\beta : \alpha < \gamma \rangle$ is a partition construction, then T_I is a tree under the relation $\supset$, with $\mathrm{Lev}_\beta(T_I) = \mathbb{J}_\beta$ for each $\beta < \gamma$; T_I has height γ.*

Proof. By induction on $\beta \leq \gamma$, $\left(\bigcup_{\alpha \leq \beta} \mathbb{J}_\alpha, \supset\right)$ is a tree whose α-th level is $\mathbb{J}_\alpha$, for each $\alpha \leq \beta$. $\square$

Proposition 22.17. *If L is an Aronszajn line, then so is $L \times \mathbb{Q}$ (ordered lexicographically).*

Proof. Suppose that $\langle (a_\xi, q_\xi) : \xi < \omega_1 \rangle$ is strictly increasing.

(1) $\forall \xi < \omega_1 \exists \eta < \omega_1 [a_\xi < a_\eta]$.

For, otherwise there is a $\xi < \omega_1$ such that $\forall \eta < \omega_1 [a_\eta \leq a_\xi]$. But then $\{(a_\eta, q_\eta) : \eta < \omega_1\}$ is countable, contradiction.

By (1) there is a strictly increasing $\langle \alpha(\xi) : \xi < \omega_1 \rangle$ such that $a_{\alpha(\xi)} < a_{\alpha(\eta)}$ for all $\xi < \eta$, contradiction.

Suppose that $\langle (a_\xi, q_\xi) : \xi < \omega_1 \rangle$ is strictly decreasing.

(2) $\forall \xi < \omega_1 \exists \eta < \omega_1 [a_\xi > a_\eta]$.

For, otherwise there is a $\xi < \omega_1$ such that $\forall \eta < \omega_1 [a_\xi \leq a_\eta]$. But then $\{(a_\eta, q_\eta) : \eta < \omega_1\}$ is countable, contradiction.

By (2) there is a strictly increasing $\langle \alpha(\xi) : \xi < \omega_1 \rangle$ such that $a_{\alpha(\xi)} > a_{\alpha(\eta)}$ for all $\xi < \eta$, contradiction.

Suppose that $\langle (a_\xi, q_\xi) : \xi < \omega_1 \rangle$ is a one-one sequence and $f : \{(a_\xi, q_\xi) : \xi < \omega_1\} \to \mathbb{R}$ is strictly increasing. Define $g(a_\xi) = f(a_\xi, 0)$ for all $\sigma < \omega_1$. Then $\langle a_\xi : \xi < \omega_1 \rangle$ is one-one and $g : \{a_\xi : \xi < \omega_1\} \to \mathbb{R}$ is strictly increasing, contradiction. $\square$

If S is a subset of a linear order L, a *convex component* of S is a maximal convex subset of S.

The following theorem is given because of its proof; we already know from Theorem 22.3 that there is an Aronszajn tree.

Theorem 22.18. *Let L be a dense Aronszajn line. Then there is an Aronszajn tree.*

Proof. First note:

(1) If C is a nonempty convex subset of L, then C has a cofinal and coinitial subset of one of the following types:

(a) 0, if $|C| = 1$;

(b) $\{0,1\}$, if C has both endpoints;
(c) ω, if C has a left endpoint but no right endpoint;
(d) ω^*, if C has a right endpoint but no left endpoint;
(e) $\omega^*+\omega$, if C has no endpoints.

We let $S(C)$ be a cofinal and coinitial subset of C of the indicated type. Note that if J is a convex component of $C\backslash S(C)$ then there is an $x \in S(C)$ such that $\forall y \in J[x<y]$ and $\{x\} \cup J$ is connected.

We now define by recursion on $\beta<\omega_1$ objects $\mathscr{U}(\beta)$, $L(\beta)$, and $\leq_\beta$. Let $\mathscr{U}(0)=\{L\}$, $L(0)=S(L)$, and $\leq_0=\{(a,a): a \in S(L)\}$. Let $\mathscr{U}(1)$ be the collection of all convex components of $L\backslash S(L)$, and $L(1)=\bigcup_{C \in \mathscr{U}(1)} S(C)$. Thus $L(1) \subseteq \bigcup \mathscr{U}(1) \subseteq L\backslash L(0)$, so $L(0) \cap L(1)=\emptyset$. Now we say that an ordered pair (x,y) is 1-*acceptable* if $x \in S(L)$, $y \in L(1)$, and if J is the member of $\mathscr{U}(1)$ such that $y \in J$, then $\forall z \in J[x<z]$ and $\{x\} \cup J$ is convex. Now we define

$$\leq_1=\leq_0 \cup\{(z,z): z \in L(1)\} \cup\{(x,y):(x,y) \text{ is a 1-acceptable pair}\}.$$

Note that if $y \in L(1)$ then there is a unique $x \in L(0)$ such that $x \leq_1 y$. Namely, there is a unique $J \in \mathscr{U}(1)$ such that $y \in J$, and then there is a unique $x \in L(0)$ such that $\forall z \in J[x<z]$ and $\{x\} \cup J$ is connected. So (x,y) is 1-acceptable, and hence $x \leq_1 y$.

Clearly $(L(0) \cup L(1), \leq_1)$ is a partial order.

Now suppose that $2 \leq \alpha<\omega_1$ and the following hold for all $\beta<\alpha$:

(2) $\mathscr{U}(\beta)$ is the collection of all convex components of $L\backslash \bigcup_{\gamma<\beta} L(\gamma)$.

(3) $L(\beta)=\bigcup_{C \in \mathscr{U}(\beta)} S(C) \subseteq \bigcup \mathscr{U}(\beta)=L\backslash \bigcup_{\gamma<\beta} L(\gamma)$.

(4) $\leq_\beta$ is a partial order on $\bigcup_{\gamma \leq \beta} L(\gamma)$.

(5) $\leq_\gamma \subseteq \leq \beta$ for all $\gamma<\beta$.

(6) $\leq_\beta=\bigcup_{\gamma<\beta} \leq_\gamma \cup\{(z,z): z \in L(\beta)\} \cup\{(x,y):(x,y)$ is a β-acceptable pair$\}$, where (x,y) is β-acceptable iff there is a $\gamma<\beta$ such that $x \in L(\gamma)$, $y \in L(\beta)$, and if J is the member of $\mathscr{U}(\gamma+1)$ containing y, then $\forall z \in J[x<z]$ and $\{x\} \cup J$ is convex. (Note that if $y \in L(\beta)$, then by (3) there is a $C \in \mathscr{U}(\beta)$ such that $y \in S(C)$, hence $y \in C$, and by (2) $C \subseteq L\backslash \bigcup_{\theta<\beta} L(\theta) \subseteq L\backslash \bigcup_{\theta \leq \gamma+1} L(\theta)$, so $y \in L\backslash \bigcup_{\theta \leq \gamma+1} L(\theta)$, and hence by (2) $y \in J$ for some $J \in \mathscr{U}(\gamma+1)$.)

(7) $\mathscr{U}(\beta) \neq \emptyset$.

(8) $\mathscr{U}(\beta)$ is countable.

(9) $L(\beta) \neq \emptyset$.

(10) $L(\beta)$ is countable.

(11) If $\gamma<\delta<\beta$, $x \in L(\gamma)$, $y \in L(\delta)$, $z \in L(\beta)$, $x \leq_\beta z$ and $y \leq_\beta z$, then $x \leq_\delta y$.

(12) If $\gamma<\beta$ and $y \in L(\beta)$, then there is an $x \in L(\gamma)$ such that $x \leq_\beta y$.

Clearly (2)–(12) hold for $\beta=0$ and $\beta=1$. Now we define $\mathscr{U}(\alpha)$ to be the collection of all convex components of $L\backslash \bigcup_{\gamma<\alpha} L(\gamma)$. Let $L(\alpha)=\bigcup_{C \in \mathscr{U}(\alpha)} S(C)$. Note that $L(\alpha) \subseteq$

$\bigcup \mathscr{U}(\alpha) = L \backslash \bigcup_{\gamma<\alpha} L(\gamma)$. Hence if $\delta \leq \alpha$ and $y \in L(\alpha)$, then $y \in L \backslash \bigcup_{\gamma<\alpha} L(\gamma) \subseteq L \backslash \bigcup_{\gamma<\delta} L_\delta$, and so y is a member of exactly one $C \in \mathscr{U}(\delta)$. Let $\leq_\alpha = \bigcup_{\gamma<\alpha} \leq_\gamma \cup \{(z,z) : z \in L(\alpha)\} \cup \{(x,y) : (x,y)$ is an α-acceptable pair$\}$, where (x,y) is α-acceptable iff there is a $\gamma < \alpha$ such that $x \in L(\gamma)$, $y \in L(\alpha)$, and if J is the member of $\mathscr{U}(\gamma+1)$ containing y, then $\forall z \in J[x < z]$ and $\{x\} \cup J$ is convex. Clearly (2)–(6) hold for α. By (10) for all $\beta < \alpha$ it follows that (7) and (9) hold for α. Now suppose that $\mathscr{U}(\alpha)$ is uncountable. For each $C \in \mathscr{U}(\alpha)$ choose $x_C \in C$. Let $D = \bigcup_{\beta<\alpha} L(\beta)$ and $M = D \cup \{x_C : C \in \mathscr{U}(\alpha)\}$. Then D is countable and M is uncountable. We claim that D is weakly dense in M. For, suppose that $a, b \in M$ with $a < b$. If one of a, b is in D that gives the desired result. Suppose that $a = x_C$ and $b = x_D$. Then (a,b) has some element $y \in D$, as desired. So D is weakly dense in M. By Proposition 21.48, M can be isomorphically embedded in $\mathbb{R}$, contradiction. So $\mathscr{U}(\alpha)$ is countable, giving (8) for α. Then (10) for α follows. Now assume the hypothesis of (11) with α in place of β. Then (x,z) and (y,z) are α-acceptable. Hence there exist $J \in \mathscr{U}(\delta+1)$ and $K \in \mathscr{U}(\gamma+1)$ such that $z \in J$, $\forall w \in J[y < w]$ and $\{y\} \cup J$ is connected, and $z \in K$, $\forall w \in K[x < w]$ and $\{x\} \cup K$ is connected. Now $J \subseteq L \backslash \bigcup_{\theta\leq\delta} L(\theta) \subseteq L \backslash \bigcup_{\theta\leq\gamma} L(\theta)$ and $z \in J \cap K$, so $J \cup K$ is connected. Thus $\{y\} \cup J \cup K$ is connected. Since $\{y\} \cup J \subseteq L \backslash \bigcup_{\theta\leq\gamma} L(\theta)$, it follows that $\{y\} \cup J \subseteq K$. Hence (x,y) is δ-acceptable, and hence $x \leq_\delta y$. So (11) holds. Finally, for (12) assume that $\gamma < \alpha$ and $y \in L(\alpha)$. Then $y \in L \backslash \bigcup_{\theta\leq\gamma} L(\theta)$, so we can let J be the member of $\mathscr{U}(\gamma+1)$ such that $y \in J$. Thus J is a convex component of

$$\begin{aligned}
L \backslash \bigcup_{\theta\leq\gamma} L(\theta) &= \left(L \backslash \bigcup_{\theta<\gamma} L(\theta) \right) \backslash L(\gamma) \\
&= \left(\bigcup_{C \in \mathscr{U}(\gamma)} C \right) \backslash \bigcup_{C \in \mathscr{U}(\gamma)} S(C) \\
&= \bigcup_{C \in \mathscr{U}(\gamma)} (C \backslash S(C)).
\end{aligned}$$

It follows that there is a $C \in \mathscr{U}(\gamma)$ such that J is a convex component of $C \backslash S(C)$. Hence there is an $x \in S(C)$ such that $\forall w \in J[x < w]$ and $\{x\} \cup J$ is connected. So $x \in L(\gamma)$ and (x,y) is α-acceptable. So $x \leq_\alpha y$. This proves (12).

This completes the recursive construction.

Let $T = \bigcup_{\alpha<\omega_1} L(\alpha)$ and $\leq_T = \bigcup_{\alpha<\omega_1} \leq_\alpha$.

(13) $\forall \gamma, \alpha < \omega_1[\gamma \leq \alpha \rightarrow \forall C \in \mathscr{U}(\alpha) \exists D \in \mathscr{U}(\gamma)[C \subseteq D]]$

For, suppose that $C \in \mathscr{U}(\alpha)$. Then C is a convex subset of $L \backslash \bigcup_{\delta<\delta} L(\delta) \subseteq L \backslash \bigcup_{\delta<\gamma} L(\delta)$, so it is a convex subset of $L \backslash \bigcup_{\delta<\gamma} L(\delta)$ and hence is contained in a convex component D of $L \backslash \bigcup_{\delta<\gamma} L(\delta)$; so $D \in \mathscr{U}_\gamma$.

(14) If $x \leq_\alpha y$, $x \in L(\gamma)$, $y \in L(\delta)$, and $x \neq y$, then $\gamma < \delta \leq \alpha$ and $x \leq_\delta y$.

We prove this by induction on α. Suppose that it holds for all $\beta < \alpha$. If $x \leq_\beta y$ for some $\beta < \alpha$, the inductive hypothesis applies. Otherwise we have $y \in L(\alpha)$, hence $\delta = \alpha$, and $\gamma < \alpha$ by definition.

(15) For each $\alpha < \omega_1$, $\leq_\alpha$ is transitive.

For, suppose that $x \leq_\alpha y \leq_\alpha z$. We may assume that $x \in L(\gamma)$, $y \in L(\delta)$, $z \in L(\alpha)$, and $\gamma < \delta < \alpha$. Thus (x, y) is δ-acceptable and (y, z) is α-acceptable. Let J be the member of $\mathscr{U}(\gamma+1)$ such that $y \in J$. Then $\forall w \in J[x < w]$, and $\{x\} \cup J$ is convex. Also, let K be the member of $\mathscr{U}(\delta + 1)$ such that $z \in K$. Then $\forall w \in K[y < w]$, and $\{y\} \cup K$ is convex. By (7) let $D \in \mathscr{U}(\gamma + 1)$ be such that $K \subseteq D$. Since $\{y\} \cup K$ is convex, $y \in (L \backslash \bigcup_{\theta \leq \gamma} L(\theta))$, and D is a convex component of $L \backslash \bigcup_{\theta \leq \gamma} L(\theta)$, it follows that $y \in D$. Hence $D = J$. Thus J is the member of $\mathscr{U}(\gamma + 1)$ such that $z \in J$, and $\forall w \in J[x < w]$, and $\{x\} \cup J$ is convex. So (x, z) is α-acceptable, and so $x \leq_\alpha z$.

(16) For all $\alpha < \omega_1$ the system $(\bigcup_{\beta \leq \alpha} L(\beta), \leq_\alpha)$ is a tree of height $\alpha + 1$ such that $L(\beta)$ is the set of all elements of height β, for each $\beta \leq \alpha$.

This is easily seen by induction on α.

(17) $(T, \leq_T)$ is a tree of height ω_1 such that $L(\beta)$ is the set of all elements of height β, for each $\beta < \omega_1$.

(18) T does not have a branch of length ω_1.

For, suppose that $\langle x_\alpha : \alpha < \omega_1 \rangle$ is a branch. Then by construction $x_\xi <_L x_\beta$ if $\alpha < \beta$, contradiction. $\square$

Theorem 22.19. *If there is a Suslin line, then there is a Suslin tree.*

Proof. Assume that there is a Suslin line. Then by Theorem 21.17 we may assume that we have a linear order L satisfying the following conditions:

(1) L is dense, with no first or last elements.

(2) No nonempty open subset of L is separable.

(3) L is ccc.

(We do not need the other condition given in Theorem 21.17.)

Now we define by recursion elements a_α, b_α of L, for $\alpha < \omega_1$. If these have already been defined for all $\beta < \alpha$, then the set $\{a_\beta : \beta < \alpha\} \cup \{b_\beta : \beta < \alpha\}$ is countable, and hence by (2) it is not dense in L. Let (c, d) be an open interval disjoint from this set, and pick a_α, b_α so that $c < a_\alpha < b_\alpha < d$ Thus for any $\xi < \alpha$ one of these conditions holds:

$a_\xi < a_\alpha < b_\alpha < b_\xi$;
$a_\alpha < b_\alpha < a_\xi < b_\xi$;
$a_\xi < b_\xi < a_\alpha < b_\alpha$.

Hence

$$\forall \xi, \alpha < \omega_1[\xi < \alpha \rightarrow [[a_\alpha, b_\alpha] \subseteq (a_\xi, b_\xi) \text{ or } [a_\alpha, b_\alpha] \cap (a_\xi, b_\xi) = \emptyset]. \tag{1}$$

Now we define a relation $\prec$ on ω_1 as follows: for any $\xi, \alpha < \omega_1$,

$$\xi \prec \alpha \text{ iff } \xi < \alpha \text{ and } [a_\alpha, b_\alpha] \subseteq (a_\xi, b_\xi).$$

If $\xi \prec \eta \prec \alpha$, then $\xi < \eta < \alpha$, hence $\xi < \alpha$, $[a_\eta, b_\eta] \subseteq (a_\xi, b_\xi)$, and $[a_\alpha, b_\beta] \subseteq (a_\eta, b_\eta)$, hence $[a_\alpha, b_\beta] \subseteq (a_\xi, b_\xi)$; so $\xi \prec \alpha$. Thus $\prec$ is transitive. Clearly it is irreflexive. So $\prec$ is a partial order on ω_1.

Now suppose that $\xi \prec \alpha$, $\eta \prec \alpha$, and $\xi \neq \eta$. We show that $\xi \prec \eta$ or $\eta \prec \xi$; hence $(\omega_1, \prec)$ is a tree. Wlog $\xi < \eta$. Now $[a_\alpha, b_\alpha] \subseteq (a_\xi, b_\xi) \cap (a_\eta, b_\eta)$, so $(a_\xi, b_\xi) \cap (a_\eta, b_\eta) \neq \emptyset$, so by (1) $[a_\eta, b_\eta] \subseteq (a_\xi, b_\xi)$. Thus $\xi \prec \eta$, as desired.

Now suppose that $\langle \alpha(\xi) < \xi < \omega_1 \rangle$ is $\prec$-increasing. Thus $\langle \alpha(\xi) < \xi < \omega_1 \rangle$ is $<$-increasing and $\forall \xi, \eta < \omega_1 [\xi < \eta \rightarrow [[a_{\alpha(\eta)}, b_{\alpha(\eta)}] \subseteq (a_{\alpha(\xi)}, b_{\alpha(\xi)})$. Then

$$\langle (a_{\alpha(\xi)}, b_{\alpha(\xi)}) \backslash [a_{\alpha(\xi+1)}, b_{\alpha(\xi+1)}] : \xi < \omega_1 \rangle$$

is a system of ω_1 pairwise disjoint open sets in L, contradiction.

Finally, if $\langle \alpha(\xi) : \xi < \omega_1 \rangle$ is a system of pairwise incomparable elements undet $\prec$, then by (1), $\langle (a_{\alpha(\xi)}, b_{\alpha(\xi)}) : \sigma < \omega_1 \rangle$ is a system of pairwise disjoint open intervals in L, contradiction. □

We mention without proof a result for higher cardinals. Assuming $V = L$, for each uncountable regular cardinal κ, there is a κ-Suslin tree iff κ is not weakly compact. It is a probably difficult open problem to show that it is consistent (relative to ZFC or even ZFC plus some large cardinals) that for each uncountable cardinal κ there is no κ^+-Aronszajn tree.

Theorem 22.20. *If there is a Kurepa line, then there is a Kurepa tree.*

Proof. Let L be a Kurepa line, and let $\langle a_\alpha : \alpha < \omega_1 \rangle$ be an enumeration of a dense subset of L. We now define the levels $L(\alpha)$ of a tree for $\alpha \leq \omega_1$, by recursion. Let $L(0) = \{L\}$. Suppose that $\alpha < \omega_1$ and $L(\alpha)$ has been defined so that the following conditions hold:

(1) $L(\alpha)$ is a countable set of pairwise disjoint convex subsets of L.

(2) $\{\xi < \omega_1 : a_\xi \notin \bigcup L(\alpha)\}$ is countable.

(3) If $C, D \in L(\alpha)$ and $C < D$, then there is a $\xi < \omega_1$ such that $a_\xi \notin \bigcup L(\alpha)$ and $C < a_\xi < D$.

(4) $L \backslash \{a_\xi : a_\xi \notin \bigcup L(\alpha)\} \subseteq \bigcup L(\alpha)$.

(5) $a_\alpha \notin \bigcup L(\alpha + 1)$.

Obviously these hold for $\alpha = 0$. Now for each $C \in L(\alpha)$ we define a set C':

(6) If $a_\alpha \in C$ and $(-\infty, a_\alpha) \cap C \neq \emptyset \neq (a_\alpha, \infty) \cap C$, let $C' = \{(-\infty, a_\alpha) \cap C, (a_\alpha, \infty) \cap C\}$.

(7) If a_α is the first element of C let $C' = \{C \backslash \{a_\alpha\}\}$.

(8) If a_α is the last element of C let $C' = \{C \backslash \{a_\alpha\}\}$.

(9) If $a_\alpha \notin C$ let $C' = \{C\}$.

Now let $L(\alpha + 1) = (\bigcup_{C \in L(\alpha)} C') \backslash \{\emptyset\}$. Clearly (1)–(5) hold for $\alpha + 1$.

Now suppose that $\gamma \leq \omega_1$ is limit and $L(\alpha)$ has been defined so that (1)–(5) hold for all $\alpha < \gamma$. For each $\alpha < \gamma$ let $M_\alpha = \{a_\xi : a_\xi \notin \bigcup L(\alpha)\}$. Let $N = \bigcup_{\alpha<\gamma} M_\alpha$. Then for any $\alpha < \gamma$ we have

$$L\backslash N \subseteq L\backslash M_\alpha \subseteq \bigcup L(\alpha),$$

and so

$$L\backslash N \subseteq \bigcap_{\alpha<\gamma} \bigcup L(\alpha) = \bigcup_{f\in F} \bigcap_{\alpha<\gamma} f(\alpha), \tag{10}$$

where $F = \prod_{\alpha<\gamma} L(\alpha)$. Let $L(\gamma) = \left\{\bigcap_{\alpha<\gamma} f(\alpha) : f \in F \text{ and } \bigcap_{\alpha<\gamma} f(\alpha) \neq \emptyset\right\}$. Clearly $L(\gamma)$ is a collection of pairwise disjoint convex subsets of L. Clearly (3) holds. For (4), $L\backslash \bigcup L(\gamma) = L\backslash \bigcup_{f\in F} \bigcap_{\alpha<\gamma} f(\alpha) \subseteq N$ by (10). Hence if $b \in L\backslash \bigcup L(\gamma)$ then there is an $\alpha < \gamma$ such that $b = a_\xi$ with $a_\xi \notin \bigcup L(\alpha)$. So for all $C \in L(\alpha)$ we have $b \notin C$, and hence for all $f \in F$ $b \notin \bigcap_{\beta<\gamma} f(\beta)$. Hence $b = a_\xi \notin \bigcup L(\gamma)$. This proves (4). Now suppose that $\lambda < \omega_1$. We claim that $L(\gamma)$ is countable; i.e. (1) holds. In fact, suppose not. If $\bigcap_{\alpha<\gamma} f(\alpha)$ and $\bigcap_{\alpha<\gamma} g(\alpha)$ are distinct members of $L(\gamma)$, then there is an $\alpha < \gamma$ such that $f(\alpha) \neq g(\alpha)$. So $f(\alpha), g(\alpha) \in L(\alpha)$. Say $f(\alpha) < g(\alpha)$. Then by (3) there is a $\xi < \omega_1$ such that $a_\xi \notin L(\alpha)$ and $f(\alpha) < a_\xi < g(\alpha)$. Thus if we choose $x_f \in \bigcap_{\beta<\gamma} f(\beta)$ for each $\bigcap_{\beta<\gamma} f(\beta) \in L(\gamma)$, the set N is dense in $N \cup \{x_f : \bigcap_{\beta<\gamma} f(\beta) \in L(\gamma)\}$, contradiction. So $L(\gamma)$ is countable. For (2), if $a_\xi \notin \bigcup L(\gamma)$ then $a_\xi \in N$ by (10); so (2) holds.

Thus we have a tree of height $\omega_1 + 1$. Note that $a_\xi \notin \bigcup L(\omega_1)$ for all $\xi < \omega_1$, by (5). Suppose that $b, c \in L\backslash\{a_\xi : \xi < \omega_1\}$ and $b < c$. Then by (4) there are $C, D \in L(\omega_1)$ such that $b \in C$ and $c \in D$. Choose a_α with $b < a_\alpha < c$. Then $C \neq D$. This shows that $L(\omega_1)$ has more than ω_1 members, and hence our tree restricted to levels less than ω_1 has more than ω_1 branches of length ω_1. ☐

Proposition 22.21. *Let $T \subseteq {}^{<\omega}\omega$ consist of all finite strictly decreasing sequences. Then there is no path through T.*

Proof. Suppose that P is a path through T. Choose m so that $\langle m\rangle \in P$. Then there are at most m elements of P above $\langle m\rangle$, contradiction. ☐

Proposition 22.22. (III.5.11) *If T is an ω_1-tree and there is an order preserving map from T into $\mathbb{R}$, then T is an Aronszajn tree.* ☐

Theorem 22.23. (III.5.12) *There is an Aronszajn tree that has an order preserving map into $\mathbb{Q}$.*

Proof. We first define sets which will be the levels of the tree:

$$\begin{aligned}
\mathscr{L}_0 &= \{0\};\\
\mathscr{L}_1 &= \omega\backslash\{0\};\\
\mathscr{L}_{n+1} &= \{\omega\cdot n + k : k \in \omega\} \quad \text{for } 0 < n < \omega;\\
\mathscr{L}_\alpha &= \{\omega\cdot\alpha + k : k \in \omega\} \quad \text{for } \omega \leq \alpha < \omega_1
\end{aligned}$$

Also, for each $\alpha < \omega_1$ we take any partition $\langle E^\xi_{\alpha+1} : \xi \in \mathscr{L}_\alpha \rangle$ of $\mathscr{L}_{\alpha+1}$ into infinite sets. Now the tree is $\bigcup_{\alpha<\omega_1} \mathscr{L}_\alpha$, and for each $\alpha < \omega_1$, $T_\alpha = \bigcup_{\beta<\alpha} \mathscr{L}_\alpha$.

Now we define the order $< \cap (T_\alpha \times T_\alpha)$ and $\varphi \restriction T_\alpha$ by induction α. Let $< \cap (T_0 \times T_0) = \emptyset$ and $\varphi \restriction T_0 = \emptyset$. Let $< \cap (T_1 \times T_1) = \emptyset$ and $\varphi \restriction T_1 = \{(0,0)\}$. Let $< \cap (T_2 \times T_2) = \{(0,m) : m \in \omega\backslash 1\}$ and $\varphi \restriction T_2 = \{(0,0)\} \cup \psi$, where ψ is a bijection from $\omega\backslash 1$ onto the rationals in $(0,\infty)$. Now suppose that $1 \le \alpha$ and $< \cap (T_{\alpha+1} \times T_{\alpha+1})$ and $\varphi \restriction T_\alpha$ have been defined so that the following condition holds:

$$\forall x \in T_\alpha \forall q \in \mathbb{Q}[\varphi(x) < q \to \exists y \in \mathscr{L}_\alpha[x < y \wedge \varphi(y) = q]]. \tag{1}$$

Note that (1) holds vacuously if $\alpha = 0$, since $T_0 = \emptyset$. For $\alpha = 1$, if $x \in T_1$ then $x \in \mathscr{L}_0$, so $x = 0$. Then if $q \in \mathbb{Q}$ and $\varphi(x) < q$, it follows that $\varphi(x) = 0$, so $q > 0$, and the existence of the desired y follows from the definition of $\varphi \restriction T_2$.

Now for each $\xi \in \mathscr{L}_\alpha$ we put each member of $E^\xi_{\alpha+1}$ directly above ξ. This defines $< \cap (T_{\alpha+2} \times T_{\alpha+2})$. We extend φ to $T_{\alpha+2}$ by mapping each set $E^\xi_{\alpha+1}$ one-one onto $\mathbb{Q} \cap (\varphi(\xi), \infty)$. To check (1) for $\alpha + 1$, suppose that $x \in T_{\alpha+1}$ and $\varphi(x) < q \in \mathbb{Q}$.

Case 1. $x \in \mathscr{L}_\alpha$. Then there is a $z \in E^x_{\alpha+1}$ such that $x < z$ and $\varphi(z) = q$, as desired.

Case 2. $x \in T_\alpha$. Choose $r \in \mathbb{Q}$ such that $\varphi(x) < r < q$. Hence by (1) for α, there is a $y \in \mathscr{L}_\alpha$ such that $x < y$ and $\varphi(y) = r$. Then by definition there is a $z \in E^y_{\alpha+1}$ such that $y < z$ and $\varphi(z) = q$, as desired.

Now we assume that γ is a countable limit ordinal and $< \cap (T_\gamma \times T_\gamma)$ and $\varphi \restriction T_\gamma$ have been defined. Let $\{(x_k, q_k) : k \in \omega\}$ enumerate all pairs (x, q) such that $x \in T_\gamma$ and $\varphi(x) < q \in \mathbb{Q}$.

(2) For each $k \in \omega$ there is a path P_k through T_γ such that $x_k \in P_k$ and $\sup\{\varphi(y) : y \in P_k\} = q_k$.

For, let $k \in \omega$. Let $\langle \alpha_n : n \in \omega \rangle$ be a strictly increasing sequence of ordinals such that $\text{height}(x_k) < \alpha_0$ and $\sup_{n\in\omega} \alpha_n = \gamma$. Let $\langle r_n : n \in \omega \rangle$ be a strictly increasing sequence of rationals such that $\varphi(x_k) = r_0$ and $\sup_{n\in\omega} r_n = q_k$. Then we define $\langle z_n : n \in \omega \rangle$ as follows. Let $z_0 = x_k$. If z_n has been defined so that $\text{ht}(z_n) < \alpha_n$ and $\varphi(z_n) = r_n$, we apply (1) to obtain $z_n < z_{n+1} \in \mathscr{L}_{\alpha_n}$ and $\varphi(z_{n+1}) = r_{n+1}$. Now let P_k be the path through T_γ determined by the z_ns. This proves (2).

Now we put $\omega \cdot \gamma + k$ directly above P_k, for each $k \in \omega$, and set $\varphi(\omega \cdot \gamma + k) = q_k$. $\square$

We define $(y\downarrow') = \{x : x \le y\}$ and $(y\uparrow') = \{x : y \le x\}$.

Proposition 22.24. *Let T be a tree. We call $U \subseteq T$* **open** *iff for all $t \in U$ of limit height there is an $x < t$ such that $(x\uparrow) \cap (t\downarrow') \subseteq U$. Then the collection of all open sets forms a topology on T.*

Proof. Clearly $\emptyset$ and T are open. Suppose that U_1 and U_2 are open and $t \in U_1 \cap U_2$ is of limit height. Choose $x < t$ such that $(x\uparrow) \cap (t\downarrow') \subseteq U_1$, and choose $y < t$ such that $(y\uparrow) \cap (t\downarrow') \subseteq U_2$. Wlog $x \le y$. Then $(y\uparrow) \cap (t\downarrow') \subseteq U_1 \cap U_2$, so $U_1 \cap U_2$ is open. Finally, suppose that $\mathscr{A}$ is a collection of open sets and $t \in \bigcup \mathscr{A}$ is of limit height. Say $t \in U \in \mathscr{A}$. Choose $x < t$ such that $(x\uparrow) \cap (t\downarrow') \subseteq U$. Then $(x\uparrow) \cap (t\downarrow') \subseteq \bigcup \mathscr{A}$. So $\bigcup \mathscr{A}$ is open. $\square$

Proposition 22.25. *If T is a tree and $\varphi : T \to \mathbb{R}$ is order preserving, then φ is continuous iff for each limit γ and each $t \in \mathrm{Lev}_\gamma(T)$, $\varphi(t) = \sup_{x<t} \varphi(x)$.*

Proof. $\Rightarrow$: Assume that φ is continuous and t has limit level. By order preserving, $\varphi(x) < \varphi(t)$ for all $x < t$; so $\sup_{x<t} \varphi(x) \le \varphi(t)$. Suppose that $\sup_{x<t} \varphi(x) < \varphi(t)$, and let $\varepsilon = \varphi(t) - \sup_{x<t} \varphi(x)$. Now $t \in \varphi^{-1}[(\varphi(t) - \varepsilon, \varphi(t) + \varepsilon)]$, and $\varphi^{-1}[(\varphi(t) - \varepsilon, \varphi(t) + \varepsilon)]$ is open, so there exist z, s such that s has limit level, $z < s$, and $t \in (z\uparrow) \cap (s\downarrow') \subseteq \varphi^{-1}[(\varphi(t) - \varepsilon, \varphi(t) + \varepsilon)]$. Now take any y with $z < y < t$. Then $y \in (z\uparrow) \cap (s\downarrow')$, so $y \in \varphi^{-1}[(\varphi(t) - \varepsilon, \varphi(t) + \varepsilon)]$. Thus $\varphi(t) - \varepsilon < \varphi(y)$. But $\varphi(y) \le \sup_{x<t} \varphi(x) = \varphi(t) - \varepsilon$, contradiction.

$\Leftarrow$: Assume the limit condition, and suppose that $t \in \varphi^{-1}[(u, v)]$. If t is not of limit level, then $t \in \{t\} \subseteq \varphi^{-1}[(u, v)]$, and $\{t\}$ is open. If t is of limit level, by the limit condition choose $x < t$ such that $u < \varphi(x)$. Then $t \in (x\uparrow) \cap (t\downarrow) \subseteq \varphi^{-1}[(u, v)]$. $\square$

Lemma 22.26. (III.5.14) *The function $\varphi : T \to \mathbb{Q}$ constructed in the proof of Theorem 22.23 is continuous.* $\square$

An Aronszajn tree T is *special* iff there are antichains A_n for $n \in \omega$ such that $T = \bigcup_{n\in\omega} A_n$.

Lemma 22.27. (III.5.17) *If T is an Aronszajn tree, then the following conditions are equivalent:*

(i) T is special.

(ii) There is an order preserving map $\varphi : T \to \mathbb{Q}$.

Proof. (ii)$\Rightarrow$(i): Assume (ii). Then $\varphi^{-1}[\{q\}]$ is an antichain for every $q \in \mathbb{Q}$.

(i)$\Rightarrow$(ii): Assume that T is special; say $T = \bigcup_{n\in\omega} A_n$, each A_n an antichain. Let $B_n = A_n \setminus \bigcup_{i<n} A_i$ for all $n \in \omega$. Then each B_n is an antichain, the B_ns are pairwise disjoint and $T = \bigcup_{n\in\omega} B_n$. We now define $\varphi \restriction B_n$ by recursion on n. For each $x \in T$ let $a(x)$ be the n such that $x \in B_n$. Let $\varphi(x) = 0$ for all $x \in B_0$. Now assume inductively that $n > 0$. For each $t \in B_n$ we define

$$p_t = \max(\{-1\} \cup \{\varphi(x) : x < t \text{ and } a(x) < n\});$$
$$q_t = \min(\{1\} \cup \{\varphi(x) : t < x \text{ and } a(x) < n\}).$$

Note that $p_t < q_t$. We define $\varphi(t) = (p_t + q_t)/2$. If $x \in \bigcup_{m<n} B_m$ and $x < t$, then $\varphi(x) \le p_t < \varphi(t)$. If $x \in \bigcup_{m<n} B_m$ and $t < x$, then $\varphi(t) < q_t \le \varphi(x)$. $\square$

Lemma 22.28. (III.5.18) *Suppose that T is an uncountable tree with no uncountable chains, and $\mathscr{E} \subseteq [T]^{<\omega}$ is pairwise disjoint and uncountable. Then there exist $a, b \in \mathscr{E}$ such that x and y are incomparable for all $x \in a$ and $y \in b$.*

Proof. We may assume that $|\mathscr{E}| = \omega_1$ and there is a positive integer n such that $|a| = n$ for all $a \in \mathscr{E}$. Write $\mathscr{E} = \{a^\alpha : \alpha < \omega_1\}$ without repetitions, and $a^\alpha = \{x_i^\alpha : i < n\}$. The case $n = 1$ is trivial, since $\langle x_i^\alpha : \alpha < \omega_1\rangle$ cannot be a chain. Hence assume that $n > 1$.

Let U be an ultrafilter on ω_1 which contains all co-countable sets.

Suppose the assertion fails. Then for all distinct $\alpha, \beta < \omega_1$ there exist $i_{\alpha\beta}, j_{\alpha\beta} < n$ such that $x^\alpha_{i_{\alpha\beta}}$ and $x^\beta_{j_{\alpha\beta}}$ are comparable. For each $\alpha < \omega_1$ we have

$$\omega_1 \backslash \{\alpha\} = \bigcup_{i,j<n} \{\beta \in \omega_1 \backslash \{\alpha\} : (i_{\alpha\beta}, j_{\alpha\beta}) = (i, j)\}.$$

Hence there exist $i_\alpha, j_\alpha < n$ such that $M_\alpha \stackrel{\text{def}}{=} \{\beta \in \omega_1 \backslash \{\alpha\} : (i_{\alpha\beta}, j_{\alpha\beta}) = (i_\alpha, j_\alpha)\} \in U$. Now $\langle (i_\alpha, j_\alpha) : \alpha < \omega_1 \rangle$ is a function mapping ω_1 into $n \times n$, so there exist an uncountable S and $i, j < n$ such that $\forall \alpha \in S[i_\alpha = i$ and $j_\alpha = j]$. Take any distinct α, β in S. Then $M_\alpha \cap M_\beta \in U$ and hence $M_\alpha \cap M_\beta$ is uncountable. Now $(x^\alpha_i \downarrow') \cup (x^\beta_i \downarrow')$ is countable, while for each $\gamma \in M_\alpha \cap M_\beta$ the element $x^\gamma_j = x^\gamma_{j_{\alpha\gamma}}$ is comparable with $x^\alpha_{i_{\alpha\gamma}} = x^\alpha_i$ and similarly x^γ_j is comparable with x^β_j. If follows that some x^γ_j is above x^α_i and x^β_i, so x^α_i and x^β_i are comparable. Thus $\langle x^\alpha_i : \alpha \in S \rangle$ is an uncountable chain, contradiction. $\square$

Proposition 22.29. *If $\varphi : T \to \mathbb{Q}$ is order preserving, then φ is continuous iff for every limit $t \in T$ and every positive $q \in \mathbb{Q}$ there is a $s \sqsubset t$ such that $\varphi(t) - q < \varphi(w) < \varphi(t)$ for all w such that $s \sqsubset w \sqsubset t$.*

Proof. $\Leftarrow$: Let V be open in $\mathbb{Q}$; we want to show that $\varphi^{-1}[V]$ is open in T. Take any limit $t \in \varphi^{-1}[V]$. Choose a positive $q \in \mathbb{Q}$ such that $\varphi(t) \in (\varphi(t) - q, \varphi(t) + q) \subseteq V$. Choose $s \sqsubset t$ such that $\varphi(t) - q < \varphi(w) < \varphi(t)$ for all w such that $s \sqsubset w \sqsubset t$. Then $(s\uparrow) \cap (t\downarrow') \subseteq \varphi^{-1}[(\varphi(t) - q, \varphi(t) + q)] \subseteq \varphi^{-1}[V]$.

$\Rightarrow$: Assume that φ is continuous, $t \in T$ is limit, and $q \in \mathbb{Q}$ is positive. Then $t \in \varphi^{-1}[\varphi(t) - q, \varphi(t) + q)]$ and $\varphi^{-1}[\varphi(t) - q, \varphi(t) + q)]$ is open. Choose $s \sqsubset t$ such that $(s\uparrow) \cap (t\downarrow') \subseteq \varphi^{-1}[\varphi(t) - q, \varphi(t) + q)]$. Then $\varphi(t) - q < \varphi(w) < \varphi(t)$ for all w such that $s \sqsubset w \sqsubset t$.

Lemma 22.30. (III.5.20) *If T is a tree and φ is an order preserving map from T into $\mathbb{R}$, then there is an order preserving continuous map $\psi : T \to \mathbb{R}$.*

Proof. Define

$$\psi(t) = \begin{cases} \varphi(t) & \text{if } t \text{ does not have limit height,} \\ \sup_{x<t} \varphi(x) & \text{if } t \text{ has limit height.} \end{cases}$$ $\square$

A tree T is *rooted* iff $|\mathrm{Lev}_0(T)| = 1$.
T is *Hausdorff* iff for all limit γ and all $x, y \in \mathrm{Lev}_\gamma(T)$, if $x\downarrow = y\downarrow$ then $x = y$.

Proposition 22.31. *T is Hausdorff in the tree topology iff T is Hausdorff in the above sense.*

Proof. $\Leftarrow$. Let $s \neq t$ be elements of T. If neither is limit, then $\{s\}$ and $\{t\}$ are disjoint open neighborhoods. Suppose, e.g., that s is non-limit but t is limit. Then $\{s\}$ and $T \backslash \{s\}$ are disjoint neighborhoods. Now suppose that both s and t are limit. If they are comparable, say $s \sqsubset t$. Then for any $x < s$, $(x\uparrow) \cap (s\downarrow')$ and $(s\uparrow') \cap (t\downarrow')$ are disjoint neighborhoods. Suppose that s and t are not comparable. If they are at different levels, clearly disjoint open neighborhoods exist. Suppose that they are at the same level. Then by assumption $(s\downarrow) \neq (t\downarrow)$. If $(s\downarrow)\cap(t\downarrow) = \emptyset$, then clearly there are disjoint neighborhoods. If $(s\downarrow)\cap(t\downarrow) \neq \emptyset$, then there is a largest x below both, using Hausdorffness. Then $(x\uparrow)\cap(s\downarrow')$ and $(x\uparrow) \cap (t\downarrow')$ are disjoint open neighborhoods.

$\Rightarrow$. Suppose that x and y are both at the same limit level, $x \neq y$. Let U, V be disjoint neighborhoods. Choose $s \sqsubset x$ with $(s\uparrow)\cap(x\downarrow') \subseteq U$ and choose $t \sqsubset y$ with $(t\uparrow)\cap(y\downarrow') \subseteq V$. Choose w with $s \sqsubset w \sqsubset x$ and $\mathrm{ht}(t) < \mathrm{ht}(w)$. Then $w \in (x\downarrow)$. If $w \in (y\downarrow)$, then w and t are comparable; hence $t \sqsubset w$. But then $w \in (t\uparrow) \cap (y\downarrow')$, so $w \in U \cap V$, contradiction. It follows that $(x\downarrow) \neq (y\downarrow)$. □

Proposition 22.32. *There is a non-Hausdorff Aronszajn tree that has an order preserving map into the rational numbers.*

Proof. We modify the proof of Theorem 22.23. Let $s : \omega \to \omega{\times}2$ be a bijection. Define $P'_{k,\varepsilon} = P_k$ for all $k \in \omega$ and $\varepsilon \in 2$. Define $P''_l = P'_{s(l)}$ for any $l \in \omega$. Put $\omega \cdot \gamma + l$ directly above P''_l for each $l \in \omega$. Note that $P''_{s^{-1}(k,0)} = P'_{k,0} = P_k$ and $P''_{s^{-1}(k,1)} = P'_{k,1} = P_k$. □

Proposition 22.33. *There is a Hausdorff Aronszajn tree that has an order preserving map into the rationals.*

Proof. We modify the proof of Theorem 22.23. Define $k \equiv l$ iff $P_k = P_l$. Note that $\langle \omega + k : k \in \omega\rangle$ is a pairwise incomparable set in T_γ. Hence there are infinitely many equivalence classes. Let $s : \omega \to T_\gamma/\equiv$ be a bijection. For each $k \in \omega$ let $P'_k = P_l$ where l is any integer equivalent to k. Put $\omega \cdot \gamma + k$ directly above P'_k. □

If T is a κ-tree, then we define $T^p = \{x \in T : |x\uparrow'| = \kappa\}$.

Lemma 22.34. (III.5.26) *If κ is regular and T is a κ-tree, then:*

(i) T^p is a normal subtree of T.

(ii) T is well-pruned iff T is rooted and $T^p = T$.

(iii) T^p is a κ-tree.

(iv) $(T^p)^p = T^p$.

(v) For each $y \in \mathrm{Lev}_0(T^p)$ the set $(y\uparrow') \cap T^p$ is a well-pruned κ-tree and is a subtree of T with y as only root.

Proof. (i) and (ii) are obvious. For (iii), note that for any $\alpha < \kappa$ we have

$$T = \bigcup_{x \in \mathrm{Lev}(\alpha)(T)} (x\uparrow') \cup \bigcup_{\xi \leq \alpha} \mathrm{Lev}(\alpha)(T).$$

Since T is a κ-tree, there is an $x \in \mathrm{Lev}_\alpha(T)$ such that $|(x\uparrow')| = \kappa$. Hence T^p has elements of each level, and (iii) follows.

For (iv), $(T^p)^p = \{x \in T^p : |(x\uparrow')| = \kappa\} = T^p$.

(v) is obvious. □

Lemma 22.35. (III.5.27) *Let κ be a regular cardinal.*

(i) If there is a κ-Aronszajn tree, then there is well-pruned κ-Aronszajn tree.

(i) If there is a κ-Suslin tree, then there is well-pruned κ-Suslin tree. $\square$

Proposition 22.36. *Let κ be a regular cardinal.*

(i) If there is a κ-Aronszajn tree, then there is well-pruned κ-Aronszajn Hausdorff tree.

(i) If there is a κ-Suslin tree, then there is well-pruned κ-Suslin Hausdorff tree.

Proof. For each limit ordinal $\gamma < \kappa$, define $x \equiv_\gamma y$ iff $x, y \in \mathrm{Lev}_\gamma(T)$ and $(x\downarrow) = (y\downarrow)$. This is an equivalence relation on $\mathrm{Lev}_\gamma(T)$. We introduce new elements $z_{\gamma t}$ for each equivalence class t, and set $z_{\gamma t} < u$ for each $u \in t$, with other obvious stipulations. $\square$

If κ is regular, we say that T is a κ-Kurepa tree iff T is a κ-tree with at least κ^+ branches of length κ.

Proposition 22.37. *For $\kappa = \omega$ or κ strongly inaccessible, the tree $2^{<\kappa}$ is a κ-Kurepa tree.* $\square$

For any regular κ, a *κ-Kurepa family* is a set $\mathscr{F} \subseteq \mathscr{P}(\kappa)$ such that $|\mathscr{F}| \geq \kappa^+$ and $\forall \alpha < \kappa[|\{X \cap \alpha : X \in \mathscr{F}\}| < \kappa]$.

Lemma 22.38. (III.5.40) *If κ is regular, then there is a κ-Kurepa tree iff there is a κ-Kurepa family.*

Proof. $\Leftarrow$: suppose that $\mathscr{F}$ is a κ-Kurepa family. Let

$$T = \bigcup_{\alpha<\kappa} \{\chi_{X\cap\alpha} : X \in \mathscr{F}\},$$

where $\chi_{X\cap\alpha}$ is the characteristic function of $X \cap \alpha$ within α. Clearly this gives a κ-Kurepa tree.

$\Rightarrow$: Assume that T is a κ-Kurepa tree. Now $|T| = \kappa$, so we may assume that $T = \kappa$. Let $\mathscr{F}$ be the set of all branches of length κ in T; $|\mathscr{F}| \geq \kappa^+$. For each $\alpha < \kappa$ fix $\delta < \kappa$ so that $\alpha \subseteq T_\delta$. Now for each $X \in \mathscr{F}$ let $\langle x^X_\xi : \xi < \kappa\rangle$ enumerate X in increasing order. Define $f(x^X_\delta) = \{x^X_\beta : \beta < \delta\} = X \cap T_\delta$. Since $\{x^X_\delta : X \in \mathscr{F}\}$ is a subset of Lev_δ, which has size less than κ, it follows that $\{X \cap T_\delta : X \in \mathscr{F}\}$ has size less than κ. For each $Y \in \{X \cap T_\delta : X \in \mathscr{F}\}$ let $g(y) = Y \cap \alpha$. Then g maps $\{X \cap T_\delta : X \in \mathscr{F}\}$ onto $\{X \cap \alpha : X \in \mathscr{F}\}$. Hence $|\{X \cap \alpha : X \in \mathscr{F}\}| \leq |\{X \cap T_\delta : X \in \mathscr{F}\}| < \kappa$. $\square$

Proposition 22.39. (III.5.41) *If κ is regular and there is a κ-Kurepa tree, then there is one which is Hausdorff and well-pruned.*

Proof. Suppose that κ is regular and T is a κ-Kurepa tree. For each limit ordinal $\alpha < \kappa$ define $s \equiv_\alpha t$ iff $\mathrm{height}(s) = \alpha = \mathrm{height}(t)$ and $s\downarrow = t\downarrow$. Let $\{u_{\alpha,x} : \alpha$ a limit ordinal less than κ and x is an $\equiv_\alpha$-class$\}$ be a system of elements not in T. We define T' to be T together with all such elements. We define $<$ on T' by extending the order on T and setting

$$\begin{aligned} s < u_{\alpha,x} &\quad \text{iff} \quad \exists t \in x[s < t]; \\ u_{\alpha,x} < s &\quad \text{iff} \quad \exists t \in x[t \leq s]; \\ u_{\alpha,x} < u_{\beta,y} &\quad \text{iff} \quad \exists s \in x \exists t \in y[s < t]. \end{aligned}$$

Then T' is a tree, with

$$\begin{aligned}
\mathrm{ht}_m(T') &= \mathrm{ht}_m(T) \quad \text{for all } m \in \omega,\\
\mathrm{ht}_\omega(T') &= \{u_{\omega,x} : x \text{ an } \equiv_\omega\text{-class}\};\\
\mathrm{ht}_{\omega+m}(T') &= \mathrm{ht}_{\omega+m-1}(T)\text{for all } m \in \omega\backslash 1;\\
\mathrm{ht}_{\omega\cdot\alpha}(T') &= \{u_{\omega\cdot\alpha,x} : x \text{ an } \equiv_{\omega\cdot\alpha}\text{-class}\} \quad \text{for all } \alpha > 0;\\
\mathrm{ht}_{\omega\cdot\alpha+m}(T') &= \mathrm{ht}_{\omega\cdot\alpha+m-1}(T) \quad \text{for all } \alpha > 0 \text{ and } m \in \omega\backslash 1.
\end{aligned}$$

Clearly T' is a Hausdorff κ-Kurepa tree.

Since $|\mathrm{ht}_0(T')| < \kappa$ and κ is regular, there is a $t \in \mathrm{ht}_0(T')$ with κ^+ branches of length κ through it.

Let $T'' = \{s \in T' : t \le s$ and there are κ^+ branches through t of length $\kappa\}$. Clearly T'' is a normal subtree of T'. Let C be the set of all branches through t in T' of length κ. Thus $|C| = \kappa^+$. Now for any $\alpha < \kappa$ we have

$$C = \bigcup_{s \in \mathrm{ht}_\alpha(T')} \{b \in C : s \in b\},$$

and $|\mathrm{ht}_\alpha(T'')| < \kappa$, so there is an $s \in \mathrm{ht}_\alpha(T')$ such that $|\{b \in C : s \in b\}| \ge \kappa^+$. this shows that T'' is well-pruned and of height κ. Clearly T'' is a Hausdorff κ-Kurepa tree. $\square$

Proposition 22.40. (III.5.42) *A tree T is Hausdorff and rooted iff T is isomorphic to a subtree of some sequence tree $B^{<\alpha}$.*

Proof. $\Leftarrow$: $\emptyset$ is the root. Suppose that β is limit less than α and $f \restriction \beta = f \restriction \beta$. Then $f = g$.

$\Rightarrow$: We define $\varphi : T \to {}^{<\mathrm{ht}(T)}T$ by recursion on height(t). For the root r of T let $\varphi(r) = \emptyset$. Now suppose that $\beta = \gamma + 1 < \mathrm{ht}(T)$ and $t \in T$ has height β. We define $\varphi(t) = \varphi(t \restriction \gamma)^\frown\langle t\rangle$. For β limit less than height(T) and t of height β let $\varphi(t) = \bigcup_{s \sqsubset t} \varphi(s)$; this is unambiguous since T is Hausdorff.

Clearly φ is the desired isomorphism. $\square$

Proposition 22.41. *Every tree is T_1 in the tree topology.*

Proof. Let t_1, t_2 be distinct members of T.

Case 1. Neither one is of limit level. Then $\{t_1\}$ and $\{t_2\}$ are disjoint open neighborhoods.

Case 2. t_1 is of limit level, but t_2 is not. Then $\{t_2\}$ is open containing t_2 but not t_1, and there clearly is an open set containing t_1 but not t_2.

Case 3. t_2 is of limit level, but t_1 is not. Ok by symmetry.

Case 4. Both have limit level.

Subcase 4.1. The levels are different; say by symmetry $t_1 \in \mathrm{Lev}_\alpha(T)$ and $t_2 \in \mathrm{Lev}_\beta(T)$ with $\alpha < \beta$. Choose γ with $\alpha < \gamma < \beta$, and let $s < t_2$ have level γ. Then $(s\uparrow) \cap (t_2\downarrow')$ is an open neighborhood of t_2 disjoint from any standard open neighborhood of t_1.

Subcase 4.2. The levels are the same, but $(t_1\downarrow) \neq (t_2\downarrow)$. Then there is an $x < t_1$ such that $x \not< t_2$. With $y < t_2$ such that level(x) =level(y) we then have disjoint open neighborhoods $(x\uparrow) \cap (t_1\downarrow')$ and $(y\uparrow) \cap (t_2\downarrow')$.

Subcase 4.3. The levels are the same, and $(t_1\downarrow) = (t_2\downarrow)$. Take any $x < t_1$. Then $(x\uparrow) \cap (t_1\downarrow')$ is open containing t_1 but not t_2, and $(x\uparrow) \cap (t_2\downarrow')$ is open containing t_2 but not t_1. □

Proposition 22.42. *If T is a Hausdorff tree, then T is regular.*

Proof. Suppose $x \in U$, U open. We want to find disjoint open sets U_1, U_2 such that $x \in U_1$ and $T\backslash U \subseteq U_2$.

Case 1. x is not at a limit level. Let $U_1 = \{x\}$ and $U_2 = T\backslash\{x\}$.

Case 2. x is at a limit level. Then there is a $y < x$ such that $(y\uparrow) \cap (x\downarrow') \subseteq U$. Note that if w is at a limit level and $w \not\leq x$ then there is a $t < w$ such that $t \not< x$. In fact, this is clear if level(x) <level(w). If level(w) =level(x), then $(w\downarrow) \not\subseteq (x\downarrow)$ using Hausdorffness, and the existence of t follows. If level(w) <level(x), then $(w\downarrow) \not\subseteq (x\downarrow)$ since $w \not\leq x$, again using Hausdorffness. This proves the existence of such a t in any case. We let t_w be such a t.

If w is limit and $y \not< w$, then there is an $s < w$ such that $y \not< s$. In fact, if level(y) <level(w), then we can take $s < w$ with level(s) = level(y). If level(w) $\leq$level(y), any $s < w$ works. An s such that $s < w$ and $y \not< s$ is denoted by s_w. Now we set

$$\begin{aligned}
U_1 &= (y\uparrow) \cap (x\downarrow');\\
U_2 &= \{z : z \text{ non-limit and } y \not< z\} \cup \{z : z \text{ non-limit and } z \not\leq x\}\\
&\quad \cup \bigcup\{(t_w\uparrow) \cap (w\downarrow') : w \text{ is limit and } w \not\leq x\}\\
&\quad \cup \bigcup\{(s_w\uparrow) \cap (w\downarrow') : w \text{ is limit and } y \not< w\}.
\end{aligned}$$

Thus $x \in U_1$. Suppose that $z \in X\backslash U$. Then

$$z \in X\backslash((y\uparrow) \cap (x\downarrow')) = \{w : y \not< w\} \cup \{w : w \not\leq x\}.$$

Clearly then $z \in U_2$. Finally, clearly $U_1 \cap U_2 = \emptyset$. □

Proposition 22.43. *If T is Hausdorff, then it is zero-dimensional.*

Proof. A base for the topology consists of all sets $\{t\}$ with t not of limit level, and all sets $(y\uparrow) \cap (x\downarrow')$ where x is of limit level and $y < x$. Clearly each $\{t\}$, t not of limit level, is clopen. We claim that $(y\uparrow) \cap (x\downarrow')$ is clopen, where x is of limit level, and $y < x$. It suffices to show that its complement

$$U \stackrel{\text{def}}{=} \{z : y \not< z\} \cup \{z : z \not\leq x\}$$

is open. First suppose that $y \not< z$. If z is not of limit level, then $\{z\} \subseteq (T\backslash U)$ and $\{z\}$ is open. If z is of limit level, take any $w < z$. Then $z \in (w\uparrow) \cap (z\downarrow')$, and this set is contained in $\{u : y \not< u\}$. Second suppose that $z \not\leq x$. If z is not of limit level then the desired conclusion is clear. Suppose that z is of limit level. Then $(z\downarrow) \not\subseteq (x\downarrow)$, using Hausdorffness. So there is $y < z$ such that $y \not< x$. Hence $z \in (y\uparrow) \cap (z\downarrow')$, and this set is contained in $T\backslash U$. □

Proposition 22.44. *Assume GCH. If κ is regular then there is a κ^+-Aronszajn tree.*

Proof. Write $\kappa = \aleph_\alpha$, and consider the set H_α defined on page 315. By Corollary 21.7, H_α is an η_α-set of size κ. Thus

(1) H_α is a linear order, and if $A, B \subseteq H_\alpha$ with $|A|, |B| < \kappa$ and $A < B$, then there is a $c \in H_\alpha$ such that $A < c < B$.

We note also the following properties of H_α:

(2) H_α is dense.

For, suppose that $f, g \in H_\alpha$ and $f < g$. Let $\beta = \chi(f, g)$, and choose $\gamma < \kappa$ such that $f(\gamma) = 1$ and $\forall\delta \in (\gamma, \kappa)[f(\delta) = 0]$. Note that $f(\beta) = 0$ and $g(\beta) = 1$.

Case 1. $\beta < \gamma$. Let $h \restriction (\gamma + 1) = f \restriction (\gamma + 1)$, $h(\gamma + 1) = 1$, $\forall\delta \in (\gamma + 1, \kappa)[h(\delta) = 0]$. Then $f < h < g$.

Case 2. $\gamma \le \beta$. Then actually $\gamma < \beta$. Let $h \restriction (\beta + 1) = f \restriction (\beta + 1)$, $h(\beta + 1) = 1$, and $\forall\delta \in (\beta + 1, \kappa)[h(\delta) = 0]$. Then $f < h < g$.

(3) For all $f \in H_\alpha$ the set (f, ∞) has size κ.

In fact, say $\xi < \kappa$ and $f(\xi) = 1$ while $\forall\eta \in (\xi, \kappa)[f(\eta) = 0]$. Then

$$\begin{aligned}
\kappa \ge |(f, \infty)| &\ge |\{g \in H_\alpha : f \restriction (\xi + 1) = g \restriction (\xi + 1) \text{ and} \\
&\qquad \exists\eta \in (\xi + 1, \kappa)[g(\eta) = 1 \text{ and } \forall\rho \in (\eta, \kappa)[g(\rho) = 0]]\}| \\
&= \sum_{\eta \in (\xi+1,\kappa)} |\{g \in H_\alpha : f \restriction (\xi + 1) = g \restriction (\xi + 1) \text{ and} \\
&\qquad [g(\eta) = 1 \text{ and } \forall\rho \in (\eta, \kappa)[g(\rho) = 0]]\}| \\
&= \sum_{\eta \in (\xi+1,\kappa)} \left|{}^{(\xi+1,\eta)}2\right| \\
&= \left|\bigcup_{\eta \in (\xi+1,\kappa)} ({}^{(\xi+1,\eta)}2 \times \{\eta\})\right| \\
&= \left|\bigcup_{0<\rho<\kappa} ({}^{\rho}2 \times \{\rho\})\right| \\
&= \sum_{0<\rho<\kappa} 2^\rho = \kappa.
\end{aligned}$$

We will define the tree T and an order preserving map from T into H_α. First we define sets which will be the levels of the tree:

$$\mathrm{Lev}_0 = \{0\};$$

$$
\begin{aligned}
\mathrm{Lev}_1 &= \kappa\backslash\{0\};\\
\mathrm{Lev}_{n+1} &= \{\kappa\cdot n+k : k\in\kappa\} \quad \text{for } 0<n<\omega;\\
\mathrm{Lev}_\beta &= \{\kappa\cdot\beta+k : k\in\kappa\} \quad \text{for } \omega\le\beta<\kappa^+
\end{aligned}
$$

Also, for each $\beta<\kappa^+$ we take any partition $\langle E^\xi_{\beta+1} : \xi\in\mathrm{Lev}_\beta\rangle$ of $\mathrm{Lev}_{\beta+1}$, each $E^\xi_{\beta+1}$ of size κ.

Now we define the tree, its order, and the order preserving function φ by induction on the level. We let 0 be the root of the tree, and we define $\varphi(0)=0$. Now suppose that the tree and φ have been defined through level β, giving $T_{\beta+1}$ and $\varphi : T_{\beta+1}\to H_\alpha$, so that the following condition holds:

(4) $\quad \forall x\in T_\beta\forall q\in H_\alpha[\mathrm{height}(x)<\beta\wedge\varphi(x)<q\to\exists y\in\mathrm{Lev}_\beta[x<y\wedge\varphi(y)=q]].$

Note that (4) holds vacuously if $\beta=0$. Now we put each member of $E^\xi_{\beta+1}$ directly above ξ, for each $\xi\in\mathrm{Lev}_\beta$. We extend φ to $\mathrm{Lev}_{\beta+1}$ by mapping each set $E^\xi_{\beta+1}$ one-one onto $(\varphi_\beta(\xi),\infty)$. This is possible by (3). To check (4) for $\beta+1$, suppose that $x\in T_{\beta+1}$, $q\in H_\alpha$, $\mathrm{height}(x)<\beta+1$, and $\varphi(x)<q$.

Case 1. $x\in T_\beta$. Then by (1) choose $r\in H_\alpha$ such that $\varphi(x)<r<q$. Thus $\mathrm{height}(x)<\beta$ and $\varphi(x)<r$. Hence by (4) for β, there is a $y\in\mathscr{L}_\beta$ such that $x<y$ and $\varphi(y)=r$. Then by definition there is a $z\in E^y_{\beta+1}$ such that $y<z$ and $\varphi(z)=q$, as desired.

Case 2. $x\in\mathrm{Lev}_\beta$. Then there is a $z\in E^x_{\beta+1}$ such that $x<z$ and $\varphi(z)=q$, as desired.

Now we assume that γ is a limit ordinal $<\kappa^+$ and T_γ and φ have been defined for all $\beta<\gamma$. Let $\{(x_k,q_k) : k\in\kappa\}$ enumerate all pairs (x,q) such that $x\in T_\gamma$ and $\varphi(x)<q\in H_\alpha$.

(5) For each $k\in\kappa$ there is a path P_k through T_γ such that $x_k\in P_k$ and $\forall y\in P_k[\varphi(y)<q_k]$.

For, let $k\in\kappa$. Let $\langle\beta_n : n\in\kappa\rangle$ be a strictly increasing sequence of ordinals such that $\mathrm{height}(x_k)<\beta_0$ and $\sup_{n\in\kappa}\beta_n=\gamma$. Let $\langle r_n : n\in\kappa\rangle$ be a strictly increasing sequence of members of H_α with $\varphi(x_k)=r_0$ and $\forall n\in\kappa[r_n<q_k]$, using the η_α property. Then we define $\langle z_n : n\in\kappa\rangle$ as follows Let $z_0=x_k$. If z_n has been defined so that $\mathrm{height}(z_n)<\beta_n$ and $\varphi(z_n)=r_n$, we apply (4) to obtain $z_n<z_{n+1}\in\mathrm{Lev}_{\beta_n}$ and $\varphi'(z_{n+1})=r_{n+1}$. If m is limit $<\kappa$ and z_n has been defined for all $n<m$, we use the η_α property to choose z_m such that $z_n<z_m$ for all $n<m$, and set $\varphi(z_m)=r_m$. Now let P_k be the path through T_γ determined by the z_ns. This proves (5).

Now we put $\kappa\cdot\gamma+k$ directly above P_k, for each $k\in\kappa$, and set $\varphi(\kappa\cdot\gamma+k)=q_k$. $\square$

. . . .

Proposition 22.45. (III.5.45) *Let T be κ-Suslin tree. For $x\in T$ and $D\subseteq T$ call D dense above x iff $\forall y\ge x\exists z\ge y[z\in D]$. Then for any $D\subseteq T$ there is an $\alpha<\kappa$ such that for all $x\in\mathrm{Lev}_\alpha$ one of the following conditions holds:*

(i) D is dense above x.

(ii) $D\cap(x\uparrow'))=\emptyset$.

Proof. Fix $D\subseteq T$. Let $\mathscr{A}=\{A\subseteq T : A$ is an antichain and $\forall x\in A[D$ is dense above x or $D\cap(x\uparrow')=\emptyset]\}$. Then $\mathscr{A}$ is nonempty, for suppose that $x\in T$. If D is dense above x, then $\{x\} \in \mathscr{A}$. If D is not dense above x, then there is a $y \geq x$ such that $\forall z \geq y[z \notin D]$. Then $\{y\} \in \mathscr{A}$. Clearly if $\mathscr{B} \subseteq \mathscr{A}$ is linearly ordered by $\subseteq$, then $\bigcup \mathscr{B} \in \mathscr{A}$. So by Zorn's lemma $\mathscr{A}$ has a maximal member A under $\subseteq$. Then $|A| < \kappa$, so we can choose $\alpha < \kappa$ greater than the height of any member of A. Take any $x \in \mathrm{ht}_\alpha$, and suppose that D is not dense above x. Then there is a $y \geq x$ such that $\forall z \geq y[z \notin D]$. Thus $D \cap (y\uparrow') = \emptyset$. By the maximality of A it follows that there is a $z \in A$ such that z and y are comparable. Since $\mathrm{level}(z) < \mathrm{level}(x) \leq \mathrm{level}(y)$, it follows that $z \leq y$. Since also $x \leq y$, we have that x and z are comparable. Since $\mathrm{level}(z) < \mathrm{level}(x)$, we have $z \leq x$. Hence D is not dense above z, so $D \cap (z\uparrow') = \emptyset$. It follows that $D \cap (x\uparrow') = \emptyset$, as desired. $\square$

Proposition 22.46. (III.5.46) *If T is a κ-Suslin tree, then $|T\backslash T^p| < \kappa$.*

Proof. Applying III.5.45 to T^p, let $\alpha < \kappa$ be such that for all $x \in \mathrm{ht}_\alpha(T)$ either T^p is dense above x or $T^p \cap (x\uparrow') = \emptyset$. If $x \in \mathrm{ht}_\alpha(T)$ and T^p is dense above x, then for any $y \geq x$ there is a $z \geq y$ such that $z \in T^p$, hence $|z\uparrow| = \kappa$, so also $|y\uparrow'| = \kappa$. So for $x \in \mathrm{ht}_\alpha(T)$ with T^p dense above x we have $(x\uparrow') \cap (T\backslash T^p) = \emptyset$. It follows that $T\backslash T^p$ is a subset of

$$(*) \qquad T_\alpha \cup \bigcup \{x\uparrow' \colon x \in \mathrm{ht}_\alpha(T) \text{ and } T^p \cap (x\uparrow') = \emptyset\}.$$

In fact, if $y \in T\backslash T^p$ and $y \notin T_\alpha$, choose $x \in T_\alpha$ so that $y \in x\uparrow'$. If T^p is dense above x, then $(x\uparrow') \cap (T\backslash T^p) = \emptyset$ by the above, contradicting y in this set. So T^p is not dense above x, and hence by Proposition 22.45 $T^p \cap (x\uparrow') = \emptyset$, proving $(*)$.

Now $|T_\alpha| < \kappa$, and if $x \in \mathrm{ht}_\alpha(T)$ and $T^p \cap (x\uparrow') = \emptyset$, then $|x\uparrow'| < \kappa$, so the second set in $(*)$ also has size less than κ. So $|T\backslash T^p| < \kappa$. $\square$

If S and T are κ-trees, we define $S \circledcirc T = \{(s,t) \in S \times T : \mathrm{height}(s) = \mathrm{height}(t)\}$, with $(s,t) \leq (s',t')$ iff $s \leq s'$ and $t \leq t'$.

Proposition 22.47. (III.5.47) *If S and T are κ-Aronsajn trees, then $S \circledcirc T$ is a κ-Aronsajn tree.* $\square$

Proposition 22.48. (III.5.47) *If S and T are κ-Kurepa trees, then $S \circledcirc T$ is a κ-Kurepa tree.* $\square$

Proposition 22.49. (III.5.48) *If T is a tree of regular height κ, then $T \circledcirc T$ is not a κ-Suslin tree.*

Proof. First we note:

(1) Suppose in T that for $i \in 2$ we have $x_i < y_i, z_i$ and $y_i \neq z_i$ and

$$\mathrm{ht}(x_0) < \mathrm{ht}(y_0) = \mathrm{ht}(z_0) < \mathrm{ht}(x_1) < \mathrm{ht}(y_1) = \mathrm{ht}(z_1).$$

Then (y_0, z_0) and (y_1, z_1) are incomparable in $T \circledcirc T$.

In fact, assume the hypotheses, but suppose that (y_0, z_0) and (y_1, z_1) are comparable in $T \circledcirc T$. Since $\mathrm{height}(y_0) < \mathrm{ht}(y_1)$, we must have $(y_0, z_0) < (y_1, z_1)$. So $y_0 < y_1$ and

$z_0 < z_1$. Since $x_1, y_0 < y_1$ and $\mathrm{ht}(y_0) < \mathrm{ht}(x_1)$, it follows that $y_0 < x_1$. Also $x_1, z_0 < z_1$ and $\mathrm{ht}(z_0) < \mathrm{ht}(x_1)$, so $z_0 < x_1$. Since $y_0, z_0 < x_1$ and $\mathrm{ht}(y_0) = \mathrm{ht}(z_0)$, this contradicts $y_0 \neq z_0$.

Now we consider two cases.

Case 1. $\forall\alpha < \kappa\exists x, y, z[\mathrm{ht}(x) \geq \alpha,\ x < y, z$ and $y \neq z$, and $\mathrm{ht}(y) = \mathrm{ht}(z)]$. Then

$$\exists x, y, z \in {}^{\kappa}T\forall\alpha < \kappa[x_\alpha < y_\alpha, z_\alpha, \mathrm{height}(y_\alpha) = \mathrm{height}(z_\alpha), \quad \text{and} \\ y_\alpha \neq z_\alpha \text{ and } \forall\alpha, \beta < \kappa[\alpha < \beta \to \mathrm{height}(y_\alpha) < \mathrm{ht}(x_\beta)]].$$

Then by (1), $\{(y_\alpha, z_\alpha) : \alpha < \kappa\}$ is an antichain in $T \circledcirc T$.

Case 2. $\exists\alpha < \kappa\forall x, y, z[\mathrm{ht}(x) \geq \alpha, x < y, z,$ and $\mathrm{ht}(y) = \mathrm{ht}(z)$ implies that $y = z]$. That is, for each x of height $\geq \alpha$ there do not exist incomparable $y, z > x$. If $|\mathrm{Lev}_\alpha(T)| = \kappa$ then $|\mathrm{Lev}_\alpha(T \circledcirc T)| = \kappa$ and so $T \circledcirc T$ is not Suslin. Assume that $|\mathrm{Lev}_\alpha(T)| < \kappa$. If there is a path through T, then there is a path through $T \circledcirc T$, and $T \circledcirc T$ is not Suslin. Assume that there is no path through T. Then for each element x of height α the tree above x is a chain of length t_x less than κ. Hence T does not have any elements of height greater than $\sup_{x\in\mathscr{L}_\alpha}(\alpha + t_x)$, contradiction. □

Proposition 22.50. (III.5.49) *If $(X, <)$ is a ccc uncountable linearly ordered set, then X has a dense subset of size ω_1.*

Proof. Suppose that X does not have a dense subset of size $\aleph_1$. Then clearly it does not have a dense subset of size $\leq \aleph_1$. We now define $\langle a_\alpha : \alpha < \omega_2\rangle$ and $\langle b_\alpha : \alpha < \omega_2\rangle$ by recursion, so that

(1) $a_\alpha < b_\alpha$ for all $\alpha < \omega_2$.

(2) If $\alpha < \beta < \omega_2$, then $a_\alpha \notin [a_\beta, b_\beta]$ and $b_\alpha \notin [a_\beta, b_\beta]$.

Suppose these have been constructed for all $\alpha < \gamma$ so that (1) and (2) hold, where $\gamma < \omega_2$. Then $\{a_\alpha : \alpha < \gamma\} \cup \{b_\alpha : \alpha < \gamma\} \cup \{x : x \text{ isolated}\}$ is not dense, so there exist c, d with (c, d) disjoint from this set. Then we take a_γ, b_γ with $c < a_\gamma < b_\gamma < d$. Then (1) and (2) hold.

Now

$$\forall\xi, \alpha < \omega_2[\xi < \alpha \to [[a_\alpha, b_\alpha] \subseteq (a_\xi, b_\xi) \text{ or } [a_\alpha, b_\alpha] \cap [a_\xi, b_\xi] = \emptyset]]. \tag{3}$$

In fact, by (1) and (2) we have $(a_\xi < b_\xi < a_\alpha < b_\alpha)$ or $(a_\xi < a_\alpha < b_\alpha < b_\xi)$ or $(a_\alpha < b_\alpha < a_\xi < b_\xi)$, and (3) follows.

Now we define a relation $\prec$ on ω_2. We say that $\xi \prec \alpha$ iff $\xi < \alpha$ and $[a_\alpha, b_\alpha] \subseteq (a_\xi, b_\xi)$. Clearly $\prec$ is well-founded and transitive. Now suppose that $\xi, \eta \prec \alpha$ with $\xi \neq \eta$; we show that $\xi \prec \eta$ or $\eta \prec \xi$. We have $[a_\alpha, b_\alpha] \subseteq (a_\xi, b_\xi), (a_\eta, b_\eta)$ and $[a_\alpha, b_\alpha] \neq \emptyset$, so by (3), $\xi \prec \eta$ or $\eta \prec \xi$. Thus $\prec$ is a tree order.

If $\langle c_\xi : \xi < \omega_1\rangle$ is a chain in increasing order under $\prec$, then $(a_{c_\xi}, b_{c_\xi})\backslash(a_{c_{\xi+1}}, b_{c_{\xi+1}}) \neq \emptyset$, contrading ccc.

Also, if C is an antichain under $\prec$, then by (3), $\{[a_\xi, b_\xi] : \xi \in C\}$ is disjoint, so C is countable by ccc.

Hence we have a tree of height at most ω_1 in which all levels are countable, so this contradicts the tree having universe ω_2. $\square$

Proposition 22.51. (III.5.50) *(In ZFC.) There is a densely ordered set $(X, <)$ such that every separable subspace is nowhere dense and such that there are no strictly increasing or strictly decreasing sequences of type ω_1.*

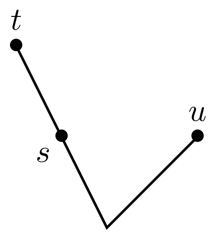
Proof. We take a Hausdorff Aronszajn tree $(T, <)$ in which each element has infinitely many immediate successors; see Propositions 22.23 and 22.33. Take a dense linear order $\prec$ with no endpoints on each set of successors of a given member of T. For each $t \in T$ and each $\alpha \leq \mathrm{ht}(t)$ let t_α be the unique element at height α which is $\leq t$. For incomparable elements s, t let $d(s,t)$ be the least α such that $s_\alpha \neq t_\alpha$. Since T is Hausdorff, $d(s,t)$ is a successor ordinal. Now we define a relation $\ll$ on T. $s \ll t$ iff $s < t$ or (s and t are incomparable in T and $s_{d(s,t)} \prec t_{d(s,t)}$). Clearly $\ll$ is irreflexive. To see that it is transitive, suppose that $s \ll t \ll u$.

Case 1. $s < t < u$. Then $s < u$ and so $s \ll u$.

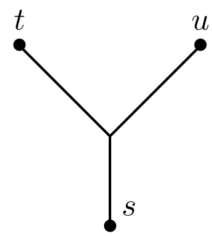
Case 2. $s < t$, t and u are incomparable, and $t_{d(t,u)} \prec u_{d(t,u)}$.

Subcase 2.1. $\mathrm{ht}(s) < d(t,u)$. Then $s < u$, hence $s \ll u$.

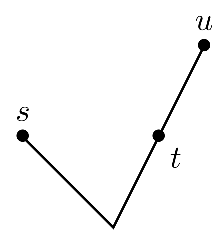
Subcase 2.2. $d(t,u) \leq \mathrm{ht}(s)$. Then s and u are incomparable, $d(s,u) = d(t,u)$, and $s_{d(s,u)} = t_{d(t,u)} \prec u_{d(s,u)}$ and hence $s \ll u$.



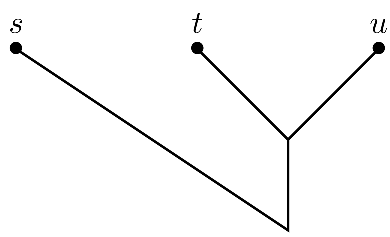
Case 2.2



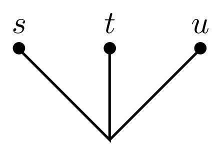
Case 2.1



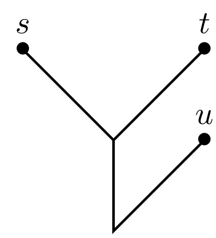
Case 3



Case 4.1



Case 4.2



Case 4.3

Case 3. s and t are incomparable, $s_{d(s,t)} \prec t_{d(s,t)}$, and $t < u$. Then s and u are incomparable, $d(s,u) = d(s.t)$, and $s_{d(s,u)} = s_{d(s,t)} \prec t_{d(s,t)} = u_{d(s,u)}$, hence $s \ll u$.

Case 4. s and t are incomparable, $s_{d(s,t)} \prec t_{d(s,t)}$, t and u are incomparable, and $t_{d(t,u)} \prec u_{d(t,u)}$.

Subcase 4.1. $d(s,t) < d(t,u)$. Then $d(s,u) = d(s,t)$ and $s_{d(s,u)} = s_{d(s,t)} \prec t_{d(s,t)} = u_{d(s,u)}$), and hence $s \ll u$.

Subcase 4.2. $d(s,t) = d(t,u)$. Then $d(s,u) = d(s,t)$ and $s_{d(s,u)} \prec t_{d(s,t)} \prec u_{d(s,u)}$ and so $s \ll u$.

Subcase 4.3. $d(t,u) < d(s,t)$. Then $d(s,u) = d(t,u)$, and $s_{d(s,u)} = t_{d(t,u)} \prec u_{d(s,u)}$ and so $s \ll u$.

Now suppose that $\langle t_\alpha : \alpha < \omega_1 \rangle$ is strictly increasing under $\ll$.

(1) If $\beta < \omega_1$, v, u both have height β, and $\{\alpha < \omega_1 : v \le t_\alpha\}$ and $\{\alpha < \omega_1 : u \le t_\alpha\}$ are uncountable, then $v = u$.

Suppose not. Take any α such that $\beta < \alpha$ and $v \le t_\alpha$. Then take γ such that $\beta < \gamma$, $t_\alpha \ll t_\gamma$, and $u \le t_\gamma$. Now since v and u have height β and are different, and $v \le t_\alpha$ and $u \le t_\gamma$, it follows that t_α and t_γ are incomparable, and $d_{t_\alpha t_\gamma} = d_{t_\alpha u} = d_{vu} = d_{vt_\gamma}$. Since $t_\alpha \ll t_\gamma$ we have $(t_\alpha)_{d(t_\alpha,t_\gamma)} = v_{d(v,u)} \prec u_{d(v,u)} = (t_\gamma)_{d(t_\alpha,t_\gamma)}$. Thus

(2) $v_{d(v,u)} \prec u_{d(v,u)}$,

Similarly, take any γ' such that $\beta < \gamma'$ and $u \le t_{\gamma'}$. Then take α' such that $\beta < \alpha'$, $t_{\gamma'} \ll t_{\alpha'}$, and $v \le t_{\alpha'}$. Now since v and u have height β and are different, and $v \le t_{\alpha'}$ and $u \le t_{\gamma'}$, it follows that $t_{\alpha'}$ and $t_{\gamma'}$ are incomparable, and $d_{t'_\alpha t'_\gamma} = d_{t'_\alpha u} = d_{vu} = d_{vt'_\gamma}$. Since $t_{\gamma'} \ll t_{\alpha'}$ we have $(t_{\gamma'})_{d(t_{\alpha'},t_{\gamma'})} = u_{d(v,u)} \prec v_{d(v,u)} = (t_{\alpha'})_{d(t_{\alpha'},t_{\gamma'})}$, contradicting (2). So (1) holds.

(3) For any $\beta < \omega_1$ there is a u of height β such that $\{\alpha < \omega_1 : u \le t_\alpha\}$ is uncountable.

In fact,

$$\omega_1 = \{\alpha < \omega_1 : t_\alpha \text{ has height less than } \beta\} \cup \bigcup_{\mathrm{ht}(s)=\beta} \{\alpha < \omega_1 : s \le t_\alpha\}.$$

Since the first set here is countable and height β is countable, there is an u of height β such that $\{\alpha < \omega_1 : u \le t_\alpha\}$ is uncountable. Thus (3) holds

By (1) and (3), for each $\beta < \omega_1$ let s_β be the unique element of T of height β such that $\{\alpha < \omega_1 : s_\beta \le t_\aleph\}$ is uncountable. Suppose that $\beta < \gamma < \omega_1$. Then

$$\begin{aligned}\{\alpha < \omega_1 : s_\beta \le t_\alpha\} =& \{\alpha < \omega_1 : s_\beta \le t_\alpha, \mathrm{ht}(t_\alpha) < \gamma\} \\ & \cup \bigcup_{\mathrm{ht}(u)=\gamma} \{\alpha < \omega_1 : s_\beta \le t_\alpha \wedge u \le t_\alpha\}.\end{aligned}$$

Since the first set here is countable and height γ is countable, there is an u of height γ such that $\{\alpha < \omega_1 : s_\beta \le t_\alpha \wedge u \le t_\alpha\}$ is uncountable. Hence also $\{\alpha < \omega_1 : u \le t_\alpha\}$ is uncountable. Hence by (1), $u = s_\gamma$. Since $s_\beta, s_\gamma < t_\alpha$ for some α, it follows that $s_\beta < s_\gamma$. Hence So $\langle s_\beta : \beta < \omega_1 \rangle$ is strictly increasing, contradiction.

Similarly, there does not exist a strictly decreasing sequence of type ω_1 in the ordering $\ll$.

The ordering $\ll$ is dense. For suppose that $s \ll t$.

Case 1. $s < t$. The set of immediate successors of s is densely ordered by $\prec$, so we can choose one, say u, incomparable with t and less in the order $\prec$ than $t_{\mathrm{ht}(s)+1}$. Note that $d_{ut} = \mathrm{height}(s) + 1$. Then there are elements of height $\mathrm{ht}(s)+1$ between u and $t_{d(u,t)}$ in the order $\prec$, giving elements in (s,t).

Case 2. s and t are incomparable. There is an element between $s_{d(s,t)}$ and $t_{d(s,t)}$ in the order $\prec$, giving an element in (s,t).

Now suppose that $X \subseteq T$ is countable. Take any $a \ll b$; we will find c, e with $a \ll c \ll e \ll b$ and $(c,e)_{\ll} \cap X = \emptyset$. This will show that $T \backslash \overline{X}$ is dense, so that $\overline{X}$ is nowhere dense.

Case 1. $a < b$. Let u be an immediate successor of a such that $u \prec b_{\mathrm{ht}(a)+1}$. Choose $v > u$ with also $\mathrm{ht}(x) < \mathrm{ht}(v)$ for all $x \in X$. Let c, e be immediate successors of v with $c \prec e$. Now $a < u < v < c$, so $a < c$ and hence $a \ll c$. Obviously $c \ll e$. We have $d(e,b) = \mathrm{ht}(a)+1$ and $e_{d(e,b)} = u \prec b_{\mathrm{ht}(a)+1} = b_{d(e,b)}$; so $e \ll b$. Thus $a \ll c \ll e \ll b$.

Suppose that $x \in X$ and $c \ll x \ll e$. Since $\mathrm{ht}(x) < \mathrm{height}(v) < \mathrm{ht}(c)$, we have $c \not< x$. If $x < e$ then $x < v < c$ hence $x \ll c$, contradiction. So x and e are incomparable. Also x and c are incomparable. Clearly $d(x,c) = d(x,e) < v$. Now $c \ll x$ implies that $v_{d(x,c)} = c_{d(x,c)} \prec x_{d(x,c)}$, and $x \ll e$ implies that $x_{d(x,c)} = x_{d(x,e)} \prec e_{d(x,e)} = v_{d(x,c)}$, contradiction.

Case 2. a and b are incomparable. So $a_{d(a,b)} \prec b_{d(a,b)}$. Choose u with $a_{d(a,b)} \prec u \prec b_{d(a,b)}$. Take $v > u$ such that $\mathrm{ht}(x) < \mathrm{ht}(v)$ for all $x \in X$. Let c, e be immediate successors of v with $c \prec e$. Clearly $a \ll c \ll e \ll b$.

Suppose that $x \in X$ and $c \ll x \ll e$. Since $\mathrm{ht}(x) < \mathrm{height}(v) < \mathrm{ht}(c)$, we have $c \not< x$. If $x < e$ then $x < v < c$ hence $x \ll c$, contradiction. So x and e are incomparable. Also x and c are incomparable. Clearly $d(x,c) = d(x,e) < v$. Now $c \ll x$ implies that $v_{d(x,c)} = c_{d(x,c)} \prec x_{d(x,c)}$, and $x \ll e$ implies that $x_{d(x,c)} = x_{d(x,e)} \prec e_{d(x,e)} = v_{d(x,c)}$, contradiction. $\square$

Proposition 22.52. *Let κ be an uncountable regular cardinal, and suppose that there is a κ-Aronszajn tree. Then there is one which is a normal subtree of ${}^{<\kappa}2$.*

Proof. Let T be a κ-Aronszajn tree. We may assume that it is well-pruned. For $s \in T$ and $\alpha < \mathrm{ht}(s,T)$ let s^α be the unique element of height α below s. For each $\alpha < \kappa$, let g_α be an injection from $\mathrm{Lev}_\alpha(T)$ into ${}^{|\mathrm{Lev}_\alpha(T)|}2$.

We define by recursion sequences $\langle \mu_\alpha : \alpha < \kappa \rangle$ and $\langle F_\alpha : \alpha < \kappa \rangle$. Let $\mu_0 = |\mathrm{Lev}_0(T)|$, and let $F_0 = g_0$. Now suppose that μ_α, F_α have been defined so that the following conditions hold:

(1_α) $\mu_\alpha < \kappa$.

(2_α) F_α is a function with domain $\bigcup_{\beta \le \alpha} \mathrm{Lev}_\beta(T)$.

(3_α) for all $\beta, \gamma < \alpha$, if $\beta < \gamma$ then $\mu_\beta \le \mu_\gamma$ and $F_\beta \subseteq F_\gamma$.

(Clearly these conditions hold for $\alpha = 0$.) Now let $\mu_{\alpha+1} = \mu_\alpha + |\mathrm{Lev}_{\alpha+1}(T)|$ (ordinal addition). Let $F_{\alpha+1}$ be the extension of F_α such that for every $t \in \mathrm{Lev}_\alpha(T)$, every immediate successor s of t, and every $\beta < \mu_{\alpha+1}$,

$$(F_{\alpha+1}(s))(\beta) = \begin{cases} (F_\alpha(t))(\beta) & \text{if } \beta < \mu_\alpha, \\ (g_{\alpha+1}(s))(\xi) & \text{if } \beta = \mu_\alpha + \xi \text{ with } \xi < |\mathrm{Lev}_{\alpha+1}(T)|. \end{cases}$$

Clearly $(1_{\alpha+1})$–$(3_{\alpha+1})$ hold.

Now suppose that α is a limit ordinal and μ_β, F_β have been defined for all $\beta < \alpha$ so that (1_β)–(3_β) hold. Let $\nu = \sum_{\beta<\alpha} \mu_\beta$, $G = \bigcup_{\beta<\alpha} F_\beta$, and set $\mu_\alpha = \nu + |\mathrm{Lev}_\alpha(T)|$. Let F_α be the extension of G such that for every $s \in \mathrm{Lev}_\alpha(T)$ and every $\beta < \mu_\alpha$,

$$(F_\alpha(s))(\beta) = \begin{cases} (G(s^\gamma))(\beta) & \text{if } \beta < \mu_\gamma \text{ with } \gamma < \alpha, \\ (g_\alpha(s))(\xi) & \text{if } \beta = \nu + \xi \text{ with } \xi < |\mathrm{Lev}_\alpha(T)|. \end{cases}$$

Clearly (1_α)–(3_α) hold.

Let $H = \bigcup_{\alpha<\kappa} F_\alpha$. Clearly

(4) If $s \in \mathrm{Lev}_\alpha(T)$, then $H(s) \in {}^{\mu_\alpha}2$,

(5) If $u < s$ then $H(u) \subset H(s)$.

We prove (5) by induction on the level of s. It is vacuously true for level 0. Now suppose inductively that s has level $\alpha + 1$. Say that t is the immediate predecessor of s. Let γ be the level of u. Then for any $\beta < \mu_\gamma$ we have

$$(H(s))(\beta) = (F_{\alpha+1}(s))(\beta) = (F_\alpha(t))(\beta) = (H(t))(\beta) = (H(u))(\beta).$$

Finally, suppose inductively that s has limit level α. Then for any $\beta < \gamma$ we have

$$(H(s))(\beta) = (F_\gamma(s^\gamma))(\beta) = (H(u))(\beta).$$

Hence (5) holds.

(6) If $s, t \in T$ have the same height and $s \neq t$, then $H(s) \neq H(t)$.

We prove (6) by induction on the common height α of s and t. If $\alpha = 0$ the conclusion is clear since g_0 is one-one. Suppose inductively that they both have height $\alpha + 1$. Let s', t' be their immediate predecessors. If $s' \neq t'$, then $H(s') \neq H(t')$, so $H(s) \neq H(t)$ by (5). Suppose that $s' = t'$. Then $H(s) \neq H(t)$ since g_α is one-one. Finally, suppose inductively that α is limit. Then $H(s) \neq H(t)$ since g_α is one-one. So (6) holds.

Now let $T' = \{h \in {}^{<\kappa}2 : h \subseteq H(s) \text{ for some } s \in T\}$. We claim that T' is as desired. Clearly it is a normal subtree of ${}^{<\kappa}2$. Now consider any $\alpha < \kappa$. Choose β minimum such that $\alpha \leq \mu_\beta$.

(7) If $h \in T'$ with $\mathrm{dmn}(h) = \alpha$, then there is an $s \in T$ of height β such that $h \subseteq H(s)$.

In fact, choose $t \in T$ such that $h \subseteq H(t)$. Then $\mathrm{dmn}(H(t)) \geq \alpha \geq \mu_\beta$, so t has height $\geq \beta$. Let $s \in T$ of height β with $s \leq t$. Then $H(s), h \subseteq H(t)$, so $h \subseteq H(s)$, as desired.

It follows from (7) that each level of T' has size less than κ. From (5) and (7) it follows that T' does not have a chain of size κ. $\square$

Proposition 22.53. *Let κ be an uncountable regular cardinal, and suppose that there is a κ-Suslin tree. Show that there is one which is a normal subtree of ${}^{<\kappa}2$.*

Proof. We use the same construction as in Proposition 22.52. Thus our new tree T' does not have any chain of size κ. Suppose that A is an antichain of size κ. For each $a \in A$ choose $s_a \in T$ such that $a \subseteq H(s_a)$. Since T is a Suslin tree, choose distinct $a, b \in A$ such that s_a and s_b are comparable. Say $s_a \leq s_b$. Then $H(s_a) \subseteq H(s_b)$. Since $a, b \subseteq H(s_b)$, it follows that a and b are comparable, contradiction. $\square$

Proposition 22.54. *A tree T is everywhere branching iff every $t \in T$ has at least two immediate successors. Every everywhere branching tree has at least 2^ω branches.*

Proof. We define a branch b_f for every $f \in {}^\omega 2$ by defining elements a_h for every $h \in {}^{<\omega}2$ by recursion on $\mathrm{dmn}(h)$. Let $a_\emptyset$ be a root of the tree. Suppose that a_h has been defined for every $h \in {}^n2$. For each $h \in {}^n2$, let a_{h0} and a_{h1} be two immediate successors of a_h. This finishes the definition of the a_h's. Now let b_f be an extension of $\langle a_{f\restriction n} : n \in \omega\rangle$ to a branch. Clearly this is as desired. $\square$

Proposition 22.55. *The hypothesis that all levels are finite is necessary in König's theorem.*

Proof. For each $n \in \omega$ let $f_n \in {}^{n+1}\omega$ be any function such that $f_n(0) = n$, and let T be the tree consisting of all $g \in {}^{<\omega}\omega$ such that $g \subseteq f_n$ for some n. Then two elements $g, h \in T$ are comparable iff they are both contained in the same f_n. If C is a maximal chain in T, it must be a subset of some f_n, and hence is finite. $\square$

Proposition 22.56. *If κ is singular with $\mathrm{cf}(\kappa) = \omega$, then there is no κ-Aronszajn tree with all levels finite.*

Proof. Let $\langle \alpha_n : n \in \omega\rangle$ be a strictly increasing sequence of ordinals with supremum κ. Suppose that T is a κ-Aronszajn tree with all levels finite. Define

$$T' = \{t \in T : \text{there is an } n \in \omega \text{ such that } t \text{ has height } \alpha_n\}.$$

Then T', with the order induced by T, is a tree of height ω with all levels finite. Hence by König's theorem it has an infinite branch B. Let $B' = \{t \in T : t \leq s$ for some $s \in T'$. Then B' is a branch in T of size κ, contradiction. $\square$

Proposition 22.57. *If κ is singular and there is a $\mathrm{cf}(\kappa)$-Aronszajn tree, then there is a κ-Aronszajn tree with all levels of power less than $\mathrm{cf}(\kappa)$.*

Proof. Let T be a $\mathrm{cf}(\kappa)$-Aronszajn tree. Let $\langle \mu_\alpha : \alpha < \mathrm{cf}(\kappa)\rangle$ be a strictly increasing continuous sequence of cardinals with supremum κ, and with $\mu_0 = 0$. We define

$$\begin{aligned}
&T' = \{(t, \beta) : \text{there is an } \alpha < \mathrm{cf}(\kappa) \text{ such that } t \in \mathrm{Lev}_\alpha(T) \text{ and } \mu_\alpha \leq \beta < \mu_{\alpha+1};\\
&\quad (t, \beta) < (t', \beta') \text{ iff } t < t', \text{ or } t = t' \text{ and } \beta < \beta'.
\end{aligned}$$

Clearly this gives a partial order. To show that it is a tree, suppose that $(t, \beta) \in T'$. We define a function f from β into the set of predecessors of (t, β) as follows. Let $\mathrm{ht}(t) = \alpha$. Suppose that $\gamma < \beta$. then there is a δ such that $\mu_\delta \leq \gamma < \mu_{\delta+1}$. Clearly $\delta \leq \alpha$. We define $f(\gamma) = (t', \gamma)$, where t' is the predecessor of t at level δ. Clearly then $f(\gamma) < (t, \beta)$. Clearly also f is order preserving and maps onto the set of all predecessors of (t, β). Thus T' is a tree. For each $\beta < \kappa$, say with $\mu_\alpha \leq \beta < \mu_{\beta+1}$ we have

$$\mathrm{Lev}_\beta(T') = \{(t, \beta) : t \in \mathrm{Lev}_\alpha(T)\}.$$

Thus each level of T' has size less than $\mathrm{cf}(\kappa)$. If B is a branch of size κ, then for each $\beta < \kappa$ it has an element of height at least β, and hence for each $\alpha < \mathrm{cf}(\kappa)$ it has an element whose first coordinate has height at least α. These first coordinates are linearly ordered. This contradicts T being a $\mathrm{cf}(\kappa)$-Aronszajn tree.

Thus T' is as desired. □

Proposition 22.58. *For every infinite cardinal κ there is an eventually branching tree T of height κ such that for every subset S of T, if S is a tree under the order induced by T and every element of S has at least two immediate successors, then S has height ω.*

The idea is to put a copy of ${}^{<\omega}2$ on top of longer and longer chains. More precisely, define

$$T = \{(\alpha, \xi, \emptyset) : \alpha < \kappa, \xi < \alpha\} \cup \{(\alpha, \alpha, f) : \alpha < \kappa, f \in {}^{<\omega}2\};$$
$$(\alpha, \xi, f) < (\beta, \eta, g) \text{ iff } \alpha = \beta \text{ and either } \xi < \eta, \text{ or } \xi = \eta = \alpha \text{ and } f \subset g.$$

Clearly T is a tree. The height of an element $(\alpha, \xi, \emptyset)$ is ξ, and the height of an element (α, α, f) is $\alpha + n$, where $f \in {}^n2$. In particular, T has height κ.

Now suppose that S is as indicated in the propositionn, and take any element (α, ξ, f) of S. (α, ξ, f) is a root of S iff $\alpha = \xi$ and $f = \emptyset$. It follows that all the non-root elements of S have the form (α, α, f), and so in S the height of every element is finite. □

Proposition 22.59. *If κ is an uncountable regular cardinal and T is a κ-Aronszajn tree, then T has a subset S such that under the order induced by T, S is a well-pruned κ-Aronszajn tree in which every element has at least two immediate successors.*

By Theorem 18.7, we may assume that T is well-pruned. Now we construct a strictly increasing sequence $\langle \alpha_\xi : \xi < \kappa \rangle$ of ordinals less than κ. Let $\alpha_0 = 0$. Suppose that α_ξ has been defined. Now T is eventually branching. (See the remark before Theorem 18.7.) Hence for each $t \in \mathrm{Lev}_{\alpha_\xi}(T)$ there is an ordinal $\beta_t > \alpha_\xi$ such that t has at least two successors at level β_t. Let $\alpha_{\xi+1}$ be any ordinal less than κ such that $\beta_t < \alpha_{\xi+1}$ for all $t \in \mathrm{Lev}_{\alpha_\xi}(T)$. Note by the well-prunedness condition, each $t \in \mathrm{Lev}_{\alpha_\xi}$ has at least two successors at level $\alpha_{\xi+1}$. Finally, suppose that η is a limit ordinal less than κ, and α_ξ has been constructed for all $\xi < \eta$. Let $\alpha_\eta = \sup_{\xi<\eta} \alpha_\xi$.

Let $S = \bigcup_{\xi<\kappa} \mathrm{Lev}_{\alpha_\xi}(T)$. Clearly S is as desired. □

23. Clubs and stationary sets

Here we introduce the important notions of clubs and stationary sets. A basic result here is Fodor's theorem. We also give a combinatorial principle $\diamondsuit$, later proved consistent with ZFC, and use $\diamondsuit$ to construct a Suslin tree.

A subset Γ of an ordinal is *unbounded* iff for every $\beta < \alpha$ there is a $\gamma \in \Gamma$ such that $\beta \le \gamma$. A subset C of α is *closed* in α provided that for every limit ordinal $\beta < \alpha$, if $C \cap \beta$ is unbounded in β then $\beta \in C$. Closed and unbounded subsets of α are called *clubs* of α.

The following simple fact about ordinals will be used below.

Lemma 23.1. *If α is an ordinal and $\Gamma \subseteq \alpha$, then o.t.$(\Gamma) \le \alpha$.*

Proof. Let $\beta = \text{o.t.}(\Gamma)$, and let f be the isomorphism of β onto Γ. For all $\gamma < \beta$ we have $\gamma \le f(\gamma) < \alpha$, so $\beta \subseteq \alpha$ and hence $\beta \le \alpha$. $\square$

Note that $\emptyset$ is club in 0. If $\alpha = \beta + 1$, then $\{\beta\}$ is club in α. We are mainly interested in limit ordinals α. Then an equivalent way of looking at clubs is as follows.

Theorem 23.2. *Let α be a limit ordinal.*

(i) If C is club in α, then there exist an ordinal β and a normal function $f : \beta \to \alpha$ such that $\text{rng}(f) = C$.

(ii) If β is an ordinal and $f : \beta \to \alpha$ is a normal function such that $\text{rng}(f)$ is unbounded in α, then $\text{rng}(f)$ is club in α.

Proof. (i): Let β be the order type of C, and let $f : \beta \to C$ be the isomorphism of β onto C. Thus $f : \beta \to \alpha$, and f is strictly increasing. To show that f is continuous, suppose that $\gamma < \beta$ is a limit ordinal; we want to show that $f(\gamma) = \bigcup_{\delta<\gamma} f(\delta)$. Let $\varepsilon = \bigcup_{\delta<\gamma} f(\delta)$. Clearly ε is a limit ordinal. Now $C \cap \varepsilon$ is unbounded in ε. For, suppose that $\varphi < \varepsilon$. Then there is a $\delta < \gamma$ such that $\varphi < f(\delta)$. Since $\delta + 1 < \gamma$ and $f(\delta) < f(\delta + 1)$, we thus have $f(\delta) \in C \cap \varepsilon$. So, as claimed, $C \cap \varepsilon$ is unbounded in ε. Hence $\varepsilon \in C$. Since ε is the lub of $f[\gamma]$, it follows that $f(\gamma) = \varepsilon$, as desired. This proves (i).

(ii): Let $C = \text{rng}(f)$. We just need to show that C is closed in α. Suppose that $\gamma < \alpha$ is a limit ordinal, and $C \cap \gamma$ is unbounded in γ. We are going to show that $\psi \stackrel{\text{def}}{=} \bigcup f^{-1}[\gamma]$ is a limit ordinal less than β and $f(\psi) = \gamma$, thereby proving that $\gamma \in C$.

Choose $\delta \in C$ such that $\gamma < \delta$. Say $f(\varphi) = \delta$. Then $f^{-1}[\gamma] \subseteq \varphi$, since for every ordinal ε, if $\varepsilon \in f^{-1}[\gamma]$ then $f(\varepsilon) \in \gamma < \delta = f(\varphi)$ and so $\varepsilon < \varphi$. It follows that also $\bigcup f^{-1}[\gamma] \le \varphi < \beta$.

Next, $\bigcup f^{-1}[\gamma]$ is a limit ordinal. For, if $\beta < \bigcup f^{-1}[\gamma]$, choose $\varepsilon \in f^{-1}[\gamma]$ such that $\beta \in \varepsilon$. Thus $f(\varepsilon) < \gamma$. Since γ is a limit ordinal and $C \cap \gamma$ is unbounded in γ, there is a θ such that $f(\varepsilon) < f(\theta) < \gamma$. Hence $\varepsilon < \theta \in f^{-1}[\gamma]$, so $\varepsilon \in \bigcup f^{-1}[\gamma]$. This shows that $\bigcup f^{-1}[\gamma]$ is a limit ordinal.

We have $f(\psi) = \bigcup_{\beta<\psi} f(\beta)$ by continuity. If $\beta < \psi$, choose $\varepsilon \in f^{-1}[\gamma]$ such that $\beta < \varepsilon$. then $f(\beta) < f(\varepsilon) \in \gamma$. This shows that $f(\psi) \le \gamma$.

Finally, suppose that $\delta < \gamma$. Since $C \cap \gamma$ is unbounded in γ, choose θ such that $\delta < f(\theta) < \gamma$. Then $\theta \in f^{-1}[\gamma]$, so $\delta \in \bigcup f^{-1}[\gamma]$, i.e., $\delta < \psi$. Since ψ is a limit ordinal, say that $\delta < \varphi < \psi$. Then $\delta < \varphi \le f(\varphi) \le f(\psi)$. This shows that $\gamma \subseteq f(\psi)$, hence $f(\psi) = \gamma$. $\square$

Corollary 23.3. *If κ is a regular cardinal and $C \subseteq \kappa$, then the following conditions are equivalent:*

(i) C is club in κ.

(ii) There is a normal function $f : \kappa \to \kappa$ such that $\mathrm{rng}(f) = C$.

Proof. (i)⇒(ii): Suppose that C is club in κ. By Theorem 23.2(i) let β be an ordinal and $f : \beta \to \kappa$ a normal function with $\mathrm{rng}(f) = C$. Thus β is the order type of C, and so by Lemma 23.1, $\beta \leq \kappa$. The regularity of κ together with C being unbounded in κ imply that $\beta = \kappa$. Thus (ii) holds.

(ii)⇒(i): Suppose that $f : \kappa \to \kappa$ is a normal function such that $\mathrm{rng}(f) = C$. Then by Theorem 23.2(i), C is club in κ. □

Corollary 23.4. *If α is a limit ordinal, then there is club of α with order type* $\mathrm{cf}(\alpha)$.

Proof. By Theorem 11.48, let $f : \mathrm{cf}(\alpha) \to \alpha$ be a strictly increasing function with $\mathrm{rng}(f)$ unbounded in α. Define $g : \mathrm{cf}(\alpha) \to \alpha$ by recursion, as follows:

$$g(\xi) = \begin{cases} 0 & \text{if } \xi = 0, \\ \max(f(\eta), g(\eta) + 1) & \text{if } \xi = \eta + 1 \text{ for some } \eta, \\ \sup_{\eta<\xi} g(\eta) & \text{if } \xi \text{ is a limit ordinal.} \end{cases}$$

Clearly then g is a normal function from $\mathrm{cf}(\alpha)$ into α, with $\mathrm{rng}(g)$ unbounded in α. By Theorem 23.2(ii), the existence of the desired set C follows. □

If $\mathrm{cf}(\alpha) = \omega$, then Corollary 23.4 yields a strictly increasing function $f : \omega \to \alpha$ with $\mathrm{rng}(f)$ unbounded in α. Then $\mathrm{rng}(f)$ is club in α. The condition on limit ordinals in the definition of club is trivial in this case. Most of our results concern limit ordinals of uncountable cofinality.

If α is any limit ordinal and $\beta < \alpha$, then the interval $[\beta, \alpha)$ is a club of α. Another simple fact about clubs is that if C is club in a limit ordinal α of uncountable cofinality, then the set D of all limit ordinals which are in C is also club in α. (We need α of uncountable cofinality in order to have D unbounded.) Also, if C is club in α with $\mathrm{cf}(\alpha) > \omega$, then the set E of all limit points of members of C is also club in α. This set E is defined to be $\{\beta < \alpha : \beta \text{ is a limit ordinal and } C \cap \beta \text{ is unbounded in } \beta\}$; clearly $E \subseteq C$.

Now we give the first major fact about clubs.

Theorem 23.5. *If α is a limit ordinal with $\mathrm{cf}(\alpha) > \omega$, then the intersection of fewer than $\mathrm{cf}(\alpha)$ clubs of α is again a club.*

Proof. Suppose that $\beta < \mathrm{cf}(\alpha)$ and $\langle C_\xi : \xi < \beta \rangle$ is a system of clubs of α. Let $D = \bigcap_{\xi<\beta} C_\xi$. First we show that D is closed. To this end, suppose that $\gamma < \alpha$ is a limit ordinal, and $D \cap \gamma$ is unbounded in γ. Then for each $\xi < \beta$, the set C_ξ is unbounded in γ, and hence $\gamma \in C_\xi$ since C_ξ is closed in α. Therefore $\gamma \in D$.

To show that D is unbounded in α, take any $\gamma < \alpha$; we want to find $\delta > \gamma$ such that $\delta \in D$. We make a simple recursive construction of a sequence $\langle \varepsilon_n : n \in \omega \rangle$ of ordinals less than α. Let $\varepsilon_0 = \gamma$. Suppose that ε_n has been defined. Using the fact that each C_ξ is unbounded in α, for each $\xi < \beta$ choose $\theta_{n,\xi} \in C_\xi$ such that $\varepsilon_n < \theta_{n,\xi}$. Then let

$$\varepsilon_{n+1} = \sup_{\xi<\beta} \theta_{n,\xi};$$

we have $\varepsilon_{n+1} < \alpha$ since $\beta < \mathrm{cf}(\alpha)$. This finishes the recursive construction. Let $\delta = \sup_{n\in\omega} \varepsilon_n$. Then $\delta < \alpha$ since $\mathrm{cf}(\alpha) > \omega$. Clearly $C_\xi \cap \delta$ is unbounded in δ for each $\xi < \beta$, and hence $\delta \in C_\xi$. So $\delta \in D$, as desired. □

Again let α be any limit ordinal, and suppose that $\langle C_\xi : \xi < \alpha\rangle$ is a system of subsets of α. We define the *diagonal intersection* of this system:

$$\triangle_{\xi<\alpha} C_\xi = \{\beta \in \alpha : \forall \xi < \beta(\beta \in C_\xi)\}.$$

This construction is used often in discussion of clubs, in particular in the definition of some of the large cardinals.

Theorem 23.6. *Suppose that* $\mathrm{cf}(\alpha) > \omega$. *Assume that* $\langle C_\xi : \xi < \alpha\rangle$ *is a system of clubs of* α.

(i) If $\bigcap_{\xi<\beta} C_\xi$ *is unbounded in* α *for each* $\beta < \alpha$, *then* $\triangle_{\xi<\alpha} C_\xi$ *is club in* α.

(ii) If α *is regular, then* $\triangle_{\xi<\alpha} C_\xi$ *is club in* α.

Proof. Clearly (ii) follows from (i) (using Theorem 23.5 to verify the hypothesis of (i)), so it suffices to prove (i). Assume the hypothesis of (i).

For brevity set $D = \triangle_{\xi<\alpha} C_\xi$ First we show that D is closed in α. So, assume that β is a limit ordinal less than α, and $D \cap \beta$ is unbounded in β. To show that $\beta \in D$, take any $\xi < \beta$; we show that $\beta \in C_\xi$. Let $E = \{\gamma \in D \cap \beta : \xi < \gamma\}$. Then E is unbounded in β, and for each $\gamma \in E$ we have $\gamma \in C_\xi$, by the definition of D. So $\beta \in C_\xi$ since C_ξ is closed.

Second we show that D is unbounded in α. So, take any $\beta < \alpha$. We define a sequence $\langle \gamma_i : i < \omega\rangle$ of ordinals less than α by recursion. Let $\gamma_0 = \beta$. If γ_i has been defined, by the hypothesis of (i) let γ_{i+1} be a member of $\bigcap_{\xi<\gamma_i} C_\xi$ which is greater than γ_i. Finally, let $\delta = \sup_{i\in\omega} \gamma_i$. So $\delta < \alpha$ since $\mathrm{cf}(\alpha) > \omega$. We claim that $\delta \in D$. To see this, take any $\xi < \delta$. Choose $i \in \omega$ such that $\xi < \gamma_i$. Then $\gamma_j \in C_\xi$ for all $j \geq i$, and hence $C_\xi \cap \delta$ is unbounded in δ, so $\delta \in C_\xi$. This argument shows that $\delta \in D$. □

We give one more general fact about closed and unbounded sets; this one is frequently useful in showing that specific sets are closed and unbounded.

A *finitary partial operation* on a set A is a nonempty function whose domain is a subset of ${}^m A$ for some positive integer m and whose range is a subset of A. We say that a subset B of A is *closed* under such an operation iff for every $a \in ({}^m B) \cap \mathrm{dmn}(f)$ we have $f(a) \in B$.

Theorem 23.7. *Suppose that* κ *is an uncountable regular cardinal,* $X \in [\kappa]^{<\kappa}$, *and* $\mathscr{F}$ *is a collection of finitary partial operations on* κ, *with* $|\mathscr{F}| < \kappa$. *Then* $\{\alpha < \kappa : X \subseteq \alpha$ *and* α *is closed under each* $f \in \mathscr{F}\}$ *is club in* κ.

Proof. Denote the indicated set by C. To show that it is closed, suppose that α is a limit ordinal less than κ, and $C \cap \alpha$ is unbounded in α. To show that α is closed under any partial operation $f \in \mathscr{F}$, suppose that $\mathrm{dmn}(f) \subseteq {}^m\kappa$ and $a \in ({}^m\alpha) \cap \mathrm{dmn}(f)$. For each $i < m$ choose $\beta_i < \alpha$ such that $a_i \in \beta_i$. Since α is a limit ordinal, the ordinal $\gamma \stackrel{\text{def}}{=} \bigcup_{i<m} \beta_i$ is still less than α. Since $C \cap \alpha$ is unbounded in α, choose $\delta \in C \cap \alpha$ such that $\gamma < \delta$. Then $a \in {}^m\delta$ so, since $\delta \in C$, we have $f(a) \in \delta \subseteq \alpha$. Thus α is closed under f. Hence $\alpha \in C$; so C is closed in κ.

To show that C is unbounded in κ, take any $\alpha < \kappa$. We now define a sequence $\langle \beta_n : n \in \omega \rangle$ by recursion. Let $\beta_0 = \alpha$. Having defined $\beta_i < \kappa$, consider the set

$$\{f(a) : f \in \mathscr{F}, a \in \mathrm{dmn}(f), \text{ and each } a_j \text{ is in } \beta_i\}.$$

This set clearly has fewer than κ members. Hence we can take β_{i+1} to be some ordinal less than κ and greater than each member of this set. This finishes the construction.

Let $\gamma = \bigcup_{i \in \omega} \beta_i$. We claim that $\gamma \in C$, as desired. For, suppose that $f \in \mathscr{F}$, f has domain $\subseteq {}^n\kappa$, and $a \in ({}^n\gamma) \cap \mathrm{dmn}(f)$. Then for each $i < n$ choose $m_i \in \omega$ such that $a_i \in \beta_{m_i}$. Let p be the maximum of all the β_i's. Then $a \in ({}^n\beta_p) \cap \mathrm{dmn}(f)$, so by construction $f(a) \in \beta_{p+1} \subseteq \gamma$. $\square$

Let α be a limit ordinal. A subset S of α is *stationary* iff S intersects every club of α. There are some obvious but useful facts about this notion. Assume that $\mathrm{cf}(\alpha) > \omega$. Then any club in α is stationary. An intersection of a stationary set with a club is again stationary. Any superset of a stationary set is again stationary. The union of fewer than $\mathrm{cf}(\alpha)$ nonstationary sets is again nonstationary. Every stationary set is unbounded in α. The following important fact is not quite so obvious:

Proposition 23.8. *If α is a limit ordinal and κ is a regular cardinal less than* $\mathrm{cf}(\alpha)$, *then the set*

$$S \stackrel{\text{def}}{=} \{\beta < \alpha : \mathrm{cf}(\beta) = \kappa\}$$

is stationary in α.

Proof. Let C be club in α. Let $f : \mathrm{cf}(\alpha) \to \alpha$ be strictly increasing, continuous, and with range cofinal in α. We define $g : \mathrm{cf}(\alpha) \to C$ by recursion. Let $g(0)$ be any member of C. For β a limit ordinal less than $\mathrm{cf}(\alpha)$, let $g(\beta) = \bigcup_{\gamma<\beta} g(\gamma)$. If $\beta < \mathrm{cf}(\alpha)$ and $g(\beta)$ has been defined, let $g(\beta+1)$ be a member of C greater than both $g(\beta)$ and $f(\beta)$. Clearly g is a strictly increasing continuous function mapping $\mathrm{cf}(\alpha)$ into C, and the range of g is cofinal in α. Thus $\mathrm{rng}(g)$ is club in α. Now $g(\kappa) \in C \cap S$, as desired. $\square$

Let S be a set of ordinals. A function $f \in {}^S\mathbf{On}$ is *regressive* iff $f(\gamma) < \gamma$ for every $\gamma \in S \backslash \{0\}$. This is a natural notion, and leads to an important fact which is used in many of the deeper applications of stationary sets.

Theorem 23.9. (Fodor; also called the **pressing down lemma**) *Suppose that α is a limit ordinal of uncountable cofinality, S is a stationary subset of α, and $f : S \to \alpha$ is regressive. Then there is an $\beta < \alpha$ such that $f^{-1}[\beta]$ is stationary in α.*

In case α is regular, there is a $\gamma < \alpha$ such that $f^{-1}[\{\gamma\}]$ is stationary.

Proof. Assume the hypothesis of the first part of the theorem, but suppose that there is no β of the type indicated. So for every $\beta < \alpha$ we can choose a club C_β in α such that $C_\beta \cap f^{-1}[\beta] = \emptyset$. Let D be a club in α of order type $\mathrm{cf}(\alpha)$. Now for each $\beta < \alpha$ let $\tau(\beta)$ be the least member of D greater than β. For each $\beta < \alpha$ we define

$$E_\beta = \bigcap_{\gamma \in D \cap (\tau(\beta)+1)} C_\gamma.$$

We claim then that for every $\beta < \alpha$,

$$E_\beta \cap f^{-1}[\beta] = \emptyset. \tag{1}$$

In fact, $\beta < \tau(\beta) \in D \cap (\tau(\beta)+1)$, so $E_\beta \cap f^{-1}[\beta] \subseteq C_{\tau(\beta)} \cap f^{-1}[\tau(\beta)] = \emptyset$. So (1) holds.

Now by Theorem 23.5, each set E_β is club in α. Moreover, clearly $E_\beta \supseteq E_\delta$ if $\beta < \delta < \alpha$. Hence we can apply Theorem 23.6(i) to infer that $F \overset{\text{def}}{=} \triangle_{\beta<\alpha} E_\beta$ is club in α. Hence also the set G of all limit ordinals which are in F is club in α. Choose $\delta \in G \cap S$. Now $f(\delta) < \delta$; since δ is a limit ordinal, choose $\xi < \delta$ such that $f(\delta) < \xi$. But $\delta \in G \subseteq F$, so it follows by the definition of diagonal intersection that $\delta \in E_\xi$. From (1) we then see that $\delta \notin f^{-1}[\xi]$. This contradicts $f(\delta) < \xi$.

For the second part of the theorem, assume that α is regular. Note that, with β as in the first part, $f^{-1}[\beta] = \bigcup_{\gamma<\beta} f^{-1}[\{\gamma\}]$. Hence the second part follows from the fact mentioned above that a union of fewer than α nonstationary sets is nonstationary. $\square$

To illustrate the use of Fodor's theorem we give the following result about Aronszajn trees which answers a natural question.

Theorem 23.10. *Suppose that κ is an uncountable regular cardinal, T is a κ-Aronszajn tree, and λ is an infinite cardinal less than κ. Further, suppose that $x \in T$ and $|\{y \in T : x < y\}| = \kappa$. Then there is an $\alpha > \mathrm{ht}(x)$ such that*

$$|\{y \in \mathrm{Lev}_\alpha(T) : x < y\}| \geq \lambda.$$

Proof. By Theorem 22.7 we may assume that T is well-pruned, and by taking $\{y \in T : x \leq y\}$ we may assume that x is the root of T. So now we want to find a level α such that $|\mathrm{Lev}_\alpha(T)| \geq \lambda$. We assume that this is not the case. So $|\mathrm{Lev}_\alpha(T)| < \lambda$ for all $\alpha < \kappa$.

Suppose that λ is singular. Then

$$\kappa = \bigcup_{\substack{\mu<\lambda \\ \mu \text{ a cardinal}}} \{\alpha < \kappa : |\mathrm{Lev}_\alpha(T)| < \mu^+\},$$

so there is a $\mu < \lambda$ such that $\Gamma \overset{\text{def}}{=} \{\alpha < \kappa : |\mathrm{Lev}_\alpha(T)| < \mu^+\}$ has power κ. Because T is well-pruned, we have $|\mathrm{Lev}_\alpha(T)| \leq |\mathrm{Lev}_\beta(T)$ whenever $\alpha < \beta$. It follows that $|\mathrm{Lev}_\alpha(T)| < \mu^+$ for all $\alpha < \kappa$, since Γ is clearly unbounded in κ. Thus we may assume that λ is regular.

For each $s \in T$ and each $\beta < \mathrm{ht}(s)$ let s_β be the unique element of height β less than s.

Let $\Delta = \{\alpha < \kappa : \mathrm{cf}(\alpha) = \lambda\}$. So Δ is stationary in κ. Now we claim

(1) For every $\alpha \in \Delta$ and every $s \in \mathrm{Lev}_\alpha(T)$ there is a $\beta < \alpha$ such that the set $\{t \in T : s_\beta \le t,\ \beta \le \mathrm{ht}(t) < \alpha\}$ is a chain.

To prove this, suppose not. Thus we can choose $\alpha \in \Delta$ and $s \in \mathrm{Lev}_\alpha(T)$ such that

(2) For all $\beta < \alpha$ there is a $\gamma \in [\beta, \alpha)$ and a $t \in \mathrm{Lev}_\alpha(T)$ such that $s_\gamma < t \ne s$ and $s_{\gamma+1} \not\le t$.

Now we use (2) to construct by recursion two sequences $\langle \gamma_\xi : \xi < \lambda\rangle$ and $\langle t_\xi : \xi < \lambda\rangle$. Suppose that these have been defined for all $\xi < \eta$, where $\eta < \lambda$, so that each $\gamma_\xi < \alpha$. Let $\delta = \bigcup_{\xi<\eta} \gamma_\xi$. So $\delta < \alpha$ since $\mathrm{cf}(\alpha) = \lambda$. By (2), choose $\gamma_\eta \in [\delta + 1, \alpha)$ and $t_\eta \in \mathrm{Lev}_\alpha(T)$ such that $s_{\gamma_\eta} < t_\eta \ne s$ and $s_{\gamma_\eta+1} \not\le t_\eta$. Since $\mathrm{Lev}_\alpha(T)$ has size less than λ, there exist ξ, η with $\xi < \eta$ and $t_\xi = t_\eta$. Then $s_{\gamma_\xi+1} \le s_{\gamma_\eta} < t_\eta = t_\xi$, contradiction. Hence (1) holds.

(3) For every $\alpha \in \Delta$ there is a $\beta < \alpha$ such that for each $s \in \mathrm{Lev}_\alpha(T)$ the set $\{t \in T : s_\beta \le t,\ \beta \le \mathrm{ht}(t) < \alpha\}$ is a chain.

To prove this, let $\alpha \in \Delta$. By (1), for each $s \in \mathrm{Lev}_\alpha(T)$ choose $\gamma_s < \alpha$ such tha the set $\{t \in T : s_{\gamma_s} \le t,\ \gamma_s \le \mathrm{ht}(t) < \alpha\}$ is a chain. Let $\beta = \sup_{\mathrm{ht}(s)=\alpha} \gamma_s$. Clearly β is as desired in (3).

Now for each $\alpha \in \Delta$ choose $f(\alpha)$ to be a β as in (3). So f is a regressive function defined on the stationary set Δ. Hence there is a $\beta < \alpha$ such that $f^{-1}[\{\beta\}]$ is stationary, and hence of size κ. So T does not branch beyond β, and hence has a branch of size κ because it is well-pruned, contradiction. $\square$

For the next result we need another important construction. Suppose that λ is an infinite cardinal, $f = \langle f_\rho : \rho < \lambda^+\rangle$ is a family of injections $f_\rho : \rho \to \lambda$, and S is a cofinal subset of λ^+. The (λ, f, S)-*Ulam matrix* is the function $A : \lambda \times \lambda^+ \to \mathscr{P}(\kappa)$ defined for any $\xi < \lambda$ and $\alpha < \lambda^+$ by

$$A^\xi_\alpha = \{\rho \in S\backslash(\alpha + 1) : f_\rho(\alpha) = \xi\}.$$

Theorem 23.11. (Ulam) *Let λ be an infinite cardinal, S is a stationary subset of λ^+, and I a collection of subsets of λ^+ having the following properties:*

(i) $\emptyset \in I$.

(ii) If $X \in [I]^{\le\lambda}$, then $\bigcup X \in I$.

(iii) If $Y \subseteq X \in I$, then $Y \in I$.

(iv) If $\alpha < \lambda^+$, then $\{\alpha\} \in I$.

(v) $S \notin I$.

Then there is a system $\langle X_\alpha : \alpha < \lambda^+\rangle$ of subsets of S such that $X_\alpha \cap X_\beta = \emptyset$ for distinct $\alpha, \beta < \lambda^+$, and $X_\alpha \notin I$ for all $\alpha < \lambda^+$.

Proof. Let $f = \langle f_\rho : \rho < \lambda^+\rangle$ be a family of injections $f_\rho : \rho \to \lambda$, and let A be the (λ, f, S)-Ulam matrix. If $\xi < \lambda$, then for distinct $\alpha, \beta < \lambda^+$ we have $A^\xi_\alpha \cap A^\xi_\beta = \emptyset$, since the functions f_ρ are one-one. Moreover, for any $\alpha < \lambda^+$ we have

$$S\backslash \bigcup_{\xi<\lambda} A^\xi_\alpha \subseteq S \cap (\alpha + 1) \in I$$

by (ii)–(iv). By conditions (ii) and (v) it then follows that for each $\alpha < \lambda^+$ there is an $h(\alpha) < \lambda$ such that $A_\alpha^{h(\alpha)} \notin I$. Thus $h : \lambda^+ \to \lambda$, so there is a $\xi < \lambda$ such that $|h^{-1}[\{\xi\}]| = \lambda^+$. Hence $\{A_\alpha^\xi : \alpha < \lambda^+,\ h(\alpha) = \xi\}$ is as desired in the theorem. □

Theorem 23.12. *(i) If λ is an infinite cardinal and S is a stationary subset of λ^+, then we can partition S into λ^+-many stationary subsets.*

(ii) If κ is weakly inaccessible, then κ can be partitioned into κ many stationary subsets.

Proof. (i): Let I be the collection of all nonstationary subsets of λ^+. The conditions of Theorem 23.11 are all clear, and so by it we get a system $\langle X_\alpha : \alpha < \lambda^+\rangle$ of subsets of S such that $X_\alpha \cap X_\beta = \emptyset$ for distinct $\alpha, \beta < \lambda^+$, and $X_\alpha \notin I$ for all $\alpha < \lambda^+$. We can union $S \backslash \bigcup_{\alpha<\lambda^+} X_\alpha$ with X_0 to get the desired partition of S.

(ii) For each regular cardinal $\lambda < \kappa$, let $S_\lambda = \{\alpha < \kappa : \mathrm{cf}(\alpha) = \lambda\}$. Thus S_λ is stationary by Proposition 23.8. By induction it is clear that if $\alpha < \kappa$, then $\aleph_{\alpha+1} < \kappa$. Hence there are κ regular cardinals less than κ. Thus we have κ many pairwise disjoint stationary subsets of κ, and these can be extended to a partition of κ as in the proof of (i). □

The first part of Theorem 23.12 can actually be extended to weak inaccessibles too, but the proof is longer.

Next we introduce an important combinatorial principle and show that it implies the existence of Suslin trees. $\diamondsuit$ is the following statement:

There exists a sequence $\langle A_\alpha : \alpha < \omega_1\rangle$ of sets with the following properties:

(i) $A_\alpha \subseteq \alpha$ for each $\alpha < \omega_1$.

(ii) For every subset A of ω_1, the set $\{\alpha < \omega_1 : A \cap \alpha = A_\alpha\}$ is stationary in ω_1.

A sequence as in $\diamondsuit$ is called a $\diamondsuit$-sequence. Such a sequence in a sense captures all subsets of ω_1 in a sequence of length ω_1. Later in these notes we will show that $\diamondsuit$ follows from $V = L$.

Theorem 23.13. $\diamondsuit \Rightarrow$ CH.

Proof. Let $\langle A_\alpha : \alpha < \omega_1\rangle$ be a $\diamondsuit$-sequence. Then for every $A \subseteq \omega$ the set $\{\alpha < \omega_1 : A \cap \alpha = A_\alpha\}$ is stationary in ω_1, and hence it has an infinite member; for such a member α we have $A = A_\alpha$. So we can let $f(A)$ be the least $\alpha < \omega_1$ such that $A = A_\alpha$, and we thus define an injection of $\mathscr{P}(\omega)$ into ω_1. □

Since $\diamondsuit$ is formulated in terms of subsets of ω_1, to construct a Suslin tree using $\diamondsuit$ it is natural to let the tree be ω_1 with some tree-order. The following lemma will be useful in doing the construction.

Lemma 23.14. *Suppose that $T = (\omega_1, \prec)$ is an ω_1-tree and A is a maximal antichain in T. Then*

$$\{\alpha < \omega_1 : (T_\alpha = \alpha \text{ and } A \cap \alpha \text{ is a maximal antichain in } T_\alpha\}$$

is club in ω_1.

Proof. Let C be the indicated set. Suppose that $A \subseteq \omega_1$ is a maximal antichain in T. To see that C is closed in ω_1, let $\alpha < \omega_1$ be a limit ordinal, and suppose that $C \cap \alpha$ is unbounded in α. If $\beta \in T_\alpha$, then there is a $\gamma < \alpha$ such that $\beta \in T_\gamma$. Choose $\delta \in (C \cap \alpha)$ such that $\gamma < \delta$. Then $\beta \in T_\delta = \delta$, so also $\beta \in \alpha$. This shows that $T_\alpha \subseteq \alpha$. Conversely, suppose that $\beta \in \alpha$. Choose $\gamma \in C \cap \alpha$ such that $\beta < \gamma$. Then $\beta \in \gamma = T_\gamma \subseteq T_\alpha$. Thus $T_\alpha = \alpha$.

To show that $A \cap \alpha$ is a maximal antichain in T_α, note first that at least it is an antichain. Now take any $\beta \in T_\alpha$; we show that β is comparable under $\prec$ to some member of $A \cap \alpha$, which will show that $A \cap \alpha$ is a maximal antichain in T_α. Choose $\gamma < \alpha$ such that $\beta \in T_\gamma$, and then choose $\delta \in (C \cap \alpha)$ such that $\gamma < \delta$. Thus $\beta \in T_\delta$. Now $A \cap \delta$ is a maximal antichain in T_δ since $\delta \in C$, so β is comparable with some $\varepsilon \in (A \cap \delta) \subseteq (A \cap \alpha)$, as desired.

To show that C is unbounded in κ we will apply Theorem 23.7 to the following three functions $f, g, h : \kappa \to \kappa$:

$$\begin{aligned} f(\beta) &= \mathrm{ht}(\beta, T); \\ g(\beta) &= \sup(\mathrm{Lev}_\beta(T)); \\ h(\beta) &= \text{ some member of } A \text{ comparable with } \beta \text{ under } \prec . \end{aligned}$$

By Theorem 23.7, the set D of all $\alpha < \kappa$ which are closed under each of f, g, h is club in κ. We now show that $D \subseteq C$, which will prove that C is unbounded in κ. So, suppose that $\alpha \in D$. If $\beta \in T_\alpha$, let $\gamma = \mathrm{ht}(\beta, T)$. Then $\gamma < \alpha$ and $\beta \in \mathrm{Lev}_\gamma(T)$, and so $\beta \le g(\gamma) < \alpha$. Thus $T_\alpha \subseteq \alpha$. Conversely, suppose that $\beta < \alpha$. Then $f(\beta) < \alpha$, i.e., $\mathrm{ht}(\beta, T) < \alpha$, so $\beta \in T_\alpha$. Therefore $T_\alpha = \alpha$. Now suppose that $\beta \in T_\alpha$; we want to show that β is comparable with some member of $A \cap \alpha$, as this will prove that $A \cap \alpha$ is a maximal antichain in T_α. Since $\beta \in \alpha$ by what has already been shown, we have $h(\beta) < \alpha$, and so the element $h(\beta)$ is as desired. $\square$

Another crucial lemma for the construction is as follows.

Lemma 23.15. *Let $T = (\omega_1, \prec)$ be an eventually branching ω_1-tree and let $\langle A_\alpha : \alpha < \omega_1 \rangle$ be a $\diamondsuit$-sequence. Assume that for every limit $\alpha < \omega_1$, if $T_\alpha = \alpha$ and A_α is a maximal antichain in T_α, then for every $x \in \mathrm{Lev}_\alpha(T)$ there is a $y \in A_\alpha$ such that $y \prec x$.*

Then T is a Suslin tree.

Proof. By Proposition 22.8 it suffices to show that every maximal antichain A of T is countable. By Lemma 23.14, the set

$$C \stackrel{\text{def}}{=} \{\alpha < \omega_1 : T_\alpha = \alpha \text{ and } A \cap \alpha \text{ is a maximal antichain in } T_\alpha\}$$

is club in ω_1. Now by the definition of the $\diamondsuit$-sequence, the set $\{\alpha < \omega_1 : A \cap \alpha = A_\alpha\}$ is stationary, so we can choose $\alpha \in C$ such that $A \cap \alpha = A_\alpha$. Now if $\beta \in T$ and $\mathrm{ht}(\beta, T) \ge \alpha$, then there is a $\gamma \in \mathrm{Lev}(\alpha, T)$ such that $\gamma \preceq \beta$, and the hypothesis of the lemma further yields a $\delta \in A_\alpha$ such that $\delta \prec \gamma$. Since $\delta \prec \beta$, it follows that $\beta \notin A$. So we have shown that for all $\beta \in T$, if $\mathrm{ht}(\beta, T) \ge \alpha$ then $\beta \notin A$. Hence for any $\beta \in T$, if $\beta \in A$ then $\beta \in T_\alpha = \alpha$. So $A \subseteq \alpha$ and hence $A = A_\alpha$, so that A is countable. $\square$

Theorem 23.16. $\diamondsuit$ *implies that there is a Suslin tree.*

Proof. Assume $\diamondsuit$, and let $\langle A_\alpha : \alpha < \omega_1\rangle$ be a $\diamondsuit$-sequence. We are going to construct a Suslin tree of the form $(\omega_1, \prec)$ in which for each $\alpha < \omega_1$ the α-th level is the set $\{\omega\cdot\alpha + m : m \in \omega\}$. We will do the construction by completely defining the tree up to heights $\alpha < \omega_1$ by recursion. Thus we define by recursion trees $(\omega\cdot\alpha, \prec_\alpha)$, so that really we are just defining the partial orders $\prec_\alpha$ by recursion.

We let $\prec_0=\prec_1=\emptyset$. Now suppose that $\beta > 1$ and $\prec_\alpha$ has been defined for all $\alpha < \beta$ so that the following conditions hold whenever $0 < \alpha < \beta$:

(1) $(\omega\cdot\alpha, \prec_\alpha)$ is a tree, denoted by T_α for brevity.

(2) If $\gamma < \alpha$ and $\xi, \eta \in T_\gamma$, then $\xi \prec_\gamma \eta$ iff $\xi \prec_\alpha \eta$.

(3) For each $\gamma < \alpha$, $\mathrm{Lev}_\gamma(T_\alpha) = \{\omega\cdot\gamma + m : m \in \omega\}$.

(4) If $\gamma < \delta < \alpha$ and $m \in \omega$, then there is an $n \in \omega$ such that $\omega\cdot\gamma + m \prec_\alpha \omega\cdot\delta + n$.

(5) If $\delta < \alpha$, δ is a limit ordinal, $\omega\cdot\delta = \delta$, and A_δ is a maximal antichain in T_δ, then for every $x \in \mathrm{Lev}_\delta(T_\alpha)$ there is a $y \in A_\delta$ such that $y \prec_\alpha x$.

Note that conditions (1)–(3) just say that the trees constructed have the special form indicated at the beginning, and are an increasing chain of trees. Condition (4) is to assure that the final tree is well-pruned. Condition (5) is connected to Lemma 23.15, which will be applied after the construction to verify that our tree is Suslin. Conditions (1)–(5) imply that if $x \in T_\alpha$, then it has the form $\omega\cdot\beta + m$ for some $\beta < \alpha$, and then $x \in \mathrm{Lev}_\beta(T_\alpha)$ and for each $\gamma < \beta$ there is a unique element $\omega\cdot\gamma + n$ in T_α such that $\omega\cdot\gamma + n \prec_\alpha x$.

If β is a limit ordinal, let $\prec_\beta=\bigcup_{\alpha<\beta}\prec_\alpha$. Conditions (1)–(5) are then clear for any $\alpha \le \beta$.

Next suppose that $\beta = \gamma + 2$ for some ordinal γ. Then we define

$$\begin{aligned}\prec_\beta=\prec_{\gamma+1} &\cup \{(\xi, \omega\cdot(\gamma+1) + 2m) : \xi \preceq_{\gamma+1} \omega\cdot\gamma + m,\ m \in \omega\}\\ &\cup \{(\xi, \omega\cdot(\gamma+1) + 2m + 1) : \xi \preceq_{\gamma+1} \omega\cdot\gamma + m,\ m \in \omega\}.\end{aligned}$$

Clearly (1)–(5) hold for all $\alpha < \beta$.

The most important case is $\beta = \gamma + 1$ for some limit ordinal γ. To treat this case, we first associate with each $x \in T_\gamma$ a chain $B(x)$ in T_γ, and to do this we define by recursion a sequence $\langle y^x_n : n \in \omega\rangle$ of elements of T_γ. To define y^x_0 we consider two cases.

Case 1. $\omega\cdot\gamma = \gamma$ and A_γ is a maximal antichain in T_γ. Then x is comparable with some member z of A_γ, and we let y^x_0 be some element of T_γ such that $x, z \prec_\gamma y^x_0$.

Case 2. Otherwise, we just let $y^x_0 = x$.

Now let $\langle \xi_m : m \in \omega\rangle$ be a strictly increasing sequence of ordinals less than γ such that $\xi_0 = \mathrm{ht}(y^x_0, T_\gamma)$ and $\sup_{m\in\omega} \xi_m = \gamma$. Now if y^x_i has been defined of height ξ_i, by (4) let y^x_{i+1} be an element of height ξ_{i+1} such that $y^x_i \prec_\gamma y^x_{i+1}$. Then we define

$$B(x) = \{z \in \omega\cdot\gamma : z \prec_\gamma y^x_i \text{ for some } i \in \omega\}.$$

Finally, let $\langle x(n) : n \in \omega\rangle$ be a one-one enumeration of $\omega\cdot\gamma$, and set

$$\prec_\beta=\prec_\gamma \cup\{(z, \omega\cdot\gamma + n) : n \in \omega,\ z \in B(x_n)\}.$$

Clearly (1)–(3) hold with γ in place of α. For (4), suppose that $\delta < \gamma$ and $m \in \omega$. Let $z = \omega \cdot \delta + m$. Thus $z \in \omega \cdot \gamma$, and hence there is an $n \in \omega$ such that $z = x(n)$. Hence $z \in B(x(n))$ and $z \prec_\beta \omega \cdot \gamma + n$, as desired.

For (5), suppose that $\omega \cdot \gamma = \gamma$, and A_γ is a maximal antichain in T_γ. Suppose that $w \in \mathrm{Lev}_\gamma(T_\beta)$. Choose n so that $w = \omega \cdot \gamma + n$. Then there is an $s \in A_\gamma$ such that $s < y_0^{x(n)}$. So $s \in B(x(n))$ and $s \prec_\beta \omega \cdot \gamma + n = w$, as desired.

Thus the construction is finished. Now we let $\prec = \bigcup_{\alpha<\omega_1} \prec_\alpha$. Clearly $T \stackrel{\text{def}}{=} (\omega_1, \prec)$ is an ω_1-tree. It is eventually branching by (4) and the $\beta = \gamma + 2$ step in the construction. The hypothesis of Lemma 23.15 holds by the step $\beta = \gamma + 1$, γ limit, in the construction. Therefore T is a Suslin tree by Lemma 23.15. $\square$

We now introduce a generalization of clubs and stationary sets. Suppose that κ is an uncountable regular cardinal and A is a set such that $|A| \geq \kappa$. Then a subset X of $[A]^{<\kappa}$ is *closed* iff for every system $\langle a_\xi : \xi < \alpha \rangle$ of elements of X, with $\alpha < \kappa$ and with $a_\xi \subseteq a_\eta$ for all $\xi < \eta < \alpha$, also the union $\bigcup_{\xi<\alpha} a_\xi$ is in X. And we say that X is *unbounded* in $[A]^{<\kappa}$ iff for every $x \in [A]^{<\kappa}$ there is a $y \in X$ such that $x \subseteq y$. Club means closed and unbounded.

Theorem 23.17. *Suppose that κ is an uncountable regular cardinal, $|A| \geq \kappa$, and $a \in [A]^{<\kappa}$. Then $\{x \in [A]^{<\kappa} : a \subseteq x\}$ is club in $[A]^{<\kappa}$.*

Proof. Let C be the indicated set. Clearly C is closed. To show that it is unbounded, suppose that $y \in [A]^{<\kappa}$. Then $y \subseteq a \cup y \in C$, as desired. $\square$

Theorem 23.18. *Suppose that κ is an regular cardinal $> \aleph_1$ and $|A| \geq \kappa$. Then $\{x \in [A]^{<\kappa} : |x| \geq \aleph_1\}$ is club in $[A]^{<\kappa}$.*

Proof. Let C be the indicated set. For closure, suppose that $\langle a_\xi : \xi < \alpha \rangle$ is a system of members of C, with $\alpha < \kappa$ and $a_\xi \subseteq a_\eta$ if $\xi < \eta < \alpha$. Since each a_ξ has size at least $\aleph_1$, so does $\bigcup_{\xi<\alpha} a_\xi$, and so $\bigcup_{\xi<\alpha} a_\xi \in C$. So C is closed. Given $x \in [A]^{<\kappa}$, let y be a subset of A of size $\aleph_1$. Then $x \subseteq x \cup y \in C$. So C is club in $[A]^{<\kappa}$. $\square$

Theorem 23.19. *Suppose that κ is an uncountable regular cardinal and λ is a cardinal $> \kappa$. Then $\{x \in [\lambda]^{<\kappa} : x \cap \kappa \in \kappa\}$ is club in $[\lambda]^{<\kappa}$.*

Proof. Let C be the indicated set. To show that C is closed, suppose that $\langle a_\xi : \xi < \alpha \rangle$ is a system of members of C, with $\alpha < \kappa$ and $a_\xi \subseteq a_\eta$ if $\xi < \eta < \alpha$. Then $a_\xi \cap \kappa$ is an ordinal $\beta_\xi < \kappa$ for every $\xi < \alpha$. Since $\alpha < \kappa$ and κ is regular, it follows that

$$\left(\bigcup_{\xi<\alpha} a_\xi\right) \cap \kappa = \bigcup_{\xi<\alpha} (a_\xi \cap \kappa) = \bigcup_{\xi<\alpha} \beta_\xi$$

is an ordinal less than κ. So $\bigcup_{\xi<\alpha} a_\xi \in C$. Thus C is closed. To show that it is unbounded, let $y \in [\lambda]^{<\kappa}$. Let $x = (y \backslash \kappa) \cup (\bigcup(y \cap \kappa) + 1)$. Since κ is regular and $|y| < \kappa$, we have $|\bigcup(y \cap \kappa)| < \kappa$, and hence $|x| < \kappa$. Clearly $x \cap \kappa = \bigcup(y \cap \kappa) + 1 \in \kappa$. So $y \subseteq x \in C$, as desired. $\square$

Theorem 23.20. *Suppose that κ is an uncountable regular cardinal and $|A| \geq \kappa$. Then the intersection of two clubs of $[A]^{<\kappa}$ is a club.*

Proof. Let C and D be club in $[A]^{<\kappa}$. Clearly $C \cap D$ is closed. To show that it is unbounded, take any $x \in [A]^{<\kappa}$. We define a sequence $\langle y_i : i \in \omega \rangle$ of members of $[A]^{<\kappa}$ by recursion. Let $y_0 = x$. Having defined y_{2i}, choose y_{2i+1} such that $y_{2i+1} \in C$ and $y_{2i} \subseteq y_{2i+1}$; and then choose y_{2i+2} such that $y_{2i+2} \in D$ and $y_{2i+1} \subseteq y_{2i+2}$. Then $x \subseteq \bigcup_{i \in \omega} y_i \in C \cap D$. $\square$

Theorem 23.21. *Suppose that κ is an uncountable regular cardinal and $|A| \geq \kappa$. Then the intersection of fewer than κ clubs of $[A]^{<\kappa}$ is a club.*

Proof. Let $\langle C_\alpha : \alpha < \lambda \rangle$ be a system of clubs in κ, with $\lambda < \kappa$. We may assume that λ is an infinite cardinal. Clearly $\bigcap_{\alpha<\lambda} C_\alpha$ is closed in $[A]^{<\kappa}$. To show that it is unbounded, suppose that $x \in [A]^{<\kappa}$. We define a sequence $\langle y_\alpha : \alpha < \lambda \cdot \omega \rangle$ by recursion, where $\cdot$ is ordinal multiplication. Let $y_0 = x$. Suppose that y_α has been defined for all $\alpha < \beta$, with $\beta < \lambda \cdot \omega$, such that if $\alpha < \gamma < \beta$ then $y_\alpha \subseteq y_\gamma \in [A]^{<\kappa}$. If β is a successor ordinal $\lambda \cdot i + \gamma + 1$ with $i \in \omega$ and $\gamma < \lambda$, choose $y_\beta \in C_\gamma$ with $y_\gamma \subseteq y_\beta$. If β is a limit ordinal, let $y_\beta = \bigcup_{\alpha<\beta} y_\alpha$; so $y_\beta \in [A]^{<\kappa}$ by the regularity of κ. Finally, let $z = \bigcup_{\alpha<\lambda\cdot\omega} y_\alpha$. We claim that $x \subseteq z \in \bigcap_{\alpha<\lambda} C_\alpha$. Clearly $x \subseteq z$. Take any $\gamma < \lambda$. To show that $z \in C_\gamma$, it suffices to prove the following two things:

(1) $y_{\lambda\cdot i+\gamma+1} \in C_\gamma$ for all $i \in \omega$.

This is clear by construction.

(2) $z = \bigcup_{i\in\omega} y_{\lambda\cdot i+\gamma+1}$.

Since $\{\lambda \cdot i + \gamma + 1 : i \in \omega\}$ is cofinal in $\lambda \cdot \omega$, this is clear too. $\square$

If κ is an uncountable regular cardinal, $|A| \geq \kappa$, and $\langle X_a : a \in A \rangle$ is a system of subsets of $[A]^{<\kappa}$, then the *diagonal intersection* of this system is the set

$$\triangle_{a\in A} X_a \stackrel{\text{def}}{=} \left\{ x \in [A]^{<\kappa} : x \in \bigcap_{a\in x} X_a \right\}.$$

Theorem 23.22. *Suppose that κ is an uncountable regular cardinal, $|A| \geq \kappa$, and $\langle X_a : a \in A \rangle$ is a system of clubs of $[A]^{<\kappa}$. Then $\triangle_{a\in A} X_a$ is club in $[A]^{<\kappa}$.*

Proof. For brevity let $D = \triangle_{a\in A} X_a$. To show that D is closed, suppose that $\langle x_\alpha : \alpha < \gamma \rangle$ is a system of members of D, with $\gamma < \kappa$, such that $x_\alpha \subseteq x_\beta$ if $\alpha < \beta < \gamma$. We want to show that $b \stackrel{\text{def}}{=} \bigcup_{\alpha<\gamma} x_\alpha$ is in D. To do this, by the definition of diagonal intersection we need to take any $a \in b$ and show that $b \in X_a$. Say $a \in x_\beta$ with $\beta < \gamma$. Then for any $\delta \in [\beta, \gamma)$ we have $a \in x_\delta$, and hence, since $x_\delta \in D$, by definition we get $x_\delta \in X_a$. Say $\beta + \tau = \gamma$. Then $\langle x_{\beta+\varepsilon} : \varepsilon < \tau \rangle$ is a system of elements of X_a, and $x_{\beta+\varepsilon} \subseteq x_{\beta+\xi}$ if $\varepsilon < \xi < \tau$. So because X_a is closed, we get

$$b = \bigcup_{\varepsilon<\tau} x_{\beta+\varepsilon} \in X_a.$$

So D is closed.

To show that D is unbounded, let $x \in [A]^{<\kappa}$ be given. We now define a sequence $\langle y_i : i \in \omega\rangle$ by recursion. Let $y_0 = x$. Having defined $y_i \in [A]^{<\kappa}$, by Theorem 23.21 the set $\bigcap_{a\in y_i} X_a$ is club in $[A]^{<\kappa}$. Hence we can choose y_{i+1} in this set such that $y_i \subseteq y_{i+1}$. This finishes the construction. Now let $z = \bigcup_{i\in\omega} y_i$. We claim that $x \subseteq z \in D$, as desired. For, clearly $x \subseteq z$. Now suppose that $a \in z$; we want to show that $z \in X_a$. Choose $i \in \omega$ so that $a \in y_i$. Then for any $j \geq i$ we have $a \in y_j$, and so by construction $y_{j+1} \in X_a$. Hence $z = \bigcup_{i\leq j} y_j \in X_a$, as desired. □

Given an uncountable regular cardinal κ and a set A with $|A| \geq \kappa$, we say that a subset X of $[A]^{<\kappa}$ is *stationary* iff it intersects every club of $[A]^{<\kappa}$.

Theorem 23.23. *Suppose that κ is an uncountable regular cardinal, $|A| \geq \kappa$, S is a stationary subset of $[A]^{<\kappa}$, and f is a function with domain S such that $f(x) \in x$ for every nonempty $x \in S$. Then there exist a stationary subset T of S and an element $a \in A$ such that $f(x) = a$ for all $x \in T$.*

Proof. It suffices to show that there is an $a \in A$ such that $f^{-1}[\{a\}]$ is stationary. Suppose to the contrary that for each $a \in A$ there is a club C_a in $[A]^{<\kappa}$ such that $C_a \cap f^{-1}[\{a\}] = \emptyset$. By Theorem 23.22 choose $x \in S \cap \triangle_{a\in A} C_a$. Thus $x \in \bigcap_{a\in x} C_a$. In particular, $x \in C_{f(x)}$. So $x \in C_{f(x)} \cap f^{-1}[\{f(x)\}]$, contradiction. □

Theorem 23.24. *Suppose that λ is regular, $\kappa^+ \leq \lambda$, and $S \subseteq [\lambda]^{<\kappa^+}$ is stationary. Then S is the disjoint union of λ stationary sets.*

Proof. For each nonempty $P \in [\lambda]^{<\kappa^+}$ write $P = \{\alpha^P_\xi : \xi < \kappa\}$.

(1) There is an $\eta < \kappa$ such that for all $\beta < \lambda$ the set $\{P \in S : \alpha^P_\eta \geq \beta\}$ is stationary.

Otherwise for every $\eta < \kappa$ there is a $\beta_\eta < \lambda$ such that $\{P \in S : \alpha^P_\eta \geq \beta_\eta\}$ is non-stationary. So there is a club C_η such that $C_\eta \cap \{P \in S : \alpha^P_\eta \geq \beta_\eta\} = \emptyset$. Let $\gamma = \sup_{\eta<\kappa} \beta_\eta$ and $D = \bigcap_{\eta<\kappa} C_\eta$. Note that D is club by Theorem 23.21. For all $P \in D\cap S$ and $\eta < \kappa$ we have $\alpha^P_\eta < \beta_\eta \leq \gamma$, so $P \subseteq \gamma$. Now by Theorem 23.17 the set $E \stackrel{\text{def}}{=} \{P \in [\lambda]^{<\kappa^+} : \gamma + 1 \subseteq P\}$ is club. So $E \cap D \cap S = \emptyset$, contradicting S stationary. So (1) holds.

Take $\eta < \kappa$ as in (1). For each $P \in S$ let $f(P) = \alpha^P_\eta$. Now for each $\beta < \lambda$ the set $T_\beta \stackrel{\text{def}}{=} \{P \in S : \alpha^P_\eta \geq \beta\}$ is stationary. For $P \in T_\beta$ we have $f(P) \in P$, so by Theorem 23.23 there is a stationary subset U_β of T_β and a $\delta_\beta < \lambda$ such that $f(P) = \delta_\beta$ for all $P \in U_\beta$. Let $V_\beta = \{P \in S : f(P) = \delta_\beta\}$. So $U_\beta \subseteq V_\beta$, hence V_β is stationary. We now define $\langle \varepsilon_\xi : \xi < \lambda\rangle$ by recursion. Suppose defined for all $\xi < \eta$. Let $\beta = \sup_{\xi<\eta}(\delta_{\varepsilon_\xi} + 1)$, and set $\varepsilon_\eta = \delta_\beta$. Clearly $V_{\varepsilon_\xi} \cap V_{\varepsilon_\eta} = \emptyset$ for $\xi \neq \eta$. □

Theorem 23.25. *Let κ be an uncountable regular cardinal. Thus $\kappa \subseteq [\kappa]^{<\kappa}$. Suppose that $C \subseteq [\kappa]^{<\kappa}$ is club. Then $C \cap \kappa$ is club in the usual sense.*

Proof. To show that $C\cap\kappa$ is closed, suppose that $\alpha < \kappa$ and $C\cap\kappa$ is unbounded in α in the usual sense. Let $\langle \beta_\xi : \xi < \mathrm{cf}(\alpha)\rangle$ be a system of elements of $C\cap\kappa$ with supremum α. Thus $\alpha = \bigcup_{\xi<\mathrm{cf}(\alpha)} \beta_\xi \in C$ since X is closed. This union is also in κ because κ is regular.

To show that $C \cap \kappa$ is unbounded in the usual sense, suppose that $\alpha < \kappa$. Since C is unbounded, choose $y_0 \in C$ such that $\alpha \subseteq y_0$. Now $y_0 \in [\kappa]^{<\kappa}$, so $\beta_0 \stackrel{\text{def}}{=} \bigcup y_0 < \kappa$. Then choose $y_1 \in C$ such that $\beta_0 \subseteq y_1$. Continuing, we obtain $\alpha \subseteq y_0 \subseteq \beta_0 \subseteq y_1 \subseteq \beta_1 \subseteq \ldots$. The union of this sequence is in C since C is closed, and it is an ordinal $< \kappa$ since κ is regular, as desired. □

Theorem 23.26. *Let κ be an uncountable regular cardinal, and let $C \subseteq \kappa$ be club in the old sense. Then $\{X \in [\kappa]^{<\kappa} : \bigcup X \in C\}$ is club in the new sense.*

Proof. Let $C' = \{X \in [\kappa]^{<\kappa} : \bigcup X \in C\}$. Suppose that $\langle X_\xi : \xi < \alpha\rangle$ is an increasing sequence of members of C', with $\alpha < \kappa$. Then $\langle \bigcup X_\xi : \xi < \alpha\rangle$ is an increasing sequence of members of C, and so $\bigcup\bigcup_{\xi<\alpha} X_\xi \in C$. It follows that $\bigcup_{\xi<\alpha} X_\xi \in C'$.

Suppose that $X \in [\kappa]^{<\kappa}$. Then $\bigcup X$ is an ordinal less than κ, and so there is a limit ordinal $\alpha \in C$ such that $\bigcup X < \alpha$. Hence $X \subseteq \alpha = \bigcup \alpha$. So $\alpha \in C'$ is as desired. □

Theorem 23.27. *Let κ be an uncountable regular cardinal, and let $S \subseteq [\kappa]^{<\kappa}$ be stationary in the new sense. Then $\{\bigcup X : X \in S\}$ is stationary in the old sense.*

Proof. Let $S' = \{\bigcup X : X \in S\}$. Let C be a club in the old sense. With C' as in the proof of Theorem 23.26, choose $X \in S \cap C'$. Then $\bigcup X \in S' \cap C$, as desired. □

Theorem 23.28. *Let κ be an uncountable regular cardinal, and $S \subseteq \kappa$ be stationary in the old sense. Then S is stationary as a subset of $[\kappa]^{<\kappa}$.*

Proof. Let $X \subseteq [\kappa]^{<\kappa}$ be club. Then by Theorem 23.25, $X \cap \kappa$ is club in the old sense. Hence $S \cap X \cap \kappa \neq \emptyset$. □

Proposition 23.29. *If γ is a limit ordinal, then there is a family of $\mathrm{cf}(\gamma)$ clubs of γ whose intersection is empty.*

Proof. Let $f : \mathrm{cf}(\gamma) \to \gamma$ be strictly increasing and continuous. For each $\alpha < \mathrm{cf}(\gamma)$ let $C_\alpha = \{f(\beta) : \alpha \leq \beta < \mathrm{cf}(\gamma)\}$. Then each C_α is club in γ, and $\bigcap_{\alpha<\mathrm{cf}(\gamma)} C_\alpha = \emptyset$. □

Proposition 23.30. *If γ is a limit ordinal and $\mathrm{cf}(\gamma) = \omega$, then there are two clubs of γ whose intersection is empty.*

Proof. Let f be as above. Let $D_0 = \{f(2n) : n \in \omega\}$ and $D_1 = \{f(2n+1) : n \in \omega\}$. Then D_0 and D_1 are club in γ, and $D_0 \cap D_1 = \emptyset$. □

For a limit ordinal γ with $\mathrm{cf}(\gamma) > \omega$ we define

$$\mathrm{club}_\gamma = \{X \subseteq \gamma : \exists C[C \text{ club and } C \subseteq X]\};$$
$$\mathrm{nonstat}_\gamma = \{X \subseteq \gamma : \exists C[C \text{ club and } X \cap C = \emptyset]\}$$

Note that club_γ is a filter on γ and $\mathrm{nonstat}_\gamma$ is an ideal on γ.

Proposition 23.31. *If γ is a limit ordinal with $\mathrm{cf}(\gamma) > \omega$, then*

$$\mathrm{add}(\mathrm{nonstat}_\gamma) = \mathrm{cov}(\mathrm{nonstat}_\gamma) = \mathrm{non}(\mathrm{nonstat}_\gamma) = \mathrm{cf}(\gamma).$$

Proof. If $\mathscr{A} \subseteq \text{nonstat}_\gamma$ and $|\mathscr{A}| < \text{cf}(\gamma)$, for each $A \in \mathscr{A}$ let C_A be club such that $A \cap C_A = \emptyset$. Then $\bigcap_{A\in\mathscr{A}} C_A$ is club, and $\bigcup \mathscr{A} \cap \bigcap_{A\in\mathscr{A}} = \emptyset$, so $\bigcup \mathscr{A} \in \text{nonstat}_\gamma$. With $\langle \alpha_\xi : \xi < \text{cf}(\gamma)\rangle$ strictly increasing with supremum γ, for each $\xi < \text{cf}(\gamma)$ the set $D_\xi = [\alpha_\xi, \gamma)$ is club, hence α_ξ is nonstationary, and $\bigcup_{\xi<\text{cf}(\gamma)} \alpha_\xi = \gamma \notin \text{nonstat}_\gamma$. Therefore $\text{add}(\text{nonstat}_\gamma) = \text{cf}(\gamma)$.

If $\mathscr{A} \subseteq \text{nonstat}_\gamma$ and $|\mathscr{A}| < \text{cf}(\gamma)$ then as above $\bigcup \mathscr{A} \in \text{nonstat}_\gamma$, hence $\bigcup \mathscr{A} \neq \gamma$. As above, $\{\alpha_\xi : \xi < \text{cf}(\gamma)\} \subseteq \text{nonstat}_\gamma$ and $\bigcup\{\alpha_\xi : \xi < \text{cf}(\gamma)\} = \gamma$. So $\text{cov}(\text{nonstat}_\gamma) = \text{cf}(\gamma)$.

If $X \in [\gamma]^\delta$ with δ a cardinal less than $\text{cf}(\gamma)$, then X is bounded below γ and hence is in nonstat_γ. By Corollary 23.4 there is a club $\subseteq \gamma$ with order type $\text{cf}(\gamma)$; it is stationary. Hence $\text{non}(\text{nonstat}_\gamma) = \text{cf}(\gamma)$. □

Lemma 23.32. (III.6.7) *Suppose that κ is uncountable and regular, and $T \subseteq \kappa$ is stationary. Suppose that $f : T \to E$ with $|E| < \kappa$. Then there is an $e \in E$ such that $f^{-1}[\{e\}]$ is stationary.*

Proof. Otherwise $T = \bigcup_{e\in E} f^{-1}[\{e\}]$ contradicts $\text{add}(\text{nonstat}_\kappa) = \kappa$. □

Theorem 23.33. (III.6.10) *Let κ be a successor cardinal, and I an ideal on κ such that $\text{add}(I) = \kappa$ and I contains all singletons. Then there are disjoint $X_\delta \subseteq \kappa$ for $\delta < \kappa$ such that each $X_\delta \notin I$.*

Proof. If $M \in [\kappa]^{<\kappa}$, then $M = \bigcup_{\alpha\in M}\{\alpha\} \in I$. Thus $[\kappa]^{<\kappa} \subseteq I$. Say $\kappa = \lambda^+$. For each $\rho < \kappa$ let f_ρ be an injection of ρ into λ. For $\xi < \lambda$ and $\alpha < \kappa$ let $X^\xi_\alpha = \{\rho : \alpha < \rho < \kappa$ and $f_\rho(\alpha) = \xi\}$. Note:

(1) $\forall \xi < \lambda \forall \alpha, \beta < \kappa[\alpha \neq \beta \to X^\xi_\alpha \cap X^\xi_\beta = \emptyset]$.

(2) $\forall \alpha < \kappa[\bigcup_{\xi<\lambda} X^\xi_\alpha = \kappa\backslash(\alpha + 1)]$.

By (2), for each $\alpha < \kappa$ choose $\xi_\alpha < \lambda$ such that $X^{\xi_\alpha}_\alpha \notin I$. Since κ is regular, there is a $\xi < \lambda$ such that $A \stackrel{\text{def}}{=} \{\alpha < \kappa : \xi_\alpha = \xi\}$ has size κ. Then by (1), $\langle X^\xi_\alpha : \alpha \in A\rangle$ is a system of pairwise disjoint subsets of κ none of which is in I. □

Proposition 23.34. (III.6.13) *If κ is regular and uncountable and $f : \kappa \to \kappa$, then $D \stackrel{\text{def}}{=} \{\beta < \kappa : \forall \alpha < \beta[f(\alpha) < \beta]\}$ is a club.*

Proof. For each $\alpha < \kappa$ let $C_\alpha = \{\beta < \kappa : f(\alpha) < \beta\}$. Then for any $\beta < \kappa$, $\beta \in \triangle_{\alpha<\kappa} C_\alpha$ iff $\forall \alpha < \beta[\beta \in C_\alpha]$ iff $\forall \alpha < \beta[f(\alpha) < \beta]$ iff $\beta \in D$. So the result follows. □

Proposition 23.35. *If κ is strongly inaccessible, then $\{\alpha < \kappa : \alpha = \beth_\alpha\}$ is club in κ.*

Proof. By induction, if $\alpha < \kappa$ then $\beth_\alpha < \kappa$. Take $f(\alpha) = \beth_\alpha$ in Lemma 23.34, obtaining a club D. Let $\beta \in D\backslash\{0\}$. Then $\forall \alpha < \beta[\beth_\alpha < \beta]$. Hence $\beta = \bigcup_{\alpha<\beta} \beth_\alpha = \beth_\beta$. □

Proposition 23.36. (III.6.16) *Let $\lambda = \omega_{\omega_1}$. A set $\mathscr{F} \subseteq {}^{\omega_1}\lambda$ is an* **eventually different family (edf)** *iff $\{\alpha < \omega_1 : f(\alpha) = g(\alpha)\}$ is countable whenever $f, g \in \mathscr{F}$ and $f \neq g$. Assume $\forall \alpha < \omega_1[2^{\aleph_\alpha} = \aleph_{\alpha+1}]$.*

Then there is an edf $\mathscr{F}$ of size 2^λ such that $\forall f \in \mathscr{F} \forall \alpha < \omega_1[f(\alpha) < \omega_{\alpha+1}]$.

Proof. For any $X \subseteq \lambda$ and $\alpha < \omega_1$ let $f_X(\alpha) = X \cap \omega_\alpha$. For each $\alpha < \omega_1$ let $g_\alpha : \mathscr{P}(\omega_\alpha) \to \omega_{\alpha+1}$ be a bijection. For each $X \subseteq \lambda$ let $h_X : \omega_1 \to \lambda$ be defined by $h_X(\alpha) = g_\alpha(f_X(\alpha))$. Thus $h_X(\alpha) < \omega_{\alpha+1}$ for all $\alpha < \omega_1$. We claim that $\langle h_X : X \subseteq \lambda\rangle$ is an eventually different family. For, suppose that $X, Y \in \mathscr{P}(\lambda)$ and $X \neq Y$. Say $\beta \in X\backslash Y$. Then for any $\alpha < \omega_1$ such that $\beta < \omega_\alpha$ we have $f_X(\alpha) = X \cap \omega_\alpha \neq Y \cap \omega_\alpha = f_Y(\alpha)$; hence $h_X(\alpha) \neq h_Y(\alpha)$. $\square$

Proposition 23.37. (III.6.17) *Assume that $\lambda = \omega_{\omega_1}$, $\forall \alpha < \omega_1[2^{\aleph_\alpha} = \aleph_{\alpha+1}]$, $\mathscr{F} \subseteq {}^{\omega_1}\lambda$ is an edf, $g : \omega_1 \to \lambda$, $\forall \alpha < \omega_1[g(\alpha) < \omega_{\alpha+1}]$, and $\forall f \in \mathscr{F}[\{\alpha < \omega_1 : f(\alpha) < g(\alpha)\}$ is stationary$]$.*

Then $|\mathscr{F}| \leq \lambda$.

Proof. Suppose that $|\mathcal{F}| > \lambda$. Note that there are fewer than λ stationary subsets of ω_1. For, $|\{S \subseteq \omega_1 : S \text{ stationary}\}| \leq |\mathscr{P}(\omega_1)| = 2^{\omega_1} = \omega_2 < \lambda$. So there is a stationary set S such that $|\{f \in \mathcal{F} : \{\alpha < \omega_1 : f(\alpha) < g(\alpha)\} = S\}| > \lambda$. Let $\mathcal{F}' = \{f \in \mathcal{F} : \{\alpha < \omega_1 : f(\alpha) < g(\alpha)\} = S\}$. Let $S' = S\backslash\{0\}$.

For each $\alpha < \omega_1$ let h_α be an injection of $g(\alpha)$ into ω_α. If $f \in \mathcal{F}'$ and $\alpha \in S'$, let $l_f(\alpha) = h_\alpha(f(\alpha))$. Note that $\alpha \neq 0$ and $l_f(\alpha) < \omega_\alpha$; so there is an $f'(\alpha) < \alpha$ such that $|l_f(\alpha)| \leq \omega_{f'(\alpha)}$. Now f' is regressive on the stationary set S', so it takes a constant value β_f on a stationary subset S''_f of S'. Now the function $f \mapsto \beta_f$ maps $\mathcal{F}'$, a set of size $> \lambda$, into λ. Hence there exist a set $\mathcal{F}'' \in [\mathcal{F}']^{\geq\lambda^+}$ and a $\gamma < \lambda$ such that $\beta_f = \gamma$ for all $f \in \mathcal{F}''$. Now there are fewer than λ stationary sets, so there exist a subset $\mathcal{F}'''$ of $\mathcal{F}''$ of size more than λ and a stationary set T such that $S''_f = T$ for all $f \in \mathcal{F}'''$. Now for all $f \in \mathcal{F}''$ and $\alpha \in T$ we have $|l_f(\alpha)| \leq \omega_\gamma$ and so $l_f(\alpha) < \omega_{\gamma+1}$. So $l_f \in {}^T\omega_{\gamma+1}$. Since $|{}^T\omega_{\gamma+1}| < \lambda$, it follows that there are distinct $f, g \in \mathcal{F}''$ such that $l_f = l_g$. Hence $f \restriction T = g \restriction T$. This contradicts the edf property. $\square$

Proposition 23.38. (III.6.18) *If $\lambda = \omega_{\omega_1}$ and $\forall \alpha < \omega_1[2^{\aleph_\alpha} = \aleph_{\alpha+1}]$, then $2^\lambda = \lambda^+$.*

Proof. We follow Baumgartner, Prikry. First we prove

Claim. *Suppose that S is a stationary subset of ω_1 and $f : S \to \omega_{\omega_1}$ is such that $f(\alpha) < \omega_\alpha$ for all $\alpha \in S$. Then there exist a stationary $T \subseteq S$ and a $\gamma < \omega_1$ such that $f(\alpha) < \omega_\gamma$ for all $\alpha \in T$.*

Proof. Let C be the collection of all limit ordinals less than ω_1; so C is club in ω_1, and hence $C \cap S$ is stationary in ω_1. For each $\alpha \in C \cap S$ we have $f(\alpha) < \omega_\alpha$, so there is a $\beta < \alpha$ such that $f(\alpha) < \omega_\beta$; let $g(\alpha)$ be the least such β. So $g(\alpha) < \alpha$. Thus g is a regressive function defined on the stationary set $S \cap C$, and so g is constant on a stationary subset T of $S \cap C$. $\square$

Proof of Proposition 23.38 Let $\lambda = \omega_{\omega_1}$, assume that $\forall \alpha < \omega_1[2^{\omega_\alpha} = \omega_{\alpha+1}]$, and suppose that $2^\lambda > \lambda^+$. We want to get a contradiction.

Let $\langle A^\alpha_\xi : \xi < \omega_{\alpha+1}\rangle$ be a listing without repetitions of $\mathscr{P}(\omega_\alpha)$. For each $B \subseteq \omega_{\omega_1}$ and each $\alpha < \omega_1$ let $f_B(\alpha)$ be the $\xi < \omega_{\alpha+1}$ such that $B \cap \omega_\alpha = A^\alpha_\xi$. If $B, C \in \mathscr{P}(\omega_{\omega_1})$ with $B \neq C$, then there is an $\alpha < \omega_1$ such that $B \cap \omega_\alpha \neq C \cap \omega_\alpha$, and then $f_B(\beta) \neq f_C(\beta)$ for all $\beta \geq \alpha$. Thus

(1) For any distinct $B, C \subseteq \omega_{\omega_1}$, the set $\{\beta < \omega_1 : f_B(\beta) = f_C(\beta)\}$ is bounded.

Define BRC iff $B, C \subseteq \omega_{\omega_1}$ and $\{\alpha < \omega_1 : f_B(\alpha) < f_C(\alpha)\}$ is stationary.

(2) If $B, C \subseteq \omega_1$ and $B \neq C$, then BRC or CRB.

For,

$$\omega_1 = \{\alpha < \omega_1 : f_B(\alpha) < f_C(\alpha)\} \cup \{\alpha < \omega_1 : f_B(\alpha) = f_C(\alpha)\} \cup \{\alpha < \omega_1 : f_C(\alpha) < f_B(\alpha)\}.$$

Now if $\{\alpha < \omega_1 : f_B(\alpha) < f_C(\alpha)$ and $\{\alpha < \omega_1 : f_C(\alpha) < f_B(\alpha)\}$ are both nonstationary, let C, D be clubs such that $C \cap \{\alpha < \omega_1 : f_B(\alpha) < f_C(\alpha) = \emptyset$ and $D \cap \{\alpha < \omega_1 : f_A(\alpha) < f_B(\alpha) = \emptyset$. Then $C \cap D \subseteq \{\alpha < \omega_1 : f_B(\alpha) = f_C(\alpha)\}$ and hence $C \cap D$ is bounded by (1), contradiction. Hence (2) follows.

(3) There is a $B \subseteq \omega_{\omega_1}$ such that $|\{C : CRB\}| \geq \lambda^+$.

In fact, let X be a subset of $\mathscr{P}(\omega_{\omega_1})$ of size λ^+. We may assume that $\forall B \in X[|\{C : CRB\}| \leq \lambda]$. Let $Y = \{C \subseteq \omega_{\omega_1} : \exists B \in X[CRB]\}$. Then $|Y| \leq |X| \cdot \lambda = \lambda^+$. Since $2^\lambda > \lambda^+$, choose $C \subseteq \omega_{\omega_1}$ such that $C \notin Y$. Then $\forall B \in X[\text{not}(CRB)]$, so by (1), $\forall B \in X[BRC]$. This proves (3).

We fix B as in (3). Now for each $\alpha < \omega_1$ we have $f_B(\alpha) < \omega_{\alpha+1}$, so there is a one-one function $g_\alpha : f_B(\alpha) \to \omega_\alpha$. Now suppose that CRB. Then $S_C \stackrel{\text{def}}{=} \{\alpha < \omega_1 : f_C(\alpha) < f_B(\alpha)\}$ is stationary. For any $\alpha \in S_C$ we have $g_\alpha(f_C(\alpha)) < \omega_\alpha$. Hence by the claim there exist a stationary $T_C \subseteq S_C$ and a $\gamma_C < \omega_1$ such that $g_\alpha(f_C(\alpha)) < \omega_{\gamma_C}$ for all $\alpha \in T_C$. The number of pairs (T_C, γ_C) is at most $2^{\omega_1} \cdot \omega_1 < \omega_{\omega_1}$, so by (3) There is a pair (U, δ) such that the set

$$V \stackrel{\text{def}}{=} \{C : CRB \text{ and } T_C = U \text{ and } \gamma_C = \delta\}$$

has size λ^+. Now for each $C \in V$ we define a function k_C with domain U. For any $\alpha \in U$ let $k_C(\alpha) = g_\alpha(f_C(\alpha))$. So k maps U into ω_δ. The number of functions mapping U into ω_δ is at most $\omega_\delta^{\omega_1} < \lambda$, so there are distinct $C, D \in V$ such that $k_C = k_D$. Since each g_α is one-one, it follows that $f_C \restriction U = f_D \restriction U$. This contradicts (1). $\square$

Proposition 23.39. (III.6.19) *Suppose that λ is a singular cardinal with $\mathrm{cf}(\lambda) > \omega$, and $\{\theta < \lambda : 2^\theta = \theta^+\}$ is stationary in λ.*

Then $2^\lambda = \lambda^+$.

Proof. First we prove:

Claim. *Suppose that λ is a singular cardinal with $\mathrm{cf}(\lambda) > \omega$. Let $\langle \mu_\alpha : \alpha < \mathrm{cf}(\lambda)\rangle$ be a strictly increasing continuous sequence of infinite cardinals with supremum λ. Suppose that S is a stationary subset of $\mathrm{cf}(\lambda)$ and $f : S \to \lambda$ is such that $f(\alpha) < \mu_\alpha$ for all $\alpha \in S$. Then there exist a stationary $T \subseteq S$ and a $\gamma < \mathrm{cf}(\lambda)$ such that $f(\alpha) < \mu_\gamma$ for all $\alpha \in T$.*

Proof. Let C be the collection of all limit ordinals less than $\mathrm{cf}(\lambda)$; so C is club in $\mathrm{cf}(\lambda)$, and hence $C \cap S$ is stationary in $\mathrm{cf}(\lambda)$. For each $\alpha \in C \cap S$ we have $f(\alpha) < \mu_\alpha$, so there is a $\beta < \alpha$ such that $f(\alpha) < \mu_\beta$. As in the proof of the claim for the proof of Proposition 23.38, this gives the desired result. $\square$

Proof of Proposition 23.39 Let λ be a singular cardinal with $\mathrm{cf}(\lambda) > \omega$, assume that $S \stackrel{\text{def}}{=} \{\theta < \lambda : \theta \text{ is a cardinal and } 2^\theta = \theta^+\}$ is stationary, while $2^\lambda > \lambda^+$; we want to get a contradiction. Let $\langle \mu_\alpha : \alpha < \mathrm{cf}(\lambda)\rangle$ be a strictly increasing continuous sequence of infinite cardinals with supremum λ. Then $C \stackrel{\text{def}}{=} \{\mu_\alpha : \alpha < \mathrm{cf}(\lambda)\}$ is club in λ, so $S \cap C$ is stationary in λ. Let $S' = \{\alpha < \mathrm{cf}(\lambda) : \mu_\alpha \in S \cap C\}$. Then S' is stationary in $\mathrm{cf}(\lambda)$, since if D is club in $\mathrm{cf}(\lambda)$ then $\{\mu_\alpha : \alpha \in D\}$ is club in λ, hence there is an $\alpha \in D$ such that $\mu_\alpha \in C \cap S$, hence $\alpha \in S' \cap D$. For each $\alpha \in S'$ let $\langle A^\alpha_\xi : \xi < \mu^+_\alpha\rangle$ be a listing without repetitions of $\mathscr{P}(\mu_\alpha)$. For each $B \subseteq \lambda$ and each $\alpha \in S'$ let $f_B(\alpha)$ be the $\xi < \mu^+_\alpha$ such that $B \cap \mu_\alpha = A^\alpha_\xi$. If $B, C \subseteq \lambda$ with $B \neq C$, then there is an $\alpha \in S$ such that $B \cap \mu_\alpha \neq C \cap \mu_\alpha$, and then $B \cap \mu_\beta \neq C \cap \mu_\beta$ for all $\beta \geq \alpha$, so that $f_B(\beta) \neq f_C(\beta)$ for all $\beta \in S$ such that $\alpha \leq \beta$. Thus

(1) For any distinct $B, C \subseteq \lambda$ the set $\{\alpha \in S' : f_B(\alpha) = f_C(\alpha)\}$ is bounded in λ.

Define BRC iff $B, C \subseteq \lambda$ and $\{\alpha \in S' : f_B(a) < f_C(\alpha)\}$ is stationary in $\mathrm{cf}(\lambda)$.

(2) If $B, C \subseteq \lambda$ and $B \neq C$, then BRC or CRB.

The proof of (2) is like that of (2) in the proof of Proposition 23.38.

(3) There is a $B \subseteq \lambda$ such that $|\{C : CRB\}| \geq \lambda^+$.

In fact, let X be a subset of $\mathscr{P}(\lambda)$ of size λ^+. We may assume that $\forall B \in X[|\{C : CRB\}| \leq \lambda]$. Let $Y = \{C \subseteq \lambda : \exists B \in X[CRB]\}$. Then $|Y| \leq |X| \cdot \lambda = \lambda^+$. Since $2^\lambda > \lambda^+$, choose $C \subseteq \lambda$ such that $C \notin Y$. Then $\forall B \in X[\mathrm{not}(CRB)]$, so by (1), $\forall B \in X[BRC]$. This proves (3).

We fix B as in (3). Now for each $\alpha < \mathrm{cf}(\lambda)$ we have $f_B(\alpha) < \mu^+_\alpha$, so there is a one-one function $g_\alpha : f_B(\alpha) \to \mu_\alpha$. Now suppose that CRB. Then $S_C \stackrel{\text{def}}{=} \{\alpha \in S' : f_C(\alpha) < f_B(\alpha)\}$ is stationary. For any $\alpha \in S_C$ we have $g_\alpha(f_C(\alpha)) < \mu_\alpha$. Hence by the claim there exist a stationary $T_C \subseteq S_C$ and a $\gamma_C < \mathrm{cf}(\lambda)$ such that $g_\alpha(f_C(\alpha)) < \mu_{\gamma_C}$ for all $\alpha \in T_C$. The number of pairs (T_C, γ_C) is at most $2^{\mathrm{cf}(\lambda)} \cdot \mathrm{cf}(\lambda) < \lambda$, so by (3) There is a pair (U, δ) such that the set

$$V \stackrel{\text{def}}{=} \{C : CRB \text{ and } T_C = U \text{ and } \gamma_C = \delta\}$$

has size λ^+. Now for each $C \in V$ we define a function k_C with domain U. For any $\alpha \in U$ let $k_C(\alpha) = g_\alpha(f_C(\alpha))$. So k maps U into μ_δ. The number of functions mapping U into μ_δ is at most $\mu_\delta^{\mathrm{cf}(\lambda)} < \lambda$, so there are distinct $C, D \in V$ such that $k_C = k_D$. Since each g_α is one-one, it follows that $f_C \restriction U = f_D \restriction U$. This contradicts (1). $\square$

Proposition 23.40. (III.6.21) *If $\mathfrak{B}$, with universe $B = \omega_1$, is a structure for a countable language $\mathscr{L}$, then $C \stackrel{\text{def}}{=} \{\alpha \subseteq \omega_1 : \alpha \preceq \mathfrak{B}\}$ is a club in ω_1.*

Proof. Fix $c \in B$. Let g be a choice function for nonempty subsets of B. For each formula of $\mathscr{L}$ of the form $\exists y\psi(\overline{x}, y)$ let $f_{\exists y\psi(\overline{x},y)}$ be the m-ary function defined as follows, where m is the length of $\overline{x}$. For any $\overline{a} \subseteq B$,

$$f_{\exists y\psi(\overline{x},y)}(\overline{a}) = \begin{cases} g(\{b \in B : \mathfrak{B} \models \psi(\overline{a}, b)\}) & \text{if this set is nonempty,} \\ c & \text{otherwise.} \end{cases}$$

Let $\mathscr{F}$ consist of all functions $f_{\exists y\psi(\overline{x},y)}$, and let D be the set of all $\alpha < \omega_1$ which are closed under $\mathscr{F}$. So by Proposition 23.7, D is club in ω_1. By the Tarski-Vaught Lemma 15.3, $\alpha \preceq \mathfrak{B}$ for all nonzero $\alpha \in D$. Clearly the converse holds, so $C = D$. $\square$

Recall from before Proposition 23.31 the definition of club_γ. We also define

$$\mathrm{club}^B = \{X \subseteq [B]^{<\kappa} : \exists C \subseteq [B]^{<\kappa}[C \text{ is club and } C \subseteq X\}.$$

Proposition 23.41. *For any $C \subseteq [\omega_1]^{\leq\omega}$, $C \in \mathrm{club}^{\omega_1}$ iff $(C \cap \omega_1) \in \mathrm{club}_{\omega_1}$.*

Proof. $\Rightarrow$: Suppose that C is club in $[\omega_1]^{\leq\omega}$, α is a countable limit ordinal, and $C \cap \omega_1 \cap \alpha$ is unbounded in α. Then there is a strictly increasing sequence $\langle \gamma_n : n \in \omega\rangle$ with union α with each $\gamma_n \in C \cap \alpha$. Then because C is club in $[\omega_1]^{\leq\omega}$, $\alpha = \bigcup_{n\in\omega} \gamma_n \in C$. Thus $C \cap \omega_1$ is closed in the usual sense. To show that it is unbounded, suppose that $\alpha < \omega_1$. Note that ω_1 is club in $[\omega_1]^{\leq\omega_1}$, and so $C \cap \omega_1$ is club in $[\omega_1]^{\leq\omega_1}$. Choose $\gamma \in C \cap \omega_1$ such that $\alpha \subseteq \gamma$. This shows that $C \cap \omega_1$ is unbounded in the usual sense.

Now if $X \in \mathrm{club}^{\omega_1}$, choose $C \subseteq [\omega_1]^{\leq\omega}$ such that C is club and $C \subseteq X$. Then $(C \cap \omega_1) \subseteq (X \cap \omega_1$, So $(X \cap \omega_1) \in \mathrm{club}_{\omega_1}$.

$\Leftarrow$: Suppose that $(X \cap \omega_1) \in \mathrm{club}_{\omega_1}$. Choose a club C in ω_1 such that $C \subseteq (X \cap \omega_1)$. We claim that C is club in $[\omega_1]^{\leq\omega}$. If $A \in [\omega_1]^{\leq\omega}$, let $\alpha = \sup A$, and choose $\beta \in C$ such that $\alpha < \beta$. Then $A \subseteq \beta$. So C is unbounded in $[\omega_1]^{\leq\omega}$. Now suppose that $A \in {}^\omega C$ and $A_n \subseteq A_{n+1}$ for all $n \in \omega$. Thus $A_n \leq A_{n+1}$ for all $n \in \omega$. Hence $\bigcup_{n\in\omega} A_n \in C$, as desired. Since $C \subseteq X$, it follows that $X \in \mathrm{club}^{\omega_1}$. $\square$

Proposition 23.42. (III.6.25) *Let $\mathscr{F}$ be a countable collection of finitary functions on B, and let C be the set of all $A \in [B]^{\leq\omega}$ such that A is closed under $\mathscr{F}$. Then C is a club subset of $[B]^{\leq\omega}$.*

Proof. If X is a countable subset of B, then the closure of X under F is a member of C containing X. Suppose that $A \in {}^\omega C$ and $\forall n \in \omega[A_n \subseteq A_{n+1}]$. Then $\bigcup_{n\in\omega} A_n$ is closed under $\mathscr{F}$ and hence is in C. $\square$

Proposition 23.43. (III.6.26) *Let $\mathfrak{B}$, with universe B, be a structure for a countable language $\mathscr{L}$. Then $C \stackrel{\text{def}}{=} \{A \in [B]^{\leq\omega} : A \preceq \mathfrak{B}\}$ is a club subset of $[B]^{\leq\omega}$.* $\square$

Proposition 23.44. (III.6.28) *Suppose that κ is an uncountable regular cardinal, X is a metric space, and $h : \kappa \to X$ is continuous (with κ having the order topology). Then there is a $\xi < \kappa$ such that $\forall\alpha \in (\xi, \kappa)[h(\alpha) = h(\xi)]$.*

Proof. Let S be the set of all limit ordinals less than κ. Fix a positive integer n. For each $\xi \in S$ we have $\xi \in h^{-1}[B_{1/n}(h(\xi))]$. so there is an $f(\xi) < \xi$ such that $\forall\alpha \in (f(\xi), \xi+1)[\alpha \in h^{-1}[B_{1/n}(h(\xi))]]$, i.e., $\forall\alpha \in (f(\xi), \xi+1)[d(h(\alpha), h(\xi)) < \frac{1}{n}]$. Then there is a stationary subset T_n of S and a $\gamma_n < \kappa$ such that $\forall\xi \in T_n\forall\alpha \in (\gamma_n, \xi + 1)[d(h(\alpha), h(\xi)) < \frac{1}{n}]$. Since T_n is cofinal in κ, we have $\forall\alpha \in (\gamma_n, \kappa)[d(h(\alpha), h(\xi)) < \frac{1}{n}]$. Let $\xi = \bigcup_{n\in\omega\setminus\{0\}} \gamma_n$. □

A cardinal κ is *weakly Mahlo* iff κ is regular limit and the collection of regular limit cardinals less than κ is stationary in κ. κ is *strongly Mahlo* iff κ is weakly Mahlo and strongly inaccessible.

Proposition 23.45. (III.6.30.1) *If κ is strongly inaccessible, then κ is strongly Mahlo iff the set of strongly inaccessible cardinals less than κ is stationary in κ.*

Proof. Assume that κ is strongly inaccessible.

$\Rightarrow$: Let C be the set of all strong limit cardinals less than κ. Clearly C is club in κ. With D the set of all regular limit cardinals less than κ, D is stationary by assumption, so $C \cap D$ is stationary, and it is the set of all strongly inaccessible cardinals less than κ.

$\Leftarrow$: Let E be the set of all strongly inaccessible cardinals less than κ; so E is stationary by assumption. It is contained in the set of all regular limit cardinals less than κ, so that set is stationary too. □

Proposition 23.46. (III.6.30.2) *Define*

$$\begin{aligned} E_0 &= \{\kappa : \kappa \textit{ is regular limit}\};\\ E_{\alpha+1} &= \{\kappa \in E_\alpha : |\kappa \cap E_\alpha| = \kappa\};\\ E_\gamma &= \bigcap_{\alpha<\gamma} E_\alpha \quad \textit{for } \gamma \textit{ limit.} \end{aligned}$$

Assume that κ is weakly Mahlo. Then $\kappa \in E_\kappa$.

Proof. Clearly $\kappa \in E_0$ and $\kappa \in E_1$. Suppose that $\kappa \in E_{\alpha+1}$. Thus $|\kappa \cap E_\alpha| = \kappa$. Let F consist of $\kappa \cap E_\alpha$ together with every supremum of members of $\kappa \cap E_\alpha$. Let $\langle \nu_\xi : \xi < \kappa\rangle$ enumerate F in increasing order. Let $M = \{\xi < \kappa : \nu_\xi = \xi\}$. We claim that M is club in κ. For closure, suppose that $\alpha < \kappa$ is a limit ordinal and $M \cap \alpha$ is unbounded in α. If $\eta < \alpha$, choose $\xi \in (\eta, \alpha)$ with $\nu_\xi = \xi$. Then $\nu_\eta < \nu_\xi = \xi < \alpha$. Hence $\nu_\alpha = \bigcup_{\eta<\alpha} \nu_\eta \le \alpha \le \nu_\alpha$; so $\alpha = \nu_\alpha$. So $\alpha \in M$. For unbounded, let $\eta < \kappa$. Define $\rho_0 = \eta$ and $\rho_{n+1} = \nu_{\rho_n} + 1$ for any $n \in \omega$. Let $\rho_\omega = \bigcup_{n\in\omega} \rho_n$. If $\eta < \rho_\omega$, choose $n \in \omega$ so that $\eta < \rho_n$. Then $\nu_\eta < \nu_{\rho_n} = \rho_{n+1} < \rho_\omega$. Hence $\nu_{\rho_\omega} = \bigcup_{\eta<\rho_\omega} \nu_\eta \le \rho_\omega \le \nu_{\rho_\omega}$. So M is unbounded.

Thus M is club in κ. Hence $M' \stackrel{\text{def}}{=} \{\alpha \in M : \alpha \text{ is regular limit}\}$ is stationary in κ. If $\lambda \in M'$, then $\nu_\lambda = \lambda$ and so $|\lambda \cap E_\alpha| \ge |\{\nu_\eta : \eta < \lambda\}| = \lambda$. Thus $\forall\lambda \in M'[\lambda \in E_{\alpha+1}]$. It follows that $|\kappa \cap E_{\alpha+1}| = \kappa$, so that $\kappa \in E_{\alpha+2}$.

By induction, $\kappa \in E_\alpha$ for all α, in particular $\kappa \in E_\kappa$. □

Let $\langle \mathscr{A}_\alpha : \alpha < \omega_1\rangle$ be a sequence of sets, with each $\mathscr{A}_\alpha$ a countable subset of $\mathscr{P}(\alpha)$. The sequence is a $\diamondsuit^*$*-sequence* iff

$$\forall A \subseteq \omega_1 \exists C \subseteq \omega_1[C \text{ is club and } \forall\alpha \in C[A \cap \alpha \in \mathscr{A}_\alpha]].$$

The sequence is a $\diamondsuit^+$-*sequence* iff

$$\forall A \subseteq \omega_1 \exists C \subseteq \omega_1 [C \text{ is club and } \forall \alpha \in C[A \cap \alpha \in \mathscr{A}_\alpha \text{ and } C \cap \alpha \in \mathscr{A}_\alpha]].$$

$\diamondsuit^*$ is the statement that a $\diamondsuit^*$-sequence exists, and $\diamondsuit^+$ is the statement that a $\diamondsuit^+$-sequence exists.

Theorem 23.47. (III.7.6) *$\diamondsuit^+$ implies that there is a ω_1-Kurepa family $\mathscr{F} \subseteq \mathscr{P}(\omega_1)$ such that*

$$(*) \qquad \forall A \in [\omega_1]^{\omega_1} \exists X \in [A]^{\omega_1}[X \in \mathscr{F}].$$

Proof. For $C \subseteq \omega_1$ and $\xi, \eta < \omega_1$, we call ξ, η *adjacent in C* iff $\xi < \eta$, $\xi, \eta \in C$, and $C \cap (\xi, \eta) = \emptyset$. If $A, C \subseteq \omega_1$, then $\mathrm{thin}(A, C)$ is the set of all ordinals $\min(A \cap [\xi, \eta))$ such that ξ, η are adjacent in C and $A \cap [\xi, \eta) \neq \emptyset$.

(1) If C is club in ω_1 and $|A| = \omega_1$, then $|\mathrm{thin}(A, C)| = \omega_1$.

For, let $\langle \alpha_\xi : \xi < \omega_1 \rangle$ be the strictly increasing enumeration of C. Thus $\alpha_\xi, \alpha_{\xi+1}$ are adjacent in C for all $\xi < \omega_1$. Now $\bigcup_{\xi<\omega_1}[\alpha_\xi, \alpha_{\xi+1}) = \omega_1 \backslash \alpha_0$, since for any $\gamma \in \omega_1 \backslash \alpha_0$ choose $\delta \in C$ minimum such that $\gamma < \delta$; then $\delta = \alpha_{\xi+1}$ for some ξ, and hence $\alpha_\xi \leq \gamma < \alpha_{\xi+1}$. Now (1) follows.

Now let $\langle \mathscr{A}_\alpha : \alpha < \omega_1 \rangle$ be a $\diamondsuit^+$-sequence. For each $\beta < \omega_1$ let

$$\mathscr{F}_\beta = \left\{ a \cup \mathrm{thin}(A, C) : a \in [\beta]^{<\omega} \text{ and } A, C \in \left(\{\emptyset\} \cup \bigcup_{\alpha \leq \beta} \mathscr{A}_\alpha \right) \right\}.$$

Note that $\mathscr{F}_\beta$ is a countable subset of $\mathscr{P}(\beta)$. Let $\mathscr{F} = \{X \subseteq \omega_1 : \forall \beta < \omega_1[X \cap \beta \in \mathscr{F}_\beta]\}$. Hence $\forall \beta < \omega_1[|\{X \cap \beta : X \in \mathscr{F}\}| \leq \omega]$.

Now we verify $(*)$. So, suppose that $A \in [\omega_1]^{\omega_1}$. By the $\diamondsuit^+$ property, choose a club $C \subseteq \omega_1$ such that $\forall \alpha \in C[A \cap \alpha \in \mathscr{A}_\alpha$ and $C \cap \alpha \in \mathscr{A}_\alpha]$. Let $X = \mathrm{thin}(A, C)$. By (1), $|X| = \omega_1$. To show that $X \in \mathscr{F}$, take any $\beta < \omega_1$; we show that $X \cap \beta \in \mathscr{F}_\beta$. Now

(2) If $C \cap \beta$ is finite, then $X \cap \beta$ is finite.

In fact, suppose that $X \cap \beta$ is infinite. For each $\rho \in X \cap \beta$ choose ξ_ρ, η_ρ adjacent in C such that $A \cap [\xi_\rho, \eta_\rho) \neq \emptyset$ and $\rho = \min(A \cap [\xi_\rho, \eta_\rho)$. Then $\xi_\rho \leq \rho < \beta$, and there are infinitely many pairs (ξ_ρ, η_ρ), so $C \cap \beta$ is infinite.

So now if $C \cap \beta$ is finite, then $X \cap \beta \in \mathscr{F}_\beta$, since $\mathrm{thin}(\emptyset, \emptyset) = \emptyset$. Suppose that $C \cap \beta$ is infinite. Now there is at least one limit point of $C \cap (\beta + 1)$. For, since $C \cap \beta$ is infinite, say $\alpha_0 < \alpha_1 < \cdots$ are in $C \cap \beta$. Then the supremum of the α_is is in $C \cap (\beta + 1)$. Let α be the supremum of all limit points of $C \cap (\beta + 1)$. So there exists finitely many ordinals

$\gamma_0 < \gamma_1 < \cdots < \gamma_{n-1}$ such that $C \cap (\beta+1) = (C \cap \alpha) \cup \{\alpha, \gamma_0, \ldots, \gamma_{n-1}\}$; possibly $n = 0$. Hence

$$\begin{aligned}
X \cap \beta &= \mathrm{thin}(A, C) \cap \beta \\
&= \{\min(A \cap [\xi, \eta)) : \xi, \eta \text{ adjacent in } C \\
&\qquad \text{and } A \cap [\xi, \eta) \neq \emptyset \text{ and } \min(A \cap [\xi, \eta)) < \beta\} \\
&= a \cup \{\min(A \cap [\xi, \eta)) : \xi, \eta \text{ adjacent in } (C \cap \alpha) \\
&\qquad \text{and } A \cap [\xi, \eta) \neq \emptyset\} \quad \text{for some finite } a \\
&= a \cup \mathrm{thin}(A \cap \alpha, C \cap \alpha).
\end{aligned}$$

Since $A \cap \alpha, C \cap \alpha \in \mathscr{A}_\alpha$, it follows that $X \cap \beta \in \mathscr{F}_\beta$. This proves $(*)$.

Finally, we show that $|\mathscr{F}| = \omega_2$. Taking a partition of ω_1 into ω_1 sets of size ω_1, we see from $(*)$ that $|\mathscr{F} \cap [\omega_1]^{\omega_1}| \geq \omega_1$. Suppose that $|\mathscr{F} \cap [\omega_1]^{\omega_1}| = \omega_1$. Write $\mathscr{F} \cap [\omega_1]^{\omega_1} = \{B_\xi : \xi < \omega_1\}$. By recursion we can define an increasing sequence $\langle \alpha_\xi : \xi < \omega_1 \rangle$ such that for each $\xi < \omega_1$ there is a $b \in B_\xi$ such that $\alpha_\xi < b < \alpha_{\xi+1}$. Let $A = \{\alpha_\xi : \xi < \omega_1\}$. Then $A \in [\omega_1]^{\omega_1}$, but there is no $X \in [A]^{\omega_1}$ with $X \in \mathscr{F}$, contradicting $(*)$. $\square$

A $\diamondsuit^-$-sequence is a sequence $\langle \mathscr{A}_\alpha : \alpha < \omega_1 \rangle$ such that each $\mathscr{A}_\alpha$ is a countable subset of $\mathscr{P}(\alpha)$ and for all $A \subseteq \omega_1$ the set $\{\alpha < \omega_1 : A \cap \alpha \in \mathscr{A}_\alpha\}$ is stationary. $\diamondsuit^-$ is the statement that a $\diamondsuit^-$-sequence exists.

Theorem 23.48. (III.7.8) $\diamondsuit^+ \to \diamondsuit^* \to \diamondsuit^- \leftrightarrow \diamondsuit$.

Proof. Clearly $\diamondsuit^+ \to \diamondsuit^* \to \diamondsuit^-$ and $\diamondsuit \to \diamondsuit^-$. To prove that $\diamondsuit^- \to \diamondsuit$, let $\langle \mathscr{A}_\alpha : \alpha < \omega_1 \rangle$ be a $\diamondsuit^-$-sequence.

For A any set of ordinals and $n \in \omega$, let $\psi_n(A) = \{\xi : \omega \cdot \xi + n \in A\}$. Note that if $A \subseteq \alpha$, then $\psi_n(A) \subseteq \alpha$.

We may assume that each $\mathscr{A}_\alpha \neq \emptyset$. Write $\mathscr{A}_\alpha = \{A^n_\alpha : n \in \omega\}$. Let $B^n_\alpha = \psi_n(A^n_\alpha)$. Thus $B^n_\alpha \subseteq \alpha$. We claim that there is an $n \in \omega$ such that $\langle B^n_\alpha : \alpha < \omega_1 \rangle$ is a $\diamondsuit$-sequence.

Suppose that this is not true. For each $n \in \omega$ let $B^n \subseteq \omega_1$ be such that $\{\alpha < \omega_1 : B^n \cap \alpha = B^n_\alpha\}$ is non-stationary. Let $C \subseteq \omega_1$ be club such that $B^n \cap \alpha \neq B^n_\alpha$ for all $\alpha \in C$ and $n \in \omega$. Now $D \stackrel{\text{def}}{=} \{\alpha < \omega_1 : \omega \cdot \alpha = \alpha\}$ is club. In fact, suppose that γ is limit and $D \cap \gamma$ is unbounded in γ. If $\alpha < \gamma$, choose β with $\alpha < \beta < \gamma$ and $\beta \in D$. Then $\omega \cdot \alpha < \omega \cdot \beta = \beta < \gamma$. Hence $\gamma \leq \omega \cdot \gamma = \bigcup_{\alpha < \gamma} \omega \cdot \alpha \leq \gamma$. So D is closed. To show that it is unbounded, take any $\alpha < \omega_1$. Let $\beta_0 = \alpha$ and $\beta_{n+1} = \omega \cdot \beta_n$. Then set $\gamma = \bigcup_{n \in \omega} \beta_n$. Then $\alpha < \gamma \in D$.

Let $C' = C \cap D \backslash \{0\}$. Then C' is club in ω_1, and for all $\alpha \in C'$ and all $\xi < \omega_1$ and $n \in \omega$, if $\xi < \alpha$ then $\omega \cdot \xi + n < \alpha$.

Define $B = \{\omega \cdot \xi + n : n \in \omega \text{ and } \xi \in B^n\}$. Then $\psi_n(B) = B^n$ and $\psi_n(B \cap \alpha) = B^n \cap \alpha$ for all $n \in \omega$ and $\alpha \in C'$. Thus for all $n \in \omega$ and $\alpha \in C'$, $\psi_n(B \cap \alpha) = B^n \cap \alpha \neq B^n_\alpha = \psi_n(A^n_\alpha)$, so $B \cap \alpha \neq A^n_\alpha$. Hence $B \cap \alpha \notin \mathscr{A}_\alpha$ for all $\alpha \in C'$. This contradicts the definition of the $\diamondsuit^-$-sequence. $\square$

Proposition 23.49. (III.7.9.1) *Assume $\diamondsuit$. Then there exist sets $A_\alpha \subseteq \alpha \times \alpha$ such that for every $A \subseteq \omega_1 \times \omega_1$ the set $\{\alpha < \omega_1 : A \cap (\alpha \times \alpha) = A_\alpha\}$ is stationary.*

Proof. Let $f : \omega_1 \to \omega_1 \times \omega_1$ be a bijection. Let $C = \{\alpha < \omega_1 : f[\alpha] = (\alpha \times \alpha)\}$. Then C is club in ω_1: to prove closure, suppose that $\gamma < \omega_1$ is a limit ordinal and $C \cap \gamma$ is unbounded in γ. Take any $\beta < \gamma$. Choose $\alpha \in C$ with $\beta < \alpha < \gamma$. Then $f(\beta) \in f[\alpha] = (\alpha \times \alpha) \subseteq (\gamma \times \gamma)$. This shows that $f[\gamma] \subseteq (\gamma \times \gamma)$. Now take any $(\varepsilon, \delta) \in (\gamma \times \gamma)$. Choose $\alpha \in C$ so that $\varepsilon, \delta < \alpha < \gamma$. Then $f[\alpha] = (\alpha \times \alpha)$, so there is a $\psi < \alpha$ such that $f(\psi) = (\varepsilon, \delta)$. This shows that $(\gamma \times \gamma) \subseteq f[\gamma]$. So C is closed.

To prove that C is unbounded, take any $\alpha < \omega_1$. Define $\beta_0 = \alpha$. Choose β_{2n+1} so that $\beta_{2n} < \beta_{2n+1}$ and $f[\beta_{2n}] \subseteq (\beta_{2n+1} \times \beta_{2n+1})$. Then choose $\beta_{2n+2} > \beta_{2n+1}$ so that $(\beta_{2n+1} \times \beta_{2n+1}) \subseteq f[\beta_{2n+2}]$. Let $\gamma = \bigcup_{n\in\omega} \beta_n$. Then $\alpha < \gamma \in C$.

Let $\langle A_\alpha : \alpha < \omega_1 \rangle$ be a $\diamondsuit$-sequence. For each $\alpha < \omega_1$ let $A'_\alpha = f[A_\alpha] \cap (\alpha \times \alpha)$. Take any $A \subseteq \omega_1 \times \omega_1$. To show that $D \stackrel{\text{def}}{=} \{\alpha < \omega_1 : A \cap (\alpha \times \alpha) = A'_\alpha\}$ is stationary it suffices to show that $D \cap C$ is stationary. Let $A' = f^{-1}[A]$. Then $E \stackrel{\text{def}}{=} \{\alpha < \omega_1 : A' \cap \alpha = A_\alpha\}$ is stationary So also $C \cap E$ is stationary. Now note that if $\alpha \in C$, then

$$\begin{aligned} A' \cap \alpha = A_\alpha \quad &\text{iff} \quad f^{-1}[A] \cap f^{-1}[\alpha \times \alpha] = f^{-1}[f[A_\alpha]] \cap f^{-1}[\alpha \times \alpha] \\ &\text{iff} \quad f^{-1}[A \cap (\alpha \times \alpha)] = f^{-1}[f[A_\alpha] \cap (\alpha \times \alpha)] \\ &\text{iff} \quad A \cap (\alpha \times \alpha) = f[A_\alpha] \cap (\alpha \times \alpha) \\ &\text{iff} \quad A \cap (\alpha \times \alpha) = A'_\alpha. \end{aligned}$$

Hence

$$\begin{aligned} \alpha \in C \cap E \quad &\text{iff} \quad \alpha \in C \text{ and } A' \cap \alpha = A_\alpha \\ &\text{iff} \quad \alpha \in C \text{ and } A \cap (\alpha \times \alpha) = A'_\alpha \\ &\text{iff} \quad \alpha \in C \text{ and } \alpha \in D \\ &\text{iff} \quad \alpha \in C \cap D. \end{aligned}$$

So $C \cap D$ is stationary. ☐

Proposition 23.50. (III.7.9.2) *Assume $\diamondsuit$. Then there exist functions $g_\alpha : \alpha \to \alpha$ for each $\alpha < \omega_1$ such that for every $g : \omega_1 \to \omega_1$ the set $\{\alpha < \omega_1 : g \restriction \alpha = g_\alpha\}$ is stationary.*

Proof. By Proposition 23.49 choose $A_\alpha \subseteq \alpha \times \alpha$ for $\alpha < \omega_1$ so that for every $A \subseteq \omega_1 \times \omega_1$ the set $\{\alpha < \omega_1 : A \cap (\alpha \times \alpha) = A_\alpha\}$ is stationary. If $A_\alpha : \alpha \to \alpha$ let $g_\alpha = A_\alpha$; otherwise let $g_\alpha = \emptyset$. Suppose that $g : \omega_1 \to \omega_1$. Let $C = \{\alpha < \omega_1 : g \restriction \alpha = g \cap (\alpha \times \alpha)\}$. We claim that C is club. Closed: suppose that γ is a limit ordinal and $C \cap \gamma$ is unbounded in γ. If $\beta < \gamma$, choose $\alpha \in C$ with $\beta < \alpha < \gamma$. Then $g \restriction \beta \subseteq g \restriction \alpha = g \cap (\alpha \times \alpha) \subseteq g \cap (\gamma \times \gamma)$. This shows that $g \restriction \gamma \subseteq g \cap (\gamma \times \gamma)$. Now suppose that $(\alpha, \beta) \in g \cap (\gamma \times \gamma)$. Choose $\delta \in C$ so that $\alpha, \beta < \delta < \gamma$. Then $(\alpha, \beta) \in g \cap (\delta \times \delta) = g \restriction \delta \subseteq g \restriction \gamma$. This shows that $g \cap (\gamma \times \gamma) \subseteq g \restriction \gamma$. So C is closed.

Unbounded: Let $\alpha < \omega_1$. Define $\beta_0 = \alpha$. Let $\beta_{2n+1} > \beta_{2n}$ be such that $g \restriction \beta_{2n} \subseteq (\beta_{2n+1} \times \beta_{2n+1})$. Let $\beta_{2n+2} > \beta_{2n+1}$ be such that $g \cap (\beta_{2n+1} \times \beta_{2n+1}) \subseteq g \restriction \beta_{2n+2}$. Let $\gamma = \bigcup_{n\in\omega} \beta_n$. Then $\alpha < \gamma \in C$.

Now $D \stackrel{\text{def}}{=} \{\alpha < \omega_1 : g \cap (\alpha \times \alpha) = A_\alpha\}$ is stationary. Hence so is $D \cap C$. For any $\alpha \in D \cap C$ we have $g \restriction \alpha = g \cap (\alpha \times \alpha) = A_\alpha = g_\alpha$, as desired. ☐

Proposition 23.51. (III.7.10) *Assume $\diamondsuit$. Then there are Suslin trees S, T such that $S \circledcirc T$ is also Suslin.*

Proof. Let $\langle A_\alpha : \alpha < \omega_1\rangle$ be a $\diamondsuit$-sequence with each $A_\alpha \subseteq \alpha \times \alpha$; see Proposition 23.49. Define $\mathcal{L}_0 = \{0\}$, $\mathcal{L}_1 = \omega\backslash\{0\}$, $\mathcal{L}_{n+1} = \{\omega \cdot n + k : k \in \omega\}$ for $0 < n < \omega$, $\mathcal{L}_\alpha = \omega \cdot \alpha + k : k \in \omega\}$ for $\omega \le \alpha < \omega_1$. Thus these sets are pairwise disjoint with union ω_1. $S = T$ is the union of all of these sets, and $|\mathcal{L}_\alpha| = \omega$ for all $\alpha \in (0, \omega_1)$. Now we define a relation $\prec^\varepsilon_\alpha$ on $\bigcup_{\beta\le\alpha} \mathcal{L}_\beta$ by induction on α so that the following conditions hold ($\varepsilon = 0$ for S, $\varepsilon = 1$ for T):

(1_α) If $\beta < \gamma \le \alpha$, then $\prec^\varepsilon_\gamma$ is an end extension of $\prec^\varepsilon_\beta$.

(2_α) $\prec^\varepsilon_\alpha$ is a tree order on $\bigcup_{\beta\le\alpha} \mathcal{L}_\beta$ with $\mathcal{L}_\beta$ the set of elements of height β.

(3_α) If $\beta < \gamma \le \alpha$ and $x \in \mathcal{L}_\beta$, then there is a $y \in \mathcal{L}_\gamma$ such that $x \prec^\varepsilon_\gamma y$.

Let $\prec^\varepsilon_0 = \emptyset$. Clearly (1_0)-(3_0) hold. $\prec^\varepsilon_1$ extends $\prec^\varepsilon_0$ by putting each member of $\mathcal{L}_1$ above 0. Clearly (1_1)-(3_1) hold. Now suppose that $\prec^\varepsilon_\alpha$ has been defined so that (1_α)-(3_α) hold, with $1 \le \alpha < \omega_1$. Let $\mathcal{L}_{\alpha+1} = \bigcup_{\xi\in\mathcal{L}_\alpha} E^\xi_{\alpha+1}$ with the $E^\xi_{\alpha+1}$'s pairwise disjoint and infinite. We extend $\prec^\varepsilon_\alpha$ to $\prec^\varepsilon_{\alpha+1}$ by putting all members of $E^\xi_{\alpha+1}$ directly above ξ, for each $\xi \in \mathcal{L}_\alpha$. Clearly $(1_{\alpha+1})$-$(3_{\alpha+1})$ hold.

Now suppose that γ is limit, and $\prec^\varepsilon_\alpha$ has been defined for all $\alpha < \gamma$ so that (1_α)-(3_α) hold. Let $\{x_k : k \in \omega\}$ enumerate all of the elements of $\bigcup_{\alpha<\gamma}(\mathcal{L}_\alpha \times \mathcal{L}_\alpha)$. Let $\langle \alpha_n : n \in \omega\rangle$ be a strictly increasing sequence of ordinals with supremum γ. For each $k \in \omega$, if $\omega \cdot \gamma = \gamma$ and A_γ is a maximal antichain in $\bigcup_{\alpha<\gamma}(\mathcal{L}_\alpha \times \mathcal{L}_\alpha)$ with respect to $\prec^0 \circledcirc \prec^1$, then there is a $z_k \in A_\gamma$ such that x_k and z_k are comparable; in this case, let t^k_0 be the maximum of x_k and z_k. If $\omega \cdot \gamma \ne \gamma$, or A_γ is not a maximal antichain in $\bigcup_{\alpha<\gamma}(\mathcal{L}_\alpha \times \mathcal{L}_\alpha)$, let $t^k_0 = x_k$. Say that the height of t^k_0 in $\bigcup_{\alpha<\gamma}(\mathcal{L}_\alpha \times \mathcal{L}_\alpha)$ is less than α_m. Then we use (3_{α_n}) for $m \le n$ to get elements $t^k_1, t^k_2 \ldots$ of $\bigcup_{\alpha<\gamma} \mathcal{L}_\alpha$ of heights $\alpha_m . \alpha_{m+1}, \ldots$ such that $t^k_0 \prec_{\alpha_m} t^k_1 \prec_{\alpha_{m+1}} \cdots$ with respect to $\circledcirc$. Finally, we put $(\omega \cdot \gamma + k, \omega \cdot \gamma + k)$ directly above $\langle t^k_0, t^k_1, \ldots\rangle$.

Let $\prec = \bigcup_{\alpha<\omega_1} \prec_\alpha$ with respect to $\circledcirc$. Clearly $(S \circledcirc T, \prec)$ is an ω_1-tree. To show that it is Suslin, let A be a maximal antichain in $S \circledcirc T$. Let

$$C = \left\{\gamma < \omega_1 : \omega \cdot \gamma = \gamma > 0 \text{ and } A \cap (\gamma \times \gamma) \text{ is a maximal antichain in } \bigcup_{\alpha<\gamma} \prec^{\circledcirc}_\alpha\right\}.$$

We claim that C is club in ω_1. For closure, suppose that γ is a limit ordinal less than ω_1 and $C \cap \gamma$ is unbounded in γ. If $\alpha < \gamma$, choose $\beta \in C \cap \gamma$ so that $\alpha < \beta$. Then $\omega \cdot \alpha < \omega \cdot \beta = \beta < \gamma$. This shows that $\omega \cdot \gamma \subseteq \gamma$. The other inclusion holds in general, so $\omega \cdot \gamma = \gamma$. To see that $A \cap (\gamma \times \gamma)$ is a maximal antichain in $\bigcup_{\alpha<\gamma} \prec^{\circledcirc}_\alpha$, take any member x of $\bigcup_{\alpha<\gamma}(\mathcal{L}_\alpha \times \mathcal{L}_\alpha)$. Say $x \in (\mathcal{L}_\alpha \times \mathcal{L}_\alpha)$ with $\alpha < \gamma$. Choose $\beta \in C \cap \gamma$ with $\alpha < \beta$. Then $A \cap (\beta \times \beta)$ is a maximal antichain in $\bigcup_{\delta<\beta} \prec^{\circledcirc}_\delta$, so there is a $y \in \bigcup_{\delta<\beta}(\mathcal{L}_\delta \times \mathcal{L}_\delta)$ such that x and y are comparable. Since $y \in \bigcup_{\delta<\gamma}(\mathcal{L}_\delta \times \mathcal{L}_\delta)$, this shows that $A \cap \gamma$ is a maximal antichain in $y \in \bigcup_{\delta<\gamma}(\mathcal{L}_\delta \times \mathcal{L}_\delta)$, and finishes the proof that C is closed.

To show that C is unbounded, suppose that $\alpha < \omega_1$. Let $\beta_0 = \alpha$. If β_n has been defined, choose γ_{n0} so that $\omega \cdot \beta_n < \gamma_{n0}$, then choose $\gamma_{n1} > \beta_{n0}$ so that $\gamma_{n0} < \omega \cdot \gamma_{n1}$.

Now for each $x \in (\mathcal{L}_{\beta_n} \times \mathcal{L}_{\beta_n})$ there is a $y \in A$ such that x and y are comparable; say $y \in (\mathcal{L}_{\delta(x)} \times \mathcal{L}_{\delta(x)})$. Choose $\beta_{n+1} > \gamma_{n1}$ so that $\delta_x < \beta_{n+1}$ for all $x \in (\mathcal{L}_{\beta_n} \times \mathcal{L}_{\beta_n})$. Let $\varepsilon = \bigcup_{n\in\omega} \beta_n$. We claim that $\varepsilon \in C$. For, suppose that $\alpha < \omega \cdot \varepsilon$. Choose n so that $\alpha < \omega \cdot \beta_n$. Then $\alpha < \gamma_{n0} < \beta_{n+1} < \delta$. Thus $\omega \cdot \varepsilon \leq \delta$. The converse holds in general. To show that $A \cap \delta$ is a maximal antichain in $\bigcup_{\varepsilon<\delta} \prec_\varepsilon^{\circledcirc}$, take any $x \in \bigcup_{\varepsilon<\delta}(\mathcal{L}_\varepsilon \times \mathcal{L}_\varepsilon)$. Say $x \in (\mathcal{L}_{\beta_n} \times \mathcal{L}_{\beta_n})$. Then by construction there is a $y \in (\mathcal{L}_{\beta_{n+1}} \times \mathcal{L}_{\beta_{n+1}}) \cap A$ which is comparable with x. This completes the proof that C is club in ω_1.

Now $\{\alpha \in \omega_1 : A \cap (\alpha \times \alpha) = A_\alpha\} \cap C$ is stationary. Choose $\alpha \in \omega_1$ with $A \cap \alpha = A_\alpha$ and $\alpha \in C$. Then $\alpha = \omega \cdot \alpha > 0$ and $A_\alpha = A \cap (\alpha \times \alpha)$ is a maximal antichain in $\bigcup_{\beta<\alpha}(\mathcal{L}_\beta \times \mathcal{L}_\beta)$. By construction, every element of T of height α or higher is above some member of A_α. So A_α is a maximal antichain in T, and it follows that $A = A_\alpha$. So A is countable. $\square$

A family $\mathscr{A} \subseteq [\kappa]^\kappa$ is *κ-almost disjoint* iff $|X \cap Y| < \kappa$ for all distinct $X, Y \in \mathscr{A}$.

Proposition 23.52. (III.7.11) *There is a ω_1-almost disjoint family of size ω_2.*

Proof. Let D be a nonempty ω_1-almost disjoint family of size at most ω_1. We claim that D is not maximal. Write $D = \{X_\alpha : \alpha < \omega_1\}$. We now define $a_\alpha \in \omega_1$ by recursion. Suppose defined for all $\beta < \alpha$. For each $\beta < \alpha$ the set $X_\alpha \cap X_\beta$ is countable Hence $X_\alpha \cap \bigcup_{\beta<\alpha} X_\beta$ is countable. We choose

$$\begin{aligned} a_\alpha \in & X_\alpha \backslash \left(\left(X_\alpha \cap \bigcup_{\beta<\alpha} X_\beta \right) \cup \{x_\beta : \beta < \alpha\} \right) \\ = & X_\alpha \backslash \left(\left(\bigcup_{\beta<\alpha} X_\beta \right) \cup \{x_\beta : \beta < \alpha\} \right) \end{aligned}$$

Clearly $\{x_\alpha : \alpha < \omega_1\}$ has size ω_1 and is almost disjoint from each X_β. $\square$

Proposition 23.53. (III.7.11a) *Assume CH. Then there is an ω_1-almost disjoint family of size 2^{ω_1}.*

Proof. Assume CH. Let $X = {}^{<\omega_1}2$. Thus by CH, $|X| = \omega_1$. For each $f \in {}^{\omega_1}2$ let $Y_f = \{f \restriction \alpha : \alpha < \omega_1\}$. Thus $Y_f \subseteq X$ and $Y_f \cap Y_g$ is countable for $f \neq g$. A bijection from ω_1 onto X transfers this property to ω_1 itself. $\square$

Proposition 23.54. (III.7.11b) ($\diamondsuit^+$) *The family $\mathscr{F}$ satisfying the condition of Theorem 23.47 has size 2^{ω_1}.*

Proof. $\diamondsuit^+$ implies CH by Propositions 23.13 and 23.48. Applying $(*)$ of Theorem 23.47 to members of the family given in Proposition 23.53 gives pairwise different ω_1-almost disjoint members of $\mathscr{F}$. $\square$

Lemma 23.55. (III.7.12) *Let θ be an uncountable cardinal, and suppose that $M \preceq H(\theta)$. Then $\omega \subseteq M$.*

Proof. We have $\emptyset \in H(\theta)$, so $H(\theta) \models \exists x\forall y[y \notin x]$. Hence $M \models \exists x\forall y[y \notin x]$. Choose $m \in M$ so that $M \models \forall y[y \notin m]$. Hence $H(\theta) \models \forall y[y \notin m]$. If $n \in m$, then $n \in H(\theta)$, contradiction. So $m = \emptyset$. Thus $\emptyset \in M$.

Suppose we have shown that $m \in M$, with $m \in \omega$. Now $m+1 \in H(\theta)$, so $H(\theta) \models \exists x\forall y[y \in x \leftrightarrow y \in m \text{ or } y = m]$. Hence this holds in M. Choose $u \in M$ such that $M \models \forall y[y \in u \leftrightarrow y \in m \text{ or } y = m]$. Then $H(\theta) \models \forall y[y \in u \leftrightarrow y \in m \text{ or } y = m]$. So $u = m+1 \in M$. □

Lemma 23.56. (III.7.12a) *Let θ be an uncountable cardinal, and suppose that $M \preceq H(\theta)$. Then $\omega \in M$.*

Proof. We have $\omega \in H(\theta)$. Thus

$$H(\theta) \models \exists x[\emptyset \in x \text{ and } \forall y \in x[y \cup \{y\} \in x] \text{ and } \forall z[\emptyset \in z \text{ and } \forall y \in z[y \cup \{y\} \in z] \to x \subseteq z]$$

Hence there is an $a \in M$ such that

$$H(\theta) \models \emptyset \in a \text{ and } \forall y \in a[y \cup \{y\} \in a] \text{ and } \forall z[\emptyset \in z \text{ and } \forall y \in z[y \cup \{y\} \in z] \to a \subseteq z]$$

It follows that $a = \omega \in M$. □

Lemma 23.57. (III.7.12b) *Let θ be an uncountable cardinal, and suppose that $M \preceq H(\theta)$. Then $\{\{a,b\} : a,b \in M\} \subseteq M$.*

Proof. Suppose that $a,b \in M$. We have $\{a,b\} \in H(\theta)$, so $H(\theta) \models \exists x\forall y[y \in x \text{ iff } y = a \text{ or } y = b]$. Hence $M \models \exists x\forall y[y \in x \text{ iff } y = a \text{ or } y = b]$. Choose $c \in M$ so that $M \models \forall y[y \in c \text{ iff } y = a \text{ or } y = b]$. Hence $H(\theta) \models \forall y[y \in c \text{ iff } y = a \text{ or } y = b]$. Hence $c = \{a,b\}$. □

Lemma 23.58. (III.7.12c) *Let θ be an uncountable cardinal, and suppose that $M \preceq H(\theta)$. Then $M \times M \subseteq M$.* □

Lemma 23.59. (III.7.12d) *Let θ be an uncountable cardinal, and suppose that $M \preceq H(\theta)$. Suppose that $R \in M$ and $M \models [R$ is a relation$]$. Then R is a relation.*

Proof. Assume the hypotheses. Then also $H(\theta) \models [R$ is a relation$]$. Thus $H(\theta) \models \forall x \in R\exists u,v[x = (u,v)]$. If $x \in R$, then also $x \in H(\theta)$, so there are $u,v \in H(\theta)$ such that $x = (u,v)$. So R is a relation. □

Lemma 23.60. (III.7.12e) *Let θ be an uncountable cardinal, and suppose that $M \preceq H(\theta)$. Suppose that $R \in M$ and $M \models [R$ is a function$]$. Then R is a function.*

Proof. Assume the hypotheses. Then

$$H(\theta) \models [R \text{ is a relation and } \forall u,v,w[(u,v),(u,w) \in R \to v = w]].$$

Hence by Lemma 23.59, R is a relation. Clearly it is a function. □

Lemma 23.61. (III.7.12f) *Let θ be an uncountable cardinal, and suppose that $M \preceq H(\theta)$. If $a \in M$ and a is countable, then $a \subseteq M$.*

Proof. Assume the hypothesis. Wlog $a \neq \emptyset$. There is a surjection f from ω onto a. Then $f \in H(\theta)$. Thus $H(\theta) \models \exists f[f$ is a surjection of ω onto $a]$. Since $\omega \in M$ by Lemma 23.56, it follows that $M \models \exists f[f$ is a surjection of ω onto $a]$. So, choose $g \in M$ such that $M \models [g$ is a surjection of ω onto $a]$. If $n \in \omega$, then $n \in M$ by Lemma 23.56, so $M \models \exists y[(n,y) \in g]$. Choose $b \in M$ such that $M \models [(n,b) \in g]$. Then $H(\theta) \models [(n,b) \in g]$, so $b \in a$. Every element of a is obtained as a b in this way, so $a \subseteq M$. $\square$

Lemma 23.62. (III.7.12g) *If $M \preceq H(\omega_1)$, then M is transitive.*

Proof. If $a \in M$, then a is countable, so $a \subseteq M$ by Lemma 23.61. $\square$

Lemma 23.63. (III.7.12h) *Let θ be an uncountable cardinal, and suppose that $M \preceq H(\theta)$. If $x \in M$, then $x \cup \{x\} \in M$.*

Proof. Assume the hypotheses. Now $H(\theta) \models \exists y \forall u[u \in y[\leftrightarrow u \in x$ or $u = x]$. Hence there is a $y \in M$ such that $M \models \forall u[u \in y[\leftrightarrow u \in x$ or $u = x]$. Hence $H(\theta) \models \forall u[u \in y[\leftrightarrow u \in x$ or $u = x]$. Clearly $y = x \cup \{x\}$. $\square$

Lemma 23.64. (III.7.12i) *Let θ be an uncountable cardinal, and suppose that $M \preceq H(\theta)$, with M countable. Then $M \cap \omega_1$ is a countable limit ordinal.*

Proof. $M \cap \omega_1$ is countable since M is countable. If $\alpha < \beta \in M \cap \omega_1$, then $\beta \subseteq M$ by Lemma 23.61, so $\alpha \in M \cap \omega_1$. Hence $M \cap \omega_1$ is an ordinal. It is a limit ordinal by Lemma 23.63. $\square$

Lemma 23.65. (III.7.12j) *Let θ be an uncountable cardinal, and suppose that $M \preceq H(\theta)$, with M countable. Suppose that $\omega_1 < \theta$. Then $\omega_1 \in M$.*

Proof. Assume the hypotheses. Then

$$\begin{aligned} M(\theta) \models \exists x[&x \text{ is an ordinal and } x \neq \emptyset \text{ and} \\ &\qquad \neg\exists f[f : \omega \to x \text{ is a surjection}] \text{ and} \\ &\forall y[y \text{ is an ordinal and } y \neq \emptyset \text{ and} \\ &\qquad \neg\exists f[f : \omega \to y \text{ is a surjection}] \to x \in y \text{ or } x = y]] \end{aligned}$$

By $M \preceq H(\theta)$ we get $x \in M$ such that

$$\begin{aligned} M(\theta) \models &x \text{ is an ordinal and } x \neq \emptyset \text{ and} \\ &\qquad \neg\exists f[f : \omega \to x \text{ is a surjection}] \text{ and} \\ &\forall y[y \text{ is an ordinal and } y \neq \emptyset \text{ and} \\ &\qquad \neg\exists f[f : \omega \to y \text{ is a surjection}] \to x \in y \text{ or } x = y] \end{aligned}$$

Clearly $x = \omega_1$. $\square$

Lemma 23.66. (III.7.12k) *Let θ be an uncountable cardinal, and suppose that $M \preceq H(\theta)$, with M countable. Then $\in$ is extensional on M.* (See after Lemma 12.29.)

Proof. We have $H(\theta) \models \forall x, y[\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y]$. □

Lemma 23.67. (III.7.12l) *Let θ be an uncountable cardinal, and suppose that $M \preceq H(\theta)$, with M countable. Let mos_{AR} be the Mostowski isomorphism from M onto a transitive set T.* (See before Lemma 12.29.) *Suppose that $\alpha \in M$ is an ordinal, with $\alpha \subseteq M$. Then $mos_{AR}(\alpha) = \alpha$.*

Proof. See Lemma 12.33. □

Lemma 23.68. (III.7.12m) *Let θ be an uncountable cardinal, and suppose that $M \preceq H(\theta)$, with M countable. Suppose that $\omega_1 < \theta$. Let mos_{AR} be the isomorphism from M onto a transitive T. Then $\mathrm{mos}_{AR}(\alpha) = \alpha$ for all $\alpha < M \cap \omega_1$.*

Proof. By Lemma 12.33. □

Lemma 23.69. (III.7.12n) *Let θ be an uncountable cardinal, and suppose that $M \preceq H(\theta)$, with M countable. Suppose that $\omega_1 < \theta$ and $\omega_1 \in M$. Let mos_{AR} be the isomorphism from M onto a transitive T. Then $\mathrm{mos}_{AR}(\omega_1) = M \cap \omega_1$.*

Proof. Assume the hypotheses. Then $\mathrm{mos}_{AR}(\omega_1) = \{\mathrm{mos}_{AR}(\alpha) : \alpha \in M$ and $\alpha < \omega_1\} = \{\alpha : \alpha \in M$ and $\alpha < \omega_1\} = M \cap \omega_1$. □

Lemma 23.70. (III.7.12o) *Let θ be an uncountable cardinal, and suppose that $M \preceq H(\theta)$, with M countable. Suppose that $\omega_1 < \theta$ and $\omega_1 \in M$. Let mos_{AR} be the isomorphism from M onto a transitive T. Then $\omega_1^T = M \cap \omega_1$.* □

Theorem 23.71. (III.7.13) *Assume $V = L$. Then $\diamondsuit$ holds.*

Proof. We define a $\diamondsuit$-sequence $A \stackrel{\text{def}}{=} \langle A_\alpha : \alpha < \omega_1\rangle$ as well as a sequence $\langle C_\alpha : \alpha < \omega_1\rangle$ by recursion. Suppose that A_ξ and C_ξ have been defined for all $\xi < \alpha$. If α is a limit ordinal, define

$$P(\alpha, A, C) \quad \text{iff} \quad A, C \subseteq \alpha \text{ and } C \text{ is club in } \alpha \text{ and } \neg\exists\xi \in C[A \cap \xi = A_\xi].$$

With $<_L$ the well-order of L, let (A_α, C_α) be the first pair (A, C) such that $P(\alpha, A, C)$. If there is no such pair, or if α is not a limit ordinal, let $A_\alpha = C_\alpha = \emptyset$.

Suppose that $\langle A_\alpha : \alpha < \omega_1\rangle$ is not a $\diamondsuit$-sequence. Then there is an $A \subseteq \omega_1$ such that $\{\alpha < \omega_1 : A \cap \alpha = A_\alpha\}$ is not stationary, hence there is a club C such that for all $\alpha \in C$ we have $A \cap \alpha \neq A_\alpha$. Thus $P(\omega_1, A, C)$ holds. So also $P(\omega_1, A_{\omega_1}, C_{\omega_1})$ holds. Fix a countable $M \preceq H(\omega_2)$ with $\omega_1 \in M$. By Theorem 17.32, $H(\omega_2) = L(\omega_2)$. Let $\beta = M \cap \omega_1$. By Lemma 23.64, β is a countable limit ordinal. Let mos_{AR} be the Mostowski isomorphism of M onto a transitive set T. Then $\mathrm{mos}_{AR}(\omega_1) = \beta$ and $\mathrm{mos}_{AR}(\xi) = \xi$ for all $\xi < \beta$, by Lemmas 23.68 and 23.69. Moreover, $\beta = \omega_1^T$ by Lemma 23.70. Now $H(\omega_2) = L(\omega_2)$, $H(\omega_2) \models ZFC - P$ by Theorem 14.13, so $H(\omega_2) \models V = L$ by Lemma 17.29. Hence

$M \models ZFC - P + V = L$, so also $T \models ZFC - P + V = L$. By Lemma 17.29, $T = L(\gamma)$ for some countable ordinal γ.

By $M \preceq H(\omega_2)$ and absoluteness, for each $\alpha \in M$ we have $A_\alpha, C_\alpha \in M$, $\mathrm{mos}_{AR}(A_\alpha) = A_{\mathrm{mos}_{AR}(\alpha)}$, and $\mathrm{mos}_{AR}(C_\alpha) = C_{\mathrm{mos}_{AR}(\alpha)}$. In particular, $A_{\omega_1}, C_{\omega_1} \in M$, $\mathrm{mos}_{AR}(A_{\omega_1}) = A_\beta$, and $\mathrm{mos}_{AR}(C_{\omega_1}) = C_\beta$. Now

$$A_\beta = \mathrm{mos}_{AR}(A_{\omega_1}) = \{\mathrm{mos}_{AR}(\xi) : \xi \in A_{\omega_1} \cap M\} = \{\xi : \xi \in A_{\omega_1} \cap \beta\} = A_{\omega_1} \cap \beta;$$

Similarly, $C_\beta = \mathrm{mos}_{AR}(C_{\omega_1}) = C_{\omega_1} \cap \beta$. Now C_{ω_1} is club in ω_1 because $P(\omega_1, A_{\omega_1}, C_{\omega_1})$, so $\mathrm{mos}_{AR}(C_{\omega_1})$ is club in $\mathrm{mos}_{AR}(\omega_1)$, so C_β is club in β. Since C_{ω_1} is closed, it follows that $\beta \in C_{\omega_1}$. Also, $A_{\omega_1} \cap \beta = A_\beta$. This contradicts $P(\omega_1, A_{\omega_1}, C_{\omega_1})$. □

Theorem 23.72. (III.7.14) *Assume $V = L$. Then $\diamondsuit^+$ holds.*

Proof. First we claim

(1) For each $\alpha < \omega_1$ there is a δ such that $\alpha < \delta < \omega_1$ and $L(\delta) \preceq L(\omega_1)$.

In fact, given $\alpha < \omega_1$ let M be such that M is countable, $M \preceq L(\omega_1)$, and $\alpha \cup \{\alpha\} \in M$. Then M is transitive by Lemma 23.62, since $L(\omega_1) = H(\omega_1)$ by Theorem 17.32. Then there is a countable ordinal δ such that $M = L(\delta)$ by Lemma 17.29. Since $\alpha + 1 \in M$, we have $\alpha < \delta$. So (1) holds.

For each $\alpha < \omega_1$ let α^* be the least δ such that (1) holds. Define $\mathscr{A}_\alpha = L(\alpha^*) \cap \mathscr{P}(\alpha)$. So $\mathscr{A}_\alpha$ is a countable subset of $\mathscr{P}(\alpha)$. To see that this gives a $\diamondsuit^+$-sequence, let $A \subseteq \omega_1$. Since $\mathrm{trcl}(A) \subseteq \omega_1$ and $L(\omega_2) = H(\omega_2)$, we have $A \in L(\omega_2)$. For each $\sigma < \omega_1$ let $M_{A\sigma}$ be the set of all elements of $L(\omega_2)$ definable in $(L(\omega_2), \in)$ using parameters in $\{A\} \cup \sigma$. By Lemma 17.3, $M_{A\sigma} \preceq L(\omega_2)$. Note that $M_{A\sigma}$ is countable, $A \in M_{A\sigma}$, and $\sigma \subseteq M_{A\sigma}$. By Lemma 23.64, $M_{A\sigma} \cap \omega_1$ is a countable limit ordinal. Clearly $M_{A\sigma} \cap \omega_1 \geq \sigma$. Let $C_A = \{\sigma < \omega_1 : M_{A\sigma} \cap \omega_1 = \sigma\}$.

(2) C_A is club in ω_1.

Closure: suppose that $\gamma < \omega_1$ is a limit ordinal and $C_A \cap \gamma$ is unbounded in γ. Suppose that $\tau \in M_{A\gamma} \cap \omega_1$. Then $\tau \in M_{A\delta} \cap \omega_1$ for some $\delta < \gamma$ with $\delta \in C_A$. So $M_{A\delta} \cap \omega_1 = \delta$. Hence $\tau \in \delta \subseteq \gamma$. This shows that $M_{A\gamma} \cap \omega_1 \subseteq \gamma$. Conversely, suppose that $\tau \in \gamma$. Then there is a $\delta \in C_A \cap \gamma$ such that $\tau < \delta$. Thus $M_{A\delta} \cap \omega_1 = \delta$, so $\tau \in M_{A\delta} \cap \omega_1$. Since $M_{A\delta} \cap \omega_1 \subseteq M_{A\gamma} \cap \omega_1$, we have $\tau \in M_{A\gamma} \cap \omega_1$. This proves closure.

Unbounded: suppose that $\alpha < \omega_1$. Let $\beta_0 = \alpha + 1$. Choose $\beta_{2n+1} > \beta_{2n}$ so that $M_{A\beta_{2n}} \cap \omega_1 \subseteq \beta_{2n+1}$. Let $\beta_{2n+2} = \beta_{2n+1} + 1$. Note that $\beta_{2n+1} < M_{A\beta_{2n+2}} \cap \omega_1$. Let $\gamma = \bigcup_{n\in\omega} \beta_n$. Then $\alpha < \gamma \in C_A$. So C_A is unbounded.

Now take any $\alpha \in C_A$; we want to show that $A \cap \alpha \in \mathscr{A}_\alpha$ and $C_A \cap \alpha \in \mathscr{A}_\alpha$. Let mos be the Mostowski isomorphism from $M_{A\alpha}$ onto some transitive set T. By Lemma 17.30, $T = L(\gamma)$ for some countable γ. Since $\alpha \in C_A$ we have $M_{A\alpha} \cap \omega_1 = \alpha$. So by Lemma 23.68 we have $\mathrm{mos}_{AR}(\omega_1) = \alpha$ and by Lemma 23.68 we have $\mathrm{mos}_{AR}(\beta) = \beta$ for all $\beta < \alpha$. Moreover, $\alpha = \omega_1^T$ by Lemma 23.70. Now $\alpha = \omega_1^T \in T = L(\gamma)$, so $\alpha < \gamma$. Now there is a surjection $g : \omega \to \alpha$ since $0 < \alpha < \omega_1$. Clearly $g \in H(\omega_1)$. Since $L(\alpha^*) \preceq L(\omega_1)$ and

$\alpha < \alpha^*$, it follows that there is a surjection $f : \omega \to \alpha$ with $f \in L(\alpha^*)$. Since $\alpha = \omega_1^{L(\gamma)}$, there is no such function in $L(\gamma)$. It follows that $\gamma < \alpha^*$. Now

$$\begin{aligned}
\mathrm{mos}_{AR}(A) &= \{\mathrm{mos}_{AR}(\beta) : \beta \in M_{A\alpha} \wedge \beta \in A\} \\
&= \{\mathrm{mos}_{AR}(\beta) : \beta \in M_{A\alpha} \wedge \beta \in A \wedge \beta \in \omega_1\} \\
&= \{\beta : \beta \in M_{A\alpha} \wedge \beta \in \omega_1 \text{ and } \beta \in A\} \\
&= \mathrm{mos}_{AR}(\omega_1) \cap A = \alpha \cap A.
\end{aligned}$$

Hence $A \cap \alpha = \mathrm{mos}_{AR}(A) \in L(\gamma) \subset L(\alpha^*)$, so $A \cap \alpha \in \mathscr{A}_\alpha$.

For each $\sigma < \alpha$ let $\mathscr{H}(\sigma)$ be the collection of all elements definable in $L(\gamma)$ with parameters in $\{A \cap \alpha\} \cup \sigma$. We define $\hat{C} = \{\sigma < \alpha : \mathscr{H}(\sigma) \cap \alpha = \sigma\}$. Clearly $\hat{C} \in L(\alpha^*)$, so $\hat{C} \in \mathscr{A}_\alpha$. So it suffices to show that $\hat{C} = C_A \cap \alpha$. Fix $\sigma < \alpha$. We want to show that $\mathscr{H}(\sigma) \cap \alpha = \sigma$ iff $M_{A\sigma} \cap \omega_1 = \sigma$.

Assume that $\mathscr{H}(\sigma) \cap \alpha = \sigma$. First suppose that $\zeta \in M_{A\sigma} \cap \omega_1$. Thus ζ is definable in $(L(\omega_2), \in)$ using parameters from $\{A\} \cup \sigma$. Now $\sigma < \alpha \in C_A$. Since $\alpha \in C_A$, we have $\zeta \in M_{A\sigma} \cap \omega_1 \subseteq M_{A\alpha} \cap \omega_1 = \alpha$; so $\zeta < \alpha$. Since $M_{A\alpha} \preceq L(\omega_2)$, it follows that ζ is definable in $M_{A\alpha}$ with parameters from $\{A\} \cup \sigma$. Now mos fixes ζ and all elements of σ, and $\mathrm{mos}_{AR}(A) = A \cap \alpha$. Hence ζ is definable in $L(\gamma)$ using parameters from $(A \cap \alpha) \cup \sigma$. Thus $\zeta \in \mathscr{H}(\sigma) \cap \alpha$, so $\zeta \in \sigma$. This shows that $M_{A\sigma} \cap \omega_1 \subseteq \sigma$. Second suppose that $\zeta \in \sigma$. So $\zeta \in \mathscr{H}(\sigma) \cap \alpha$. Thus ζ is definable in $L(\gamma)$ with parameters in $\{A \cap \alpha\} \cup \sigma$. Applying mos_{AR}^{-1}, ζ is definable in $M_{A\alpha}$ with parameters in $\{A\} \cup \sigma$. So ζ is definable in $M_{A\sigma}$ with parameters in $\{A\} \cup \sigma$. Hence $\zeta \in M_{A\sigma} \cap \omega_1$.

Now assume that $M_{A\sigma} \cap \omega_1 = \sigma$. First suppose that $\zeta \in \mathscr{H}(\sigma) \cap \alpha$. So ζ is definable in $L(\gamma)$ with parameters in $\{A \cap \alpha\} \cup \sigma$. Applying mos_{AR}^{-1}, ζ is definable in $M_{A\alpha}$ with parameters in $\{A\} \cup \sigma$. Hence $\zeta \in M_{A\sigma} \cap \omega_1$. Hence $\zeta \in \sigma$. Second suppose that $\zeta \in \sigma$. Then $\zeta \in M_{A\sigma} \cap \omega_1$. So ζ is definable in $(L(\omega_2), \in)$ with parameters in $\{A\} \cup \sigma$. As above, ζ is definable in $L(\gamma)$ using parameters from $(A \cap \alpha) \cup \sigma$. So $\zeta \in \mathscr{H}(\sigma) \cap \alpha$. □

Proposition 23.73. *Assume that κ is an uncountable regular cardinal and $\langle A_\alpha : \alpha < \kappa \rangle$ is a sequence of subsets of κ. Let $D = \triangle_{\alpha<\kappa} A_\alpha$. Then:*

(i) For all $\alpha < \kappa$, the set $D \backslash A_\alpha$ is nonstationary.

(ii) Suppose that $E \subseteq \kappa$ and for every $\alpha < \kappa$, the set $E \backslash A_\alpha$ is nonstationary. Then $E \backslash D$ is nonstationary.

Proof. (i): For any $\beta < \kappa$,

$$\begin{aligned}
D \backslash A_\beta &= \{\alpha < \kappa : \forall \gamma < \alpha (\alpha \in A_\gamma) \text{ and } \alpha \notin A_\beta\} \\
&\subseteq \{\alpha < \kappa : \alpha \le \beta\};
\end{aligned}$$

Hence $D \backslash A_\beta$ is nonstationary.

(ii): For each $\beta < \kappa$ let C_β be club in κ such that $(E \backslash A_\beta) \cap C_\beta = \emptyset$. Let F be the diagonal intersection of the C_β's; thus

$$F = \{\gamma < \kappa : \forall \alpha < \gamma (\gamma \in C_\alpha)\}.$$

Thus F is club. We claim that $F \cap (E \backslash D) = \emptyset$ (as desired). For, suppose that $\gamma \in F \cap (E \backslash D)$. Since $\gamma \notin D$, there is a $\beta < \gamma$ such that $\gamma \notin A_\beta$. Since $\gamma \in F$, we have $\gamma \in C_\beta$. Since $(E \backslash A_\beta) \cap C_\beta = \emptyset$, this is a contradiction. □

Proposition 23.74. *Let $\kappa > \omega$ be regular. Then there is a sequence $\langle S_\alpha : \alpha < \kappa \rangle$ of stationary subsets of κ such that $S_\beta \subseteq S_\alpha$ whenever $\alpha < \beta < \kappa$, and $\triangle_{\alpha<\kappa} S_\alpha = \{0\}$.*

Proof. By Theorem 23.12, let $\langle A_\alpha : \alpha < \kappa \rangle$ be pairwise disjoint stationary subsets of κ. Then $A_\alpha \backslash (\alpha+1)$ is also stationary. Let $S_\alpha = \bigcup_{\beta>\alpha}(A_\beta \backslash (\beta+1))$. Then S_α is stationary and $\alpha < \beta$ implies that $S_\beta \subseteq S_\alpha$. Let D be the diagonal intersection of the S_α's:

$$D = \{\gamma < \kappa : \forall \alpha < \gamma(\gamma \in S_\alpha)\}.$$

Thus $0 \in D$. Suppose that $0 \neq \gamma \in D$. Then $0 < \gamma$, so $\gamma \in S_0$. Hence there is a $\beta > 0$ such that $\gamma \in A_\beta \backslash (\beta + 1)$. Hence $\beta < \gamma$. So $\gamma \in S_\beta$. Choose $\delta > \beta$ such that $\gamma \in A_\delta \backslash (\delta + 1)$. So $\gamma \in A_\beta \cap A_\delta = \emptyset$, contradiction. □

Proposition 23.75. *Suppose that κ is uncountable and regular, and for each limit ordinal $\alpha < \kappa$ we are given a function $f_\alpha \in {}^\omega\alpha$. Suppose that S is a stationary subset of κ. Let $n \in \omega$. Then there exist a $t \in {}^n\kappa$ and a stationary $S' \subseteq S$ such that for all $\alpha \in S'$, $f_\alpha \restriction n = t$.*

Proof. We define by recursion sequences $\langle S_0, S_1, \ldots, S_n \rangle$ of stationary subsets of S and $\langle \beta_0, \ldots, \beta_{n-1} \rangle$ of ordinals less than κ. Let $S_0 = S$. Suppose that S_i has been defined, $i < n$. Let $g(\alpha) = f_\alpha(i)$ for all $\alpha \in S_i$. Then g is a regressive function, and hence there exist a stationary subset S_{i+1} of S_i and an ordinal β_i such that $g(\alpha) = \beta_i$ for all $\alpha \in S_{i+1}$. This finishes the construction.

If $\alpha \in S_n$, then for any $i < n$ we have $\alpha \in S_{i+1}$, and hence $f_\alpha(i) = \beta_i$. Hence we can let $t(i) = \beta_i$ for all $i < n$, and the property of the proposition holds. □

Proposition 23.76. *Suppose that $\mathrm{cf}(\kappa) > \omega$, $C \subseteq \kappa$ is club of order type $\mathrm{cf}(\kappa)$, and $\langle c_\beta : \beta < \mathrm{cf}(\kappa) \rangle$ is the strictly increasing enumeration of C. Let $X \subseteq \kappa$. Then X is stationary in κ iff $\{\beta < \mathrm{cf}(\kappa) : c_\beta \in X\}$ is stationary in $\mathrm{cf}(\kappa)$.*

Proof. Assume the hypotheses. Let $X' = \{\beta < \mathrm{cf}(\kappa) : c_\beta \in X\}$.

$\Rightarrow$: Assume that X is stationary in κ. We want to show that X' is stationary in $\mathrm{cf}(\kappa)$. Let D' be club in $\mathrm{cf}(\kappa)$. Define $D = \{c_\beta : \beta \in D'\}$. We claim that D is club in κ. For closure, suppose that $\alpha < \kappa$ is a limit ordinal and $D \cap \alpha$ is unbounded in α. Let $\gamma = \bigcup\{\beta \in D' : c_\beta < \alpha\}$.

(1) $\gamma < \mathrm{cf}(\kappa)$.

For, since C is unbounded in κ, there is a $\xi < \mathrm{cf}(\kappa)$ such that $\alpha < c_\xi$. If $\xi < \gamma$, then there is a $\beta \in D'$ such that $c_\beta < \alpha$ and $\xi < \beta$. Then $c_\xi < c_\beta < \alpha$, contradiction. Hence $\gamma \leq \xi < \mathrm{cf}(\kappa)$, proving (1).

(2) γ is a limit ordinal, and $D' \cap \gamma$ is unbounded in γ.

For, suppose that $\delta < \gamma$. Choose $\beta \in D'$ such that $c_\beta < \alpha$ and $\delta < \beta$. Since $D \cap \alpha$ is unbounded in α, choose $c_\varepsilon \in D \cap \alpha$ such that $c_\beta < c_\varepsilon$. Since $c_\varepsilon \in D$ we have $\varepsilon \in D'$ and so $\varepsilon \leq \gamma$. Thus $\delta < \beta < \varepsilon \leq \gamma$. Since $\beta \in D'$, this proves (2).

(3) $c_\gamma = \alpha$.

For, suppose that $\delta < c_\gamma$. Now clearly $c_\gamma = \bigcup_{\beta<\gamma} c_\beta$, so there is a $\beta < \gamma$ such that $\delta < c_\beta$. By the definition of γ, there is a $\beta' \in D'$ such that $\beta < \beta'$ and $c_{\beta'} < \alpha$. Thus $\delta < c_\beta < c_{\beta'} < \alpha$, so $\delta < \alpha$. This proves that $c_\gamma \leq \alpha$. On the other hand, suppose that $\delta < \alpha$. Since $D \cap \alpha$ is unbounded in α, there is a $\beta \in D'$ such that $\delta < c_\beta < \alpha$. Thus $\beta \leq \gamma$, so $\delta < c_\gamma$. This proves that $\alpha \leq c_\gamma$, and finishes the proof of (3).

Now by (2) we have $\gamma \in D'$, and hence (3) yields $\alpha \in D$, as desired; we have now proved that D is closed in κ.

To show that D is unbounded in κ, let $\alpha < \kappa$ be arbitrary. Choose $\beta < \mathrm{cf}(\kappa)$ such that $\alpha < c_\beta$. Since D' is unbounded in $\mathrm{cf}(\kappa)$, choose $\beta' \in D'$ such that $\beta < \beta'$. Thus $\alpha < c_\beta < c_{\beta'} \in D$, as desired.

So we have shown that D is club in κ. Since X is stationary in κ, choose $\beta \in D'$ such that $c_\beta \in X$. thus $\beta \in D' \cap X'$, as desired.

$\Leftarrow$: Assume that X' is stationary in $\mathrm{cf}(\kappa)$. Let D be club in κ, and let $D' = \{\beta < \mathrm{cf}(\kappa) : c_\beta \in D\}$. We claim that D' is club in $\mathrm{cf}(\kappa)$. To show that it is closed, suppose that $\alpha < \mathrm{cf}(\kappa)$ is a limit ordinal and $D' \cap \alpha$ is unbounded in α. We claim that $D \cap c_\alpha$ is unbounded in c_α. For, suppose that $\gamma < c_\alpha$. Then there is a $\beta < \alpha$ such that $\gamma < c_\beta$. Choose $\delta \in D' \cap \alpha$ such that $\beta < \delta$; this is possible since $D' \cap \alpha$ is unbounded in α. Thus $\gamma < c_\beta < c_\delta \in D \cap c_\alpha$, as desired. So D' is closed in $\mathrm{cf}(\kappa)$. To show that it is unbounded, suppose that $\alpha < \mathrm{cf}(\kappa)$. Now $C \cap D$ is club in κ, so there is a β such that $c_\alpha < c_\beta \in D$. So $\alpha < \beta \in D'$. This shows that D' is unbounded in $\mathrm{cf}(\kappa)$. Hence D' is club in $\mathrm{cf}(\kappa)$.

Choose $\beta \in D' \cap X'$. Then $c_\beta \in D \cap X$, as desired. □

Proposition 23.77. *Suppose that κ is regular and uncountable, and $S \subseteq \kappa$ is stationary. Also, suppose that every $\alpha \in S$ is an uncountable regular cardinal. Then*

$$T \stackrel{\text{def}}{=} \{\alpha \in S : S \cap \alpha \textit{ is non-stationary in } \alpha\}$$

is stationary in κ.

Proof. Assume the hypotheses. Let C be club in κ; we want to show that $T \cap C \neq \emptyset$. Let C' be the set of all limit points of C, i.e., the set of all limit ordinals $\alpha \in \kappa$ such that $C \cap \alpha$ is unbounded in α. Clearly $C' \subseteq C$, and C' is club in κ. Since S is stationary in κ, let α be the least element of $S \cap C'$. Clearly $C' \cap \alpha$ is closed in α; we claim that it is also unbounded in α. For, suppose that $\beta < \alpha$. Now $C \cap \alpha$ is unbounded in α, so we can construct a sequence $\langle \gamma_i : i < \omega \rangle$ of members of C such that $\beta < \gamma_0 < \gamma_1 < \cdots < \alpha$. Let $\delta = \sup_{i\in\omega} \gamma_i$. Then $\delta \in C'$, and $\delta < \alpha$ since α is uncountable and regular. So $C' \cap \alpha$ is club in α. Now $S \cap C' \cap \alpha = \emptyset$ by the minimality of α, so $C' \cap \alpha$ is a club in α which shows that $S \cap \alpha$ is non-stationary in α. So $\alpha \in T \cap C$, as desired. □

Proposition 23.78. *Suppose that κ is uncountable and regular, and $\kappa \leq |A|$. Suppose that C is a closed subset of $[A]^{<\kappa}$ and D is a directed subset of C with $|D| < \kappa$. (Directed means that if $x, y \in D$ then there is a $z \in D$ such that $x \cup y \subseteq z$.) Then $\bigcup D \in C$.*

Proof. If D is finite, then $\bigcup D \in D$; so $\bigcup D \in C$. Suppose that $|D| = \omega$; say $D = \{x_n : n \in \omega\}$. For each $n \in \omega$ choose $y_n \in D$ so that $\{x_m : m \leq n\} \cup \{y_m : m < n\} \subseteq y_n$. Then $\bigcup_{n\in\omega} y_n \in C$ since C is closed, and $\bigcup D = \bigcup_{n\in\omega} y_n$.

Now suppose inductively that $\kappa > |D| > \omega$. Let $|D| = \lambda$ and write $D = \{x_\alpha : \alpha < \lambda\}$. For all $y, z \in D$ let $f(y, z) \in D$ be such that $y, z \subseteq f(y, z)$. We now define $\langle E_\alpha : \alpha < \kappa\rangle$ by recursion. Suppose that E_β has been defined for all $\beta < \alpha$ so that $E_\beta \subseteq D$, E_β is directed, and $|E_\beta| \leq |\beta| + \omega$, with $\alpha < \lambda$. Let $F_0 = \{x_\alpha\} \cup \bigcup_{\beta<\alpha} E_\beta$. So $|F_0| \leq |\alpha| + \omega$. Let $F_{n+1} = F_n \cup \{f(x, y) : x, y \in F_n\}$. Then $|F_{n+1}| \leq |\alpha| + \omega$. Let $E_\alpha = \bigcup_{n\in\omega} F_n$. Then $E_\alpha \subseteq D$, E_α is directed, and $|E_\alpha| \leq |\alpha| + \omega$.

By the inductive hypothesis, $y_\alpha \stackrel{\text{def}}{=} \bigcup E_\alpha$ is in C, and $y_\alpha \subseteq y_\beta$ for $\alpha < \beta$. Hence $\bigcup D = \bigcup_{\alpha<\lambda} y_\alpha \in C$. □

Proposition 23.79. *Let κ be uncountable and regular, and $\kappa \leq |A|$. If $f : [A]^{<\omega} \to [A]^{<\kappa}$ let $C_f = \{x \in [A]^{<\kappa} : \forall s \in [x]^{<\omega}[f(s) \subseteq x]\}$. Then C_f is club in $[A]^{<\kappa}$.*

Proof. Suppose that $x_\xi \in C_f$ for all $\xi < \alpha$, with $\alpha < \kappa$ and $x_\xi \subseteq x_\eta$ for $\xi < \eta$. Clearly $\bigcup_{\xi<\alpha} x_\xi \in C_f$ if α is a successor ordinal. Suppose that α is a limit ordinal. Take any $s \in \left[\bigcup_{\xi<\alpha} x_\xi\right]^{<\omega}$. Then there is a $\xi < \alpha$ such that $s \in [x_\xi]^{<\omega}$, and hence $f(s) \subseteq x_\xi \subseteq \bigcup_{\eta<\alpha} x_\eta$. Thus C_f is closed.

To show that C_f is unbounded, let $y \in [A]^{<\kappa}$. Define $z_0 = y$ and $z_{n+1} = z_n \cup \{f(s) : s \in [z_n]^{<\omega}\}$. By induction, $z_n \in [A]^{<\kappa}$ for all $n \in \omega$. Now $\bigcup_{n\in\omega} z_n \in C_f$, showing that C_f is unbounded. □

Proposition 23.80. *(Continuing Proposition 23.79) Let κ be uncountable and regular, and $\kappa \leq |A|$. Let D be club in $[A]^{<\kappa}$. Then there is an $f : [A]^{<\omega} \to [A]^{<\kappa}$ such that $C_f \subseteq D$.*

Proof. We claim that there is an $f : [A]^{<\omega} \to D$ such that $\forall e \in [A]^{<\omega}[e \subseteq f(e)]$ and $\forall e_1, e_2 \in [A]^{<\omega}[e_1 \subseteq e_2 \to f(e_1) \subseteq f(e_2)]$. We define f by induction on $|e|$. Let $f(\emptyset)$ be any member of D. Suppose that $f(e)$ has been defined for all $e \in [A]^{<\omega}$ such that $|e| < m$, and suppose that $e \in [A]^{<\omega}$ with $|e| = m$. Let $f(e)$ be a member of D such that $e \subseteq f(e)$ and $f(e\backslash\{a\}) \subseteq f(e)$ for all $a \in e$. Clearly f is as desired.

Now we show that $C_f \subseteq D$. Let $x \in C_f$. Note that $\{f(e) : e \in [x]^{<\omega}\}$ is directed and has union x. Hence $x \in D$ by Proposition 23.78. □

Proposition 23.81. *Let κ be uncountable and regular, $\kappa \leq |A|$, and $A \subseteq B$. If $Y \in [A]^{<\kappa}$, let $Y^B = \{x \in [B]^{<\kappa} : x \cap A \in Y\}$. Then if Y is club in $[A]^{<\kappa}$, then Y^B is club in $[B]^{<\kappa}$.*

Proof. Assume the hypotheses. Suppose that $\alpha < \kappa$ and $\langle x_\xi : \xi < \alpha\rangle$ is a sequence of members of Y^B with $x_\xi \subseteq x_\eta$ for $\xi < \eta$. Then $x_\xi \cap A \in Y$ for all $\xi < \alpha$, and $x_\xi \cap A \subseteq x_\eta \cap A$ for $\xi < \eta$. Hence $\bigcup_{\xi<\alpha}(x_\xi \cap A) \in Y$. So $\bigcup_{\xi<\alpha} x_\xi \in Y^B$. Thus Y^B is closed.

To show that Y^B is unbounded, let $b \in [B]^{<\kappa}$. Then $b \cap A \in [A]^{<\kappa}$, so there is a $c \in Y$ such that $b \cap A \subseteq c$. Then $b \subseteq c \cup (b\backslash A)$, and $(c \cup (b\backslash A)) \cap A = c \in Y$. So $c \cup (b\backslash A) \in Y^B$. □

Proposition 23.82. *Let κ be uncountable and regular, $\kappa \le |A|$, and $A \subseteq B$. If $Y \in [B]^{<\kappa}$, let $Y \restriction A = \{y \cap A : y \in Y\}$. Then if Y is stationary in $[B]^{<\kappa}$ then $Y \restriction A$ is stationary in $[A]^{<\kappa}$.*

Proof. Assume the hypotheses. Suppose that C is club in $[A]^{<\kappa}$. Then by Proposition 23.81, C^B is club in $[B]^{<\kappa}$. Choose $y \in Y \cap C^B$. Then $y \cap A \in (Y \restriction A) \cap C$. □

Proposition 23.83. *With κ, A, B as in Proposition 23.81, suppose that $f : [B]^{<\omega} \to [B]^{<\kappa}$. For each $e \in [A]^{<\omega}$ define*

$$\begin{aligned} x_0(e) &= e; \\ x_{n+1}(e) &= x_n(e) \cup \{f(s) : s \in [x_n(e)]^{<\omega}\}; \\ w(e) &= \bigcup_{n\in\omega} x_n(e). \end{aligned}$$

Also, for each $y \in [A]^{<\kappa}$ let $v(y) = \bigcup\{w(e) : e \in [y]^{<\omega}\}$.

Then $w(e) \in C_f$ for all $e \in [A]^{<\omega}$ and $v(y) \in C_f$ for all $y \in [A]^{<\kappa}$.

Proof. If $z \in [w(e)]^{<\omega}$, then $z \in [x_n(e)]^{<\omega}$ for some n, and hence $f(z) \subseteq x_{n+1}(e) \subseteq w(e)$. Thus $w(e) \in C_f$. Note that if $e_1 \subseteq e_2$ then $x_n(e_1) \subseteq x_n(e_2)$ for all n (by induction), and so $w(e_1) \subseteq w(e_2)$. Now suppose that $z \in [v(y)]^{<\omega}$. Then there is a finite $F \subseteq [y]^{<\omega}$ such that $z \subseteq \bigcup_{e\in F} w(e)$. Let $e' = \bigcup_{e\in F} e$. Then $\bigcup_{e\in F} w(e) \subseteq w(e')$. So $z \in [w(e')]^{<\omega}$. It follows from the first part of this proof that $f(z) \subseteq w(e') \subseteq v(y)$. Thus $v(y) \in C_f$. □

Proposition 23.84. *With κ, A, B as in Proposition 23.81, suppose that S is stationary in $[A]^{<\kappa}$. Then S^B is stationary in $[B]^{<\kappa}$.*

Proof. Assume the hypotheses. Suppose that D is club in $[B]^{<\kappa}$. By Proposition 23.80 there is an $f : [B]^{<\omega} \to [B]^{<\kappa}$ such that $C_f \subseteq D$. For any $e \in [A]^{<\omega}$ let $g(e) = w(e) \cap A$. We claim that $C_f \restriction A = C_g$. Suppose that $y \in C_f$, so that $y \cap A \in C_f \restriction A$. To show that $y \cap A \in C_g$, let $e \in [y \cap A]^{<\omega}$. Then $x_n(e) \subseteq y$ by induction on n. Since $x_n(e) = e$, it is true for $n = 0$. Suppose that $x_n(e) \subseteq y$. If $s \in [x_n(e)]^{<\omega}$, then $s \in [y]^{<\omega}$, so $f(s) \subseteq y$ since $y \in C_f$. Hence $x_{n+1}(e) \subseteq y$. It follows that $w(e) \subseteq y$, and so $g(e) \subseteq y \cap A$. This shows that $y \cap A \in C_g$, and proves that $C_f \restriction A \subseteq C_g$. Now suppose that $y \in C_g$. Hence $\forall e \in [y]^{<\omega}[g(e) \subseteq y]$. We claim that $v(y) \cap A = y$. For, if $e \in [y]^{<\omega}$, then $g(e) \subseteq y$, i.e., $w(e) \cap A \subseteq y$. So $v(y) \cap A \subseteq y$. If $a \in y$, then $a \in w(\{a\}) \subseteq v(y)$; so $v(y) \cap A = y$. This shows that $C_g \subseteq C_f \restriction A$. Thus $C_g = C_f \restriction A$.

Choose $z \in C_g \cap S$. Then $z \in (C_f \restriction A) \cap S$, so there is a $y \in C_f$ such that $z = y \cap A$. Thus $y \cap A \in S$, so $y \in S^B \cap D$. □

24. Infinite combinatorics

We have already given several theorems concerning infinite combinatorcs: the Dushnik-Miller Theorem 12.58, the Δ-system Theorem 21.29, Ramsey's Theorem 20.12, and Theorem 21.35 concerning an independent family of functions. We now give some additional results of this sort.

Two sets A, B are *almost disjoint* iff $|A| = |B|$ while $|A \cap B| < |A|$. Of course we are mainly interested in this notion if A and B are infinite.

Theorem 24.1. *There is a family of 2^ω pairwise almost disjoint infinite sets of natural numbers.*

Proof. Let $X = \bigcup_{n\in\omega} {}^n2$. Then $|X| = \omega$, since X is clearly infinite, while

$$|X| \leq \sum_{n\in\omega} 2^n \leq \omega \cdot \omega = \omega.$$

Let f be a bijection from ω onto X. Then for each $g \in {}^\omega 2$ let $x_g = \{g \restriction n : n \in \omega\}$. So x_g is an infinite subset of X. If $g, h \in {}^\omega 2$ and $g \neq h$, choose n so that $g(n) \neq h(n)$. Then clearly $x_g \cap x_h \subseteq \{g \restriction i : i \leq n\}$, and so this intersection is finite. Thus we have produced 2^ω pairwise almost disjoint infinite subsets of X. That carries over to ω. Namely, $\{f^{-1}[x_g] : g \in {}^\omega 2\}$ is a family of 2^ω pairwise almost disjoint infinite subsets of ω, as is easily checked. $\square$

Let X be an infinite set. A collection $\mathscr{A}$ of subsets of X is *independent* iff for any two finite disjoint subsets $\mathscr{B}, \mathscr{C}$ of $\mathscr{A}$ we have

$$\left(\bigcap_{Y\in\mathscr{B}} Y\right) \cap \left(\bigcap_{Z\in\mathscr{C}} (X\backslash Z)\right) \neq \emptyset.$$

Theorem 24.2. (Fichtenholz, Kantorovitch, Hausdorff) *For any infinite cardinal κ there is an independent family $\mathscr{A}$ of subsets of κ such that each member of $\mathscr{A}$ has size κ and $|\mathscr{A}| = 2^\kappa$; moreover, each of the above intersections has size κ.*

Proof. Let $\mathscr{F}$ be the family of all finite subsets of κ; thus $|\mathscr{F}| = \kappa$. Let Φ be the set of all finite subsets of $\mathscr{F}$; thus also $|\Phi| = \kappa$. It suffices now to work with $\mathscr{F} \times \Phi$ rather than κ itself.

For each $\Gamma \subseteq \kappa$ let

$$b_\Gamma = \{(\Delta, \varphi) \in \mathscr{F} \times \Phi : \Delta \cap \Gamma \in \varphi\}.$$

Note that each b_Γ has size κ; for example, $(\emptyset, \{\emptyset, \{\alpha\}\}) \in b_\Gamma$ for every $\alpha < \kappa$. So to finish the proof it suffices to take any two finite disjoint subsets H and K of $\mathscr{P}(\kappa)$ and show that

$$(*) \qquad \left(\bigcap_{A\in H} b_A\right) \cap \left(\bigcap_{B\in K} ((\mathscr{F} \times \Phi)\backslash b_B)\right)$$

has size κ. For distinct $A, B \in H \cup K$ pick $\alpha_{AB} \in A \triangle B$, and let $\Delta = \{\alpha_{AB} : A, B \in H \cup K,\ A \neq B\}$. Now it suffices to show that if $\beta \in \kappa \backslash \Delta$ and $\varphi = \{\Delta \cap A : A \in H\} \cup \{\{\beta\}\}$, then (Δ, φ) is a member of $(*)$. If $A \in H$, then $\Delta \cap A \in \varphi$, and so $(\Delta, \varphi) \in b_A$. Now suppose, to get a contradiction, that $B \in K$ and $(\Delta, \varphi) \in b_B$. Then $\Delta \cap B \in \varphi$. Since $\beta \notin \Delta$, it follows that there is an $A \in H$ such that $\Delta \cap B = \Delta \cap A$. Since $A \neq B$, we have $\alpha_{AB} \in A \triangle B$ and $\alpha_{AB} \in \Delta$, contradiction. $\square$

We now give a generalization of the Δ-system theorem.

Theorem 24.3. *Suppose that κ and λ are cardinals, $\omega \leq \kappa < \lambda$, λ is regular, and for all $\alpha < \lambda$, $|[\alpha]^{<\kappa}| < \lambda$. Suppose that $\mathscr{A}$ is a collection of sets, with each $A \in \mathscr{A}$ of size less than κ, and with $|\mathscr{A}| \geq \lambda$. Then there is a $\mathscr{B} \in [\mathscr{A}]^\lambda$ which is a Δ-system.*

Proof.

(1) There is a regular cardinal μ such that $\kappa \leq \mu < \lambda$.

In fact, if κ is regular, we may take $\mu = \kappa$. If κ is singular, then $\kappa^+ \leq |[\kappa]^{<\kappa}| < \lambda$, so we may take $\mu = \kappa^+$.

We take μ as in (1). Let $S = \{\alpha < \lambda : \alpha \text{ is a limit ordinal and } \mathrm{cf}(\alpha) = \mu\}$. Then S is a stationary subset of λ.

Let $\mathscr{A}_0$ be a subset of $\mathscr{A}$ of size λ. Now $\left|\bigcup_{A \in \mathscr{A}_0} A\right| \leq \lambda$ since $\kappa < \lambda$. Let a be an injection of $\bigcup_{A \in \mathscr{A}_0} A$ into λ, and let A be a bijection of λ onto $\mathscr{A}_0$. Set $b_\alpha = a[A_\alpha]$ for each $\alpha < \lambda$. Now if $\alpha \in S$, then $|b_\alpha \cap \alpha| \leq |b_\alpha| = |A_\alpha| < \kappa \leq \mu = \mathrm{cf}(\alpha)$, so there is an ordinal $g(\alpha)$ such that $\sup(b_\alpha \cap \alpha) < g(\alpha) < \alpha$. Thus g is a regressive function on S. By Fodor's theorem, there exist a stationary $S' \subseteq S$ and a $\beta < \lambda$ such that $g[S'] = \{\beta\}$. For each $\alpha \in S'$ let $F(\alpha) = b_\alpha \cap \alpha$. Thus $F(\alpha) \in [\beta]^{<\kappa}$, and $|[\beta]^{<\kappa}| < \lambda$, so there exist an $S'' \in [S']^\lambda$ and a $B \in [\beta]^{<\kappa}$ such that $b_\alpha \cap \alpha = B$ for all $\alpha \in S''$.

Now we define $\langle \alpha_\xi : \xi < \lambda \rangle$ by recursion. For any $\xi < \lambda$, α_ξ is a member of S'' such that

(2) $\alpha_\eta < \alpha_\xi$ for all $\eta < \xi$, and
(3) $\delta < \alpha_\xi$ for all $\delta \in \bigcup_{\eta < \xi} b_{\alpha_\eta}$.

Since $\left|\bigcup_{\eta < \xi} b_{\alpha_\eta}\right| < \lambda$, this is possible by the regularity of λ.

Now let $\mathscr{A}_1 = A[\{\alpha_\xi : \xi < \lambda\}]$ and $r = a^{-1}[B]$. We claim that $C \cap D = r$ for distinct $C, D \in \mathscr{A}_1$. For, write $C = A_{\alpha_\xi}$ and $D = A_{\alpha_\eta}$. Without loss of generality, $\eta < \xi$. Suppose that $x \in r$. Thus $a(x) \in B \subseteq b_{\alpha_\xi}$, so by the definition of b_{α_ξ} we have $x \in A_{\alpha_\xi} = C$. Similarly $x \in D$. Conversely, suppose that $x \in C \cap D$. Thus $x \in A_{\alpha_\xi} \cap A_{\alpha_\eta}$, and hence $a(x) \in b_{\alpha_\xi} \cap b_{\alpha_\eta}$. By the definition of α_ξ, since $a(x) \in b_{\alpha_\eta}$ we have $a(x) < \alpha_\xi$. So $a(x) \in b_{\alpha_\xi} \cap \alpha_\xi = B$, and hence $x \in r$.

Clearly $|\mathscr{A}_1| = \lambda$. $\square$

Another form of this theorem is as follows. An *indexed Δ-system* is a system $\langle A_i : i \in I \rangle$ of sets such that there is a set r (the *root*) such that $A_i \cap A_j = r$ for all distinct $i, j \in I$. Some, or even all, the A_i's can be equal.

Theorem 24.4. *Suppose that κ and λ are cardinals, $\omega \le \kappa < \lambda$, λ is regular, and for all $\alpha < \lambda$, $|[\alpha]^{<\kappa}| < \lambda$. Suppose that $\langle A_i : i \in I\rangle$ is a system of sets, with each A_i of size less than κ, and with $|I| \ge \lambda$. Then there is a $J \in [I]^\lambda$ such that $\langle A_i : i \in J\rangle$ is an indexed Δ-system.*

Proof. Define $i \equiv j$ iff $i, j \in I$ and $A_i = A_j$. If some equivalence class has λ or more elements, a subset J of that class of size λ is as desired. If every equivalence class has fewer than λ elements, then there are at least λ equivalence classes. Let $\mathscr{A}$ have exactly one element in common with λ equivalence classes. We apply Theorem 24.3 to get a subset $\mathscr{B}$ of $\mathscr{A}$ of size λ which is a Δ-system, say with kernel r. Say $\mathscr{B} = \{A_i : i \in J\}$ with $J \in [I]^\lambda$ and $A_i \ne A_j$ for $i \ne j$. Then $\langle A_i : i \in J\rangle$ is an indexed Δ-system with root r. $\square$

Now we turn to the possibility of generalizing Ramsey's theorem; see the definitions on page 311. According to the following theorem, the most obvious generalization of Ramsey's theorem does not hold.

Theorem 24.5. *For any infinite cardinal κ we have $2^\kappa \not\to (\kappa^+, \kappa^+)^2$.*

Proof. We consider ${}^\kappa 2$ under the lexicographic order; see the beginning of Chapter 21. Let $\langle f_\alpha : \alpha < 2^\kappa\rangle$ be a one-one enumeration of ${}^\kappa 2$. Define $F : 2^\kappa \to 2$ by setting, for any $\alpha < \beta < \kappa$,

$$F(\{\alpha, \beta\}) = \begin{cases} 0 & \text{if } f_\alpha < f_\beta, \\ 1 & \text{if } f_\beta < f_\alpha. \end{cases}$$

If $2^\kappa \to (\kappa^+, \kappa^+)^2$ holds, then there is a set $\Gamma \in [2^\kappa]^{\kappa^+}$ which is homogeneous for F. If $F(\{\alpha, \beta\}) = 0$ for all distinct $\alpha < \beta$ in Γ, then $\langle f_\alpha : \alpha \in \Gamma\rangle$ is a strictly increasing sequence of length o.t.(Γ), contradicting Theorem 21.5. A similar contradiction is reached if $F(\{\alpha, \beta\}) = 1$ for all distinct $\alpha < \beta$ in Γ. $\square$

Corollary 24.6. *$\kappa^+ \not\to (\kappa^+, \kappa^+)^2$ for every infinite cardinal κ.*

Proof. Given $F : [\kappa^+]^2 \to 2$, extend F in any way to a function $G : [2^\kappa]^2 \to 2$. A homogeneous set for F yields a homogeneous set for G. So our corollary follows from Theorem 24.5. $\square$

To formulate another generalization of Ramsey's theorem it is convenient to introduce a notation for a special form of the arrow notation. We write

$$\begin{aligned} &\kappa \to (\lambda)^\nu_\mu \quad \text{iff} \\ &\kappa \to (\langle \lambda : \alpha < \mu\rangle)^\nu \end{aligned}$$

In direct terms, then, $\kappa \to (\lambda)^\nu_\mu$ means that for every $f : [\kappa]^\nu \to \mu$ there is a $\Gamma \in [\kappa]^\lambda$ such that $|f[\Gamma]| = 1$.

The following cardinal notation is also needed for our next result: for any infinite cardinal κ we define

$$\begin{aligned} 2^\kappa_0 &= \kappa; \\ 2^\kappa_{n+1} &= 2^{(2^\kappa_n)} \quad \text{for all } n \in \omega. \end{aligned}$$

Theorem 24.7. (Erdös-Rado) *For every infinite cardinal κ and every positive integer n, $(2^\kappa_{n-1})^+ \to (\kappa^+)^n_\kappa$.*

Proof. Induction on n. For $n = 1$ we want to show that $\kappa^+ \to (\kappa^+)^1_\kappa$, and this is obvious. Now assume the statement for $n \geq 1$, and suppose that $f : [(2^\kappa_n)^+]^{n+1} \to \kappa$. For each $\alpha \in (2^\kappa_n)^+$ define $F_\alpha : [(2^\kappa_n)^+ \backslash \{\alpha\}]^n \to \kappa$ by setting $F_\alpha(x) = f(x \cup \{\alpha\})$.

(1) There is an $A \in [(2^\kappa_n)^+]^{2^\kappa_n}$ such that for all $C \in [A]^{2^\kappa_{n-1}}$ and all $u \in (2^\kappa_n)^+ \backslash C$ there is a $v \in A \backslash C$ such that $F_u \restriction [C]^n = F_v \restriction [C]^n$.

To prove this, we define a sequence $\langle A_\alpha : \alpha < (2^\kappa_{n-1})^+ \rangle$ of subsets of $(2^\kappa_n)^+$, each of size 2^κ_n. Let $A_0 = 2^\kappa_n$, and for α limit let $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$. Now suppose that A_α has been defined, and $C \in [A_\alpha]^{2^\kappa_{n-1}}$. Define $u \equiv v$ iff $u, v \in (2^\kappa_n)^+ \backslash C$ and $F_u \restriction [C]^n = F_v \restriction [C]^n$. Now $|{}^{[C]^n}\kappa| = 2^\kappa_n$, so there are at most 2^κ_n equivalence classes. Let K_C have exactly one element in common with each equivalence class. Let $A_{\alpha+1} = A_\alpha \cup \{K_C : C \in [A_\alpha]^{2^\kappa_{n-1}}\}$. Since $(2^\kappa_n)^{2^\kappa_{n-1}} = 2^\kappa_n$, we still have $|A_{\alpha+1}| = 2^\kappa_n$. This finishes the construction. Clearly $A \stackrel{\text{def}}{=} \bigcup_{\alpha \leq (2^\kappa_{n-1})^+} A_\alpha$ is as desired in (1).

Take A as in (1), and fix $a \in (2^\kappa_n)^+ \backslash A$. We now define a sequence $\langle x_\alpha : \alpha < (2^\kappa_{n-1})^+ \rangle$ of elements of A. Given $C \stackrel{\text{def}}{=} \{x_\beta : \beta < \alpha\}$, by (1) let $x_\alpha \in A \backslash C$ be such that $F_{x_\alpha} \restriction [C]^n = F_a \restriction [C]^n$. This defines our sequence. Let $X = \{x_\alpha : \alpha < (2^\kappa_{n-1})^+\}$.

Now define $G : [X]^n \to \kappa$ by $G(x) = F_a(x)$. Suppose that $\alpha_0 < \cdots < \alpha_n < (2^\kappa_{n-1})^+$. Then

$$\begin{aligned} f(\{x_{\alpha_0}, \ldots, x_{\alpha_n}\}) &= F_{x_{\alpha_n}}(\{x_{\alpha_0}, \ldots, x_{\alpha_{n-1}}\}) \\ &= F_a(\{x_{\alpha_0}, \ldots, x_{\alpha_{n-1}}\}) \\ &= G(\{x_{\alpha_0}, \ldots, x_{\alpha_{n-1}}\}). \end{aligned}$$

Now by the inductive hypothesis there is an $H \in [X]^{\kappa^+}$ such that G is constant on $[H]^n$. By the above, f is constant on $[H]^{n+1}$. □

Corollary 24.8. $(2^\kappa)^+ \to (\kappa^+)^2_\kappa$ *for any infinite cardinal κ.* □

Theorem 24.9. *For any infinite cardinal κ we have $2^\kappa \not\to (3)^2_\kappa$.*

Proof. Define $F : [{}^\kappa 2]^2 \to \kappa$ by setting $F(\{f, g\}) = \chi(f, g)$ for any two distinct $f, g \in {}^\kappa 2$. If $\{f, g, h\}$ is homogeneous for F with f, g, h distinct, let $\alpha = \chi(f, g)$. Then $f(\alpha), g(\alpha), h(\alpha)$ are distinct members of 2, contradiction. □

Corollary 24.10. *For any infinite cardinal κ we have $2^\kappa \not\to (\kappa^+)^2_\kappa$.* □

Our final result in the partition calculus indicates that infinite exponents are in general hopeless.

Theorem 24.11. $\omega \not\to (\omega, \omega)^\omega$.

Proof. Let $<$ well-order $[\omega]^\omega$. We define for any $X \in [\omega]^\omega$

$$F(X) = \begin{cases} 0 & \text{if } Y < X \text{ for some } Y \in [X]^\omega, \\ 1 & \text{otherwise.} \end{cases}$$

Suppose that $H \in [\omega]^\omega$ is homogeneous for F. Let X be the $<$-least element of $[H]^\omega$. Thus $F(X) = 1$. So we must have $F(Y) = 1$ for all $Y \in [H]^\omega$. Write $H = \{m_i : i \in \omega\}$ without repetitions. For each $k \in \omega$ let

$$I_k = \{m_0, m_2, \ldots, m_{2k}\} \cup \{m_{2i+1} : i \in \omega\}.$$

Thus these are infinite subsets of H. Choose k_0 so that I_{k_0} is minimum among all of the I_k's. Then $I_{k_0} \subset I_{k_0+1}$ and $I_{k_0} < I_{k_0+1}$, so $F(I_{k_0+1}) = 0$, contradiction. □

The following theorem of Comfort and Negrepontis is similar to Theorem 21.35.

Theorem 24.12. *Suppose that $\lambda = \lambda^{<\kappa}$. Then there is a system $\langle f_\alpha : \alpha < 2^\lambda \rangle$ of members of ${}^\lambda\lambda$ such that*

$$\forall M \in \left[2^\lambda\right]^{<\kappa} \forall g \in {}^M\lambda \exists \beta < \lambda \forall \alpha \in M[f_\alpha(\beta) = g(\alpha)].$$

Proof. Let

$$\mathscr{F} = \left\{(F, G, s) : F \in [\lambda]^{<\kappa},\ G \in [\mathscr{P}(F)]^{<\kappa},\ \text{and } s \in {}^G\lambda\right\}.$$

Now if $F \in [\lambda]^{<\kappa}$, say $|F| = \mu$, then

$$\left|[\mathscr{P}(F)]^{<\kappa}\right| \leq (2^\mu)^{<\kappa}| \leq (\lambda^\mu)^{<\kappa} \leq \lambda^{<\kappa} = \lambda,$$

and if $G \in [\mathscr{P}(F)]^{<\kappa}$ then $|{}^G\lambda| \leq \lambda^{<\kappa} = \lambda$. It follows that $|\mathscr{F}| = \lambda$. Let h be a bijection from λ onto $\mathscr{F}$, and let k be a bijection from 2^λ onto $\mathscr{P}(\lambda)$. Now for each $\alpha < 2^\lambda$ we define $f_\alpha \in {}^\lambda\lambda$ by setting, for each $\beta < \lambda$, with $h(\beta) = (F, G, s)$,

$$f_\alpha(\beta) = \begin{cases} s(k(\alpha) \cap F) & \text{if } k(\alpha) \cap F \in G, \\ 0 & \text{otherwise.} \end{cases}$$

Now to prove that $\langle f_\alpha : \alpha < 2^\lambda \rangle$ is as desired, suppose that $M \in [2^\lambda]^{<\kappa}$ and $g \in {}^M\lambda$. For distinct members α, β of M choose $\gamma(\alpha, \beta) \in k(\alpha) \triangle k(\beta)$. Then let

$$F = \{\gamma(\alpha, \beta) : \alpha, \beta \in M, \alpha \neq \beta\} \quad \text{and} \quad G = \{k(\alpha) \cap F : \alpha \in M\}.$$

Moreover, define $s : G \to \lambda$ by setting $s(k(\alpha) \cap F) = g(\alpha)$ for any $\alpha \in M$. This is possible since $k(\alpha) \cap F) \neq k(\beta) \cap F)$ for distinct $\alpha, \beta \in M$. Finally, let $\beta = h^{-1}(F, G, s)$. Then for any $\alpha \in M$ we have

$$f_\alpha(\beta) = s(k(\alpha) \cap F) = g(\alpha).$$ □

Proposition 24.13. *Suppose that $\kappa^\omega > \kappa$. Then there is a family $\mathscr{A}$ of subsets of κ, each of size ω, with $|\mathscr{A}| = \kappa^+$ and the intersection of any two members of $\mathscr{A}$ is finite.*

Proof. Let $K = {}^{<\omega}\kappa$. Thus $|K| = \kappa$. Let F be a bijection from K to κ. For each $f \in {}^{\omega}\kappa$ let

$$X_f = F[\{f \restriction m : m \in \omega\}].$$

Clearly each X_f has size ω. If $f, g \in {}^{\omega}\kappa$ and $f \neq g$, choose $p \in \omega$ such that $f(p) \neq g(p)$. Then

$$\begin{aligned} X_f \cap X_g &= F[\{f \restriction m : m \in \omega\}] \cap F[\{g \restriction m : m \in \omega\}] \\ &= F[\{f \restriction m : m \in \omega\} \cap \{g \restriction m : m \in \omega\}] \\ &\subseteq F[\{f \restriction m : m \leq p\}] \end{aligned}$$

and hence $X_f \cap X_g$ is finite. Since $\kappa^\omega \geq \kappa^+$, the desired result follows. $\square$

Proposition 24.14. *Suppose that κ is any infinite cardinal, and λ is minimum such that $\kappa^\lambda > \kappa$. Then there is a family $\mathscr{A}$ of subsets of κ, each of size λ, with the intersection of any two members of $\mathscr{A}$ being of size less than λ, and with $|\mathscr{A}| = \lambda^+$.*

Proof. Let $K = {}^{<\lambda}\kappa$. Thus $|K| = \kappa$. Let F be a bijection from K to κ. For each $f \in {}^{\lambda}\kappa$ let

$$X_f = F[\{f \restriction m : m \in \lambda\}].$$

Clearly each X_f has size λ. If $f, g \in {}^{\lambda}\kappa$ and $f \neq g$, choose $p \in \lambda$ such that $f(p) \neq g(p)$. Then

$$\begin{aligned} X_f \cap X_g &= F[\{f \restriction m : m \in \lambda\}] \cap F[\{g \restriction m : m \in \lambda\}] \\ &= F[\{f \restriction m : m \in \lambda\} \cap \{g \restriction m : m \in \lambda\}] \\ &\subseteq F[\{f \restriction m : m \leq p\}] \end{aligned}$$

and hence $X_f \cap X_g$ has size less than λ. Since $\kappa^\lambda > \kappa$, the desired result follows. $\square$

The following proposition generalizes Theorem 24.1.

Proposition 24.15. *Suppose that κ is uncountable and regular. Then there is a family $\mathscr{A}$ of subsets of κ, each of size κ with the intersection of any two members of $\mathscr{A}$ of size less than κ, and with $|\mathscr{A}| = \kappa^+$.*

Proof. First of all, recall that κ can be partitioned into κ sets, each of size κ. Namely, if $f : \kappa \times \kappa \to \kappa$ is a bijection, let $X_\alpha = f[\{(\alpha, \beta) : \beta < \kappa\}]$; then clearly $\langle X_\alpha : \alpha < \kappa \rangle$ is as claimed.

Thus we can apply Zorn's lemma to get a maximal collection $\mathscr{A} \subseteq [\kappa]^\kappa$ such that the members of $\mathscr{A}$ are pairwise almost disjoint, and $|\mathscr{A}| \geq \kappa$.

Hence we just have to get a contradiction from the assumption that $|\mathscr{A}| = \kappa$. Making this assumption, let $\langle Y_\alpha : \alpha < \kappa \rangle$ be a one-one enumeration of $\mathscr{A}$. Note that for any $\alpha < \kappa$,

$$Y_\alpha \backslash \bigcup_{\beta<\alpha} Y_\beta = Y_\alpha \backslash \bigcup_{\beta<\alpha} (Y_\alpha \cap Y_\beta)$$

has size κ. This enables us to define by recursion a sequence $\langle z_\alpha : \alpha < \kappa\rangle$ like this: having defined z_β for all $\beta < \alpha$, choose

$$z_\alpha \in Y_\alpha \backslash \left(\{z_\beta : \beta < \alpha\} \cup \bigcup_{\beta<\alpha} Y_\beta\right).$$

Then $Z \stackrel{\text{def}}{=} \{z_\alpha : \alpha < \kappa\}$ is a set of size κ, and for any $\alpha < \kappa$,

$$Z \cap Y_\alpha \subseteq \{z_\beta : \beta \leq \alpha\},$$

so that $|Z \cap Y_\alpha| < \kappa$. This contradicts the maximality of $\mathscr{A}$. $\square$

Proposition 24.16. *If $\mathscr{F}$ is an uncountable family of finite functions each with range $\subseteq \omega$, then there are distinct $f, g \in \mathscr{F}$ such that $f \cup g$ is a function.*

Proof. We apply Theorem 24.4 to the indexed system $\langle \mathrm{dmn}(f) : f \in \mathscr{F}\rangle$ and get an uncountable subset $\mathscr{G}$ of $\mathscr{F}$ such that $\langle \mathrm{dmn}(f) : f \in \mathscr{G}\rangle$ is an indexed Δ-system; say that $\mathrm{dmn}(f) \cap \mathrm{dmn}(g) = D$ for all distinct $f, g \in \mathscr{G}$. Then

$$\mathscr{G} = \bigcup_{h \in {}^D\omega} \{f \in \mathscr{G} : f \restriction D = h\};$$

since the index set ${}^D\omega$ is countable and $\mathscr{G}$ is uncountable, there exist an $h \in {}^D\omega$ for which there are two distinct $f, g \in \mathscr{G}$ such that $f \restriction D = g \restriction D = h$. Then $f \cup g$ is a function. $\square$

Proposition 24.17. *(Double Δ-system theorem) Suppose that κ is a singular cardinal with $\mathrm{cf}(\kappa) > \omega$. Let $\langle \lambda_\alpha : \alpha < \mathrm{cf}(\kappa)\rangle$ be a strictly increasing sequence of successor cardinals with supremum κ, with $\mathrm{cf}(\kappa) < \lambda_0$, and such that for each $\alpha < \mathrm{cf}(\kappa)$ we have $(\sum_{\beta<\alpha} \lambda_\beta)^+ \leq \lambda_\alpha$. Suppose that $\langle A_\xi : \xi < \kappa\rangle$ is a system of finite sets. Then there exist a set $\Gamma \in [\mathrm{cf}(\kappa)]^{\mathrm{cf}(\kappa)}$, a sequence $\langle \Xi_\alpha : \alpha \in \Gamma\rangle$ of subsets of κ, a sequence $\langle F_\alpha : \alpha \in \Gamma\rangle$ of finite sets, and a finite set G, such that the following conditions hold:*

(i) $\langle \Xi_\alpha : \alpha \in \Gamma\rangle$ is a pairwise disjoint system, and $|\Xi_\alpha| = \lambda_\alpha$ for every $\alpha \in \Gamma$.

(ii) $\langle A_\xi : \xi \in \Xi_\alpha\rangle$ is a Δ-system with root F_α for every $\alpha \in \Gamma$.

(iii) $\langle F_\alpha : \alpha \in \Gamma\rangle$ is a Δ-system with root G.

(iv) If $\xi \in \Xi_\alpha$, $\eta \in \Xi_\beta$, and $\alpha \neq \beta$, then $A_\xi \cap A_\eta = G$.

Proof. Let $\kappa = \bigcup_{\alpha<\mathrm{cf}(\kappa)} \Xi'_\alpha$ where the Ξ'_α's are pairwise disjoint, with $|\Xi'_\alpha| = \lambda_\alpha$ for every $\alpha < \mathrm{cf}(\kappa)$. For each $\alpha < \mathrm{cf}(\kappa)$ let $\Xi''_\alpha \in [\Xi'_\alpha]^{\lambda_\alpha}$ be such that $\langle A_\eta : \eta \in \Xi''_\alpha\rangle$ is a Δ-system, say with root F_α. Choose $\Gamma \in [\mathrm{cf}(\kappa)]^{\mathrm{cf}(\kappa)}$ such that $\langle F_\alpha : \alpha \in \Gamma\rangle$ is a Δ-system, say with root G. For each $\alpha \in \Gamma$ let

$$B_\alpha = \bigcup \left\{ \bigcup_{\xi \in \Xi''_\beta} A_\xi : \beta \in \Gamma, \beta < \alpha \right\}.$$

We claim

(1) $|B_\alpha| < \lambda_\alpha$.

In fact,

$$|B_\alpha| \leq \sum_{\substack{\beta<\alpha\\ \beta\in\Gamma}} \left| \bigcup_{\xi\in\Xi''_\beta} A_\xi \right| \leq \sum_{\substack{\beta<\alpha\\ \beta\in\Gamma}} \lambda_\beta < \lambda_\alpha.$$

So (1) holds. Now for any $\alpha \in \Gamma$,

$$\Xi''_\alpha = \bigcup_{J\in[B_\alpha]^{<\omega}} \{\xi \in \Xi''_\alpha : A_\xi \cap B_\alpha = J\},$$

so by (1) there is a $C_\alpha \in [B_\alpha]^{<\omega}$ such that $\Xi'''_\alpha \stackrel{\text{def}}{=} \{\xi \in \Xi''_\alpha : A_\xi \cap B_\alpha = C_\alpha\}$ has size λ_α. Note that $C_\alpha \subseteq F_\alpha$, since for distinct $\xi, \eta \in \Xi'''_\alpha$ we have $C_\alpha = A_\xi \cap A_\eta \cap B_\alpha \subseteq F_\alpha$. Next note that $\langle A_\xi \backslash F_\alpha : \xi \in \Xi'''_\alpha\rangle$ is a system of pairwise disjoint sets; hence for each $\beta \in \Gamma$, the set $\{\xi \in \Xi'''_\alpha : (A_\xi\backslash F_\alpha) \cap F_\beta \neq 0\}$ is finite. Since $|\Gamma| = \mathrm{cf}(\kappa) < \lambda_\alpha$, it follows that the set

$$\Xi_\alpha \stackrel{\text{def}}{=} \Xi'''_\alpha \backslash \{\xi \in \Xi''_\alpha : (A_\xi\backslash F_\alpha) \cap F_\beta \neq \emptyset \text{ for some } \beta \in \Gamma\}.$$

has size λ_α.

Now we can verify the conditions of the proposition. Conditions (i)–(iii) are clear. Now suppose that $\xi \in \Xi_\alpha$, $\eta \in \Xi_\beta$, and $\alpha \neq \beta$. Say $\beta < \alpha$. Suppose that $\gamma \in A_\xi \cap A_\eta \backslash G$; we want to get a contradiction. Since $F_\alpha \cap F_\beta = G$, we have two possibilities.

Case 1. $\gamma \notin F_\alpha$. But $\gamma \in A_\xi \cap B_\alpha = C_\alpha \subseteq F_\alpha$, contradiction.

Case 2. $\gamma \in F_\alpha\backslash F_\beta$. Thus $\gamma \in (A_\eta\backslash F_\beta) \cap F_\alpha$, contradicting the definition of Ξ_β. □

Proposition 24.18. *Suppose that $\mathscr{F}$ is a collection of countable functions, each with range $\subseteq 2^\omega$, and with $|\mathscr{F}| = (2^\omega)^+$. Then there are distinct $f, g \in \mathscr{F}$ such that $f \cup g$ is a function.*

Proof. Let $\kappa = \omega_1$, $\lambda = (2^\omega)^+$, and apply Theorem 24.4 with $\langle A_i : i \in I\rangle$ replaced by $\langle \mathrm{dmn}(f) : f \in \mathscr{F}\rangle$. We get $J \in [\mathscr{F}]^\lambda$ such that $\langle \mathrm{dmn}(f) : f \in J\rangle$ is an indexed Δ-system, say with root r. Now

$$J = \bigcup_{h:r\to 2^\omega} \{f \in J : f \restriction r = h\},$$

and $|{}^r(2^\omega)| = 2^\omega$, so there is an $h : r \to 2^\omega$ such that $K \stackrel{\text{def}}{=} \{f \in J : f \restriction r = h\}$ has size $(2^\omega)^+$. For any two $f, g \in K$, the set $f \cup g$ is a function. □

Proposition 24.19. *For any infinite cardinal κ, any linear order of size at least $(2^\kappa)^+$ has a subset of order type κ^+ or one similar to $(\kappa^+, >)$.*

Proof. Let L be a linear order of size $(2^\kappa)^+$, and let $\prec$ be a well-order of L. Define $f : [L]^2 \to 2$ by setting, for any $\{a, b\} \in [L]^2$, say with $a < b$,

$$f(\{a,b\}) = \begin{cases} 0 & \text{if } a \prec b, \\ 1 & \text{if } b \prec a. \end{cases}$$

By the Erdös-Rado theorem let $A \in [L]^{\kappa^+}$ such that A is homogeneous for f. If f takes the value 0 on $[A]^2$, then A is well-ordered under $<$, and since its size is κ^+, it has a subset of order type κ^+. Similarly if f takes the value 1 on $[A]^2$. $\square$

Proposition 24.20. *For any infinite cardinal κ, any tree of size at least $(2^\kappa)^+$ has a branch or an antichain of size at least κ^+.*

Proof. Suppose that T is a tree of size at least $(2^\kappa)^+$. Let S be a subset of T of size $(2^\kappa)^+$. Define $f : [S]^2 \to 2$ by setting, for distinct $s, t \in S$,

$$f(\{s,t\}) = \begin{cases} 1 & \text{if } s \text{ and } t \text{ are comparable,} \\ 0 & \text{otherwise.} \end{cases}$$

By the Erdös-Rado theorem, let $X \subseteq S$ be homogeneous for f of size κ^+. So if f has constant value 1 on $[X]^2$, then X is a chain of size κ^+, hence extends to a branch of size at least κ^+, while if f has constant value 0 on $[X]^2$, then X is an antichain of size κ^+. $\square$

Proposition 24.21. *Any uncountable tree either has an uncountable branch or an infinite antichain.*

Proof. Suppose that T is an uncountable tree. Let $S \in [T]^{\aleph_1}$. We define $f : [S]^2 \to 2$ by setting, for any distinct $s, t \in S$,

$$f(\{s,t\}) = \begin{cases} 1 & \text{if } s \text{ and } t \text{ are comparable,} \\ 0 & \text{otherwise.} \end{cases}$$

Then the desired conclusion follows from the Dushnik-Miller theorem. $\square$

Proposition 24.22. *Suppose that m is a positive integer. Then any infinite set X of positive integers contains an infinite subset Y such that one of the following conditions holds:*

(i) The members of Y are pairwise relatively prime.

(ii) There is a prime $p < m$ such that for any two $a, b \in Y$, p divides $a - b$.

(iii) If a, b are distinct members of Y, then a, b are not relatively prime, but the smallest prime divisor of $a - b$ is at least equal to m.

Proof. Let $p_0, \ldots, p_{i-1}$ list all of the primes $< m$, in order. Thus $i = 0$ if $m = 1$. Define $f : [X]^2 \to i + 2$ by setting, for any distinct $x, y \in X$,

$$f(\{x,y\}) = \begin{cases} i & \text{if } x \text{ and } y \text{ are relatively prime,} \\ j & \text{if } j < i \text{ and } p_j \text{ is the smallest prime dividing } x - y, \\ i+1 & \text{otherwise.} \end{cases}$$

Applying Ramsey's theorem in the form

$$\omega \to (\underbrace{\omega, \ldots, \omega}_{i+1 \text{ times}})^2,$$

we get an infinite homogeneous subset Y of X. If $f[[Y]^2] = \{i\}$, then any two members of Y are relatively prime. If $f[[Y]^2] = \{j\}$ with $j < i$, then p_j divides $x - y$ for any two members x, y of Y. If $f[[Y]^2] = \{i + 1\}$, then for any two members x, y of Y, the least prime dividing $x - y$ is at least as big as m. □

Proposition 24.23. *Suppose that X is an infinite set, and $(X, <)$ and $(X, \prec)$ are two well-orderings of X. Then there is an infinite subset Y of X such that for all $y, z \in Y$, $y < z$ iff $y \prec z$.*

Proof. Define $f : [X]^2 \to 2$ by setting, for any distinct $x, y \in X$, say with $x < y$,

$$f(\{x, y\}) = \begin{cases} 1 & \text{if } x \prec y \\ 0 & \text{otherwise.} \end{cases}$$

By Ramsey's theorem, let Y be an infinite subset of X which is homogeneous for f. If $f[[Y]^2 = \{0\}$, then $x < y$ implies $x \succ y$ for all distinct $x, y \in Y$. Now since Y is infinite and $<$ is a well-order of Y, the order type of Y under $<$ is an infinite ordinal. Hence there is a system $\langle y_n : n \in \omega \rangle$ of elements of Y such that $y_0 < y_1 < \cdots$. Hence $y_0 \succ y_1 \succ \cdots$, contradicting the fact that $\prec$ is a well-order.

Hence $f[[Y]^2 = 1$. This means that for any distinct $x, y \in Y$ we have $x < y$ iff $x \prec y$, as desired. □

Proposition 24.24. *Let S be an infinite set of points in the plane. Then S has an infinite subset T such that all members of T are on the same line, or else no three distinct points of T are collinear.*

Proof. Define $f : [S]^3 \to 2$ by

$$f(\{s, t, u\}) = \begin{cases} 1 & \text{if } s, t, u \text{ are on a line,} \\ 0 & \text{otherwise.} \end{cases}$$

Let T be an infinite subset of S homogeneous for f. If $f[[T]^2] = \{1\}$, then all points of T are on a line. If $f[[T]^2] = \{0\}$, then no three points of T are on a line. □

Proposition 24.25. *We consider the following variation of the arrow relation. For cardinals $\kappa, \lambda, \mu, \nu$, we define*

$$\kappa \to [\lambda]^\mu_\nu$$

to mean that for every function $f : [\kappa]^\mu \to \nu$ there exist an $\alpha < \nu$ and a $\Gamma \in [\kappa]^\lambda$ such that $f[[\Gamma]^\mu] \subseteq \nu \backslash \{\alpha\}$. In coloring terminology, we color the μ-element subsets of κ with ν colors, and then there is a set which is anti-homogeneous for f, in the sense that there is a color α and a subset of size λ all of whose μ-element subsets do not get the color α.

Then for any infinite cardinal κ,

$$\kappa \not\to [\kappa]^\kappa_{2^\kappa}.$$

Proof.

(1) There is an enumeration $\langle X_\alpha : \alpha < 2^\kappa\rangle$ of $[\kappa]^\kappa$ in which every member of $[\kappa]^\kappa$ is repeated 2^κ times.

In fact, let $f : 2^\kappa \to [\kappa]^\kappa$ be a surjection and let $g : 2^\kappa \times 2^\kappa \to 2^\kappa$ be a bijection. For each $\alpha < 2^\kappa$ let $X_\alpha = f(1^{\mathrm{st}}(g^{-1}(\alpha)))$. Then for all $\alpha, \beta < 2^\kappa$,

$$X_{g(\alpha,\beta)} = f(1^{\mathrm{st}}(g^{-1}(g(\alpha,\beta)))) = f(1^{\mathrm{st}}(\alpha,\beta)) = f(\alpha),$$

and (1) follows.

(2) $|[\kappa]^\kappa| = 2^\kappa$.

To prove (2), let $f : \kappa \times \kappa \to \kappa$ be a bijection. For each $g \in {}^\kappa 2$ let

$$Z_g = \bigcup_{\substack{\alpha<\kappa,\\ g(\alpha)=1}} f[\kappa \times \{\alpha\}].$$

Then $|Z_g| = \kappa$ provided that g is not identically 0, and $Z_g \neq Z_h$ if $g \neq h$. (If $g(\alpha) \neq h(\alpha)$, say $g(\alpha) = 1$ and $h(\alpha) = 0$; then $f[\kappa \times \alpha] \subseteq Z_g$ but $f[\kappa \times \{\alpha\}] \cap Z_h = \emptyset$.) Thus $2^\kappa \leq |[\kappa]^\kappa| + 1 \leq 2^\kappa + 1$ with cardinal addition, and so (2) follows.

(3) There is a one-one $\langle Y_\alpha : \alpha < 2^\kappa\rangle$ such that $Y_\alpha \in [X_\alpha]^\kappa$ for all $\alpha < 2^\kappa$.

We construct Y_α by recursion. If Y_β has been constructed for all $\beta < \alpha$, where $\alpha < 2^\kappa$, choose $Y_\alpha \in [X_\alpha]^\kappa \backslash \{Y_\beta : \beta < \alpha\}$; this is possible by (2). So (3) holds.

Now we define $f : [\kappa]^\kappa \to 2^\kappa$ so that for all $\alpha < 2^\kappa$ one has

$$f(Y_\alpha) = \mathrm{o.t.}(\{\beta < \alpha : X_\beta = X_\alpha\}).$$

This defines f on $\{Y_\alpha : \alpha < 2^\kappa\}$, and it can be extended to all of $[\kappa]^\kappa$ in any fashion.

Now we show that f is the desired counterexample. For, suppose that $\beta < 2^\kappa$, $\Gamma \in [\kappa]^\kappa$, and $f[[\Gamma]^\kappa] \subseteq 2^\kappa \backslash \{\beta\}$. Choose $\alpha < 2^\kappa$ such that $X_\alpha = \Gamma$ and $\{\gamma < \alpha : X_\gamma = \Gamma\}$ has order type β. Then $Y_\alpha \in [\Gamma]^\kappa$ and $f(Y_\alpha) = \beta$, contradiction. □

We prove Hindman's theorem, following Graham, Rothschild, Spencer. A *semigroup* is an algebraic structure $(A, \cdot)$ where $\cdot$ is associative. A *topological semigroup* is a semigroup $(A, \cdot)$ together with a Hausdorff topology on A under which $\cdot$ is continuous.

Theorem 24.26. *Let E be a semigroup with a topology which is compact. Define $\mathscr{R}_b(a) = a \cdot b$ for all $a, b \in E$. Assume that $\forall b \in E[\mathscr{R}_b$ is continuous$]$. Then there is an $e \in E$ such that $e^2 = e$.*

Proof. Let $\mathscr{A}$ be the set of all subsemigroups of E which are compact under the relativized topology. If $\mathscr{C} \subseteq \mathscr{A}$ is a chain, then $\bigcap \mathscr{C} \in \mathscr{A}$. Note that compact subspaces are closed, and hence $\bigcap \mathscr{C} \neq \emptyset$. By Zorn's lemma there is a minimal element A of $\mathscr{A}$. Fix

$e \in A$. Then Ae is a subsemigroup, since $a_1ea_1e = (a_1ea_2)e \in Ae$ for $a_1, a_2 \in A$. Now $\mathscr{R}_e$ is a continuous mapping of A onto Ae, so Ae is compact. Now $Ae \subseteq A$, so $Ae = A$ by minimality. Let $B = \{f \in A : fe = e\}$. Since $Ae = A$, $B \neq \emptyset$. B is a subsemigroup, since if $f_1, f_2 \in B$ then $f_1f_2e = f_1e = e$. Now B is closed in A, and hence is compact. For, suppose that $f \in A\backslash B$. Then $f \in \mathscr{R}_f^{-1}[A\backslash\{e\}] \subseteq A\backslash B$. Hence $A = B$ by minimality. Since $e \in B$ it follows that $e^2 = e$. □

For brevity let $\omega' = \omega\backslash\{0\}$. If $S \subseteq \omega'$, let $\Sigma(S)$ be the set of all finite sums of members of S.

Theorem 24.27. (Hindman) *If $m \in \omega$ and $f : \omega' \to m$, then there exist an $i < m$ and an infinite $S \subseteq \omega'$ such that $f(x) = i$ for all $x \in \Sigma(S)$.*

Proof. ${}^{\omega'}2$ is a compact Hausdorff space under the product topology. This transfers to a compact Hausdorff topology on $\mathscr{P}(\omega')$. Namely, let χ_A be the characteristic function of $A \subseteq \omega'$. Then χ is a bijection from $\mathscr{P}(\omega')$ onto ${}^{\omega'}2$, and we call $U \subseteq \mathscr{P}(\omega')$ open iff $\chi[U]$ is open. A basic open set in $\mathscr{P}(\omega')$ has the form U_{FG}, where F and G are finite disjoint subsets of $\mathscr{P}(\omega')$ and $U_{FG} = \{X \subseteq \mathscr{P}(\omega') : F \subseteq X \text{ and } X \cap G = \emptyset\}$.

Let $\mathscr{U}$ be the set of all ultrafilters on ω'.

(1) $\mathscr{U}$ is a closed subset of $\mathscr{P}(\omega')$.

In fact, suppose that $X \subseteq \mathscr{P}(\omega')$ and $X \notin \mathscr{U}$.

Case 1. $\omega' \notin X$. Let $G = \{\omega'\}$. Then $X \in U_{\emptyset G}$ and $U_{\emptyset G} \cap \mathscr{U} = \emptyset$.

Case 2. $a \in X$, $a \subseteq b$, $b \notin X$. Then $X \in U_{\{a\}\{b\}}$ and $U_{\{a\}\{b\}} \cap \mathscr{U} = \emptyset$.

Case 3. $a, b \in X$ but $a \cap b \notin X$. Then $X \in U_{\{a,b\}\{a\cap b\}}$ and $U_{\{a,b\}\{a\cap b\}} \cap \mathscr{U} = \emptyset$.

Case 4. $a, \omega'\backslash a \notin X$. Then $X \in U_{\emptyset\{a,\omega'\backslash a\}}$ and $U_{\emptyset\{a,\omega'\backslash a\}} \cap \mathscr{U} = \emptyset$.

Thus (1) holds.

Now for $F, G \in \mathscr{U}$ we define

$$F + G = \{A \subseteq \omega' : \{n : \{m : m + n \in A\} \in G\} \in F\}.$$

(2) $F + G$ is an ultrafilter.

In fact, for any n, $\{m : m + n \in \omega'\} = \omega' \in G$, and hence $\omega' \in F + G$. Now for any n, $\{m : m + n \in \emptyset\} = \emptyset \notin G$, so $\{n : \{m : m + n \in \emptyset\} \in G\} = \emptyset \notin F$. So $\emptyset \notin F + G$. Suppose that $A \in F + G$ and $A \subseteq B$. Then $H = \{n : \{m : m+n \in A\} \in G\} \in F$, and for $n \in H$ we have $\{m : m+n \in A\} \in G$, hence $\{m : m+n \in B\} \in G$, so $H \subseteq \{n : \{m : m+n \in B\} \in G\}$ and so $B \in F + G$. Now suppose that $A, A' \in F + G$. Then

$$\begin{aligned} F \ni &\{n : \{m : m + n \in A\} \in G\} \cap \{n : \{m : m + n \in A'\} \in G\} = \\ &\{n : \{m : m + n \in A \cap A'\} \in G\}, \end{aligned}$$

so $A \cap A' \in F + G$. Now suppose that $A \subseteq \omega'$ and $A \notin F + G$. Then $\{n : \{m : m + n \in A\} \in G\} \notin F\}$, so $\omega'\backslash\{n : \{m : m + n \in A\} \in G\} \in F$. Now $\omega'\backslash\{n : \{m : m + n \in A\} = \{n : \{m : m + n \in \omega'\backslash A\}\}$. Hence $\omega'\backslash A \in F + G$.

(3) $+$ is associative.

For, if $A \subseteq \omega'$ let $A_n = \{m : m + n \in A\}$. Then

$$\begin{aligned}
F + (G + H) &= \{A \subseteq \omega' : \{n : \{m : m + n \in A\} \in G + H\} \in F\} \\
&= \{A \subseteq \omega' : \{n : A_n \in G + H\} \in F\} \\
&= \{A \subseteq \omega' : \{n : \{p : \{m : m + p \in A_n\} \in H\} \in G\} \in F\} \\
&= \{A \subseteq \omega' : \{n : \{p : \{m : m + p + n \in A\} \in H\} \in G\} \in F\}
\end{aligned}$$

Also, let $B = \{n : \{m : m + n \in A\} \in H\}$. Then

$$\begin{aligned}
(F + G) + H &= \{A \subseteq \omega' : \{n : \{m : m + n \in A\} \in H\} \in F + G\} \\
&= \{A \subseteq \omega' : B \in F + G\} \\
&= \{A \subseteq \omega' : \{n : \{p : p + n \in B\} \in G\} \in F\} \\
&= \{A \subseteq \omega' : \{n : \{p : \{m + p + q \in A\} \in H\} \in G\} \in F\},
\end{aligned}$$

which is the same as the above. So (3) holds.

(4) For each ultrafilter G define $\mathscr{R}_G : \mathscr{U} \to \mathscr{U}$ by $\mathscr{R}_G(F) = F + G$. Then $\mathscr{R}_G$ is continuous.

Let $\mathscr{S} = \{H \in \mathscr{U} : A \in H, A \subseteq \omega'\} \cup \{H \in \mathscr{U} : A \notin H, A \subseteq \omega'\}$. $\mathscr{S}$ is a subbase for the topology on $\mathscr{U}$, and it suffices to show that if $B \in \mathscr{S}$ and $F \in \mathscr{R}_G^{-1}[B]$ then there is an open set V such that $F \in V \subseteq \mathscr{R}_G^{-1}[B]$.

Case 1. $B = \{H \in \mathscr{U} : A \in H\}$. Thus $A \in F + G$, so $\{n : \{m : m + n \in A\} \in G\} \in F\}$. Then $F \in \{H \in \mathscr{U} : \{n : \{m : m + n \in A\} \in G\} \in H\}$, and if $K \in \{n : \{m : m + n \in A\} \in G\} \in H$ then $K + G \in B$, as desired.

Case 2. $B = \{H \in \mathscr{U} : A \notin H\}$. Thus $A \notin F + G$, so $\{n : \{m : m + n \in A\} \in G\} \notin F\}$. Then $F \in \{H \in \mathscr{U} : \{n : \{m : m + n \in A\} \in G\} \notin H\}$, and if $K \in \{n : \{m : m + n \in A\} \in G\} \notin H$ then $K + G \in B$, as desired.

So (4) holds.

Now by Theorem 24.26 there is an ultrafilter F such that $F + F = F$.

(5) For each $i \in \omega'$ let $K_i = \{A \subseteq \omega' : i \in A\}$. Then K_i is a principal ultrafilter, and $K_i + K_i = K_{2i} \neq K_i$.

In fact,

$$\begin{aligned}
K_i + K_i &= \{A \subseteq \omega' : \{n : \{m : m + n \in A\} \in K_i\} \in K_i\} \\
&= \{A \subseteq \omega' : \{n : i + n \in A\} \in K_i\} \\
&= \{A \subseteq \omega' : 2i \in A\} = K_{2i}.
\end{aligned}$$

So (5) holds.

Hence F is nonprincipal.

Now suppose that $f : \omega' \to m$. Then $\omega' = \bigcup_{i<m} f^{-1}[\{i\}]$, so there is an $i < m$ such that $A_0 \stackrel{\text{def}}{=} f^{-1}[\{i\}] \in F$. For each $B \subseteq \omega'$ and $n \in \omega'$ let $B - n = \{m : n + m \in B\}$ and $B^* = \{n : B - n \in F\}$.

(6) If $B \in F$, then $B^* \in F$ and so $B \cap B^* \in F$.

In fact, if $B \in F$, then $B \in F + F$, so $\{n : \{m : m + n \in B\} \in F\} \in F\}$, hence $\{n : B - n \in F\} \in F$, hence $B^* \in F$.

Now if $A_n \in F$ has been defined, pick $a_{n+1} \in A_n \cap A_n^*$. Since $a_{n+1} \in A_n^*$, we have $A_n - a_{n+1} \in F$ Let $A_{n+1} = (A_n \cap (A_n - a_{n+1}))\backslash\{a_{n+1}\}$. So $A_{n+1} \in F$.

(7) $a_{n+1} + A_{n+1} \subseteq A_n$.

In fact, if $m \in A_{n+1}$, then $m \in A_n - a_{n+1}$, so $a_{n+1} + m \in A_n$.

Now let $S = \{a_n : n \in \omega'\}$. We claim that $x \in A_0$, hence $f(x) = i$, for all $x \in \Sigma(S)$. Take any $x \in \Sigma(S)$. Say $x = a_{i_0} + \cdots + a_{i_m}$ with $0 < i_0 < \ldots < i_m$. We prove that $x \in A_0$ by induction on m. It is clear for $m = 0$, since each $a_i \in A_0$ because $A_0 \supseteq A_1 \supseteq \cdots$. Assume it for m and suppose that $x = a_{i_0} + \cdots + a_{i_{m+1}}$. Then $a_{i_0} + \cdots + a_{i_m} \in A_0$ and $a_{i_{m+1}} \in A_1$, so $x \in A_0$ by (7). □

We prove van der Waerden's theorem, following Mauro de Nasso. A set $A \subseteq \omega'$ is *thick* iff $\forall m \in \omega \exists a \in \omega'[[a, a + m] \subseteq A]$. Recall that for any $A \subseteq \omega'$ and $n \in \omega$, $A - n = \{m : m + n \in A\} = \{p - n : p \in A, p > n\}$.

Proposition 24.28. *A is thick iff for every finite set $F \subseteq \omega'$ there is an $x \in \omega'$ such that $F + x \subseteq A$.*

Proof. $\Rightarrow$: Assume that A is thick and $F \in [\omega']^{<\omega}$. Say $F = \{b_0, \ldots, b_{m-1}\}$ with $b_0 < \cdots < b_{m-1}$. Choose $a \in \omega'$ such that $[a, a + b_{m-1}] \subseteq A$. Then $F + a = \{b_0 + a, \ldots, b_{m-1} + a\} \subseteq [a, a + b_{m-1}] \subseteq A$.

$\Leftarrow$: Assume the indicated condition, and suppose that $m \in \omega$. Now $[1, m]$ is a finite subset of ω', so there is an $x \in \omega'$ such that $[1, m] + x \subseteq A$. Thus $[x + 1, x + m] \subseteq A$. □

Proposition 24.29. *A is thick iff for every finite set $F \subseteq \omega'$ there is an $x \in A$ such that $F + x \subseteq A$.*

Proof. $\Rightarrow$: Assume that A is thick, and $F = \{b_0, \ldots, b_{m-1}\} \subseteq \omega'$ with $b_0 < \cdots < b_{m-1}$. Apply the condition in Proposition 24.28 to $[1, b_{m-1} + 1]$; this gives $x \in \omega'$ such that $1 + x, \ldots, b_{m-1} + 1 + x] \subseteq A$. Then $1 + x \in A$ and $\{b_0 + 1 + x, \ldots, b_{m-1} + 1 + x\} \subseteq A$.

$\Leftarrow$: see the proof of Proposition 24.28. □

Proposition 24.30. *A is thick iff $\forall n_0, \ldots, n_{k-1} \in \omega[\bigcap_{i<k}(A - n_i) \neq \emptyset]$.*

Proof. $\Rightarrow$: Assume that A is thick, and $n_0, \ldots, n_{k-1} \in \omega$. Say $n_0 < \cdots < n_{k-1}$. Note that $A - 0 = A$. Hence we may assume that $0 < n_0$. Choose m so that $\{n_0 + m, \ldots, n_{k-1} + m\} \subseteq A$. Then $m \in \bigcap_{i<k}(A - n_i)$.

$\Leftarrow$: clear by reversing the above argument. □

A is *syndetic* iff $\exists k \in \omega' \forall l[l, l + k - 1] \cap A \neq \emptyset]$.

Corollary 24.31. *A is thick, then so is $A - n$.* □

Proposition 24.32. *A is syndetic iff there are $n_0, \ldots, n_{k-1} \in \omega$ such that $\omega' = \bigcup_{i<k}(A - n_i)$.*

Proof. $\Rightarrow$: Assume that A is syndetic. Choose $k \in \omega'$ such that $\forall l[l, l+k-1] \cap A \neq \emptyset]$. Let $n_0 = 0, n_1 = 1, \ldots, n_{k-1} = k-1$. Suppose that $s \in \omega'$. Now $[s, s+k-1] \cap A \neq \emptyset$. Choose $i < k$ with $s + i \in A$. Then $s \in (A - n_i)$.

$\Leftarrow$: Assume the indicated condition. Assume that $n_0 < \cdots < n_{k-1}$. Given l, choose $i < k$ such that $l \in (A - n_i)$. Then $n_i + l \in A$. So $[l, l + n_{k-1} + 1 - 1] \cap A \neq \emptyset$. □

Proposition 24.33. *A is syndetic iff $A \cap B \neq \emptyset$ for every thick B.*

Proof. $\Rightarrow$: Assume that A is syndetic and B is thick. By Proposition 24.32 let $n_0, \ldots, n_{k-1} \in \omega$ be such that $\omega' = \bigcup_{i<k}(A - n_i)$. By Proposition 24.32 choose $b \in \bigcap_{i<k}(B - n_i)$. Choose $i < k$ such that $b + n_i \in A$. Also $b + n_i \in B$.

$\Leftarrow$: Suppose that A is not syndetic. By Proposition 24.32, for every finite subset F of ω' we have $\omega' \neq \bigcup_{x \in F}(A - x)$. Then

(1) For every finite subset F of ω' we have $\bigcap_{x \in F}((\omega' \backslash A) - x) \neq \emptyset$.

In fact, for F a finite subset of ω' choose $y \notin \bigcup_{x \in F}(A - x)$. Thus $\forall x \in F[x + y \in (\omega' \backslash A)]$, and hence $y \in ((\omega' \backslash A) - x)$. So (1) holds.

By (1) and Proposition 24.32, $\omega' \backslash A$ is thick. This proves $\Leftarrow$. □

Proposition 24.34. *A is syndetic iff $\omega' \backslash A$ is not thick.*

Proof. $\Rightarrow$: by Proposition 24.33. $\Leftarrow$: see the proof of Proposition 24.33, second part. □

A is *piecewise syndetic* iff there exist a thick B and a syndetic C such that $A = B \cap C$.

Proposition 24.35. *The following are equivalent:*

(i) A is piecewise syndetic.

(ii) There is a finite $F \subseteq \omega'$ such that for every finite $G \subseteq \omega'$ there is an $s \in \omega'$ such that for every $t \in G$ there is an $x \in F$ such that $s + t + x \in A$.

(iii) There is a finite $F \subseteq \omega$ such that $\bigcup_{x \in F}(A - x)$ is thick.

Proof. (i)$\Rightarrow$(ii): Choose $k \in \omega'$ such that $\forall l[[l, l+k-1] \cap C \neq \emptyset]$, and let $F = [0, k-1]$. Suppose that $G \subseteq \omega'$ is finite. Let $H = \{t+i : t \in G, i < k\}$. So H is finite. By Proposition 24.28 choose $s \in \omega'$ such that $H + s \subseteq B$. Suppose that $t \in G$. Then $t \in G$, so $t + s \in B$. Choose $x \in F$ such that $t + s + x \in C$. Now $t + x \in H$, so $t + x + s \in B$. So $s + t + x \in A$.

(ii)$\Rightarrow$(iii): Assume (ii), and choose F as indicated. We claim that $\bigcup_{x \in F}(A - x)$ is thick. To prove this we use Proposition 24.28. Suppose that G is a finite subset of ω'. Choose s as in the indicated condition. Then we claim that $G + s \subseteq \bigcup_{x \in F}(A - x)$. For, take any $t \in G$. By the indicated condition there is an $x \in F$ such that $s + t + x \in A$, as desired.

(iii)$\Rightarrow$(i): Assume (iii). Let $F' = F \cup \{0\}$. Then $\bigcup_{x \in F'}(A - x)$ is thick, and $A \subseteq \bigcup_{x \in F'}(A - x)$. We claim that $A \cup (\omega' \backslash B)$ is syndetic; its intersection with B is A, as desired. Suppose that $A \cup (\omega' \backslash B)$ is not syndetic. Say $F = \{0, \ldots, k\}$. There is an l such that $[l, l+k+1] \cap (A \cup (\omega' \backslash B)) = \emptyset$. So $[l, l+k] \cap A = \emptyset$ and $[l, l+k] \subseteq B$. Since $l \notin A$, there is an $m \in F'$ with $m \neq 0$ such that $l \in A - m$. So $l + m \in A$. But $l + m \leq l + k$, contradiction. $\square$

Corollary 24.36. *If A is piecewise syndetic, then so is $A - n$.* $\square$

Lemma 24.37. *Suppose that $\mathscr{S} \subseteq \mathscr{P}(\omega')$ is closed upwards. Let $\mathscr{T} = \{T \subseteq \omega' : \forall S \in \mathscr{S}[T \cap S \neq \emptyset]\}$. Let $\mathscr{A} = \{S \cap T : S \in \mathscr{S}, T \in \mathscr{T}\}$.*

Suppose that $A \in \mathscr{A}$ and $A = B \cup C$ with $B \cap C = \emptyset$. Then $B \in \mathscr{A}$ or $C \in \mathscr{A}$.

Proof. Say $A = S \cap T$ with $S \in \mathscr{S}$ and $T \in \mathscr{T}$. Let $\tilde{S} = B \cup (S \backslash A)$.

(1) $B = \tilde{S} \cap T$.

In fact, $\tilde{S} \cap T = (B \cap T) \cup ((S \cap T) \backslash A) = B \cap T = B$ since $B \subseteq A \subseteq T$.

So if $\tilde{S} \in \mathscr{S}$ we have $B \in \mathscr{A}$, as desired.

Suppose that $\tilde{S} \notin \mathscr{S}$. Let $\tilde{T} = \omega' \backslash \tilde{S}$.

(2) $\tilde{T} \in \mathscr{T}$.

In fact, if $U \in \mathscr{S}$ and $U \cap \tilde{T} = \emptyset$, then $U \subseteq \tilde{S}$ and so $\tilde{S} \in \mathscr{S}$, contradiction.

(3) $C = \tilde{T} \cap S$.

For, $\tilde{T} \cap S = S \backslash \tilde{S} = S \backslash (B \cup (S \backslash A)) = (S \backslash B) \cap (S \cap A) = A \subseteq B = C$. $\square$

Theorem 24.38. *If A is piecewise syndetic and $A = B \cup C$ with $B \cap C = \emptyset$, then B is piecewise syndetic or C is piecewise syndetic.*

Proof. Let $\mathscr{S}$ be the collection of all thick subsets of $\mathscr{P}(\omega')$ and $\mathscr{T}$ be the collection of all syndetic subsets of $\mathscr{P}(\omega')$. By Proposition 24.32, the hypotheses of Lemma 24.37 hold. $\square$

A set $\mathscr{C} \subseteq \mathscr{P}(\omega')$ is *translation invariant* iff $\forall A \in \mathscr{C}[(A-1) \in \mathscr{C}]$.

Proposition 24.39. *If $\mathscr{C}$ is a collection of translation invariant set algebras $(A, \cup, \omega' \backslash)$ on ω', then $\bigcap \mathscr{C}$ is a translation invariant set algebra on ω'.* $\square$

A filter F on a translation invariant set algebra A on ω' is *translation invariant* iff $\forall A \in F[(A-1) \in F]$. TIF abbreviates translation invariant filter.

Proposition 24.40. *A is thick iff $A \in F$ for some TIF F.*

Proof. $\Rightarrow$: Assume that A is thick. By Proposition 24.32, $\mathscr{A} \stackrel{\text{def}}{=} \{A - n : n \in \omega\}$ has fip. Let F be the filter generated by $\mathscr{A}$. So

$$F = \left\{ B : \exists G \in [\omega']^{<\omega} \left[\bigcap_{n \in G} (A - n) \subseteq B \right] \right\}.$$

Suppose that $p \in \omega$ and $B \in F$. Say $G \in [\omega']^{<\omega}$ and $\bigcap_{n \in G}(A - n) \subseteq B$. Let $H = G + p$. Suppose that $q \in \bigcap_{n \in H}(A - n)$. Thus $\forall n \in H[n + q \in A]$, so for all $n \in G[n + p + q \in A]$.

Hence $\forall n \in G[p+q \in (A-n)]$, hence $p+q \in \bigcap_{n\in G}(A-n)$, hence $p+q \in B$, hence $q \in (B-p)$. So we have shown that $\bigcap_{n\in H}(A-n) \subseteq (B-p)$, and so $(B-p) \in F$. Thus $A \in F$ and F is a TIF.

$\Leftarrow$: Suppose that $A \in F$ with F a TIF. If $G \subseteq \omega$ is finite, then for all $n \in G$, $A-n \in F$. Hence $\bigcap_{n\in G}(A-n) \in F$, and so it is nonempty. By Proposition 24.32, A is thick. □

Proposition 24.41. *Every TIF is a subset of a maximal TIF.*

Proof. Zorn's lemma. □

Proposition 24.42. *If $F \subseteq \mathscr{P}(\omega')$ and $n \in \omega$, then $(\bigcap F) - n = \bigcap_{A\in F}(A-n)$.*

Proof.

$$\begin{aligned}\forall m \in \omega'\Bigg[m \in \left(\left(\bigcap F\right) - n\right) \quad &\text{iff} \quad n+m \in \bigcap F \\ &\text{iff} \quad \forall A \in F[n+m \in A] \\ &\text{iff} \quad \forall A \in F[m \in A-n] \\ &\text{iff} \quad m \in \bigcap_{A\in F}(A-n)\Bigg].\end{aligned}$$

□

Proposition 24.43. *Suppose that $F \subseteq \mathscr{P}(\omega')$. Let $\mathscr{F} = \{(Y,n) : Y \in F,\ n \in \omega\}$. Then the TIF generated by F is*

$$\left\{X : \exists G \in [\mathscr{F}]^{<\omega}\left[\bigcap_{(Y,n)\in G}(Y-n) \subseteq X\right]\right\}.$$

Proof. Let K be the indicated set. Note that $F \subseteq K$, since if $Y \in F$ then we can take $G = \{(y,0)\}$. Clearly K is closed upwards and is also closed under $\cap$. Now suppose that G is as indicated, and $m \in \omega$. Then by Proposition 24.42,

$$\begin{aligned}\left(\bigcap_{(Y,n)\in G}(Y-n)\right) - m &= \bigcap_{(Y,n)\in G}((Y-n)-m) = \bigcap_{(Y,n)\in G}(Y-(m+n)) \\ &= \bigcap_{(Y,n)\in G'}(Y-n),\end{aligned}$$

where $G' = \{(Y, n+m) : (Y,n) \in G\}$. It follows that K is closed under $-$. □

Proposition 24.44. *Let B be a translation invariant field of subsets of ω', let M be a maximal TIF, and let U be an ultrafilter extending M. Then every $B \in U$ is piecewise syndetic.*

Proof. First we claim

(1) $\forall B \in U \exists F \in [\omega']^{<\omega}[\bigcup_{x\in F}(B-x) \in M]$.

In fact, let $\Lambda = \{(\omega'\backslash B) - n : n \in \omega\}$. Then $M \cup \Lambda$ does not have fip. For, suppose that it has fip. Then the translation invariant filter M' generated by it is proper. Otherwise, by Proposition 24.43 we get a finite subset F of ω and a $Y \in M$ such that $Y \cap \bigcap_{n\in F}((\omega'\backslash B) - n) = \emptyset$, contradiction. But $\omega'\backslash B \in M'$ while $\omega'\backslash B \notin M$, as otherwise $\omega'\backslash B \in U$, contradiction. So $M \subset M'$, contradicion. So, this proves that $M \cup \Lambda$ does not have fip. Hence there exist $Y \in M$ and a finite $F \subseteq \omega'$ such that $Y \cap \bigcap_{n\in F}((\omega'\backslash B) - n)) = \emptyset$. Thus $Y \subseteq \bigcup_{n\in F}(\omega'\backslash((\omega'\backslash B) - n))$. Now $\omega'\backslash((\omega'\backslash B) - n) = \{m : m + n \notin (\omega'\backslash B) = \{m : m + n \in B\} = \{m : m \in (B - n)\}$, so $\bigcup_{n\in F}(B - n) \in M$. So (1) holds.

Now for any $B \in U$ we take F as in (1). By Proposition 24.40, $\bigcup_{x\in F}(B - x)$ is thick. By Proposition 24.35, B is piecewise syndetic. □

Proposition 24.45. *Let B be a translation invariant field of subsets of ω', let M be a maximal TIF, and let U be an ultrafilter extending M. Then for every $B \in U$, the set $B_U = \{n \in \omega : (B - n) \in U\}$ is syndetic.*

Proof. By (1) in the proof of Proposition 24.40, there is an $F \in [\omega']^{<\omega}$ such that $\bigcup_{x\in F}(B - x) \in M$. Now the proposition follows by Proposition 24.32 from

(1) $\omega' = \bigcup_{x\in F}(B_U - x)$

For, by translation invariance of M, for every $m \in \omega$ we have $(\bigcap_{x\in F}(B - x)) - m = \bigcap_{x\in F}(B - x - m) \in M \in U$, so there is an $x \in F$ such that $(B - x - m) \in U$, so that $m \in (B_U - x)$. This proves (1). □

Lemma 24.46. *Let A be a translation invariant field of subset of ω', let M be a maximal TIF contained in A, and let U be an ultrafilter on A extending M. Suppose that $B \subseteq \omega'$, $l \in \omega$ and $B - l \in U$. Then for every $k \in \omega'$, $B_U - l$ contains an arithmetic progression of length k.*

Proof. Induction on k. For $k = 1$, we just need to show that $B_U - l$ is nonempty. Now $\forall n \in \omega[n \in (B_U - l)$ iff $n + l \in B_U$ iff $B - l - n \in U$ iff $n \in (B - l)_U$. Since $(B - l)_U$ is syndetic by Proposition 24.45, it follows that $B_U - l$ is syndetic, and hence is nonempty.

Now we assume that $B_U - l$ contains an arithmetic progression of length k; we want to show that it contains one of length $k + 1$. Let $l_0 = l$. Since $B_U - l_0$ is syndetic, by Proposition 24.31 there is a finite $F \subseteq \omega$ such that $\omega' = \bigcup_{x\in F}(B_U - l_0 - x)$. We may assume that $0 \in F$. Thus

(1) $\forall n \in \omega' \exists x \in F[l_0 + x + n \in B_U]$.

Now by the inductive hypothesis choose $l_1 \in \omega$ and $y_1 \in \omega'$ such that

(2) $l_1 + iy_1 \in (B_U - l)$ for $i = 1, \ldots, k$.

(If $k = 1$ take $l_1 = 0$ and y_1 any member of $B_U - l$.) Let $x_0 = 0$. Thus

(3) For all $i = 1, \ldots, k$, $l_0 + l_1 + x_0 + iy_1 \in B_U$.

By (1) choose $x_1 \in F$ so that $l_0 + l_1 + x_1 \in B_U$. Thus

(4) $B - (l_0 + l_1 + x_0 + iy_1) \in U$ for all $i = 1, \ldots, k$ and

(5) $B - (l_0 + l_1 + x_1) \in U$.

Case 1. $x_1 = 0$. Then

(6) $l_1 + iy_1 \in (B_U - l)$ for $i = 0, \ldots, k$ is an arithmetic progression of length $k + 1$.

Case 2. $x_1 \neq 0$.

We now define sequences $B_1, B_2, \ldots$ and $x_1, x_2, \ldots \in F$ and $l_0, l_1, \ldots$ so that for each $s = 1, 2, \ldots$ we have

$$B_s = (B - x_s) \cap \bigcap_{i=1}^{k} (B - x_{s-1} - iy_s) \cap \bigcap_{i=1}^{k} (B - x_{s-2} - i(y_{s-1} + y_s))$$
$$\cap \ldots \cap \bigcap_{i=1}^{k} (B - x_0 - i(y_1 + y_2 + \cdots + y_s)) \tag{$*$}$$

and

$$(B_s - (l_0 + \cdots + l_s)) \in U. \tag{$**$}$$

Let

$$B_1 = (B - x_1) \cap \bigcap_{i=1}^{k} (B - x_0 - iy_1).$$

So ($*$) and ($**$) hold for $s = 1$. Assume that they hold for s. Then

$$B_s - (l_0 + \cdots + l_s) = (B - (l_0 + \cdots + l_s + x_s) \cap$$
$$\bigcap_{i=1}^{k} (B - (l_0 + \cdots + l_s + x_{s-1} + iy_s)$$
$$\cap \ldots \cap \bigcap_{i=1}^{k} (B - (x_0 + i(y_1 + \cdots + y_s) + l_0 + \cdots + l_s)$$

By the inductive hypothesis there are $l_{s+1} \in \omega$ and $y_{s+1} \in \omega'$ such that

(7) $l_{s+1} + iy_{s+1} \in (B_s)_U - (l_0 + \cdots + l_s)$ for $i = 1, \ldots, k$. Thus

(8) $(l_0 + \cdots + l_{s+1} + iy_{s+1}) \in (B_s)_U$ for $i = 1, \ldots, k$.

Hence

(9) $l_0 + \cdots + l_{s+1} + x_{s-t} + i(y_{s-t+1} + \cdots + y_{s+1}) \in B_U$ for $i = 1, \ldots, k$ and $0 \leq t \leq s$.

By (1) choose $x_{s+1} \in F$ so that $l_0 + \cdots + l_{s+1} + x_{s+1} \in B_U$.

Subcase 2.1. $x_{s+1} = x_0$. Then $x_0 + l_0 + \cdots + l_{s+1} + i(y_1 + \cdots + y_s)$ is a $(k+1)$-ary arithmetic progression in $B_U - l$ for $i = 0, \ldots, k$.

Subcase 2.2. $x_{s+1} = x_j$ for some $j = 1, \ldots, s$. Then we have $l_0 + l_1 + \cdots + l_{s+1} + x_j + i(y_j + \cdots + y_{s+1}) \in B_U$ for $i = 0, \ldots, k$.

it Subcase 2.3. $x_{s+1} \neq x_j$ for all $j \leq s$. Then the construction continues.

Since F is finite, the construction eventually stops. $\square$

Theorem 24.47. *Suppose that A is a piecewise syndetic set. Then for every $k \in \omega'$, $\{x \in A : \exists y \in \omega' \forall i = 1, \ldots, k[x + iy \in A]\}$ is piecewise syndetic.*

Proof. Let B be the translation invariant field of subsets of ω' generated by $\{A - n : n \in \omega\}$. By Proposition 29.12 there is a finite subset F of ω such that $T \stackrel{\text{def}}{=} \bigcup_{n \in F}(A - n)$ is thick. By Proposition 24.32, $\mathscr{G} \stackrel{\text{def}}{=} \{T - m : m \in \omega\}$ has fip and so it is contained in a maximal TIF M on B. Let U be an ultrafilter on B with $M \subseteq U$. Now $T \in \mathscr{G} \subseteq M \subseteq U$, so there is an $n \in F$ such that $(A - n) \in U$. By Lemma 24.46, $A_U - n$ has an arithmetic progression of every length $k \in \omega$. Now for each $k \in \omega$ choose x and y so that $x + iy \in (A_U - n)$ for all $i = 1, \ldots, k$. Thus $C \stackrel{\text{def}}{=} \bigcap_{i=1}^{k}(A - (n + x + iy)) \in U$.

(1) $C \subseteq \{z : \exists y \in \omega' \forall i = 1, \ldots, k[z + iy \in A]\} - n - x$.

In fact,

$$\begin{aligned}
&\{z : \exists y \in \omega' \forall i = 1, \ldots, k[z + iy \in A]\} - n - x \\
&= \{m : m + n + x \in \{z : \exists y \in \omega' \forall i = 1, \ldots, k[z + iy \in A]\} \\
&= \{m : \exists y \in \omega' \forall i = 1, \ldots, k[m + n + x + iy \in A]\}
\end{aligned}$$

and

$$C = \{m : \forall i = 1, \ldots, k[m + n + x + iy \in A\},$$

so (1) holds.

Now by Proposition 24.32 and Corollary 24.35 the theorem follows. $\square$

Theorem 24.48. *If $\omega' = C_1 \cup \ldots \cup C_n$ is a partition of ω', then there is an i such that for every $k \in \omega'$, $\{x \in C_i : \exists y \in \omega' : \forall i = 1, \ldots, k[x + iy \in C_i]\}$ is piecewise syndetic.*

Proof. Clearly, for example by Proposition 24.33, ω' is piecewise syndetic. By Theorem 29.15 there is an i such that C_i is piecewise syndetic. Now apply Theorem 24.45. $\square$

Theorem 24.49. (van der Waerden) If $\omega' = C_1 \cup \ldots \cup C_n$ is a partition of ω', then there is an i such that for every $k \in \omega'$, C_i has an arithmetic progression of length k. $\square$

We prove the Hales-Jewitt theorem, following the proof in the book of Stasys Jukna, which is based on a sketch of Alon Nilli, which in turn is a simplified version of Shelah's proof.

For t a positive integer let $[t] = \{1, \ldots, t\}$. With t, n, r positive integers, we are going to consider colorings of ${}^n[t]$ with r colors, i.e., functions $f : {}^n[t] \to r$. A (t, n)-*root* is a member of ${}^n([t] \cup \{*\})$ taking the value $*$ at least once. For τ a (t,n)-root and $i < t$, we denote by $\tau(i)$ the result of replacing all $*$ in τ by i. A subset $L \subseteq {}^n[t]$ is a *line* iff there is a (t,n)-root τ such that $L = \{\tau(1), \ldots, \tau(t)\}$. For example, the following is a line in ${}^3[4]$, given by the root $(*, 1, *)$:

$(1, 1, 1)$
$(2, 1, 2)$
$(3, 1, 3)$
$(4, 1, 4)$
$(5, 1, 5)$

Theorem 24.50. (Hales-Jewitt) *For all positive integers t and r there is a positive integer $n \stackrel{\text{def}}{=} HJ(r,t)$ such every coloring of ${}^n[t]$ with r colors has a monochromatic line.*

Proof. We go by induction on t. For $t = 1$ the only line is $\{(1, 1, \ldots, 1)\}$ and the desired conclusion is obvious: $HJ(r, 1) = 1$.

Now suppose the result is true for $t - 1 \geq 1$. Let

$$\begin{aligned} n &= HJ(r, t-1); \\ N_i &= r^{t^{n+\sum_{j=1}^{i-1} N_j}} \qquad \text{for } i = 1, \ldots, n; \\ N &= N_1 + \cdots + N_n. \end{aligned}$$

We suppose that $\chi : {}^N[t] \to r$. If $\tau = \tau_1 \ldots \tau_n$ is a sequence of n roots, with each τ_i of length N_i, and $a \in {}^n[t]$, we define

$$\tau(a) = \tau_1(a_1) \ldots \tau_n(a_n).$$

Thus $\tau(a) \in {}^N[t]$.

Two members $a, b \in {}^n[t]$ are *neighbors* iff they differ at exactly one place, where one of them has value 1 and the other has value 2.

(1) There is a sequence $\tau = \tau_1 \ldots \tau_n$ of roots, with each τ_i of length N_i, such that $\chi(\tau(a)) = \chi(\tau(b))$ for any two neighbors a, b.

We define τ_i by downward induction on i. First we take the case $i = n$. Let $L_{n-1} = N_1 + \cdots + N_{n-1}$. For $k = 0, \ldots, N_n$ let W_k be the following member of ${}^{N_n}[t]$:

$$W_k = \underbrace{1 \ldots 1}_{k} \underbrace{2 \ldots 2}_{N_n - k}$$

For each $k = 0, \ldots, N_n$ we define a coloring $\chi_k : {}^{L_{n-1}}[t] \to r$ by

$$\chi_k(x_1, \ldots, x_{L_{n-1}}) = \chi(x_1, \ldots, x_{L_{n-1}} W_k).$$

Now the number of colorings of ${}^{L_{n-1}}[t]$ is

$$r^{t^{L_{n-1}}} < N_n,$$

and we have $N_n + 1$ colorings. So there exist $s < k \leq N_n$ such that $\chi_s = \chi_k$. We then define

$$\tau_n = \underbrace{1 \ldots 1}_{s}\underbrace{* \ldots *}_{k-s}\underbrace{2 \ldots 2}_{N_n - k}$$

The inductive step from τ_{i+1} to τ_i is similar. Let $L_{i-1} = N_1 + \cdots + N_{i-1}$. For $k = 0, \ldots, N_i$ let W_k be the following member of ${}^{N_i}[t]$:

$$W_k = \underbrace{1 \ldots 1}_{k}\underbrace{2 \ldots 2}_{N_i - k}$$

For each $k = 0, \ldots, N_i$ we define a coloring $\chi_k : {}^{L_{i-1}+n-i}[t] \to r$ by

$$\chi_k(x_1, \ldots, x_{L_{i-1}}, y_{i+1}, \ldots, y_n) = \chi(x_1, \ldots, x_{L_{i-1}} W_k, \tau_{i+1}(y_{i+1}), \ldots, \tau_n(y_n)).$$

Now the number of colorings of ${}^{L_{i-1}+n-i}[t]$ is

$$r^{t^{L_{n-1}+n-i}} \leq N_i,$$

and we have $N_i + 1$ colorings. So there exist $s < k \leq N_n$ such that $\chi_s = \chi_k$. We then define

$$\tau_i = \underbrace{1 \ldots 1}_{s}\underbrace{* \ldots *}_{k-s}\underbrace{2 \ldots 2}_{N_i - k}$$

Now to check that this works, suppose that a and b are neighbors in the i-th place. Say $a_i = 1$ and $b_i = 2$.

Case 1. $i = n$. Thus

$$\begin{aligned} a &= a_0, \ldots, a_{n-2}, 1; \\ b &= a_0, \ldots, a_{n-2}, 2 \end{aligned}$$

Then

$$\begin{aligned} \tau(a) &= \tau_1(a_0) \ldots \tau_{n-1}(a_{n-2})\tau_n(1); \\ \tau(b) &= \tau_1(a_0) \ldots \tau_{n-1}(a_{n-2})\tau_n(2). \end{aligned}$$

Now

$$\tau_n(1) = \underbrace{1 \ldots 1}_{k}\underbrace{2 \ldots 2}_{N_n - k} = W_k \quad \text{and} \quad \tau_n(2) = \underbrace{1 \ldots 1}_{s}\underbrace{2 \ldots 2}_{N_n - s} = W_s.$$

Hence

$$\begin{aligned} \chi(\tau(a)) &= \chi(\tau_1(a_0) \ldots \tau_{n-1}(a_{n-2})\tau_n(1)) = \chi(\tau_1(a_0) \ldots \tau_{n-1}(a_{n-2})W_k) \\ &= \chi_k(\tau_1(a_0) \ldots \tau_{n-1}(a_{n-2})) = \chi_s(\tau_1(a_0) \ldots \tau_{n-1}(a_{n-2})) \\ &= \chi(\tau_1(a_0) \ldots \tau_{n-1}(a_{n-2})W_s) = \chi(\tau(b)). \end{aligned}$$

Case 2. $i < n$. Similar to Case 1.

Now to prove the theorem, let τ be as in (1). Define $\chi' : {}^n\{2,\ldots,t\} \to r$ by defining $\chi'(a) = \chi(\tau(a))$ for any $a \in {}^n\{2,\ldots,t\}$. By the inductive hypothesis there is a root $\nu \in {}^n(\{2,\ldots,t\} \cup \{*\})$ such that $\{\nu(2),\ldots,\nu(t)\}$ is monochromatic under χ'. Now the string $\rho \stackrel{\text{def}}{=} \tau_1(\nu_0)\cdots\tau_n(\nu_{n-1})$ has length N and it is a root since ν is. We claim that

$$M \stackrel{\text{def}}{=} \{\rho(1),\ldots,\rho(t)\}$$

is a monochromatic line under χ. Note that for any $i = 1,\ldots,t$, $\rho(i) = \tau(\nu(i))$. Now $\chi'(\nu(2)) = \cdots = \chi'(\nu(t))$; hence $\chi(\tau(\nu(2))) = \cdots = \chi(\tau(\nu(t)))$. We claim that also $\chi(\tau(\nu(1))) = \chi(\tau(\nu(2)))$. Clearly there are members $\sigma^1,\ldots,\sigma^s$ of ${}^n[t]$ such that $\nu(1) = \sigma^1$, $\nu(2) = \sigma^s$, and successive members of $\sigma^1,\ldots,\sigma^s$ are neighbors. Hence by (1), $\chi(\tau(\nu(1))) = \chi(\tau(\nu(2)))$. □

Next we treat the Halpern-Läuchli theorem, using their original proof. We deal with trees of height ω, finitely branching, with a unique root, and with no maximal nodes. A *vector tree* is a sequence of such trees. A set S of nodes is (h,k)-*dense* iff there is a node x of height h such that S dominates the nodes of height $h+k$ which are above x. k-*dense* means $(0,k)$-dense, and ∞-*dense* means k-dense for all k.

Proposition 24.51. *S is k-dense iff S dominates the nodes of height k.* □

Proposition 24.52. *S is ∞-dense iff S dominates all nodes of T.* □

We define $T \uparrow t = \{s : t \le s\}$. For each $n \in \omega$, $n(T) = \{T \uparrow x : |x| = n\}$. For $B \subseteq T$, $n(T,B) = \{(T \uparrow t) \cap B : |t| = n\}$. If $\mathbf{T} = (T_1,\ldots,T_d)$ is a system of trees, then an (h,k)-*matrix* for $\mathbf{T}$ is a product $\prod_{i=1}^d A_i$ with each A_i (h,k)-dense in T_i. A k-*matrix* is a $(0,k)$-matrix.

Theorem 24.53. (Halpern-Läuchli) *Let $\mathbf{T} = (T_1,\ldots,T_d)$ be a system of trees, each finitely branching, with a single root, and of height ω. Suppose that $Q \subseteq \prod_{i=1}^d T_i$. Then one of the following conditions holds:*

(i) For all $k \in \omega$ there is a k-matrix contained in Q.

(ii) There is an $h \in \omega$ such that for each k there is an (h,k)-matrix contained in $(\prod_{i=1}^d T_i)\backslash Q$.

Proof. We first introduce a certain algebra of symbols. *Atomic symbols* are

$$\exists A_i, \forall x_i, \forall a_i, \exists x_i \quad \text{for each positive integer } i.$$

For each positive integer d we define

$L_d = \{\sigma : \sigma$ is a function with domain $\{1,\ldots,2d\}$, and for each $i \in \{1,\ldots,d\}$ exactly one of the following holds:

(i) Each of $\exists A_i$ and $\forall x_i$ occurs exactly once in σ, with $\exists A_i$ before $\forall x_i$.

(ii) Each of $\forall a_i$ and $\exists x_i$ occurs exactly once in σ, with $\forall a_i$ before $\exists x_i$.

Examples:

$L_1 = \{\langle \exists A_1, \forall x_1\rangle, \langle \forall a_1 \exists x_1\rangle\}$.
$L_2 = \{\langle \exists A_1, \forall x_1, \exists A_2, \forall x_2\rangle, \langle \exists A_1, \exists A_2, \forall x_1, \forall x_2\rangle, \ldots\}$.

Now we define a relation $\vdash_d$ on L_d. α, β stand for A_i, a_i, x_i and U and V are strings of atomic symbols which are in L_d.

Rules 1.
$U\exists\alpha\exists\beta V \vdash_d U\exists\beta\exists\alpha V$.
$U\forall\alpha\forall\beta V \vdash_d U\forall\beta\forall\alpha V$.
$U\exists\alpha\forall\beta V \vdash_d U\forall\beta\exists\alpha V$, if $U\forall\beta\exists\alpha V \in L_d$.
Rules 2.
$U\forall a_i \exists x_i V \vdash_d U\exists A_i \forall x_i V$ for all $i = 1, \ldots, d$.
$U\exists A_i \forall x_i V \vdash_d U\forall a_i \exists x_i V$ for all $i = 1, \ldots, d$.

To state rules 3, we first define, if $\langle V_i : r \le i \le k\rangle$ is a sequence of strings of atomis symbols, then $(V_i)_r^k$ is the concatenation $V_r \cdots V_k$.

Rules 3.
If σ is a permutation of $\{1, \ldots, d\}$, then
$(\forall a_{\sigma(i)})_1^r (\exists A_{\sigma(i)})_{r+1}^d V \vdash_d (\exists A_{\sigma(i)})_{r+1}^d (\forall a_{\sigma(i)})_1^r V$ for $r = 1 \ldots d-1$.

Example

$$d = 4, \quad r = 2, \quad V = \exists x_3 \forall x_1 \forall x_4 \exists x_2, \quad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$$

gives

$$\forall a_2 \forall a_3 \exists A_1 \exists A_4 \exists x_3 \forall x_1 \forall x_4 \exists x_2 \vdash_d \exists A_1 \exists A_4 \forall a_2 \forall a_3 \exists x_3 \forall x_1 \forall x_4 \exists x_2.$$

$\models_d$ is the transitive closure of $\vdash_d$.

(1) $\forall a_d (\exists A_i)_1^{d-1} (\forall x_i)_1^{d-1} \exists x_d \models_d \exists A_d (\forall a_i)_1^{d-1} (\exists x_i)_1^{d-1} \forall x_d$.

Proof of (1): Let $\sigma(1) = d$ and $\sigma(i+1) = i$ for $i = 2, \ldots, d-1$. Then an instance of rules 3 is

$$(\forall a_{\sigma(i)})_1^1 (\exists A_{\sigma(i)})_2^d (\forall x_i)_1^{d-1} \exists x_d \models_d (\exists A_{\alpha(i)})_2^d (\forall a_{\sigma(i)})_1^1 (\forall x_i)_1^{d-1} \exists x_d,$$

or

$$(1a) \qquad \forall a_d (\exists A_i)_1^{d-1} (\forall x_i)_1^d \exists x_d \models_d (\exists A_i)_1^{d-1} \forall a_d (\forall x_i)_1^{d-1} \exists x_d$$

By Rules 1,

$$(1b) \qquad (\exists A_i)_1^{d-1} \forall a_d (\forall x_i)_1^{d-1} \exists x_d \models_d (\exists A_i)_1^{d-1} (\forall x_i)_1^{d-1} \forall a_d \exists x_d$$

Also,

$$(1c) \qquad (\exists A_i)_1^{d-1} (\forall x_i)_1^{d-1} \forall a_d \exists x_d \models_d (\exists A_i \forall x_i)_1^{d-1} \forall a_d \exists x_d$$

By rules 2,

$$(\exists A_i \forall x_i)_1^{d-1} \forall a_d \exists x_d \models_d (\forall a_i \exists x_i)_1^{d-1} \exists A_d \forall x_d \tag{1d}$$

By rules 1,

$$(\forall a_i \exists x_i)_1^{d-1} \exists A_d \forall x_d \models_d (\forall a_i)_1^{d-1} \exists A_d (\exists x_i)_1^{d-1} \forall x_d \tag{1e}$$

Now with σ the identity and $r = d - 1$, rules 3 give

$$(\forall a_i)_1^{d-1} \exists A_d (\exists x_i)_1^{d-1} \forall x_d \models_d \exists A_d (\forall a_i)_1^{d-1} (\exists x_i)_1^{d-1} \forall x_d \tag{1f}$$

Now (1a)–(1f) give (1).

(2) Suppose that $UV \in L_d$, U has length d, no atoms of the forms $\forall x_i, \exists x_i$ occur in U, and $\bar{U}$ is any rearrangement of U. Then $UV \models_d \bar{U}V$.

In fact, assume the hypotheses. So only $\exists A_i$ and $\forall a_i$ occur in U. If no $\forall a_i$ occurs, or no $\exists A_i$ occurs, the conclusion is clear by rules 1. So suppose some $\forall a_i$ occurs and some $\exists A_i$ occurs. By rules 1 we have

$$UV \models_d (\forall a_{\sigma(i)})_1^r (\exists A_{\sigma(i)})_{r+1}^d V$$

By rules 3,

$$(\forall a_{\sigma(i)})_1^r (\exists A_{\sigma(i)})_{r+1}^d V \models_d (\exists A_{\sigma(i)})_{r+1}^d (\forall a_{\sigma(i)})_1^r V$$

Then by rules 1,

$$(\forall a_{\sigma(i)})_1^r (\exists A_{\sigma(i)})_{r+1}^d V \models_d \bar{U}V.$$

Thus (2) holds.

(3) If $W \models_{d-1} \bar{W}$, then $\forall a_d W \exists x_d \models_d \forall a_d \bar{W} \exists x_d$.

For, assume that $W \models_{d-1} \bar{W}$. Say

$$W = S_0 \vdash_{d-1} S_1 \vdash_{d-1} S_2 \cdots \vdash_{d-1} S_n = \bar{W}.$$

We claim that

$$\forall a_d W \exists x_d = \forall a_d S_0 \exists x_d \vdash_d \forall a_d S_1 \exists x_d \cdots \vdash_d \forall a_d S_n \exists x_d = \forall a_d \bar{W} \exists x_d.$$

Consider the step from S_i to S_{i+1}. If rules (1) or rules (2) are used in going from S_i to S_{i+1}, clearly the same rules go from $\forall a_d S_i \exists x_d$ to $\forall a_d S_{i+1} \exists x_d$. Suppose that rules (3) are used. Say S_i is $(a_{\sigma(i)})_1^r (\exists A_{\sigma(i)})_{r+1}^{d-1} V$ and S_{i+1} is $(\exists A_{\sigma(i)})_{r+1}^{d-1} (\forall a_{\sigma(i)})_1^r V$. Then $\forall a_d S_i \exists x_d$ is $\forall a_d (a_{\sigma(i)})_1^r (\exists A_{\sigma(i)})_{r+1}^{d-1} V \exists x_d$ and $\forall a_d S_{i+1} \exists x_d$ is $\forall a_d (\exists A_{\sigma(i)})_{r+1}^{d-1} (\forall a_{\sigma(i)})_1^r V \exists x_d$. Hence $\forall a_d S_i \exists x_d \models_d \forall a_d S_{i+1} \exists x_d$ by (2). Hence (3) holds.

(4) $(\forall a_i)_1^d (\exists x_i)_1^d \models_d (\exists A_i)_1^d (\forall x_i)_1^d$.

In fact, we prove this by induction on d. For $d=1$ the assertion is that $\forall a_1 \exists x_1 \models_d \exists A_1 \forall x_1$, which is an instance of rules 2. Now assume (4) for $d-1 \geq 1$. Then rules 1 give

$$(4a) \qquad (\forall a_i)_1^d (\exists x_i)_1^d \models_d \forall a_d (\forall a_i)_1^{d-1} (\exists x_i)_1^{d-1} \exists x_d$$

By the inductive hypothesis and (3) we have

$$(4b) \qquad \forall a_d (\forall a_i)_1^{d-1} (\exists x_i)_1^{d-1} \exists x_d \models_d \forall a_d (\exists A_i)_1^{d-1} \forall x_i)_1^{d-1} \exists x_d$$

By (1) we have

$$(4c) \qquad \forall a_d (\exists A_i)_1^{d-1} (\forall x_i)_1^{d-1} \exists x_d \models_d \exists A_d (\forall a_i)_1^{d-1} (\exists x_i)_1^{d-1} \forall x_d$$

By the inductive hypothesis and (3) we have

$$(4d) \qquad \exists A_d (\forall a_i)_1^{d-1} (\exists x_i)_1^{d-1} \forall x_d \models_d \exists A_d (\exists A_i)_1^{d-1} (\forall x_i)_1^{d-1} \forall x_d$$

Now by rules (1) we get

$$(4e) \qquad \exists A_d (\exists A_i)_1^{d-1} (\forall x_i)_1^{d-1} \forall x_d \models_d (\exists A_i)_1^d (\forall x_i)_1^d$$

Now (4a)–(4e) give (4).

Now suppose that $\mathbf{T} = \langle T_i : 1 \leq i \leq d \rangle$ is a vector tree and $Q \subseteq \prod_{i=1}^d T_i$. We define a $(d+1)$-sorted language $\mathscr{L}$. The sorts are $S_1, \ldots, S_{d+1}$. Additional constants are as follows.

A d-ary function symbol Seq acting on d-tuples from $S_1 \times \cdots S_d$ with values in S_{d+1}.
For each $i = 1, \ldots, d$, a binary relation symbol $<_i$ acting on S_i.
$x_1, \ldots, x_d$ are variables ranging over $S_1, \ldots, S_d$ respectively.
$B_1, \ldots, B_d$ are constants for subsets of $S_1, \ldots, S_d$ respectively.
$A_1, \ldots, A_d$ are variables ranging over subsets of $S_1, \ldots, S_d$ respectively.
v_{ik} for $i = 1, \ldots, d$ and $k \in \omega$ are variables ranging over S_i,
$a_1, \ldots, a_d$ are variables ranging over subsets of $S_1, \ldots, S_d$ respectively.
Q, a constant for a subset of S_{d+1}

A structure for this language assigns T_i to S_i for $i = 1, \ldots, d$, the product $\prod_{i=1}^d T_i$ to S_{d+1}, and subsets B_i of T_1 for $i = 1, \ldots, d$, with Q assigned to Q.

Now with each sequence $\mathbf{n} = (n_1, \ldots, n_d)$ of positive integers and each sequence W of atomic symbols we associate a formula $\varphi = \varphi_{W\mathbf{n}}$. This is done by induction on the length of W

If W is empty, we let $\varphi_{W\mathbf{n}}$ be the formula $Seq(x_1, \ldots, x_d) \in Q$.

If $W = \exists A_i W'$, then we let $\varphi_{W\mathbf{n}}$ be the formula $\exists A_i [A_i \subseteq B_i \wedge A_i$ is n_i-dense in $S_i \wedge \varphi_{W'\mathbf{n}}]$. Here "$A_i$ is n_i-dense in S_i" is the formula

$$\forall t \in S_i [|t| = n_i \rightarrow \exists s \in A_i [t \leq_i s]].$$

We use the variables v_{ik} to express this.

If $W = \forall x_i W'$, then we let $\varphi_{W\mathbf{n}}$ be the formula $\forall x_i[x_i \in A_i \to \varphi_{W'\mathbf{n}}]$.

If $W = \forall a_i W'$, then we let $\varphi_{W\mathbf{n}}$ be the formula $\forall a_i[a_i \in n_i(T_i, B_i) \to \varphi_{W'\mathbf{n}}]$ Here $a_i \in n_i(T_i, B_i)$ is the formula

$$\exists t \in S_i[|t| = n_i \wedge \forall s[s \in a_i \leftrightarrow [t \leq s \wedge s \in B_i]]].$$

If $W = \exists x_i W'$, then we let $\varphi_{W\mathbf{n}}$ be the formula $\exists x_i[x_i \in a_i \wedge \varphi_{W'\mathbf{n}}]$.

(5) If $W \in L_d$ and $\mathbf{n} = (n_1, \ldots, n_d)$ with each $n_i > 0$, then $\varphi_{W\mathbf{n}}$ is a sentence, where each B_i is considered as an individual constant.

In fact, for each $i = 1, \ldots, d$, either $\exists x_i$ or $\forall x_i$ is an entry in W. Since $\exists A_i$ comes before $\forall x_i$ if either occurs, A_i is not free in $\varphi_{W\mathbf{n}}$. Since $\forall a_i$ comes before $\exists x_i$ if either occurs, a_i is not free in $\varphi_{W\mathbf{n}}$. So (5) holds.

Now we let $\psi(W, n, p)$ be the statement "$\forall B_1 \subseteq S_1 \cdots \forall B_d \subseteq S_d[\forall i = 1, \ldots, d[B_i$ is p-dense in $S_i \to \varphi_{W\mathbf{n}}]]$"

Example: W is $\exists A_i \forall a_2 \exists x_2 \forall x_1$. Then $\psi(W, n, p)$ is

$$\forall B_1 \subseteq S_1 \cdots \forall B_d \subseteq S_d[\forall i = 1, \ldots, d[B_i \text{ is } p\text{-dense in } S_i \to \varphi_{W\mathbf{n}}]];$$

We expand $\varphi_{W\mathbf{n}}$:

$$\exists A_1[A_1 \subseteq B_1 \wedge A_1 \text{ is } n_i\text{-dense in } S_i \wedge \varphi_{W'\mathbf{n}}]$$

with $W' = \forall a_2 \exists x_2 \forall x_1$. Expanding further,

$$\exists A_1[A_1 \subseteq B_1 \wedge A_1 \text{ is } n_1\text{-dense in } S_1 \wedge \forall a_2[a_2 \in n_i(S_i, B_i) \to \varphi_{W''\mathbf{n}}]]$$

with $W'' = \exists x_2 \forall x_1$. Another expansion gives

$$\exists A_1[A_1 \subseteq B_1 \wedge A_1 \text{ is } n_1\text{-dense in } S_1 \wedge \forall a_2[a_2 \in n_i(S_i, B_i) \to \\ \exists x_2[x_2 \in a_2 \wedge \varphi_{W'''\mathbf{n}}]]]$$

with $W''' = \forall x_1$. A further expansion:

$$\exists A_1[A_1 \subseteq B_1 \wedge A_1 \text{ is } n_1\text{-dense in } S_1 \wedge \forall a_2[a_2 \in n_i(S_i, B_i) \to \\ \exists x_2[x_2 \in a_2 \wedge \forall x_1[x_1 \in A_1 \varphi_{W^{iv}\mathbf{n}}]]]]$$

with W^{iv} empty. A final expansion, gives $\psi(W, n, p)$ as follows:

$$\forall B_1 \subseteq S_1 \cdots \forall B_d \subseteq S_d[\forall i = 1, \ldots, d[B_i \text{ is } p\text{-dense in } S_i \to \\ \exists A_1[A_1 \subseteq B_1 \wedge A_1 \text{ is } n_1\text{-dense in } S_1 \wedge \forall a_2[a_2 \in n_i(S_i, B_i) \to \\ \exists x_2[x_2 \in a_2] \wedge \forall x_1[x_1 \in A_1 \wedge Seq(x_1, x_2) \in Q]]]]]$$

(6) Suppose that W, W', ρ are sequences of atomic symbols. Suppose that under every assignment of values to the variables, φ_{Wn} implies $\varphi_{W'n}$. Then $\varphi_{\rho Wn}$ under any assignment implies $\varphi_{\rho W'n}$ under that assignment.

We prove this by induction on ρ. If ρ is empty, it is obvious. The induction step is clear upon looking at what $\varphi_{\rho W n}$ is.

Case 1. $\rho = \exists A_i \rho'$. Then $\varphi_{\rho W n}$ is

$$\exists A_i[A_i \subseteq B_i \wedge A_i \text{ is } n_i\text{-dense in } S_i \wedge \varphi_{\rho' W n}]$$

Case 2. $\rho = \forall x_i \rho'$. Then $\varphi_{\rho W n}$ is

$$\forall x_i[x_i \in A_i \to \varphi_{\rho' W n}].$$

Case 3. $\rho = \forall a_i \rho'$. Then $\varphi_{\rho W n}$ is

$$\forall a_i[a_i \in n_i(S_i, B_i) \to \varphi_{\rho' W n}]$$

Case 4. $\rho = \exists x_i \rho'$. Then $\varphi_{\rho W n}$ is

$$\exists x_i[x_i \in a_i \wedge \varphi_{\rho' W n}]$$

This proves (6).

Now we note that $\forall n \exists p \psi(\overline{W}, n, p)$ has the form

(7) $$\forall n \exists p[\forall B_1 \subseteq T_1 \cdots \forall B_d \subseteq T_d[\forall i = 1, \ldots, d[B_i \text{ is } p\text{-dense in } T_i \to \varphi_{W n}]]].$$

(8) If $W \models_d \overline{W}$ and $\forall n \exists p \psi(W, n, p)$, then $\forall n \exists p \psi(\overline{W}, n, p)$.

To prove (8), note by (7) that it suffices to prove

(8′) If $W \models_d \overline{W}$ then for all n, $\varphi_{W n}$ implies that $\varphi_{\overline{W} n}$.

Case 1. $W = U \exists A_i \exists A_j V$ and $\overline{W} = U \exists A_j \exists A_i V$. Then for some ρ, $\varphi_{W n}$ is

$$\rho \exists A_i \subseteq B_i[A_i \text{ is } n_i\text{-dense in } T_i \wedge \varphi_{\exists A_j V n}];$$

expanding the portion involving A_j, this gives for $\varphi_{W n}$

$$\rho \exists A_i \subseteq B_i[A_i \text{ is } n_i\text{-dense in } T_i \wedge \exists A_j \subseteq B_j[A_j \text{ is } n_j\text{-dense in } T_j \wedge \varphi_{V n}]].$$

For purely logical reasons this is equivalent to

$$\rho \exists A_i \exists A_j[A_i \subseteq B_i \wedge A_j \subseteq B_j \wedge A_i \text{ is } n_i\text{-dense in } T_i \wedge A_j \text{ is } n_j\text{-dense in } T_j \wedge \varphi_{V n}].$$

Clearly $\varphi_{\overline{W} n}$ is equivalent to the same thing.

Case 2. $W = U \exists A_i \exists x_j V$ and $\overline{W} = U \exists x_j \exists A_i V$. Then for some ρ, $\varphi_{W n}$ is

$$\rho \exists A_i \subseteq B_i[A_i \text{ is } n_i\text{-dense in } T_i \wedge \varphi_{\exists x_j V n}];$$

expanding the portion involving x_j, this gives for $\varphi_{W n}$

$$\rho \exists A_i \subseteq B_i[A_i \text{ is } n_i\text{-dense in } T_i \wedge \exists x_j[x_j \in a_j \wedge \varphi_{V n}]].$$

For purely logical reasons this is equivalent to

$$\rho\exists A_i \exists x_j [A_i \subseteq B_i \wedge x_j \in a_j \wedge [A_i \text{ is } n_i\text{-dense in } T_i] \wedge \varphi_{Vn}$$

Clearly $\varphi_{\overline{W}n}$ is equivalent to the same thing.

Case 3. $W = U\exists x_i \exists x_j V$ and $\overline{W} = U\exists x_j \exists x_i V$. Then for some ρ, φ_{Wn} is

$$\rho\exists x_i [x_i \in a_i \wedge \varphi_{\exists x_j V}];$$

expanding the portion involving x_j, this gives for φ_{Wn}

$$\rho\exists x_i [x_i \in a_i \wedge \exists x_j [x_j \in a_j \wedge \varphi_{Vn}]].$$

For purely logical reasons this is equivalent to

$$\rho\exists x_i \exists x_j [x_i \in a_i \wedge x_j \in a_j \wedge \varphi_{Vn}]$$

Clearly $\varphi_{\overline{W}n}$ is equivalent to the same thing.

Case 4. $W = U\exists A_i \forall x_j V$ and $\overline{W} = U\forall x_j \exists A_i V$. Then for some ρ, φ_{Wn} is

$$\rho\exists A_i [A_i \subseteq B_i \wedge A_i \text{ is } n_i\text{-dense in } S_i \wedge \varphi_{\forall x_j Vn}];$$

expanding the portion involving x_j, this gives for φ_{Wn}

$$\rho\exists A_i [A_i \subseteq B_i \wedge A_i \text{ is } n_i\text{-dense in } S_i \wedge \forall x_j [x_j \in A_j \to \varphi_{Vn}]].$$

For purely logical reasons this is equivalent to

$$(8'a) \qquad \rho\exists A_i \forall x_j [A_i \subseteq B_i \wedge A_i \text{ is } n_i\text{-dense in } S_i \wedge [x_j \in A_j \to \varphi_{Vn}]].$$

Now $\varphi_{\overline{W}n}$ is

$$\rho\forall x_j [x_j \in A_j \to \exists A_i [A_i \subseteq B_i \wedge \text{ is } n_i\text{-dense in } S_i \wedge \varphi_{Vn}]]$$

For purely logical reasons this is equivalent to

$$\rho\forall x_j \exists A_i [x_j \in A_j \to A_i \subseteq B_i \wedge \text{ is } n_i\text{-dense in } S_i \wedge \varphi_{Vn}]$$

Now using (6) the desired conclusion follows.

Case 5. $W = U\exists A_i \forall a_j V$ and $\overline{W} = U\forall a_j \exists A_i V$. Then for some ρ, φ_{Wn} is

$$\rho\exists A_i [A_i \subseteq B_i \wedge A_i \text{ is } n_i\text{-dense in } S_i \wedge \varphi_{\forall a_j Vn}];$$

expanding the portion involving a_j, this gives for φ_{Wn}

$$\rho\exists A_i [A_i \subseteq B_i \wedge A_i \text{ is } n_i\text{-dense in } S_i \wedge \forall a_j [a_j \in n_j(S_j, b_j) \to \varphi_{Vn}]]$$

For purely logical reasons this is equivalent to

$$\rho\exists A_i\forall a_j[A_i \subseteq B_i \wedge A_i \text{ is } n_i\text{-dense in } S_i \wedge [a_j \in n_j(S_j, b_j) \to \varphi_{Vn}]]$$

Now $\varphi_{\overline{W}n}$ is

$$\rho\forall a_j[a_j \in n_j(S_j, b_j) \to \exists A_i[A_i \subseteq B_i \wedge A_i \text{ is } n_i\text{-dense in } S_i \wedge \varphi_{Vn}]]$$

For purely logical reasons this is equivalent to

$$\rho\forall a_j\exists A_i[a_j \in n_j(S_j, b_j) \to [A_i \subseteq B_i \wedge A_i \text{ is } n_i\text{-dense in } S_i \wedge [\varphi_{Vn}]]$$

Again, using (6) the desired conclusion follows.

Case 6. $W = U\exists x_i\forall x_j V$ and $\overline{W} = U\forall x_j\exists x_i V$. Then for some ρ, φ_{Wn} is

$$\rho\exists x_i[x_i \in a_i \wedge \forall x_j[x_j \in a_j \wedge \varphi_{Vn}]]$$

This is equivalent to

$$\rho\exists x_i\forall x_j[x_i \in a_i \wedge x_j \in a_j \wedge \varphi_{Vn}]$$

Clearly the desired conclusion follows.

Case 7. $W = U\exists x_i\forall a_j V$. Then for some ρ, φ_{Wn} is

$$\rho\exists x_i[x_i \in a_i \wedge \forall a_j[a_j \in n_j(T_j, B_j) \to \varphi_{Vn}]];$$

this is equivalent to

$$\rho\exists x_i\forall a_j[x_i \in a_i \wedge [a_j \in n_j(T_j, B_j) \to \varphi_{Vn}]].$$

Now $\varphi_{\overline{W}n}$ is

$$\rho\forall a_j[a_j \in n_j(T_j, B_j) \to \exists x_i[x_i \in a_i \wedge \varphi_{Vn}]],$$

which is equivalent to

$$\rho\forall a_j\exists x_i[a_j \in n_j(T_j, B_j) \to x_i \in a_i \wedge \varphi_{Vn}].$$

The desired conclusion is clear.

This takes care of rules 1. Before getting to rules 2, we note:

(9) The following are equivalent:

(9a) $A_i \subseteq B_i$ and A_i is n_i-dense in T_i.

(9b) $A_i \subseteq B_i$ and $\forall a_i \in n_i(T_i, B_i)[a_i \cap A_i \neq \emptyset]$.

(9a)$\Rightarrow$(9b): Assume that (9a) holds, and $a_i \in n_i(T_i, B_i)$. Say $a_i = (T_i \uparrow t) \cap B_i$ with $|t| = n_i$. By (9a) there is a $u \in A_i$ with $t \leq u$. so $u \in a_i \cap A_i$.

(9b)$\Rightarrow$(9a): Assume (9b), and suppose that $t \in T_i$ with $|t| = n_i$. Let $a_i = (T \uparrow t) \cap B_i$. By (9b) choose $u \in a_i \cap A_i$. Then $t \leq u$, proving (9a).

Case 8. $1 \le i \le d$, $W = U\forall a_i \exists x_i V$, and $\overline{W} = U\exists A_i \forall x_i V$. Then for some ρ, φ_{Wn} is

$$\rho\forall a_i[a_i \in n_i(T_i, B_i) \to \exists x_i[x_i \in a_i \wedge \varphi_{Vn}]],$$

which is equivalent to

$$\rho\forall a_i \exists x_i[a_i \in n_i(T_i, B_i) \to [x_i \in a_i \wedge \varphi_{Vn}]]. \tag{9c}$$

On the other hand, $\varphi_{\overline{W}n}$ is

$$\rho\exists A_i[A_i \subseteq B_i \wedge A_i \text{ is } n_i\text{-dense in } T_i \wedge \forall x_i[x_i \in A_i \to \varphi_{Vm}]],$$

which is equivalent to

$$\rho\exists A_i \forall x_i[A_i \subseteq B_i \wedge A_i \text{ is } n_i\text{-dense in } T_i \wedge [x_i \in A_i \to \varphi_{Vm}]].$$

Now assume (9c). For each $a_i \in n_i(T_i, B_i)$ choose $x_i(a_i) \in a_i$. Let $A_i = \{x_i(a_i) : a_i \in n_I(T_i, B_i)\}$. Note that $\forall a_i \in n_i(T_i, B_i)[a_i \subseteq B_i]$. Hence $A_i \subseteq B_i$. Hence by (9), A_i is n_i-dense in T_i. This is as desired in Case 8.

Case 9. $1 \le i \le d$, $W = U\exists A_i \forall x_i V$, and $\overline{W} = U\forall a_i \exists x_i V$. Then for some ρ, φ_{Wn} is

$$\rho\exists A_i[A_i \subseteq B_i \text{ and } A_i \text{ is } n_i\text{-dense in } T_i \text{ and } \forall x_i[x_i \in A_i \to \varphi_{Vn}]],$$

which is equivalent to

$$\rho\exists A_i \forall x_i[A_i \subseteq B_i \text{ and } A_i \text{ is } n_i\text{-dense in } T_i \text{ and } [x_i \in A_i \to \varphi_{Vn}]], \tag{9d}$$

On the other hand, $\varphi_{\overline{W}n}$ is

$$\rho\forall a_i[a_i \in n_i(T_i, B_i) \to \exists x_i[x_i \in a_i \wedge \varphi_{Vn}]],$$

which is equivalent to

$$\rho\forall a_i \exists x_i[a_i \in n_i(T_i, B_i) \to [x_i \in a_i \wedge \varphi_{Vn}]].$$

Assume (9d), let A_i be as indicated, and suppose that $a_i \in n_i(T_i, B_i)$. By (9), choose $x_i \in a_i \cap A_i$, as desired.

This takes care of rules 2. Now for rules 3, suppose that σ is a permutation of $\{1, \ldots, d\}$. Let $W = (\forall a_{\sigma(i)})_1^r (\exists A_{\sigma(i)})_{r+1}^d V$ and $\overline{W} = (\exists A_{\sigma(i)})_{r+1}^d (\forall a_{\sigma(i)})_1^r V$ with $r \in \{1 \ldots d-1\}$. Now for simplicity we assume that σ is the identity. Note that V is a string of length d whose entries are $\forall x_i$ for $r+1 \le i \le d$ and $\exists x_j$ for $1 \le j \le r$; moreover, only A_i for $i = r+1, \ldots, d$ and a_i for $i = 1 \ldots, r$ are free. If V is such a string, $\mathbf{a}$ is an assignment of values to the a_i's, $A_{r+1}, \ldots, A_d$ an assignment of values to the A_i's, then the assertion $\varphi_{Vn}[\mathbf{a}, A_{r+1}, \ldots, A_d]$ has the natural meaning.

(10) If V is such a string, $\mathbf{a}$ assigns values to the a_i for $i = 1, \ldots, r$, $A_{r+1}, \ldots, A_d$ an assignment of values to the A_i's, $A'_{r+1} \subseteq A_{r+1}, \ldots A'_d \subseteq A_d$, and $\varphi_{Vn}[\mathbf{a}, A_{r+1}, \ldots, A_d]$, then $\varphi_{Vn}[\mathbf{a}, A'_{r+1}, \ldots, A'_d]$.

We prove (10) by induction on the length of V. It is trivial for the empty string. Now suppose that the string is $\forall x_i V$. Then $\varphi_{\forall x_i V n}[\mathbf{a}, A_{r+1}, \ldots, A_d]$ is

$$\forall x_i[x_i \in A_i \to \varphi_{Vn}[\mathbf{a}, A_{r+1}, \ldots, A_d]],$$

so

$$\forall x_i[x_i \in A'_i \to \varphi_{Vn}[\mathbf{a}, A'_{r+1}, \ldots, A'_d]],$$

If the string is $\exists x_i V$, then $\varphi_{\exists x_i V n}[\mathbf{a}, A_{r+1}, \ldots, A_d]$ is

$$\exists x_i[x_i \in a_i \wedge \varphi_{Vn}[\mathbf{a}, A_{r+1}, \ldots, A_d]],$$

and the conclusion is obvious. So (10) holds

Now for rules 3 we prove (8) directly rather than using (8′). So, suppose that $\forall \mathbf{n} \exists p \psi(W, \mathbf{n}, p)$. Let F be such that $\forall \mathbf{n} \psi(W, \mathbf{n}, F(\mathbf{n}))$. Thus

(11) $\quad \forall \mathbf{n}[\forall B_1 \subseteq T_1 \cdots \forall B_d \subseteq T_d[\forall i = 1, \ldots, d[B_i \text{ is } F(\mathbf{n})\text{-dense in } T_i \to \varphi_{W\mathbf{n}}]]]$.

Since p'-density implies p-density for $p < p'$, we may assume that for all $\mathbf{n}$ and all $i = 1, \ldots, d$, $F(\mathbf{n}) > n_i$.

Now fix a sequence $\mathbf{n} = (n_1, \ldots, n_d)$ of positive integers. We want to find p such that $\psi(\overline{W}, \mathbf{n}, p)$. Define G by induction, as follows.

$$G(0) = \max\{n_i : r < i \leq d\};$$
$$G(j+1) = F(\mathbf{k}), \text{ where } k_i = \begin{cases} n_i & \text{if } 1 \leq i \leq r, \\ G(j) & \text{if } r < i \leq d. \end{cases}$$

Now for each $i = 1, \ldots, r$ let z_i be the number of elements of T_i of height n_i, and let $m = \prod_{i=1}^r z_i$. For each $j \leq m$ let $p_j = G(m - j)$.

(12) If $j < m$, then $p_{j+1} \leq p_j$.

For,

$$p_j = G(m-j) = G(m-j-1+1) = F(\mathbf{k}^j), \text{ where } k_i^j = \begin{cases} n_i & \text{if } 1 \leq i \leq r, \\ G(m-j-1) & \text{if } r < i \leq d \end{cases}$$
$$= \begin{cases} n_i & \text{if } 1 \leq i \leq r, \\ p_{j+1} & \text{if } r < i \leq d \end{cases}$$

Since p_{j+1} is an entry of $\mathbf{k}^j$, (12) holds.

It follows that

(13) If a set is p_j-dense in T_i, then it is also p_{j+1}-dense in T_i.

We claim that $\psi(\overline{W}, \mathbf{n}, p_0)$. Now $\psi(\overline{W}, \mathbf{n}, p_0)$ is

$$\forall B_1 \subseteq T_1 \cdots \forall B_d \subseteq T_d[\forall i = 1, \ldots, d[B_i \text{ is } p_0\text{-dense in } T_i \rightarrow \varphi_{\overline{W}\mathbf{n}}]].$$

So, assume that $B_1 \subseteq T_1 \cdots \forall B_d \subseteq T_d$ and $\forall i = 1, \ldots, d[B_i$ is p_0-dense in $T_i]$.

(14) If $a_1 \in n_1(T_1, B_1) \wedge \ldots \wedge a_r \in n_r(T_r, B_r)$, $0 \leq j < m$, $A_{r+1} \subseteq B_{r+1}, \ldots, A_d \subseteq B_d$, and $A_{r+1}, \ldots, A_d$ are p_j-dense in $T_{r+1}, \ldots, T_d$ respectively, then there exist $A'_{r+1} \subseteq A_{r+1}, \ldots, A'_d \subseteq A_d$ which are p_{j+1}-dense such that $\varphi_{V\mathbf{k}^j}[\vec{a}, A'_{r+1}, \ldots, A'_d]$.

By (11), $\varphi_{W\mathbf{k}^j}[A_{r+1}, \ldots, A_d]$, and hence by the form of W, there are $A'_{r+1} \subseteq A_{r+1}, \ldots, A'_d \subseteq A_d$ such that $A'_{r+1}, \ldots, A'_d$ are k^j_{r+1}-$, \ldots, k^j_d$-dense and $\varphi_{V\mathbf{k}^j}[\vec{a}, A'_{r+1}, \ldots, A'_d]$. Now $k^j_{r+1} = \cdots = k^j_d = p_{j+1}$, as desired.

Now clearly $|\prod_{i=1}^r n_i(T_i, B_i)| \leq |\prod_{i=1}^r z_i| = m$.

(15) For any $J \subseteq \prod_{i=1}^r n_i(T_i, B_i)$ with $|J| = j \leq m$, there are $A_{r+1} \subseteq B_{r+1}, \ldots, A_d \subseteq B_d$ such that each A_i is p_j-dense in T_i, and for every $\mathbf{a} \in J$, $\varphi_{V\mathbf{n}}[\mathbf{a}, A_{r+1}, \ldots, A_d]$.

We prove this by induction on j. It is obvious for $j = 0$. Now assume that $\mathbf{b} \notin J$ and $J \cup \{\mathbf{b}\} \subseteq \prod_{i=1}^r n_i(T_i, B_i)$ and the assertion is true for J. So $j < m$ and there are $A_{r+1} \subseteq B_{r+1}, \ldots, A_d \subseteq B_d$ such that each A_i is p_j-dense in T_i, and for every $\mathbf{a} \in J$, $\varphi_{V\mathbf{n}}[\mathbf{a}, A_{r+1}, \ldots, A_d]$. Now by (14) there exist $A'_{r+1} \subseteq A_{r+1}, \ldots, A'_d \subseteq A_d$ such that $A'_{r+1}, \ldots, A'_d$ are p_{j+1}-dense in $T_{r+1}, \ldots, T_d$ respectively, and $\varphi_{V\mathbf{k}^j}[\mathbf{b}, A'_{r+1}, \ldots, A'_d]$. By (10), $\varphi_{V\mathbf{n}}[\mathbf{c}, A'_{r+1}, \ldots, A'_d]$ for all $\mathbf{c} \in J \cup \{\mathbf{b}\}$. This proves (15).

This completes the proof of (8)

Now the proof of the theorem goes as follows. Let $W_0 = (\forall a_i)_1^d(\exists x_i)_1^d$, $W_1 = (\exists A_i)_1^d(\forall x_i)_1^d$. By (4), $W_0 \models_d W_1$. By (8), $\forall n \exists p \psi(W_0, n, p)$ implies $\forall n \exists p \psi(W_1, n, p)$.

Case 1. $\forall n \exists p \psi(W_0, n, p)$. Hence $\forall n \exists p \psi(W_1, n, p)$. For any $k \in \omega$ let $\mathbf{n}$ be constantly k. Then choose p so that $\psi(W_1, \mathbf{n}, p)$. Then there exist A_i for $i = 1, \ldots, d$ such that $A_i \subseteq B_I$ for all $i = 1, \ldots, d$, A_i is n_i-dense in T_i for all $i = 1, \ldots, d$, and for all $i = 1, \ldots, d$, $\forall x_i \in A_i$, $\varphi_{V\mathbf{n}}[A_1, \ldots, A_d, x_1, \ldots, x_d]$. Then $\forall i = 1, \ldots, d[A_i$ is k-dense in T_i and $(x_1, \ldots, x_d) \in Q$. Thus (i) in the theorem holds.

Case 2. There is an n such that for all p, $\neg\psi(W_0, n, p)$. Thus for every p there are $B_1, \ldots, B_d$ which are p-dense in their respective trees, and $a_i \in n_i(T_i, B_i)$ for $i = 1, \ldots d$ such that $\prod_{i=1}^d a_i \subseteq \prod_{i=1}^d T_i \backslash Q$. Let $h = \max\{n_i : 1 \leq i \leq d\}$. For any k, take $p = h + k$. Then $a_1, \ldots, a_d$ is an (m, k)-matrix contained in $\prod_{i=1}^d T_i \backslash Q$. $\square$

We now treat gaps in $[\omega]^\omega$. For $a, b \in [\omega]^\omega$ we define $a \subseteq^* b$ iff $a \backslash b$ is finite. It is convenient to use Boolean algebra terminology. The set $[\omega]^{<\omega}$ is an ideal in $\mathscr{P}(\omega)$, and the quotient $\mathscr{P}(\omega)/[\omega]^{<\omega}$ is a BA which we use. The equivalence class of $X \subseteq \omega$ under $[\omega]^{<\omega}$ is denoted by $[X]$. Note that $[a] \leq [b]$ iff $a \subseteq^* b$.

Proposition 24.54. *If $\mathscr{A}, \mathscr{B}$ are nonempty countable subsets of $[\omega]^\omega$ and $a \subseteq^* b$ whenever $a \in \mathscr{A}$ and $b \in \mathscr{B}$, then there is a $c \in [\omega]^\omega$ such that $a \subseteq^* c \subseteq^* b$ whenever $a \in \mathscr{A}$ and $b \in \mathscr{B}$.*

Proof. Write $\mathscr{A} = \{a_n : n \in \omega\}$ and $\mathscr{B} = \{b_n : n \in \omega\}$. Let

$$c = \bigcup_{n\in\omega}\left[\left(\bigcup_{m\le n} a_m\right) \cap \bigcap_{m\le n} b_m\right].$$

Now suppose that $p \in \omega$. Then

$$\begin{aligned}
a_p\backslash c &= \bigcap_{n\in\omega}\left[a_p \cap \left(\bigcap_{m\le n}(\omega\backslash a_m) \cup \bigcup_{m\le n}(\omega\backslash b_m)\right)\right]\\
&= \bigcap_{n<p}\left[a_p \cap \left(\bigcap_{m\le n}(\omega\backslash a_m) \cup \bigcup_{m\le n}(\omega\backslash b_m)\right)\right]\\
&\quad\cap \bigcap_{n\ge p}\left[a_p \cap \left(\bigcap_{m\le n}(\omega\backslash a_m) \cup \bigcup_{m\le n}(\omega\backslash b_m)\right)\right]\\
&\subseteq \bigcap_{n<p}\left[a_p \cap \left(\bigcap_{m\le n}(\omega\backslash a_m) \cup \bigcup_{m\le n}(\omega\backslash b_m)\right)\right]\\
&\quad\cap \bigcap_{n\ge p}\left[a_p \cap \bigcup_{m\le n}(\omega\backslash b_m)\right]\\
&\subseteq a_p \cap \bigcup_{m\le p}(\omega\backslash b_m),
\end{aligned}$$

and this last set is finite.

Furthermore,

$$\begin{aligned}
c\backslash b_p &= \bigcup_{n<p}\left[\left(\bigcup_{m\le n} a_m\right) \cap \bigcap_{m\le n} b_m \cap (\omega\backslash b_p)\right]\\
&\subseteq \left(\bigcup_{m<p} a_m\right)\backslash b_p,
\end{aligned}$$

and this last set is finite.

The set c is infinite, as otherwise $a_0 = (a_0 \cap c) \cup (a_0\backslash c)$ would be finite. $\square$

Proposition 24.55. *If $\mathscr{A}$ is an infinite countable collection of almost disjoint members of $[\omega]^\omega$, then $\mathscr{A}$, then $\mathscr{A}$ is not maximal.*

Proof. Say $\mathscr{A} = \{a_n : n \in \omega\}$. Note that $a_n \cap \bigcup_{m<n} a_m$ is finite, for any $n \in \omega$. Let $x_n \in (a_n\backslash \bigcup_{m<n} a_m)\backslash\{x_m : m < n\}$. Then $\{x_n : n \in \omega\} \in [\omega]^\omega$ and $a_n \cap \{x_n : n \in \omega\}$ is finite, for each $n \in \omega$. $\square$

Proposition 24.56. *Suppose that $\mathscr{A}$ is a nonempty countable family of members of $[\omega]^\omega$, and $\forall a, b \in \mathscr{A}[a \subseteq^* b$ or $b \subseteq^* a]$. Also suppose that $\forall a \in \mathscr{A}[a \subset^* d]$, where $d \in [\omega]^\omega$. Then there is a $c \in [\omega]^\omega$ such that $\forall a \in \mathscr{A}[a \subseteq^* c \subset^* d]$.*

Proof. If $\exists a \in \mathscr{A} \forall b \in \mathscr{A}[b \subseteq^* a]$, then the conclusion is obvious. So suppose that no such a exists. Then there is a sequence $\langle a_n : n \in \omega \rangle$ of elements of $\mathscr{A}$ such that $a_n \subset^* a_m$ for $n < m$, and the sequence is cofinal in $\mathscr{A}$ in the $\subseteq^*$-sense. In fact, let $\langle a'_n : n \in \omega \rangle$ be a list of all of the elements of $\mathscr{A}$. Let $a_0 = a'_0$. If a_n has been defined, then by hypothesis $a_n \subseteq^* a'_n$ or $a'_n \subseteq^* a_n$; choose $a_{n+1} \in \mathscr{A}$ such that $a_n, a'_n \subset^* a_{n+1}$. Let $\mathscr{C} = \{a_0\} \cup \{a_{m+1} \backslash a_m : m \in \omega\} \cup \{\omega \backslash d\}$. Then $\mathscr{C}$ is an almost disjoint family, except that possibly $\omega \backslash d$ is finite. By Proposition 24.53, let $e \subseteq \omega$ be infinite and almost disjoint from each member of $\mathscr{C}$. Let $c = d \backslash e$. Then for any $n \in \omega$,

$$\begin{aligned} a_{n+1} \backslash c &= (a_{n+1} \backslash d) \cup (a_{n+1} \cap e) \\ &\subseteq (a_{n+1} \backslash d) \cup \left[\bigcup_{i \le n} (a_{i+1} \backslash a_i) \cup a_0 \right] \cap e, \end{aligned}$$

and the last set is finite. Thus $a_{n+1} \subseteq^* c$, hence $b \subseteq^* c$ for all $b \in \mathscr{A}$.

Since $c \subseteq d$, we have $c \subseteq^* d$. Also, $d \backslash c = d \cap e$, and this is infinite since $e \backslash d$ is finite. Thus $c \subset^* d$

Note that c is infinite, since $a \subseteq^* c$ for all $a \in \mathscr{A}$.

Proposition 24.57. *If $a, b \in [\omega]^\omega$ and $a \subset^* b$, then there is a $c \in [\omega]^\omega$ such that $a \subset^* c \subset^* b$.*

Proof. Write $b \backslash a = d \cup e$ with d, e infinite and disjoint. Let $c = a \cup d$. $\square$

Proposition 24.58. *Suppose that $\mathscr{A}$ and $\mathscr{B}$ are nonempty countable subsets of $[\omega]^\omega$, $\forall x, y \in \mathscr{A}[x \subseteq^* y$ or $y \subseteq^* x]$, $\forall x, y \in \mathscr{B}[x \subseteq^* y$ or $y \subseteq^* x]$, and $\forall x \in \mathscr{A} \forall y \in \mathscr{B}[a \subset^* b]$. Then there is a $c \in [\omega]^\omega$ such that $a \subset^* c \subset^* b$ for all $a \in \mathscr{A}$ and $b \in \mathscr{B}$.*

Proof. By Proposition 24.52 choose $d \subseteq \omega$ such that $\forall a \in \mathscr{A} \forall b \in \mathscr{B}[a \subseteq^* d \subseteq^* b]$. Thus either $\forall a \in \mathscr{A}[a \subset^* d]$ or $\forall b \in \mathscr{B}[d \subset^* b]$.

Case 1. $\forall a \in \mathscr{A}[a \subset^* d]$. By Proposition 24.54 choose $e \subseteq \omega$ such that $\forall a \in \mathscr{A}[a \subseteq^* e \subset^* d]$. By Proposition 24.55 choose $c \in [\omega]^\omega$ such that $e \subset^* c \subset^* d$.

Case 2. $\forall b \in \mathscr{B}[d \subset^* b]$. Then $\forall b \in \mathscr{B}[(\omega \backslash b) \subset^* (\omega \backslash d)]$. By Proposition 24.54 choose $e \subseteq \omega$ such that $\forall b \in \mathscr{B}[(\omega \backslash b) \subseteq^* e \subset^* (\omega \backslash d)]$. By Proposition 24.55 choose $c \subseteq \omega$ such that $e \subset^* c \subset^* (\omega \backslash d)$. Then $\forall a \in \mathscr{A} \forall b \in \mathscr{B}[a \subset^* (\omega \backslash c) \subset^* b]$. $\square$

Now we need some more terminology. Let $\mathscr{A} \subseteq [\omega]^\omega$, $b \in [\omega]^\omega$, and $\forall a \in \mathscr{A}[a \subset^* b]$. We say that b is *near* to $\mathscr{A}$ iff for all $m \in \omega$ the set $\{a \in \mathscr{A} : a \backslash b \subseteq m\}$ is finite.

Proposition 24.60. *Suppose that $a_m \in [\omega]^\omega$ for all $m \in \omega$, $a_m \subset^* a_n$ whenever $m < n \in \omega$, $b \in [\omega]^\omega$, and $a_m \subset^* b$ for all $m \in \omega$. Then there is a $c \in [\omega]^\omega$ such that $\forall m \in \omega[a_m \subset^* c \subset^* b]$ and c is near to $\{a_n : n \in \omega\}$.*

Proof. By Proposition 24.55 choose $d \subseteq \omega$ such that $\forall m \in \omega[a_n \subset^* d \subset^* b]$. Now for each $m \in \omega$, $\bigcup_{i \leq m}(a_i \backslash a_m)$ is finite, and $a_{m+1} \backslash \bigcup_{i \leq m} a_i = (a_{m+1} \backslash a_m) \backslash \bigcup_{i \leq m}(a_i \backslash a_m)$, so $a_{m+1} \backslash \bigcup_{i \leq m} a_i$ is infinite. Choose $e_m \subseteq a_{m+1} \backslash \bigcup_{i \leq m} a_i$ such that $|e_m| = m$. Let $c = d \backslash \bigcup_{m \in \omega} e_m$. Thus $c \subseteq^* d \subset^* b$.

If $n \in \omega$, then

$$a_n \backslash c = (a_n \backslash d) \cup \bigcup_{m \in \omega} (a_n \cap e_m) = (a_n \backslash d) \cup \bigcup_{m < n} (a_n \cap e_m),$$

and this last set is finite. Hence $a_n \subseteq^* c$. Since n is arbitrary, it follows that $a_n \subset^* c$ for all $n \in \omega$.

Also for any $m \in \omega$ we have $a_{m+1} \backslash c \supseteq a_{m+1} \cap e_m = e_m$, and so $|a_{m+1} \backslash c| \geq m$. So if $p \in \omega$ then $|a_{p+1} \backslash c| \geq p$ and so $\{a_m : a_m \backslash c \subseteq n\} \subseteq \{a_0, \ldots, a_n\}$. So c is near to $\{a_m : m \in \omega\}$. $\square$

Proposition 24.61. *Suppose that $\mathscr{A} \subseteq [\omega]^\omega$, $\forall x, y \in \mathscr{A}[x \subset^* y$ or $y \subset^* x]$, $b \in [\omega]^\omega$, $\forall x \in \mathscr{A}[x \subset^* b]$, and $\forall a \in \mathscr{A}[b$ is near to $\{d \in \mathscr{A} : d \subset^* a\}]$.*

Then there is a $c \in [\omega]^\omega$ such that $\forall a \in \mathscr{A}[a \subset^* c \subset^* b]$ and c is near to $\mathscr{A}$.

Proof. We consider several cases.

Case 1. $\exists a \in \mathscr{A} \forall d \in \mathscr{A}[d \subseteq^* a]$. By Proposition 24.55, choose c such that $a \subset^* c \subset^* b$. Choose $n \in \omega$ such that $c \backslash b \subseteq n$. Then for any $m \in \omega$ and any $d \in \mathscr{A}$, if $d \backslash c \subseteq m$ then $d \backslash b \subseteq (d \backslash c) \cup (c \backslash b) \subseteq \max(m, n)$. Hence

$$\{d \in \mathscr{A} : d \backslash c \subseteq m\} \subseteq \{a\} \cup \{d \in \mathscr{A} : d \subset^* a \text{ and } d \backslash b \subseteq \max(m, n)\},$$

and the later set is finite, since b is near to $\{d \in \mathscr{A} : d \subset^* a\}$. Thus c is as desired.

Case 2. $\forall a \in \mathscr{A} \exists d \in \mathscr{A}[a \subset^* d]$ and b is near to $\mathscr{A}$. By Proposition 24.55 choose c so that $\forall a \in \mathscr{A}[a \subset^* c \subset^* b]$. Choose $n \in \omega$ such that $c \backslash b \subseteq n$. Then for any $m \in \omega$ and any $d \in \mathscr{A}$, if $d \backslash c \subseteq m$ then $d \backslash b \subseteq (d \backslash c) \cup (c \backslash b) \subseteq \max(m, n)$. Hence

$$\{d \in \mathscr{A} : d \backslash c \subseteq m\} \subseteq \{a\} \cup \{d \in \mathscr{A} : d \backslash b \subseteq \max(m, n)\},$$

and the later set is finite, since b is near to $\mathscr{A}$. Thus c is as desired.

Case 3. $\forall a \in \mathscr{A} \exists d \in \mathscr{A}[a \subset^* d]$ and b is not near to $\mathscr{A}$. For each $m \in \omega$ let $\mathscr{B}_m = \{a \in \mathscr{A} : a \backslash b \subseteq m\}$. Since b is not near to $\mathscr{A}$, choose m so that $\mathscr{B}_m$ is infinite. Note that $p < q \to \mathscr{B}_p \subseteq \mathscr{B}_q$. Hence $\mathscr{B}_n$ is infinite for every $n \geq m$. Now we claim

$$(1) \qquad \forall n \geq m \forall a \in \mathscr{A} \exists d \in \mathscr{B}_n[a \subseteq^* d].$$

In fact, otherwise we get $n \geq m$ and $a \in \mathscr{A}$ such that $\forall d \in \mathscr{B}_n[d \subset^* a]$. Now b is near to $\{d \in \mathscr{A} : d \subset^* a\}$ by a hypothesis of the lemma, so $\{d \in \mathscr{A} : d \subset^* a$ and $d \backslash b \subseteq n\}$ is finite. But $\mathscr{B}_n \subseteq \{d \in \mathscr{A} : d \subset^* a$ and $d \backslash b \subseteq n\}$, contradiction. So (1) holds.

Next we claim

$$(2) \qquad \forall n \geq m \forall d \in \mathscr{B}_n[\{e \in \mathscr{B}_n : e \subset^* d\} \text{ is finite}].$$

In fact, suppose that $n \geq m$, $d \in \mathscr{B}_n$ and $\{e \in \mathscr{B}_n : e \subset^* d\}$ is infinite. Since b is near to $\{a \in \mathscr{A} : a \subset^* d\}$, the set $\{a \in \mathscr{A} : a \subset^* d \text{ and } a \backslash b \subseteq n\}$ is finite. But $\{e \in \mathscr{B}_n : e \subset^* d\} \subseteq \{a \in \mathscr{A} : a \subset^* d \text{ and } a \backslash b \subseteq n\}$, contradiction. So (2) holds.

From (2) it follows that $\mathscr{B}_n$ has order type ω under $\subset^*$, for each $n \geq m$. Now clearly $\mathscr{A} = \bigcup_{p \in \omega} \mathscr{B}_p$, so $\mathscr{A}$ is countable.

Now by Proposition 24.59, choose c_m such that $\forall d \in \mathscr{B}_m[d \subset^* c_m \subset^* b]$ and c_m is near to $\mathscr{B}_m$. By (1), $a \subset^* c_m$ for each $a \in \mathscr{A}$. Now suppose that $n \geq m$ and c_n has been defined so that $a \subset^* c_n$ for each $a \in \mathscr{A}$. Again by Proposition 24.57 choose c_{n+1} such that $\forall d \in \mathscr{B}_{n+1}[d \subset^* c_{n+1} \subset^* c_n]$ and c_{n+1} is near to $\mathscr{B}_{n+1}$. Thus we have

$$\forall a \in \mathscr{A}[a \subset^* \cdots \subset^* c_{n+1} \subset^* c_n \subset^* \cdots \subset^* c_m \subset^* b].$$

By Proposition 24.55, choose d so that $\forall a \in \mathscr{A} \forall n \geq m[a \subset^* d \subset^* c_n]$. We claim that d is near to $\mathscr{A}$, completing the proof. For, let $n \in \omega$. Let $p = \max(m, n)$, and choose $q \geq p$ such that $d \backslash c_p \subseteq q$. Then

$$\begin{aligned}\{a \in \mathscr{A} : a \backslash d \subseteq n\} &\subseteq \{a \in \mathscr{A} : a \backslash d \subseteq p\}\\ &= \{a \in \mathscr{B}_p : a \backslash d \subseteq p\}\\ &\subseteq \{a \in \mathscr{B}_p : a \backslash c_p \subseteq q\},\end{aligned}$$

where the last inclusion holds since $a \backslash c_p = (a \backslash d) \cup (d \backslash c_p)$. The last set is finite since c_p is near to $\mathscr{B}_p$, as desired. □

Proposition 24.62. *(The Hausdorff gap) There exist sequences $\langle a_\alpha : \alpha < \omega_1 \rangle$ and $\langle b_\alpha : \alpha < \omega_1 \rangle$ of members of $[\omega]^\omega$ such that $\forall \alpha, \beta < \omega_1[\alpha < \beta \to a_\alpha \subset^* a_\beta$ and $b_\beta \subset^* b_\alpha]$, $\forall \alpha, \beta < \omega_1[a_\alpha \subset^* b_\beta]$, and there does not exist a $c \subseteq \omega$ such that $\forall \alpha < \omega_1[a_\alpha \subset^* c$ and $c \subset^* b_\alpha]$.*

Proof. We construct by recursion $a_\alpha, b_\alpha \subseteq \omega$ for $\alpha < \omega_1$ so that $a_\alpha \subset^* b_\alpha$, $\alpha < \beta \to a_\alpha \subset^* a_\beta$ and $b_\beta \subset^* b_\alpha$, and for all $\alpha < \omega_1$, b_β is near to $\{a_\alpha : \alpha < \beta\}$.

Let $a_0 = \emptyset$, $b_0 = \omega$. Suppose that a_α and b_α have been constructed for all $\alpha < \beta$ so that $a_\alpha \subset^* b_\alpha$, $\alpha < \gamma < \beta \to a_\alpha \subset^* a_\gamma$ and $b_\gamma \subset^* b_\beta$, and $\alpha < \beta \to b_\alpha$ is near to $\{a_\gamma : \gamma < \alpha\}$. By Proposition 24.55 choose c such that $\forall \alpha < \beta[a_\alpha \subset^* c \subset^* b_\alpha]$. Suppose that $\alpha < \beta$. We claim that c is near to $\{a_\gamma : \gamma < \alpha\}$. In fact, suppose that $m \in \omega$. Choose $n \geq m$ such that $c \backslash b_\alpha \subseteq n$. Now for any $\gamma < \alpha$ we have $a_\gamma \backslash b_\alpha \subseteq (a_\gamma \backslash c) \cup (c \backslash b_\alpha)$, so

$$\{a_\gamma : \gamma < \alpha \text{ and } a_\gamma \backslash c \subseteq m\} \subseteq \{a_\gamma : \gamma < \alpha \text{ and } a_\gamma \backslash b_\alpha \subseteq n\},$$

and the latter set is finite since b_α is near to $\{a_\gamma : \gamma < \alpha\}$. Thus indeed c is near to $\{a_\gamma : \gamma < \alpha\}$. Now by Proposition 24.58 there is a b_β such that $\forall \alpha < \beta[a_\alpha \subset^* b_\beta \subset^* c]$ and b_β is near to $\{a_a : \alpha < \beta\}$. By Proposition 24.55 choose a_β so that $\forall \alpha < \beta[a_\alpha \subset^* a_\beta \subset^* b_\beta]$. This finishes the construction.

Now suppose that $d \subseteq \omega$ and $\forall \alpha < \omega_1[a_\alpha \subset^* d \subset^* b_\alpha]$. Now $\omega_1 = \bigcup_{m \in \omega}\{\alpha < \omega_1 : a_\alpha \backslash d \subseteq m\}$, so we can choose $m \in \omega$ such that $|\{\alpha < \omega_1 : a_\alpha \backslash d \subseteq m\}| = \omega_1$. Hence there is an $\alpha < \omega_1$ such that $\{\beta < \alpha : a_\beta \backslash d \subseteq m\}$ is infinite. Choose $p \geq m$ such that $d \backslash b_\alpha \subseteq p$. Now $a_\beta \backslash b_\alpha \subseteq (a_\beta \backslash d) \cup (d \backslash b_\alpha)$, so $\{\beta < \alpha : a_\beta \backslash d \subseteq m\} \subseteq \{\beta < \alpha : a_\beta \backslash b_\alpha \subseteq p\}$, contradicting b_α near to $\{a_\beta : \beta < \alpha\}$. □

25. Martin's axiom

Martin's axiom is not an axiom of ZFC, but it can be added to those axioms. It has many important consequences. Actually, the continuum hypothesis implies Martin's axiom, so it is of most interest when combined with the negation of the continuum hypothesis. The consistency of $\mathrm{MA}+\neg\mathrm{CH}$ involves iterated forcing, and is prove much later in these notes.

• A *forcing poset* is a triple $(P, \leq, 1)$ such that $\leq$ is reflexive on P and transitive, and $\forall p \in P[p \leq 1]$.

• If P is a forcing poset, a subset D of P is *dense* iff $\forall p \in P \exists q \in D[q \leq p]$.

• If P is a forcing poset, a subset G of P is a *filter* iff $1 \in G$, $\forall p, q \in P[p \in G$ and $p \leq q$ imply that $q \in G]$, and $\forall p, q \in G \exists r \in G[r \leq p, q]$.

• If P is a forcing poset, elements $p, q \in P$ are *compatible* iff there is an $r \in P$ such that $r \leq p, q$. $p \perp q$ abbreviates that p and q are incompatible.

• An *antichain* in a forcing poset P is a collection of pairwise incompatible elements of P. Note that this notion is apparently different from the notion of an antichain in a linear order (Page 340) and an antichain in a tree (Page 370).

• P has the *countable chain condition*, ccc, iff every subset of pairwise incompatible elements is countable

• For any infinite cardinal κ, the notation $\mathrm{MA}(\kappa)$ abbreviates the statement that for any ccc forcing poset $\mathbb{P}$ and any family $\mathscr{D}$ of dense sets in $\mathbb{P}$, with $|\mathscr{D}| \leq \kappa$, there is a filter G on $\mathbb{P}$ such that $G \cap D \neq \emptyset$ for every $D \in \mathscr{D}$.

• *Martin's axiom*, abbreviated MA, is the statement that $\mathrm{MA}(\kappa)$ holds for every infinite $\kappa < 2^\omega$.

Clearly if $\kappa < \lambda$ and $\mathrm{MA}(\lambda)$, then also $\mathrm{MA}(\kappa)$.

Theorem 25.1. $\mathrm{MA}(\omega)$ *holds.*

Proof. Let $\mathbb{P}$ be a ccc forcing poset and $\mathscr{D}$ a countable collection of dense sets in $\mathbb{P}$. If $\mathscr{D}$ is empty, we can fix any $p \in P$ and let $G = \{q \in \mathbb{P} : p \leq q\}$. Then G is a filter on $\mathbb{P}$, which is all that is required in this case.

Now suppose that $\mathscr{D}$ is nonempty, and let $\langle D_n : n \in \omega\rangle$ enumerate all the members of $\mathscr{D}$; repetitions are needed if $\mathscr{D}$ is finite. We now define a sequence $\langle p_n : n \in \omega\rangle$ of elements of P by recursion. Let p_0 be any element of P. If p_n has been defined, by the denseness of D_n let p_{n+1} be such that $p_{n+1} \leq p_n$ and $p_{n+1} \in D_n$. This finishes the construction. Let $G = \{q \in P : p_n \leq q$ for some $n \in \omega\}$. Clearly G is as desired. $\square$

Note that ccc was not used in this proof.

Corollary 25.2. CH *implies* MA. $\square$

Theorem 25.3. (III.3.13) $\mathrm{MA}(2^\omega)$ *does not hold.*

Proof. Suppose that it does hold. Let

$$\begin{aligned}
P &= \{f : f \text{ is a finite function with } \mathrm{dmn}(f) \subseteq \omega \text{ and } \mathrm{rng}(f) \subseteq 2\};\\
f \leq g &\text{ iff } f, g \in P \text{ and } f \supseteq g;\\
\mathbb{P} &= (P, \leq).
\end{aligned}$$

Then $\mathbb{P}$ has ccc, since P itself is countable. Now for each $n \in \omega$ let

$$D_n = \{f \in P : n \in \mathrm{dmn}(f)\}.$$

Each such set is dense in $\mathbb{P}$. For, if $g \in P$, either g is already in D_n, or $n \notin \mathrm{dmn}(g)$, and then $g \cup \{(n, 0)\}$ is in D_n and it is $\leq g$.

For each $h \in {}^{\omega}2$ let

$$E_h = \{f \in P : \text{there is an } n \in \mathrm{dmn}(f) \text{ such that } f(n) \neq h(n)\}.$$

Again, each such set E_h is dense in $\mathbb{P}$. For, let $f \in P$. If $f \not\subseteq h$, then already $f \in E_h$, so suppose that $f \subseteq h$. Take any $n \in \omega \backslash \mathrm{dmn}(f)$, and let $g = f \cup \{(n, 1 - h(n))\}$. Then $g \in E_h$ and $g \leq f$, as desired.

So, by $\mathrm{MA}(2^{\omega})$ let G be a filter on $\mathbb{P}$ which intersects each of the sets D_n and E_h. Let $k = \bigcup G$.

(*) $k : \omega \to \omega$.

In fact, k is obviously a relation. Suppose that $(m, \varepsilon), (m, \delta) \in k$. Choose $f, g \in G$ such that $(m, \varepsilon) \in f$ and $(m, \delta) \in g$. Then choose $s \in G$ such that $s \leq f, g$. So $f, g \subseteq s$, and s is a function. It follows that $\varepsilon = \delta$. Thus k is a function.

If $n \in \omega$, choose $f \in G \cap D_n$. So $n \in \mathrm{dmn}(f)$, and so $n \in \mathrm{dmn}(k)$. So we have proved $(*)$.

Now take any $f \in G \cap E_k$. Choose $n \in \mathrm{dmn}(f)$ such that $f(n) \neq k(n)$. But $f \subseteq k$, contradiction. $\square$

There is one more fact concerning the definition of MA which should be mentioned. Namely, for $\kappa > \omega$ the assumption of ccc is essential in the statement of $\mathrm{MA}(\kappa)$. (Recall our comment above that ccc is not needed in order to prove that $\mathrm{MA}(\omega)$ holds.) To see this, define

$$\begin{aligned}
P &= \{f : f \text{ is a finite function, } \mathrm{dmn}(f) \subseteq \omega, \text{ and } \mathrm{rng}(f) \subseteq \omega_1\};\\
f \leq g &\text{ iff } f, g \in P \text{ and } f \supseteq g;\\
\mathbb{P} &= (P, \leq).
\end{aligned}$$

This example is similar to two of the forcing posets above. Note that $\mathbb{P}$ does not have ccc, since for example $\{\{(0, \alpha)\} : \alpha < \omega_1\}$ is an uncountable antichain. Defining D_n as in the proof of Theorem 25.3, we clearly get dense subsets of $\mathbb{P}$. Also, for each $\alpha < \omega_1$ let

$$F_\alpha = \{f \in P : \alpha \in \mathrm{rng}(f)\}.$$

Then F_α is dense in $\mathbb{P}$. For, suppose that $g \in P$. If $\alpha \in \text{rng}(g)$, then g itself is in F_α, so suppose that $\alpha \notin \text{rng}(g)$. Choose $n \in \omega \backslash \text{dmn}(g)$. Let $f = g \cup \{(n, \alpha)\}$. Then $f \in F_\alpha$ and $f \leq g$, as desired. Now if $\text{MA}(\omega_1)$ holds without the assumption of ccc, then we can apply it to our present forcing poset. Suppose that G is a filter on $\mathbb{P}$ which intersects each of these sets D_n and F_α. As in the proof of Theorem 25.3, $k \stackrel{\text{def}}{=} \bigcup G$ is a function mapping ω into ω_1. For any $\alpha < \omega_1$ choose $f \in G \cap F_\alpha$. Thus $\alpha \in \text{rng}(f)$, and so $\alpha \in \text{rng}(k)$. Thus k has range ω_1. This is impossible.

Now we proceed beyond the discussion of the definition of MA in order to give several typical applications of it. First we consider again almost disjoint sets of natural numbers. Our result here will be used to derive some important implications of MA for cardinal arithmetic. We proved in Theorem 24.1 that there is a family of size 2^ω of almost disjoint sets of natural numbers. Considering this further, we may ask what the size of maximal almost disjoint families can be; and we may consider the least such size. This is one of many min-max questions concerning the natural numbers which have been considered recently. There are many consistency results saying that numbers of this sort can be less than 2^ω; in particular, it is consistent that there is a maximal family of almost disjoint subsets of ω which has size less than 2^ω. MA, however, implies that this size, and most of the similarly defined min-max functions, is 2^ω.

Let $\mathscr{A} \subseteq \mathscr{P}(\omega)$. The *almost disjoint partial order for* $\mathscr{A}$ is defined as follows:

$$P_{\mathscr{A}} = \{(s, F) : s \in [\omega]^{<\omega} \text{ and } F \in [\mathscr{A}]^{<\omega}\};$$
$$(s', F') \leq (s, F) \text{ iff } s \subseteq s',\ F \subseteq F', \text{ and } x \cap s' \subseteq s \text{ for all } x \in F;$$
$$\mathbb{P}_{\mathscr{A}} = (P_{\mathscr{A}}, \leq).$$

We give some useful properties of this construction.

Lemma 25.4. *Let* $\mathscr{A} \subseteq \mathscr{P}(\omega)$.

(i) $\mathbb{P}_{\mathscr{A}}$ *is a forcing poset.*

(ii) Let $(s, F), (s', F') \in P_{\mathscr{A}}$. *Then the following conditions are equivalent:*

(a) (s, F) *and* (s', F') *are compatible.*

(b) $\forall x \in F(x \cap s' \subseteq s)$ *and* $\forall x \in F'(x \cap s \subseteq s')$.

(c) $(s \cup s', F \cup F') \leq (s, F), (s', F')$.

(iii) Suppose that $x \in \mathscr{A}$, *and let* $D_x = \{(s, F) \in P_{\mathscr{A}} : x \in F\}$. *Then* D_x *is dense in* $\mathbb{P}_{\mathscr{A}}$.

(iv) $\mathbb{P}_{\mathscr{A}}$ *has ccc.*

Proof. (i): Clearly $\leq$ is reflexive on $P_{\mathscr{A}}$ and it is antisymmetric, i.e. $(s, F) \leq (s', F') \leq (s, F)$ implies that $(s, F) = (s', F')$. Now suppose that $(s'', F'') \leq (s', F') \leq (s, F)$. Thus $s \subseteq s' \subseteq s''$, so $s \subseteq s''$. Similarly, $F \subseteq F''$. Now take any $x \in F$. Then $x \in F'$, so $x \cap s'' \subseteq s'$ because $(s'', F'') \leq (s', F')$. Hence $x \cap s'' \subseteq x \cap s'$. And $x \cap s' \subseteq s$ because $(s', F') \leq (s, F)$. So $x \cap s'' \subseteq s$, as desired.

(ii): For (a)$\Rightarrow$(b), assume (a). Choose $(s'', F'') \leq (s, F), (s', F')$. Now take any $x \in F$. Then $x \cap s' \subseteq x \cap s''$ since $s' \subseteq s''$, and $x \cap s'' \subseteq s$ since $(s'', F'') \leq (s, F)$; so $x \cap s' \subseteq s''$. The other part of (b) follows by symmetry.

(b)⇒(c): By symmetry it suffices to show that $(s \cup s', F \cup F') \leq (s, F)$, and for this we only need to check the last condition in the definition of $\leq$. So, suppose that $x \in F$. Then $x \cap (s \cup s') = (x \cap s) \cup (x \cap s') \subseteq s$ by (b).

(c)⇒(a): Obvious.

(iii): For any $(s, F) \in P_{\mathscr{A}}$, clearly $(s, F \cup \{x\}) \leq (s, F)$.

(iv) Suppose that $\langle (s_\xi, F_\xi) : \xi < \omega_1 \rangle$ is a pairwise incompatible system of elements of $P_{\mathscr{A}}$. Clearly then $s_\xi \neq s_\eta$ for distinct $\xi, \eta < \omega_1$, contradiction. □

Theorem 25.5. *Let κ be an infinite cardinal, and assume* MA(κ). *Suppose that $\mathscr{A}, \mathscr{B} \subseteq \mathscr{P}(\omega)$, and $|\mathscr{A}|, |\mathscr{B}| \leq \kappa$. Also assume that*

(i) For all $y \in \mathscr{B}$ and all $F \in [\mathscr{A}]^{<\omega}$ we have $|y \backslash \bigcup F| = \omega$.

Then there is a $d \subseteq \omega$ such that $|d \cap x| < \omega$ for all $x \in \mathscr{A}$ and $|d \cap y| = \omega$ for all $y \in \mathscr{B}$.

Proof. For each $y \in \mathscr{B}$ and each $n \in \omega$ let

$$E_n^y = \{(s, F) \in P_{\mathscr{A}} : s \cap y \not\subseteq n\}.$$

We claim that each such set is dense. For, suppose that $(s, F) \in \mathbb{P}_{\mathscr{A}}$. Then by assumption, $|y \backslash \bigcup F| = \omega$, so we can pick $m \in y \backslash \bigcup F$ such that $m > n$. Then $(s \cup \{m\}, F) \leq (s, F)$, since for each $z \in F$ we have $z \cap (s \cup \{m\}) \subseteq s$ because $m \notin z$. Also, $m \in y \backslash n$, so $(s \cup \{m\}, F) \in E_n^y$. This proves our claim.

There are clearly at most κ sets E_n^y; and also there are at most κ sets D_x with $x \in \mathscr{A}$, with D_x as in Lemma 25.4(iii). Hence by MA(κ) we can let G be a filter on $\mathbb{P}_{\mathscr{A}}$ intersecting all of these dense sets. Let $d = \bigcup_{(s,F) \in G} s$.

(1) For all $x \in \mathscr{A}$, the set $d \cap x$ is finite.

For, by the denseness of D_x, choose $(s, F) \in G \cap D_x$. Thus $x \in F$. We claim that $d \cap x \subseteq s$. To prove this, suppose that $n \in d \cap x$. Choose $(s', F') \in G$ such that $n \in s'$. Now (s, F) and (s', F') are compatible. By Lemma 25.4(ii), $\forall y \in F (y \cap s' \subseteq s)$; in particular, $x \cap s' \subseteq s$. Since $n \in x \cap s'$, we get $n \in s$. This proves our claim, and so (1) holds.

The proof will be finished by proving

(2) For all $y \in \mathscr{B}$, the set $d \cap y$ is infinite.

To prove (2), given $n \in \omega$ choose $(s, F) \in E_n^y \cap G$. Thus $s \cap y \not\subseteq n$, so we can choose $m \in s \cap y \backslash n$. Hence $m \in d \cap y \backslash n$, proving (2). □

Corollary 25.6. *Let κ be an infinite cardinal and assume* MA(κ). *Suppose that $\mathscr{A} \subseteq \mathscr{P}(\omega)$ is an almost disjoint set of infinite subsets of ω of size κ. Then $\mathscr{A}$ is not maximal.*

Proof. If F is a finite subset of $\mathscr{A}$, then we can choose $a \in \mathscr{A} \backslash F$; then $a \cap \bigcup F = \bigcap_{b \in F} (a \cap b)$ is finite. Thus $\omega \backslash \bigcup F$ is infinite. Hence we can apply Theorem 25.5 to $\mathscr{A}$ and $\mathscr{B} \stackrel{\text{def}}{=} \{\omega\}$ to obtain the desired result. □

Corollary 25.7. *Assuming MA, every maximal almost disjoint set of infinite sets of natural numbers has size 2^ω.* □

Lemma 25.8. *Suppose that $\mathscr{B} \subseteq \mathscr{P}(\omega)$ is an almost disjoint family of infinite sets, and $|\mathscr{B}| = \kappa$, where $\omega \le \kappa < 2^\omega$. Also suppose that $\mathscr{A} \subseteq \mathscr{B}$. Assume* $\mathrm{MA}(\kappa)$.

Then there is a $d \subseteq \omega$ such that $|d \cap x| < \omega$ for all $x \in \mathscr{A}$ and $|d \cap x| = \omega$ for all $x \in \mathscr{B} \backslash \mathscr{A}$.

Proof. We apply 25.5 with $\mathscr{B} \backslash \mathscr{A}$ in place of $\mathscr{B}$. If $y \in \mathscr{B} \backslash \mathscr{A}$ and $F \in [\mathscr{A}]^{<\omega}$, then $y \cup F \subseteq \mathscr{B}$, and hence $y \cap z$ is finite for all $y \in F$. Hence also $y \cap \bigcup F$ is finite. Since y itself is infinite, it follows that $y \backslash \bigcup F$ is infinite.

Thus the hypotheses of 25.5 hold, and it then gives the desired result. $\square$

We now come to two of the most striking consequences of Martin's axiom.

Theorem 25.9. *If κ is an infinite cardinal and* $\mathrm{MA}(\kappa)$ *holds, then $2^\kappa = 2^\omega$.*

Proof. By Theorem 24.1 let $\mathscr{B}$ be an almost disjoint family of infinite subsets of ω such that $|\mathscr{B}| = \kappa$. For each $d \subseteq \omega$ let $F(d) = \{b \in \mathscr{B} : |b \cap d| < \omega\}$. We claim that F maps $\mathscr{P}(\omega)$ onto $\mathscr{P}(\mathscr{B})$; from this it follows that $2^\kappa \le 2^\omega$, hence $2^\kappa = 2^\omega$. To prove the claim, suppose that $\mathscr{A} \subseteq \mathscr{B}$. A suitable d with $F(d) = \mathscr{A}$ is then given by Lemma 25.8. $\square$

Corollary 25.10. MA *implies that 2^ω is regular.*

Proof. Assume MA, and suppose that $\omega \le \kappa < 2^\omega$. Then $2^\kappa = 2^\omega$ by Theorem 25.9, and so $\mathrm{cf}(2^\omega) = \mathrm{cf}(2^\kappa) > \kappa$ by Corollary 11.55. $\square$

Another important application of Martin's axiom is to the existence of Suslin trees; in fact, Martin's axiom arose out of the proof of this theorem:

Theorem 25.11. $\mathrm{MA}(\omega_1)$ *implies that there are no Suslin trees.*

Proof. Suppose that $(T, \le)$ is a Suslin tree. By Theorem 22.7 we may assume that T is well-pruned. We are going to apply $\mathrm{MA}(\omega_1)$ to the forcing poset $(T, \ge)$, i.e., to T turned upside down. Because T has no uncountable antichains in the tree sense, $(T, \ge)$ has no uncountable antichains in the incompatibility sense. Now for each $\alpha < \omega_1$ let

$$D_\alpha = \{t \in T : \mathrm{ht}(t, T) > \alpha\}.$$

Then each D_α is dense in $(T, \ge)$. For, suppose that $s \in T$. By well-prunedness, choose $t \in T$ such that $s < t$ and $\mathrm{ht}(t, T) > \alpha$. Thus $t \in D_\alpha$ and $t > s$, as desired.

Now we let G be a filter on $(T, \ge)$ which intersects each D_α. Any two elements of G are compatible in $(T, \ge)$, so they are comparable in $(T, \le)$. Since $G \cap D_\alpha \ne \emptyset$ for all $\alpha < \omega_1$, G has a member of T of height greater than α, for each $\alpha < \omega_1$. Hence G is an uncountable chain, contradiction. $\square$

Our next application of Martin's axiom involves Lebesgue measure.

Theorem 25.12. *Suppose that κ is an infinite cardinal and* $\mathrm{MA}(\kappa)$ *holds. If $\langle M_\alpha : \alpha < \kappa \rangle$ is a system of subsets of $\mathbb{R}$ each of Lebesgue measure 0, then also $\bigcup_{\alpha<\kappa} M_\alpha$ has Lebesgue measure 0.*

Proof. Let $\varepsilon > 0$. We are going to find an open set U such that $\bigcup_{\alpha<\kappa} M_\alpha \subseteq U$ and $\mu(U) \leq \varepsilon$; this will prove our result. Let

$$\mathbb{P} = \{p \subseteq \mathbb{R} : p \text{ is open and } \mu(p) < \varepsilon\}.$$

The ordering, as usual, is $\supseteq$.

(1) Elements $p, q \in \mathbb{P}$ are compatible iff $\mu(p \cup q) < \varepsilon$.

In fact, the direction $\Leftarrow$ is clear, while if p and q are compatible, then there is an $r \in \mathbb{P}$ with $r \supseteq p, q$, hence $p \cup q \subseteq r$ and $\mu(r) < \varepsilon$, hence $\mu(p \cup q) < \varepsilon$.

Next we check that $\mathbb{P}$ has ccc. Suppose that $\langle p_\alpha : \alpha < \omega_1 \rangle$ is a system of pairwise incompatible elements of $\mathbb{P}$. Now

$$\omega_1 = \bigcup_{n\in\omega} \left\{\alpha < \omega_1 : \mu(p_\alpha) \leq \varepsilon - \frac{1}{n+1}\right\},$$

so there exist an uncountable $\Gamma \subseteq \omega_1$ and a positive integer m such that $\mu(p_\alpha) \leq \varepsilon - \frac{1}{m}$ for all $\alpha \in \Gamma$. Let $\mathscr{C}$ be the collection of all finite unions of open intervals with rational coefficients. Note that $\mathscr{C}$ is countable. By Lemma 18.99 for each $\alpha \in \Gamma$ let C_α be a member of $\mathscr{C}$ such that $\mu(p_\alpha \triangle C_\alpha) \leq \frac{1}{3m}$. Now take any two distinct members $\alpha, \beta \in \Gamma$. Then

$$\varepsilon \leq \mu(p_\alpha \cup p_\beta) = \mu(p_\alpha \cap p_\beta) + \mu(p_\alpha \triangle p_\beta) \leq \varepsilon - \frac{1}{m} + \mu(p_\alpha \triangle p_\beta),$$

and hence $\mu(p_\alpha \triangle p_\beta) \geq \frac{1}{m}$. Thus, using Lemma 18.95,

$$\frac{1}{m} \leq \mu(p_\alpha \triangle p_\beta) \leq \mu(p_\alpha \triangle C_\alpha) + \mu(C_\alpha \triangle C_\beta) + \mu(C_\beta \triangle p_\beta) \leq \frac{1}{3m} + \mu(C_\alpha \triangle C_\beta) + \frac{1}{3m};$$

Hence $\mu(C_\alpha \triangle C_\beta) \geq \frac{1}{3m}$. It follows that $C_\alpha \neq C_\beta$. But this means that $\langle C_\alpha : \alpha \in \Gamma \rangle$ is a one-one system of members of $\mathscr{C}$, contradiction. So $\mathbb{P}$ has ccc.

Now for each $\alpha < \kappa$ let

$$D_\alpha = \{p \in \mathbb{P} : M_\alpha \subseteq p\}.$$

To show that D_α is dense, take any $p \in \mathbb{P}$. Thus $\mu(p) < \varepsilon$. By Lemma 18.97(i), let U be an open set such that $M_\alpha \subseteq U$ and $\mu(U) < \varepsilon - \mu(p)$. Then $\mu(p \cup U) \leq \mu(p) + \mu(U) < \varepsilon$; so $p \cup U \in D_\alpha$ and $p \cup U \supset p$, as desired.

Now let G be a filter on $\mathbb{P}$ which intersects each D_α. Set $V = \bigcup G$. So V is an open set. For each $\alpha < \kappa$, choose $p_\alpha \in G \cap D_\alpha$. Then $M_\alpha \subseteq p_\alpha \subseteq V$. It remains only to show that $\mu(V) \leq \varepsilon$. Let $\mathscr{B}$ be the set of all open intervals with rational endpoints. We claim that $V = \bigcup(G \cap \mathscr{B})$. In fact, $\supseteq$ is clear, so suppose that $x \in V$. Then $x \in p$ for some $p \in G$, hence there is a $U \in \mathscr{B}$ such that $x \in U \subseteq p$, since p is open. Then $U \in G$ since G is a filter and the forcing poset is $\supseteq$. So we found a $U \in G \cap \mathscr{B}$ such that $x \in U$; hence $x \in \bigcup(G \cap \mathscr{B})$. This proves our claim. Now if F is a finite subset of G, then $\bigcup F \in G$ since G is a filter. In particular, $\bigcup F \in \mathbb{P}$, so its measure is less than ε. Now $G \cap \mathscr{B}$ is countable;

let $\langle p_i : i \in \omega\rangle$ enumerate it. Define $q_i = p_i \backslash \bigcup_{j<i} p_j$ for all $i \in \omega$. Then by induction one sees that $\bigcup_{i<m} p_i = \bigcup_{i<m} q_i$, and hence $\bigcup(G \cap \mathscr{B}) = \bigcup_{i<\omega} q_i$. So

$$\begin{aligned}\mu(V) &= \mu\left(\bigcup(G\cap\mathscr{B})\right) = \mu\left(\bigcup_{i<\omega} q_i\right)\\ &= \sum_{i<\omega}\mu(q_i) = \lim_{m\to\infty}\sum_{i<m}\mu(q_i) = \lim_{m\to\infty}\mu\left(\bigcup_{i<m} q_i\right) = \lim_{m\to\infty}\mu\left(\bigcup_{i<m} p_i\right) \le \varepsilon. \qquad \square\end{aligned}$$

Proposition 25.13. *Assume* MA(κ). *Suppose that X is a compact Hausdorff space, and any pairwise disjoint collection of open sets in X is countable. Suppose that U_α is dense open in X for each $\alpha < \kappa$. Then $\bigcap_{\alpha<\kappa} U_\alpha \neq \emptyset$.*

Proof. Let P consist of all nonempty open subsets of X, with $\subseteq$ as the order. Note that for $U, V \in P$, U is compatible with V iff $U \cap V \neq \emptyset$. Hence ccc holds for P. For each $\alpha < \kappa$ let $D_\alpha = \{p \in P : \overline{p} \subseteq U_\alpha\}$. We claim that D_α is dense in the sense of P. For, suppose that $p \in P$. Since U_α is (topologically) dense, we have $p \cap U_\alpha \neq \emptyset$. By regularity of the space there is a nonempty open set q such that $\overline{q} \subseteq p \cap U_\alpha$. Thus $q \in D_\alpha$ and $q \subseteq p$, as desired.

So, we apply MA(κ) and obtain a filter G intersecting each D_α. Because G is a filter, it has the fip as a collection of open sets. Hence by compactness, $\bigcap_{p\in G}\overline{p} \neq \emptyset$. For any $\alpha < \kappa$ there is a $p \in G \cap D_\alpha$, and hence $\overline{p} \subseteq U_\alpha$. This implies that $\bigcap_{p\in G}\overline{p} \subseteq \bigcap_{\alpha<\kappa} U_\alpha$. $\square$

Proposition 25.14. (III.3.35) *A forcing poset P is said to have ω_1 as a pre-caliber iff for every system $\langle p_\alpha : \alpha < \omega_1\rangle$ of elements of P there is an $X \in [\omega_1]^{\omega_1}$ such that for every finite subset F of X there is a $q \in P$ such that $q \le p_\alpha$ for all $\alpha \in F$.*

Then MA(ω_1) implies that every ccc forcing poset P has ω_1 as a pre-caliber.

Proof. Let $\langle p_\alpha : \alpha < \omega_1\rangle$ be a system of elements of P; we want to come up with a set X as indicated. For each $\alpha < \omega_1$ let

$$W_\alpha = \{q \in P : \exists\beta > \alpha(q \text{ and } p_\alpha \text{ are compatible})\}.$$

Clearly if $\alpha < \beta < \omega_1$ then $W_\beta \subseteq W_\alpha$. Now we claim

$$\exists\alpha\forall\beta \in (\alpha, \omega_1)[W_\alpha = W_\beta]. \tag{1}$$

In fact, otherwise we get a strictly increasing sequence $\langle \alpha_\xi : \xi < \omega_1\rangle$ of ordinals such that $W_{\alpha_{\xi+1}} \subset W_{\alpha_\xi}$ for all $\xi < \omega_1$. Choose $q_\xi \in W_{\alpha_\xi}\backslash W_{\alpha_{\xi+1}}$ for all $\xi < \omega_1$. Then there is an ordinal β_ξ such that $\alpha_\xi < \beta_\xi \le \alpha_{\xi+1}$ and q_ξ and p_{β_ξ} are compatible; say $r_\xi \le q_\xi, p_{\beta_\xi}$. We claim that r_ξ and r_η are incompatible for $\xi < \eta < \omega_1$ (contradicting ccc for P). For, if $s \le r_\xi, r_\eta$, then q_ξ and p_{β_η} are compatible, and hence $q_\xi \in W_{\alpha_{\xi+1}}$, contradiction. Thus (1) holds.

We are going to apply MA(ω_1) to W_α. The dense sets are as follows. For each $\beta \in (\alpha, \omega_1)$, let

$$D_\beta = \{q \in W_\alpha : \exists\gamma \in (\beta, \omega_1)[q \le p_\gamma]\}.$$

To prove density, suppose that $r \in W_\alpha$. Then, since $W_\alpha = W_\beta$ it follows that r and p_γ are compatible for some $\gamma > \beta$, as desired.

So, let G be a filter on W_α intersecting each set D_β. It follows that there exist a strictly increasing sequence $\langle \beta_\xi : \xi < \omega_1 \rangle$ and a sequence $\langle q_\xi : \xi < \omega_1 \rangle$ such that $q_\xi \leq p_{\beta_\xi}$ with $q_\xi \in G$ for all $\xi < \omega_1$. Clearly then $\{p_{\beta_\xi} : \xi < \omega_1\}$ has the desired property. □

Proposition 25.15. *Call a topological space X ccc iff every collection of pairwise disjoint open sets in X is countable. Then $\prod_{i \in I} X_i$ is ccc iff $\forall F \in [I]^{<\omega}[\prod_{i \in F} X_i$ is ccc].*

Proof. $\Rightarrow$: Suppose that $\prod_{i \in I} X_i$ is ccc and $F \in [I]^{<\omega}$. Also suppose that $\mathscr{A}$ is a pairwise disjoint collection of open sets in $\prod_{i \in F} X_i$. Then

$$\mathscr{A}' \stackrel{\text{def}}{=} \left\{ \left\{ x \in \prod_{i \in I} X_i : \langle x_i : i \in T \rangle \in U \right\} : U \in \mathscr{A} \right\}$$

is a collection of pairwise disjoint open sets in $\prod_{i \in I} X_i$, and hence $\mathscr{A}'$ is countable. Clearly this implies that $\mathscr{A}$ is countable.

$\Leftarrow$: See Theorem 21.31. □

Proposition 25.16. *Assuming* MA(ω_1)*, any product of ccc spaces is ccc.*

Proof. By Proposition 25.15 it suffices to show that any product of two ccc spaces X, Y is ccc. Suppose that $\mathscr{A}$ is an uncountable collection of pairwise disjoint open subsets of $X \times Y$; we want to get a contradiction. We may assume that each member of $\mathscr{A}$ has the form $U \times V$, with U open in X and V open in Y. Let $\langle U_\alpha \times V_\alpha : \alpha < \omega_1 \rangle$ be a one-one enumeration of a subset of $\mathscr{A}$. Let P be the poset consisting of all nonempty open subsets of X, ordered by $\subseteq$. Thus by Proposition 25.14, P has ω_1 as a pre-caliber. Hence let $M \in [\omega_1]^{\omega_1}$ be such that for every finite subset F of M there is a $V \in P$ such that $V \subseteq U_\alpha$ for all $\alpha \in F$. Take any distinct $\alpha, \beta \in M$. Then $U_\alpha \cap U_\beta \neq \emptyset$, while $(U_\alpha \times V_\alpha) \cap (U_\beta \times V_\beta) = \emptyset$, so $V_\alpha \cap V_\beta = \emptyset$. This contradicts ccc for Y. □

Proposition 25.17. *Assume* MA(ω_1)*. Suppose that P and Q are ccc posets. Define $\leq$ on $P \times Q$ by setting $(a, b) \leq (c, d)$ iff $a \leq c$ and $b \leq d$. Then $<$ is a ccc forcing poset on $P \times Q$.*

Proof. Suppose that $\langle (p_\alpha, q_\alpha) : \alpha < \omega_1 \rangle$ is a system of elements of $P \times Q$; we want to find distinct $\alpha, \beta < \omega_1$ such that (p_α, q_α) and (p_β, q_β) are compatible. By Proposition 25.14, let $\Gamma \in [\omega_1]^{\omega_1}$ be such that for every finite subset F of Γ there is an $r \in P$ such that $r \leq p_\alpha$ for all $\alpha \in F$. Since Q has ccc, there exist distinct $\alpha, \beta \in \Gamma$ such that q_α and q_β are compatible. Also, p_α and p_β are compatible. So (p_α, q_α) and (p_β, q_β) are compatible. □

Proposition 25.18. *We define $<^*$ on ${}^\omega\omega$ by setting $f <^* g$ iff $f, g \in {}^\omega\omega$ and $\exists n \forall m > n(f(m) < g(m)$. Suppose that $MA(\kappa)$ holds and $\mathscr{F} \in [{}^\omega\omega]^\kappa$. Then there is a $g \in {}^\omega\omega$ such that $f <^* g$ for all $f \in \mathscr{F}$.*

Proof. Let $\mathbb{P} = \{(p, F) : p \in \text{Fn}(\omega, \omega, \omega),\ F \in [{}^\omega\omega]^{<\omega}\}$ with $(p, F) \leq (q, G)$ iff $q \subseteq p$, $G \subseteq F$, and $\forall f \in G \forall n \in \text{dmn}(p) \backslash \text{dmn}(q)[f(n) < p(n)]$. Now $\leq$ is clearly reflexive and antisymmetric. Suppose that $(p,F)\le(q,G)\le(r,H)$. So $H\subseteq G\subseteq F$, hence $H\subseteq F$, and $r\subseteq q\subseteq p$, hence $r\subseteq p$. Now suppose that $f\in H$ and $n\in \mathrm{dmn}(p)\backslash\mathrm{dmn}(r)$. If $n\in\mathrm{dmn}(q)$, then $f(n)<q(n)=p(n)$ since $(q,G)\le(r,H)$. If $n\notin\mathrm{dmn}(q)$ then $f(n)<p(n)$ since $(p,F)\le(q,G)$. So $\le$ is transitive.

$\mathbb{P}$ is ccc: assume that X is an uncountable subset of $\mathbb{P}$. There are only countably many finite functions from ω to ω, so there are distinct $(p,F),(p,G)\in\mathbb{P}$. Then $(p,F\cup G)\le(p,F),(p,G)$. So X is not an incompatible set.

For each $h\in\mathscr{F}$ let $D_h=\{(p,F)\in\mathbb{P}:h\in F\}$. Then D_h is dense, since for any $(p,F)\in\mathscr{P}$ we have $(p,F\cup\{h\})\le(p,F)$.

For each $n\in\omega$ let $E_n=\{(p,F):n\in\mathrm{dmn}(p)\}$. Then E_n is dense. For, suppose that $(p,F)\in\mathbb{P}$ and $n\notin\mathrm{dmn}(p)$. Choose $m\in\omega$ greater than each member of $\{f(n):f\in F\}$. Then clearly $(p\cup\{(n,m)\},F)\le(p,F)$.

t G be a filter on $\mathbb{P}$ intersecting each D_h and E_n. Let $g=\bigcup_{(p,F)\in F}p$. Then g is a function since G is a filter. Moreover, $g\in{}^{\omega}\omega$ since $G\cap E_n\neq\emptyset$ for all n. Now let $f\in\mathscr{F}$. Choose $(p,F)\in G\cap D_f$. Thus $p\in F$. We claim that if m is greater than each member of $\mathrm{dmn}(p)$ then $g(m)>f(m)$ (as desired).

Since $m\in\mathrm{dmn}(g)$, choose $(q,H)\in G$ such that $m\in\mathrm{dmn}(q)$. Choose $(r,K)\in G$ with $(r,K)\le(p,F),(q,H)$. Then $m\in\mathrm{dmn}(q)\subseteq\mathrm{dmn}(r)$, so $m\in\mathrm{dmn}(r)$. Hence $m\in\mathrm{dmn}(r)\backslash\mathrm{dmn}(p)$, and $(r,K)\le(p,F)$, so $g(m)=r(m)>f(m)$. □

Proposition 25.19. *Let $\mathscr{B}\subseteq[\omega]^\omega$ be almost disjoint of size κ, with $\omega\le\kappa<2^\omega$. Let $\mathscr{A}\subseteq\mathscr{B}$ with $\mathscr{A}$ countable. Assume $MA(\kappa)$. Then there is a $d\subseteq\omega$ such that $|d\cap x|<\omega$ for all $x\in\mathscr{A}$, and $|x\backslash d|<\omega$ for all $x\in\mathscr{B}\backslash\mathscr{A}$.*

Proof. Let $\langle a_i:i\in\omega\rangle$ enumerate $\mathscr{A}$. Then let

$$\begin{gathered}\mathbb{P}=\{(s,F,m):s\in[\omega]^{<\omega},F\in[\mathscr{B}\backslash\mathscr{A}]^{<\omega},\text{ and }m\in\omega\};\\ (s',F',m')\le(s,F,m)\text{ iff }s\subseteq s',F\subseteq F',m\le m',\text{ and}\\ \forall x\in F\left[\left(x\backslash\bigcup_{i\in m}a_i\right)\cap s'\subseteq s\right].\end{gathered}$$

Clearly $\le$ is reflexive and antisymmetric. Now suppose that $(s'',F'',m'')\le(s',F',m')\le(s,F,m)$. Clearly $s\subseteq s''$, $F\subseteq F''$, and $m\le m''$. Now suppose that $x\in F$. Then $(x\backslash\bigcup_{i\in m}a_i)\cap s''\subseteq s'$, hence $(x\backslash\bigcup_{i\in m}a_i)\cap s''\subseteq(x\backslash\bigcup_{i\in m}a_i)\cap s'\subseteq s$. So $\le$ is transitive.

$\mathbb{P}$ is ccc: assume that X is an uncountable subset of $\mathbb{P}$. Now $[\omega]^{<\omega}$ is countable, so there are distinct $(s,F,m),(s,F',m')\in X$. Then $(s,F\cup F'),\max(m,m'))\le(s,F,m),(s,F',m')$, as desired.

For each $x\in\mathscr{B}\backslash\mathscr{A}$ let $D_x=\{(s,F,m)\in\mathbb{P}:x\in F\}$. Then D_x is dense, since clearly $(s,F\cup\{x\},m)\le(s,F,m)$ for any $(s,F,m)\in\mathbb{P}$.

Let $\mathscr{D}=\{(s,F,m,i,n):(s,F,m)\in\mathbb{P},i<m,\text{ and }n\in a_i\backslash s\}$. Clearly $|\mathscr{D}|=\kappa$. For each $(s,F,m,i,n)\in\mathscr{D}$ let

$$\begin{aligned}E_{(s,F,m,i,n)}=\{(s',F',m')\in\mathbb{P}:{}&(s,F,m)\text{ and }(s',F',m')\text{ are incompatible}\\ &\text{or }(s',F',m')\le(s,F,m)\text{ and }n\in s'\}.\end{aligned}$$

Now $E_{(s,F,m,i,n)}$ is dense for each $(s, F, m, i, n) \in \mathscr{D}$. For, suppose that (s', F', m') is given. We may assume that (s, F, m) and (s', F', m') are compatible; say $(s'', F'', m'') \leq (s, F, m), (s', F', m')$. We may assume that $n \notin s''$. We claim that $(s'' \cup \{n\}, F'', m'') \leq (s'', F'', m'')$ (as desired). This is true since for any $x \in F''$ we have $n \notin (x \backslash \bigcup_{j \in m''} a_j)$, by virtue of $n \in a_i$ and $i \in m \leq m''$.

Next, for any $i < \omega$ let $H_i = \{(s, F, m) \in \mathbb{P} : i < m\}$. Then H_i is dense, since $(s, F, i+1) \leq (s, F, m)$ for any $m \leq i$.

Now let G be a filter on $\mathbb{P}$ intersecting all of these dense sets. Let $d = \bigcup_{(s,F,m) \in G} s$. Take any $x \in \mathscr{B} \backslash \mathscr{A}$, and choose $(s, F, m) \in G \cap D_x$; so $x \in F$. We claim that $x \cap d \subseteq \bigcup_{i \in m} (x \cap a_i) \cup s$, so that $x \cap d$ is finite. For, suppose that $n \in x \cap d$. choose $(s', F', m') \in G$ such that $n \in s'$. Take $(s'', F'', m'') \in G$ with $(s'', F'', m'') \leq (s, F, m), (s', F', m')$. Then $n \in s''$. Since $(x \backslash \bigcup_{i \in m} a_i) \cap s'' \subseteq s$ and $n \in x \cap s''$, it follows that $n \in \bigcup_{i \in m} a_i \cup s$, as desired.

Next, take any $i < \omega$. Choose $(s, F, m) \in G \cap H_i$. Thus $i < m$. We claim that $a_i \backslash s \subseteq d$, so that $a \backslash d$ is finite. To prove this, take any $n \in a_i \backslash s$. Then $(s, F, m, i, n) \in \mathscr{D}$, so we can choose $(s', F', m') \in G \cap E_{(s,F,m,i,n)}$. Since (s, F, m) and (x', F', m') are compatible as elements of G, it follows that $(s', F', m') \leq (s, F, m)$ and $n \in s'$. Thus $n \in d$, as desired. This proves the claim. $\square$

Proposition 25.20. *The condition that $\mathscr{A}$ is countable is needed in Proposition 25.19. In fact, there exist $\mathscr{A}, \mathscr{B}$ such that $\mathscr{B}$ is an almost disjoint family of infinite subsets of ω, $\mathscr{A} \subseteq \mathscr{B}$, $|\mathscr{A}| = |\mathscr{B} \backslash \mathscr{A}| = \omega_1$, and there does not exist a $d \subseteq \omega$ such that $|x \backslash d| < \omega$ for all $x \in \mathscr{A}$, and $|x \cap d| < \omega$ for all $x \in \mathscr{B} \backslash \mathscr{A}$.*

Proof. We claim that there are $\mathscr{A} = \{a_\alpha : \alpha < \omega_1\}$ and $\mathscr{B} \backslash \mathscr{A} = \{b_\alpha : \alpha < \omega_1\}$ such that the elements are infinite and pairwise almost disjoint, and also $a_\alpha \cap b_\alpha = \emptyset$, while for $\alpha \neq \beta$ we have $a_\alpha \cap b_\beta \neq \emptyset$.

First assume that the claim holds. Suppose that we have constructed $\langle a_\alpha : \alpha < \omega_1 \rangle$ and $\langle b_\alpha : \alpha < \omega_1 \rangle$, so that they are infinite and pairwise almost disjoint, with the additional indicated property. Suppose that d exists as indicated. Wlog $\forall \alpha < \omega_1 (a_\alpha \backslash d = F$ and $b_\alpha \cap d = G)$. Choose $m \in a_0 \cap b_1$. If $m \in d$, then $m \in b_1 \cap d = G \subseteq b_0$, so $m \in a_0 \cap b_0$, contradiction. If $m \notin d$, then $m \in a_0 \backslash d = F \subseteq a_1$, so $m \in a_1 \cap b_1$, contradiction.

Now we prove ths claim. Let $\langle c_i : i < \omega \rangle$ be a system of pairwise disjoint infinite subsets of ω. Let $\langle c_{i,j} : j < \omega \rangle$ be a one-one enumeration of c_i. Then we define for each $m \in \omega$

$$\begin{aligned} a_m &= c_{2m} \cup \{c_{2n+1,0} : n < m\}; \\ b_m &= c_{2m+1} \cup \{c_{2n,0} : n < m\}. \end{aligned}$$

Clearly this defines infinite pairwise almost disjoint subsets of ω, $a_m \cap b_m = \emptyset$ for all $m \in \omega$, and for $n < m$ we have $c_{2n,0} \in a_n \cap b_m$ and $c_{2n+1,0} \in a_m \cap b_n$.

Now suppose that a_β and b_β have been defined for all $\beta < \alpha$, with $\omega \leq \alpha < \omega_1$. Let f be a function from ω onto α. By recursion, choose

$$x_m \in b_{f(m)} \backslash \left(\bigcup_{n<m} b_{f(n)} \cup \bigcup_{n<m} a_{f(n)} \cup \{x_n : n < m\} \cup \{y_n : n < m\} \right);$$

$$y_m \in a_{f(m)} \backslash \left(\bigcup_{n<m} b_{f(n)} \cup \bigcup_{n<m} a_{f(n)} \cup \{x_n : n \le m\} \cup \{y_n : n < m\} \right).$$

Let $a_\alpha = \{x_m : m \in \omega\}$ and $b_\alpha = \{y_m : m \in \omega\}$. Then $a_\alpha \cap b_\alpha = \emptyset$ and

$$\begin{aligned} x_m \in a_\alpha \cap b_{f(m)} &\subseteq \{x_0, \ldots, x_m\}; \\ y_m \in b_\alpha \cap a_{f(m)} &\subseteq \{y_0, \ldots, y_m\}; \\ a_\alpha \cap a_{f(m)} &\subseteq \{x_0, \ldots, x_m\}; \\ b_\alpha \cap b_{f(m)} &\subseteq \{y_0, \ldots, y_m\}. \end{aligned}$$

Hence the construction is complete. □

Proposition 25.21. *Suppose that $\mathscr{A}$ is a family of infinite subsets of ω such that $\bigcap F$ is infinite for every finite subset F of $\mathscr{A}$. Suppose that $|\mathscr{A}| \le \kappa$. Assuming $MA(\kappa)$, Then there is an infinite $X \subseteq \omega$ such that $X \backslash A$ is finite for every $A \in \mathscr{A}$.*

Proof. Let $\mathscr{A}' = \{X \subseteq \omega : \omega \backslash X \in \mathscr{A}\}$, and let $\mathscr{B} = \{\omega\}$. Clearly the hypothesis of Theorem 25.5 holds for $\mathscr{A}'$ and $\mathscr{B}$. Hence by Theorem 25.5 there is a $d \subseteq \omega$ such that $|d \cap X| < \omega$ for all $X \in \mathscr{A}'$ and $|d| = |d \cap \omega| = \omega$. Clearly d is as desired. □

Proposition 25.22. *$MA(\kappa)$ is equivalent to $MA(\kappa)$ restricted to ccc forcing posets of cardinality $\le \kappa$.*

Proof. We assume the indicated special form of MA(κ), and assume given a ccc poset P and a family $\mathscr{D}$ of at most κ dense sets in P; we want to find a filter on P intersecting each member of $\mathscr{D}$. We introduce some operations on P. For each $D \in \mathscr{D}$ define $f_D : P \to P$ by setting, for each $p \in P$, $f_D(p)$ to be some element of D which is $\le p$. Also we define $g : P \times P \to P$ by setting, for all $p, q \in P$,

$$g(p,q) = \begin{cases} p & \text{if } p \text{ and } q \text{ are incompatible,} \\ r & \text{with } r \le p, q \text{ if there is such an } r. \end{cases}$$

Here, as in the definition of f_D, we are implicitly using the axiom of choice; for g, we choose any r of the indicated form.

We may assume that $\mathscr{D} \neq \emptyset$. Choose $D \in \mathscr{D}$, and choose $s \in D$. Now let Q be the intersection of all subsets of P which have s as a member and are closed under all of the operations f_D and g. We take the order on Q to be the order induced from P.

(1) $|Q| \le \kappa$.

To prove this, we give an alternative definition of Q. Define

$$\begin{aligned} R_0 &= \{s\}; \\ R_{n+1} &= R_n \cup \{g(a,b) : a, b \in R_n\} \cup \{f_D(a) : D \in \mathscr{D} \text{ and } a \in R_n\}. \end{aligned}$$

Clearly $\bigcup_{n\in\omega} R_n = Q$. By induction, $|R_n| \le \kappa$ for all $n \in \omega$, and hence $|Q| \le \kappa$, as desired in (1).

We also need to check that Q is ccc. Suppose that X is a collection of pairwise incompatible elements of Q. Then these elements are also incompatible in P, since $x, y \in X$ with x, y compatible in P implies that $g(x, y) \leq x, y$ and $g(x, y) \in Q$, so that x, y are compatible in Q. It follows that X is countable. So Q is ccc.

Now if $D \in \mathscr{D}$, then $D \cap Q$ is dense in Q. In fact, take any $q \in Q$. Then $f_D(q) \in Q$ and $f_D(q) \leq q$, as desired.

Now we can apply our special case of $\mathrm{MA}(\kappa)$ to Q and $\{D \cap Q : D \in \mathscr{D}\}$; we obtain a filter G on Q such that $G \cap D \cap Q \neq \emptyset$ for all $D \in \mathscr{D}$. Let

$$G' = \{p \in P : q \leq p \text{ for some } q \in G\}.$$

We claim that G' is the desired filter on P intersecting each $D \in \mathscr{D}$.

Clearly if $p \in G'$ and $p \leq r$, then $r \in G'$.

Suppose that $p_1, p_2 \in G'$. Choose $q_1, q_2 \in G$ such that $q_i \leq p_1$ for each $i = 1, 2$. Then there is an $r \in G$ such that $r \leq q_1, q_2$. Then $r \in G'$ and $r \leq p_1, p_2$. So G' is a filter on P.

Now take any $D \in \mathscr{D}$. Then as proved above, $D \cap Q$ is dense in Q. It follows that $G \cap D \cap Q \neq \emptyset$; say $q \in G \cap D \cap Q$. Then $q \in G' \cap D$, as desired. $\square$

Proposition 25.23. *Define $x \subset^* y$ iff $x, y \subseteq \omega$, $x \backslash y$ is finite, and $y \backslash x$ is infinite. Assume $\mathrm{MA}(\kappa)$, and suppose that $(L, \triangleleft)$ is a linear ordering of size at most κ. Then there is a system $\langle a_x : x \in L \rangle$ of infinite subsets of ω such that for all $x, y \in L$, $x \triangleleft y$ iff $a_x \subset^* a_y$.*

Proof. First note that it is enough to do the construction so that $\forall x, y \in L[x \triangleleft y \rightarrow a_x \subset^* a_y]$. In fact, knowing this, if $a_x \subset^* a_y$, then we must have $x \triangleleft y$, as otherwise $y \triangleleft x$ or $y = x$, and hence $a_y \subset^* a_x$ or $a_y = a_x$, both of which are ruled out by $a_x \subset^* a_y$.

Let P consist of all triples $p = (S_p, k_p, \sigma_p)$ such that $S_p \in [L]^{<\omega}$, $k_p \in \omega$, and $\sigma_p : S_p \times k_p \rightarrow 2$. We define $q \leq p$ iff $S_p \subseteq S_q$, $k_p \leq k_q$, $\sigma_p \subseteq \sigma_q$, and $\forall x, y \in S_p \forall n[x \triangleleft y$ and $n \in k_q \backslash k_p \rightarrow \sigma_q(x, n) \leq \sigma_q(y, n)]$. Clearly $\leq$ is reflexive and antisymmetric. Now suppose that $p \leq q \leq r$. Clearly then $S_p \subseteq S_r$, $k_p \leq k_r$, and $\sigma_p \subseteq \sigma_r$. Now suppose that $x, y \in S_p$, $x \triangleleft y$, and $n \in k_r \backslash k_p$. If $n \in k_q$, then $\sigma_r(x, n) = \sigma_q(x, n) \leq \sigma_q(y, n) = \sigma_r(y, n)$. If $n \notin k_q$, then $\sigma_r(x, n) \leq \sigma_r(y, n)$. So $\leq$ is transitive.

Now we show that P has ccc. Suppose that $\mathscr{A}$ is an uncountable subset of P. By the indexed Δ-system theorem, Theorem 24.4, there is an uncountable subset $\mathscr{B}$ of $\mathscr{A}$ such that $\langle S_p : p \in \mathscr{B} \rangle$ is an indexed Δ-system. Say $M \in [L]^{<\omega}$ with $S_p \cap S_q = M$ for any distinct $(p, n), (q, m) \in \mathscr{B}$. Next, there exist an uncountable $\mathscr{C} \subseteq \mathscr{B}$ and an $l \in \omega$ such that $k_p = l$ for all $p \in \mathscr{C}$. Then there is an uncountable $\mathscr{D} \subseteq \mathscr{C}$ such that $\sigma_p \restriction (M \times l) = \sigma_q \restriction (M \times l)$ for all $p, q \in \mathscr{D}$. Now we claim that any $p, q \in \mathscr{D}$ are compatible. For, define r as follows: $S_r = S_p \cup S_q$; $k_r = l$; for any $(x, i) \in S_r \times l$ let

$$\sigma_r(x, i) = \begin{cases} \sigma_p(x, i) & \text{if } (x, i) \in S_p \times l, \\ \sigma_q(x, i) & \text{otherwise.} \end{cases}$$

Thus $S_p \subseteq S_r$, $k_p = l = k_r$, and $\sigma_p \subseteq \sigma_r$. The final condition is clear, so $r \leq p$. Similarly $r \leq q$. Thus P has ccc.

Now after defining certain dense sets we are going to take a filter G with respect to them and then define

$$a_x = \{n \in \omega : \exists p \in G[x \in S_p,\ n < k_p,\ \text{and } \sigma_p(x, n) = 1]\}$$

for each $x \in L$.

To show that for a given $x \in L$ the set a_x is infinite, we consider for each $i \in \omega$ the set

$$E_{ix} \stackrel{\text{def}}{=} \{p \in P : x \in S_p \text{ and } \exists j \in [i, k_p)[\sigma_p(x, j) = 1]\}.$$

To show that this set is dense, let $q \in P$. Let $S_p = S_q \cup \{x\}$, $k_p = \max(i+1, k_q)$, and for any $(y, j) \in S_p \times k_p$ let

$$\sigma_p(y, j) = \begin{cases} \sigma_q(y, j) & \text{if } (y, j) \in S_q \times k_q, \\ 1 & \text{if } y \in S_r \text{ and } j \in k_p \backslash k_q, \\ 1 & \text{if } y = x \notin S_q. \end{cases}$$

Thus $S_q \subseteq S_p$, $k_q \leq k_p$, and $\sigma_q \subseteq \sigma_p$. Suppose that $y, z \in S_q$, $y \triangleleft z$, and $n \in k_p \backslash k_q$. Then $\sigma_r(y, n) = 1 = \sigma_r(z, n)$. So $r \leq q$. Clearly $r \in E_{ix}$.

If $E_{ix} \in G$ for each $i \in \omega$ and each $x \in L$, then a_x is infinite for each $x \in L$.

Next, for each $x \in L$ let $D_x = \{p \in P : x \in S_p\}$. Clearly D_x is dense in P. Now suppose that $D_x \in G$ for each $x \in L$. Suppose that $x, y \in L$ and $x \triangleleft y$. Let $p \in D_x \cap D_y \cap G$. We claim that $a_x \backslash a_y \subseteq k_p$. For, suppose that $n \in a_x \backslash a_y$. Choose $q \in G$ such that $x \in S_q$, $n < k_q$, and $\sigma_q(x, n) = 1$. Choose $r \in G$ with $r \leq p, q$. Now $n < k_q$, so $n < k_r$. If $k_p \leq n$ then $r \leq p$ implies that $\sigma_r(y, n) = 1$, hence $n \in a_y$, contradiction. So $n < k_p$. This shows that $a_x \subseteq^* a_y$.

Next, for any $x, y \in L$ such that $x \triangleleft y$ and any $i \in \omega$ let

$$F_{ixy} = \{p \in P : x, y \in S_p \text{ and } \exists n \in [i, k_p)[\sigma_p(x, n) = 0 \text{ and } \sigma_p(y, n) = 1]\}.$$

To show that this set is dense, let $q \in P$. Using the sets D_x and D_y let $r \leq q$ with $x, y \in S_r$. Take $n \geq i$ such that $(x, n), (y, n) \notin S_r$. Now define $S_p = S_r$, $k_p = \max(k_r, n+1)$, and define σ_p as follows. For any $(z, i) \in S_p \times k_p$ let

$$\sigma_p(z, i) = \begin{cases} \sigma_r(z, i) & \text{if } i < k_r, \\ 0 & \text{if } z \triangleleft x \text{ or } z = x, \text{ and } i = n, \\ 1 & \text{otherwise.} \end{cases}$$

Then $S_q \subseteq S_r = S_p$, $k_q \leq k_r \leq k_p$, and $\sigma_q \subseteq \sigma_r \subseteq \sigma_p$. Now suppose that $u, v \in L$, $u \triangleleft v$, and $i \in k_p \backslash k_q$. If $i < k_r$, then $\sigma_p(u, i) = \sigma_r(u, i) \leq \sigma_r(v, i) = \sigma_p(v, i)$. If $k_r \leq i$ and $\sigma_p(v, i) = 0$, then $i = n$ and $v \triangleleft x$ or $v = x$, hence also $u \triangleleft x$, and so $\sigma_p(u, i) = 0$. Hence $p \leq q$. Clearly $p \in F_{ixy}$. So F_{ixy} is dense.

Now if G is a filter containing all these sets, we claim that for $x \triangleleft y$ the set $a_y \backslash a_x$ is infinite. For, let $i \in \omega$ and choose $p \in F_{ixy} \cap G$. Choose $n \in [i, k_p)$ such that $\sigma_p(x, n) = 0$ while $\sigma_p(y, n) = 1$. Then $n \in a_y$. Suppose that $n \in a_x$. Choose $q \in G$ such that $x \in S_q$, $n < k_q$, and $\sigma_q(x, n) = 1$. Choose $r \in G$ with $r \leq p, q$. Then because $r \leq p$, we have $\sigma_r(x, n) = 0$. But because $r \leq q$ we have $\sigma_r(x, n) = 1$, contradiction. □

Proposition 25.24. *If $\mathscr{A}, \mathscr{B}$ are nonempty countable subsets of $[\omega]^\omega$ and $a \subseteq^* b$ whenever $a \in \mathscr{A}$ and $b \in \mathscr{B}$, then there is a $c \in [\omega]^\omega$ such that $a \subseteq^* c \subseteq^* b$ whenever $a \in \mathscr{A}$ and $b \in \mathscr{B}$.*

Proof. Write $\mathscr{A} = \{a_n : n \in \omega\}$ and $\mathscr{B} = \{b_n : n \in \omega\}$. Let

$$c = \bigcup_{n\in\omega}\left[\left(\bigcup_{m\le n} a_m\right) \cap \bigcap_{m\le n} b_m\right].$$

Now suppose that $p \in \omega$. Then

$$\begin{aligned}
a_p\backslash c &= \bigcap_{n\in\omega}\left[a_p \cap \left(\bigcap_{m\le n}(\omega\backslash a_m) \cup \bigcup_{m\le n}(\omega\backslash b_m)\right)\right]\\
&= \bigcap_{n<p}\left[a_p \cap \left(\bigcap_{m\le n}(\omega\backslash a_m) \cup \bigcup_{m\le n}(\omega\backslash b_m)\right)\right]\\
&\quad \cap \bigcap_{n\ge p}\left[a_p \cap \left(\bigcap_{m\le n}(\omega\backslash a_m) \cup \bigcup_{m\le n}(\omega\backslash b_m)\right)\right]\\
&= \bigcap_{n<p}\left[a_p \cap \left(\bigcap_{m\le n}(\omega\backslash a_m) \cup \bigcup_{m\le n}(\omega\backslash b_m)\right)\right]\\
&\quad \cap \bigcap_{n\ge p}\left[a_p \cap \bigcup_{m\le n}(\omega\backslash b_m)\right]\\
&\subseteq a_p \cap \bigcup_{m\le p}(\omega\backslash b_m),
\end{aligned}$$

and this last set is finite.

Furthermore,

$$\begin{aligned}
c\backslash b_p &= \bigcup_{n<p}\left[\left(\bigcup_{m\le n} a_m\right) \cap \bigcap_{m\le n} b_m \cap (\omega\backslash b_p)\right]\\
&\subseteq \left(\bigcup_{m<p} a_m\right)\backslash b_p,
\end{aligned}$$

and this last set is finite.

The set c is infinite, as otherwise $a_0 = (a_0 \cap c) \cup (a_0\backslash c)$ would be finite. $\square$

Proposition 25.25. *Suppose that $\mathscr{A}$ is a nonempty countable family of members of $[\omega]^\omega$, and $\forall a, b \in \mathscr{A}[a \subseteq^* b$ or $b \subseteq^* a]$. Also suppose that $\forall a \in \mathscr{A}[a \subset^* d]$, where $d \in [\omega]^\omega$. Then there is a $c \in [\omega]^\omega$ such that $\forall a \in \mathscr{A}[a \subseteq^* c \subset^* d]$.*

Proof. If $\exists a \in \mathscr{A} \forall b \in \mathscr{A}[b \subseteq^* a]$, then the conclusion is obvious. So suppose that no such a exists. Then there is a sequence $\langle a_n : n \in \omega\rangle$ of elements of $\mathscr{A}$ such that $a_n \subset^* a_m$ for $n < m$, and the sequence is cofinal in $\mathscr{A}$ in the $\subseteq^*$-sense. Let $\mathscr{C} = \{a_0\} \cup \{a_{m+1}\backslash a_m : m \in \omega\} \cup \{\omega\backslash d\}$. Then $\mathscr{C}$ is an almost disjoint family, except that possibly $\omega\backslash d$ is finite. By Corollary 25.6, let $e \subseteq \omega$ be infinite and almost disjoint from each member of $\mathscr{C}$. Let $c = d\backslash e$. Then for any $n \in \omega$,

$$\begin{aligned} a_{n+1}\backslash c &= (a_{n+1}\backslash d) \cup (a_{n+1} \cap e) \\ &\subseteq (a_{n+1}\backslash d) \cup \left[\bigcup_{i\leq n}(a_{i+1}\backslash a_i) \cup a_0\right] \cap e, \end{aligned}$$

and the last set is finite. Thus $a_{n+1} \subseteq^* c$, hence $b \subseteq^* c$ for all $b \in \mathscr{A}$.

Since $c \subseteq d$, we have $c \subseteq^* d$. Also, $d\backslash c = d \cap e$, and this is infinite since $e\backslash d$ is finite. Thus $c \subset^* d$

Note that c is infinite, since $a \subseteq^* c$ for all $a \in \mathscr{A}$.

Proposition 25.26. *If $a, b \in [\omega]^\omega$ and $a \subset^* b$, then there is a $c \in [\omega]^\omega$ such that $a \subset^* c \subset^* b$.*

Proof. Write $b\backslash a = d \cup e$ with d, e infinite and disjoint. Let $c = a \cup d$. □

Proposition 25.27. *Suppose that $\mathscr{A}$ and $\mathscr{B}$ are nonempty countable subsets of $[\omega]^\omega$, $\forall x, y \in \mathscr{A}[x \subseteq^* y$ or $y \subseteq^* x]$, $\forall x, y \in \mathscr{B}[x \subseteq^* y$ or $y \subseteq^* x]$, and $\forall x \in \mathscr{A} \forall y \in \mathscr{B}[a \subset^* b]$. Then there is a $c \in [\omega]^\omega$ such that $a \subset^* c \subset^* b$ for all $a \in \mathscr{A}$ and $b \in \mathscr{B}$.*

Proof. By Proposition 25.24 choose $d \subseteq \omega$ such that $\forall a \in \mathscr{A} \forall b \in \mathscr{B}[a \subseteq^* d \subseteq^* b]$. Thus either $\forall a \in \mathscr{A}[a \subset^* d]$ or $\forall b \in \mathscr{B}[d \subset^* b]$.

Case 1. $\forall a \in \mathscr{A}[a \subset^* d]$. By Proposition 25.25 choose $e \subseteq \omega$ such that $\forall a \in \mathscr{A}[a \subseteq^* e \subset^* d]$. By Proposition 25.26 choose $c \in [\omega]^\omega$ such that $e \subset^* c \subset^* d$.

Case 2. $\forall b \in \varepsilon B[d \subset^* b]$. Then $\forall b \in \mathscr{B}[(\omega\backslash b) \subset^* (\omega\backslash d)]$. By Proposition 25.25 choose $e \subseteq \omega$ such that $\forall b \in \mathscr{B}[(\omega\backslash b) \subseteq^* e \subset^* (\omega\backslash d)]$. By Proposition 25.26 choose $c \subseteq \omega$ such that $e \subset^* c \subset^* (\omega\backslash d)$. Then $\forall a \in \mathscr{A} \forall b \in \mathscr{B}[a \subset^* (\omega\backslash c) \subset^* b]$. □

Now we need some more terminology. Let $\mathscr{A} \subseteq [\omega]^\omega$, $b \in [\omega]^\omega$, and $\forall a \in \mathscr{A}[a \subset^* b]$. We say that b is *near* to $\mathscr{A}$ iff for all $m \in \omega$ the set $\{a \in \mathscr{A} : a\backslash b \subseteq m\}$ is finite.

Proposition 25.28. *Suppose that $a_m \in [\omega]^\omega$ for all $m \in \omega$, $a_m \subset^* a_n$ whenever $m < n \in \omega$, $b \in [\omega]^\omega$, and $a_m \subset^* b$ for all $m \in \omega$. Then there is a $c \in [\omega]^\omega$ such that $\forall m \in \omega[a_m \subset^* c \subset^* b]$ and c is near to $\{a_n : n \in \omega\}$.*

Proof. By Proposition 25.27 choose $d \subseteq \omega$ such that $\forall m \in \omega[a_n \subset^* d \subset^* b]$. Now for each $m \in \omega$, $\bigcup_{i<m}(a_i\backslash a_m)$ is finite, and $a_{m+1}\backslash \bigcup_{i\leq m} a_i = (a_{m+1}\backslash a_m)\backslash \bigcup_{i<m}(a_i\backslash a_m)$, so $a_{m+1}\backslash \bigcup_{i\leq m} a_i$ is infinite. Choose $e_m \subseteq a_{m+1}\backslash \bigcup_{i\leq m} a_i$ such that $|e_m| = m$. Let $c = d\backslash \bigcup_{m\in\omega} e_m$. Thus $c \subseteq^* d \subset^* b$.

If $n \in \omega$, then

$$a_n \backslash c = (a_n \backslash d) \cup \bigcup_{m\in\omega} (a_n \cap e_m) = (a_n \backslash d) \cup \bigcup_{m<n} (a_n \cap e_m),$$

and this last set is finite. Hence $a_n \subseteq^* c$. Since n is arbitrary, it follows that $a_n \subset^* c$ for all $n \in \omega$.

Also for any $m \in \omega$ we have $a_{m+1} \backslash c \supseteq a_{m+1} \cap e_m = e_m$, and so $|a_{m+1} \backslash c| \geq m$. It follows that for any $n \in \omega$, $\{a_m : a_m \backslash c \subseteq n\} \subseteq \{a_0, \ldots, a_n\}$. So c is near to $\{a_m : m \in \omega\}$. $\square$

Proposition 25.29. *Suppose that* $\mathscr{A} \subseteq [\omega]^\omega$, $\forall x, y \in \mathscr{A}[x \subset^* y$ *or* $y \subset^* x]$, $b \in [\omega]^\omega$, $\forall x \in \mathscr{A}[x \subset^* b]$, *and* $\forall a \in \mathscr{A}[b$ *is near to* $\{d \in \mathscr{A} : d \subset^* a\}]$.

Then there is a $c \in [\omega]^\omega$ *such that* $\forall a \in \mathscr{A}[a \subset^* c \subset^* b]$ *and* c *is near to* $\mathscr{A}$.

Proof. We consider several cases.

Case 1. $\exists a \in \mathscr{A} \forall d \in \mathscr{A}[d \subseteq^* a]$. By Proposition 25.26, choose c such that $a \subset^* c \subset^* b$. Choose $n \in \omega$ such that $c \backslash b \subseteq n$. Then for any $m \in \omega$ and any $d \in \mathscr{A}$, if $d \backslash c \subseteq m$ then $d \backslash b \subseteq (d \backslash c) \cup (c \backslash b) \subseteq \max(m, n)$. Hence

$$\{d \in \mathscr{A} : d \backslash c \subseteq m\} \subseteq \{a\} \cup \{d \in \mathscr{A} : d \subset^* a \text{ and } d \backslash b \subseteq \max(m, n)\},$$

and the later set is finite, since b is near to $\{d \in \mathscr{A} : d \subset^* a\}$. Thus c is as desired.

Case 2. $\forall a \in \mathscr{A} \exists d \in \mathscr{A}[a \subset^* d]$ and b is near to $\mathscr{A}$. By Proposition 25.27, choose c so that $\forall a \in \mathscr{A}[a \subset^* c \subset^* b]$. Choose $n \in \omega$ such that $c \backslash b \subseteq n$. Then for any $m \in \omega$ and any $d \in \mathscr{A}$, if $d \backslash c \subseteq m$ then $d \backslash b \subseteq (d \backslash c) \cup (c \backslash b) \subseteq \max(m, n)$. Hence

$$\{d \in \mathscr{A} : d \backslash c \subseteq m\} \subseteq \{a\} \cup \{d \in \mathscr{A} : d \backslash b \subseteq \max(m, n)\},$$

and the later set is finite, since b is near to $\mathscr{A}$. Thus c is as desired.

Case 3. $\forall a \in \mathscr{A} \exists d \in \mathscr{A}[a \subset^* d]$ and b is not near to $\mathscr{A}$. For each $m \in \omega$ let $\mathscr{B}_m = \{a \in \mathscr{A} : a \backslash b \subseteq m\}$. Since b is not near to $\mathscr{A}$, choose m so that $\mathscr{B}_m$ is infinite. Note that $p < q \to \mathscr{B}_p \subseteq \mathscr{B}_q$. Hence $\mathscr{B}_n$ is infinite for every $n \geq m$. Now we claim

$$(1) \qquad \forall n \geq m \forall a \in \mathscr{A} \exists d \in \mathscr{B}_n[a \subseteq^* d].$$

In fact, otherwise we get $n \geq m$ and $a \in \mathscr{A}$ such that $\forall d \in \mathscr{B}_n[d \subset^* a]$. Now b is near to $\{d \in \mathscr{A} : d \subset^* a\}$ by a hypothesis of the lemma, so $\{d \in \mathscr{A} : d \subset^* a$ and $d \backslash b \subseteq n\}$ is finite. But $\mathscr{B}_n \subseteq \{d \in \mathscr{A} : d \subset^* a$ and $d \backslash b \subseteq n\}$, contradiction. So (1) holds.

Next we claim

$$(2) \qquad \forall n \geq m \forall d \in \mathscr{B}_n[\{e \in \mathscr{B}_n : e \subset^* d\} \text{ is finite}].$$

In fact, suppose that $n \geq m$, $d \in \mathscr{B}_n$ and $\{e \in \mathscr{B}_n : e \subset^* d\}$ is infinite. Since b is near to $\{a \in \mathscr{A} : a \subset^* d\}$, the set $\{a \in \mathscr{A} : a \subset^* d$ and $a \backslash b \subseteq n\}$ is finite. But $\{e \in \mathscr{B}_n : e \subset^* d\} \subseteq \{a \in \mathscr{A} : a \subset^* d$ and $a \backslash b \subseteq n\}$, contradiction. So (2) holds.

From (2) it follows that $\mathscr{B}_n$ has order type ω under $\subset^*$, for each $n \geq m$. Now clearly $\mathscr{A} = \bigcup_{p\in\omega} \mathscr{B}_p$, so $\mathscr{A}$ is countable.

Now by Proposition 25.28, choose c_m such that $\forall d \in \mathscr{B}_m[d \subset^* c_m \subset^* b]$ and c_m is near to $\mathscr{B}_m$. By (1), $a \subset^* c_m$ for each $a \in \mathscr{A}$. Now suppose that $n \geq m$ and c_n has been defined so that $a \subset^* c_n$ for each $a \in \mathscr{A}$. Again by Proposition 25.28 choose c_{n+1} such that $\forall d \in \mathscr{B}_{n+1}[d \subset^* c_{n+1} \subset^* c_n]$ and c_{n+1} is near to $\mathscr{B}_{n+1}$. Thus we have

$$\forall a \in \mathscr{A}[a \subset^* \cdots \subset^* c_{n+1} \subset^* c_n \subset^* \cdots \subset^* c_m \subset^* b].$$

By Proposition 25.27, choose d so that $\forall a \in \mathscr{A} \forall n \geq m[a \subset^* d \subset^* c_n]$. We claim that d is near to $\mathscr{A}$, completing the proof. For, let $n \in \omega$. Let $p = \max(m, n)$, and choose $q \geq p$ such that $d \backslash c_p \subseteq q$. Then

$$\begin{aligned}\{a \in \mathscr{A} : a\backslash d \subseteq n\} &\subseteq \{a \in \mathscr{A} : a\backslash d \subseteq p\}\\ &= \{a \in \mathscr{B}_p : a\backslash d \subseteq p\}\\ &\subseteq \{a \in \mathscr{B}_p : a\backslash c_p \subseteq q\},\end{aligned}$$

where the last inclusion holds since $a\backslash c_p = (a\backslash d) \cup (d\backslash c_p)$. The last set is finite since c_p is near to $\mathscr{B}_p$, as desired.

Proposition 25.30. *(The Hausdorff gap) There exist sequences $\langle a_\alpha : \alpha < \omega_1\rangle$ and $\langle b_\alpha : \alpha < \omega_1\rangle$ of members of $[\omega]^\omega$ such that $\forall \alpha, \beta < \omega_1[\alpha < \beta \rightarrow a_\alpha \subset^* a_\beta$ and $b_\beta \subset^* b_\alpha]$, $\forall \alpha, \beta < \omega_1[a_\alpha \subset^* b_\beta]$, and there does not exist a $c \subseteq \omega$ such that $\forall \alpha < \omega_1[a_\alpha \subset^* c$ and $c \subset^* b_\alpha]$.*

Proof. We construct by recursion $a_\alpha, b_\alpha \subseteq \omega$ for $\alpha < \omega_1$ so that $a_\alpha \subset^* b_\alpha$, $\alpha < \beta \rightarrow a_\alpha \subset^* a_\beta$ and $b_\beta \subset^* b_\alpha$, and for all $\alpha < \omega_1$, b_β is near to $\{a_\alpha : \alpha < \beta\}$.

Let $a_0 = \emptyset$, $b_0 = \omega$. Suppose that a_α and b_α have been constructed for all $\alpha < \beta$ so that $a_\alpha \subset^* b_\alpha$, $\alpha < \gamma < \beta \rightarrow a_\alpha \subset^* a_\gamma$ and $b_\gamma \subset^* b_\beta$, and $\alpha < \beta \rightarrow b_\alpha$ is near to $\{a_\gamma : \gamma < \alpha\}$. By Proposition 25.27 choose c such that $\forall \alpha < \beta[a_\alpha \subset^* c \subset^* b_\alpha]$. Suppose that $\alpha < \beta$. We claim that c is near to $\{a_\gamma : \gamma < \alpha\}$. In fact, suppose that $m \in \omega$. Choose $n \geq m$ such that $c\backslash b_\alpha \subseteq n$. Now for any $\gamma < \alpha$ we have $a_\gamma\backslash b_\alpha \subseteq (a_\gamma\backslash c) \cup (c\backslash b_\alpha)$, so

$$\{a_\gamma : \gamma < \alpha \text{ and } a_\gamma\backslash c \subseteq m\} \subseteq \{a_\gamma : \gamma < \alpha \text{ and } a_\gamma\backslash b_\alpha \subseteq n\},$$

and the latter set is finite since b_α is near to $\{a_\gamma : \gamma < \alpha\}$. Thus indeed c is near to $\{a_\gamma : \gamma < \alpha\}$. Now by Proposition E25.29 there is a b_β such that $\forall \alpha < \beta[a_\alpha \subset^* b_\beta \subset^* c]$ and b_β is near to $\{a_a : \alpha < \beta\}$. By Proposition E25.27 choose a_β so that $\forall \alpha < \beta[a_\alpha \subset^* a_\beta \subset^* b_\beta]$. This finishes the construction.

Now suppose that $d \subseteq \omega$ and $\forall \alpha < \omega_1[a_\alpha \subset^* d \subset^* b_\alpha]$. Now $\omega_1 = \bigcup_{m\in\omega}\{\alpha < \omega_1 : a_\alpha\backslash d \subseteq m\}$, so we can choose $m \in \omega$ such that $|\{\alpha < \omega_1 : a_\alpha\backslash d \subseteq m\}| = \omega_1$. Hence there is an $\alpha < \omega_1$ such that $\{\beta < \alpha : a_\beta\backslash d \subseteq m\}$ is infinite. Choose $p \geq m$ such that $d\backslash b_\alpha \subseteq p$. Now $a_\beta\backslash b_\alpha \subseteq (a_\beta\backslash d) \cup (d\backslash b_\alpha)$, so $\{\beta < \alpha : a_\beta\backslash d \subseteq m\} \subseteq \{\beta < \alpha : a_\beta\backslash b_\alpha \subseteq p\}$, contradicting b_α near to $\{a_\beta : \beta < \alpha\}$. $\square$

$\mathfrak{m}$ is the least cardinal κ such that $\neg\mathrm{MA}(\kappa)$. An *atom* in a forcing poset P is an element $r \in P$ such that there do not exist $p, q \leq r$ such that $p \perp q$.

Proposition 25.31. (III.3.20) *If P has an atom, then $\mathrm{MA}_P(\kappa)$ holds for all infinite κ.*

Proof. Let $r \in P$ be an atom. Let $G = \{p \in P : p \text{ and } r \text{ are compatible}\}$. Then G is a filter. For, clearly $1 \in G$. If $p \in G$ and $p \leq q$, clearly $q \in G$. Suppose that $p, q \in G$. Say $s \leq p, r$ and $t \leq q, r$. Then $s, t \leq r$, so s, t are compatible. Say $u \leq s, t$. So also $u \leq r$. So $u \leq p, q$ and $u \in G$. So G is a filter.

If D is a dense set, choose $q \in D$ such that $q \leq p$. Then $q \in D \cap G$. □

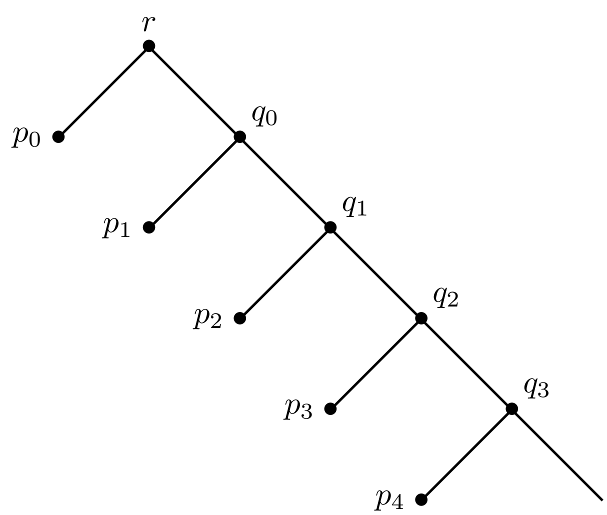
Proposition 25.32. (III.3.20) *If $\mathbb{P}$ is atomless, then $\mathrm{MA}_{\mathbb{P}}(\kappa)$ is false for $\kappa = 2^{|\mathbb{P}|}$.*

Proof. Assume the contrary, and let G be a filter on $\mathbb{P}$ which meets each dense subset of $\mathbb{P}$. Now $\mathbb{P}\backslash G$ is dense. For, if $p \in \mathbb{P}$, then there are $q, r \leq p$ such that $q \perp r$. At least one of q, r is not in G. So, $\mathbb{P}\backslash G$ is dense. Hence $G \cap (\mathbb{P}\backslash G) \neq \emptyset$, contradiction. □

Lemma 25.33. (III.3.21) *If $\mathbb{P}$ is atomless and $r \in \mathbb{P}$, then there is an infinite antichain of elements $\leq r$.*

Proof. We define $\langle p_n : n \in \omega \rangle$ and $\langle q_n : n \in \omega \rangle$ by recursion. Let p_0, q_0 be such that $p_0, q_0 \leq r$ and $p_0 \perp q_0$. If p_n and q_n have been defined, choose $p_{n+1}, q_{n+1} \leq q_n$ such that $p_{n+1} \perp q_{n+1}$. Then $\{p_n : n \in \omega\}$ is an infinite antichain of elements $\leq r$. □

This construction can be visualized as follows:



Theorem 25.34. (III.3.22) $\mathfrak{m} \leq \mathfrak{p}$.

Proof. Let κ be an infinite cardinal such that $\kappa < \mathfrak{m}$; we show that $\kappa < \mathfrak{p}$. Thus we are assuming that $\mathrm{MA}(\kappa)$ holds. Let $\mathscr{E} \subseteq [\omega]^\omega$ have SFIP, with $|\mathscr{E}| = \kappa$. We want to find a pseudo-intersection of $\mathscr{E}$. Let

$$\mathbb{P} = \{(s_p, W_p) : s_p \in [\omega]^{<\omega} \text{ and } W_p \in [\mathscr{E}]^{<\omega}\}.$$

We define $q \leq p$ iff the following hold:

(1) $s_p \subseteq s_q$.
(2) $W_p \subseteq W_q$.
(3) $\forall Z \in W_p[(s_q \backslash s_p) \subseteq Z]$.

This is a forcing order. For transitivity, suppose that $r \leq q \leq p$. Clearly (1) and (2) for r and p hold. Now suppose that $Z \in W_p$. Then $Z \in W_q$, and $(s_r \backslash s_p) = (s_r \backslash s_q) \cup (s_q \backslash s_p) \subseteq Z$.

If $s_p = s_q$, then p and q are compatible, since $(s_p, W_p \cup W_q)$ extends both of them. Since $[\omega]^{<\omega}$ is countable, it follows that $\mathbb{P}$ has ccc.

Now for each $n \in \omega$ let $D_n = \{p \in \mathbb{P} : |s_p| \geq n\}$. Then D_n is dense, for if $p \in \mathbb{P}$, then $\bigcap W_p$ is infinite, so we can choose $t \subseteq \bigcap W_p$ with $|t| = n$, and then $(s_p \cup t, W_p) \in D_n$ and $(s_p \cup t, W_p) \leq p$.

For any $Z \in \mathscr{E}$ let $E_Z = \{p \in \mathbb{P} : Z \in W_p\}$. Then E_Z is dense, since if $p \in \mathbb{P}$, then $(s_p, W_p \cup \{Z\}) \in E_Z$ and $(s_p, W_p \cup \{Z\}) \leq p$.

Let G be a filter intersecting all of these dense sets. Let $K_G = \bigcup_{p \in G} s_p$. Then G intersecting all sets D_n for $n \in \omega$ implies that K_G is infinite.

Given $Z \in \mathscr{E}$, choose $p \in G \cap E_Z$. Suppose that $m \in K_G \backslash Z$. Say $m \in s_q$ with $q \in G$. Choose $r \in G$ such that $r \leq p, q$. Then $m \in s_r$ since $r \leq q$. If $m \notin s_p$, then $m \in Z$ since $r \leq p$. Thus $K_G \subseteq Z \subseteq s_p$ and hence $K_G \backslash Z$ is finite. □

A subset C of a forcing order $\mathbb{P}$ is *centered* iff for every finite subset F of C there is a $p \in \mathbb{P}$ such that $\forall q \in F[p \leq q]$.

$\mathbb{P}$ is σ*-centered* iff there is a countable $\mathscr{C} \subseteq \mathscr{P}(\mathbb{P})$ such that $\mathbb{P} = \bigcup \mathscr{C}$ and each member of $\mathscr{C}$ is centered.

Lemma 25.35. (III.3.24) *Every σ-centered forcing order $\mathbb{P}$ has ccc.* □

Lemma 25.36. (III.3.25) *If G is a filter and F is a finite subset of G, then there is a $p \in G$ such that $\forall q \in F[p \leq q]$.*

In particular, every filter is centered. □

Lemma 25.37. (III.3.26) *The forcing order defined in the proof of Lemma 25.34 is σ-centered.*

Proof. For each $t \in {}^{<\omega}\omega$ let $C_t = \{p \in \mathbb{P} : s_p = t\}$. Clearly C_t is centered, and $\mathbb{P} = \bigcup_{t \in {}^{<\omega}\omega} C_t$. □

Proposition 25.38. (III.3.27) *If X is a compact Hausdorff space, then the following conditions are equivalent:*

(i) X is separable.

(ii) The collection $\mathbb{O}_X$ of nonempty open subsets of X is σ-centered.

(iii) $\mathbb{O}_X$ is a countable union of filters.

Proof. (i)⇒(ii): Suppose that X is separable; say D is a countable dense subset. For each $d \in D$ let $\mathbb{O}_d = \{U : U \text{ open}, d \in U\}$. Clearly $\mathbb{O}_X = \bigcup_{d \in D} \mathbb{O}_d$. Thus $\mathbb{O}_X$ is σ-centered.

(ii)⇒(iii): Suppose that $\mathbb{O}_X$ is σ-centered; say $\mathbb{O}_X = \bigcup_{n\in\omega} \mathbb{O}_n$ with each $\mathbb{O}_n$ centered. For each $n \in \omega$ let

$$F_n = \{U \in \mathbb{O}_X : \exists m \in \omega \exists V_1, \ldots, V_m \in \mathbb{O}_n[V_1 \cap \ldots \cap V_m \subseteq U]\}.$$

Clearly F_n is closed upwards and has X as a member. Suppose $U, W \in F_n$. Say $V_1, \ldots, V_m \in \mathbb{O}_n$ with $V_1 \cap \ldots \cap V_m \subseteq U$ and $Y_1, \ldots, Y_n \in \mathbb{O}_n$ with $Y_1 \cap \ldots \cap Y_n \subseteq W$. Then $V_1 \cap \ldots \cap V_m \cap Y_1 \cap \ldots \cap Y_n \subseteq U \cap W$,so $U \cap W \in F_n$. Thus F_n is a filter. Since $\mathbb{O}_n \subseteq F_n$, we have $\mathbb{O}_X = \bigcup_{n\in\omega} F_n$.

(iii)⇒(i): Suppose that $\mathbb{O}_X = \bigcup_{n\in\omega} F_n$ each F_n a filter. For each n choose $d_n \in \bigcap_{U\in F_n} \overline{U}$; this is possible by compactness. We claim that $D \overset{\text{def}}{=} \{d_n : n \in \omega\}$ is dense. For, let U be a nonempty open set. Let V be a nonempty open set such that $\overline{V} \subseteq U$. This is possible since X is compact, hence regular. Choose $n \in \omega$ such that $V \in F_n$. Then $d_n \in \bigcap_{Y\in F_n} \overline{Y} \subseteq \overline{V} \subseteq U$, as desired.

For X a topological space, $\mathbb{O}_X$ is the forcing poset of open subsets of X under inclusion.

Proposition 25.39. *For $X = {}^\kappa 2$ with $\kappa > 2^\omega$, $\mathbb{O}_X$ is ccc but not σ-centered.*

Proof. By Lemma 21.34 ${}^\kappa 2$ is not separable; applying Lemma 25.38, $\mathbb{O}_X$ is not σ-centered. By Corollary 21.33, $\mathbb{O}_X$ is ccc. □

Theorem 25.40. (III.3.28) $\mathfrak{m} \le \text{add(null)}$.

Proof. This is just a reformulation of Theorem 25.12. □

Proposition 25.41. *If $\mathbb{P}$ is σ-centered, then $\mathbb{P}$ has ω_1 as a pre-caliber.* □

Proposition 25.42. (III.3.32) *If $2 \le |J| \le \omega$, then* $\text{Fn}(I, J, \omega)$ *has ω_1 as a pre-caliber.*

Proof. Suppose that $\langle p_\alpha : \alpha < \omega_1\rangle$ is a system of elements of $\text{Fn}(I, J, \omega)$. Let $\langle \text{dmn}(f_\alpha) : \alpha \in M\rangle$ be a Δ-system with kernel N, $|M| = \omega_1$. Then

$$M = \bigcup_{g\in {}^N J} \{\alpha \in M : f_\alpha \restriction N = g\},$$

so there is a $g \in {}^N J$ such that $M' \overset{\text{def}}{=} \{\alpha \in M : f_\alpha \restriction N = g\}$ has size ω_1. Clearly $\langle p_\alpha : \alpha \in M'\rangle$ is centered. □

Proposition 25.43. (III.3.32) *If $2 \le |J| \le \omega$, then* $\text{Fn}(I, J, \omega)$ *is σ-centered iff $|I| \le 2^\omega$.*

Proof. Suppose that $\text{Fn}(I, J, \omega)$ is σ-centered: $\text{Fn}(I, J, \omega) = \bigcup_{n\in\omega} K_n$, with each K_n centered; but suppose also that $|I| > 2^\omega$. For each $n \in \omega$ let $f_n \in {}^I J$ be such that $\bigcup K_n \subseteq f_n$; this is possible since K_n is centered. Then for each $i \in I$ we have $\langle f_n(i) : n \in \omega\rangle \in {}^\omega J$ so, since $|I| > 2^\omega$, there are distinct $j, k \in I$ such that $\langle f_n(j) : n \in \omega\rangle = \langle f_n(k) : n \in \omega\rangle$. Let $h = \{(j, 0), (k, 1)\}$. Choose n so that $h \in K_n$. Then $f_n(j) = 0 \ne 1 = f_n(k)$, contradiction.

Now suppose that $|I| \leq 2^\omega$. Wlog $J \subseteq \omega$. Fix $j \in J$. Let $\langle f_i : i \in I\rangle$ be a system of independent functions; see Theorem 21.35. For each $e \in \omega$ define $x_e \in {}^I J$ by setting

$$x_e(i) = \begin{cases} f_i(e) & \text{if } f_i(e) \in J, \\ j & \text{otherwise.} \end{cases}$$

Let $K_e = \{h \in \mathrm{Fn}(I, J, \omega) : h \subseteq x_e\}$. Clearly each K_e is centered. We claim that $\mathrm{Fn}(I, J, \omega) = \bigcup_{e\in\omega} K_e$. For, suppose that $h \in \mathrm{Fn}(I, J, \omega)$. Choose $e \in \omega$ so that $f_i(e) = h(i)$ for each $i \in \mathrm{dmn}(h)$. Then $x_e(i) = f_i(e) = h(i)$ for each $i \in \mathrm{dmn}(h)$, so $h \subseteq x_e$ and hence $h \in K_e$. $\square$

If $\langle \mathbb{P}_i : i \in I\rangle$ is a system of forcing posets, then $\prod_{i\in I} \mathbb{P}_i$ is their product, with $x \leq y$ iff $\forall i \in I[x_i \leq y_i]$.

Lemma 25.44. (III.3.37) *If X and Y are topological spaces, then $X \times Y$ is ccc iff the forcing poset $\{O : O$ open and nonempty in $X\} \times \{O : O$ open and nonempty in $Y\}$ is ccc.*

Proof. $\Rightarrow$: If $\langle (U_\alpha, V_\alpha) : \alpha < \omega_1\rangle$ is pairwise disjoint in $\{O : O$ open and nonempty in $X\} \times \{O : O$ open and nonempty in $Y\}$, then $\langle U_\alpha \times V_\alpha : \alpha < \omega_1\rangle$ is pairwise disjoint in $X \times Y$.

$\Leftarrow$: If $\langle W_\alpha : \alpha < \omega_1\rangle$ is a system of pairwise disjoint open sets in $X \times Y$, for each $\alpha < \omega_1$ there is a nonempty $U_\alpha \times V_\alpha \subseteq W_\alpha$; then $\langle (U_\alpha, V_\alpha) : \alpha < \omega_1\rangle$ is pairwise disjoint in $\{O : O$ open and nonempty in $X\} \times \{O : O$ open and nonempty in $Y\}$. $\square$

Proposition 25.45. (III.3.38) *If there are ccc spaces X, Y such that $X \times Y$ is not ccc, then there is a ccc space Z such that $Z \times Z$ is not ccc. If there are ccc forcing posets $\mathbb{P}, \mathbb{Q}$ such that $\mathbb{P} \times \mathbb{Q}$ is not ccc, then there is a ccc forcing poset $\mathbb{R}$ such that $\mathbb{R} \times \mathbb{R}$ is not ccc.*

Proof. First suppose that X and Y are ccc but $X \times Y$ is not ccc. Let $Z = (X \times \{0\}) \cup (Y \times \{1\})$. Let

$$M = \{(U \times \{0\}) : U \text{ nonempty, open in } X\} \cup \{(V \times \{1\}) : V \text{ nonempty, open in } Y\}$$

Clearly M is a base for a topology on Z.

Z is ccc, for suppose that $\mathscr{A}$ is an uncountable set of pairwise disjoint nonempty open subsets of Z. Wlog each member of $\mathscr{A}$ is in M. So there are uncountably many of the first kind, or of the second kind, contradicting X ccc or Y ccc.

$Z \times Z$ is not ccc. For, let $\langle W_\alpha : \alpha < \omega_1\rangle$ be a system of nonempty pairwise disjoint open subsets of $X \times Y$. We may assume that $W_\alpha = U_\alpha \times V_\alpha$ with U_α open in X and V_α open in Y. Then $\langle (U_\alpha \times \{0\}) \times (V_\alpha \times \{1\}) : \alpha < \omega_1$ is a system of nonempty pairwise disjoint elements of $Z \times Z$.

Second, suppose that $\mathbb{P}$ and $\mathbb{Q}$ are ccc posets such that $\mathbb{P} \times \mathbb{Q}$ is not ccc. Let $\mathbb{R} = (\mathbb{P} \times \{0\}) \cup (\mathbb{Q} \times \{1\})$, with

$$\leq_{\mathbb{R}} = \{((p, 0), (p', 0)) : p \leq p'\} \cup \{((q, 1), (q', 1)) : q \leq q'\}.$$

If $\langle r_\alpha : \alpha < \omega_1\rangle$ is a system of pairwise incompatible members of $\mathbb{R}$, then we get a contradiction depending on whether there are countably many of type 1 or of type 2.

Say $\langle (p_\alpha, q_\alpha) : \alpha < \omega_1 \rangle$ is a system of pairwise incompatable elements. Then clearly $\langle ((p_\alpha, 0), (q_\alpha, 1)) : \alpha < \omega_1 \rangle$ is a system of pairwise incompatible elements of $\mathbb{R} \times \mathbb{R}$. $\square$

Proposition 25.46. (III.3.39) *Let $\mathbb{P}_i$ be a forcing poset for all $i \in I$, and suppose that $p_i, q_i \in \mathbb{P}_i$ with $p_i \perp q_i$. Then $\prod_{i \in I} \mathbb{P}_i$ has an antichain of size $2^{|I|}$.*

Proof. Clearly $\prod_{i \in I} \{p_i, q_i\}$ is pairwise incompatible. $\square$

$\prod_{i \in I}^{\text{fin}} \mathbb{P}_i$ is the set of all $p \in \prod_{i \in I} \mathbb{P}_i$ such that $\{i \in I : p_i \neq \mathbb{1}_i\}$ is finite.

Theorem 25.47. (III.3.41) *Let $\mathbb{P}_i$ be a forcing poset for each $i \in I$. Suppose that $\prod_{i \in I}^{\text{fin}} \mathbb{P}_i$ is not ccc. Then there is a finite $F \subseteq I$ such that $\prod_{i \in F}^{\text{fin}} \mathbb{P}_i$ is not ccc.*

Proof. Suppose that $\langle p^\alpha : \alpha < \omega_1 \rangle$ is pairwise incompatible. For each $q \in \prod_{i \in I}^{\text{fin}} \mathbb{P}_i$ let $\text{supp}(q) = \{i \in I : q_i \neq \mathbb{1}_i\}$. Let $\langle \text{supp}(p^\alpha) : \alpha \in M \rangle$ be a Δ-system with kernel D, M uncountable. Then $\langle p^\alpha \restriction D : \alpha \in M \rangle$ is pairwise incompatible. For, if $\alpha, \beta \in M$ with $\alpha \neq \beta$, then there exists $i \in I$ such that p_i^α and p_i^β are incompatible. Clearly $i \in D$. $\square$

Lemma 25.48. (III.3.42) *If $\mathbb{P}$ is ccc and ω_1 is a pre-caliber for $\mathbb{Q}$, then $\mathbb{P} \times \mathbb{Q}$ has ccc.*

Proof. Suppose that $\langle (p_\alpha, q_\alpha) : \alpha < \omega_1 \rangle$ is an antichain. Fix an uncountable $B \subseteq \omega_1$ such that $\{q_\alpha : \alpha \in B\}$ is centered. If α, β are distinct members of B, then $p_\alpha \perp p_\beta$, contradiction. $\square$

Proposition 25.49. *If $\langle X_i : i \in I \rangle$ is a system of topological spaces, then $\prod_{i \in I} X_i$ has ccc iff $\prod_{i \in I}^{\text{fin}} \mathbb{O}_{X_i}$ has ccc.*

Proof. Suppose that $\prod_{i \in I} X_i$ does not have ccc. Then there is a pairwise disjoint system $\langle U_\alpha : \alpha < \omega_1 \rangle$ of nonempty open basic open sets. Say $F_\alpha \subseteq I$ is a finite set with $U_\alpha = \{f \in \prod_{i \in I} X_i : f \restriction F_\alpha \in W_\alpha\}$ where $W_\alpha = \prod_{i \in F_\alpha} V_{i\alpha}$, each $V_{i\alpha}$ open in X_i. Let $\langle F_\alpha : \alpha \in M \rangle$ be a Δ-system with kernel G, M uncountable. Then $\langle \langle V_{i\alpha} : i \in G \rangle : \alpha \in M \rangle$ is an antichain in $\prod_{i \in G} \mathbb{O}_{X_i}$, and so $\prod_{i \in I}^{\text{fin}} \mathbb{O}_{X_i}$ does not have ccc.

If $\prod_{i \in I}^{\text{fin}} \mathbb{O}_{X_i}$ does not have ccc, clearly $\prod_{i \in I} X_i$ does not have ccc. $\square$

Theorem 25.50. (III.3.43) *Assume MA(ω_1). If $\mathbb{P}_i$ is ccc for each $i \in I$, then $\prod_{i \in I}^{\text{fin}} \mathbb{P}_i$ is ccc.*

Proof. Assume MA(ω_1) and $\mathbb{P}_i$ is ccc for each $i \in I$. By Lemma 25.14, each $\mathbb{P}_i$ has ω_1 as a pre-caliber. Hence by Lemma 25.48, each finite product of the $\mathbb{P}_i$ is ccc. So by Theorem 25.47, $\prod_{i \in I}^{\text{fin}} \mathbb{P}_i$ is ccc. $\square$

Theorem 25.51. (III.3.43) *Assume MA(ω_1). If X_i is a ccc space for each $i \in I$, then $\prod_{i \in I} X_i$ is ccc.*

Proof. Assume MA(ω_1) and suppose that X_i is a ccc space for each $i \in I$. Then each $\mathbb{O}_I$ is a ccc poset. By Theorem 25.50, $\prod_{i \in I}^{\text{fin}} \mathbb{O}_i$ is ccc. Hence by Proposition 25.49, $\prod_{i \in I} X_i$ is ccc. $\square$

Proposition 25.52. (III.3.45) *(i) If X is a compact Hausdorff space, then X has ω_1 as a caliber iff $\mathbb{O}_X$ has ω_1 as a pre-caliber.*

(ii) MA(ω_1) implies that every ccc compact Hausdorff space has ω_1 as a caliber.

(iii) Under ¬SH, there is a compact Hausdorff space without ω_1 as a caliber.

Proof. (i): $\Rightarrow$: obvious. $\Leftarrow$: Suppose that $U_\alpha : \alpha < \omega_1\rangle$ is a system of nonempty open sets. For each α let V_α be nonempty, open, with $\overline{V_\alpha} \subseteq U_\alpha$. Choose an uncountable M such that $\langle V_\alpha : \alpha \in M\rangle$ is centered. Then $\emptyset \neq \bigcap_{\alpha\in M} \overline{V_\alpha} \subseteq \bigcap_{\alpha\in M} U_\alpha$.

(ii): Assuming MA(ω_1), if X is ccc compact Hausdorff, then by Lemma 25.14, $\mathbb{O}_X$ has ω_1 as a pre-caliber, hence by the above has ω_1 as a caliber.

(iii): Under ¬SH, a Suslin line gives a ccc X with $X \times X$ not ccc. By Proposition 21.39 we may assume that X is compact. By Lemma 25.48, X does not have ω_1 as a pre-caliber. □

Lemma 25.53. (III.3.46) *Every ccc poset of size less than $\mathfrak{m}$ is a countable union of filters and hence is σ-centered.*

Proof. Assume that $|\mathbb{P}| = \kappa$, $\mathbb{P}$ is ccc, and MA(κ). The conclusion is obvious if $\kappa = \omega$, so suppose that $\kappa > \omega$. Let $\mathbb{Q} = {}^{\omega}\mathbb{P}^{\text{fin}}$. Then $\mathbb{Q}$ is ccc by Theorem 25.50. Let π_i be the natural mapping of $\mathbb{Q}$ onto $\mathbb{P}$, given by $\pi_i(p) = p_i$. Note that if G is a filter on $\mathbb{Q}$, then $\pi_i[G]$ is a filter on $\mathbb{P}$. In fact, if $p \in G$ and $\pi_i(p) \leq q$, let $r \restriction (\omega\backslash\{i\}) = p \restriction (\omega\backslash\{i\})$ and $r_i = q$. Then $p \leq r$, so $r \in G$, hence $q \in \pi_i[G]$. If $p, q \in G$, choose $r \in G$ with $r \leq p, q$; then $r_i \leq p_i, q_i$. If $r \in \mathbb{P}$, then $D_r \stackrel{\text{def}}{=} \{q \in \mathbb{Q} : \exists i[q_i = r]\}$ is dense in $\mathbb{Q}$. For, let $s \in \mathbb{Q}$. Choose $i \notin \text{supp}(s)$ and let t be like s except that $t_i = r$. Let G intersect each D_r. Then $\bigcup_{i\in\omega} \pi_i[G] = \mathbb{P}$. □

We write $\mathbb{Q} \leq \mathbb{P}$ for the usual notion of substructure. For $q, q' \in \mathbb{Q}$ we write $q \not\perp_{\mathbb{P}} q'$ for $\exists r \in \mathbb{P}[r \leq q, q']$.

Proposition 25.54. (III.3.48) *The poset $\mathbb{Q} = \{\{(\alpha, 0)\} : \alpha < \omega_1\} \cup \{\emptyset\}$ is not ccc, but is a subposet of $\text{Fn}(\omega_1, 2, \omega)$, which is ccc.* □

We write $\mathbb{Q} \subseteq_{\text{ctr}} \mathbb{P}$ iff $\mathbb{Q} \subseteq \mathbb{P}$ and for every finite $F \subseteq \mathbb{Q}[\exists p \in \mathbb{P}\forall q \in F[p \leq q] \to \exists p \in \mathbb{Q}\forall q \in F[p \leq q]$.

Proposition 25.55. *If $I \subseteq L$, then $\text{Fn}(I, J, \lambda) \subseteq_{\text{ctr}} \text{Fn}(L, J, \lambda)$.*

Proof. Clearly $\text{Fn}(I, J, \lambda) \subseteq \text{Fn}(L, J, \lambda)$. Now suppose that $F \in [\text{Fn}(I, J, \lambda)]^{<\omega}$, $p \in \text{Fn}(L, J, \lambda)$, and $\forall q \in F[p \leq q]$. Then $\bigcup F \subseteq p$, $\bigcup F \in \text{Fn}(I, J, \lambda)$, and $\forall q \in F[\bigcup F \leq q]$. □

Lemma 25.56. (III.3.50) *If $\mathbb{Q} \subseteq_{\text{ctr}} \mathbb{P}$, then:*

(i) If $\mathbb{P}$ is ccc, then $\mathbb{Q}$ is ccc.

(ii) If $\mathbb{P}$ is σ-centered, then $\mathbb{Q}$ is σ-centered.

(iii) If $\mathbb{P}$ has κ as a pre-caliber, then $\mathbb{Q}$ has κ as a pre-caliber.

Proof. (i): Given $q \in {}^{\omega_1}\mathbb{Q}$, there exist distinct $\alpha, \beta < \omega_1$ such that q_α and q_β are compatible in $\mathbb{P}$, hence in $\mathbb{Q}$.

(ii): Assume that $\mathbb{P}$ is σ-centered; say $\mathbb{P} = \bigcup_{n\in\omega} A_n$, each A_n centered in $\mathbb{P}$. Then $\mathbb{Q} = \bigcup_{n\in\omega}(A_n \cap \mathbb{Q})$, and each $A_n \cap \mathbb{Q}$ is centered in $\mathbb{P}$, hence also in $\mathbb{Q}$.

(iii): Given $q \in {}^{\kappa}\mathbb{Q}$, choose $A \in [\kappa]^{\kappa}$ such that $\langle q_\alpha : \alpha \in A\rangle$ is centered in $\mathbb{P}$, hence in $\mathbb{Q}$. □

Lemma 25.57. (III.3.51) *If κ is an infinite cardinal and $MA_{\mathbb{P}}(\kappa)$ is false, then there is a $\mathbb{Q} \subseteq_{\rm ctr} \mathbb{P}$ such that $|\mathbb{Q}| \le \kappa$ and $MA_{\mathbb{Q}}(\kappa)$ is false.*

Proof. Fix dense sets D_α in $\mathbb{P}$ for $\alpha < \kappa$ such that there is no filter intersecting each D_α. Consider the structure $(\mathbb{P}, \le, \mathbb{1}, D_\alpha)_{\alpha<\kappa}$. By the downward Löwenheim-Skolem theorem let $(\mathbb{Q}, \le, \mathbb{1}, D'_\alpha)_{\alpha<\kappa}$ be an elementary substructure of $(\mathbb{P}, \le, \mathbb{1}, D_\alpha)_{\alpha<\kappa}$ such that $|\mathbb{Q}| \le \kappa$. Then D'_α is dense in $\mathbb{Q}$ for each $\alpha < \kappa$. For any finite $F \subseteq \mathbb{Q}$, if $\exists p \in \mathbb{P}\forall q \in F[p \le q]$, then $\exists p \in \mathbb{Q}\forall q \in F[p \le q]$. Thus $\mathbb{Q} \subseteq_{\rm ctr} \mathbb{P}$.

Suppose that $\mathrm{MA}_{\mathbb{Q}}(\kappa)$ holds. Let G be a filter on $\mathbb{Q}$ which intersects each D'_α. Let $G^+ = \{p \in \mathbb{P} : \exists q \in G[q \le p]\}$. Clearly G^+ is a filter on $\mathbb{P}$, and it intersects each D_α, contradiction. □

For the next result, recall that $s{\downarrow}' = \{x : x \le s\}$.

Proposition 25.58. *For any forcing poset $\mathbb{P}$ the set $\{U \subseteq \mathbb{P} : \forall s \in U[s{\downarrow}' \subseteq U]\}$ is a topology on $\mathbb{P}$.* □

The topology given in this proposition is called the *poset topology* on $\mathbb{P}$.

Proposition 25.59. *$D \subseteq \mathbb{P}$ is dense in the topological sense iff it is dense in the poset sense.*

Proof. ⇒: Suppose that $D \subseteq \mathbb{P}$ is dense in the topological sense, and $p \in \mathbb{P}$. Choose $s \in D \cap (p{\downarrow}')$. So $s \le p$.

⇐: Suppose that $D \subseteq \mathbb{P}$ is dense in the poset sense, and U is a nonempty open set. Say $p \in U$. Choose $q \in D$ such that $q \le p$. Since $(p{\downarrow}') \subseteq U$, we have $q \in U$. Thus $D \cap U \ne \emptyset$. □

Proposition 25.60. *$\mathbb{P}$ has ccc in the topological sense iff it has ccc in the poset sense.*

Proof. ⇒: Suppose that $\mathbb{P}$ has ccc in the topological sense, and $\langle p_\alpha : \alpha < \omega_1\rangle$ is pairwise incompatible. Then $\langle p_\alpha{\downarrow}' : \alpha < \omega_1\rangle$ is a pairwise disjoint system of open sets.

⇐: reverse this argument. □

For any $S \subseteq \mathbb{P}$ let $S{\downarrow}' = \bigcup_{p\in S}(p{\downarrow}')$. This is the smallest open set containing S.

Lemma 25.61. (III.3.55) *If G is a filter on $\mathbb{P}$ and $S \subseteq P$, then $S \cap G = \emptyset$ iff $(S{\downarrow}') \cap G = \emptyset$.* □

$L \subseteq \mathbb{P}$ is *linked* iff any two elements of L are compatible.

A forcing poset P is said to have *property* K iff for every system $\langle p_\alpha : \alpha < \omega_1\rangle$ of elements of P there is an $X \in [\omega_1]^{\omega_1}$ which is linked.

For $E \subseteq \mathbb{P}$ and $p \in \mathbb{P}$, $p \perp E$ means $\forall q \in E[p \perp q]$. E is *predense* iff $\neg\exists p[p \perp E]$.

Proposition 25.62. *E is predense iff $\forall p \exists q \in E[p$ and q are compatible$]$.* ☐

Lemma 25.63. (III.3.59) *E is predense iff $E{\downarrow}'$ is dense iff $E{\downarrow}'$ is open dense.*

Proof. Suppose that E is predense. Given p, choose $q \in E$ such that p and q are compatible. Say $r \leq p, q$. Then $r \in E{\downarrow}'$ and $r \leq p$. So $E{\downarrow}'$ is dense. Other implications trivial. ☐

Theorem 25.64. (III.3.60) *Let κ be an infinite cardinal, $\mathbb{P}$ a forcing poset, and $E_\alpha \subseteq \mathbb{P}$ for all $\alpha < \kappa$. Let χ be the statement that if each E_α has property φ then there is an $F \subseteq \mathbb{P}$ with property ψ such that $\forall \alpha < \kappa[F \cap E_\alpha \neq \emptyset]$.*

Here φ is any of the following:

1. *dense set*
2. *dense open set*
3. *maximal antichain*
4. *predense set*

ψ is one of

a. *filter*
b. *linked family*

This gives 8 possibilities for χ, and the assertion of this theorem is that they are all equivalent.

Proof. (A) (xa)$\Rightarrow$(xb). For, Assume (xa). If $E_\alpha \subseteq \mathbb{P}$ for all $\alpha < \kappa$ and all of them have the property x, then there is an $F \subseteq \mathbb{P}$ with property a such that $F \cap E_\alpha \neq \emptyset$ for all α. Then F has property b. So (xb) holds.

(B) (4y)$\Rightarrow$(3y). For, assume (4y). If $E_\alpha \subseteq \mathbb{P}$ for all $\alpha < \kappa$ and all of them have the property 3; then all of them have property 4, and the desired conclusion follows.

(C) (3y)$\Rightarrow$(1y). For, assume (3y). Assume that $E_\alpha \subseteq \mathbb{P}$ for all $\alpha < \kappa$ and all of them have the property 1. For each $\alpha < \kappa$ let E'_α be maximal such that $E'_\alpha \subseteq E_\alpha$ and E'_α is an antichain. Then E'_α is actually a maximal antichain. So E'_α has property 3. The desired conclusion follows.

(D) (1y)$\Rightarrow$(2y). This is clear since 2 implies 1.

(E) (2a)$\Rightarrow$(4a). Assume (2a), and suppose that $E_\alpha \subseteq \mathbb{P}$ for all $\alpha < \kappa$ and all of them are predense. Hence each $E_\alpha{\downarrow}'$ is dense open by Lemma 25.63. Hence there is a filter $F \subseteq \mathbb{P}$ such that $F \cap E_\alpha{\downarrow}' \neq \emptyset$ for all α. By Lemma 25.61, $F \cap E_\alpha \neq \emptyset$ for all α.

(F) (1a), (2a), (3a), (4a) are all equivalent. This is true by (B)–(E).

(G) (2b)$\Rightarrow$(2a). Assume (2b), and suppose that $E_\alpha \subseteq \mathbb{P}$ for all $\alpha < \kappa$ and all of them are dense open. We will get a filter which meets each of them. For each α let $A_\alpha \subseteq E_\alpha$ be maximal among antichains contained in E_α. So A_α is a maximal antichain. Let $F_0 = \{A_\alpha{\downarrow}' \colon \alpha < \kappa\}$. Suppose that F_n has been defined and consists of dense open sets. For each pair (B, C) of elements of F_n the set $B \cap C$ is dense open, and we can find a maximal antichain $D_{BC} \subseteq B \cap C$. Let $F_{n+1} = F_n \cup \{D_{BC}{\downarrow}' \colon B, C \in F_n\}$. Finally, let $G = \bigcup_{n \in \omega} F_n$. Thus for every pair B, C of elements of G there is a maximal antichain $D \subseteq B \cap C$ such that $D{\downarrow}' \in G$. By (2b), let L be linked such that $L \cap B \neq \emptyset$ for all $B \in G$. Each element B of G has the form $A_B{\downarrow}'$ for some maximal antichain A_B. For each $B \in G$

choose $q_B \in L \cap B$, and then choose $p_B \in A_B$ such that $q_B \leq p_B$. Since $\{q_B : B \in G\}$ is a pairwise compatible set, so is $\{p_B : B \in G\}$.

(1) For any $B, C \in G$ there is a $D \in G$ such that $p_D \leq p_B, p_C$.

In fact, write $B = B'{\downarrow}'$ with B' a maximal antichain, and $C = C'{\downarrow}'$ with C' a maximal antichain. Let E be a maximal antichain such that $E \subseteq B \cap C$ and $D \stackrel{\text{def}}{=} E{\downarrow}' \in G$. Suppose that $p_D \not\leq p_B$. Now $p_D \in E \subseteq B = B'{\downarrow}'$, so there is an $r \in B'$ such that $p_D \leq r$. But also $p_B \in B'$, so $r \neq p_B$ and hence p_B and p_D are incompatible, contradiction. By symmetry, (1) follows.

Now let $H = \{s : p_B \leq s$ for some $B \in G$. Then H is a filter. It intersects each E_α. For, $B \stackrel{\text{def}}{=} A_\alpha{\downarrow}' \subseteq E_\alpha$, and $B \in G$, so $p_B \in H \cap E_\alpha$. So (G) holds.

Now all are equivalent. For, (1a)–(4a) are equivalent by (F), (4a)⇒(4b) by (A), (4b)⇒(3b)⇒(1b)⇒(2b) by (B), (C), (D), and (2b)⇒(2a) by (G). □

Theorem 25.65. (III.3.61) *Let $\mathfrak{m}_\sigma$ be the least cardinal κ such that $MA_{\mathbb{P}}(\kappa)$ is false for some σ-centered $\mathbb{P}$. Then $\mathfrak{m}_\sigma = \mathfrak{p}$.*

Proof. $\mathfrak{m}_\sigma \leq \mathfrak{p}$ by Lemma 25.37 and Theorem 25.34. To prove that $\mathfrak{p} \leq \mathfrak{m}_\sigma$, take any infinite $\kappa < \mathfrak{p}$ and fix a σ-centered forcing poset $\mathbb{P}$. We show that $\mathrm{MA}_{\mathbb{P}}(\kappa)$ holds. By Lemma 25.57 we may assume that $|\mathbb{P}| \leq \kappa$. Fix dense open $D_\alpha \subseteq \mathbb{P}$ for each $\alpha < \kappa$. By Theorem 25.64 it suffices to find a linked $L \subseteq \mathbb{P}$ such that $\forall \alpha < \kappa[L \cap D_\alpha \neq \emptyset]$.

Let $\mathbb{P} = \bigcup_{l \in \omega} C_l$, with each C_l centered. We may assume that $\mathbb{1} \in C_l$ for each $l \in \omega$. By Proposition 25.31 we may assume that $\mathbb{P}$ is atomless.

For $\alpha < \kappa$ and $p \in \mathbb{P}$, let

$$B_\alpha(p) = \{l \in \omega : D_\alpha \cap C_l \cap p{\downarrow}' \neq \emptyset\}. \tag{1}$$

Now we claim that for each $m \in \omega$, the set $\{B_\alpha(p) : p \in C_m$ and $\alpha < \kappa\}$ has SFIP. For, suppose that $\langle p_i : i \in n\rangle$ is a system of elements of C_m and $\langle \alpha_i : i < n\rangle$ is a system of members of κ; we want to show that $\bigcap_{i<n} B_{\alpha_i}(p_i)$ is infinite. Let $E = \bigcap_{i<n} D_{\alpha_i}$. So E is dense open. Since C_m is centered, choose q so that $q \leq p_i$ for each i. Then

$$I \stackrel{\text{def}}{=} \{l \in \omega : E \cap C_l \cap q{\downarrow}' \neq \emptyset\} \subseteq \bigcap_{i<n} B_{\alpha_i}(p_i). \tag{2}$$

In fact, if $l \in I$ and $i < n$, then $E \cap C_l \cap q{\downarrow}' \subseteq D_{\alpha_i} \cap C_l \cap p_i{\downarrow}'$, and so $D_{\alpha_i} \cap C_l \cap p_i{\downarrow}' \neq \emptyset$, as desired.

Since $\mathbb{P}$ is atomless, let $\langle a_j : j \in \omega\rangle$ be a system of pairwise incompatible elements $\leq q$; see Lemma 25.33. Choose $b_j \leq a_j$ with $b_j \in E$ for every $j \in \omega$. Say $b_j \in C_{l_j}$ for all $j \in \omega$. Since the b_j's are pairwise incompatible, the sequence $\langle l_j : j \in \omega\rangle$ is one-one. Clearly each l_j is in I. So our claim follows from (2).

Now by the claim, since $\kappa < \mathfrak{p}$, for each $m \in \omega$ let $Z_m \in [\omega]^\omega$ be such that $Z_m \subseteq^* B_\alpha(p)$ for all $p \in C_m$ and all $\alpha < \kappa$.

For each $\tau \in {}^{<\omega}\omega$ let $\Lambda(\tau) = \emptyset$ if $\tau = \emptyset$, and otherwise let $\Lambda(\tau) = \tau(\mathrm{dmn}(\tau) - 1)$.

Let $T = \{\tau \in {}^{<\omega}\omega : \forall n < \mathrm{dmn}(\tau)[\tau(n) \in Z_{\Lambda(\tau \restriction n)}]\}$. Thus $\emptyset \in T$. If $\tau \in T$ and $m \in Z_{\Lambda(\tau)}$, then $\tau^\frown\langle m\rangle \in T$.

For each $\alpha < \kappa$ fix $\Delta_\alpha : T \to \mathbb{P}$ with the following properties:

(3) $\Delta_\alpha(\emptyset) = \mathbb{1}$.

(4) If $\tau^\frown\langle l\rangle \in T$, then
(a) If $l \in B_\alpha(\Delta_\alpha(\tau))$, then $\Delta_\alpha(\tau^\frown\langle l\rangle) \in D_\alpha \cap C_l \cap \Delta_\alpha(\tau)\downarrow'$.
(b) If $l \notin B_\alpha(\Delta_\alpha(\tau))$, then $\Delta_\alpha(\tau^\frown\langle l\rangle) = \mathbb{1}$.

Note that

$$\forall\tau \in T[\Delta_\alpha(\tau) \in C_{\Lambda(\tau)}], \tag{5}$$

In fact, if $\tau \neq \emptyset$ and $\tau(\mathrm{dmn}(\tau) - 1) \in B_\alpha(\Delta_\alpha(\tau \restriction (\mathrm{dmn}(\tau) - 1)))$, then this follows from (4)(a). Otherwise it follows since $\mathbb{1} \in C_{\Lambda(\tau)}$.

$$\forall\alpha\forall\tau \in T[Z_{\Lambda(\tau)} \subseteq^* B_\alpha(\Delta_\alpha(\tau))]. \tag{6}$$

In fact, suppose that $\alpha < \kappa$ and $\tau \in T$. Since $\Delta_\alpha(\tau) \in C_{\Lambda(\tau)}$ by (5), it follows that $Z_{\Lambda(\tau)} \subseteq^* B_\alpha(\Delta_\alpha(\tau))$, proving (6).

$$\forall\alpha\forall\tau \in T\forall l \in \omega[\tau^\frown\langle l\rangle \in T \to l \in Z_{\Lambda(\tau)}]. \tag{7}$$

This follows from the definition of T.

By (6), for each $\alpha < \kappa$ there is a function $\Phi_\alpha : T \to \omega$ such that:

$$\forall\tau \in T\forall l \in Z_{\Lambda(\tau)}[l \geq \Phi_\alpha(\tau) \to l \in B_\alpha(\Delta_\alpha(\tau))]. \tag{8}$$

Now $|T| = \omega$; let $h : \omega \to T$ be a bijection. Also, $\kappa < \mathfrak{p} \leq \mathfrak{b}$, so there is a $\Gamma : \omega \to \omega$ such that $\Phi_\alpha \circ h \leq^* \Gamma$ for all $\alpha < \kappa$.

$$\forall\tau \in T\forall m \in \omega\exists l \geq m[\tau^\frown\langle l\rangle \in T]. \tag{9}$$

In fact, $Z_{\Lambda(\tau)}$ is infinite, so (9) follows.

By (9), there is a function $g : \omega \to \omega$ such that $g \restriction n \in T$ for all $n \in \omega$ and $g(n)$ is the least $l \geq \Gamma(h^{-1}(g \restriction n)), n + 1$ such that $(g \restriction n)^\frown\langle l\rangle \in T$.

Since $\Phi_\alpha \circ h \leq^* \Gamma$, there is a $k : \kappa \to \omega$ such that $\Phi_\alpha(h(n)) \leq \Gamma(n)$ for all $n \geq k_\alpha$. Hence if $h^{-1}(g \restriction n) \geq k_\alpha$ then $g(n) \geq \Gamma(h^{-1}(g \restriction n)) \geq \Phi_a(g \restriction n)$; hence by (8), $g(n) \in B_\alpha(\Delta_a(g \restriction n))$.

Now $(g \restriction n)^\frown\langle g(n)\rangle = g \restriction (n + 1) \in T$ and $g(n) \in B_\alpha(\Delta_\alpha(g \restriction n))$, so by (4)(a) we get

$$\Delta_\alpha(g \restriction (n + 1)) \in D_\alpha \cap C_{g(n)} \cap \Delta_\alpha(g \restriction n)\downarrow', \tag{10}$$

and in particular

$$\Delta_\alpha(g \restriction (n + 1)) \leq \Delta_\alpha(g \restriction n). \tag{11}$$

Let $p_\alpha = \Delta_\alpha(g \restriction (k_\alpha + 1))$. So $p_\alpha \in D_\alpha \cap C_{g(k_\alpha)} \cap \Delta_\alpha(g \restriction k_\alpha)\downarrow'$. Let $L = \{p_\alpha : \alpha < \kappa\}$. Hence $L \cap D_\alpha \neq \emptyset$ for all α. To show that L is linked, suppose that $\alpha, \beta < \kappa$. Say

$k_\alpha \le k_\beta$. If $k_\alpha = k_\beta$, then $p_\alpha, p_\beta \in C_{g(k_\alpha)}$, hence $p_\alpha \not\perp p_\beta$. Suppose that $k_\alpha < k_\beta$. By (11), $\Delta_\alpha(g \restriction (k_\beta+1)) \le \Delta_\alpha(g \restriction (k_\alpha+1)) = p_\alpha$. Now $\Delta_\alpha(g \restriction (k_\beta+1)), \Delta_\beta(g \restriction (k_\beta+1)) \in C_{g(k_\beta)}$ by (10), so $p_\alpha \not\perp p_\beta$. □

$\mathbb{Q} \subseteq_c \mathbb{P}$ ($\mathbb{Q}$ is a *complete* subposet of $\mathbb{P}$ iff $\mathbb{Q} \subseteq_{\mathrm{ctr}} \mathbb{P}$ and if $A \subseteq \mathbb{Q}$ is a maximal antichain of $\mathbb{Q}$ then it is also a maximal antichain of $\mathbb{P}$.

Proposition 25.66. (III.3.64) *If $I \subseteq L$, then* $\mathrm{Fn}(I, J, \lambda) \subseteq_c \mathrm{Fn}(L.J, \lambda)$.

Proof. By Proposition 25.55 we have $\mathrm{Fn}(I, J, \lambda) \subseteq_{\mathrm{ctr}} \mathrm{Fn}(L.J, \lambda)$. Now suppose that $A \subseteq \mathrm{Fn}(I, J, \lambda)$ is a maximal antichain of $\mathrm{Fn}(I, J, \lambda)$. Take any $p \in \mathrm{Fn}(L, J, \lambda)$. Let $q = p \restriction I$. By the maximality of A, there is an $r \in A$ such that q and r are compatible; say $s \in \mathrm{Fn}(I, J, \lambda)$ and $q, r \subseteq s$. Then $p, r \subseteq s \cup (p \restriction (L \backslash J))$. □

If $i : \mathbb{Q} \to \mathbb{P}$, then i is a *complete embedding* iff

(i) $i(\mathbb{1}_\mathbb{Q}) = \mathbb{1}_\mathbb{P}$.

(ii) $\forall q_1, q_2 \in \mathbb{Q}[q_1 \le q_2 \to i(q_1) \le i(q_2)$.

(iii) $\forall q_1, q_2 \in \mathbb{Q}[q_1 \perp q_2 \leftrightarrow i(q_1) \perp i(q_2)$.

(iv) $\forall A \subseteq \mathbb{Q}[A$ a maximal antichain in $\mathbb{Q} \to i[A]$ is a maximal antichain in $\mathbb{P}]$.

i is a *dense embedding* iff (i)–(iii) hold together with

(v) $i[\mathbb{Q}]$ is a dense subset of $\mathbb{P}$.

Proposition 25.67. *If $i : \mathbb{Q} \to \mathbb{P}$, i is onto, and $\forall q_1, q_2 \in \mathbb{Q}|[q_1 \le q_2$ iff $i(q_1) \le i(q_2)]$, then (ii), (iii) of the definition of complete embedding hold.*

Proof. Clearly (ii) holds, and $\Leftarrow$ in (iii) holds. Now suppose that $i(q_1)$ and $i(q_2)$ are compatible. Say $i(q_3) \le i(q_1), i(q_2)$. Then $q_3 \le q_1, q_2$. □

Proposition 25.68. *If $i : \mathbb{Q} \to \mathbb{P}$, (i), (v) of the definition of dense embedding hold, and and $\forall q_1, q_2 \in \mathbb{Q}|[q_1 \le q_2$ iff $i(q_1) \le i(q_2)]$, then (ii), (iii) of the definition of dense embedding hold.*

Proof. (ii) is obvious, and $\Leftarrow$ of (iii) is clear. Now suppose that $i(q_1)$ and $i(q_2)$ are compatible. Say $p \le i(q_1), i(q_2)$. Choose $q_3 \in \mathbb{Q}$ so that $i(q_3) \le p$, Then $i(q_3) \le i(q_1), i(q_2)$. Hence $q_3 \le q_1, q_2$. □

Lemma 25.69. (III.3.66) *Suppose that $\mathbb{Q}$ is a subset of $\mathbb{P}$ and $i : \mathbb{Q} \to \mathbb{P}$ is the inclusion map. Then i is a complete embedding iff $\mathbb{Q} \subseteq_c \mathbb{P}$.*

Proof. First assume that i is a complete embedding. To show that $\mathbb{Q} \le_{\mathrm{ctr}} \mathbb{P}$, suppose that $q_1, \ldots, q_n \in \mathbb{Q}$ have a common extension in $\mathbb{P}$. By Zorn's lemma, let A be maximal among subsets of $\mathbb{Q}$ satisfying

$$(*) \qquad A \text{ is an antichain and } \forall a \in A \exists j (a \perp q_j).$$

Then A is not a maximal antichain in $\mathbb{P}$, since if $p \in \mathbb{P}$ is a common extension of $q_1, \ldots, q_n$ and $a \in A$, then $a \perp p$. It follows by (iv) that A is not a maximal antichain in $\mathbb{Q}$. So there is an $r \in \mathbb{Q}$ such that $r \perp a$ for all $a \in A$. Hence

$$(**) \qquad \forall r' \in \mathbb{Q}[r' \le r \to \forall j[r' \not\perp q_j]].$$

In fact, otherwise we get $r' \in \mathbb{Q}$ and j such that $r' \leq r$ and $r' \perp q_j$, and then $r' \perp a$ for all $a \in A$, and $A \cup \{r'\}$ satisfies $(*)$, contradicting maximality.

By $(**)$ we can inductively construct $p_1, \ldots, p_n \in \mathbb{Q}$ such that $r \geq p_1 \geq \cdots \geq p_n$ and $p_j \leq q_j$ for each j. Then p_n is a common extension of $q_1, \ldots, q_n$. Hence $\mathbb{Q} \leq_{\text{ctr}} \mathbb{P}$.

It follows that $\mathbb{Q} \subseteq_c \mathbb{P}$.

Second, assume that $\mathbb{Q} \subseteq_c \mathbb{P}$. Then only (iii) is problematic. If $q_1 \not\perp_{\mathbb{Q}} q_2$, obviously $q_1 \not\perp_{\mathbb{P}} q_2$. The other direction follows from $\mathbb{Q} \subseteq_{\text{ctr}} \mathbb{P}$. □

Proposition 25.70. *Suppose that $i : \mathbb{Q} \to \mathbb{P}$ is a complete embedding, and $\forall q_1, q_2 \in \mathbb{Q}[q_1 \leq q_2$ iff $i(q_1) \leq i(q_2)]$. Let $\mathbb{R} = i[\mathbb{Q}]$, and define $i(q_1) \leq_{\mathbb{R}} i(q_2)$ iff $q_1 \leq q_2$. Then*

(i) $\mathbb{R}$ is a forcing poset.

(ii) $\mathbb{R} \subseteq_{\text{c}} \mathbb{P}$.

Proof. (i) is clear. Clearly also $\mathbb{R}$ is a subposet of $\mathbb{P}$. To show that $\mathbb{R} \leq_{\text{ctr}} \mathbb{P}$, suppose that $q_1, \ldots, q_n \in \mathbb{Q}$, $p \in \mathbb{P}$, and $\forall i[p \leq i(q_i)]$. Let A be maximal among subsets of $\mathbb{Q}$ satisfying

$$(*) \qquad A \text{ is an antichain and } \forall a \in A \exists j (a \perp q_j).$$

Then $i[A]$ is not a maximal antichain in $\mathbb{P}$, since $\forall a \in A[i(a) \perp p]$. It follows by (iv) in the definition of complete embedding that A is not a maximal antichain in $\mathbb{Q}$. So there is an $r \in \mathbb{Q}$ such that $r \perp a$ for all $a \in A$. Hence

$$(**) \qquad \forall r' \in \mathbb{Q}[r' \leq r \to \forall j[r' \not\perp q_j]].$$

In fact, otherwise we get $r' \in \mathbb{Q}$ and j such that $r' \leq r$ and $r' \perp q_j$, and then $r' \perp a$ for all $a \in A$, and $A \cup \{r'\}$ satisfies $(*)$, contradicting maximality.

By $(**)$ we can inductively construct $p_1, \ldots, p_n \in \mathbb{Q}$ such that $r \geq p_1 \geq \cdots \geq p_n$ and $p_j \leq q_j$ for each j. Then p_n is a common extension of $q_1, \ldots, q_n$. Hence $\mathbb{Q} \leq_{\text{ctr}} \mathbb{P}$.

Now suppose that $A \subseteq \mathbb{Q}$ and $i[A]$ is a maximal antichain in $\mathbb{R}$. Clearly then A is an antichain in $\mathbb{Q}$. Suppose that $q \in \mathbb{Q}$ and $\forall a \in A[q \perp a]$. Then $\forall a \in A[i(q) \perp i(a)]$, contradiction. Hence A is a maximal antichain in $\mathbb{Q}$. It follows from (iv) in the definition of complete embedding that $i[A]$ is a maximal antichain in $\mathbb{P}$. This proves that $\mathbb{R} \subseteq_{\text{c}} \mathbb{P}$. □

Lemma 25.71. (III.3.67) *Every dense embedding is a complete embedding.*

Proof. Suppose that $i : \mathbb{Q} \to \mathbb{P}$ is a dense embedding. To verify (iv), suppose that $A \subseteq \mathbb{Q}$ is a maximal antichain in $\mathbb{Q}$, but $i[A]$ is not a maximal antichain in $\mathbb{P}$. By (iii), $i[A]$ is an antichain in $\mathbb{P}$. So there is a $p \in \mathbb{P}$ such that $p \perp i(q)$ for all $q \in A$. By (v) choose $r \in \mathbb{Q}$ such that $i(r) \leq p$. Then $i(r) \perp i(q)$ for all $q \in A$, so by (iii), $r \perp q$ for all $q \in A$, contradiction. □

Proposition 25.72. *If $i : \mathbb{Q} \to \mathbb{P}$ is a dense embedding and $D \subseteq \mathbb{P}$ is dense open, then $i^{-1}[D]$ is dense in $\mathbb{Q}$.* (For the definition of an open subset of a forcing poset, see Proposition 25.58.)

Proof. Suppose that $q \in \mathbb{Q}$. Choose $r \in D$ such that $r \leq i(q)$. Then choose $q' \in \mathbb{Q}$ such that $i(q') \leq r$. Then $i(q') \in D$ since D is open. Since $i(q') \leq i(q)$, it follows that $i(q)$ and $i(q')$ are compatible, hence by (iii), q and q' are compatible. Let $s \leq q, q'$. Then $i(s) \leq r$, so $i(s) \in D$ since D is open. So $s \in i^{-1}[D]$ and $s \leq q$. $\square$

Lemma 25.73. *Let $i : \mathbb{Q} \to \mathbb{P}$ be a complete embedding, and assume that $MA_{\mathbb{P}}(\kappa)$. Then $MA_{\mathbb{Q}}(\kappa)$.*

Proof. By Theorem 25.64 it suffices to fix a family $\mathscr{A}$ of maximal antichains in $\mathbb{Q}$ with $|\mathscr{A}| \leq \kappa$ and produce a linked set $L \subseteq \mathbb{Q}$ such that $L \cap A \neq \emptyset$ for all $A \in \mathscr{A}$. Again by Theorem 25.64 there is a filter G on $\mathbb{P}$ such that $i[A] \cap G \neq \emptyset$ for all $A \in \mathscr{A}$. Now $i^{-1}[G]$ is linked, since if $q_1, q_2 \in i^{-1}[G]$ then $i(q_1), i(q_2) \in G$, hence they are compatible, so by (iii) also q_1 and q_2 are compatible. Now if $q \in A$ and $i(q) \in G$, then $q \in i^{-1}[G] \cap A$. So $i^{-1}[G]$ is a linked set intersecting each $A \in \mathscr{A}$. $\square$

Lemma 25.74. *Let $i : \mathbb{Q} \to \mathbb{P}$ be a dense embedding, and assume that $MA_{\mathbb{Q}}(\kappa)$. Then $MA_{\mathbb{P}}(\kappa)$.*

Proof. By Theorem 25.64 it suffices to fix a family $\mathscr{A}$ of dense open sets in $\mathbb{P}$ with $|\mathscr{A}| \leq \kappa$ and produce a linked set $L \subseteq \mathbb{P}$ such that $L \cap A \neq \emptyset$ for all $A \in \mathscr{A}$. Now for each $A \in \mathscr{A}$ the set $i^{-1}[A]$ is dense in $\mathbb{Q}$, by Proposition 25.72. Again by Theorem 25.64, let H be a filter on $\mathbb{Q}$ such that $i^{-1}[A] \cap H \neq \emptyset$ for all $A \in \mathscr{A}$. Now $i[H]$ is linked, since if $q_1, q_2 \in H$ then $i(q_1)$ and $i(q_2)$ are compatible by (iii). If $q \in i^{-1}[A] \cap H$, then $i(q) \in A \cap i[H]$. So $i[H]$ is a linked set intersecting each $A \in \mathscr{A}$. $\square$

Theorem 25.75. *If $i : \mathbb{Q} \to \mathbb{P}$ be a dense embedding and $\mathbb{Q}$ has ccc, then $\mathbb{P}$ has ccc.*

Proof. Assume that $\mathbb{Q}$ has ccc. Suppose that $\langle p_\alpha : \alpha < \omega_1 \rangle$ is an antichain in $\mathbb{P}$. For each $\alpha < \omega_1$ choose $q_\alpha \in \mathbb{Q}$ such that $i(q_\alpha) \leq p_\alpha$. Then for $\alpha \neq \beta$ we have $i(q_\alpha) \perp i(q_\beta)$, so $q_\alpha \perp q_\beta$, contradiction. $\square$

Let $\mathbb{T} = {}^{<\omega}\omega = \bigcup_{m \in \omega} {}^m\omega$.

Proposition 25.76. (III.3.69) *Let* $\mathbb{P} = \mathrm{Fn}(\omega, \omega, \omega)$. *Then $\mathbb{T}$ is a dense subposet of $\mathbb{P}$.*

Proof. Clearly $\mathbb{T}$ is a subposet of $\mathbb{P}$. (i), (ii), and (v) in the definition of dense embedding (for the inclusion map) are clear. $\Leftarrow$ in (iii) is clear. Now suppose that $t_1 \perp_{\mathbb{T}} t_2$. Then there is an $i \in \mathrm{dmn}(t_1) \cap \mathrm{dmn}(t_2$ such that $t_1(i) \neq t_2(i)$. So $t_1 \perp_{\mathbb{P}} t_2$. $\square$

Proposition 25.77. (III.3.69) *Let* $\mathbb{P} = \mathrm{Fn}(\omega, \omega, \omega)$. *Then* $\forall\kappa[\mathrm{MA}_{\mathbb{T}}(\kappa)$ *iff* $\mathrm{MA}_{\mathbb{P}}(\kappa)]$.

Proof. By Lemmas 25.71, 25.73, 25.74, 25.76. $\square$

Proposition 25.78. (III.3.70) *$MA_{\mathbb{P}}(\kappa) \leftrightarrow MA_{\mathbb{Q}}(\kappa)$ if $\mathbb{P}$ and $\mathbb{Q}$ are countable atomless forcing posets.*

Proof. Let $\langle p_k : k \in \omega \rangle$ enumerate $\mathbb{P}$. We define $i(t) \in \mathbb{P}$ with $t \in \mathbb{T}$ and $\mathrm{dmn}(t) = m$ by induction on m. Let $i(\emptyset) = \mathbb{1}$. Suppose that $i(t)$ has been defined for all t having domain

m, such that $\langle i(t) : \mathrm{dmn}(t) = m\rangle$ is a maximal antichain. Let $\langle t_j : j \in \omega\rangle$ enumerate all $s \in \mathbb{T}$ with domain m. Then there is a $j \in \omega$ such that $i(t_j)$ and p_m are compatible. Say $r \leq i(t_j), p_m$. Let X be a maximal antichain in $i(t_j){\downarrow}'$ with r as a member, and let $\langle q^j_k : k \in \omega\rangle$ enumerate X. Then for any $k \in \omega$ we define $i(t_j^\frown\langle k\rangle) = q^j_k$. For $s \neq j$, let $\langle q^s_k : k \in \omega\rangle$ enumerate a maximal antichain below $i(t_s)$, and for any $k \in \omega$ define $i(t_s^\frown\langle k\rangle) = q^s_k$. We claim that $\{i(u) : \mathrm{dmn}(u) = m + 1\}$ is a maximal antichain. For, if $v \in \mathbb{P}$ choose s so that v and $i(t_s)$ are compatible; say $w \leq v, i(t_s)$. Then choose k so that w and q^s_k are compatible. Then v and $i(t_s^\frown\langle k\rangle)$ are compatible. This completes the inductive construction.

We check that i is a dense embedding of $\mathbb{T}$ into $\mathbb{P}$. (i) follows from $i(\emptyset) = \mathbb{1}$. (ii) holds since if s has domain m, then for any k we have $s^\frown\langle k\rangle \leq s$, and $i(s^\frown\langle k\rangle)$ is defined to be below $i(s)$. Also, this means that $i(s^\frown\langle k\rangle) \leq i(s)$ implies that $s^\frown\langle k\rangle \leq s$. Clearly $i[\mathbb{T}]$ is dense in $\mathbb{P}$. So i is a dense embedding of $\mathbb{T}$ into $\mathbb{P}$ by Proposition 25.68.

Now the assertion of the proposition follows by Lemmas 25.71, 25.73, 25.74. $\square$

If $\mathbb{Q}, \mathbb{P}, i$ satisfy (i)–(iii) in the definition of complete embedding and $p \in \mathbb{P}$. a *reduction* of p to $\mathbb{Q}$ is an element $\boxed{p} \in \mathbb{Q}$ such that $\forall q \in \mathbb{Q}\left[i(q) \perp p \rightarrow q \perp \boxed{p}\ \right]$.

Lemma 25.79. (III.3.72) *If $\mathbb{Q}, \mathbb{P}, i$ satisfy (i)–(iii) in the definition of complete embedding, then i is a complete embedding of $\mathbb{Q}$ into $\mathbb{P}$ iff every element of $\mathbb{P}$ has a reduction to $\mathbb{Q}$.*

Proof. $\Leftarrow$: assume that every element of $\mathbb{P}$ has a reduction to $\mathbb{Q}$, and A is a maximal antichain in $\mathbb{Q}$. Given $p \in \mathbb{P}$ let q be a reduction of p to $\mathbb{Q}$. Then there is an $r \in A$ which is compatible with q. It follows that $i(r)$ is compatible with p. This shows that $i[A]$ is a maximal antichain in $\mathbb{P}$.

$\Rightarrow$: suppose that i is a complete embedding of $\mathbb{Q}$ into $\mathbb{P}$. Thus for every maximal antichain A in $\mathbb{Q}$ the set $i[A]$ is a maximal antichain in $\mathbb{P}$. Take any $p \in \mathbb{P}$; we want to find a reduction of p to $\mathbb{Q}$. An antichain A in $\mathbb{Q}$ is *nice* iff $i(q) \perp p$ for all $q \in A$. Thus $\emptyset$ is nice. By Zorn's lemma let A be a maximal nice antichain. Then $i[A] \cup \{p\}$ is an antichain in $\mathbb{P}$, so by i being complete, A is not a maximal antichain in $\mathbb{Q}$. Let $r \in \mathbb{Q}$ be such that $r \perp q$ for all $q \in A$. We claim that r is a reduction of p to $\mathbb{Q}$. For, suppose that $q \in \mathbb{Q}$ and $i(q) \perp p$ but q and r are compatible. Say $s \leq q, r$. Then $s \perp t$ for all $t \in A$, since $s \leq r$. But also $i(s) \perp p$ since $i(s) \leq i(q)$. Hence $A \cup \{s\}$ is an antichain properly containing A, contradiction. This shows that $i(q) \perp p$ implies that $q \perp r$. $\square$

Proposition 25.80. *Suppose that $\mathbb{Q} \subseteq_{\mathrm{ctr}} \mathbb{P}$ and for all $p \in \mathbb{P}$ there is a $p' \in \mathbb{Q}$ such that $\forall q \in \mathbb{Q}[q \perp_{\mathbb{P}} p$ implies that $q \perp_{\mathbb{Q}} p']$.*

Then $\mathbb{Q} \subseteq_c \mathbb{P}$.

Proof. Suppose that A is a maximal antichain in $\mathbb{Q}$, but A is not maximal in $\mathbb{P}$. Say $p \in P$ and $p \perp_{\mathbb{P}} q$ for all $q \in A$. Then by definition, $q \perp_{\mathbb{Q}} p'$ for all $q \in A$, contradicting the maximality of A. $\square$

Lemma 25.81. (III.3.73) *If $\mathbb{P} = \mathbb{Q} \times \mathbb{R}$ and $i : \mathbb{Q} \rightarrow \mathbb{P}$ is defined by $i(q) = (q, \mathbb{1})$, then i is a complete embedding.*

Proof. Clearly (i)–(iii) in the definition of complete embedding hold. For any $(q,s) \in \mathbb{Q} \times \mathbb{R}$ let $\boxed{(q,s)} = q$. Then $\boxed{(q,s)}$ is a reduction of (q,s), for if $i(u) \perp (q,s)$, then $(u, \mathbb{1}) \perp (q,s)$, hence $u \perp q$. $\square$

Lemma 25.82. *If $\mathbb{P}$ and $\mathbb{Q}$ have ω_1 as a pre-caliber, then so does $\mathbb{P} \times \mathbb{Q}$.*

Proof. Suppose that $p \in {}^{\kappa}(\mathbb{P} \times \mathbb{Q})$. Let $B \in [\omega_1]^{\omega_1}$ be such that $\langle p(\alpha)_0 : \alpha \in B \rangle$ is centered. Then let $C \in [B]^{\omega_1}$ be such that $\langle p(\alpha)_1 : \alpha \in C \rangle$ is centered. Then $\langle p(\alpha) : \alpha \in C \rangle$ is centered. $\square$

Proposition 25.83. (III.3.74) *Let $\mathbb{Q} = \prod_{i \in I}^{\text{fin}} \mathbb{P}_i$. If each $\mathbb{P}_i$ has ω_1 as a pre-caliber, then so does $\mathbb{Q}$.*

Proof. Suppose that $p \in {}^{\kappa}\mathbb{Q}$. For each $\alpha < \kappa$ let $F_\alpha = \{i \in I : p_a(i) \neq \mathbb{1}\}$. Thus each F_α is finite. Let $\langle F_\alpha : \alpha \in B \rangle$ be a Δ-system, say with kernel G, with $B \in [\omega_1]^{\omega_1}$. Choose $C \in [B]^{\omega_1}$ so that $\langle p_\alpha \restriction G : \alpha \in C \rangle$ is centered. Then $\langle p_\alpha : \alpha \in C \rangle$ is centered. $\square$

Proposition 25.84. (III.3.74) *Let $\mathbb{Q} = \prod_{i \in I}^{\text{fin}} \mathbb{P}_i$. If each $\mathbb{P}_i$ is σ-centered and $|I| \leq 2^\omega$, then $\mathbb{Q}$ is σ-centered.*

Proof. Assume that each $\mathbb{P}_i$ is σ-centered; say $\mathbb{P}_i = \bigcup_{n \in \omega} M_{in}$ with each M_{in} centered. Let $\mathscr{F} \subseteq {}^{\omega}\omega$ be a collection of independent functions, with $|\mathscr{F}| = 2^\omega$, and let $f : I \to \mathscr{F}$ be an injection; see Theorem 21.35. For each $n \in \omega$ let $C_n = \{p \in \mathbb{Q} : \forall i \in I[p(i) \in M_{if_i(n)}]\}$. Then C_n is centered; for suppose that $p_1, \ldots p_k \in C_n$. Let $F = \{i \in I : p_j(i) \neq \mathbb{1}$ for some $j = 1, \ldots, k\}$. So F is finite. Let q be a member of $\mathbb{Q}$ such that $q(i) \leq p_1(i), \ldots, p_k(i)$ for each $i \in F$. Then $q \leq p_1, \ldots p_k$, showing that C_n is centered.

To show that $\mathbb{Q} = \bigcup_{n \in \omega} C_n$, let $q \in \mathbb{Q}$. Let $G = \{i \in I : q_i \neq \mathbb{1}\}$. Say $q_i \in M_{i,m(i)}$ for all $i \in G$. Choose $n \in \omega$ such that $f_i(n) = m(i)$ for all $i \in G$. Then $q \in C_n$. $\square$

Proposition 25.85. (III.3.74) *Let $\mathbb{Q} = \prod_{i \in I}^{\text{fin}} \mathbb{P}_i$. If $|I| > 2^\omega$ and for each $i \in I$ there are $p_i, q_i \in \mathbb{P}_i$ such that $p_i \perp q_i$, then $\mathbb{Q}$ is not σ-centered.*

Proof. Suppose that $\mathbb{Q} = \bigcup_{n \in \omega} M_n$ with each M_n centered. We define $g_n : I \to 2$ by setting, for any $i \in I$,

$$g_n(i) = \begin{cases} 1 & \text{if there is an } x \in M_n \text{ such that } x_i = p_i, \\ 0 & \text{otherwise.} \end{cases}$$

Now for each $i \in I$, the sequence $\langle g_n(i) : n \in \omega \rangle$ is in ${}^{\omega}2$. Since $|I| > 2^\omega$, there are distinct $i, j \in I$ such that $\langle g_n(i) : n \in \omega \rangle = \langle g_n(j) : n \in \omega \rangle$. Let $x \in \mathbb{Q}$ be such that $x_i = p_i$ and $x_j = q_j$. Say $x \in M_n$. Then $g_n(i) = 1$, and hence $g_n(j) = 1$. So there is a $y \in M_n$ such that $y_j = p_j$. Now $x, y \in M_n$ and $x_j = q_j$ while $y_j = p_j$, contradiction. $\square$

Proposition 25.86. (III.3.76) $\mathrm{cf}(\mathfrak{m}) > \omega$.

Proof. Suppose to the contrary that $\mathrm{cf}(\mathfrak{m}) = \omega$. Since $\mathfrak{m} > \omega$, there is a strictly increasing sequence $\langle \kappa_n : n \in \omega \rangle$ of infinite cardinals with supremum $\mathfrak{m}$. We now show that MA($\mathfrak{m}$), contradiction. Let $\mathbb{P}$ be a ccc poset and $\mathscr{D}$ a family of dense subsets, with $|\mathscr{D}| \le \mathfrak{m}$. Let $\langle D_\alpha : \alpha < \mathfrak{m}\rangle$ enumerate $\mathscr{D}$. Now we construct a sequence $\langle(\mathbb{Q}_n, D'_{\alpha n})_{\alpha<\kappa_n}$ by recursion. Choose $(\mathbb{Q}_0, D'_{\alpha 0}) \preceq (\mathbb{P}, D_\alpha)_{\alpha<\kappa_0}$ with $|\mathbb{Q}_0| \le \kappa_0$. If $(\mathbb{Q}_n, D'_{\alpha n})_{\alpha<\kappa_n}$ has been constructed so that $(\mathbb{Q}_n, D'_{\alpha n})_{\alpha<\kappa_n} \preceq (\mathbb{P}, D_\alpha)_{\alpha<\kappa_n}$ and $|\mathbb{Q}_n| \le \kappa_n$, choose $(\mathbb{Q}_{n+1}, D'_{\alpha,n+1})_{\alpha<\kappa_{n+1}} \preceq (\mathbb{P}, D_\alpha)_{\alpha<\kappa_{n+1}}$ so that $|\mathbb{Q}_{n+1}| \le \kappa_{n+1}$ and $(\mathbb{Q}_n, D'_{\alpha n})_{\alpha<\kappa_n} \preceq (\mathbb{Q}_{n+1}, D'_{\alpha,n+1})_{\alpha<\kappa_n}$. Let $\mathbb{R} = \bigcup_{n\in\omega} \mathbb{Q}_n$ and

$$D''_\alpha = \bigcup_{\substack{\alpha<\kappa_n\\ n\in\omega}} D'_{\alpha n}.$$

Now each $\mathbb{Q}_n$ is σ-centered by Lemma 25.53, and so also $\mathbb{R}$ is σ-centered. Now $\mathfrak{m} < \mathfrak{p}$ by our supposition on $\mathfrak{m}$, and by Theorem 25.34 and Lemma 20.54. Hence by Theorem 25.65, Martin's axiom applies to $\mathbb{R}$, and hence the desired conclusion follows. □

Theorem 25.87. (III.3.78) *Every linear order of size $\le \omega_1$ can be isomorhically embedded in $\mathscr{P}(\omega)/\mathrm{fin}$.*

Proof. We may assume that our linear order has the form $(\omega_1, \triangleleft)$. Let A_0 be such that $|A_0| = \omega = |\omega\backslash A_0|$. Now suppose that A_ξ has been defined for all $\xi < \alpha$, where $\alpha < \omega_1$, such that if $\xi, \eta < \alpha$ and $\xi \triangleleft \eta$, then $A_\xi \subseteq^* A_\eta$ and $A_\xi \ne^* A_\eta$. Furthermore, A_ξ and $\omega\backslash A_\xi$ are infinite for all $\xi < \alpha$. Let $L = \{\xi < \alpha : \xi \triangleleft \alpha\}$ and $R = \{\xi < \alpha : \alpha \triangleleft \xi\}$. Thus $A_\xi \subseteq^* A_\eta$ and $A_\xi \ne^* A_\eta$ for $\xi \in L$ and $\eta \in R$. If L is infinite, write $\{A_\xi : \xi \in L\} = \{C_n : n \in \omega\}$, and for R infinite write $\{A_\xi : \xi \in R\} = \{D_n : n \in \omega\}$.

Case 1. $L \ne \emptyset \ne M$, L does not have a largest element, and R does not have a smallest element (under $\triangleleft$). Let

$$A_\alpha = \bigcup_{n\in\omega}\left[\left(\bigcup_{m\le n} C_m\right) \cap \bigcap_{m\le n} D_m\right].$$

Now suppose that $p \in \omega$. Then

$$\begin{aligned}
C_p\backslash A_\alpha &= \bigcap_{n\in\omega}\left[C_p \cap \left(\bigcap_{m\le n}(\omega\backslash C_m) \cup \bigcup_{m\le n}(\omega\backslash D_m)\right)\right]\\
&= \bigcap_{n<p}\left[C_p \cap \left(\bigcap_{m\le n}(\omega\backslash C_m) \cup \bigcup_{m\le n}(\omega\backslash D_m)\right)\right]\\
&\quad \cap \bigcap_{n\ge p}\left[C_p \cap \left(\bigcap_{m\le n}(\omega\backslash C_m) \cup \bigcup_{m\le n}(\omega\backslash D_m)\right)\right]\\
&= \bigcap_{n<p}\left[C_p \cap \left(\bigcap_{m\le n}(\omega\backslash C_m) \cup \bigcup_{m\le n}(\omega\backslash D_m)\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&\cap \bigcap_{n\geq p}\left[C_p \cap \bigcup_{m\leq n}(\omega\backslash D_m)\right]\\
&\subseteq C_p \cap \bigcup_{m\leq p}(\omega\backslash D_m),
\end{aligned}
$$

and this last set is finite.

Furthermore,

$$
\begin{aligned}
A_\alpha\backslash D_p &= \bigcup_{n<p}\left[\left(\bigcup_{m\leq n} C_m\right) \cap \bigcap_{m\leq n} D_m \cap (\omega\backslash D_p)\right]\\
&\subseteq \left(\bigcup_{m<p} C_m\right)\backslash D_p,
\end{aligned}
$$

and this last set is finite.

Case 2. L has a largest element ξ under $\triangleleft$, and R has a smallest element η under $\triangleleft$. Now $A_\xi \subseteq^* A_\eta$ and $A_\xi \neq^* A_\eta$. Hence $A_\eta\backslash A_\xi$ is infinite. Write $A_\eta\backslash A_\xi = B \cup C$ with $B \cap C = \emptyset$ and B, C infinite. Let $A_\alpha = A_\xi \cup C$.

Case 3. L has a largest element ξ under $\triangleleft$, but R does not have a smallest element under $\triangleleft$.

(1) $\forall n \in \omega[(\bigcap_{m\leq n} D_m)\backslash A_\xi$ is infinite].

In fact, write $D_m = A_{\eta(m)}$ for all $m \leq n$, where each $\eta(m) \in R$. Let $i \leq n$ be such that $\eta(i)$ is $\triangleleft$-smallest among all $\eta(m)$ for $m \leq n$. Then $A_{\eta(i)}\backslash A_{\eta(m)}$ is finite for all $m \leq n$. Hence $\bigcup_{m\leq n, m\neq i}(A_{\eta(i)}\backslash A_{\eta(m)}$ is finite. Now

$$
\begin{aligned}
A_{\eta(i)}\backslash A_\xi &= \left(\left(\bigcap_{m\leq n} D_m\right)\backslash A_\xi\right) \cup \left(\left(A_{\eta(i)}\backslash\left(\bigcap_{\substack{m\leq n\\ m\neq i}} A_{\eta(m)}\right)\right)\backslash A_\xi\right)\\
&= \left(\left(\bigcap_{m\leq n} D_m\right)\backslash A_\xi\right) \cup \left(\left(\bigcup_{\substack{m\leq n\\ m\neq i}} (A_{\eta(i)}\backslash A_{\eta(m)})\right)\backslash A_\xi\right).
\end{aligned}
$$

Since the last entry in the union is finite and the left side of the equation is infinite, (1) follows.

Hence we can define by induction

$$
a_n \in \bigcap_{m\leq n} D_m\backslash(A_\xi \cup \{a_m : m < n\}).
$$

Let $A_\alpha = A_\xi \cup \{a_n : n \in \omega\}$. Then obviously $A_\xi \subseteq A_\alpha$, and $A_\alpha\backslash A_\xi$ is infinite. If $n \in \omega$, then there is a $k \in \omega$ such that $A_\xi\backslash D_n \subseteq k$, and $A_\alpha\backslash D_n \subseteq k \cup n$.

Case 4. L does not have a largest element under $\triangleleft$, but R has a smallest element η under $\triangleleft$. Then for any $n \in \omega$, the set

$$\left(\bigcap_{m \le n} (\omega \backslash C_m)\right) \cap A_\eta \text{ is infinite.} \tag{2}$$

In fact, for each $m \le n$ choose $\xi(m) \in L$ such that $C_m = A_{\xi(m)}$. Let $i \le m$ be such that $\xi(i)$ is $\triangleleft$-largest among the $\xi(m)$'s. Then $\bigcup_{m \le n,\ m \ne i} (A_{\xi(m)} \backslash A_{\xi(i)})$ is finite. Now

$$\begin{aligned} A_\eta \cap \bigcap_{m \le n} (\omega \backslash C_m) &= A_\eta \cap (\omega \backslash A_{\xi(i)}) \cap \bigcap_{\substack{m \le n \\ m \ne i}} (\omega \backslash A_{\xi(m)}) \\ &= A_\eta \cap (\omega \backslash A_{\xi(i)}) \cap \bigcap_{\substack{m \le n \\ m \ne i}} ((\omega \backslash A_{\xi(m)}) \cup A_{\xi(i)}) \\ &= A_\eta \cap (\omega \backslash A_{\xi(i)}) \backslash \bigcup_{m \le n,\ m \ne i} (A_{\xi(m)} \backslash A_{\xi(i)}) \end{aligned}$$

Now (2) follows. Hence we can define by induction

$$a_n \in \left(\bigcap_{m \le n} (\omega \backslash C_m)\right) \cap A_\eta \backslash \{a_m : m < n\}.$$

Let $A_\alpha = A_\eta \backslash \{a_n : n \in \omega\}$. Thus clearly $A_\alpha \subseteq A_\eta$ and $A_\eta \backslash A_\alpha$ is infinite. Now suppose that $n \in \omega$. Then

$$C_n \backslash A_\alpha = (C_n \backslash A_\eta) \cup (C_n \cap \{a_m : m \in \omega\}) = (C_n \backslash A_\eta) \cup (C_n \cap \{a_m : m < n\},$$

and the set on the right is finite.

Case 5. L does not have a largest element under $\triangleleft$, and $R = \emptyset$. For any $n \in \omega$ the set

$$\bigcap_{m \le n} (\omega \backslash D_m) \text{ is infinite.} \tag{3}$$

In fact, write $D_m = A_{\xi(m)}$ for all $m \le n$, with each $\xi(m) \in L$. Let $\xi(i)$ be $\triangleleft$-maximum among all $\xi(m)$'s. Then

$$\begin{aligned} \bigcap_{m \le n} (\omega \backslash D_m) &\supseteq (A_{\xi(i)+1} \backslash A_{\xi(i)}) \cap \bigcap_{\substack{m \le n \\ m \ne i}} ((\omega \backslash A_{\xi(m)}) \cup A_{\xi(i)}) \\ &= (A_{\xi(i)+1} \backslash A_{\xi(i)}) \backslash \bigcup_{\substack{m \le n \\ m \ne i}} (A_{\xi(m)} \backslash A_{\xi(i)}); \end{aligned}$$

since $\bigcup_{\substack{m\leq n\\ m\neq i}}(A_{\xi(m)}\backslash A_{\xi(i)})$ is finite, (3) follows.

Using (3), define by induction

$$a_n\in\bigcap_{m\leq n}(\omega\backslash D_m)\backslash\{a_m:m<n\}.$$

Let $A_\alpha=\omega\backslash\{a_{2n}:n\in\omega\}$. Thus clearly $\omega\backslash A_\alpha$ is infinite. Now suppose that $n\in\omega$. Then

$$D_n\backslash A_\alpha=D_n\cap\{a_{2m}:m\in\omega\})=D_n\cap\{a_{2m}:2m<n\},$$

and the set on the right is finite.

Case 6. L has a largest element ξ under $\triangleleft$, and $R=\emptyset$. Let $\omega\backslash A_\xi=C\cup B$ with $|C|=|B|=\omega$ and $C\cap B=\emptyset$. Then let $A_\alpha=A_\xi\cup C$.

Case 7. $L=\emptyset$ and R has a smallest element η under $\triangleleft$. Let $A_\eta=A_\alpha\cup D$ with $|A_\alpha|=|D|=\omega$ and $A_\alpha\cap D=\emptyset$.

Case 8. $L=\emptyset$ and R does not have a smallest element under $\triangleleft$. Then

(4) $\forall n\in\omega[\bigcap_{m\leq n}D_m$ is infinite].

In fact, write $D_m=A_{\eta(m)}$ for all $m\leq n$. Let $i\leq n$ be such that $\eta(i)$ is $\triangleleft$-minimum among all the η_m's. Then $\bigcup_{m\leq n,m\neq i}(A_{\eta(i)}\backslash A_{\eta(m)})$ is finite, and

$$A_{\eta(i)}\backslash\bigcup_{m\leq n,m\neq i}(A_{\eta(i)}\backslash A_{\eta(m)})=A_{\eta(i)}\cap\bigcap_{\substack{m\leq n\\ m\neq i}}((\omega\backslash A_{\eta(i)})\cup A_{\eta(m)})=\bigcap_{m\leq n}A_{\eta(m)},$$

and (4) follows.

Hence we can define by induction

$$a_n\in\bigcap_{m\leq n}D_m\backslash\{a_m:m<n\}.$$

Let $A_\alpha=\{a_{2n}:n\in\omega\}$. If $n\in\omega$, then $A_\alpha\backslash D_n\subseteq n$.

Proposition 25.88. (III.3.79) $\kappa<\mathfrak{p}$ *implies that every linear order of size κ can be embedded in $\mathscr{P}(\omega)/fin$.*

Proof. Let $\mathbb{P}$ be as in the proof of Proposition 25.23. So we are given a linear order $(L,<)$ with $|L|\leq\kappa$. By Theorem 25.63 it suffices to show that $\mathbb{P}$ is σ-centered. Note that $\kappa<2^\omega$ by Propositions 20.51 and 20.52. Let $\langle f_{ij}:i\in L,j\in\omega\rangle$ be a system of independent functions; see Theorem 21.35. For each $e\in\omega$ define $x_e\in{}^{L\times\omega}2$ by

$$x_e(u,j)=\begin{cases}1 & \text{if } f_{uj}(e)=1,\\ 0 & \text{otherwise.}\end{cases}$$

Then for $e,l\in\omega$ let

$$K_{el}=\{p\in\mathbb{P}:k_p=l\text{ and }\forall u\in S_p\forall i<k_p[\sigma_p(u,i)=x_e(u,i)]\}.$$

Then K_{el} is centered. For, suppose that $p, q \in K_{el}$. Define $S_r = S_p \cup S_q$, $k_r = l$, and for $u \in S_r$ and $i < l$ let $\sigma_r(u,i) = x_e(u,i)$. Then if $u \in S_p$ and $i < l$ then $\sigma_r(u,i) = x_e(u,i) = \sigma_p(u,i)$. So $\sigma_p \subseteq \sigma_r$. Similarly, $\sigma_q \subseteq \sigma_r$. The final condition for $r \leq p$ and $r \leq q$ is clear. So K_{el} is centered.

We claim that $\mathbb{P} = \bigcup_{e,l\in\omega} K_{el}$. For, suppose that $p \in \mathbb{P}$. Choose $e \in \omega$ such that $\forall(x,i) \in S_p \times k_p[f_{xi}(e) = \sigma_p(x,i)]$. Then $p \in K_{ek_p}$. □

Proposition 25.89. (III.3.80) *Assume that $A_n, B_n \in [\omega]^\omega$ for all $n \in \omega$ and $A_n \cap B_m$ is finite for all $m, n \in \omega$. Then there is a $C \in [\omega]^\omega$ such that $A_n \subseteq^* C$ and $B_n \cap C$ is finite for all $n \in \omega$.*

Proof. See the Handbook of Boolean algebras, page 79. □

Proposition 25.90. (III.3.81) *There are almost disjoint $A_\alpha \in [\omega]^\omega$ for $\alpha < \omega_1$ such that whenever X, Y are disjoint uncountable subsets of ω_1, there is no $C \in [\omega]^\omega$ such that $\forall\alpha \in X[A_\alpha \subseteq^* C]$ and $\forall\beta \in Y[A_\beta \cap C$ is finite$]$.*

Proof. Let $\langle A_n : n \in \omega\rangle$ be a partition of ω into infinite subsets. Now suppose that $\omega \leq \alpha < \omega_1$ and A_ξ has been defined for all $\xi < \alpha$ so that each A_ξ is infinite and $A_\xi \cap A_\eta$ is finite for $\xi \neq \eta$. Write $\{A_\xi : \xi < \alpha\} = \{B_m : m \in \omega\}$ without repetitions. Then for any n the set $B_{n+1}\backslash\bigcap_{m\leq n}(\omega\backslash B_m)$ is infinite, since $B_{n+1} \cap \bigcup_{m\leq n} B_m$ is finite. Hence we can let F_n be a subset of $B_{n+1}\backslash\bigcap_{m\leq n}(\omega\backslash B_m)$ of size n. Let $A_\alpha = \bigcup_{n\in\omega} F_n$. Then $A_\alpha \cap B_n \subseteq \bigcup_{m<n} F_m$, since $B_n \cap F_m = \emptyset$ for $m \geq n$. Thus $A_\alpha \cap A_\xi$ is finite for all $\xi < \alpha$. Clearly A_α is infinite. Also, $\{m \in \omega : |B_m \cap A_\alpha| \leq n\} \subseteq n+1$, since for $m > n$ we have $B_{m+1} \cap A_\alpha \supseteq F_m$ and F_m has size $m > n$. It follows that $\{m \in \omega : |B_m \cap A_\alpha| \leq n\}$ is finite and hence also $\{\xi < \alpha : |A_\xi \cap A_\alpha| \leq n\}$ is finite.

Now suppose that X and Y are disjoint uncountable subsets of ω_1 and $C \in [\omega]^\omega$ is such that $A_\alpha \subseteq^* C$ for all $\alpha \in X$ and $A_\beta \cap C$ is finite for all $\beta \in Y$. For each $\alpha \in X$ let $n_\alpha \in \omega$ be such that $A_\alpha\backslash C \subseteq n_\alpha$, and let $X' \in [X]^{\omega_1}$ and $m \in \omega$ be such that $n_\alpha = m$ for all $\alpha \in X'$. For each $\beta \in Y$ let $p_\beta \in \omega$ be such that $A_\beta \cap C \subseteq p_\beta$, and let $Y' \in [Y]^{\omega_1}$ and $q \in \omega$ be such that $p_\beta = q$ for all $\beta \in Y'$. Choose $\beta \in Y'$ so that $\{\alpha \in X' : \alpha < \beta\}$ is infinite. Now if $\alpha \in X'$ and $\alpha < \beta$, then

$$A_\alpha \cap A_\beta \subseteq (A_\alpha\backslash C) \cup (C \cap A_\beta) \subseteq m \cup q,$$

contradiction. □

Proposition 25.91. (III.3.82) *Assume that $A_n, B_\alpha \in [\omega]^\omega$ for $n \in \omega$ and $\alpha < \kappa$. and $A_n \cap B_\alpha$ is finite for each α, n. Assume that $\kappa < \mathfrak{b}$. Then there is a $C \in [\omega]^\omega$ such that $A_n \subseteq^* C$ and $B_\alpha \cap C$ is finite for each n, α.*

Proof. Let $A'_n = \bigcup_{m\leq n} A_m$.

Case 1. $\exists m\forall n \geq m[A'_n =^* A'_m]$. Clearly $C \stackrel{\text{def}}{=} A'_m$ is as desired.

Case 2. $\forall m\exists n \geq m[A'_n \neq^* A'_m]$. Define $p_0 = 0$. Having defined $p_0 < \cdots < p_m$ so that $A'_{p_0} \neq^* A'_{p_1} \neq^* \cdots \neq^* A'_{p_m}$, choose $p_{m+1} > p_m$ so that $A'_{p_m} \neq^* A'_{p_{m+1}}$. Note that $A'_{p_0} \subseteq A'_{p_1} \subseteq \cdots$ and $A'_{p_{m+1}}\backslash A'_{p_m}$ is infinite for all m. Also note that $A'_{p_n} \cap B_\alpha$ is finite for all $n \in \omega$ and $\alpha < \kappa$. Then $A_n \subseteq A'_{p_n}$ for all n. So it suffices to find C such that $A'_{p_n} \subseteq^* C$ for all $n \in \omega$ and $C \cap B_\alpha$ is finite for all $\alpha < \kappa$.

For each $\alpha < \kappa$ and $n \in \omega$ let $f_\alpha(n)$ be greater than each member of $B_\alpha \cap (A'_{p_{n+1}} \backslash A'_{p_n})$. Since $\kappa < \mathfrak{b}$, choose $g \in {}^\omega\omega$ such that $f_\alpha \leq^* g$ for all $\alpha < \kappa$. Now let

$$C = A'_{p_0} \cup \bigcup_{n \in \omega} ((A'_{p_{n+1}} \backslash A'_{p_n}) \backslash g(n)).$$

By induction, $A'_{p_n} \subseteq^* C$ for every $n \in \omega$. In fact, obviously $A'_{p_0} \subseteq C$. Suppose that $A'_{p_n} \subseteq^* C$. Then $A'_{p_{n+1}} = A'_{p_n} \cup (A'_{p_{n+1}} \backslash A'_{p_n}) \subseteq^* C$.

Now suppose that $\alpha < \kappa$. Choose m so that $f_\alpha(n) \leq g(n)$ for all $n \geq m$. Then

$$\begin{aligned} C \cap B_\alpha =& (B_\alpha \cap A'_{p_0}) \\ & \cup \bigcup_{n<m} ((B_\alpha \cap A'_{p_{n+1}}) \backslash A'_{p_n}) \backslash g(n) \\ & \cup \bigcup_{m \leq n} ((B_\alpha \cap A'_{p_{n+1}} \backslash A'_{p_n}) \backslash g(n) \end{aligned}$$

Now the first two parts of the right side are finite, and the third is empty, as desired. □

Proposition 25.92. (III.3.87) *Let $\mathbb{P}$ be the set of closed subsets of $[0,1]$ of positive Lebesgue measure, under $\subseteq$. Then $\mathbb{P}$ is ccc but not σ-centered. Under CH, $\mathbb{P}$ does not have ω_1 as a pre-caliber.*

Proof. ccc: Suppose that $\mathscr{A}$ is a collection of pairwise disjoint closed sets of positive measure with $|\mathscr{A}| = \omega_1$. Then there exist a positive integer n and an uncountable subset B of $\mathscr{A}$ such that $\mu(b) > \frac{1}{n}$ for all $b \in B$, contradiction.

not σ-centered: Suppose that $\langle C_i : i \in \omega \rangle$ is a system of centered subsets of $\mathbb{P}$; we want to find $p \in \mathbb{P}$ which is not in any C_i. For each $i \in \omega$ choose $a_i \in \bigcap C_i$. Then $\{a_i : i \in \omega\}$ has measure 0, so $p' \stackrel{\text{def}}{=} [0,1] \backslash \{a_i : i \in \omega\}$ has measure 1. By Corollary 18.96(ii), there is a closed $p \subseteq p'$ of positive measure. If $p \in C_i$, then $a_i \in \bigcap C_i \subseteq p$, contradiction.

not pre-caliber ω_1 under CH: Let $[0,1] = \{x_\xi : \xi < \omega_1\}$. For each $\alpha < \omega_1$ the set $\{x_\xi : \xi < \alpha\}$ is countable, and so has measure 0. Hence $p'_\alpha \stackrel{\text{def}}{=} [0,1] \backslash \{x_\xi : \xi < \alpha\}$ has measure 1. By Corollary 18.96(ii), there is a closed $p_\alpha \subseteq p'_\alpha$ of positive measure. Suppose that $M \in [\omega_1]^{\omega_1}$ is such that $\{p_\alpha : \alpha \in M\}$ is centered. But $\emptyset \neq \bigcap_{\alpha \in M} p_\alpha = \emptyset$, contradiction. □

Proposition 25.93. (III.3.88) *Let μ be a probability measure defined on a σ-algebra $\mathscr{A}$ of subsets of X. Let $\mathbb{P} = \{p \in \mathscr{A} : \mu(p) > 0\}$, under $\subseteq$. Let $E \in [\mathbb{P}]^{\omega_1}$. Then there is a linked $L \in [E]^{\omega_1}$.*

Proof. Wlog there is a positive integer n such that $\forall p \in E[\mu(p) \geq \frac{1}{n}]$. We now proceed by induction on n. The case $n = 1$ is clear: any two elements of E intersect in a set of positive measure. Now suppose inductively that $n > 1$.

Case 1. There is a $p \in E$ such that $|\{q \in E : q \perp p\}| = \omega_1$. Let $Y = X \backslash p$, $\mathscr{A}' = \{s \backslash p : s \in \mathscr{A}\}$, and

$$\mu'(s) = \frac{\mu(s)}{\mu(X \backslash p)}$$

for each $s \in \mathscr{A}'$. Now $\mu(X\backslash p) + \mu(p) = 1$, so $\mu(X\backslash p) = 1 - \mu(p) \le 1 - \frac{1}{n} = \frac{n-1}{n}$. Hence $\frac{1}{\mu(X\backslash p)} \ge \frac{n}{n-1}$ and so for any $q \in \mathscr{A}'$ we have $\mu'(q) = \frac{\mu(q)}{\mu(X\backslash p)} \ge \frac{1}{n-1}$. Hence the inductive hypothesis applies to give the desired result.

Case 2. $\forall p \in E[|\{q \in E : q \perp p\}| \le \omega]$. Let M be a maximal linked subset of E. Suppose that M is countable. For every $p \in E$ there is a $q \in M$ such that $p \perp q$; so

$$E = \bigcup_{q \in M} \{p \in E : p \perp q\},$$

so there is a $q \in M$ such that $\{p \in E : p \perp q\}$ is uncountable, contradiction. □

Proposition 25.94. *In the proof of Theorem 25.12, if C is a centered subset of $\mathbb{P}$, then $\mu(\bigcup C) \le \varepsilon$.*

Proof. Suppose that $\mu(\bigcup C) > \varepsilon$. Now $\mu(\bigcup C) = \sup_{n\in\omega} \mu([-n, n] \cap \bigcup C)$, so there is an $n \in \omega$ such that $\mu([-n, n] \cap \bigcup C) > \varepsilon$. By Corollary 18.96(ii), there is a closed $F \subseteq [-n, n] \cap \bigcup C)$ such that $\mu(F) > \varepsilon$. By compactness of F, there is a finite $C' \subseteq C$ such that $F \subseteq [-n, n] \cap \bigcup C'$. Since C is centered, there is a $d \in C$ such that $\bigcup C' \subseteq d$. Hence $\varepsilon < \mu(F) \le \mu(d)$, contradiction. □

Proposition 25.95. (III.3.89) *For the forcing poset $\mathbb{P}$ of Theorem 25.12, $\mathbb{P}$ is not σ-centered.*

Proof. Suppose that $\langle C_i : i \in \omega\rangle$ is a system of centered subsets of $\mathbb{P}$. By Proposition 25.94 we have $\bigcup C_i \ne \mathbb{R}$, so choose $a_i \in \mathbb{R}\backslash \bigcup C_i$. Then $\{a_i : i \in \omega\}$ has measure 0, so by Corollary 18.96(i), there is an open set U with $\mu(U) < \varepsilon$ such that $\{a_i : i \in \omega\} \subseteq U$. If $U \in C_i$, then $a_i \in U \subseteq \bigcup C_i$, contradiction. □

Proposition 25.96. (III.3.90) *$MA_{\mathbb{P}}(2^\omega)$ is false if $\mathbb{P}$ is ccc and atomless.*

Proof. *Case 1.* $|\mathbb{P}| \le 2^\omega$. Note that there are at most $\mathfrak{c}^\omega = 2^\omega$ maximal antichains. We claim that $\mathbb{P}\backslash G$ is dense: let $p \in \mathbb{P}$ be arbitrary. Since $\mathbb{P}$ is atomless, choose $q, r \le p$ such that $q \perp r$. At most one of q, r is in G, as desired. Let X be a maximal antichain consisting entirely of elements of $\mathbb{P}\backslash G$, and let $p \in G \cap X$; contradiction.

Case 2. $|\mathbb{P}| > 2^\omega$. Consider the following functions. $f : {}^\omega\mathbb{P} \to \mathbb{P}$ is defined by

$$f(a) = \begin{cases} \text{an } x \text{ such that } \forall i \in \omega[a_i \perp x] & \text{if there is such an } x, \\ \mathbb{1} & \text{otherwise.} \end{cases}$$

Also, for each integer $m \ge 2$ the function $g_m : {}^m\mathbb{P} \to \mathbb{P}$ defined by

$$g_m(a) = \begin{cases} \text{an } x \text{ such that } \forall i < m[x \le a_i] & \text{if there is such an } x, \\ \mathbb{1} & \text{otherwise.} \end{cases}$$

We also consider functions $h, k : \mathbb{P} \to \mathbb{P}$ such that for any $p \in \mathbb{P}$, $h(p), k(p) \le p$ and $h(p) \perp k(p)$. Let A be a subset of $\mathbb{P}$ of size ω, and let $\mathbb{Q}$ be the closure of A under the functions f, g_m for $m \in \omega$, and h, k. Clearly $|\mathbb{Q}| \le 2^\omega$. The functions g_m assure that $\mathbb{Q} \subseteq_{\text{ctr}} \mathbb{P}$. Hence by Proposition 25.56, $\mathbb{Q}$ is ccc. The functions h, k assure that $\mathbb{C}$ is atomless. If $X \subseteq \mathbb{Q}$ is an antichain which is not maximal in $\mathbb{P}$, then it is also not maximal in $\mathbb{Q}$, because of the function f. Let G be a filter on $\mathbb{P}$ which intersects all maximal antichains of $\mathbb{Q}$. Now $\mathbb{Q}\backslash G$ is dense in $\mathbb{Q}$, as above. Let X be a maximal antichain of $\mathbb{Q}$ consisting entirely of elements of $\mathbb{Q}\backslash G$. and let $p \in G \cap X$; contradiction. □

The Suslin number $S(\mathbb{P})$ of $\mathbb{P}$ is the least κ such that $|A| < \kappa$ for every antichain of $\mathbb{P}$. For X a topological space, $S(X) = S(\mathbb{O}_X)$.

Proposition 25.97. (III.3.92) *We call $p \in \mathbb{P}$ **S-minimal** iff $S(p\downarrow') = S(q\downarrow')$ for all $q \le p$. The set of all S-minimal elements of $\mathbb{P}$ is dense.*

Proof. Note that if $q \le p$ then $(q\downarrow') \subseteq (p\downarrow')$ and hence $S(q\downarrow') \le S(p\downarrow')$. Given $p \in \mathbb{P}$, choose $q \le p$ so that $S(q\downarrow')$ is minimal. □

Proposition 25.98. (III.3.93) *There is no $\mathbb{P}$ such that $S(\mathbb{P}) = \omega$.*

Proof. Assume that $S(\mathbb{P}) = \omega$. Thus every antichain in $\mathbb{P}$ is finite, but there is no upper bound on the size of antichains. Let X be a maximal antichain all of whose elements are S-minimal.

(1) $S(p\downarrow') \le 2$ for all $p \in X$.

In fact, suppose that $p \in X$ and $S(p\downarrow') > 2$. Define $q_\emptyset = p$, and if $a \in {}^m2$ let $q_{a\frown\langle 0\rangle}$ and $q_{a\frown\langle 1\rangle}$ be incompatible elements below q_a. Then $\{q_{\langle 1\rangle}, q_{\langle 0,1\rangle}, q_{\langle 0,0,1\rangle}, \ldots\}$ is an infinite antichain, contradiction.

By (1), $p\downarrow'$ is a linear order.

Now suppose that Y is any antichain. For each $y \in Y$ choose $x_y \in X$ compatible with y; say $z_y \le y, x_y$. If $y, y' \in Y$ and $x_y = x_{y'}$, then z_y and $z_{y'}$ are comparable, and hence y and y' are compatible, so $y = y'$. It follows that $|Y| \le |X|$. So every maximal antichain has size at most $|X|$, contradiction. □

Proposition 25.99. (III.3.94) *If $S(\mathbb{P})$ is infinite, then $S(\mathbb{P})$ is uncountable and regular.*

Proof. By Proposition 25.98, $S(\mathbb{P})$ is uncountable. Suppose that $\kappa \stackrel{\text{def}}{=} S(\mathbb{P})$ is singular. Let $\langle \lambda_\alpha : \alpha < \text{cf}(\kappa)\rangle$ be a strictly increasing sequence of cardinals with supremum κ. Let X be a maximal antichain all of whose elements are S-minimal.

Case 1. There is a $p \in X$ such that $S(p\downarrow') = \kappa$. Let $\langle q_\alpha : \alpha < \text{cf}(\kappa)\rangle$ be an antichain below p, and for each $\alpha < \text{cf}(\kappa)$ let X_α be an antichain of size λ_α below q_α. Then $\bigcup_{\alpha<\text{cf}(\kappa)} X_\alpha$ is an antichain of size κ, contradiction.

Case 2. $S(p\downarrow') < \kappa$ for all $p \in X$, but $\sup_{p\in X} S(p\downarrow') = \kappa$. We now define $\langle q_\alpha : \alpha < \text{cf}(\kappa)\rangle \in {}^{\text{cf}(\kappa)}X$ by recursion. If q_β has been defined for all $\beta < \alpha$, then $\sup_{\beta<\alpha} S(q_\beta\downarrow') < \kappa$, and we let $q_\alpha \in X$ be such that $\lambda_\alpha \le S(q_\alpha\downarrow')$ and $\sup_{\beta<\alpha} S(q_\beta\downarrow') < S(q_\alpha\downarrow')$. Clearly $\langle q_\alpha : \alpha < \text{cf}(\kappa)\rangle$ is a one-one sequence, hence is an antichain. So we obtain an antichain of size κ as in Case 1, Contradiction.

Case 3. $S(p{\downarrow}') < \kappa$ for all $p \in X$, and $\sup_{p\in X} S(p{\downarrow}') < \kappa$. Let $\langle q_\alpha : \alpha < \mu^+\rangle$ be an antichain with $|X| < \mu$. For each $\alpha < \mu^+$ there is an $x_\alpha \in X$ such that q_α and x_α are compatible; hence there is a $y_\alpha \leq x_\alpha, q_\alpha$. It follows that there is a $z \in X$ such that $x_\alpha = z$ for all α in a set M of size μ^+. For distinct $\alpha, \beta \in M$ the elements y_α and y_α are incompatible, since they are $\leq q_\alpha$, q_β respectively. This contradicts $S(z{\downarrow}') < \mu^+$. □

Proposition 25.100. (III.3.95) *If $I \neq \emptyset$ and λ is infinite, then $S(\mathrm{Fn}(I, \lambda, \omega)) = \lambda^+$.*

Proof. Fix $i \in I$. Then $\{\{(i, \alpha)\} : \alpha < \lambda\}$ is an antichain of size λ; so $\lambda^+ \leq S(\mathrm{Fn}(I, \lambda, \omega))$. Now suppose that $\langle p_\alpha : \alpha < \lambda^+\rangle$ is a system of elements of $\mathrm{Fn}(I, \lambda, \omega)$. Then there exist an $M \in [\lambda^+]^{\lambda^+}$ and G such that $\langle \mathrm{dmn}(p_\alpha) : \alpha \in M\rangle$ is a Δ-system with kernel G. Then there exist an $N \in [M]^{\lambda^+}$ and $g \in {}^G\lambda$ such that $p_\alpha \restriction G = g$ for all $\alpha \in N$. Then for any two elements α, β of N we have $p_\alpha \not\perp p_\beta$. □

Proposition 25.101. (III.3.96) *If κ is weakly inaccessible and $\mathbb{P} = \prod^{\mathrm{fin}}_{\alpha<\kappa} \mathrm{Fn}(\omega, \alpha, \omega)$, then $S(\mathbb{P}) = \kappa$.*

Proof. For any infinite cardinal $\lambda < \kappa$ the set

$$\{\{p \in \mathbb{P} : p_\lambda = \{(0, \beta)\} \text{ and } \forall \alpha \in (\kappa\backslash\{\lambda\})[p_\alpha = \emptyset]\} : \beta < \lambda\}$$

is an antichain of size λ. Hence $\kappa \leq S(\mathbb{P})$. Now suppose that $p \in {}^\kappa\mathbb{P}$. For each $\alpha < \kappa$ choose a finite set $F_\alpha \subseteq \kappa$ such that $p_\alpha(\beta) = \emptyset$ for all $\beta \in \kappa\backslash F_\alpha$. Let $M \in [\kappa]^\kappa$ and G be such that $\langle F_\alpha : \alpha \in M\rangle$ is a Δ-system with kernel G. Then

$$M = \bigcup_{g\in\prod_{\alpha\in G} \mathrm{Fn}(\omega,\alpha.\omega)} \{\alpha \in M : p_\alpha \restriction G = g\}.$$

Since $|\prod_{\alpha\in G} \mathrm{Fn}(\omega, \alpha, \omega)| < \kappa$, there exist $N \in [M]^\kappa$ and $g \in \prod_{\alpha\in G} \mathrm{Fn}(\omega, \alpha, \omega)$ such that $p_\alpha \restriction G = g$ for all $\alpha \in N$. Then for any $\alpha, \beta \in N$ we have $p_\alpha \not\perp p_\beta$. □

Theorem 25.102. (III.5.19) *Let T be an infinite tree with no uncountable chains and let $\kappa = |T|$. Assume MA(κ). Then there is a continuous order preserving $\varphi : T \to \mathbb{Q}$.*

Proof. Let $\mathbb{P}$ be the set of all $p \in \mathrm{Fn}(T, \mathbb{Q}, \omega)$ such that $\forall x, y \in \mathrm{dmn}(p)[x < y \to p(x) < p(y)]$, ordered by $\supseteq$.

To show that $\mathbb{P}$ is ccc, let $A \subseteq \mathbb{P}$ be uncountable. For each $a \in [T]^{<\omega}$ the set $\{p \in \mathbb{P} : \mathrm{dmn}(p) = a\}$ is countable, so there is an uncountable $A_1 \subseteq A$ such that $\langle \mathrm{dmn}(p) : p \in A_1\rangle$ is one-one. Next, let $\langle \mathrm{dmn}(p) : p \in A_2\rangle$ be a Δ-system with kernel B, where $A_2 \in [A_1]^{\omega_1}$. Since $\{p \restriction B : p \in A_2\} \subseteq {}^B\mathbb{Q}$, there exist an uncountable $A_3 \subseteq A_2$ and an $f \in {}^B\mathbb{Q}$ such that $p \restriction B = f$ for all $p \in A_3$. Then $\langle \mathrm{dmn}(p)\backslash B : p \in A_3\rangle$ is an uncountable system of pairwise disjoint finite subsets of T. Hence by Lemma 22.19 there are distinct $p, q \in A_3$ such that x and y are incomparable for all $x \in \mathrm{dmn}(p)\backslash B$ and $y \in \mathrm{dmn}(q)\backslash B$. Then $p \cup q \in \mathbb{P}$, as desired.

For each $x \in T$ let $D_x = \{p \in \mathbb{P} : x \in \mathrm{dmn}(p)\}$. Then D_x is dense in $\mathbb{P}$; for assume that $p \in \mathbb{P}$ and $x \notin \mathrm{dmn}(p)$.

Case 1. x is not comparable with any element of $\mathrm{dmn}(p)$. Let $q = p \cup \{(x,0)\}$. Clearly $q \in D_x$ and $q \leq p$.

Case 2. $\exists y \in \mathrm{dmn}(p)[y < x]$ but not$\exists z \in \mathrm{dmn}(p)[x < z]$. Let a be a rational number greater than $p(y)$ for all $y \in \mathrm{dmn}(p)$ for which $y < x$, and set $q = p \cup \{(x,a)\}$. Clearly $q \in D_x$ and $q \leq p$.

Case 3. $\exists y \in \mathrm{dmn}(p)[x < y]$ but not$\exists z \in \mathrm{dmn}(p)[z < x]$. Let a be a rational number less than $p(y)$ for all $y \in \mathrm{dmn}(p)$ for which $x < y$, and set $q = p \cup \{(x,a)\}$. Clearly $q \in D_x$ and $q \leq p$.

Case 4. $\exists y \in \mathrm{dmn}(p)[x < y]$ and $\exists z \in \mathrm{dmn}(p)[z < x]$. Let a be a rational number less than $p(y)$ for all $y \in \mathrm{dmn}(p)$ for which $x < y$, and greater than $p(z)$ for all $z \in \mathrm{dmn}(p)$ for which $z < x$, and set $q = p \cup \{(x,a)\}$. Clearly $q \in D_x$ and $q \leq p$.

Now we define for any $n \in \omega$ and each $t \in T$ of limit level

$$E_{tn} = \{p \in \mathbb{P} : t \in \mathrm{dmn}(p) \text{ and } \exists x \in \mathrm{dmn}(p) \cap (t\downarrow)[p(x) \geq p(t) - 2^{-n}]\}.$$

To show that E_{tn} is dense, suppose that $n \in \omega$, $t \in T$ is of limit level, and $p \in \mathbb{P}$. Using the fact that D_t is dense, wlog $t \in \mathrm{dmn}(p)$. Choose $x < t$ such that $y < x$ for all $y < t$ with $y \in \mathrm{dmn}(p)$. Note that $p(y) < p(t)$ for all $y < t$ with $y \in \mathrm{dmn}(p)$. Define $q = p \cup \{(x,s)\}$, where $s < p(t)$ with $p(y) < s$ for all $y < t$ with $y \in \mathrm{dmn}(p)$, and with $s \geq p(t) - 2^{-n}$. Clearly $q \in E_{tn}$ and $q \leq p$.

Now let G be a filter intersecting all the sets D_x and E_{tn}. Let $\varphi = \bigcup G$. Then $\varphi : T \to \mathbb{Q}$ and it is increasing because $G \cap D_x \neq \emptyset$ for all $x \in T$. φ is continuous because of the sets T_{tn}, using Lemma 12.20. $\square$

26. Large cardinals

The study, or use, of large cardinals is one of the most active areas of research in set theory currently. There are many provably different kinds of large cardinals. We describe some important ones, ranging from (mere) inaccessibles to ones close to inconsistency.

Mahlo cardinals

Mahlo cardinals were mentioned briefly on page 425. As we mentioned in the elementary part of these notes, one cannot prove in ZFC that uncountable weakly inaccessible cardinals exist (if ZFC itself is consistent). But now we assume that even the somewhat stronger inaccessible cardinals exist, and we want to explore, roughly speaking, how many such there can be. The weakest kind of large cardinals beyond mere inaccessibles are the Mahlo cardinals, which were briefly discussed in Chapter 23.

We begin with some easy propositions. A *strong limit cardinal* is an infinite cardinal κ such that $2^\lambda < \kappa$ for all $\lambda < \kappa$.

Proposition 26.1. *Assume that uncountable inaccessible cardinals exist, and suppose that κ is the least such. Then every uncountable strong limit cardinal less than κ is singular.* $\square$

The inaccessibles are a class of ordinals, hence form a well-ordered class, and they can be enumerated in a strictly increasing sequence $\langle \iota_\alpha : \alpha \in O\rangle$. Here O is an ordinal, or **On**, the class of all ordinals. The definition of Mahlo cardinal is motivated by the following simple proposition.

Proposition 26.2. *If $\kappa = \iota_\alpha$ with $\alpha < \kappa$, then the set $\{\lambda < \kappa : \lambda$ is regular$\}$ is a nonstationary subset of κ.*

Proof. Since κ is regular and $\alpha < \kappa$, we must have $\sup_{\beta<\alpha} \iota_\beta < \kappa$. Let $C = \{\gamma : \sup_{\beta<\alpha} \iota_\beta < \gamma < \kappa$ and γ is a strong limit cardinal$\}$. Then C is club in κ with empty intersection with the given set. $\square$

• κ is *Mahlo* iff κ is an uncountable inaccessible cardinal and $\{\lambda < \kappa : \lambda$ is regular$\}$ is stationary in κ.

• κ is *weakly Mahlo* iff κ is an uncountable weakly inaccessible cardinal and $\{\lambda < \kappa : \lambda$ is regular$\}$ is stationary in κ.

Since the function ι is strictly increasing, we have $\alpha \le \iota_\alpha$ for all α. Hence the following is a corollary of Proposition 26.2

Corollary 26.3. *If κ is a Mahlo cardinal, then $\kappa = \iota_\kappa$.* $\square$

Thus a Mahlo cardinal κ is not only inaccessible, but also has κ inaccessibles below it.

Proposition 26.4. *For any uncountable cardinal κ the following conditions are equivalent:*
(i) κ is Mahlo.
(ii) $\{\lambda < \kappa : \lambda$ is inaccessible$\}$ is stationary in κ.

Proof. (i)⇒(ii): Let $S = \{\lambda < \kappa : \lambda \text{ is regular}\}$, and $S' = \{\lambda < \kappa : \lambda \text{ is inaccessible}\}$. Assume that κ is Mahlo. In particular, κ is uncountable and inaccessible. Suppose that C is club in κ. The set $D = \{\lambda < \kappa : \lambda \text{ is strong limit}\}$ is clearly club in κ too. If $\lambda \in S \cap C \cap D$, then λ is inaccessible, as desired.

(ii)⇒(i): obvious. $\square$

The following proposition answers a natural question one may ask after seeing Corollary 26.3.

Proposition 26.5. *Suppose that κ is minimum such that $\iota_k = \kappa$. Then κ is not Mahlo.*

Proof. Suppose to the contrary that κ is Mahlo, and let $S = \{\lambda < \kappa : \lambda \text{ is inaccessible}\}$ For each $\lambda \in S$, let $f(\lambda)$ be the $\alpha < \kappa$ such that $\lambda = \iota_\alpha$. Then $\alpha = f(\lambda) < \lambda$ by the minimality of κ. So f is regressive on the stationary set S, and hence there is an $\alpha < \kappa$ and a stationary subset S' of S such that $f(\lambda) = \alpha$ for all $\lambda \in S'$. But actually f is clearly a one-one function, contradiction. $\square$

Mahlo cardinals are in a sense larger than "ordinary" inaccessibles. Namely, below every Mahlo cardinal κ there are κ inaccessibles. But now in principle one could enumerate all the Mahlo cardinals, and then apply the same idea used in going from regular cardinals to Mahlo cardinals in order to go from Mahlo cardinals to higher Mahlo cardinals. Thus we can make the definitions

• κ is *hyper-Mahlo* iff κ is inaccessible and the set $\{\lambda < \kappa : \lambda \text{ is Mahlo}\}$ is stationary in κ.

• κ is *hyper-hyper-Mahlo* iff κ is inaccessible and the set $\{\lambda < \kappa : \lambda \text{ is hyper-Mahlo}\}$ is stationary in κ.

Of course one can continue in this vein.

Note that if strong inaccessibles exist, then the least such is not Mahlo.

Weakly compact cardinals

• A cardinal κ is *weakly compact* iff $\kappa > \omega$ and $\kappa \to (\kappa, \kappa)^2$. There are several equivalent definitions of weak compactness. The one which justifies the name "compact" involves infinitary logic, and it will be discussed later. Right now we consider equivalent conditions involving trees and linear orderings.

• A cardinal κ has the *tree property* iff every κ-tree has a chain of size κ.

Equivalently, κ has the tree property iff there is no κ-Aronszajn tree.

• A cardinal κ has the *linear order property* iff every linear order $(L, <)$ of size κ has a subset with order type κ or κ^* under $<$.

Lemma 26.6. *For any regular cardinal κ, the linear order property implies the tree property.*

Proof. We are going to go from a tree to a linear order in a different way from the branch method of Chapter 22.

Assume the linear order property, and let $(T,<)$ be a κ-tree. For each $x\in T$ and each $\alpha\le \mathrm{ht}(x,T)$ let x^α be the element of height α below x. Thus x^0 is the root which is below x, and $x^{\mathrm{ht}(x)}=x$. For each $x\in T$, let $T\restriction x=\{y\in T:y<x\}$. If x,y are incomparable elements of T, then let $\chi(x,y)$ be the smallest ordinal $\alpha\le\min(\mathrm{ht}(x),\mathrm{ht}(y))$ such that $x^\alpha\ne y^\alpha$. Let $<'$ be a well-order of T. Then we define, for any distinct $x,y\in T$,

$$x<''y \quad\text{iff}\quad x<y,\text{ or } x \text{ and } y \text{ are incomparable and } x^{\chi(x,y)}<' y^{\chi(x,y)}.$$

We claim that this gives a linear order of T. To prove transitivity, suppose that $x<''y<''z$. Then there are several possibilities. These are illustrated in diagrams below.

Case 1. $x<y<z$. Then $x<z$, so $x<''z$.

Case 2. $x<y$, while y and z are incomparable, with $y^{\chi(y,z)}<'z^{\chi(y,z)}$.

Subcase 2.1. $\mathrm{ht}(x)<\chi(y,z)$. Then $x=x^{\mathrm{ht}(x)}=y^{\mathrm{ht}(x)}=z^{\mathrm{ht}(x)}$ so that $x<z$, hence $x<''z$.

Subcase 2.2. $\chi(y,z)\le\mathrm{ht}(x)$. Then x and z are incomparable. In fact, if $z<x$ then $z<y$, contradicting the assumption that y and z are incomparable; if $x\le z$, then $y^{\mathrm{ht}(x)}=x=x^{\mathrm{ht}(x)}=z^{\mathrm{ht}(x)}$, contradiction. Now if $\alpha<\chi(x,z)$ then $y^\alpha=x^\alpha=z^\alpha$; it follows that $\chi(x,z)\le\chi(y,z)$. If $\alpha<\chi(y,z)$ then $\alpha\le\mathrm{ht}(x)$, and hence $x^\alpha=y^\alpha=z^\alpha$; this shows that $\chi(y,z)\le\chi(x,z)$. So $\chi(y,z)=\chi(x,z)$. Hence $x^{\chi(x,z)}=y^{\chi(x,z)}=y^{\chi(y,z)}<'z^{\chi(y,z)}=z^{\chi(x,z)}$, and hence $x<''z$.

Case 3. x and y are incomparable, and $y<z$. Then x and z are incomparable. Now if $\alpha<\chi(x,y)$, then $x^\alpha=y^\alpha=z^\alpha$; this shows that $\chi(x,y)\le\chi(x,z)$. Also, $x^{\chi(x,y)}<'y^{\chi(x,y)}=z^{\chi(x,y)}$, and this implies that $\chi(x,z)\le\chi(x,y)$. So $\chi(x,y)=\chi(x,z)$. It follows that $x^{\chi(x,z)}=x^{\chi(x,y)}<'y^{\chi(x,y)}=z^{\chi(x,z)}$, and hence $x<''z$.

Case 4. x and y are incomparable, and also y and z are incomparable. We consider subcases.

Subcase 4.1. $\chi(y,z)<\chi(x,y)$. Now if $\alpha<\chi(y,z)$, then $x^\alpha=y^\alpha=z^\alpha$; so $\chi(y,z)\le\chi(x,z)$. Also, $x^{\chi(y,z)}=y^{\chi(y,z)}<'z^{\chi(y,z)}$, so that $\chi(x,z)\le\chi(y,z)$. Hence $\chi(x,z)=\chi(y,z)$, and $x^{\chi(x,z)}=y^{\chi(y,z)}<'z^{\chi(y,z)}$, and hence $x<''z$.

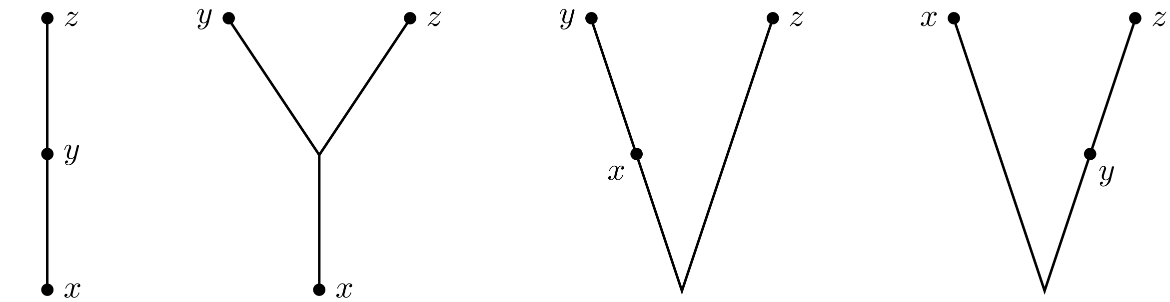
Subcase 4.2. $\chi(y,z)=\chi(x,y)$. Now $x^{\chi(x,y)}<'y^{\chi(x,y)}=y^{\chi(y,z)}<'z^{\chi(y,z)}=z^{\chi(x,y)}$. It follows that $\chi(x,z)\le\chi(x,y)$. For any $\alpha<\chi(x,y)$ we have $x^\alpha=y^\alpha=z^\alpha$ since $\chi(y,z)=\chi(x,y)$. So $\chi(x,y)=\chi(x,z)$. Hence $x^{\chi(x,z)}=x^{\chi(x,y)}<'y^{\chi(x,y)}=y^{\chi(y,z)}<'z^{\chi(y,z)}=z^{\chi(x,z)}$, so $x<''z$.

Subcase 4.3. $\chi(x,y)<\chi(y,z)$. Then $x^{\chi(x,y)}<'y^{\chi(x,y)}=z^{\chi(x,y)}$, and if $\alpha<\chi(x,y)$ then $x^\alpha=y^\alpha=z^\alpha$. It follows that $x<''z$

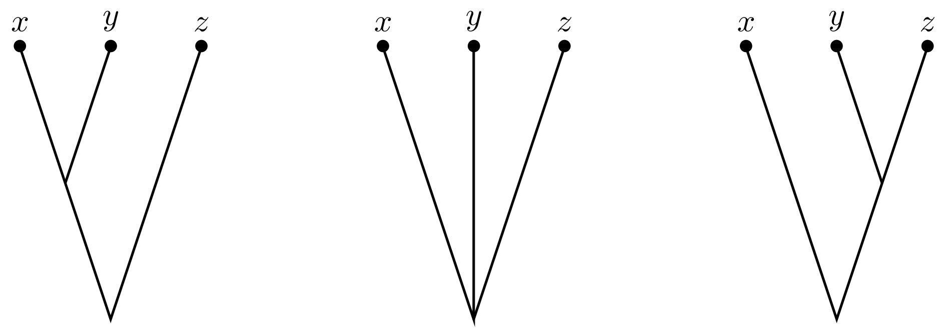
Clearly any two elements of T are comparable under $<''$, so we have a linear order. The following property is also needed.

(*) If $t<x,y$ and $x<''a<''y$, then $t<a$.

In fact, suppose not. If $a\le t$, then $a<x$, hence $a<''x$, contradiction. So a and t are incomparable. Then $\chi(a,t)\le\mathrm{ht}(t)$, and hence $x<''y<''a$ or $a<''x<''y$, contradiction.



Case 1 Subcase 2.1 Subcase 2.2 Case 3



Subcase 4.1 Subcase 4.2 Subcase 4.3

Now by the linear order property, $(T, <'')$ has a subset L of order type κ or κ^*. First suppose that L is of order type κ. Define

$$B = \{t \in T : \exists x \in L \forall a \in L[x \leq'' a \rightarrow t \leq a]\}.$$

We claim that B is a chain in T of size κ. Suppose that $t_0, t_1 \in B$ with $t_0 \neq t_1$, and choose $x_0, x_1 \in L$ correspondingly. Say wlog $x_0 <'' x_1$. Now $t_0 \in B$ and $x_0 \leq'' x_1$, so $t_0 \leq x_1$. And $t_1 \in B$ and $x_1 \leq x_1$, so $t_1 \leq x_1$. So t_0 and t_1 are comparable.

Now let $\alpha < \kappa$; we show that B has an element of height α. For each t of height α let $V_t = \{x \in L : t \leq x\}$. Then

$$\{x \in L : \mathrm{ht}(x) \geq \alpha\} = \bigcup_{\mathrm{ht}(t)=\alpha} V_t;$$

since there are fewer than κ elements of height less than κ, this set has size κ, and so there is a t such that $\mathrm{ht}(t) = \alpha$ and $|V_t| = \kappa$. We claim that $t \in B$. To prove this, take any $x \in V_t$ such that $t < x$. Suppose that $a \in L$ and $x \leq'' a$. Choose $y \in V_t$ with $a <'' y$ and $t < y$. Then $t < x$, $t < y$, and $x \leq'' a <'' y$. If $x = a$, then $t \leq a$, as desired. If $x <'' a$, then $t < a$ by (*).

This finishes the case in which L has a subset of order type κ. The case of order type κ^* is similar, but we give it. So, suppose that L has order type κ^*. Define

$$B = \{t \in T : \exists x \in L \forall a \in L[a \leq'' x \rightarrow t \leq a]\}.$$

We claim that B is a chain in T of size κ. Suppose that $t_0, t_1 \in B$ with $t_0 \neq t_1$, and choose $x_0, x_1 \in L$ correspondingly. Say wlog $x_0 <'' x_1$. Now $t_0 \in B$ and $x_0 \leq x_0$, so $t_0 \leq x_0$. and $t_1 \in B$ and $x_0 \leq'' x_1$, so $t_1 \leq x_0$. So t_0 and t_1 are comparable.

Now let $\alpha < \kappa$; we show that B has an element of height α. For each t of height α let $V_t = \{x \in L : t \leq x\}$. Then

$$\{x \in L : \mathrm{ht}(x) \geq \alpha\} = \bigcup_{\mathrm{ht}(t)=\alpha} V_t;$$

since there are fewer than κ elements of height less than κ, this set has size κ, and so there is a t such that $\mathrm{ht}(t) = \alpha$ and $|V_t| = \kappa$. We claim that $t \in B$. To prove this, take any $x \in V_t$ such that $t < x$. Suppose that $a \in L$ and $a \leq'' x$. Choose $y \in V_t$ with $y <'' a$ and $t < y$. Then $t < x$, $t < y$, and $y <'' a \leq'' x$. If $a = x$, then $t < a$, as desired. If $a <'' x$, then $t < a$ by (*). □

Theorem 26.7. *For any uncountable cardinal κ the following conditions are equivalent:*
(i) κ is weakly compact.
(ii) κ is inaccessible, and it has the linear order property.
(iii) κ is inaccessible, and it has the tree property.
(iv) For any cardinal λ such that $1 < \lambda < \kappa$ we have $\kappa \to (\kappa)^2_\lambda$.

Proof. (i)⇒(ii): Assume that κ is weakly compact. First we need to show that κ is inaccessible.

To show that κ is regular, suppose to the contrary that $\kappa = \sum_{\alpha<\lambda} \mu_\alpha$, where $\lambda < \kappa$ and $\mu_\alpha < \kappa$ for each $\alpha < \lambda$. By the definition of infinite sum of cardinals, it follows that we can write $\kappa = \bigcup_{\alpha<\lambda} M_\alpha$, where $|M_\alpha| = \mu_\alpha$ for each $\alpha < \lambda$ and the M_α's are pairwise disjoint. Define $f : [\kappa]^2 \to 2$ by setting, for any distinct $\alpha, \beta < \kappa$,

$$f(\{\alpha, \beta\}) = \begin{cases} 0 & \text{if } \alpha, \beta \in M_\xi \text{ for some } \xi < \lambda, \\ 1 & \text{otherwise.} \end{cases}$$

Let H be homogeneous for f of size κ. First suppose that $f[[H]^2] = \{0\}$. Fix $\alpha_0 \in H$, and say $\alpha_0 \in M_\xi$. For any $\beta \in H$ we then have $\beta \in M_\xi$ also, by the homogeneity of H. So $H \subseteq M_\xi$, which is impossible since $|M_\xi| < \kappa$. Second, suppose that $f[[H]^2] = \{1\}$. Then any two distinct members of H lie in distinct M_ξ's. Hence if we define $g(\alpha)$ to be the $\xi < \lambda$ such that $\alpha \in M_\xi$ for each $\alpha \in H$, we get a one-one function from H into λ, which is impossible since $\lambda < \kappa$.

To show that κ is strong limit, suppose that $\lambda < \kappa$ but $\kappa \leq 2^\lambda$. Now by Theorem 24.5 we have $2^\lambda \not\to (\lambda^+, \lambda^+)^2$. So choose $f : [2^\lambda]^2 \to 2$ such that there does not exist an $X \in [2^\lambda]^{\lambda^+}$ with $f \restriction [X]^2$ constant. Define $g : [\kappa]^2 \to 2$ by setting $g(A) = f(A)$ for any $A \in [\kappa]^2$. Choose $Y \in [\kappa]^\kappa$ such that $g \restriction [Y]^2$ is constant. Take any $Z \in [Y]^{\lambda^+}$. Then $f \restriction [Z]^2$ is constant, contradiction.

So, κ is inaccessible. Now let $(L, <)$ be a linear order of size κ. Let $\prec$ be a well order of L. Now we define $f : [L]^2 \to 2$; suppose that $a, b \in L$ with $a \prec b$. Then

$$f(\{a, b\}) = \begin{cases} 0 & \text{if } a < b, \\ 1 & \text{if } b > a. \end{cases}$$

Let H be homogeneous for f and of size κ. If $f[[H]^2] = \{0\}$, then H is well-ordered by $<$. If $f[[H]^2] = \{1\}$, then H is well-ordered by $>$.

(ii)$\Rightarrow$(iii): By Lemma 26.6.

(iii)$\Rightarrow$(iv): Assume (iii). Suppose that $F : [\kappa]^2 \to \lambda$, where $1 < \lambda < \kappa$; we want to find a homogeneous set for F of size κ. We construct by recursion a sequence $\langle t_\alpha : \alpha < \kappa\rangle$ of members of ${}^{<\kappa}\kappa$; these will be the members of a tree T. Let $t_0 = \emptyset$. Now suppose that $0 < \alpha < \kappa$ and $t_\beta \in {}^{<\kappa}\kappa$ has been constructed for all $\beta < \alpha$. We now define t_α by recursion; its domain will also be determined by the recursive definition, and for this purpose it is convenient to actually define an auxiliary function $s : \kappa \to \kappa+1$ by recursion. If $s(\eta)$ has been defined for all $\eta < \xi$, we define

$$s(\xi) = \begin{cases} F(\{\beta, \alpha\}) & \text{where } \beta < \alpha \text{ is minimum such that } s \restriction \xi = t_\beta \text{, if there is such a } \beta, \\ \\ \kappa & \text{if there is no such } \beta. \end{cases}$$

Now eventually the second condition here must hold, as otherwise $\langle s \restriction \xi : \xi < \kappa\rangle$ would be a one-one function from κ into $\{t_\beta : \beta < \alpha\}$, which is impossible. Take the least ξ such that $s(\xi) = \kappa$, and let $t_\alpha = s \restriction \xi$. This finishes the construction of the t_α's. Let $T = \{t_\alpha : \alpha < \kappa\}$, with the partial order $\subseteq$. Clearly this gives a tree.

By construction, if $\alpha < \kappa$ and $\xi < \mathrm{dmn}(t_\alpha)$, then $t_\alpha \restriction \xi \in T$. Thus the height of an element t_α is $\mathrm{dmn}(t_\alpha)$.

(2) The sequence $\langle t_\alpha : \alpha < \kappa\rangle$ is one-one.

In fact, suppose that $\beta < \alpha$ and $t_\alpha = t_\beta$. Say that $\mathrm{dmn}(t_\alpha) = \xi$. Then $t_\alpha = t_\alpha \restriction \xi = t_\beta$, and the construction of t_α gives something with domain greater than ξ, contradiction. Thus (2) holds, and hence $|T| = \kappa$.

(3) The set of all elements of T of level $\xi < \kappa$ has size less than κ.

In fact, let U be this set. Then

$$|U| \le \prod_{\eta<\xi} \lambda = \lambda^\xi < \kappa$$

since κ is inaccessible. So (3) holds, and hence, since $|T| = \kappa$, T has height κ and is a κ-tree.

(4) If $t_\beta \subset t_\alpha$, then $\beta < \alpha$ and $F(\{\beta, \alpha\}) = t_\alpha(\mathrm{dmn}(t_\beta))$.

This is clear from the definition.

Now by the tree property, there is a branch B of size κ. For each $\xi < \lambda$ let

$$H_\xi = \{\alpha < \kappa : t_\alpha \in B \text{ and } t_\alpha^\frown\langle\xi\rangle \in B\}.$$

We claim that each H_ξ is homogeneous for F. In fact, take any distinct $\alpha, \beta \in H_\xi$. Then $t_\alpha, t_\beta \in B$. Say $t_\beta \subset t_\alpha$. Then $\beta < \alpha$, and by construction $t_\alpha(\mathrm{dmn}(t_\beta)) = F(\{\alpha, \beta\})$. So $F(\{\alpha, \beta\}) = \xi$ by the definition of H_ξ, as desired. Now

$$\{\alpha < \kappa : t_\alpha \in B\} = \bigcup_{\xi<\lambda} \{\alpha < \kappa : t_\alpha \in H_\xi\},$$

so since $|B| = \kappa$ it follows that $|H_\xi| = \kappa$ for some $\xi < \lambda$, as desired.

(iv)$\Rightarrow$(i): obvious. $\square$

Now we go into the connection of weakly compact cardinals with logic, thereby justifying the name "weakly compact".

Let κ and λ be infinite cardinals. The language $L_{\kappa\lambda}$ is an extension of ordinary first order logic as follows. The notion of a model is unchanged. In the logic, we have a sequence of λ distinct individual variables, and we allow quantification over any one-one sequence of fewer than λ variables. We also allow conjunctions and disjunctions of fewer than κ formulas. It should be clear what it means for an assignment of values to the variables to satisfy a formula in this extended language. We say that an infinite cardinal κ is *logically weakly compact* iff the following condition holds:

(*) For any language $L_{\kappa\kappa}$ with at most κ basic symbols, if Γ is a set of sentences of the language and if every subset of Γ of size less than κ has a model, then also Γ has a model.

Notice here the somewhat unnatural restriction that there are at most κ basic symbols. If we drop this restriction, we obtain the notion of a strongly compact cardinal. These cardinals are much larger than even the measurable cardinals discussed later. See below for more about strongly compact cardinals.

Theorem 26.8. *An infinite cardinal is logically weakly compact iff it is weakly compact.*

Proof. Suppose that κ is logically weakly compact.

(1) κ is regular.

Suppose not; say $X \subseteq \kappa$ is unbounded but $|X| < \kappa$. Take the language with individual constants c_α for $\alpha < \kappa$ and also one more individual constant d. Consider the following set Γ of sentences in this language:

$$\{d \neq c_\alpha : \alpha < \kappa\} \cup \left\{ \bigvee_{\beta \in X} \bigvee_{\alpha < \beta} (d = c_\alpha) \right\}.$$

If $\Delta \in [\Gamma]^{<\kappa}$, let A be the set of all $\alpha < \kappa$ such that $d \neq c_\alpha$ is in Δ. So $|A| < \kappa$. Take any $\alpha \in \kappa \backslash A$, and consider the structure $M = (\kappa, \gamma, \alpha)_{\gamma < \kappa}$. There is a $\beta \in X$ with $\alpha < \beta$, and this shows that M is a model of Δ.

Thus every subset of Γ of size less than κ has a model, so Γ has a model; but this is clearly impossible.

(2) κ is strong limit.

In fact, suppose not; let $\lambda < \kappa$ with $\kappa \leq 2^\lambda$. We consider the language with distinct individual constants c_α, d^i_α for all $\alpha < \kappa$ and $i < 2$. Let Γ be the following set of sentences in this language:

$$\left\{ \bigwedge_{\alpha < \lambda} [(c_\alpha = d^0_\alpha \vee c_\alpha = d^1_\alpha) \wedge d^0_\alpha \neq d^1_\alpha] \right\} \cup \left\{ \bigvee_{\alpha < \lambda} (c_\alpha \neq d^{f(\alpha)}_\alpha) : f \in {}^\lambda 2 \right\}.$$

Suppose that $\Delta \in [\Gamma]^{<\kappa}$. We may assume that Δ has the form

$$\left\{\bigwedge_{\alpha<\lambda}[(c_\alpha = d_\alpha^0 \vee c_\alpha = d_\alpha^1) \wedge d_\alpha^0 \neq d_\alpha^1]\right\} \cup \left\{\bigvee_{\alpha<\lambda}(c_\alpha \neq d_\alpha^{f(\alpha)}) : f \in M\right\},$$

where $M \in [{}^\lambda 2]^{<\kappa}$. Fix $g \in {}^\lambda 2 \backslash M$. Let $d_\alpha^0 = \alpha$, $d_\alpha^1 = \alpha + 1$, and $c_\alpha = d_\alpha^{g(\alpha)}$, for all $\alpha < \lambda$. Clearly $(\kappa, c_\alpha, d_\alpha^i)_{\alpha<\lambda, i<2}$ is a model of Δ.

Thus every subset of Γ of size less than κ has a model, so Γ has a model, say $(M, u_\alpha, v_\alpha^i)_{\alpha<\lambda,i<2}$. By the first part of Γ there is a function $f \in {}^\lambda 2$ such that $u_\alpha = d_\alpha^{f(\alpha)}$ for every $\alpha < \lambda$. this contradicts the second part of Γ.

Hence we have shown that κ is inaccessible.

Finally, we prove that the tree property holds. Suppose that $(T, \leq)$ is a κ-tree. Let L be the language with a binary relation symbol $\prec$, unary relation symbols P_α for each $\alpha < \kappa$, individual constants c_t for each $t \in T$, and one more individual constant d. Let Γ be the following set of sentences:

all $L_{\kappa\kappa}$-sentences holding in the structure $M \stackrel{\text{def}}{=} (T, <, \mathrm{Lev}_\alpha(T), t)_{\alpha<\kappa, t\in T}$;

$\exists x[P_\alpha x \wedge x \prec d]$ for each $\alpha < \kappa$.

Clearly every subset of Γ of size less than κ has a model. Hence Γ has a model $N \stackrel{\text{def}}{=} (A, <', S'_\alpha, a_t, b)_{\alpha<\kappa,t\in T}$. For each $\alpha < \kappa$ choose $e_\alpha \in S'_\alpha$ with $e_\alpha <' b$. Now the following sentence holds in M and hence in N:

$$\forall x \left[P_\alpha x \leftrightarrow \bigvee_{s\in\mathrm{Lev}_\alpha(T)} (x = c_s)\right].$$

Hence for each $\alpha < \kappa$ we can choose $t(\alpha) \in T$ such that $e_a = a_{t(\alpha)}$. Now the sentence

$$\forall x, y, z[x < z \wedge y < z \rightarrow x \text{ and } y \text{ are comparable}]$$

holds in M, and hence in N. Now fix $\alpha < \beta < \kappa$. Now $e_\alpha, e_\beta <' b$, so it follows that e_α and e_β are comparable under $\leq'$. Hence $a_{t(\alpha)}$ and $a_{t(\beta)}$ are comparable under $\leq'$. It follows that $t(\alpha)$ and $t(\beta)$ are comparable under $\leq$. So $t(\alpha) < t(\beta)$. Thus we have a branch of size κ.

Now suppose that κ is weakly compact. Let L be an $L_{\kappa\kappa}$-language with at most κ symbols, and suppose that Γ is a set of sentences in L such that every subset Δ of Γ of size less than κ has a model M_Δ. We will construct a model of Γ by modifying Henkin's proof of the completeness theorem for first-order logic.

First we note that there are at most κ formulas of L. This is easily seen by the following recursive construction of all formulas:

$$F_0 = \text{ all atomic formulas;}$$

$$F_{\alpha+1} = F_\alpha \cup \{\neg\varphi : \varphi \in F_\alpha\} \cup \left\{\bigvee \Phi : \Phi \in [F_\alpha]^{<\kappa}\right\} \cup \{\exists \overline{x}\varphi : \varphi \in F_\alpha, \overline{x} \text{ of length } < \kappa\};$$

$$F_\alpha = \bigcup_{\beta<\alpha} F_\beta \text{ for } \alpha \text{ limit.}$$

By induction, $|F_\alpha| \le \kappa$ for all $\alpha \le \kappa$, and F_κ is the set of all formulas. (One uses that κ is inaccessible.)

Expand L to L' by adjoining a set C of new individual constants, with $|C| = \kappa$. Let Θ be the set of all subformulas of the sentences in Γ. Let $\langle \varphi_\alpha : \alpha < \kappa \rangle$ list all sentences of L' which are of the form $\exists \overline{x} \psi_\alpha(\overline{x})$ and are obtained from a member of Θ by replacing variables by members of C. Here $\overline{x}$ is a one-one sequence of variables of length less than κ; say that $\overline{x}$ has length β_α. Now we define a sequence $\langle d_\alpha : \alpha < \kappa \rangle$; each d_α will be a sequence of members of C of length less than κ. If d_β has been defined for all $\beta < \alpha$, then

$$\bigcup_{\beta<\alpha} \mathrm{rng}(d_\beta) \cup \{c \in C : c \text{ occurs in } \varphi_\beta \text{ for some } \beta < \alpha\}$$

has size less than κ. We then let d_α be a one-one sequence of members of C not in this set; d_α should have length β_α. Now for each $\alpha \le \kappa$ let

$$\Omega_\alpha = \{\exists \overline{x} \psi_\beta(\overline{x}) \to \psi_\beta(\overline{d_\beta}) : \beta < \alpha\}.$$

Note that $\Omega_\alpha \subseteq \Omega_\gamma$ if $\alpha < \gamma \le \kappa$. Now we define for each $\Delta \in [\Gamma]^{<\kappa}$ and each $\alpha \le \kappa$ a model N^Δ_α of $\Delta \cup \Omega_\alpha$. Since $\Omega_0 = \emptyset$, we can let $N^\Delta_0 = M_\Delta$. Having defined N^Δ_α, since the range of d_α consists of new constants, we can choose denotations of those constants, expanding N^Δ_α to $N^\Delta_{\alpha+1}$, so that the sentence

$$\exists \overline{x} \psi_\alpha(\overline{x}) \to \psi_\alpha(\overline{d_\alpha})$$

holds in $N^\Delta_{\alpha+1}$. For $\alpha \le \kappa$ limit we let $N^\Delta_\alpha = \bigcup_{\beta<\alpha} N^\Delta_\beta$.

It follows that N^Δ_κ is a model of $\Delta \cup \Omega_\kappa$. So each subset of $\Gamma \cup \Omega_\kappa$ of size less than κ has a model.

It suffices now to find a model of $\Gamma \cup \Omega_\kappa$ in the language L'. Let $\langle \psi_\alpha : \alpha < \kappa \rangle$ be an enumeration of all sentences obtained from members of Θ by replacing variables by members of C, each such sentence appearing κ times. Let T consist of all f satisfying the following conditions:

(3) f is a function with domain $\alpha < \kappa$.

(4) $\forall \beta < \alpha[(\psi_\beta \in \Gamma \cup \Omega_\kappa \to f(\beta) = \psi_\beta)$ and $\psi_\beta \notin \Gamma \cup \Omega_\kappa \to f(\beta) = \neg\psi_\beta)]$.

(5) $\mathrm{rng}(f)$ has a model.

Thus T forms a tree $\subseteq$.

(6) T has an element of height α, for each $\alpha < \kappa$.

In fact, $\Delta \stackrel{\text{def}}{=} \{\psi_\beta : \beta < \alpha, \psi_\beta \in \Gamma \cup \Omega_\kappa\} \cup \{\neg\psi_\beta : \beta < \alpha, \neg\psi_\beta \in \Gamma \cup \Omega_\kappa\}$ is a subset of $\Gamma \cup \Omega_\kappa$ of size less than κ, so it has a model P. For each $\beta < \alpha$ let

$$f(\beta) = \begin{cases} \psi_\beta & \text{if } P \models \psi_\beta, \\ \neg\psi_\gamma & \text{if } P \models \neg\psi_\beta. \end{cases}$$

Clearly f is an element of T with height α. So (6) holds.

Thus T is clearly a κ-tree, so by the tree property we can let B be a branch in T of size κ. Let $\Xi = \{f(\alpha) : \alpha < \kappa,\ f \in B,\ f \text{ has height } \alpha + 1\}$. Clearly $\Gamma \cup \Omega_\kappa \subseteq \Xi$ and for every $\alpha < \kappa$, $\psi_\alpha \in \Xi$ or $\neg\psi_\alpha \in \Xi$.

(7) If $\varphi, \varphi \to \chi \in \Xi$, then $\chi \in \Xi$.

In fact, say $\varphi = f(\alpha)$ and $\varphi \to \chi = f(\beta)$. Choose $\gamma > \alpha, \beta$ so that ψ_γ is χ. We may assume that $\mathrm{dmn}(f) \geq \gamma + 1$. Since $\mathrm{rng}(f)$ has a model, it follows that $f(\gamma) = \chi$. So (7) holds.

Let S be the set of all terms with no variables in them. We define $\sigma \equiv \tau$ iff $\sigma, \tau \in S$ and $(\sigma = \tau) \in \Xi$. Then $\equiv$ is an equivalence relation on S. In fact, let $\sigma \in S$. Say that $\sigma = \sigma$ is ψ_α. Since ψ_α holds in every model, it holds in any model of $\{f(\beta) : \beta \leq \alpha\}$, and hence $f(\alpha) = (\sigma = \sigma)$. So $(\sigma = \sigma) \in \Xi$ and so $\sigma \equiv \sigma$. Symmetry and transitivity follow by (7).

Let M be the collection of all equivalence classes. Using (7) it is easy to see that the function and relation symbols can be defined on M so that the following conditions hold:

(8) If F is an m-ary function symbol, then

$$F^M(\sigma_0/\equiv, \ldots, \sigma_{m-1}/\equiv) = F(\sigma_0, \ldots, \sigma_{m-1})/\equiv .$$

(9) If R is an m-ary relation symbol, then

$$\langle \sigma_0/\equiv, \ldots, \sigma_{m-1}/\equiv\rangle \in R^M \quad \text{iff} \quad R(\sigma_0, \ldots, \sigma_{m-1}) \in \Xi.$$

Now the final claim is as follows:

(10) If φ is a sentence of L', then $M \models \varphi$ iff $\varphi \in \Xi$.

Clearly this will finish the proof. We prove (10) by induction on φ. It is clear for atomic sentences by (8) and (9). If it holds for φ, it clearly holds for $\neg\varphi$. Now suppose that Q is a set of sentences of size less than κ, and (10) holds for each member of Q. Suppose that $M \models \bigwedge Q$. Then $M \models \varphi$ for each $\varphi \in Q$, and so $Q \subseteq \Xi$. Hence there is a $\Delta \in [\kappa]^{<\kappa}$ such that $Q = f[\Delta]$, with $f \in B$. Choose α greater than each member of Δ such that ψ_α is the formula $\bigwedge Q$. We may assume that $\alpha \in \mathrm{dmn}(f)$. Since $\mathrm{rng}(f)$ has a model, it follows that $f(\alpha) = \bigwedge Q$. Hence $\bigwedge Q \in \Xi$.

Conversely, suppose that $\bigwedge Q \in \Xi$. From (7) it easily follows that $\varphi \in \Xi$ for every $\varphi \in Q$, so by the inductive hypothesis $M \models \varphi$ for each $\varphi \in Q$, so $M \models \bigwedge Q$.

Finally, suppose that φ is $\exists\overline{x}\psi$, where (10) holds for shorter formulas. Suppose that $M \models \exists\overline{x}\psi$. Then there are members of S such that when they are substituted in ψ for $\overline{x}$, obtaining a sentence ψ', we have $M \models \psi'$. Hence by the inductive hypothesis, $\psi' \in \Xi$. (7) then yields $\exists\overline{x}\psi \in \Xi$.

Conversely, suppose that $\exists\overline{x}\psi \in \Xi$. Now there is a sequence $\overline{d}$ of members of C such that $\exists\overline{x}\psi \in \Xi \to \psi(\overline{d})$ is also in Ξ, and so by (7) we get $\psi(\overline{d}) \in \Xi$. By the inductive hypothesis, $M \models \psi(\overline{d})$, so $M \models \exists\overline{x}\psi \in \Xi$. □

Next we want to show that every weakly compact cardinal is a Mahlo cardinal. To do this we need two lemmas.

Lemma 26.9. *Let A be a set of infinite cardinals such that for every regular cardinal κ, the set $A \cap \kappa$ is non-stationary in κ. Then there is a one-one regressive function with domain A.*

Proof. We proceed by induction on $\gamma \stackrel{\text{def}}{=} \bigcup A$. Note that γ is a cardinal; it is 0 if $A = \emptyset$. The cases $\gamma = 0$ and $\gamma = \omega$ are trivial, since then $A = \emptyset$ or $A = \{\omega\}$ respectively.

Next, suppose that γ is a successor cardinal κ^+. Then $A = A' \cup \{\kappa^+\}$ for some set A' of infinite cardinals less than κ^+. Then $\bigcup A' < \kappa^+$, so by the inductive hypothesis there is a one-one regressive function f on A'. We can extend f to A by setting $f(\kappa^+) = \kappa$, and so we get a one-one regressive function defined on A.

Suppose that γ is singular. Let $\langle \mu_\xi : \xi < \mathrm{cf}(\gamma) \rangle$ be a strictly increasing continuous sequence of infinite cardinals with supremum γ, with $\mathrm{cf}(\gamma) < \mu_0$. Note then that for every cardinal $\lambda < \gamma$, either $\lambda < \mu_0$ or else there is a unique $\xi < \mathrm{cf}(\gamma)$ such that $\mu_\xi \le \lambda < \mu_{\xi+1}$. For every $\xi < \mathrm{cf}(\gamma)$ we can apply the inductive hypothesis to $A \cap \mu_\xi$ to get a one-one regressive function g_ξ with domain $A \cap \mu_\xi$. We now define f with domain A. In case $\mathrm{cf}(\gamma) = \omega$ we define, for each $\lambda \in A$,

$$f(\lambda) = \begin{cases} g_0(\lambda) + 2 & \text{if } \lambda < \mu_0, \\ \mu_\xi + g_{\xi+1}(\lambda) + 1 & \text{if } \mu_\xi < \lambda < \mu_{\xi+1}, \\ \mu_\xi & \text{if } \lambda = \mu_{\xi+1}, \\ 1 & \text{if } \lambda = \mu_0, \\ 0 & \text{if } \lambda = \gamma \in A. \end{cases}$$

Here the addition is ordinal addition. Clearly f is as desired in this case. If $\mathrm{cf}(\gamma) > \omega$, let $\langle \nu_\xi : \xi < \mathrm{cf}(\gamma) \rangle$ be a strictly increasing sequence of limit ordinals with supremum $\mathrm{cf}(\gamma)$. Then we define, for each $\lambda \in A$,

$$f(\lambda) = \begin{cases} g_0(\lambda) + 1 & \text{if } \lambda < \mu_0, \\ \mu_\xi + g_{\xi+1}(\lambda) + 1 & \text{if } \mu_\xi < \lambda < \mu_{\xi+1}, \\ \nu_\xi & \text{if } \lambda = \mu_\xi, \\ 0 & \text{if } \lambda = \gamma \in A. \end{cases}$$

Clearly f works in this case too.

Finally, suppose that γ is a regular limit cardinal. By assumption, there is a club C in γ such that $C \cap \gamma \cap A = \emptyset$. We may assume that $C \cap \omega = \emptyset$. Let $\langle \mu_\xi : \xi < \gamma \rangle$ be the strictly increasing enumeration of C. Then we define, for each $\lambda \in A$,

$$f(\lambda) = \begin{cases} g_0(\lambda) + 1 & \text{if } \lambda < \mu_0, \\ \mu_\xi + g_{\xi+1}(\lambda) + 1 & \text{if } \mu_\xi < \lambda < \mu_{\xi+1}, \\ 0 & \text{if } \lambda = \gamma \in A. \end{cases}$$

Clearly f works in this case too. □

Lemma 26.10. *Suppose that κ is weakly compact, and S is a stationary subset of κ. Then there is a regular $\lambda < \kappa$ such that $S \cap \lambda$ is stationary in λ.*

Proof. Suppose not. Thus for all regular $\lambda < \kappa$, the set $S \cap \lambda$ is non-stationary in λ. Let C be the collection of all infinite cardinals less than κ. Clearly C is club in κ, so

$S \cap C$ is stationary in κ. Clearly still $S \cap C \cap \lambda$ is non-stationary in λ for every regular $\lambda < \kappa$. So we may assume from the beginning that S is a set of infinite cardinals.

Let $\langle \lambda_\xi : \xi < \kappa \rangle$ be the strictly increasing enumeration of S. Let

$$T = \left\{ s : \exists \xi < \kappa \left[s \in \prod_{\eta < \xi} \lambda_\eta \text{ and } s \text{ is one-one} \right] \right\}.$$

For every $\xi < \kappa$ the set $S \cap \lambda_\xi$ is non-stationary in every regular cardinal, and hence by Lemma 26.9 there is a one-one regressive function s with domain $S \cap \lambda_\xi$. Now $S \cap \lambda_\xi = \{\lambda_\eta : \eta < \xi\}$. Hence $s \in T$.

Clearly T forms a tree of height κ under $\subseteq$. Now for any $\alpha < \kappa$,

$$\prod_{\beta < \alpha} \lambda_\beta \leq \left(\sup_{\beta < \alpha} \lambda_\beta \right)^{|\alpha|} < \kappa.$$

Hence by the tree property there is a branch B in T of size κ. Thus $\bigcup B$ is a one-one regressive function with domain S, contradicting Fodor's theorem. $\square$

Theorem 26.11. *Every weakly compact cardinal is Mahlo, hyper-Mahlo, hyper-hyper-Mahlo, etc.*

Proof. Let κ be weakly compact. Let $S = \{\lambda < \kappa : \lambda \text{ is regular}\}$. Suppose that C is club in κ. Then C is stationary in κ, so by Lemma 26.10 there is a regular $\lambda < \kappa$ such that $C \cap \lambda$ is stationary in λ; in particular, $C \cap \lambda$ is unbounded in λ, so $\lambda \in C$ since C is closed in κ. Thus we have shown that $S \cap C \neq \emptyset$. So κ is Mahlo.

Let $S' = \{\lambda < \kappa : \lambda \text{ is a Mahlo cardinal}\}$. Suppose that C is club in κ. Let $S'' = \{\lambda < \kappa : \lambda \text{ is regular}\}$. Since κ is Mahlo, S'' is stationary in κ. Then $C \cap S''$ is stationary in κ, so by Lemma 26.10 there is a regular $\lambda < \kappa$ such that $C \cap S'' \cap \lambda$ is stationary in λ. Hence λ is Mahlo, and also $C \cap \lambda$ is unbounded in λ, so $\lambda \in C$ since C is closed in κ. Thus we have shown that $S' \cap C \neq \emptyset$. So κ is hyper-Mahlo.

Let $S''' = \{\lambda < \kappa : \lambda \text{ is a hyper-Mahlo cardinal}\}$. Suppose that C is club in κ. Let $S^{iv} = \{\lambda < \kappa : \lambda \text{ is Mahlo}\}$. Since κ is hyper-Mahlo, S^{iv} is stationary in κ. Then $C \cap S^{iv}$ is stationary in κ, so by Lemma 26.10 there is a regular $\lambda < \kappa$ such that $C \cap S^{iv} \cap \lambda$ is stationary in λ. Hence λ is hyper-Mahlo, and also $C \cap \lambda$ is unbounded in λ, so $\lambda \in C$ since C is closed in κ. Thus we have shown that $S''' \cap C \neq \emptyset$. So κ is hyper-hyper-Mahlo.

Etc. $\square$

We now give another equivalent definition of weak compactness. For it we need several lemmas.

Lemma 26.12. *Suppose that* $\mathbf{R}$ *is a well-founded class relation on a class* $\mathbf{A}$*, and it is set-like and extensional. Also suppose that* $\mathbf{B} \subseteq \mathbf{A}$*,* $\mathbf{B}$ *is transitive,* $\forall a, b \in \mathbf{A}[a\mathbf{R}b \in \mathbf{B} \rightarrow a \in \mathbf{B}]$*, and* $\forall a, b \in \mathbf{B}[a\mathbf{R}b \leftrightarrow a \in b]$*. Let* $\mathbf{G}, \mathbf{M}$ *be the Mostowski collapse of* $(\mathbf{A}, \mathbf{R})$*. Then* $\mathbf{G} \restriction \mathbf{B}$ *is the identity.*

Proof. Suppose not, and let $\mathbf{X} = \{b \in \mathbf{B} : \mathbf{G}(b) \neq b\}$. Since we are assuming that $\mathbf{X}$ is a nonempty subclass of $\mathbf{A}$, choose $b \in \mathbf{X}$ such that $y \in \mathbf{A}$ and $y\mathbf{R}b$ imply that $y \notin \mathbf{X}$. Then

$$\begin{aligned}\mathbf{G}(b) &= \{\mathbf{G}(y) : y \in \mathbf{A} \text{ and } y\mathbf{R}b\}\\ &= \{\mathbf{G}(y) : y \in \mathbf{B} \text{ and } y\mathbf{R}b\}\\ &= \{y : y \in \mathbf{B} \text{ and } y\mathbf{R}b\}\\ &= \{y : y \in \mathbf{B} \text{ and } y \in b\}\\ &= \{y : y \in b\}\\ &= b,\end{aligned}$$

contradiction. □

Lemma 26.13. *Let κ be weakly compact. Then for every $U \subseteq V_\kappa$, the structure $(V_\kappa, \in, U)$ has a transitive elementary extension $(M, \in, U')$ such that $\kappa \in M$.*

(This means that $V_\kappa \subseteq M$ and a sentence holds in the structure $(V_\kappa, \in, U, x)_{x\in V_\kappa}$ iff it holds in $(M, \in, U', x)_{x\in V_\kappa}$.)

Proof. Let Γ be the set of all $L_{\kappa\kappa}$-sentences true in the structure $(V_\kappa, \in, U, x)_{x\in V_\kappa}$, together with the sentences

$$\begin{aligned}&c \text{ is an ordinal},\\ &\alpha < c \text{ (for all } \alpha < \kappa),\end{aligned}$$

where c is a new individual constant. The language here clearly has κ many symbols. Every subset of Γ of size less than κ has a model; namely we can take $(V_\kappa, \in, U, x, \beta)_{x\in V_\kappa}$, choosing β greater than each α appearing in the sentences of Γ. Hence by weak compactness, Γ has a model $(M, E, W, k_x, y)_{x\in V_\kappa}$. This model is well-founded, since the sentence

$$\neg\exists v_0 v_1 \ldots \left[\bigwedge_{n\in\omega}(v_{n+1} \in v_n)\right]$$

holds in $(V_\kappa, \in, U, x)_{x\in V_\kappa}$, and hence in $(M, E, W, k_x, y)_{x\in V_\kappa}$.

Note that k is an injection of V_κ into M. Let F be a bijection from $M\backslash\mathrm{rng}(k)$ onto $\{(V_\kappa, u) : u \in M\backslash\mathrm{rng}(k)\}$. Then $G \stackrel{\text{def}}{=} k^{-1} \cup F^{-1}$ is one-one, mapping M onto some set N such that $V_\kappa \subseteq N$. We define, for $x, z \in N$, $xE'z$ iff $G^{-1}(x)EG^{-1}(z)$. Then G is an isomorphism from $(M, E, W, k_x, y)_{x\in V_\kappa}$ onto $\overline{N} \stackrel{\text{def}}{=} (N, E', G[W], x, G(y))_{x\in V_\kappa}$. Of course $\overline{N}$ is still well-founded. It is also extensional, since the extensionality axiom holds in $(V_\kappa, \in)$ and hence in (M, E) and (N, E'). Let H, P be the Mostowski collapse of (N, E'). Thus P is a transitive set, and

(1) H is an isomorphism from (N, E') onto $(P, \in)$.

(2) $\forall a, b \in N[aE'b \in V_\kappa \to a \in b]$.

In fact, suppose that $a, b \in N$ and $aE'b \in V_\kappa$. Let the individual constants used in the expansion of $(V_\kappa, \in, U)$ to $(V_\kappa, \in, U, x)_{a \in V_\kappa}$ be $\langle c_x : x \in V_\kappa \rangle$. Then

$$(V_\kappa, \in, U, x)_{a \in V_\kappa} \models \forall z \left[z \in k_b \rightarrow \bigvee_{w \in b} (z = k_w) \right],$$

and hence this sentence holds in $(N, E', G[W], x, G(y))_{x \in V_\kappa}$ as well, and so there is a $w \in b$ such that $a = w$, i.e., $a \in b$. So (2) holds.

(3) $\forall a, b \in V_\kappa [a \in b \rightarrow aE'b]$

In fact, suppose that $a, b \in V_\kappa$ and $a \in b$. Then the sentence $k_a \in k_b$ holds in $(V_\kappa, \in, U, x)_{x \in V_\kappa}$, so it also holds in $(N, E', G[W], x, G(y))_{x \in V_\kappa}$, so that $aE'b$.

We have now verified the hypotheses of Lemma 26.12. It follows that $H \restriction V_\kappa$ is the identity. In particular, $V_\kappa \subseteq P$. Now take any sentence σ in the language of $(V_\kappa, \in, U, x)_{x \in V_\kappa}$. Then

$$\begin{aligned}
(V_\kappa, \in, U, x)_{x \in V_\kappa} \models \sigma \quad & \text{iff} \quad (M, E, W, k_x)_{x \in V_\kappa} \models \sigma \\
& \text{iff} \quad (N, E', G[W], x)_{x \in V_\kappa} \models \sigma \\
& \text{iff} \quad (P, \in, H[G[W]], x)_{x \in V_\kappa} \models \sigma.
\end{aligned}$$

Thus $(P, \in, H[G[W]])$ is an elementary extension of $(V_\kappa, \in, U)$.

Now for $\alpha < \kappa$ we have

$$\begin{aligned}
& (M, E, W, k_x, y)_{x \in V_\kappa} \models [y \text{ is an ordinal and } k_\alpha E y], \quad \text{hence} \\
& (N, E', G[W], x, G(y))_{x \in V_\kappa} \models [G(y) \text{ is an ordinal and } \alpha E' G(y)], \quad \text{hence} \\
& (P, \in, H[G[W]], x, H(G(y)))_{x \in V_\kappa} \models [H(G(y)) \text{ is an ordinal and } \alpha \in H(G(y))].
\end{aligned}$$

Thus $H(G(y))$ is an ordinal in P greater than each $\alpha < \kappa$, so since P is transitive, $\kappa \in P$. □

An infinite cardinal κ is *first-order describable* iff there is a $U \subseteq V_\kappa$ and a sentence σ in the language for $(V_\kappa, \in, U)$ such that $(V_\kappa, \in, U) \models \sigma$, while there is no $\alpha < \kappa$ such that $(V_\alpha, \in, U \cap V_\alpha) \models \sigma$.

Theorem 26.14. *If κ is infinite but not inaccessible, then it is first-order describable.*

Proof. ω is describable by the sentence that says that κ is the first limit ordinal; absoluteness is used. The subset U is not needed for this. Now suppose that κ is singular.

Let $\lambda = \mathrm{cf}(\kappa)$, and let f be a function whose domain is some ordinal $\gamma < \kappa$ with $\mathrm{rng}(f)$ cofinal in κ. Let $U = \{(\lambda, \beta, f(\beta)) : \beta < \lambda\}$. Let σ be the sentence expressing the following:

For every ordinal γ there is an ordinal δ with $\gamma < \delta$, U is nonempty, and there is an ordinal μ and a function g with domain μ such that U consists of all triples $(\mu, \beta, g(\beta))$ with $\beta < \mu$.

Clearly $(V_\kappa, \in, U) \models \sigma$. Suppose that $\alpha < \kappa$ and $(V_\alpha, \in, V_\alpha \cap U) \models \sigma$. Then α is a limit ordinal, and there is an ordinal $\gamma < \alpha$ and a function g with domain γ such that $V_\alpha \cap U$ consists of all triples $(\gamma, \beta, g(\beta))$ with $\beta < \gamma$. (Some absoluteness is used.) Now $V_\alpha \cap U$ is nonempty; choose $(\gamma, \beta, g(\beta))$ in it. Then $\gamma = \lambda$ since it is in U. It follows that $g = f$. Choose $\beta < \lambda$ such that $\alpha < f(\beta)$. Then $(\lambda, \beta, f(\beta)) \in U \cap V_\alpha$. Since $\alpha < f(\beta)$, it follows that α has rank less than α, contradiction.

Now suppose that $\lambda < \kappa \leq 2^\lambda$. A contradiction is reached similarly, as follows. Let f be a function whose domain is $\mathscr{P}(\lambda)$ with range κ. Let $U = \{(\lambda, B, f(B)) : B \subseteq \lambda\}$. Let σ be the sentence expressing the following:

For every ordinal γ there is an ordinal δ with $\gamma < \delta$, U is nonempty, and there is an ordinal μ and a function g with domain $\mathscr{P}(\mu)$ such that U consists of all triples $(\mu, B, g(B))$ with $B \subseteq \mu$.

Clearly $(V_\kappa, \in, U) \models \sigma$. Suppose that $\alpha < \kappa$ and $(V_\alpha, \in, V_\alpha \cap U) \models \sigma$. Then α is a limit ordinal, and there is an ordinal $\gamma < \alpha$ and a function g with domain $\mathscr{P}(\gamma)$ such that $V_\alpha \cap U$ consists of all triples $(\gamma, B, g(B))$ with $B \subseteq \gamma$. (Some absoluteness is used.) Clearly $\gamma = \lambda$; otherwise $U \cap V_\alpha$ would be empty. Note that $g = f$. Choose $B \subseteq \lambda$ such that $\alpha = f(B)$. Then $(\lambda, B, f(B)) \in U \cap V_\alpha$. Again this implies that α has rank less than α, contradiction. □

The new equivalent of weak compactness involves second-order logic. We augment first order logic by adding a new variable S ranging over subsets rather than elements. There is one new kind of atomic formula: Sv with v a first-order variable. This is interpreted as saying that v is a member of S.

Now an infinite cardinal κ is Π^1_1-*indescribable* iff for every $U \subseteq V_\kappa$ and every second-order sentence σ of the form $\forall S\varphi$, with no quantifiers on S within φ, if $(V_\kappa, \in, U) \models \sigma$, then there is an $\alpha < \kappa$ such that $(V_\alpha, \in, U \cap V_\alpha) \models \sigma$. Note that if κ is Π^1_1-indescribable then it is not first-order describable.

Theorem 26.15. *An infinite cardinal κ is weakly compact iff it is Π^1_1-indescribable.*

Proof. First suppose that κ is Π^1_1-indescribable. By Theorem 26.14 it is inaccessible. So it suffices to show that it has the tree property. By the proof of Theorem 26.7(iii)⇒(iv) it suffices to check the tree property for a tree $T \subseteq {}^{<\kappa}\kappa$. Note that ${}^{<\kappa}\kappa \subseteq V_\kappa$. Let σ be the following sentence in the second-order language of $(V_\kappa, \in, T)$:

$$\exists S[T \text{ is a tree under } \subset, \text{ and}$$
$$S \subseteq T \text{ and } S \text{ is a branch of } T \text{ of unbounded length}].$$

Thus for each $\alpha < \kappa$ the sentence σ holds in $(V_\alpha, \in, T \cap V_\alpha)$. Hence it holds in $(V_\kappa, \in, T)$, as desired.

Now suppose that κ is weakly compact. Let $U \subseteq V_\kappa$, and let σ be a Π^1_1-sentence holding in $(V_\kappa, \in, U)$. By Lemma 26.13, let $(M, \in, U')$ be a transitive elementary extension of $(V_\kappa, \in, U)$ such that $\kappa \in M$. Say that σ is $\forall S\varphi$, with φ having no quantifiers on S. Now

$$\forall X \subseteq V_\kappa[(V_\kappa, \in, U) \models \varphi(X)]. \tag{1}$$

Now since $\kappa \in M$ and $(M, \in)$ is a model of ZFC, V_κ^M exists, and by absoluteness it is equal to V_κ. Hence by (1) we get

$$(M, \in, U') \models \forall X \subseteq V_\kappa \varphi^{V_\kappa}(U' \cap V_\kappa).$$

Hence

$$(M, \in, U') \models \exists\alpha\forall X \subseteq V_\alpha \varphi^{V_\alpha}(U' \cap V_\alpha),$$

so by the elementary extension property we get

$$(V_\kappa, \in, U) \models \exists\alpha\forall X \subseteq V_\alpha \varphi^{V_\alpha}(U' \cap V_\alpha).$$

We choose such an α. Since $V_\kappa \cap \mathbf{On} = \kappa$, it follows that $\alpha < \kappa$. Hence $(V_\alpha, \in, U' \cap V_\alpha) \models \sigma$, as desired. $\square$

Assuming that Mahlo cardinals exist, the first such is not weakly compact. This follows from the theorem that if κ is weakly compact, then the set of Mahlo cardinals below κ is stationary; see Corollary 17.19 in Jech.

Measurable cardinals

Our third kind of large cardinal is the class of measurable cardinals. Although, as the name suggests, this notion comes from measure theory, the definition and results we give are purely set-theoretical. Moreover, similarly to weakly compact cardinals, it is not obvious from the definition that we are dealing with large cardinals.

The definition is given in terms of the notion of an ultrafilter on a set.

• Let X be a nonempty set. A *filter* on X is a family $\mathscr{F}$ of subsets of X satisfying the following conditions:

(i) $X \in \mathscr{F}$.
(ii) If $Y, Z \in \mathscr{F}$, then $Y \cap Z \in \mathscr{F}$.
(iii) If $Y \in \mathscr{F}$ and $Y \subseteq Z \subseteq X$, then $Z \in \mathscr{F}$.

• A filter $\mathscr{F}$ on a set X is *proper* or *nontrivial* iff $\emptyset \notin \mathscr{F}$.

• An *ultrafilter* on a set X is a nontrivial filter $\mathscr{F}$ on X such that for every $Y \subseteq X$, either $Y \in \mathscr{F}$ or $X \backslash Y \in \mathscr{F}$.

• A family $\mathscr{A}$ of subsets of X has the *finite intersection property*, fip, iff for every finite subset $\mathscr{B}$ of $\mathscr{A}$ we have $\bigcap \mathscr{B} \neq \emptyset$.

• If $\mathscr{A}$ is a family of subsets of X, then the *filter generated by* $\mathscr{A}$ is the set

$$\{Y \subseteq X : \bigcap \mathscr{B} \subseteq Y \text{ for some finite } \mathscr{B} \subseteq \mathscr{A}\}.$$

[Clearly this is a filter on X, and it contains $\mathscr{A}$.]

Proposition 26.16. *If $x \in X$, then $\{Y \subseteq X : x \in Y\}$ is an ultrafilter on X.* $\square$

An ultrafilter of the kind given in this proposition is called a *principal ultrafilter*. There are nonprincipal ultrafilters on any infinite set, as we will see shortly.

Proposition 26.17. *Let $\mathscr{F}$ be a proper filter on a set X. Then the following are equivalent:*

(i) $\mathscr{F}$ is an ultrafilter.

(ii) $\mathscr{F}$ is maximal in the partially ordered set of all proper filters (under $\subseteq$).

Proof. (i)⇒(ii): Assume (i), and suppose that $\mathscr{G}$ is a filter with $\mathscr{F} \subset \mathscr{G}$. Choose $Y \in \mathscr{G}\backslash\mathscr{F}$. Since $Y \notin \mathscr{F}$, we must have $X\backslash Y \in \mathscr{F} \subseteq \mathscr{G}$. So $Y, X\backslash Y \in \mathscr{G}$, hence $\emptyset = Y \cap (X\backslash Y) \in \mathscr{G}$, and so $\mathscr{G}$ is not proper.

(ii)⇒(i): Assume (ii), and suppose that $Y \subseteq X$, with $Y \notin \mathscr{F}$; we want to show that $X\backslash Y \in \mathscr{F}$. Let

$$\mathscr{G} = \{Z \subseteq X : Y \cap W \subseteq Z \text{ for some } W \in \mathscr{F}\}.$$

Clearly $\mathscr{G}$ is a filter on X, and $\mathscr{F} \subseteq \mathscr{G}$. Moreover, $Y \in \mathscr{G}\backslash\mathscr{F}$. It follows that $\mathscr{G}$ is not proper, and so $\emptyset \in \mathscr{G}$. Thus there is a $W \in \mathscr{F}$ such that $Y \cap W = \emptyset$. Hence $W \subseteq X\backslash Y$, and hence $X\backslash Y \in \mathscr{F}$, as desired. □

Theorem 26.18. *For any infinite set X there is a nonprincipal ultrafilter on X. Moreover, if $\mathscr{A}$ is any collection of subsets of X with fip, then $\mathscr{A}$ can be extended to an ultrafilter.*

Proof. First we show that the first assertion follows from the second. Let $\mathscr{A}$ be the collection of all cofinite subsets of X—the subsets whose complements are finite. $\mathscr{A}$ has fip, since if $\mathscr{B}$ is a finite subset of $\mathscr{A}$, then $X\backslash\bigcap\mathscr{B} = \bigcup_{Y\in\mathscr{B}}(X\backslash B)$ is finite. By the second assertion, $\mathscr{A}$ can be extended to an ultrafilter F. Clearly F is nonprincipal.

To prove the second assertion, let $\mathscr{A}$ be a collection of subsets of X with fip, and let $\mathscr{C}$ be the collection of all proper filters on X which contain $\mathscr{A}$. Clearly the filter generated by $\mathscr{A}$ is proper, so $\mathscr{C} \neq \emptyset$. We consider $\mathscr{C}$ as a partially ordered set under inclusion. Any subset $\mathscr{D}$ of $\mathscr{C}$ which is a chain has an upper bound in $\mathscr{C}$, namely $\bigcup\mathscr{D}$, as is easily checked. So by Zorn's lemma $\mathscr{C}$ has a maximal member F. By Proposition 26.16, F is an ultrafilter. □

• Let X be an infinite set, and let κ be an infinite cardinal. An ultrafilter F on X is *κ-complete* iff for any $\mathscr{A} \in [F]^{<\kappa}$ we have $\bigcap\mathscr{A} \in F$. We also say σ-complete synonomously with $\aleph_1$-complete.

This notion is clearly a generalization of one of the properties of ultrafilters. In fact, every ultrafilter is ω-complete, and every principal ultrafilter is κ-complete for every infinite cardinal κ.

Lemma 26.19. *Suppose that X is an infinite set, F is an ultrafilter on X, and κ is the least infinite cardinal such that there is an $\mathscr{A} \in [F]^\kappa$ such that $\bigcap\mathscr{A} \notin F$. Then there is a partition $\mathscr{P}$ of X such that $|\mathscr{P}| = \kappa$ and $X\backslash Y \in F$ for all $Y \in \mathscr{P}$.*

Proof. Let $\langle Y_\alpha : \alpha < \kappa\rangle$ enumerate $\mathscr{A}$. Let $Z_0 = X\backslash Y_0$, and for $\alpha > 0$ let $Z_\alpha = (\bigcap_{\beta<\alpha} Y_\beta)\backslash Y_\alpha$. Note that $Y_\alpha \subseteq X\backslash Z_\alpha$, and so $X\backslash Z_\alpha \in F$. Clearly $Z_\alpha \cap Z_\beta = \emptyset$ for $\alpha \neq \beta$. Let $W = \bigcap_{\alpha<\lambda} Y_\alpha$. Clearly $W \cap Z_\alpha = \emptyset$ for all $\alpha < \lambda$. Let

$$\mathscr{P} = (\{Z_\alpha : \alpha < \kappa\} \cup \{W\})\backslash\{\emptyset\}.$$

So $\mathscr{P}$ is a partition of X and $X\backslash Z \in F$ for all $Z \in \mathscr{P}$. Clearly $|\mathscr{P}| \leq \kappa$. If $|\mathscr{P}| < \kappa$, then

$$\emptyset = \bigcap_{Z\in\mathscr{P}} (X\backslash Z) \in F,$$

contradiction. So $|\mathscr{P}| = \kappa$. □

Theorem 26.20. *Suppose that κ is the least infinite cardinal such that there is a nonprincipal σ-complete ultrafilter F on κ. Then F is κ-complete.*

Proof. Assume the hypothesis, but suppose that F is not κ-complete. So there is a $\mathscr{A} \in [F]^{<\kappa}$ such that $\bigcap\mathscr{A} \notin F$. Hence by Lemma 26.19 there is a partition $\mathscr{P}$ of κ such that $|\mathscr{P}| < \kappa$ and $X\backslash P \in F$ for every $P \in \mathscr{P}$. Let $\langle P_\alpha : \alpha < \lambda\rangle$ be a one-one enumeration of $\mathscr{P}$, λ an infinite cardinal. We are now going to construct a nonprincipal σ-complete ultrafilter G on λ, which will contradict the minimality of κ.

Define $f : \kappa \to \lambda$ by letting $f(\beta)$ be the unique $\alpha < \lambda$ such that $\beta \in P_\alpha$. Then we define

$$G = \{D \subseteq \lambda : f^{-1}[D] \in F\}.$$

We check the desired conditions for G. $\emptyset \notin G$, since $f^{-1}[\emptyset] = \emptyset \notin F$. If $D \in G$ and $D \subseteq E$, then $f^{-1}[D] \in F$ and $f^{-1}[D] \subseteq f^{-1}[E]$, so $f^{-1}[E] \in F$ and hence $E \in G$. Similarly, G is closed under $\cap$. Given $D \subseteq \lambda$, either $f^{-1}[D] \in F$ or $f^{-1}[\lambda\backslash D] = \kappa\backslash f^{-1}[D] \in F$, hence $D \in G$ or $\lambda\backslash D \in G$. So G is an ultrafilter on λ. It is nonprincipal, since for any $\alpha < \lambda$ we have $f^{-1}[\{\alpha\}] = P_\alpha \notin F$ and hence $\{\alpha\} \notin G$. Finally, G is σ-complete, since if $\mathscr{D}$ is a countable subset of G, then

$$f^{-1}\left[\bigcap\mathscr{D}\right] = \bigcap_{P\in\mathscr{D}} f^{-1}[P] \in F,$$

and hence $\bigcap\mathscr{D} \in G$. □

We say that an uncountable cardinal κ is *measurable* iff there is a κ-complete nonprincipal ultrafilter on κ.

Theorem 26.21. *Every measurable cardinal is weakly compact.*

Proof. Let κ be a measurable cardinal, and let U be a nonprincipal κ-complete ultrafilter on κ.

Since U is nonprincipal, $\kappa\backslash\{\alpha\} \in U$ for every $\alpha < \kappa$. Then κ-completeness implies that $\kappa\backslash F \in U$ for every $F \in [\kappa]^{<\kappa}$.

Now we show that κ is regular. For, suppose it is singular. Then we can write $\kappa = \bigcup_{\alpha<\lambda}\Gamma_\alpha$, where $\lambda < \kappa$ and each Γ_α has size less than κ. So by the previous paragraph, $\kappa\backslash\Gamma_\alpha \in U$ for every $\alpha < \kappa$, and hence

$$\emptyset = \bigcap_{\alpha<\lambda} (\kappa\backslash\Gamma_\alpha) \in U,$$

contradiction.

Next, κ is strong limit. For, suppose that $\lambda < \kappa$ and $2^\lambda \geq \kappa$. Let $S \in [{}^\lambda 2]^\kappa$. Let $\langle f_\alpha : \alpha < \kappa \rangle$ be a one-one enumeration of S. Now for each $\beta < \lambda$, one of the sets $\{\alpha < \kappa : f_\alpha(\beta) = 0\}$ and $\{\alpha < \kappa : f_\alpha(\beta) = 1\}$ is in U, so we can let $\varepsilon(\beta) \in 2$ be such that $\{\alpha < \kappa : f_\alpha(\beta) = \varepsilon(\beta)\} \in U$. Then

$$\bigcap_{\beta<\lambda} \{\alpha < \kappa : f_\alpha(\beta) = \varepsilon(\beta)\} \in U;$$

this set clearly has only one element, contradiction.

Thus we now know that κ is inaccessible. Finally, we check the tree property. Let $(T, \prec)$ be a tree of height κ such that every level has size less than κ. Then $|T| = \kappa$, and we may assume that actually $T = \kappa$. Let $B = \{\alpha < \kappa : \{t \in T : \alpha \preceq t\} \in U\}$. Clearly any two elements of B are comparable under $\prec$. Now take any $\alpha < \kappa$; we claim that $\mathrm{Lev}_\alpha(T) \cap B \neq \emptyset$. In fact,

$$\kappa = \{t \in T : \mathrm{ht}(t, T) < \alpha\} \cup \bigcup_{t \in \mathrm{Lev}_\alpha(T)} \{s \in T : t \preceq s\}. \tag{1}$$

Now by regularity of κ we have $|\{t \in T : \mathrm{ht}(t, T) < \alpha\}| < \kappa$, and so the complement of this set is in U, and then (1) yields

$$\bigcup_{t \in \mathrm{Lev}_\alpha(T)} \{s \in T : t \preceq s\} \in U. \tag{2}$$

Now $|\mathrm{Lev}_\alpha(T)| < \kappa$, so from (2) our claim easily follows.

Thus B is a branch of size κ, as desired. $\square$

Measurable cardinals allow one to define a natural elementary embedding of V into a class M, as follows.

Let κ be a measurable cardinal, let S be a set with $\kappa \leq |S|$, and let U be a κ-complete nonprincipal ultrafilter on S. Then $\mathrm{Fcn}(S)$ is the class of all functions with domain S. We define

$$\begin{aligned} f =^* g \quad &\text{iff} \quad f, g \in \mathrm{Fcn}(S) \text{ and } \{x \in S : f(x) = g(x)\} \in U; \\ f \in^* g \quad &\text{iff} \quad f, g \in \mathrm{Fcn}(S) \text{ and } \{x \in S : f(x) \in g(x)\} \in U. \end{aligned}$$

Clearly $=^*$ is an equivalence relation on $\mathrm{Fcn}(S)$. We denote by $[f]$ the Scott equivalence class of f:

$$[f] = \{g : f =^* g \text{ and } \forall h(h =^* f \rightarrow \mathrm{rank}(g) \leq \mathrm{rank}(h))\}.$$

Then we define Ult_S to be the collection of all equivalence classes, with $\in_{\mathrm{Ult}_S} = \{([f], [g]) : f \in^* g\}$. We can write this as $\in_{\mathrm{Ult}_S} = \{(x, y) : \exists f, g[x = [f],\ y = [g], \text{ and } f \in^* g\}$.

Proposition 26.22. *If $f, g \in \mathrm{Fcn}(S)$ and $f \in^* g$, then there is an $f' \in \prod_{s \in S}(g(s) \cup \{\emptyset\})$ such that $f =^* f'$.*

Proof. For each $s \in S$, let

$$f'(s) = \begin{cases} f(s) & \text{if } f(s) \in g(s), \\ \emptyset & \text{otherwise.} \end{cases}$$

Clearly $f =^* f'$. □

Proposition 26.23. *$\in_{\mathrm{Ult}_S}$ is set-like.*

Proof. Let $x \in \mathrm{Ult}_S$. Choose $g \in \mathrm{Fcn}(S)$ such that $x = [g]$. We claim

$$(*) \qquad \{y \in \mathrm{Ult}_S : y \in_{\mathrm{Ult}_S} x\} = \left\{[f] : f \in \prod_{s \in S}(g(s) \cup \{\emptyset\}) \text{ and } f \in^* g\right\}.$$

(Clearly this will prove the proposition.) To prove $(*)$, first suppose that $y \in \mathrm{Ult}_S$ and $y \in_{\mathrm{Ult}_S} x$. Choose $f, g' \in \mathrm{Fcn}(S)$ such that $x = [g']$, $y = [f]$, and $f \in^* g'$. Then $[g] = [g']$ and $f \in^* g$. By Proposition 26.22, choose $f' \in \prod_{s \in S}(g(s) \cup \{\emptyset\})$ such that $f =^* f'$. Then $f' \in^* g$ and $y = [f] = [f']$. So y is in the right side of $(*)$.

Second suppose that $f \in \prod_{s \in S}(g(s) \cup \{\emptyset\})$ and $f \in^* g$. Then $[f] \in \mathrm{Ult}_S$ and $[f] \in_{\mathrm{Ult}_S} x$, as desired. □

Theorem 26.24. *For any formula $\varphi(x_1, \ldots, x_n)$ of set theory and any $f_1, \ldots, f_n \in Fcn(S)$ we have*

$$\mathrm{Ult}_S \models \varphi([f_1], \ldots, [f_n]) \quad \text{iff} \quad \{s \in S : \varphi(f_1(s), \ldots, f_n(s))\} \in U$$

Note that this is a theorem schema in ZFC.

Proof. Induction on φ:

$$\begin{aligned}
[f] = [g] \quad & \text{iff} \quad f =^* g \\
& \text{iff} \quad \{s \in S : f(s) = g(s)\} \in U; \\
[f] \in [g] \quad & \text{iff} \quad f \in^* g \\
& \text{iff} \quad \{s \in S : f(s) \in g(s)\} \in U; \\
\mathrm{Ult}_S \models \neg\varphi([f_1], \ldots, [f_n]) \quad & \text{iff} \quad \text{not}(\mathrm{Ult}_S \models \varphi([f_1], \ldots, [f_n])) \\
& \text{iff} \quad \text{not}(\{s \in S : \varphi(f_1(s), \ldots, f_n(s))\} \in U \\
& \text{iff} \quad \{s \in S : \varphi(f_1(s), \ldots, f_n(s))\} \notin U \\
& \text{iff} \quad \{s \in S : \neg\varphi(f_1(s), \ldots, f_n(s))\} \in U.
\end{aligned}$$

$\vee$ is treated similarly. Now suppose that $\mathrm{Ult}_S \models \exists y \varphi([f_1], \ldots, [f_n], y)$. Choose $g \in \mathrm{Fcn}(S)$ such that $\mathrm{Ult}_S \models \varphi([f_1], \ldots, [f_n], [g])$. Then by the inductive hypothesis, $\{s \in S : \varphi(f_1(s), \ldots, f_n(s), g(s))\} \in U$. Now

$$\{s \in S : \varphi(f_1(s), \ldots, f_n(s), g(s))\} \subseteq \{s \in S : \exists y \varphi(f_1(s), \ldots, f_n(s), y)\},$$

so the latter set is in U.

Conversely, suppose that $\{s \in S : \exists y\varphi(f_1(s), \ldots, f_n(s), y)\} \in U$. Then by the axiom of choice, choose $g \in \mathrm{Fcn}(S)$ so that $\{s \in S : \varphi(f_1(s), \ldots, f_n(s), g(s))\} \in U$. By the inductive hypothesis, $\mathrm{Ult}_S \models \varphi([f_1], \ldots, [f_n], [g])$ and hence $\mathrm{Ult}_S \models \exists y\varphi([f_1], \ldots, [f_n], y)$. □

Let a be any set. We define c_a^S, a function with domain S, by $c_a^S(s) = a$ for all $s \in S$.

Proposition 26.25. $\in_{\mathrm{Ult}_S}$ *is well-founded.*

Proof. Suppose that $\ldots f_{n+1} \in^* f_n \ldots \in^* f_0$. Then $\forall n \in \omega\{x \in S : f_{n+1}(x) \in f_n(x)\} \in U$, so $\bigcap_{n\in\omega}\{x \in S : f_{n+1}(x) \in f_n(x)\} \in U$. This set is hence nonempty, and for any x in it, $\cdots f_{n+1}(x) \in f_n(x) \ldots \in f_0(x)$, contradiction.

Proposition 26.26. $(**)$ $\in_{\mathrm{Ult}_S}$ *is extensional on* Ult_S*.*

For, suppose that $[f], [g] \in \mathrm{Ult}_S$ and $[f] \neq [g]$. Then $A \stackrel{\text{def}}{=} \{x \in S : f(x) \neq g(x)\} \in U$. For each $x \in A$ choose $a_x \in f(x)\triangle g(x)$. Then $A = \{x \in S : a_x \in f(x)\backslash g(x)\} \cup \{x \in S : a_x \in g(x)\backslash f(x)$. By symmetry say $\{x \in S : a_x \in f(x)\backslash g(x)\} \in U$. Then $[a] \in_{\mathrm{Ult}_S} [f]$ and $[a] \notin_{\mathrm{Ult}_S} [g]$. □

By Lemma 12.32, the Mostowski collapse mos is an isomorphism. It is defined by

$$\pi([f]) = \{\pi([g]) : [g] \in^* [f]\}$$

for any $f \in \mathrm{Fcn}(S)$. Thus $\pi([g]) \in \pi([f])$ iff $[g] \in^* [f]$ iff $g \in^* f$ iff $\{s \in S : g(s) \in f(s)\} \in U$. We denote $\mathrm{mos}_{AR}[\mathrm{Ult}_S]$ by Ult'_S.

j_U^S is the natural elementary embedding of V into Ult'_S, given by $j_U^S(a) = \pi([c_a^S])$ for any set a. That j_U^S is an elementary embedding of V into Ult'_S is expressed as follows:

Proposition 26.27. *For any formula $\varphi(x_1, \ldots, x_n)$ of set theory and any $a_1, \ldots, a_n$,*

$$\varphi(a_1, \ldots, a_n) \quad \textit{iff} \quad \mathrm{Ult}'_S \models \varphi(j_U^S(a_1), \ldots, j_U^S(a_n)).$$

Proof.

$$\begin{aligned}
\mathrm{Ult}'_S \models \varphi(j_U^S(a_1), \ldots, j_U^S(a_n)) \quad &\text{iff} \quad \mathrm{Ult}'_S \models \varphi(\pi([c_{a_1}^S]), \ldots, \pi([c_{a_n}^S]))\\
&\text{iff} \quad \mathrm{Ult}_S \models \varphi([c_{a_1}^S]), \ldots, [c_{a_n}^S])\\
&\text{iff} \quad \{s \in S : \varphi((c_{a_1}^S)(s), \ldots, (c_{a_n}^S)(s))\} \in U\\
&\text{iff} \quad \{s \in S : \varphi(a_1, \ldots, a_n)\} \in U\\
&\text{iff} \quad \varphi(a_1, \ldots, a_n).
\end{aligned}$$
□

Theorem 26.28. *(i)$\forall\alpha$[α is an ordinal$\rightarrow j_U^S(\alpha)$ is an ordinal].*
(ii) $\alpha < \beta \rightarrow j_U^S(\alpha) < j_U^S(\beta)$.

Proof. (i) holds by Proposition 26.27, taking φ to be the formula "α is an ordinal", and using absoluteness. (ii) also holds by Proposition 2.27. □

Corollary 26.29. Ult'_S *is a transitive class model of ZFC containing all ordinals.*

Proof. If φ is an axiom of ZFC, then $\{s \in S : \varphi\} = S \in U$, so by Theorem 26.24, $\mathrm{Ult}_S \models \varphi$. Since mos_{AR} is an isomorphism, $\mathrm{Ult}'_S \models \varphi$. Thus Ult'_S is a transitive class model of ZFC. Now by Theorem 26.28, $\forall\alpha[\alpha \le j^S_U(\alpha)]$. Since Ult'_S is transitive, it follows that every ordinal is in Ult'_S. □

Theorem 26.30. *If $\gamma < \kappa$, then $j^S_U(\gamma) = \gamma$.*

Proof. In fact, suppose that this is true for all $\alpha < \beta$, with $\beta < \kappa$, and suppose that $\beta < j^S_U(\beta)$. Say $\beta = j^S_U(a)$. Thus $[c^S_a] \in^* [c^S_\beta]$, so $\{s \in S : a \in \beta\} \in U$. Now

$$\{s \in S : a \in \beta\} = \bigcup_{\alpha<\beta} \{s \in S : a = \alpha\},$$

so by κ-completeness there is an $\alpha < \beta$ such that $\{s \in S : a = \alpha\} \in U$. Hence $[c^S_a] = [c^S_\alpha]$, so $\beta = j^S_U(a) = j^S_U(\alpha) = \alpha$, contradiction.

Theorem 26.31. *If U is a κ-complete nonprincipal ideal on κ, then $\kappa < j^\kappa_U(\kappa)$.*

Proof. Let $d(\alpha) = \alpha$ for all $\alpha < \kappa$. Then for any $\gamma < \kappa$, $\{\alpha < \kappa : \gamma < d(\alpha)\} = \{\alpha < \kappa : \gamma < \alpha\} = \kappa\backslash(\alpha+1) \in U$, and so by Theorem 26.24, $[c^\kappa_\gamma] \in_{\mathrm{Ult}_\kappa} [d]$. Hence $j^\kappa_U(\gamma) \in \pi([d])$. Therefore, $\kappa \le \pi([d])$. Also, $\{\alpha < \kappa : d(\alpha) < c^\kappa_\kappa(\alpha)\} = \{\alpha < \kappa : \alpha < \kappa\} = \kappa \in U$, so $[d] \in_{\mathrm{Ult}_\kappa} [c^\kappa_\kappa]$. Hence $\pi([d]) < j^\kappa_U(\kappa)$. So $\kappa < j^\kappa_U(\kappa)$. □

An *inner model of ZFC* is a transitive proper class model of ZFC containing all ordinals. Thus L is an inner model.

Theorem 26.32. *If M is an inner model of ZFC, then $L \subseteq M$.*

Proof. By absoluteness, $L^M = L \subseteq M$. □

Theorem 26.33. (Dana Scott) *If there is a measurable cardinal, then $V \neq L$.*

Proof. Suppose that κ is the least measurable cardinal, and $V = L$. Now $L \subseteq \mathrm{Ult}'_\kappa$ by Theorem 26.29. Hence $V = \mathrm{Ult}'_\kappa = L$. Since j^κ_U is an elementary embedding, it follows that $j^\kappa_U(\kappa)$ is the least measurable cardinal. Since $\kappa < j^\kappa_U(\kappa)$, this is a contradiction. □

Theorem 26.34. *The following conditions are equivalent:*

(i) There is a measurable cardinal.

(ii) There is a nontrivial elementary embedding of the universe into some transitive model of ZFC which contains all ordinals.

Proof. (i)⇒(ii): Propositions 26.27 and 26.29.

Now assume (ii); let f be a nontrivial elementary embedding of the universe V into M, where M is transitive and contains all ordinals.

(1) There is an ordinal α such that $f(\alpha) \neq \alpha$.

In fact, suppose not. Then we claim

(2) $\text{rank}(f(x)) = \text{rank}(x)$ for every set x.

For, let $\varphi(x, y)$ be the formula which defines rank; so

$$V \models \forall x, y[\varphi(x, y) \leftrightarrow y \text{ is an ordinal and } \text{rank}(x) = y].$$

Suppose that $x \in V$. Let $\text{rank}(x) = \alpha$. Then $V \models \varphi(x, \alpha)$, so $M \models \varphi(f(x), f(\alpha))$, hence by the "suppose not" above, $M \models \varphi(f(x), \alpha)$. Since $f(x), \alpha \in V$, by elementarity we have $V \models \varphi(f(x), \alpha)$, so $\text{rank}(f(x)) = \alpha$, as desired in (2).

Now since f is nontrivial, let x be such that $f(x) \neq x$, and choose such an x of minimal rank. If $y \in x$, then $f(y) = y$ by the minimality of $\text{rank}(x)$, and $f(y) \in f(x)$, so $y \in f(x)$. Thus $x \subseteq f(x)$. Since $f(x) \neq x$, we can thus choose $y \in f(x) \backslash x$. But by (2) we have $\text{rank}(f(x)) = \text{rank}(x)$, so $\text{rank}(y) < \text{rank}(x)$, and hence $f(y) = y$ by the minimality of $\text{rank}(x)$. So $f(y) \in f(x)$, hence $y \in x$, contradiction. Hence (1) holds.

Let κ be the least such α.

(3) $j_U^\kappa(x \cup \{x\}) = j_U^\kappa(x) \cup \{j_U^\kappa(x)\}$.

This is clear from Proposition 26.27 and absoluteness. The following two facts follow from (3):

(4) $j_U^\kappa(n) = n$ for all $n \in \omega$.

(5) κ is a limit ordinal.

(6) $j_U^\kappa(\omega) = \omega$.

In fact,

$$\forall x[x = \omega \leftrightarrow \forall y \in x[y \cup \{y\} \in x] \wedge \forall y \in x[\forall z[z \notin y] \vee \exists z[y = z \cup \{z\}]];$$

hence by Proposition 26.27 and absoluteness,

$$\forall x[x = j_U^\kappa(\omega) \leftrightarrow \forall y \in x[y \cup \{y\} \in x] \wedge \forall y \in x[\forall z[z \notin y] \vee \exists z[y = z \cup \{z\}]].$$

Hence (6) follows.

Now we define

$$D = \{X \subseteq \kappa : \kappa \in f(X)\}.$$

Now $\kappa < f(\kappa)$, so $\kappa \in D$. Since $f(\emptyset) = \emptyset$, we have $\emptyset \notin D$. Now $\forall y(y \in X \cap Y \leftrightarrow y \in X$ and $y \in Y)$, so by elementarity, $\forall y(y \in f(X \cup Y) \leftrightarrow y \in f(X)$ and $y \in f(Y))$. So $f(X \cap Y) = f(X) \cap f(Y)$. Also, if $X \subseteq Y$, then $\forall x(x \in X \to x \in Y)$, so by elementarity, $\forall x(x \in f(X) \to x \in f(Y))$. So $X \subseteq Y$ implies that $f(X) \subseteq f(Y)$. From these facts it follows that D is a filter. Also, for any $X \subseteq \kappa$ we have $\forall y(y \in \kappa \leftrightarrow y \in X$ or $y \in (\kappa \backslash X))$, so by elementarity $\forall y(y \in f(\kappa) \leftrightarrow y \in f(X)$ or $y \in f(\kappa \backslash X))$. Hence $f(\kappa) = f(X) \cup f(\kappa \backslash X)$. Since $\kappa \in f(\kappa)$, it follows that $\kappa \in f(X)$ or $\kappa \in f(\kappa \backslash X)$. So D is an ultrafilter.

To show that D is nonprincipal, suppose that $\alpha < \kappa$. Now $\forall x(x \in \{\alpha\} \leftrightarrow x = \alpha)$, so $\forall x(x \in f(\{\alpha\}) \leftrightarrow x = f(\alpha))$. Since $f(\alpha) = \alpha$, it follows that $f(\{\alpha\}) = \{\alpha\}$. So $\kappa \notin f(\{\alpha\})$, and consequently $\{\alpha\} \notin D$.

Next assume that $\gamma < \kappa$ and $X = \langle X_\alpha : \alpha < \gamma\rangle$ is a sequence of members of D. Thus X is a function with domain γ. Let $Y = \bigcap_{\alpha<\gamma} X_\alpha$. Now $f(X)$ is a function with domain $f(\gamma)$, which is γ.

(7) If $\alpha < \gamma$, then $(f(X))_\alpha = f(X_\alpha)$.

For, $(\alpha, X_\alpha) \in X$, so $(f(\alpha), f(X_\alpha)) \in f(X)$. Since $f(\alpha) = \alpha$, (7) follows.

(8) $f(Y) = \bigcap_{\alpha<\gamma} f(X_\alpha)$.

For, $\forall y[y \in Y \leftrightarrow \forall\alpha < \gamma(y \in X_\alpha)]$, so $\forall y[y \in f(Y) \to \forall\alpha < f(\gamma)(y \in (f(X))_\alpha)]$. By (3) and the fact that $f(\gamma) = \gamma$ it follows that $\forall y[y \in f(Y) \leftrightarrow \forall\alpha < \gamma(y \in f(X_\alpha))]$. Thus (8) holds.

Hence $\kappa \in f(Y)$ and so $Y \in D$.

Finally, we show that κ is a cardinal. For suppose it isn't. Then there is a function g mapping some ordinal $\alpha < \kappa$ onto κ. Thus $\kappa\backslash\{g(\xi)\} \in D$ for every $\xi < \alpha$, so also $\bigcap_{\xi<\alpha}(\kappa\backslash\{g(\xi)\}) \in D$. But $\bigcap_{\xi<\alpha}(\kappa\backslash\{g(\xi)\}) = \emptyset$, contradiction. $\square$

If there are weakly compact cardinals, then the first such is not measurable. This follows from Theorem 17.10 of Jech.

Strongly compact cardinals

A cardinal κ is *strongly compact* iff κ is uncountable, and for every $\lambda \geq \kappa$, every κ-complete filter on λ can be extended to a κ-complete ultrafilter on λ.

Theorem 26.35. *Every strongly compact cardinal is measurable.* $\square$

We give two equivalent definitions of strongly compact cardinal. One involves a new kind of ultrafilter.

If $\kappa \leq \lambda$ are infinite cardinals, for each $P \in [\lambda]^{<\kappa}$ let $\hat{P} = \{Q \in [\lambda]^{<\kappa} : P \subseteq Q\}$.

Proposition 26.36. *If $\kappa \leq \lambda$ are infinite cardinals, then there is a κ-complete proper filter containing $\{\hat{P} : P \in [\lambda]^{<\kappa}\}$.*

Proof. Let $\kappa \leq \lambda$ be infinite cardinals, and let $\mathscr{F} \in [[\lambda]^{<\kappa}]^{<\kappa}$. Then

$$\bigcap_{P\in\mathscr{F}} \hat{P} = \left(\bigcup \mathscr{F}\right)\hat{} \neq \emptyset,$$

and the proposition follows. $\square$

For infinite cardinals $\kappa \leq \lambda$, a *fine ultrafilter* on $[\lambda]^{<\kappa}$ is a κ-complete ultrafilter on $[\lambda]^{<\kappa}$ containing $\{\hat{P} : P \in [\lambda]^{<\kappa}\}$.

Theorem 26.37. *For κ an uncountable cardinal the following are equivalent:*

(i) For every cardinal $\lambda \geq \kappa$ there is a fine ultrafilter on $[\lambda]^{<\kappa}$.

(ii) κ is strongly compact.

(iii) For any language $L_{\kappa\omega}$, if Γ is a set of sentences of the language and if every subset of Γ of size less than κ has a model, then also Γ has a model.

Proof. (ii)$\Rightarrow$(i): clear.

(i)$\Rightarrow$(iii): Assume (i), and suppose that Γ is a set of sentences of $L_{\kappa\omega}$ such that every subset of Γ of size less than κ has a model. We may assume that $|\Gamma| \geq \kappa$. Let U be a fine ultrafilter on $[\Gamma]^{<\kappa}$. For each $\Delta \in [\Gamma]^{<\kappa}$ let $\mathbf{A}_\Delta$ be a model of Δ. As in the proof of Theorem 26.8, κ is inaccessible.

Now we need a form of Łoś's theorem on ultraproducts. For $f, g \in \prod_{\Delta \in [\Gamma]^{<\kappa}} A_\Delta$ define $f \equiv g$ iff $\{\Theta \in [\Gamma]^{<\kappa} : f(\Theta) = g(\Theta)\} \in U$.

(1) $\equiv$ is an equivalence relation on $\prod_{\Delta \in [\Gamma]^{<\kappa}} A_\Delta$.

For, clearly $\equiv$ is symmetric and is reflexive on $\prod_{\Delta \in [\Gamma]^{<\kappa}} A_\Delta$. Now suppose that $f \equiv g \equiv h$. Then

$$\{\Theta \in [\Gamma]^{<\kappa} : f(\Theta) = g(\Theta)\} \cap \{\Theta \in [\Gamma]^{<\kappa} : g(\Theta) = h(\Theta)\} \subseteq \{\Theta \in [\Gamma]^{<\kappa} : f(\Theta) = h(\Theta)\},$$

so $f \equiv h$. Thus (1) holds.

Let B be the collection of all equivalence classes. The equivalence class of $f \in \prod_{\Delta \in [\Gamma]^{<\kappa}} A_\Delta$ is denoted by $[f]$. For $\mathbf{R}$ an m-ary relation symbol of our language let

$$\begin{aligned} \mathbf{R}^{\mathbf{B}} = \Bigg\{ & x \in {}^mB : \exists f \in {}^m \left(\prod_{\Delta \in [\Gamma]^{<\kappa}} A_\Delta \right) [\forall i < m[x_i = [f_i]] \wedge \\ & \{\Delta \in [\Gamma]^{<\kappa} : \langle f_{i\Delta} : i < m\rangle \in \mathbf{R}^{A_\Delta}\} \in U \Bigg\} \end{aligned}$$

(2) For all $f \in {}^m(\prod_{\Delta \in [\Gamma]^{<\kappa}} A_\Delta)$ we have

$$[\langle [f_i] : i < m\rangle \in \mathbf{R}^{\mathbf{M}} \leftrightarrow \{\Delta \in [\Gamma]^{<\kappa} : \langle f_0(\Delta), \ldots, f_{m-1}(\Delta)\rangle \in \mathbf{R}^{A_\Delta}\} \in U]$$

In fact, take any $f \in {}^m \left(\prod_{\Delta \in [\Gamma]^{<\kappa}} A_\Delta\right)$. Then $\Leftarrow$ is clear. Now assume that $\langle [f_i] : i < m\rangle \in \mathbf{R}^{\mathbf{M}}$. Choose $g \in {}^m \left(\prod_{\Delta \in [\Gamma]^{<\kappa}} A_\Delta\right)$ such that $\forall i < m[[f_i] = [g_i]]$ and $\{\Delta \in [\Gamma]^{<\kappa} : \langle g_{i\Delta} : i < m\rangle \in \mathbf{R}^{A_\Delta}\} \in U$. Then

$$\begin{aligned} & \bigcap_{i<m} \{\Delta \in [\Gamma]^{<\kappa} : g_{i\Delta} = f_{i\Delta}\} \cap \{\Delta \in [\Gamma]^{<\kappa} : \langle g_{i\Delta} : i < m\rangle \in \mathbf{R}^{A_\Delta}\} \\ & \subseteq \{\Delta \in [\Gamma]^{<\kappa} : \langle f_{i\Delta} : i < m\rangle \in \mathbf{R}^{A_\Delta}\}, \end{aligned}$$

and the left side is in U, so the right side is too, as desired for (2).

(3) If $\mathbf{F}$ is an m-ary function symbol of our language, then there is an m-ary function $\mathbf{F}^B$ on B such that for any $f \in {}^m \prod_{\Delta \in [\Gamma]^{<\kappa}} A_\Delta$ we have

$$\mathbf{F}^B(\langle [f_i] : i < m\rangle) = [\langle F^{A_\Delta}(f_0(\Delta), \ldots, f_{m-1}(\Delta)) : \Delta \in [\Gamma]^{<\kappa}\rangle].$$

In fact, suppose that $\langle [f_i] : i < m\rangle = \langle [g_i] : i < m\rangle$. Then

$$\begin{aligned}&\bigcap_{i<m}\{\Delta \in [\Gamma]^{<\kappa} : f_i(\Delta) = g_i(\Delta)\} \subseteq \{\Delta \in [\Gamma]^{<\kappa} :\\ &\langle F^{A_\Delta}(f_0(\Delta), \ldots, f_{m-1}(\Delta)) = \langle F^{A_\Delta}(g_0(\Delta), \ldots, g_{m-1}(\Delta))\},\end{aligned}$$

and the left side is in U; so also the right side is in U, and (3) follows.

(4) If τ is a term involving at most $v_0, \ldots, v_{m-1}$, then

$$\tau^{\mathbf{B}}([f_0], \ldots, [f_{m-1}]) = [\langle \tau^{A_\Delta}(f_0(\Delta), \ldots, f_{m-1}(\Delta))\rangle : \Delta \in [\Gamma]^{<\kappa}\rangle].$$

We prove (4) by induction on τ. For $\tau = v_i$ we have

$$v_i^{\mathbf{B}}([f_0], \ldots, [f_{m-1}]) = [f_i] = [\langle v_i^{A_\Delta}(f_0(\Delta), \ldots, f_{m-1}(\Delta))\rangle : \Delta \in [\Gamma]^{<\kappa}\rangle].$$

For $\tau = \mathbf{F}(\sigma_0, \ldots, \sigma_{n-1})$ we have

$$\begin{aligned}&\tau^{\mathbf{B}}([f_0], \ldots, [f_{m-1}]) = \mathbf{F}^{\mathbf{B}}(\sigma_0^{\mathbf{B}}([f_0], \ldots, [f_{m-1}]), \ldots, \sigma_{n-1}^{\mathbf{B}}([f_0], \ldots, [f_{m-1}]))\\ &= \mathbf{F}^{\mathbf{B}}([\langle \sigma_0^{A_\Delta}(f_0(\Delta), \ldots, f_{m-1}(\Delta)) : \Delta \in [\Gamma]^{<\kappa}\rangle], \ldots,\\ &[\langle \sigma_{n-1}^{A_\Delta}(f_0(\Delta), \ldots, f_{m-1}(\Delta)) : \Delta \in [\Gamma]^{<\kappa}\rangle])\\ &= [\langle \tau^{A_\Delta}(f_0(\Delta), \ldots, f_{m-1}(\Delta))\rangle : \Delta \in [\Gamma]^{<\kappa}\rangle].\end{aligned}$$

So (4) holds.

Now we claim:

(5) For any formula $\varphi(v_0, \ldots, v_{m-1})$ and any $f_0, \ldots, f_{m-1} \in \prod_{\Delta\in[\Gamma]^{<\kappa}} A_\Delta$ we have

$$\mathbf{B} \models \varphi([f_0], \ldots, [f_{m-1}]) \quad \text{iff} \quad \{\Delta \in [\Gamma]^{<\kappa} : A_\Delta \models \varphi(f_0(\Delta), \ldots, f_{m-1}(\Delta))\} \in U.$$

We prove this by induction on φ.

Case 1. φ is $\sigma = \tau$. Then

$$\begin{aligned}&\mathbf{B} \models (\sigma = \tau)([f_0], \ldots, [f_{m-1}]) \quad \text{iff} \quad \sigma^{\mathbf{B}}([f_0], \ldots, [f_{m-1}]) = \tau^{\mathbf{B}}([f_0], \ldots, [f_{m-1}])\\ &\text{iff} \quad [\langle \sigma^{A_\Delta}(f_0(\delta), \ldots, f_{m-1}(\Delta)) : \Delta \in [\Gamma]^{<\kappa}\rangle] = [\langle \tau^{A_\Delta}(f_0(\Delta), \ldots, f_{m-1}(\Delta)) : \Delta \in [\Gamma]^{<\kappa}\rangle]\\ &\text{iff} \quad \{\Delta : \sigma^{A_\Delta}(f_0(\delta), \ldots, f_{m-1}(\Delta)) = \tau^{A_\Delta}(f_0(\delta), \ldots, f_{m-1}(\Delta))\} \in U\\ &\text{iff} \quad \{\Delta : A_D \models (\sigma = \tau)(f_0(\delta), \ldots, f_{m-1}(\Delta))\} \in U\end{aligned}$$

Case 2. φ is $\mathbf{R}(\sigma_0, \ldots, \sigma_{n-1})$. Then

$$\begin{aligned}&\mathbf{B} \models \mathbf{R}(\sigma_0, \ldots, \sigma_{n-1})([f_0], \ldots, [f_{m-1}])\\ &\text{iff} \quad \langle \sigma_0^{\mathbf{B}}([f_0], \ldots, [f_{m-1}]), \ldots, \sigma_{n-1}^{\mathbf{B}}([f_0], \ldots, [f_{m-1}]))\rangle \in \mathbf{R}^{\mathbf{B}}\\ &\text{iff} \quad \langle [\langle \sigma_0^{A_\Delta}(f_0(\Delta), \ldots, f_{m-1}(\Delta))\rangle : \Delta \in [\Gamma]^{<\kappa}\rangle], \ldots,\\ &\langle \sigma_{n-1}^{A_\Delta}(f_0(\Delta), \ldots, f_{m-1}(\Delta))\rangle : \Delta \in [\Gamma]^{<\kappa}\rangle]\rangle \in \mathbf{R}^{\mathbf{B}}\\ &\text{iff} \quad \{\Delta : \langle \sigma_0^{A_\Delta}(f_0(\Delta), \ldots, f_{m-1}(\Delta)), \ldots,\\ &\sigma_{n-1}^{A_\Delta}(f_0(\Delta), \ldots, f_{m-1}(\Delta))\rangle \in \mathbf{R}^{A_\Delta}\} \in U\\ &\text{iff} \quad \{\Delta : A_\Delta \models \varphi(f_0(\Delta), \ldots, f_{m-1}(\Delta))\} \in U.\end{aligned}$$

Case 3. φ is $\neg\psi$. Then

$$\begin{aligned}\mathbf{B} \models \neg\psi([f_0], \ldots, [f_{m-1}]) \quad &\text{iff} \quad \text{not}(\mathbf{B} \models \psi([f_0], \ldots, [f_{m-1}]))\\ &\text{iff} \quad \text{not}(\{\Delta : A_\Delta \models \psi(f_0(\Delta), \ldots, f_{m-1}(\Delta))\} \in U)\\ &\text{iff} \quad \{\Delta : A_\Delta \models \neg\psi(f_0(\Delta), \ldots, f_{m-1}(\Delta))\} \in U)\end{aligned}$$

Case 4. φ is $\bigwedge \Psi$, with $|\Psi| < \kappa$. Then

$$\mathbf{B} \models \left(\bigwedge \Psi\right)([f_0], \ldots, [f_{m-1}]) \quad \text{iff} \quad \forall \psi \in \Psi \left[\mathbf{B} \models \bigwedge \psi([f_0], \ldots, [f_{m-1}])\right]$$
$$\text{iff} \quad \forall \psi \in \Psi[\{\Delta : A_\Delta \models \psi(f_0(\Delta), \ldots, f_{m-1}(\Delta))\} \in U]$$
$$\text{iff} \quad \left\{\Delta : A_\Delta \models \left(\bigwedge \Psi\right)(f_0(\Delta), \ldots, f_{m-1}(\Delta))\right\} \in U.$$

Case 5. φ is $\exists v_i \psi$. First suppose that $\mathbf{B} \models \varphi([f_0], \ldots, [f_{m-1}])$. Choose g so that $\mathbf{B} \models \psi([f_0], \ldots, [f_{i-1}], [g], [f_{i+1}], \ldots, [f_{m-1}])$. By the inductive hypothesis,

$$\{\Delta : A_\Delta \models \psi(f_0(\Delta), \ldots, f_{i-1}(\Delta), g_i(\Delta), f_{i+1}(\Delta), \ldots, f_{m-1}(\Delta))\} \in U.$$

This set is contained in

$$\{\Delta : A_\Delta \models \varphi(f_0(\Delta), \ldots, f_{m-1}(\Delta))\},$$

which is hence in U, as desired.

Second, suppose that

$$\{\Delta : A_\Delta \models \varphi(f_0(\Delta), \ldots, f_{m-1}(\Delta))\} \in U$$

Take g so that

$$\{\Delta : A_\Delta \models \psi(f_0(\Delta), \ldots, f_{i-1}(\Delta), g(\Delta), f_{i+1}(\Delta), \ldots, f_{m-1}(\Delta))\} \in U$$

Then by the inductive hypothesis, $\mathbf{B} \models \psi([f_0], \ldots, [f_{i-1}], [g], [f_{i+1}], \ldots, [f_{m-1}])$ Hence $\mathbf{B} \models \varphi([f_0], \ldots, [f_{m-1}])$. This completes the proof of (5).

By (5), $\mathbf{B}$ is a model of Γ.

(iii)⇒(ii): Assume (iii), let $\lambda \geq \kappa$ be a cardinal, and suppose that F is a κ-complete proper filter on λ. Take the $L_{\kappa\omega}$-language which has a one-place relation symbol $\mathbf{R}_A$ for each $A \subseteq \lambda$ and an individual constant $\mathbf{c}$. Let Σ be the following set of sentences:

(6) Every sentence true in $(\lambda, A)_{A \subseteq \lambda})$.

(7) Every sentence $\mathbf{R}_A\mathbf{c}$ for $A \in F$.

(8) $\neg\mathbf{R}_\emptyset c$.

We claim:

(8) Every subset of Σ of size less than κ has a model.

(9) $\mathbf{R}_A c \vee \mathbf{R}_{\lambda \backslash A} c$ for each $A \subseteq \lambda$.

For, suppose that Δ is a subset of Σ of size less than κ. Take the structure $(\lambda, A, c)_{A \subseteq \lambda}$ with $c \in A$ for each A such that $\mathbf{R}_A \mathbf{c}$ is a member of Δ. This is possible since F is κ-complete, and we clearly have a model of Δ.

It follows that Σ has a model $\mathbf{B} = (B, \mathbf{R}_A^{\mathbf{B}}, c)_{A \subseteq \lambda}$. Now we define

$$U = \{A : A \subseteq \lambda, \mathbf{B} \models \mathbf{R}_A c\}.$$

We claim that U is the required proper κ-complete ultrafilter on λ containing F

$\emptyset \notin U$ since (8) is in Γ.

If $A \in U$ and $A \subseteq B \subseteq \lambda$, then $\forall x[\mathbf{R}_A x \rightarrow \mathbf{R}_B x]$ is in (6) and hence holds in $\mathbf{B}$, so that $\mathbf{B} \models \mathbf{R}_B c$, which implies that $B \in U$.

For any $A \subseteq \lambda$, $A \in U$ or $\lambda \backslash A \in U$. This holds by (9).

If $\alpha < \kappa$ and $A_\xi \in U$ for each $\xi < \alpha$, then $\bigcap_{\alpha < \alpha} A_\xi \in U$. This holds since $\forall x[\bigwedge_{\xi < \alpha} \mathbf{R}_{A_\xi} x \rightarrow \mathbf{R}_C x]$ is a member of Σ, where $C = \bigcap_{\xi < \alpha} A_\xi$.

$F \subseteq U$ by(7). $\square$

A diagram of large cardinals

We define some more large cardinals, and then indicate relationships between them by a diagram.

All cardinals are assumed to be uncountable.

1. **regular limit cardinals**.

2. **inaccessible**.

3. **Mahlo**.

4. **weakly compact**.

5. **indescribable**. The ω-order language is an extension of first order logic in which one has variables of each type $n \in \omega$. For n positive, a variable of type n ranges over $\mathscr{P}^n(A)$ for a given structure A. In addition to first-order atomic formulas, one has formulas $P \in Q$ with P n-th order and Q $(n+1)$-order. Quantification is allowed over the higher order variables.

κ is *indescribable* iff for all $U \subseteq V_\kappa$ and every higher order sentence σ, if $(V_\kappa, \in, U) \models \sigma$ then there is an $\alpha < \kappa$ such that $(V_\alpha, \in, U \cap V_\alpha) \models \sigma$.

6. $\kappa \rightarrow (\omega)_2^{<\omega}$. Here in general

$$\kappa \rightarrow (\alpha)_m^{<\omega}$$

means that for every function $f : \bigcup_{n \in \omega} [\kappa]^n \rightarrow m$ there is a subset $H \subseteq \kappa$ of order type α such that for each $n \in \omega$, $f \restriction [H]^n$ is constant.

7. $0^\sharp$ exists. This means that there is a non-identity elementary embedding of L into L. Thus no actual cardinal is referred to. But $0^\sharp$ implies the existence of some large cardinals, and the existence of some large cardinals implies that $0^\sharp$ exists.

8. **Jónsson** κ is a Jónsson cardinal iff every model of size κ has a proper elementary substructure of size κ.

9. **Rowbottom** κ is a Rowbottom cardinal iff for every uncountable $\lambda < \kappa$, every model of type (κ, λ) has an elementary submodel of type (κ, ω).

10. **Ramsey** $\kappa \to (\kappa)_2^{<\omega}$.

11. **measurable**

12. **strong** κ is a strong cardinal iff for every set X there exists a nontrivial elementary embedding from V to $\mathbf{M}$ with κ the first ordinal moved and with $\kappa \in \mathbf{M}$.

13. **Woodin** κ is a Woodin cardinal iff

$$\forall A \subseteq V_\kappa \forall \lambda < \kappa \exists \mu \in (\lambda, \kappa) \forall \nu < \kappa \exists j [j \text{ is a nontrivial elementary embedding of } V$$
$$\text{into some set } \mathbf{M}, \text{ with } \mu \text{ the first ordinal moved, such that}$$
$$j(\mu) > \nu, V_\nu \subseteq \mathbf{M}, A \cap V_\nu = j(A) \cap V_\nu]$$

14. **superstrong** κ is superstrong iff there is a nontrivial elementary embedding $j : V \to \mathbf{M}$ with κ the first ordinal moved, such that $V_{j(\kappa)} \subseteq \mathbf{M}$.

15. **strongly compact** κ is strongly compact iff for any $L_{\kappa\kappa}$-language, if Γ is a set of sentences and every subset of Γ of size less than κ has a model, then Γ itself has a model.

16. **supercompact** κ is supercompact iff for every A with $|A| \geq \kappa$ there is normal measure on $P_\kappa(A)$.

17. **extendible** For an ordinal η, we say that k is η-extendible iff there exist ζ and a nontrivial elementary embedding $j : V_{\kappa+\eta} \to V_\zeta$ with κ first ordinal moved, with $\eta < j(\kappa)$. κ is extendible iff it is η-extendible for every $\eta > 0$.

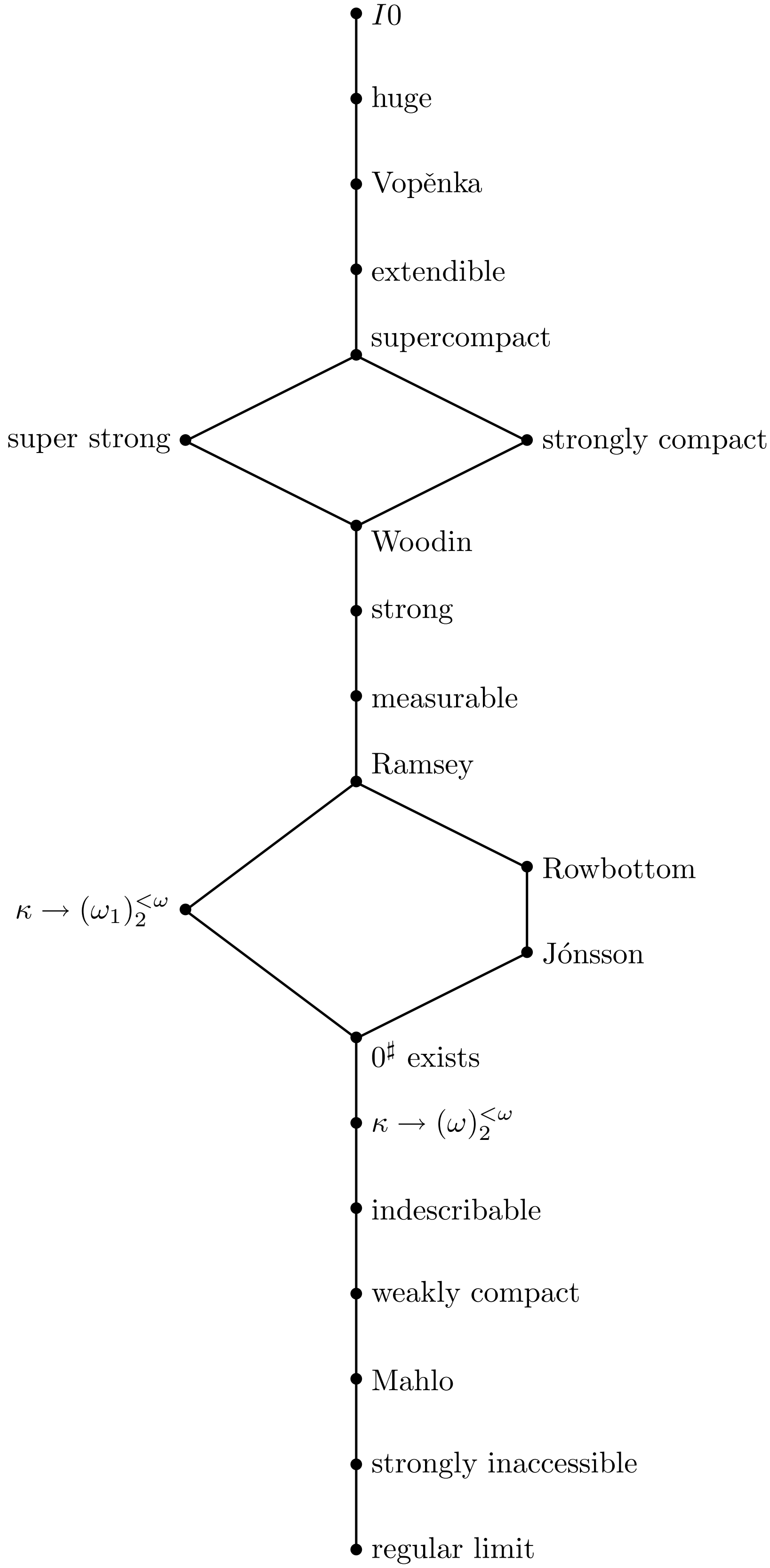
26. **Vopěnka's principle** If C is a proper class of models in a given first-order language, then there exist two distinct members $A, B \in C$ such that A can be elementarily embedded in B.

19. **huge** A cardinal κ is huge iff there is a nontrivial elementary embedding $j : V \to \mathbf{M}$ with κ the first ordinal moved, such that $\mathbf{M}^{j(\kappa)} \subseteq \mathbf{M}$.

20. *I*0. There is an ordinal δ and a proper elementary embedding j of $L(V_{\delta+1})$ into $L(V_{\delta+1})$ such that the first ordinal moved is less than δ.

Many even stronger large cardinals are described in S. Cramer [2017].

In the diagram on the next page, a line indicates that (the consistency of the) existence of the cardinal above implies (the consistency of the) existence of the one below.


$I0$
huge
Vopěnka
extendible
supercompact
super strong
strongly compact
Woodin
strong
measurable
Ramsey
Rowbottom
$\kappa \to (\omega_1)_2^{<\omega}$
Jónsson
$0^\sharp$ exists
$\kappa \to (\omega)_2^{<\omega}$
indescribable
weakly compact
Mahlo
strongly inaccessible
regular limit

Proposition 26.22. *Let κ be an uncountable regular cardinal. We define $S < T$ iff S and T are stationary subsets of κ and the following two conditions hold:*

(1) $\{\alpha \in T : \text{cf}(\alpha) \leq \omega\}$ is nonstationary in κ.

(2) $\{\alpha \in T : S \cap \alpha$ is nonstationary in $\alpha)\}$ is nonstationary in κ.

Then if $\omega < \lambda < \mu < \kappa$, all these cardinals regular, then $E^\kappa_\lambda < E^\kappa_\mu$, where

$$E^\kappa_\lambda = \{\alpha < \kappa : \text{cf}(\alpha) = \lambda\},$$

and similarly for E^κ_μ.

Proof. First of all, $\{\alpha \in E^\kappa_\mu : \text{cf}(\alpha) \leq \omega\}$ is empty, so of course it is nonstationary in κ.

For (2), let $C = (\mu, \kappa)$. We claim that

$$\{\alpha \in E^\kappa_\mu : E^\kappa_\lambda \cap \alpha \text{ is nonstationary in } \alpha\} \cap C = \emptyset;$$

this will prove (2). In fact, suppose that α is in the indicated intersection. Let D be club in α such that $E^\kappa_\lambda \cap D = \emptyset$. Now $\alpha \in E^\kappa_\mu$, so $\text{cf}(\alpha) = \mu$. Define α_ξ for all $\xi < \lambda$ as follows. Let α_0 be the least member of D. If $\alpha_\xi \in D$ has been defined, take any member $\alpha_{\xi+1}$ of D greater than α_ξ. If ξ is limit less than λ, let $\alpha_\xi = \bigcup_{\eta<\xi} \alpha_\eta$. Then $\alpha_\xi \in D$ because D is closed. Now let $\beta = \bigcup_{\xi<\lambda} \alpha_\xi$. Then $\beta \in D$ since D is closed, and $\text{cf}(\beta) = \lambda$. So $\beta \in E^\kappa_\lambda \cap D$, contradiction. $\square$

Proposition 26.23. *Continuing Proposition 26.22: Assume that κ is uncountable and regular. Then the relation $<$ is transitive.*

Proof. Suppose that $A < B < C$. Then by definition

(1) $\{\alpha \in C : \text{cf}(\alpha) \leq \omega\}$ is nonstationary in κ.

(2) $\{\alpha \in B : \alpha \cap A$ is nonstationary$\}$ is nonstationary in κ.

(3) $\{\alpha \in C : \alpha \cap B$ is nonstationary$\}$ is nonstationary in κ.

We want to show

$$\{\alpha \in C : \alpha \cap A \text{ is nonstationary}\} \text{ is nonstationary.}$$

Our assumptions give us clubs M, N in κ such that

$$\begin{aligned}&\{\alpha \in B : \alpha \cap A \text{ is nonstationary}\} \cap M = \emptyset \text{ and}\\ &\{\alpha \in C : \alpha \cap B \text{ is nonstationary}\} \cap N = \emptyset.\end{aligned}$$

Let M' be the set of all limits of members of M; so also M' is club in κ. Now it suffices to show that

$$\{\alpha \in C : \alpha \cap A \text{ is nonstationary}\} \cap M' \cap N = \emptyset.$$

So, suppose that $\alpha \in C \cap M' \cap N$; we show that $\alpha \cap A$ is stationary in α. To this end, let P be club in α, and let P' be the set of all of its limit points. Now $\alpha \in C \cap N$, so $\alpha \cap B$

is stationary. Since $\alpha \in M'$, it follows that $\alpha \cap M$ is club in α. So $M \cap P'$ is club in α, and so we can choose $\beta \in \alpha \cap B \cap M \cap P'$. Now $\beta \in B \cap M$, so $\beta \cap A$ is stationary in β. Since $\beta \in P'$, it follows that $P \cap \beta$ is club in β. So $\beta \cap A \cap P \neq \emptyset$, hence $A \cap P \neq \emptyset$, as desired. □

Proposition 26.24. *If κ is an uncountable regular cardinal and S is a stationary subset of κ, we define*

$$Tr(S) = \{\alpha < \kappa : \mathrm{cf}(\alpha) > \omega \text{ and } S \cap \alpha \text{ is stationary in } \alpha\}.$$

Suppose that A, B are stationary subsets of an uncountable regular cardinal κ and $A < B$. Then $Tr(A)$ is stationary.

Proof. Assume the conditions of the proposition. Thus by definition, $\{\alpha \in B : \mathrm{cf}(\alpha) \leq \omega\}$ is nonstationary in κ, and also $\{\alpha \in B : A \cap \alpha$ is non-stationary in $\alpha\}$ is non-stationary in κ. Hence there is a club C in κ such that $C \cap \{\alpha \in B : \mathrm{cf}(\alpha) \leq \omega\} = \emptyset$ and also $C \cap \{\alpha \in B : A \cap \alpha$ is non-stationary in $\alpha\} = \emptyset$. Thus $B \cap C \subseteq \mathrm{Tr}(A)$, and it follows that $\mathrm{Tr}(A)$ is stationary in κ. □

Proposition 26.25. *(Real-valued measurable cardinals) We describe a special kind of measure. A measure on a set S is a function $\mu : \mathscr{P}(S) \to [0, \infty)$ satisfying the following conditions:*

(1) $\mu(\emptyset) = 0$ and $\mu(S) = 1$.

(2) If $\mu(\{s\}) = 0$ for all $s \in S$,

(3) If $\langle X_i : i \in \omega \rangle$ is a system of pairwise disjoint subsets of S, then $\mu(\bigcup_{i \in \omega} X_i) = \sum_{i \in \omega} \mu(X_i)$. (The X_i's are not necessarily nonempty.)

Let κ be an infinite cardinal. Then μ is κ-additive iff for every system $\langle X_\alpha : \alpha < \gamma \rangle$ of nonempty pairwise disjoint sets, wich $\gamma < \kappa$, we have

$$\mu\left(\bigcup_{\alpha < \gamma} X_\alpha\right) = \sum_{\alpha < \gamma} \mu(X_\alpha).$$

Here this sum (where the index set γ might be uncountable), is understood to be

$$\sup_{\substack{F \subseteq \gamma, \\ F \text{ finite}}} \sum_{\alpha \in F} \mu(X_\alpha).$$

We say that an uncountable cardinal κ is real-valued measurable iff there is a κ-additive measure on κ. Then every measurable cardinal is real-valued measurable.

Proof. Suppose that κ is measurable. Thus κ is uncountable, and there is a κ-complete nonprincipal ultrafilter U on κ. Now for any $X \subseteq \kappa$ we define

$$\mu(X) = \begin{cases} 1 & \text{if } X \in U, \\ 0 & \text{otherwise.} \end{cases}$$

Conditions (1) and (2) in the definition of measure are clear. We can check (3) and κ-additivity simultaneously, by assuming that $\langle X_\alpha : \alpha < \beta \rangle$ is a system of pairwise disjoint subsets of κ, with $\beta < \kappa$. If $\mu(\bigcup_{\alpha<\beta} X_\alpha) = 0$, clearly $\mu(X_\alpha) = 0$ for all $\alpha < \beta$, and so

$$\mu\left(\bigcup_{\alpha<\gamma} X_\alpha\right) = \sum_{\alpha<\gamma} \mu(X_\alpha).$$

Suppose that $\mu(\bigcup_{\alpha<\beta} X_\alpha) = 1$. Thus $\bigcup_{\alpha<\beta} X_\alpha \in U$. If $X_\alpha \notin U$ for all $\alpha < \beta$, then $\kappa \backslash X_\alpha \in U$ for all $\alpha < \beta$, and hence by κ-completeness,

$$\kappa \backslash \left(\bigcup_{\alpha<\beta} X_\alpha\right) = \bigcap_{\alpha<\beta} (\kappa \backslash X_\alpha) \in U,$$

contradiction. Hence $X_\alpha \in U$ for some $\alpha < \beta$. There can be only one such α, since if $\gamma \neq \alpha$ and $X_\gamma \in U$, then $\emptyset = X_\alpha \cap X_\gamma \in U$, contradiction. Hence again

$$\mu\left(\bigcup_{\alpha<\gamma} X_\alpha\right) = \sum_{\alpha<\gamma} \mu(X_\alpha). \qquad \square$$

Proposition 26.26. *Suppose that μ is a measure on a set S. A subset A of S is a μ-atom iff $\mu(A) > 0$ and for every $X \subseteq A$, either $\mu(X) = 0$ or $\mu(X) = \mu(A)$. Then if κ is a real-valued measurable cardinal, μ is a κ-additive measure on κ, and $A \subseteq \kappa$ is a μ-atom, then $\{X \subseteq A : \mu(X) = \mu(A)\}$ is a κ-complete nonprincipal ultrafilter on A. It follows that κ is a measurable cardinal if there exist such μ and A.*

Proof. Let F be the indicated set. Obviously $A \in F$. Suppose that $X \in F$ and $X \subseteq Y \subseteq A$. Then

$$\mu(A) = \mu(X \cup (Y \backslash X) \cup (A \backslash Y)) = \mu(X) + \mu(Y \backslash X) + \mu(A \backslash Y) = \mu(A) + \mu(Y \backslash X) + \mu(A \backslash Y),$$

and so $\mu(A \backslash Y) = 0$. Hence $\mu(A) = \mu((A \backslash Y \cup Y) = \mu(A \backslash Y) + \mu(Y) = \mu(Y)$. So $Y \in F$. Now suppose that $Y, Z \in F$. Then

$$\begin{aligned} \mu(A) &= \mu(Y) = \mu(Y \cap Z) + \mu(Y \backslash Z) \quad \text{and} \\ \mu(A) &= \mu(Z) = \mu(Y \cap Z) + \mu(Z \backslash Y). \end{aligned}$$

It follows that $\mu(Y \backslash Z) = \mu(Z \backslash Y)$. If $\mu(Y \backslash Z) = \mu(A)$, then also $\mu(Z \backslash Y) = \mu(A)$, and hence

$$2\mu(A) = \mu(Y \backslash Z) + \mu(Z \backslash Y) = \mu((Y \backslash Z) \cup (Z \backslash Y)) \leq \mu(A),$$

contradiction. So $\mu(Y \backslash Z) = 0$, and hence $\mu(A) = \mu(Y \cap Z)$. It follows that $Y \cap Z \in F$. So, F is a filter.

Clearly $\emptyset \notin F$, so F is proper.

If $X \subseteq A$, then $\mu(A) = \mu(X) + \mu(A\backslash X)$, and hence $\mu(X) = \mu(A)$ or $\mu(A\backslash X) = \mu(A)$. So $X \in F$ or $A\backslash X \in F$. Thus F is an ultrafilter.

Finally, for κ-completeness, suppose that $\mathscr{A} \in [F]^{<\kappa}$ Suppose that $\bigcap \mathscr{A} \notin F$. Then $A\backslash \bigcap \mathscr{A} \in F$. Let $\langle X_\alpha : \alpha < \lambda\rangle$ be an enumeration of $\mathscr{A}$. For each $\alpha < \lambda$ let $Y_\alpha = \bigcap_{\beta<\alpha} X_\beta \backslash X_\alpha$.

$$(1) \qquad \bigcup_{\alpha<\lambda} Y_\alpha = \bigcup_{\alpha<\lambda} (A\backslash X_\alpha)$$

In fact, $\subseteq$ is clear. Suppose that $\xi \in \bigcup_{\alpha<\lambda}(A\backslash X_\alpha)$, and choose $\alpha < \lambda$ minimum such that $\xi \in (A\backslash X_\alpha)$. Then $\xi \in Y_\alpha$. So (1) holds.

Clearly the Y_α's are pairwise disjoint. So from (1) we get

$$\begin{aligned}
\mu(A) &= \mu\left(A\backslash \bigcap \mathscr{A}\right) \\
&= \mu\left(\bigcup_{\alpha<\lambda} (A\backslash X_\alpha)\right) \\
&= \mu\left(\bigcup_{\alpha<\lambda} Y_\alpha\right) \\
&= \sum_{\alpha<\lambda} \mu(Y_\alpha),
\end{aligned}$$

and hence there is a $\alpha < \lambda$ such that $\mu(Y_\alpha) = 1$. Hence $\mu(A\backslash X_\alpha) = \mu(A)$ also, contradiction.

Hence F is κ-complete.

Since all members of F have size κ by κ-completeness and nonprincipality, it follows that $|A| = \kappa$. So κ is a measurable cardinal. $\square$

Proposition 26.27. *If κ is real-valued measurable then either κ is measurable or $\kappa \leq 2^\omega$.*

Proof. Let μ be a κ-additive measure on κ. By Proposition 26.26, if there is a μ-atom, then κ is measurable. So, suppose that there do not exist any μ-atoms. We construct a tree under $\supset$ by constructing the levels L_α, as follows. $L_0 = \{\kappa\}$. Suppose that L_α has been constructed, and that it is a nonempty collection of subsets of κ each of positive measure. For each $X \in L_\alpha$ let Y_X be a subset of X such that $0 < \mu(Y_X) < \mu(X)$; such a set exists since X is not a μ-atom. Then we define

$$L_{\alpha+1} = \{Y_X, X\backslash Y_X : X \in L_\alpha\}.$$

If α is a limit ordinal and L_β has been constructed for every $\beta < \alpha$, then we define

$$L_\alpha = \left\{\bigcap_{\beta<\alpha} Z_\beta : Z_\beta \in L_\beta \text{ for all } \beta < \alpha \text{ and } \mu\left(\bigcap_{\beta<\alpha} Z_\beta\right) > 0\right\},$$

except that if $L_\alpha = \emptyset$ the construction stops.

Clearly this gives a tree. Let α be the least ordinal such that L_α is not defined. So α is a limit ordinal.

(1) $\alpha \leq \omega_1$, and in fact, if $\langle Z_\beta : \alpha < \gamma\rangle$ is a branch of the tree, thus with $Z_\beta \subset Z_\delta$ if $\delta < \beta < \gamma$, then γ is countable.

In fact, we have $\mu(Z_\beta\backslash Z_{\beta+1}) > 0$ for every $\beta < \gamma$, and the sets $Z_\beta\backslash Z_{\beta+1}$ are pairwise disjoint. If $\gamma \geq \omega_1$, then

$$\gamma = \bigcup_{n\in\omega}\left\{\beta < \gamma : \mu(Z_\beta\backslash Z_{\beta+1}) > \frac{1}{n+1}\right\},$$

and hence there would be an $n \in \omega$ such that

$$\left\{\beta < \gamma : \mu(Z_\beta\backslash Z_{\beta+1}) > \frac{1}{n+1}\right\}$$

is uncountable, which is not possible. So (1) holds.

Similarly each level of our tree is countable. It follows that the tree has at most 2^ω branches.

Let $\mathscr{B}$ be the collection of all branches in this tree, and for each $B \in \mathscr{B}$ let $W_B = \bigcap_{X\in B} X$. Let $\mathscr{C} = \{W_B : B \in \mathscr{B}\}\backslash\{\emptyset\}$. Now clearly $|\mathscr{C}| \leq 2^\omega$, and $\mathscr{C}$ consists of measure 0 sets.

(2) $\kappa = \bigcup\mathscr{C}$.

In fact, if $\alpha \in \kappa$, then $B = \{X \in T : \alpha \in X\}$ is a branch, and so $\alpha \in W_B$.

From (2) it follows that $\kappa \leq 2^\omega$, since the measure μ is κ-additive and $\mu(\kappa) = 1$. In fact, $2^\omega < \kappa$ would imply by (2) that $\mu(\kappa) = 0$, contradiction. □

Proposition 26.28. *Let κ be a regular uncountable cardinal. Then the diagonal intersection of the system $\langle(\alpha + 1, \kappa) : \alpha < \kappa\rangle$ is the set of all limit ordinals less than κ.*

Proof. For any $\beta \in \kappa$,

$$\begin{aligned}\beta \in \triangle_{\alpha<\kappa}(\alpha + 1, \kappa) \quad &\text{iff} \quad \forall\alpha < \beta[\beta \in (\alpha + 1, \kappa)]\\ &\text{iff} \quad \forall\alpha < \beta[\alpha + 1 < \beta]\\ &\text{iff} \quad \beta \text{ is a limit ordinal.}\end{aligned}$$

□

Proposition 26.29. *Let F be a filter on a regular uncountable cardinal κ. We say that F is normal iff it is closed under diagonal intersections. Suppose that F is normal, and $(\alpha, \kappa) \in F$ for every $\alpha < \kappa$. Then every club of κ is in F.*

Proof. Let C be a club, and let $\langle\alpha_\xi : \xi < \kappa\rangle$ be the strictly increasing enumeration of C, and let D be the set of all limit ordinals less than κ. By Proposition 26.28 suffices to show that

$$D \cap \triangle_{\xi<\kappa}(\alpha_\xi, \kappa) \subseteq C.$$

So, take any $\beta \in D \cap \triangle_{\xi<\kappa}(\alpha_\xi, \kappa)$. Thus β is a limit ordinal, and $\forall \xi < \beta[\beta \in (\alpha_\xi, \kappa)]$, i.e., $\forall \xi < \beta[\alpha_\xi < \beta]$. Now $\xi \leq \alpha_\xi$ for all ξ, so $C \cap \beta$ is unbounded in β. Hence $\beta \in C$. $\square$

Proposition 26.30. *Let F be a proper filter on a regular uncountable cardinal κ. Then the following conditions are equivalent.*

(i) F is normal

(ii) For any $S_0 \subseteq \kappa$, if $\kappa \backslash S_0 \notin F$ and f is a regressive function defined on S_0, then there is an $S \subseteq S_0$ with $\kappa \backslash S \notin F$ and f is constant on S.

Proof. (i)⇒(ii): Assume (i), and suppose that $S_0 \subseteq \kappa$, $\kappa \backslash S_0 \notin F$, and f is a regressive function on S_0. Suppose that the conclusion fails. Then for every $\gamma < \kappa$ we have $\kappa \backslash f^{-1}[\{\gamma\}] \in F$, as otherwise we could take $S = f^{-1}[\{\gamma\}]$. By (i), take $\beta \in \triangle_{\gamma<\kappa}(\kappa \backslash f^{-1}[\{\gamma\}])$. Then $\forall \gamma < \beta[\beta \in \kappa \backslash f^{-1}[\{\gamma\}]$; in particular, $\beta \in \kappa \backslash f^{-1}[\{f(\beta)\}]$, contradiction.

(ii)⇒(i): Assume (ii), and suppose that $\langle a_\alpha : \alpha < \kappa \rangle$ is a system of members of F. Suppose that $\triangle_{\alpha<\kappa} a_\alpha \notin F$. Now $\forall \alpha \in \kappa \backslash \triangle_{\alpha<\kappa} a_\alpha \exists \beta < \alpha[\alpha \notin a_\beta]$. This gives us a regressive function f defined on $\kappa \backslash \triangle_{\alpha<\kappa} a_\alpha$ such that for every α in that set, $\alpha \notin a_{f(\alpha)}$. Hence by (ii) choose $S \subseteq \kappa \backslash \triangle_{\alpha<\kappa} a_\alpha$ such that f is constant on S, say with value γ, with $\kappa \backslash S \notin F$. Since $a_\gamma \in F$, we have $a_\gamma \not\subseteq \kappa \backslash S$. Choose $\beta \in a_\gamma \cap S$. Then $\beta \notin a_{f(\beta)}$ gives a contradiction.

Proposition 26.31. *A probability measure on a set S is a real-valued function μ with domain $\mathscr{P}(S)$ having the following properties:*

(i) $\mu(\emptyset) = 0$ and $\mu(S) = 1$.

(ii) If $X \subseteq Y$, then $\mu(X) \leq \mu(Y)$.

(iii) $\mu(\{a\}) = 0$ for all $a \in S$.

(iv) If $\langle X_n : n \in \omega \rangle$ is a system of pairwise disjoint sets, then $\mu(\bigcup_{n\in\omega} X_n) = \sum_{n\in\omega} \mu(X_n)$. (Some of the sets X_n might be empty.)

There does not exist a probability measure on ω_1.

Proof. Suppose that μ is a probability measure on ω_1. Let $f = \langle f_\rho : \rho < \omega_1 \rangle$ be a family of injections $f_\rho : \rho \to \omega$. Define the function $A : \omega \times \omega_1 \to \mathscr{P}(\omega_1)$ by setting, for any $\xi < \omega$ and $\alpha < \omega_1$,

$$A^\xi_\alpha = \{\rho \in \omega_1 \backslash (\alpha + 1) : f_\rho(\alpha) = \xi\}.$$

Take any $\alpha < \omega_1$. Since $\bigcup_{n\in\omega} A^n_\alpha = \omega_1 \backslash (\alpha+1)$, $A^n_\alpha \cap A^m_\alpha = \emptyset$ for $\alpha \neq \beta$, and $\mu(\omega_1 \backslash (\alpha+1)) = 1$, choose $n(\alpha) \in \omega$ such that $\varphi(A^{n(\alpha)}_\alpha) > 0$. Then there exist $M \in [\omega_1]^{\omega_1}$ and $m \in \omega$ such that $n(\alpha) = m$ for every $\alpha \in M$. Then $\langle A^m_\alpha : \alpha \in M \rangle$ is a system of pairwise disjoint sets each of positive measure, contradiction. $\square$

Proposition 26.32. *If κ is a measurable cardinal, then there is a normal κ-complete nonprincipal ultrafilter on κ.*

Proof. Let D be a κ-complete nonprincipal ultrafilter on κ. Define $f \equiv g$ iff $f, g \in {}^\kappa\kappa$ and $\{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in D$. Then $\equiv$ is an equivalence relation on ${}^\kappa\kappa$:

$\equiv$ *is reflexive:* $\kappa = \{\alpha < \kappa : f(\alpha) = f(\alpha)\}$, hence $f \equiv f$.

$\equiv$ *is symmetric:* Assume that $f \equiv g$. Thus $\{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in D$. Hence $\{\alpha < \kappa : g(\alpha) = f(\alpha)\} = \{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in D$. Hence $g \equiv f$.

$\equiv$ *is transitive:* Assume that $f \equiv g \equiv h$. Thus $\{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in D$ and $\{\alpha < \kappa : g(\alpha) = h(\alpha)\} \in D$. Hence

$$\{\alpha < \kappa : f(\alpha) = g(\alpha)\} \cap \{\alpha < \kappa : g(\alpha) = h(\alpha)\} \in D;$$

since

$$\{\alpha < \kappa : f(\alpha) = g(\alpha)\} \cap \{\alpha < \kappa : g(\alpha) = h(\alpha)\} \subseteq \{\alpha < \kappa : f(\alpha) = h(\alpha)\},$$

Now define

$$x \prec y \quad \text{iff} \quad \exists f, g[x = [f] \text{ and } y = [g] \text{ and } \{\alpha < \kappa : f(\alpha) < g(\alpha)\} \in D].$$

(1) $\forall f, g \in {}^\kappa\kappa[[f] \prec [g]$ iff $\{\alpha < \kappa : f(\varphi) < g(\alpha)\} \in D]$.

In fact, $\Leftarrow$ is immediate from the definition. Now suppose that $[f] \prec [g]$. Choose $f', g' \in {}^\kappa\kappa$ such that $[f] = [f']$, $[g] = [g']$, and $\{\alpha < \kappa : f'(\alpha) < g'(\alpha)\} \in D$. Then

$$\begin{aligned}\{\alpha < \kappa : f(\alpha) = f'(\alpha)\} \cap \{\alpha < \kappa : f'(\alpha) < g'(\alpha)\} \cap \{\alpha < \kappa : g(\alpha) = g'(\alpha)\}\\ \subseteq \{\alpha < \kappa : f(\alpha) < g(\alpha)\};\end{aligned}$$

the left side is in D, hence also the right side is in D, so $\{\alpha < \kappa : f(\alpha) < g(\alpha)\} \in D$. Thus (1) holds.

$\prec$ *is irreflexive*: $\{\alpha < \kappa : f(\alpha) < f(\alpha)\} = \emptyset \notin D$, so $[f] \not\prec [f]$.

$\prec$ *is transitive*: Assume that $[f] \prec [g] \prec [h]$. Then

$$\{\alpha < \kappa : f(\alpha) < g(\alpha)\} \cap \{\alpha < \kappa : g(\alpha) < h(\alpha)\} \subseteq \{\alpha < \kappa : f(\alpha) < h(\alpha)\};$$

the left side is in D, hence also the right side is in D, so $[f] \prec [h]$.

$\prec$ *is a linear order*: Suppose that $f, g \in {}^\kappa\kappa$ are such that $[f] \neq [g]$ and $[f] \not\prec [g]$. Now

$$\kappa = \{\alpha < \kappa : f(\alpha) < g(\alpha)\} \cup \{\alpha < \kappa : f(\alpha) = g(\alpha\} \cup \{\alpha < \kappa : g(\alpha) < f(\alpha)\};$$

The first two sets are not in D, so the third one is in D, and hence $[g] \prec [f]$.

$\prec$ *is a well-order*: Suppose not. Then we get a sequence $\langle f^m : m \in \omega\rangle$ of members of ${}^\kappa\kappa$ such that $[f_{m+1}] \prec [f_m]$ for all $m \in \omega$. Thus $\{\alpha < \kappa : f_{m+1}(\alpha) < f_m(\alpha)\} \in D$ for all $m \in \omega$. It follows that

$$\bigcap_{m\in\omega} \{\alpha < \kappa : f_{m+1}(\alpha) < f_m(\alpha)\} \in D;$$

taking any element α in this intersection, we get $\ldots f_{m+1}(\alpha) < f_m(\alpha) \ldots$, contradiction.

Now let $k(\alpha) = \alpha$ for all $\alpha < \kappa$. Then for any $\gamma < \kappa$ we have

$$\{\alpha < \kappa : \gamma < k(\alpha)\} = \{\alpha < \kappa : \gamma < \alpha\} = \kappa \backslash (\gamma + 1) \in D.$$

It follows that we can take the smallest equivalence class $[f]$ such that for any $\gamma < \kappa$ we have $\{\alpha < \kappa : \gamma < f(\alpha)\} \in D$. Now we let $E = \{X \subseteq \kappa : f^{-1}[X] \in D\}$. We claim that E is as desired in the exercise.

$\emptyset \notin E$: This is true since $f^{-1}[\emptyset] = \emptyset \notin D$.

If $X \subseteq Y \subseteq \kappa$ and $X \in E$, then $Y \in E$: In fact, assume that $X \subseteq Y \subseteq \kappa$ and $X \in E$. Then $f^{-1}[X] \subseteq f^{-1}[Y]$ and $f^{-1}[X] \in D$, so $f^{-1}[y] \in D$, so that $Y \in E$.

If $X, Y \in E$, then $X \cap Y \in E$: In fact, $f^{-1}[X \cap Y] = f^{-1}[X] \cap f^{-1}[y]$, so this is clear.

If $X \subseteq \kappa$, then $X \in E$ or $(\kappa \backslash X) \in E$: For, suppose that $X \notin E$. Then $f^{-1}[X] \notin D$, so $f^{-1}[\kappa \backslash X] = (\kappa \backslash f^{-1}[X]) \in D$, and hence $(\kappa \backslash X) \in E$.

E is nonprincipal: for any $\alpha < \kappa$ we have $\{\beta < \kappa : \alpha < f(\beta)\} \in D$, and $\{\beta < \kappa : \alpha < f(\beta)\} \subseteq \{\beta < \kappa : \alpha \neq f(\beta)\}$, so $\{\beta < \kappa : \alpha \neq f(\beta)\} \in D$, hence $\{\beta < \kappa : \alpha = f(\beta)\} \notin D$, hence $f^{-1}[\{\alpha\}] \notin D$ and so $\{\alpha\} \notin E$.

E is κ-complete: Suppose that $\langle X_\alpha : \alpha < \beta \rangle$ is a system of subsets of κ, with $\beta < \kappa$ and with $[X_\alpha] \in E$ for all $\alpha < \beta$. Thus $f^{-1}[X_\alpha] \in D$ for all $\alpha < \beta$. Since $f^{-1}\left[\bigcap_{\alpha<\beta} X_\alpha\right] = \bigcap_{\alpha<\beta} f^{-1}[X_\alpha] \in D$, it follows that $\bigcap_{\alpha<\beta} X_\alpha \in E$.

E is normal: We apply Proposition 26.30. Suppose that $S_0 \in E$ and g is regressive on S_0. Note that $f^{-1}[S_0] \in D$. Let $h = g \circ f$. Then for any $\alpha \in f^{-1}[S_0]$ we have $h(\alpha) < f(\alpha)$, so that $[h] \prec [f]$. By the definition of f it then follows that there is a $\gamma < \kappa$ such that $\{\alpha < \kappa : \gamma < h(\alpha)\} \notin D$. Hence $\{\alpha < \kappa : h(\alpha) \leq \gamma\} \in D$. Now

$$\{\alpha < \kappa : h(\alpha) \leq \gamma\} = \bigcup_{\delta \leq \gamma} \{\alpha < \kappa : h(\alpha) = \delta\},$$

and so there is a $\delta \leq \gamma$ such that $\{\alpha < \kappa : h(\alpha) = \delta\} \in D$. Now $\{\alpha < \kappa : h(\alpha) = \delta\} = h^{-1}[\{\delta\}] = f^{-1}[g^{-1}[\{\delta\}]$, so $g^{-1}[\{\delta\}] \in E$. This checks the condition of Proposition 26.30. □

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FORCING

27. Boolean algebras and forcing orders

To introduce the apparatus of generic extensions and forcing in a clear fashion, it is necessary to go into a special set theoretic topic: Boolean algebras and their relation to certain orders.

A *Boolean algebra* (BA) is a structure $\langle A, +, \cdot, -, 0, 1\rangle$ with two binary operations $+$ and $\cdot$, a unary operation $-$, and two distinguished elements 0 and 1 such that the following axioms hold for all $x, y, z \in A$:

(A)	$x + (y + z) = (x + y) + z$;	(A′)	$x \cdot (y \cdot z) = (x \cdot y) \cdot z$;
(C)	$x + y = y + x$;	(C′)	$x \cdot y = y \cdot x$;
(L)	$x + (x \cdot y) = x$;	(L′)	$x \cdot (x + y) = x$;
(D)	$x \cdot (y + z) = (x \cdot y) + (x \cdot z)$;	(D′)	$x + (y \cdot z) = (x + y) \cdot (x + z)$;
(K)	$x + (-x) = 1$;	(K′)	$x \cdot (-x) = 0$.

The main example of a Boolean algebra is a field of sets: a set A of subsets of some set X, closed under union, intersection, and complementation with respect to X. The associated Boolean algebra is $\langle A, \cup, \cap, \backslash, 0, X\rangle$. Here $\backslash$ is treated as a one-place operation, producing $X\backslash a$ for any $a \in A$. This example is really all-encompassing—every BA is isomorphic to one of these. We will not prove this, or use it.

As is usual in algebra, we usually denote a whole algebra $\langle A, +, \cdot, -, 0, 1\rangle$ just by mentioning its *universe* A, everything else being implicit.

Some notations used in some treatments of Boolean algebras are: $\vee$ or $\cup$ for $+$; $\wedge$ or $\cap$ for $\cdot$; $'$ for $-$. These notations might be confusing if discussing logic, or elementary set theory. Our notation might be confusing if discussing ordinary algebra.

Now we give the elementary arithmetic of Boolean algebras. We recommend that the reader go through them, but then approach any arithmetic statement in the future from the point of view of seeing if it works in fields of sets; if so, it should be easy to derive from the axioms.

First we have the duality principle, which we shall not formulate carefully; our particular uses of it will be clear. Namely, notice that the axioms come in pairs, obtained from each other by interchanging $+$ and $\cdot$ and 0 and 1. This means that also if we prove some arithmetic statement, the *dual* statement, obtained by this interchanging process, is also valid.

Proposition 27.1. *$x + x = x$ and $x \cdot x = x$.*

Proof.

$$\begin{aligned} x + x &= x + x \cdot (x + x) \text{ by (L}'\text{)} \\ &= x \text{ by (L)}; \end{aligned}$$

the second statement follows by duality. □

Proposition 27.2. *$x + y = y$ iff $x \cdot y = x$.*

Proof. Assume that $x + y = y$. Then, by (L′),

$$x \cdot y = x \cdot (x + y) = x.$$

The converse follows by duality. □

In any BA we define $x \leq y$ iff $x + y = y$. Note that the dual of $x \leq y$ is $y \leq x$, by 27.2 and commutativity. (The dual of a defined notion is obtained by dualizing the original notions.)

Proposition 27.3. *On any BA, $\leq$ is reflexive, transitive, and antisymmetric; that is, the following conditions hold:*

(i) $x \leq x$;
(ii) If $x \leq y$ and $y \leq z$, then $x \leq z$;
(iii) If $x \leq y$ and $y \leq x$, then $x = y$.

Proof. $x \leq x$ means $x + x = x$, which was proved in 27.1. Assume the hypothesis of (*ii*). Then

$$\begin{aligned} x + z &= x + (y + z) \\ &= (x + y) + z \\ &= y + z \\ &= z, \end{aligned}$$

as desired. Finally, under the hypotheses of (*iii*),

$$x = x + y = y + x = y.$$ □

Note that Proposition 27.3 says that $\leq$ is a partial order on the BA A. There are some notions concerning partial orders which we need. An element z is an *upper bound* for a set Y of elements of X if $y \leq z$ for all $y \in Y$; similarly for lower bounds. And z is a least upper bound for Y if it is an upper bound for Y and is $\leq$ any other upper bound for Y; simlarly for greatest lower bounds. By antisymmetry, in any partial order least upper bounds and greatest lower bounds are unique if they exist.

Proposition 27.4. *$x + y$ is the least upper bound of $\{x, y\}$, and $x \cdot y$ is the greatest lower bound of $\{x, y\}$.*

Proof. We have $x + (x + y) = (x + x) + y = x + y$, and similarly $y + (x + y) = y + (y + x) = (y + y) + x = y + x = x + y$; so $x + y$ is an upper bound for $\{x, y\}$. If z is any upper bound for $\{x, y\}$, then

$$(x + y) + z = (x + (y + z) = x + z = z,$$

as desired. The other part follows by duality(!). □

Proposition 27.5. *(i) $x + 0 = x$ and $x \cdot 1 = x$;*

(ii) $x \cdot 0 = x$ *and* $x + 1 = 1$*;*
(iii) $0 \leq x \leq 1$.

Proof. By (K) and Proposition 27.4, 1 is the least upper bound of x and $-x$; in particular it is an upper bound, so $x \leq 1$. Everything else follows by duality, Proposition 27.2, and the definitions. □

Proposition 27.6. *For any* x *and* y*,* $y = -x$ *iff* $x \cdot y = 0$ *and* $x + y = 1$*.*

Proof. $\Rightarrow$ holds by (K) and (K'). Now suppose that $x \cdot y = 0$ and $x + y = 1$. Then

$$\begin{aligned} y &= y \cdot 1 = y \cdot (x + -x) = y \cdot x + y \cdot -x = 0 + y \cdot -x = y \cdot -x; \\ -x &= -x \cdot 1 = -x \cdot (x + y) = -x \cdot x + -x \cdot y = 0 + -x \cdot y = -x \cdot y = y. \end{aligned}$$ □

Proposition 27.7. *(i)* $- - x = x$*;*
(ii) if $-x = -y$ *then* $x = y$*;*
(iii) $-0 = 1$ *and* $-1 = 0$*;*
(iv) (DeMorgan's laws) $-(x + y) = -x \cdot -y$ *and* $-(x \cdot y) = -x + -y$*.*

Proof. If we apply Proposition 27.6 with x and y replaced respectively by $-x$ and x, we get $- - x = x$. Next, if $-x = -y$, then $x = - - x = - - y = y$. For (iii), by 27.5(iii), $0 \cdot 1 = 0$ and $0 + 1 = 1$, so by 27.6, $-0 = 1$. Then $-1 = 0$ by duality. For the first part of (iv),

$$\begin{aligned} (x + y) \cdot -x \cdot -y &= x \cdot -x \cdot -y + y \cdot -x \cdot -y \\ &= 0 + 0 = 0, \end{aligned}$$

and

$$\begin{aligned} (x + y) + -x \cdot -y &= x \cdot (y + -y) + y + -x \cdot -y \\ &= x \cdot y + x \cdot -y + y + -x \cdot -y \\ &= y + x \cdot -y + -x \cdot -y \\ &= y + -y = 1, \end{aligned}$$

so that $-(x + y) = -x \cdot -y$ by Proposition 27.6. Finally, the second part of (iv) follows by duality. □

Proposition 27.8. $x \leq y$ *iff* $-y \leq -x$*.*

Proof. Assume that $x \leq y$. Then $x + y = y$, so $-x \cdot -y = -y$, i.e., $-y \leq -x$. For the converse, use the implication just proved, plus 27.7(i). □

Proposition 27.9. *If* $x \leq x'$ *and* $y \leq y'$*, then* $x + y \leq x' + y'$ *and* $x \cdot y \leq x' \cdot y'$*.*

Proof. Assume the hypothesis. Then

$$(x + y) + (x' + y') = (x + x') + (y + y') = x' + y',$$

and so $x + y \leq x' + y'$; the second conclusion follows by duality. □

Proposition 27.10. *$x \leq y$ iff $x \cdot -y = 0$.*

Proof. If $x \leq y$, then $x = x \cdot y$ and so $x \cdot -y = 0$. Conversely, if $x \cdot -y = 0$, then

$$x = x \cdot (y + -y) = x \cdot y + x \cdot -y = x \cdot y,$$

so that $x \leq y$. $\square$

Elements $x, y \in A$ are *disjoint* if $x \cdot y = 0$. For any x, y we define

$$x \triangle y = x \cdot -y + y \cdot -x;$$

this is the *symmetric difference* of x and y.

Proposition 27.11. *(i) $x = y$ iff $x \triangle y = 0$;*
(ii) $x \cdot (y \triangle z) = (x \cdot y) \triangle (x \cdot z)$;
(iii) $x \triangle (y \triangle z) = (x \triangle y) \triangle z$.

Proof. For (i), $\Rightarrow$ is trivial. Now assume that $x \triangle y = 0$. Then $x \cdot -y = 0 = y \cdot -x$, so $x \leq y$ and $y \leq x$, so $x = y$.

For (ii), we have

$$\begin{aligned} x \cdot (y \triangle z) &= x \cdot y \cdot -z + x \cdot z \cdot -y \\ &= (x \cdot y) \cdot -(x \cdot z) + (x \cdot z) \cdot -(x \cdot y) \\ &= (x \cdot y) \triangle (x \cdot z), \end{aligned}$$

as desired.

Finally, for (iii),

$$\begin{aligned} x \triangle (y \triangle z) &= x \cdot -(y \cdot -z + -y \cdot z) + (y \cdot -z + -y \cdot -z) \cdot -x \\ &= x \cdot (-y + z) \cdot (y + -z) + -x \cdot y \cdot -z + -x \cdot -y \cdot z \\ &= x \cdot -y \cdot -z + x \cdot y \cdot z + -x \cdot y \cdot -z + -x \cdot -y \cdot z; \end{aligned}$$

if we apply the same argument to $z \triangle (y \triangle x)$ we get

$$z \triangle (y \triangle x) = z \cdot -y \cdot -x + z \cdot y \cdot x + -z \cdot y \cdot -x + -z \cdot -y \cdot x,$$

which is the same thing. So the obvious symmetry of $\triangle$ gives the desired result. $\square$

One further useful result is that axiom (D$'$) is redundant:

Proposition 27.12. *(D') is redundant.* (Assume all axioms except D'.)

Proof.

$$\begin{aligned} (x + y) \cdot (x + z) &= ((x + y) \cdot x) + ((x + y) \cdot z) \\ &= (x \cdot (x + y)) + (z \cdot (x + y)) \\ &= x + ((z \cdot x) + (z \cdot y)) \\ &= x + ((x \cdot z) + (y \cdot z)) \\ &= (x + (x \cdot z)) + (y \cdot z) \\ &= x + (y \cdot z). \end{aligned}$$

$\square$

If M is a subset of a BA A, we denote by $\sum M$ its least upper bound (if it exists), and by $\prod M$ its greatest lower bound, if it exists. A is *complete* iff these always exist. Note that frequently people use $\bigvee M$ and $\bigwedge M$ instead of $\sum M$ and $\prod M$.

Proposition 27.13. *Assume that A is a complete BA.*

(i) $-\sum_{i\in I} a_i = \prod_{i\in I} -a_i$.

(ii) $-\prod_{i\in I} a_i = \sum_{i\in I} -a_i$.

Proof. For (i), let $a = \sum_{i\in I} a_i$; we show that $-a$ is the greatest lower bound of $\{-a_i : i \in I\}$. If $i \in I$, then $a_i \le a$, and hence $-a \le -a_i$; thus $-a$ is a lower bound for the indicated set. Now suppose that x is any lower bound for this set. Then for any $i \in I$ we have $x \le -a_i$, and so $a_i \le -x$. So $-x$ is an upper bound for $\{a_i : i \in I\}$, and so $a \le -x$. Hence $x \le -a$, as desired.

(ii) is proved similarly. □

The following (possibly infinite) distributive law is frequently useful. One should be aware of the fact that more general infinite distributive laws do not hold, in general.

Proposition 27.14. *If $\sum_{i\in I} a_i$ exists, then $\sum_{i\in I}(b\cdot a_i)$ exists and*

$$b\cdot\sum_{i\in I} a_i = \sum_{i\in I}(b\cdot a_i).$$

Proof. Let $s = \sum_{i\in I} a_i$; we shall show that $b\cdot s$ is the least upper bound of $\{b\cdot a_i : i \in I\}$. If $i \in I$, then $a_i \le s$ and so $b\cdot a_i \le b\cdot s$; so $b\cdot s$ is an upper bound for the indicated set. Now suppose that x is any upper bound for this set. Then for any $i \in I$ we have $b\cdot a_i \le x$, hence $b\cdot a_i\cdot -x = 0$ and so $a_i \le -(b\cdot -x) = -b + x$; so $-b + x$ is an upper bound for $\{a_i : i \in I\}$. It follows that $s \le -b + x$, and hence $s\cdot b \le x$, as desired. □

Forcing orders

A forcing order is a triple $\mathbb{P} = (P, \le, 1)$ such that $\le$ is a reflexive and transitive relation on the nonempty set P, and $\forall p \in P(p \le 1)$. Note that we do not assume that $\le$ is antisymmetric. Partial orders are special cases of forcing orders in which this is assumed (but we do not assume the existence of 1 in partial orders). Note that we assume that every forcing order has a largest element. Many set-theorists use "partial order" instead of "forcing order".

Frequently we use just P for a forcing order; $\le$ and 1 are assumed.

We say that elements $p, q \in P$ are *compatible* iff there is an $r \le p, q$. We write $p \perp q$ to indicate that p and q are incompatible. A set A of elements of P is an *antichain* iff any two distinct members of A are incompatible. WARNING: sometimes "antichain" is used to mean pairwise incomparable, or in the case of Boolean algebras, pairwise disjoint. A subset Q of P is *dense* iff for every $p \in P$ there is a $q \in Q$ such that $q \le p$.

Now we are going to describe how to embed a forcing order into a complete BA. We take the regular open algebra of a certain topological space. We assume a very little bit of topology. To avoid assuming any knowledge of topology we now give a minimalist introduction to topology.

A *topology* on a set X is a collection $\mathscr{O}$ of subsets of X satisfying the following conditions:

(1) $X, \emptyset \in \mathscr{O}$.

(2) $\mathscr{O}$ is closed under arbitrary unions.

(3) $\mathscr{O}$ is closed under finite intersections.

The members of $\mathscr{O}$ are said to be *open*. The *interior* of a subset $Y \subseteq X$ is the union of all open sets contained in Y; we denote it by $\mathrm{int}(Y)$.

Proposition 27.15. *(i)* $\mathrm{int}(\emptyset) = \emptyset$.
(ii) $\mathrm{int}(X) = X$.
(iii) $\mathrm{int}(Y) \subseteq Y$.
(iv) $\mathrm{int}(Y \cap Z) = \mathrm{int}(Y) \cap \mathrm{int}(Z)$.
(v) $\mathrm{int}(\mathrm{int}(Y)) = \mathrm{int}(Y)$.
(vi) $\mathrm{int}(Y) = \{x \in X : x \in U \subseteq Y$ *for some open set* $U\}$.

Proof. (i)–(iii), (v), and (vi) are obvious. For (iv), if U is an open set contained in $Y \cap Z$, then it is contained in Y; so $\mathrm{int}(Y \cap Z) \subseteq \mathrm{int}(Y)$. Similarly for Z, so $\subseteq$ holds. For $\supseteq$, note that the right side is an open set contained in $Y \cap Z$. (v) holds since $\mathrm{int}(Y)$ is open. □

A subset C of X is *closed* iff $X \backslash C$ is open.

Proposition 27.16. *(i)* $\emptyset$ *and* X *are closed.*
(ii) The collection of all closed sets is closed under finite unions and intersections of any nonempty subcollection. □

For any $Y \subseteq X$, the *closure* of Y, denoted by $\mathrm{cl}(Y)$, is the intersection of all closed sets containing Y.

Proposition 27.17. *(i)* $\mathrm{cl}(Y) = X \backslash \mathrm{int}(X \backslash Y)$.
(ii) $\mathrm{int}(Y) = X \backslash \mathrm{cl}(X \backslash Y)$.
(iii) $\mathrm{cl}(\emptyset) = \emptyset$.
(iv) $\mathrm{cl}(X) = X$.
(v) $Y \subseteq \mathrm{cl}(Y)$.
(vi) $\mathrm{cl}(Y \cup Z) = \mathrm{cl}(Y) \cup \mathrm{cl}(Z)$.
(vii) $\mathrm{cl}(\mathrm{cl}(Y)) = \mathrm{cl}(Y)$.
(viii) $\mathrm{cl}(Y) = \{x \in X :$*for every open set* U*, if* $x \in U$ *then* $U \cap Y \neq \emptyset\}$.

Proof. (i): $\mathrm{int}(X \backslash Y)$ is an open set contained in $X \backslash Y$, so Y is a subset of the closed set $X \backslash \mathrm{int}(X \backslash Y)$. Hence $\mathrm{cl}(Y) \subseteq X \backslash \mathrm{int}(X \backslash Y)$. Also. $\mathrm{cl}(Y)$ is a closed set containing Y, so $X \backslash \mathrm{cl}(Y)$ is an open set contained in $X \backslash Y$. Hence $X \backslash \mathrm{cl}(Y) \subseteq \mathrm{int}(X \backslash Y)$. Hence $X \backslash \mathrm{int}(X \backslash Y \subseteq \mathrm{cl}(Y)$. This proves (i).

(ii): Using (i),

$$X\backslash \mathrm{cl}(X\backslash Y) = X\backslash(X\backslash \mathrm{int}(X\backslash(X\backslash Y))) = \mathrm{int}(Y).$$

(iii)–(v): clear.
(vi):

$$\begin{aligned}
\mathrm{cl}(Y \cup Z) &= X\backslash \mathrm{int}(X\backslash(Y \cup Z)) \quad \text{by (i)}\\
&= X\backslash \mathrm{int}((X\backslash Y) \cap (X\backslash Z))\\
&= X\backslash(\mathrm{int}(X\backslash Y) \cap \mathrm{int}(X\backslash Z)) \quad \text{by 27.15(iv)}\\
&= [X\backslash \mathrm{int}(X\backslash Y)] \cup [X\backslash \mathrm{int}(X\backslash Z)]\\
&= \mathrm{cl}(Y) \cup \mathrm{cl}(Z).
\end{aligned}$$

(vii):

$$\begin{aligned}
\mathrm{cl}(\mathrm{cl}(Y)) &= \mathrm{cl}(X\backslash \mathrm{int}(X\backslash Y))\\
&= X\backslash \mathrm{int}(X\backslash(X\backslash \mathrm{int}(X\backslash Y)))\\
&= X\backslash \mathrm{int}(\mathrm{int}(X\backslash Y))\\
&= X\backslash \mathrm{int}(X\backslash Y)\\
&= \mathrm{cl}(Y).
\end{aligned}$$

(vii): First suppose that $x \in \mathrm{cl}(Y)$, and $x \in U$, U open. By (i) and Proposition 27.15(vi) we have $U \not\subseteq X\backslash Y$, i.e., $U \cap Y \neq \emptyset$, as desired. Second, suppose that $x \notin \mathrm{cl}(Y)$. Then by (i) and 27.15(vi) there is an open U such that $x \in U \subseteq X\backslash Y$; so $U \cap Y = \emptyset$, as desired. □

Now we go beyond this minimum amount of topology and work with the notion of a regular open set, which is not a standard part of topology courses.

We say that Y is *regular open* iff $Y = \mathrm{int}(\mathrm{cl}(Y))$.

Proposition 27.18. *(i) If Y is open, then $Y \subseteq \mathrm{int}(\mathrm{cl}(Y))$.*

(ii) If U and V are regular open, then so is $U \cap V$.

(iii) $\mathrm{int}(\mathrm{cl}(Y))$ is regular open.

(iv) If U is open, then $\mathrm{int}(\mathrm{cl}(U))$ is the smallest regular open set containing U.

(v) If U is open then $U \cap \mathrm{cl}(Y) \subseteq \mathrm{cl}(U \cap Y)$.

(vi) If U is open, then $U \cap \mathrm{int}(\mathrm{cl}(Y)) \subseteq \mathrm{int}(\mathrm{cl}(U \cap Y))$.

(vii) If U and V are open and $U \cap V = \emptyset$, then $\mathrm{int}(\mathrm{cl}(U)) \cap V = \emptyset$.

(viii) If U and V are open and $U \cap V = \emptyset$, then $\mathrm{int}(\mathrm{cl}(U)) \cap \mathrm{int}(\mathrm{cl}(V)) = \emptyset$.

(ix) For any set M of regular open sets, $\mathrm{int}(\mathrm{cl}(\bigcup M)$ is the least regular open set containing each member of M.

Proof. (i): $Y \subseteq \mathrm{cl}(Y)$, and hence $Y = \mathrm{int}(Y) \subseteq \mathrm{int}(\mathrm{cl}(Y))$.

(ii): $U \cap V$ is open, and so $U \cap V \subseteq \mathrm{int}(\mathrm{cl}(U \cap V))$. For the other inclusion, $\mathrm{int}(\mathrm{cl}(U \cap V)) \subseteq \mathrm{int}(\mathrm{cl}(U)) = U$, and similarly for V, so the other inclusion holds.

(iii): $\text{int}(\text{cl}(X)) \subseteq \text{cl}(X)$, so $\text{cl}(\text{int}(\text{cl}(X))) \subseteq \text{cl}(\text{cl}(X)) = \text{cl}(X)$; hence

$$\text{int}(\text{cl}(\text{int}(\text{cl}(X)))) \subseteq \text{int}(\text{cl}(X));$$

the other inclusion is clear.

(iv): By (iii), $\text{int}(\text{cl}(U))$ is a regular open set containing U. If V is any regular open set containing U, then $\text{int}(\text{cl}(U)) \subseteq \text{int}(\text{cl}(V)) = V$.

(v):

$$\begin{aligned}
U \cap (X\backslash(U \cap Y)) &\subseteq X\backslash Y, \quad \text{hence} \\
U \cap \text{int}(X\backslash(U \cap Y)) &= \text{int}(U) \cap \text{int}(X\backslash(U \cap Y)) \\
&= \text{int}(U \cap (X\backslash(U \cap Y))) \\
&\subseteq \text{int}(X\backslash Y), \quad \text{hence} \\
X\backslash\text{int}(X\backslash Y) &\subseteq X\backslash(U \cap \text{int}(X\backslash(U \cap Y))) \\
&= (X\backslash U) \cup (X\backslash\text{int}(X\backslash(U \cap Y))), \quad \text{hence} \\
U \cap (X\backslash\text{int}(X\backslash Y)) &\subseteq (X\backslash\text{int}(X\backslash(U \cap Y))),
\end{aligned}$$

and (v) follows.

(vi):

$$\begin{aligned}
U \cap \text{int}(\text{cl}(Y)) &= \text{int}(U) \cap \text{int}(\text{cl}(Y)) \\
&= \text{int}(U \cap \text{cl}(Y)) \\
&\subseteq \text{int}(\text{cl}(U \cap Y)) \quad \text{by (v).}
\end{aligned}$$

(vii): $U \subseteq X\backslash V$, hence $\text{cl}(U) \subseteq \text{cl}(X\backslash V) = X\backslash V$, hence $\text{cl}(U) \cap V = \emptyset$, and the conclusion of (vii) follows.

(viii): Apply (vii) twice.

(ix): If $U \in M$, then $U \subseteq \bigcup M \subseteq \text{int}(\text{cl}(\bigcup M)$. Suppose that V is regular open and $U \subseteq V$ for all $U \in M$. Then $\bigcup M \subseteq V$, and so $\text{int}(\text{cl}(\bigcup M)) \subseteq \text{int}(\text{cl}(V) = V$. □

We let $\text{RO}(X)$ be the collection of all regular open sets in X. We define operations on $\text{RO}(X)$ which will make it a Boolean algebra. For any $Y, Z \in \text{RO}(X)$, let

$$\begin{aligned}
Y + Z &= \text{int}(\text{cl}(Y \cup Z)); \\
Y \cdot Z &= Y \cap Z; \\
-Y &= \text{int}(X\backslash Y).
\end{aligned}$$

Theorem 27.19. *The structure*

$$\langle \text{RO}(X), +, \cdot, -, \emptyset, X \rangle$$

is a complete BA. Moreover, the ordering $\leq$ coincides with $\subseteq$.

Proof. RO(X) is closed under $+$ by Proposition 27.18(ix), and is closed under $\cdot$ by Proposition 27.18(ii). Clearly it is closed under $-$, and $\emptyset, X \in \mathrm{RO}(X)$. Now we check the axioms. The following are completely obvious: (A$'$), (C$'$), (C). Now let unexplained variables range over RO(X). For (A), note by 27.18(i) that $U \subseteq U + V \subseteq (U+V)+W$; and similarly $V \subseteq (U+V)+W$ and $W \subseteq U+V \subseteq (U+V)+W$. If $U, V, W \subseteq Z$, then by 27.18(iv), $U + V \subseteq Z$ and hence $(U+V)+W \subseteq Z$. Thus $(U+V)+W$ is the least upper bound in RO(X) of U, V, W. This is true for all U, V, W. So $U+(V+W) = (V+W)+U$ is also the least upper bound of them; so (A) holds. For (L):

$$U + U \cdot V = \mathrm{int}(\mathrm{cl}(U \cup (U \cap V))) = \mathrm{int}(\mathrm{cl}(U)) = U.$$

(L$'$) holds by 27.18(i). For (D), first note that

$$\begin{aligned} Y \cdot (Z+W) &= Y \cap \mathrm{int}(\mathrm{cl}(Z \cup W)) \\ &\subseteq \mathrm{int}(\mathrm{cl}(Y \cap (Z \cup W))) \quad \text{by 27.18(vi)} \\ &= \mathrm{int}(\mathrm{cl}((Y \cap Z) \cup (Y \cap W))) \\ &= Y \cdot Z + Y \cdot W. \end{aligned}$$

On the other hand, $(Y \cap Z) \cup (Y \cap W) = Y \cap (Z \cup W) \subseteq Y, Z \cup W$, and hence easily

$$\begin{aligned} Y \cdot Z + Y \cdot W &= \mathrm{int}(\mathrm{cl}((Y \cap Z) \cup (Y \cap W))) \\ &\subseteq \mathrm{int}(\mathrm{cl}(Y) = Y \quad \text{and} \\ Y \cdot Z + Y \cdot W &= \mathrm{int}(\mathrm{cl}((Y \cap Z) \cup (Y \cap W))) \\ &\subseteq \mathrm{int}(\mathrm{cl}(Z \cup W) = Z + W; \end{aligned}$$

so the other inclusion follows, and (D) holds.

(K): For any regular open Y, from Proposition 27.17(ii) we get $-Y = \mathrm{int}(X \backslash Y) = X \backslash \mathrm{cl}(X \backslash (X \backslash Y)) = X \backslash \mathrm{cl}(Y)$. Hence

$$X = \mathrm{cl}(Y) \cup (X \backslash \mathrm{cl}(Y)) \subseteq \mathrm{cl}(Y) \cup \mathrm{cl}((X \backslash \mathrm{cl}(Y)) = \mathrm{cl}(Y \cup (X \backslash \mathrm{cl}(Y))),$$

and hence $X = Y + -Y$.

(K$'$): Clearly $\emptyset = Y \cap \mathrm{int}(X \backslash Y) = Y \cdot -Y$.

Thus we have now proved that $\langle \mathrm{RO}(X), +, \cdot, -, \emptyset, X \rangle$ is a BA. Since $\cdot$ is the same as $\cap$, $\leq$ is the same as $\subseteq$. Hence by Proposition 27.18(ix), $\langle \mathrm{RO}(X), +, \cdot, -, \emptyset, X \rangle$ is a complete BA. $\square$

Now we return to our task of embedding a forcing order into a complete Boolean algebra. Let P be a given forcing order. For each $p \in P$ let $P \downarrow p = \{q : q \leq p\}$. Now we define

$$\mathscr{O}_P = \{X \subseteq P : (P \downarrow p) \subseteq X \text{ for every } p \in X\}.$$

We check that this gives a topology on P. Clearly $P, \emptyset \in \mathscr{O}$. To show that $\mathscr{O}$ is closed under arbitrary unions, suppose that $\mathscr{X} \subseteq \mathscr{O}$. Take any $p \in \bigcup \mathscr{X}$. Choose $X \in \mathscr{X}$

such that $p \in X$. Then $(P \downarrow p) \subseteq X \subseteq \bigcup \mathscr{X}$, as desired. If $X, Y \in \mathscr{O}_P$, suppose that $p \in X \cap Y$. Then $p \in X$, so $(P \downarrow p) \subseteq X$. Similarly $(P \downarrow p) \subseteq Y$, so $(P \downarrow p) \subseteq X \cap Y$. Thus $X \cap Y \in \mathscr{O}_P$, finishing the proof that $\mathscr{O}_P$ is a topology on P.

We denote the complete BA of regular open sets in this topology by $\mathrm{RO}(P)$.

Now for any $p \in P$ we define

$$e(p) = \mathrm{int}(\mathrm{cl}(P \downarrow p)).$$

Thus e maps P into $\mathrm{RO}(P)$.

This is our desired embedding. Actually it is not really an embedding in general, but it has several useful properties, and for many forcing orders it really is an embedding.

The useful properties mentioned are as follows. We say that a subset X of P is *dense below* p iff for every $r \le p$ there is a $q \le r$ such that $q \in X$.

Theorem 27.20. *Let P be a forcing order. Suppose that $p, q \in P$, F is a finite subset of P, $a, b \in \mathrm{RO}(P)$, and N is a subset of* $\mathrm{RO}(P)$

(i) $e[P]$ is dense in $\mathrm{RO}(P)$, i.e., for any nonzero $Y \in \mathrm{RO}(P)$ there is a $p \in P$ such that $e(p) \subseteq Y$.

(ii) If $p \le q$ then $e(p) \subseteq e(q)$.

(iii) $p \perp q$ iff $e(p) \cap e(q) = \emptyset$.

(iv) If $e(p) \le e(q)$, then p and q are compatible.

(v) The following conditions are equivalent:

(a) $e(p) \le e(q)$.

(b) $\{r : r \le p, q\}$ is dense below p.

(vi) The following conditions are equivalent, for F nonempty:

(a) $e(p) \le \prod_{q \in F} e(q)$.

(b) $\{r : r \le q$ for all $q \in F\}$ is dense below p.

(vii) The following conditions are equivalent:

(a) $e(p) \le (\prod_{q \in F} e(q)) \cdot \sum N$.

(b) $\{r : r \le q$ for all $q \in F$ and $e(r) \le s$ for some $s \in N\}$ is dense below p.

(viii) $e(p) \le -a$ iff there is no $q \le p$ such that $e(q) \le a$.

(ix) $e(p) \le -a + b$ iff for all $q \le p$, if $e(q) \le a$ then $e(q) \le b$.

Proof. (i): Assume the hypothesis. By the definition of the topology and since Y is nonempty and open, there is a $p \in P$ such that $P \downarrow p \subseteq Y$. Hence $e(p) = \mathrm{int}(\mathrm{cl}(P \downarrow p)) \subseteq \mathrm{int}(\mathrm{cl}(Y)) = Y$.

(ii): If $p \le q$, then $P \downarrow p \subseteq P \downarrow q$, and so $e(p) = \mathrm{int}(\mathrm{cl}(P \downarrow p)) \subseteq \mathrm{int}(\mathrm{cl}(P \downarrow q) = e(q))$.

(iii): Assume that $p \perp q$. Then $(P \downarrow p) \cap (P \downarrow q) = \emptyset$, and hence by Proposition 27.18(viii), $e(p) \cap e(q) = \emptyset$.

Conversely, suppose that $e(p) \cap e(q) = \emptyset$. Then $(P \downarrow p) \cap (P \downarrow q) \subseteq e(p) \cap e(q) = \emptyset$, and so $p \perp q$.

(iv): If $e(p) \le e(q)$, then $e(p) \cdot e(q) = e(p) \ne \emptyset$, so p and q are compatible by (iii).

(v): For (a)$\Rightarrow$(b), suppose that $e(p) \le e(q)$ and $s \le p$. Then $e(s) \le e(p) \le e(q)$, so s and q are compatible by (iv); say $r \le s, q$. Then $r \le s \le p$, hence $r \le p, q$, as desired.

For (b)$\Rightarrow$(a), suppose that $e(p) \not\le e(q)$. Thus $e(p) \cdot -e(q) \ne 0$. Hence there is an s such that $e(s) \subseteq e(p) \cdot -e(q)$. Hence $e(s) \cdot e(q) = \emptyset$, so $s \perp q$ by (iii). Now $e(s) \subseteq e(p)$, so s

and p are compatible by (iv); say $t \leq s, p$. For any $r \leq t$ we have $r \leq s$, and hence $r \perp q$. So (b) fails.

(vi): We proceed by induction on $|F|$. The case $|F| = 1$ is given by (v). Now assume the result for F, and suppose that $t \in P \backslash F$. First suppose that $e(p) \leq \prod_{q \in F} e(q) \cdot e(t)$. Suppose that $s \leq p$. Now $e(p) \leq \prod_{q \in F} e(q)$, so by the inductive hypothesis there is a $u \leq s$ such that $u \leq q$ for all $q \in F$. Thus $e(u) \leq e(s) \leq e(p) \leq e(t)$, so by (iv), u and t are compatible. Take any $v \leq u, t$. then $v \leq q$ for any $q \in F \cup \{t\}$, as desired.

Second, suppose that (b) holds for $F \cup \{t\}$. In particular, $\{r : r \leq q$ for all $q \in F\}$ is dense below p, and so $e(p) \leq \prod_{q \in F} e(q)$ by the inductive hypothesis. But also clearly $\{r : r \leq t\}$ is dense below p, so $e(p) \leq e(t)$ too, as desired.

(vii): First assume that $e(p) \leq (\prod_{q \in F} e(q)) \cdot \sum N$, and suppose that $u \leq p$. By (vi), there is a $v \leq u$ such that $v \leq q$ for each $q \in F$. Now $e(v) \leq e(u) \leq e(p) \leq \sum N$, so $0 \neq e(v) = e(v) \cdot \sum N = \sum_{s \in N} (e(v) \cdot e(s))$. Hence there is an $s \in N$ such that $e(v) \cdot e(s) \neq 0$. Hence by (iii), v and s are compatible; say $r \leq v, s$. Clearly r is in the set described in (b).

Second, suppose that (b) holds. Clearly then $\{r : r \leq q$ for all $q \in F\}$ is dense below p, and so $e(p) \leq \prod_{q \in F} e(q)$ by (vi). Now suppose that $e(p) \not\leq \sum N$. Then $e(p) \cdot -\sum N \neq 0$, so there is a q such that $e(q) \leq e(p) \cdot -\sum N$. By (iv), q and p are compatible; say $s \leq p, q$. Then by (b) choose $r \leq s$ and $t \in N$ such that $e(r) \leq t$. Thus $e(r) \leq e(s) \cdot t \leq e(p) \cdot t \leq (-\sum N) \cdot \sum N = 0$, contradiction.

(viii)$\Rightarrow$: Assume that $e(p) \leq -a$. Suppose that $q \leq p$ and $e(q) \leq a$. Then $e(q) \leq -a \cdot a = 0$, contradiction.

(viii)$\Leftarrow$: Assume that $e(p) \not\leq -a$. Then $e(p) \cdot a \neq 0$, so there is a q such that $e(q) \leq e(p) \cdot a$. By (vii) there is an $r \leq p, q$ with $e(r) \leq a$, as desired.

(ix)$\Rightarrow$: Assume that $e(p) \leq -a + b$, $q \leq p$, and $e(q) \leq a$. Then $e(q) \leq a \cdot (-a + b) \leq b$, as desired.

(ix)$\Leftarrow$: Assume the indicated condition, but suppose that $e(p) \not\leq -a + b$. Then $e(p) \cdot a \cdot -b \neq 0$, so there is a q such that $e(q) \leq e(p) \cdot a \cdot -b$. By (vii) with $F = \{p\}$ and $N = \{a \cdot -b\}$ we get q such that $q \leq p$ and $e(q) \leq a \cdot -b$. So $q \leq p$ and $e(q) \leq a$, so by our condition, $e(q) \leq b$. But also $e(q) \leq -b$, contradiction. □

We now expand on the remarks above concerning when e really is an embedding. Note that if P is a simple ordering, then the closure of $P \downarrow p$ is P itself, and hence P has only two regular open subsets, namely the empty set and P itself. If the ordering on P is trivial, meaning that no two elements are comparable, then every subset of P is regular open.

An important condition satisfied by many forcing orders is defined as follows. We say that P is *separative* iff it is a partial order (thus is an antisymmetric forcing order), and for any $p, q \in P$, if $p \not\leq q$ then there is an $r \leq p$ such that $r \perp q$.

Proposition 27.21. *Let P be a forcing order.*

(i) $\mathrm{cl}(P \downarrow p) = \{q : p$ *and* q *are compatible*$\}$.

(ii) $e(p) = \{q :$ *for all* $r \leq q$, r *and* p *are compatible*$\}$.

(iii) The following conditions are equivalent:

(a) P is separative.

(b) e is one-one, and for all $p, q \in P$, $p \leq q$ iff $e(p) \leq e(q)$.

Proof. (i) and (ii) are clear. For (iii), (a)⇒(b), assume that P is separative. Take any $p, q \in P$. If $p \leq q$, then $e(p) \leq e(q)$ by 27.20(ii). Suppose that $p \not\leq q$. Choose $r \leq p$ such that $r \perp q$. Then $r \in e(p)$, while $r \notin e(q)$ by (ii). Thus $e(p) \not\leq e(q)$.

Now suppose that $e(p) = e(q)$. Then $p \leq q \leq p$ by what was just shown, so $p = q$ since P is a partial order.

For (iii), (b)⇒(a), suppose that $p \leq q \leq p$. Then $e(p) \subseteq e(q) \subseteq e(p)$, so $e(p) = e(q)$, and hence $p = q$. So P is a partial order. Suppose that $p \not\leq q$. Then $e(p) \not\subseteq e(q)$. Choose $s \in e(p) \backslash e(q)$. Since $s \notin e(q)$, by (ii) we can choose $t \leq s$ such that $t \perp q$. Since $s \in e(p)$, it follows that t and p are compatible; choose $r \leq t, p$. Clearly $r \perp q$. □

Now we prove a theorem which says that the regular open algebra of a forcing order is unique up to isomorphism.

Theorem 27.22. *Let P be a forcing order, A a complete BA, and j a function mapping P into $A \backslash \{0\}$ with the following properties:*

(i) $j[P]$ is dense in A, i.e., for any nonzero $a \in A$ there is a $p \in P$ such that $j(p) \subseteq a$.

(ii) For all $p, q \in P$, if $p \leq q$ then $j(p) \leq j(q)$.

(iii) For any $p, q \in P$, $p \perp q$ iff $j(p) \cdot j(q) = 0$.

Then there is a unique isomorphism f from $\mathrm{RO}(P)$ onto A such that $f \circ e = j$. That is, f is a bijection from $\mathrm{RO}(P)$ onto A, and for any $x, y \in \mathrm{RO}(P)$, $x \subseteq y$ iff $f(x) \leq f(y)$; and $f \circ e = j$.

Note that since the Boolean operations are easily expressible in terms of $\leq$ (as least upper bounds, etc.), the condition here implies that f preserves all of the Boolean operations too; this includes the infinite sums and products.

Proof. Before beginning the proof, we introduce some notation in order to make the situation more symmetric. Let $B_0 = \mathrm{RO}(P)$, $B_1 = A$, $k_0 = e$, and $k_1 = j$. Then for each $m < 2$ the following conditions hold:

(1) $k_m[P]$ is dense in B_m.

(2) For all $p, q \in P$, if $p \leq q$ then $k_m(p) \leq k_m(q)$.

(3) For all $p, q \in P$, $p \perp q$ iff $k_m(p) \cdot k_m(q) = 0$.

(4) For all $p, q \in P$, if $k_m(p) \leq k_m(q)$, then p and q are compatible.

In fact, (1)–(3) follow from 27.20 and the assumptions of the theorem. Condition (4) for $m = 0$, so that $k_m = e$, follows from 27.20(iv). For $m = 1$, so that $k_m = j$, it follows easily from (iii).

Now we begin the proof. For each $m < 2$ we define, for any $x \in B_m$,

$$g_m(x) = \sum \{k_{1-m}(p) : p \in P,\ k_m(p) \leq x\}.$$

The proof of the theorem now consists in checking the following, for each $m \in 2$:

(5) If $x, y \in B_m$ and $x \leq y$, then $g_m(x) \leq g_m(y)$.

(6) $g_{1-m} \circ g_m$ is the identity on B_m.

(7) $g_0 \circ k_0 = k_1$.

In fact, suppose that (5)–(7) have been proved. If $x, y \in \mathrm{RO}(P)$, then

$$x \le y \text{ implies that } g_0(x) \le g_0(y) \text{ by (5);}$$
$$g_0(x) \le g_0(y) \text{ implies that } x = g_1(g_0(x)) \le g_1(g_0(y)) = y \text{ by (5) and (6).}$$

Also, (6) holding for both $m = 0$ and $m = 1$ implies that g_0 is a bijection from $\mathrm{RO}(P)$ onto A. Moreover, by (7), $g_0 \circ e = g_0 \circ k_0 = k_1 = j$. So g_0 is the desired function f of the theorem.

Now (5) is obvious from the definition. To prove (6), assume that $m \in 2$. We first prove

(8) For any $p \in P$ and any $b \in B_m$, $k_m(p) \le b$ iff $k_{1-m}(p) \le g_m(b)$.

To prove (8), first suppose that $k_m(p) \le b$. Then obviously $k_{1-m}(p) \le g_m(b)$. Second, suppose that $k_{1-m}(p) \le g_m(b)$ but $k_m(p) \not\le b$. Thus $k_m(p) \cdot -b \ne 0$, so by the denseness of $k_m[P]$ in B_m, choose $q \in P$ such that $k_m(q) \le k_m(p) \cdot -b$. Then p and q are compatible by (4), so let $r \in P$ be such that $r \le p, q$. Hence

$$k_{1-m}(r) \le k_{1-m}(p) \le g_m(b) = \sum\{k_{1-m}(s) : s \in P,\ k_m(s) \le b\}.$$

Hence $k_{1-m}(r) = \sum\{k_{1-m}(s) \cdot k_{1-m}(r) : s \in P,\ k_m(s) \le b\}$, so there is an $s \in P$ such that $k_m(s) \le b$ and $k_{1-m}(s) \cdot k_{1-m}(r) \ne 0$. Hence s and r are compatible; say $t \le s, r$. Hence $k_m(t) \le k_m(r) \le k_m(q) \le -b$, but also $k_m(t) \le k_m(s) \le b$, contradiction. This proves (8).

Now take any $b \in B_m$. Then

$$\begin{aligned} g_{1-m}(g_m(b)) &= \sum\{k_m(p) : p \in P,\ k_{1-m}(p) \le g_m(b)\} \\ &= \sum\{k_m(p) : p \in P,\ k_m(p) \le b\} \\ &= b. \end{aligned}$$

Thus (6) holds.

For (7), clearly $k_1(p) \le g_0(k_0(p))$. Now suppose that $k_0(q) \le k_0(p)$ but $k_1(q) \not\le k_1(p)$. Then $k_1(q) \cdot -k_1(p) \ne 0$, so there is an r such that $k_1(r) \le k_1(q) \cdot -k_1(p)$. Hence q and r are compatible, but $r \perp p$. Say $s \le q, r$. Then $k_0(s) \le k_0(q) \le k_0(p)$, so s and p are compatible. Say $t \le s, p$. Then $t \le r, p$, contradiction. This proves (7).

This proves the existence of f. Now suppose that g is also an isomorphism from $\mathrm{RO}(P)$ onto A such that $g \circ e = j$, but suppose that $f \ne g$. Then there is an $X \in \mathrm{RO}(P)$ such that $f(X) \ne g(X)$. By symmetry, say that $f(X) \cdot -g(X) \ne 0$. By (ii), choose $p \in P$ such that $j(p) \le f(X) \cdot -g(X)$. So $f(e(p)) = j(p) \le f(X)$, so $e(p) \le X$, and hence $j(p) = g(e(p)) \le g(X)$. This contradicts $j(p) \le -g(X)$. $\square$

Proposition 27.23. *Let $(A, +, \cdot, -, 0, 1)$ be a Boolean algebra. Then $(A, \triangle, \cdot, 0, 1)$ is a ring with identity in which every element is idempotent. This means that $x \cdot x = x$ for all x.*

Proof. Obviously $\triangle$ is commutative, and it is associative by Proposition 13.11(iii). Clearly $x\triangle 0 = x$ for all x. Clearly $x\triangle x = 0$, so each element x has itself as additive inverse. Hence $(A, \triangle, 0)$ is an abelian group.

Clearly $\cdot$ is associative. The distributive law holds by Proposition 13.11(ii). Clearly $x \cdot 1 = x$ for all x, and clearly $x \cdot x = x$ for all x.

Hence $(A, \triangle, \cdot, 0, 1)$ is a ring with identity in which every element is idempotent. $\square$

Proposition 27.24. *Let $(A, +, \cdot, 0, 1)$ be a ring with identity in which every element is idempotent. Then A is a commutative ring, and $(A, \oplus, \cdot, -, 0, 1)$ is a Boolean algebra, where for any $x, y \in A$, $x \oplus y = x + y + xy$ and for any $x \in A$, $-x = 1 + x$.*

Proof. $x + y = (x + y)^2 = x^2 + xy + yx + y^2 = x + xy + yx + y$, and hence $0 = xy + yx$ for any x, y. Setting $x = y$, we get $0 = x + x$, and so x is its own additive inverse. Then from $0 = xy + yx$ we see that yx is the additive inverse of xy, hence $xy = yx$. Thus the ring is commutative.

To show that $(A, \oplus, \cdot, -, 0, 1)$ is a Boolean algebra, we need to check all of the axioms.

(C): Clear.

(A): For any x, y, z,

$$\begin{aligned} x \oplus (y \oplus z) &= x + (y \oplus z) + x(y \oplus z) \\ &= x + y + z + yz + x(y + z + yz) \\ &= x + y + z + yz + xy + xz + xyz; \end{aligned}$$

Hence, using (C),

$$\begin{aligned} (x \oplus y) \oplus z &= z \oplus (x \oplus y) \\ &= z + x + y + xy + zy + zxy \\ &= \text{above}. \end{aligned}$$

(A′): obvious.

(C′): obvious.

(L):

$$x \oplus xy = x + xy + xxy = x + xy + xy = x.$$

(L′):

$$x(x \oplus y) = x(x + y + xy) = xx + xy + xxy = x + xy + xy = x.$$

(D):

$$\begin{aligned} x(y \oplus z) &= x(y + z + yz) = xy + xz + xyz; \\ xy \oplus xz &= xy + xz + xyxz = xy + xz + xyz. \end{aligned}$$

(D′): See Proposition 13.12.

(K): $x + (-x) = x + 1 + x = 1$.

(K'): $x(1 + x) = x + xx = x + x = 0$.

Thus we have a BA. $\square$

Proposition 27.25. *The processes described in Propositions 27.23 and 27.24 are inverses of one another.*

Proof. For each BA $(A, +, \cdot, -, 0, 1)$ let $\mathscr{R}(A, +, \cdot, -, 0, 1) = (A, \triangle, \cdot, 0, 1)$ be the associated ring, and for each ring $(A, +, \cdot, 0, 1)$ with identity in which every element is idempotent let $\mathscr{B}(A, +, \cdot, 0, 1) = (A, \oplus, \cdot, -, 0, 1)$ be the associated Boolean algebra. We want to show that $\mathscr{R}$ and $\mathscr{B}$ are inverses of each other.

First suppose that $(A, +, \cdot, -, 0, 1)$ is a BA. Let $\mathscr{R}((A, +, \cdot, -, 0, 1)) = (A, \triangle, \cdot, 0, 1)$ be the associated ring, and let $\mathscr{B}(\mathscr{R}((A, +, \cdot, -, 0, 1)) = (A, \oplus, \cdot, -', 0, 1)$ be the BA associated with that ring; we want to show that $+ = \oplus$ and $- = -'$. We have

$$\begin{aligned}
x \oplus y &= x \triangle y \triangle (x \cdot y) \\
&= x \triangle (y \cdot -(x \cdot y) + x \cdot y \cdot -y) \\
&= x \triangle (y \cdot -x) \\
&= x \cdot -(y \cdot -x) + y \cdot -x \cdot -x \\
&= x + y \cdot -x \\
&= x + y \cdot x + y \cdot -x \\
&= x + y.
\end{aligned}$$

Also, $-'x = 1 \triangle x = -x$.

Second, suppose that $(A, +, \cdot, 0, 1)$ is a ring with identity in which every element is idempotent, let $\mathscr{B}((A, +, \cdot, 0, 1)) = (A, +', \cdot, -', 0, 1)$ be the associated BA, and let $\mathscr{R}(\mathscr{B}((A, +, \cdot, 0, 1))) = (A, \triangle', \cdot, 0, 1)$ be the ring associated with it. We want to show that $+ = \triangle'$. We have

$$\begin{aligned}
x \triangle' y &= (x \cdot -'y) +' (y \cdot -'x) \\
&= (x \cdot (1 + y)) +' (y \cdot (1 + x)) \\
&= (x + xy) +' (y + xy) \\
&= x + xy + y + xy + (x + xy)(y + xy) \\
&= x + y + xy + xy + xy + xxy + xyy + xyxy \\
&= x + y + xy + xy + xy + xy + xy + xy \\
&= x + y.
\end{aligned}$$

$\square$

Proposition 27.26. *A filter F is an ultrafilter iff F is maximal among the set of all filters G such that $0 \notin G$.*

Proof. $\Rightarrow$: Assume that F is an ultrafilter. Hence by definition $0 \notin F$. Suppose that $F \subset G$ with G a filter. Choose $x \in G \backslash F$. Since $x \notin F$ it follows that $-x \in F$, and hence $-x \in G$. So $0 = x \cdot -x \in G$. So F is maximal among the set of filters G such that $0 \notin G$.

$\Leftarrow$: Suppose that F is maximal among the set of filters G such that $0 \notin G$. Suppose that $a \in A$ and $a \notin F$. Let $G = \{x \in A : a \cdot y \le x \text{ for some } y \in F\}$. Then G is a filter on A. In fact, obviously conditions (1) and (2) hold. For (3), suppose that $x, z \in G$. Choose $y, w \in F$ such that $x = a \cdot y$ and $z = a \cdot w$. Now $y \cdot w \in F$, and $x \cdot z = a \cdot y \cdot w$. So $x \cdot z \in G$. Thus, indeed, G is a filter on A. Clearly also $F \subseteq G$. Clearly $a \in G$ (taking $y = 1$), so $F \subset G$.

It follows by supposition that $0 \in G$. Say $0 = a \cdot y$, with $y \in F$. then $y \le -a$, so $-a \in F$. Thus F is an ultrafilter.

Proposition 27.27. *For any nonzero $a \in A$ there is an ultrafilter F such that $a \in F$.*

Proof. Let $\mathscr{A} = \{G : G \text{ is a filter in } A,\ a \in G, \text{ and } 0 \notin G\}$. We consider $\mathscr{A}$ as a partially ordered set under $\subseteq$. To verify the hypothesis of Zorn's lemma, suppose that $\mathscr{B}$ is a subset of $\mathscr{A}$ linearly ordered by $\subseteq$. Now $\{x \in A : a \le x\}$ is clearly a member of $\mathscr{A}$, so we may assume that $\mathscr{B}$ is nonempty. Let $H = \bigcup \mathscr{B}$. Since $\mathscr{B}$ is nonempty, it is clear that $a \in H$. Suppose that $x \in H$ and $x \le y$. Choose $G \in \mathscr{B}$ such that $x \in G$. Then $y \in G$ since G is a filter. So $y \in H$. Suppose that $x, y \in H$. Choose $G, G' \in \mathscr{B}$ such that $x \in G$ and $y \in G'$. By symmetry say $G \subseteq G'$. Then $x, y \in G'$, so $x \cdot y \in G'$, hence $x \cdot y \in H$. Thus we have shown that H is a filter on A. Clearly $0 \notin H$. So H is a member of $\mathscr{A}$ which is an upper bound for $\mathscr{B}$.

Thus by Zorn's lemma, $\mathscr{A}$ has a maximal member G. By Proposition 27.26, G is as desired. $\square$

Proposition 27.28. (Stone's representation theorem) *Any BA is isomorphic to a field of sets.*

Proof. Let X be the collection of all ultrafilters, and let $F, G \in X$.

$F \in f(-a)$ iff $-a \in F$ iff $a \notin F$, so $f(-a) = X \backslash f(a)$.

Suppose that $F \in f(a+b)$. Then $a + b \in F$. Suppose that $F \notin f(a)$. Then $a \notin F$, so $-a \in F$, hence $-a \cdot (a+b) \in F$. Since $-a \cdot (a+b) \le b$, also $b \in F$, so $F \in f(b)$. This shows that $f(a+b) \subseteq f(a) \cup f(b)$. On the other hand, if $F \in f(a)$, then $a \in F$; but $a \le a + b$, so also $a + b \in F$; hence $F \in f(a+b)$. Altogether this shows that $f(a+b) = f(a) \cup f(b)$.

Suppose that $a \neq b$. Then $a \triangle b \neq 0$, so $a \triangle b \in F$ for some ultrafilter F, by Proposition 27.27. Hence $F \in f(a \triangle b) = [f(a) \backslash f(b)] \cup [f(b) \backslash f(a)]$, and so $f(a) \neq f(b)$. So f is one-one. $\square$

Proposition 27.29. *Suppose that F is an ultrafilter on a BA A. Let 2 be the two-element BA. (This is, up to isomorphism, the BA of all subsets of 1.) For any $a \in A$ let*

$$f(a) = \begin{cases} 1 & \text{if } a \in F, \\ 0 & \text{if } a \notin F. \end{cases}$$

Then f is a homomorphism of A into 2.

Proof. $f(a \cdot b) = 1$ iff $a \cdot b \in F$ iff $a, b \in F$ iff $f(a) \cdot f(b) = 1$. Hence $f(a \cdot b) = f(a) \cdot f(b)$.
$f(-a) = 1$ iff $-a \in F$ iff $a \notin F$ iff $f(a) = 0$. Hence $f(-a) = -f(a)$.

$f(a+b) = f(-(-a \cdot -b)) = -(-f(a) \cdot -f(b)) = f(a) + f(b)$.
$f(0) = f(a \cdot -a) = f(a) \cdot -f(a) = 0$.
$f(1) = f(a + -a) = f(a) + -f(a) = 1$. $\square$

Proposition 27.30. *Suppose that $\mathscr{L}$ is a first-order language and T is a set of sentences of $\mathscr{L}$. Define $\varphi \equiv_T \psi$ iff φ and ψ are sentences of $\mathscr{L}$ and $T \models \varphi \leftrightarrow \psi$. This is an equivalence relation on the set S of all sentences of $\mathscr{L}$. Let A be the collection of all equivalence classes under this equivalence relation. Moreover, there are operations $+$, $\cdot$, $-$ on A such that for any sentences φ, ψ,*

$$[\varphi] + [\psi] = [\varphi \vee \psi];$$
$$[\varphi] \cdot [\psi] = [\varphi \wedge \psi];$$
$$-[\varphi] = [\neg\varphi].$$

Finally, $(A, +, \cdot, -, [\exists v_0(\neg(v_0 = v_0))], [\exists v_0(v_0 = v_0)])$ is a Boolean algebra.

Proof. $\equiv_T$ is reflexive: $T \models \varphi \leftrightarrow \varphi$ for any sentence φ.
$\equiv_T$ is symmetric: If $T \models \varphi \leftrightarrow \psi$, then $T \models \psi \leftrightarrow \varphi$.
$\equiv_T$ is transitive: If $T \models \varphi \leftrightarrow \psi$ and $T \models \psi \leftrightarrow \chi$, then $T \models \varphi \leftrightarrow \chi$.
$+$ is well-defined: If $T \models \varphi \leftrightarrow \varphi'$ and $T \models \psi \leftrightarrow \psi'$, then $T \models (\varphi \vee \psi) \leftrightarrow (\varphi' \vee \psi')$.
$\cdot$ is well-defined: If $T \models \varphi \leftrightarrow \varphi'$ and $T \models \psi \leftrightarrow \psi'$, then $T \models (\varphi \wedge \psi) \leftrightarrow (\varphi' \wedge \psi')$.
$-$ is well-defined: If $T \models \varphi \leftrightarrow \varphi'$, then $T \models \neg\varphi \leftrightarrow \neg\varphi'$.
Finally, we need to check the axioms for BAs:

(A) holds since

$$[\varphi] + ([\psi] + [\chi]) = [\varphi \vee (\psi \vee \chi)] = [(\varphi \vee \psi) \vee \chi] = ([\varphi] + [\psi]) + [\chi];$$

(A$'$) holds since

$$[\varphi] \cdot ([\psi] \cdot [\chi]) = [\varphi \wedge (\psi \wedge \chi)] = [(\varphi \wedge \psi) \wedge \chi] = ([\varphi] \cdot [\psi]) \cdot [\chi];$$

(C) holds since

$$[\varphi] + [\psi] = [\varphi \vee \psi] = [\psi \vee \varphi] = [\psi] + [\varphi];$$

(C$'$) holds since

$$[\varphi] \cdot [\psi] = [\varphi \wedge \psi] = [\psi \wedge \varphi] = [\psi] \cdot [\varphi];$$

(L) holds since

$$[\varphi] + [\varphi] \cdot [\psi]) = [\varphi \vee (\varphi \wedge \psi)] = [\varphi];$$

(L$'$) holds since

$$[\varphi] \cdot [\varphi] + [\psi]) = [\varphi \wedge (\varphi \vee \psi)] = [\varphi];$$

(D) holds since

$$[\varphi] \cdot ([\psi] + [\chi]) = [\varphi \wedge (\psi \vee \chi)] = [(\varphi \wedge \psi) \vee (\varphi \wedge \chi)] = [\varphi] \cdot [\psi] + [\varphi] \cdot [\chi];$$

for (D′) see Proposition 2.12; (K) holds since

$$[\varphi] + -[\varphi] = [\varphi \vee \neg\varphi] = [\exists v_0(v_0 = v_0)];$$

(K′) holds since

$$[\varphi] \cdot -[\varphi] = [\varphi \wedge \neg\varphi] = [\exists v_0(\neg(v_0 = v_0))]. \qquad \square$$

Proposition 27.31. *Every Boolean algebra is isomorphic to one obtained as in Proposition 27.30.*

Proof. Let A be a Boolean algebra. Let $\mathscr{L}$ be the first-order language which has a unary relation symbol R_a for each $a \in A$. Let T be the following set of sentences of $\mathscr{L}$:

$$\begin{aligned}
&\forall x \forall y(x = y);\\
&\forall x[R_{-a}(x) \leftrightarrow \neg R_a(x)] \quad \text{for each } a \in A;\\
&\forall x[R_{a\cdot b}(x) \leftrightarrow R_a(x) \wedge R_b(x)] \quad \text{for all } a, b \in A;\\
&\forall x R_1(x).
\end{aligned}$$

Consider $\equiv_T$. Define $f(a) = [\forall x R_a(x)]$ for any $a \in A$. To show that f preserves $\cdot$, suppose that $a, b \in A$. Note that

$$T \models \forall x R_{a\cdot b}(x) \leftrightarrow \forall x R_a(x) \wedge \forall x R_b(x);$$

hence $f(a \cdot b) = f(a) \cdot f(b)$.

To proceed we need the following fact

(1) $T \models \forall x\varphi \leftrightarrow \varphi$ for any variable x and any formula φ.

In fact, trivially $T \models \forall x\varphi \rightarrow \varphi$, and $T \models \varphi \rightarrow \exists x\varphi$. Since $T \models x = y$, clearly $T \models \exists x\varphi \rightarrow \forall x\varphi$. So (1) holds.

Now to show that f preserves $-$, suppose that $a \in A$. Then $T \models \forall x R_{-a}(x) \leftrightarrow \forall x \neg R_a(x)$. By (1), $T \models \forall x \neg R_a(x) \leftrightarrow \neg R_a(x)$ and $T \models \neg R_a(x) \leftrightarrow \neg\forall x R_a(x)$. Putting these statements together we have $T \models \forall x R_{-a}(x) \leftrightarrow \neg\forall x R_a(x)$, and it follows that f preserves $-$.

To show that f is one-one, suppose that $a, b \in A$ and $a \neq b$; say $a \cdot -b \neq 0$. Let F be an ultrafilter on A such that $a \cdot -b \in F$. We now define an $\mathscr{L}$-structure $\mathfrak{A}$. Let $A = 1$. For each $a \in A$, let

$$R_a^{\mathfrak{A}} = \begin{cases} 1 & \text{if } a \in F,\\ 0 & \text{otherwise.}\end{cases}$$

Clearly $\mathfrak{A}$ is a model of T. Also, $\mathfrak{A} \models R_{a\cdot -b}(x)$. It follows that $[\forall x R_{a\cdot -b}(x)] \neq [\exists v_0(\neg(v_0 = v_0))]$, and so $f(a) = [\forall x R_a(x)] \neq [\forall x R_b(x)] = f(b)$, as desired.

It remains only to show that f maps onto.

(2) For any formula φ there is an $a \in A$ such that $T \models \varphi \leftrightarrow R_a(x)$.

Condition (2) is easily proved by induction on φ, using (1). Hence f is onto. $\square$

Proposition 27.32. *Let A be the collection of all subsets X of $Y \stackrel{\text{def}}{=} \{r \in \mathbb{Q} : 0 \leq r\}$ such that there exist an $m \in \omega$ and $a, b \in {}^m(Y \cup \{\infty\})$ such that $a_0 < b_0 < a_1 < b_1 < \cdots < a_{m-1} < b_{m-1} \leq \infty$ and*

$$X = [a_0, b_0) \cup [a_1, b_1) \cup \ldots \cup [a_{m-1}, b_{m-1}).$$

Note that $\emptyset \in A$ by taking $m = 0$, and $Y \in A$ since $Y = [0, \infty)$.

(i) If X is as above, $c, d \in Y \cup \{\infty\}$ with $c < d$, $c \leq a_0$, then $X \cup [c, d) \in A$, and c is the first element of $X \cup [c, d)$.

(ii) If X is as above and $c, d \in Y \cup \{\infty\}$ with $c < d$, then $X \cup [c, d) \in A$.

(iii) $(A, \cup, \cap, \backslash, \emptyset, Y)$ is a Boolean algebra.

Proof. (i): Assume the hypothesis. If $m = 0$ the desired conclusion is clear, so suppose that $m > 0$. We consider several cases.

Case 1. $b_{m-1} \leq d$. Then $X \cup [c, d) = [c, d) \in A$.

Case 2. There is an $i < m - 1$ such that $b_i \leq d < a_{i+1}$. Then

$$X \cup [c, d) = [c, d) \cup [a_{i+1}, b_{i+1}) \cup \ldots \cup [a_{m-1}, b_{m-1}) \in A.$$

Case 3. There is an $i < m$ such that $a_i \leq d < b_i$. Then

$$X \cup [c, d) = [c, b_i) \cup [a_{i+1}, b_{i+1}) \cup \ldots \cup [a_{m-1}, b_{m-1}) \in A.$$

Case 4. $d < a_0$. Then

$$X \cup [c, d) = [c, d) \cup [a_0, b_0) \cup \ldots \cup [a_{m-1}, b_{m-1}) \in A.$$

(ii): Again we consider several cases.

Case 1. $c \leq a_0$. Then $X \cup [c, d) \in A$ by (i).

Case 2. There is an $i < m$ such that $a_i \leq c \leq b_i$. Let $X' = [a_i, b_i) \cup \ldots \cup [a_{m-1}, b_{m-1})$. Then by (i) applied to X' and $[a_i, d)$ we get $X' \cup [a_i, d) \in A$, and a_i is the least element of $X' \cup [a_i, d)$. Clearly

$$X \cup [c, d) = [a_0, b_0) \cup \ldots \cup [a_{i-1}, b_{i-1}) \cup X' \cup [a_i, d) \in A.$$

Case 3. There is an $i < m - 1$ such that $b_i < c < a_{i+1}$. Then we can apply (i) to $[a_{i+1}, b_{i+1}) \cup \ldots \cup [a_{m-1}, b_{m-1})$ and $[c, d)$ to get the desired result as in Case 2.

Case 4. $c = b_{m-1}$. Then

$$X \cup [c, d) = [a_0, b_0) \cup \ldots \cup [a_{m-1}, d) \in A.$$

Case 5. $b_{m-1} < c$. This case is clear.

(iii): From (ii) it is clear that A is closed under $\cup$. Now suppose that X is given as above. To show that also $Y \backslash X \in A$, we consider several cases.

Case 1. $m = 0$. So $X = \emptyset$, and $Y = [0, \infty) \in A$.

Case 2. $m > 0$, $0 < a_0$, and $b_{m-1} < \infty$. Then

$$Y \backslash X = [0, a_0) \cup [b_0, a_1) \cup \ldots \cup [b_{m-2}, a_{m-1}) \cup [b_{m-1}, \infty) \in A.$$

Case 3. $m > 0$, $a_0 = 0$, and $b_{m-1} < \infty$. Then

$$Y \backslash X = [b_0, a - 1) \cup \ldots \cup [b_{m-2}, a_{m-1}) \cup [b_{m-1}, \infty) \in A.$$

Case 4. $m > 0$, $0 < a_0$, and $b_{m-1} = \infty$. Then

$$Y \backslash X = [0, a_0) \cup [b_0, a_1) \cup \ldots \cup [b_{m-2}, a_{m-1}) \in A.$$

Case 5. $m > 0$, $0 = a_0$, and $b_{m-1} = \infty$. Then

$$Y \backslash X = [b_0, a_1) \cup \ldots \cup [b_{m-2}, a_{m-1}) \in A.$$

Thus (iii) holds. □

Proposition 27.33. (Continuing Proposition 27.32) *For each $n \in \omega$ let $x_n = [n, n+1)$, an interval in $\mathbb{Q}$. Then $\sum_{n\in\omega} x_{2n}$ does not exist in A.*

Proof. Suppose that the sum does exist. Let $X = \sum_{n\in\omega} x_{2n}$, and assume that X is as in Proposition 27.32.

We claim that $b_{m-1} = \infty$. In fact, if $b_{m-1} < \infty$, then there is an $m \in \omega$ such that $b_{m-1} < 2m$; then $x_{2m} = [2m, 2m + 1)$ is disjoint from X according to the form of X, but $x_{2m} \leq X$ by definition, contradiction. So our claim holds.

Now choose $m \in \omega$ so that $a_{m-1} < 2m + 1$. Then $[2m + 1, 2m + 2) \cap x_{2n} = \emptyset$ for all n, hence $[2m + 1, 2m + 2) \cap X = \emptyset$. But $[2m + 1, 2m + 2) \subseteq [a_{m-1}, b_{m-1}) \subseteq X$, contradiciton. □

Proposition 27.34. *Let A be the Boolean algebra of all subsets of some nonempty set X, under the natural set-theoretic operations. If $\langle a_i : i \in I\rangle$ is a system of elements of A, then*

$$\prod_{i\in I}(a_i + -a_i) = 1 = \sum_{\varepsilon\in {}^I2} \prod_{i\in I} a_i^{\varepsilon(i)},$$

where for any y, $y^1 = y$ and $y^0 = -y$.

Proof. First note that the big products and sums are just the ordinary intersections and unions. Obviously $a_i + -a_i = a_i \cup (X\backslash a_i) = X = 1$, giving the first equality. Now suppose that $x \in X$. We define

$$\varepsilon(i) = \begin{cases} 1 & \text{if } x \in a_i, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly then $x \in a_i^{\varepsilon(i)}$ for each $i \in I$, and hence x is in the right side of the second equality, as desired. □

Proposition 27.35. *Let M be the set of all finite functions $f \subseteq \omega \times 2$. For each $f \in M$ let*

$$U_f = \{g \in {}^\omega 2 : f \subseteq g\}.$$

Let A consist of all finite unions of sets U_f.

(i) A is a Boolean algebra under the set-theoretic operations.

(ii) For each $i \in \omega$, let $x_i = U_{\{(i,1)\}}$. Then

$$^\omega 2 = \prod_{i\in\omega}(x_i + -x_i)$$

while

$$\sum_{\varepsilon\in{}^\omega 2}\prod_{i\in\omega} x_i^{\varepsilon(i)} = \emptyset,$$

where for any y, $y^1 = y$ and $y^0 = -y$.

Proof. (i): Obviously A is closed under $\cup$. Now suppose that $a \in A$; we want to show that $({}^\omega 2)\backslash a \in A$. Say $a = \bigcup_{f\in N} U_f$, where N is a finite subset of M. Let

$$P = \left\{ g \in M : \mathrm{dmn}(g) \subseteq \bigcup_{f\in N} \mathrm{dmn}(f) \text{ and } \forall f \in N \exists i \in \mathrm{dmn}(g) \cap \mathrm{dmn}(f)[f(i) \neq g(i)] \right\}.$$

Clearly P is a finite subset of M. We claim that $({}^\omega 2)\backslash a = \bigcup_{g\in P} U_g$. First suppose that $h \in ({}^\omega 2)\backslash a$. Let $g = h \restriction \bigcup_{f\in N} \mathrm{dmn}(f)$. So $g \in M$ and $h \in U_g$. We claim that $g \in P$. For, suppose that $f \in N$. Then $U_f \subseteq a$, so it follows that $h \notin U_f$. So we can choose $i \in \mathrm{dmn}(f)$ such that $f(i) \neq h(i)$. Clearly $i \in \mathrm{dmn}(g)$ and $f(i) \neq g(i)$. This shows that $g \in P$, proving $\subseteq$ of our claim.

For $\supseteq$, suppose that $g \in P$ and $h \in U_g$. Suppose that $h \in a$. Choose $f \in N$ such that $h \in U_f$, hence $f \subseteq h$. But $g \subseteq H$ too, so there is an $i \in \mathrm{dmn}(g) \cap \mathrm{dmn}(f)$ such that $f(i) \neq g(i)$. But this means that $f(i) \neq h(i)$, contradicting $f \subseteq h$. We have now shown (i).

Clearly $x_i \cup (({}^\omega 2)\backslash x_i = {}^\omega 2$ for any $i \in \omega$. Hence ${}^\omega 2 = \prod_{i\in\omega}(x_i + -x_i)$.

Now suppose that $\varepsilon \in {}^\omega 2$; we want to show that $\prod_{i\in\omega} x_i^{\varepsilon(i)} = 0$, i.e., that there is no nonzero element a of A such that $a \leq x_i^{\varepsilon(i)}$ for all $i \in \omega$. suppose that a is such an element. Then there is a $g \in M$ such that $U_g \subseteq a$. Take any $i \notin \mathrm{dmn}(g)$, and let $h \in {}^\omega 2$ be any function such that $g \subseteq h$ and $h(i) \neq \varepsilon(i)$. Then $h \in U_g$ but $h \notin x_i^{\varepsilon(i)}$, contradiction. □

Proposition 27.36. *Suppose that $(P, \leq, 1)$ is a forcing order. Define*

$$p \equiv q \quad \text{iff} \quad p, q \in P,\ p \leq q,\ \text{and } q \leq p.$$

Then $\equiv$ is an equivalence relation, and if Q is the collection of all $\equiv$-classes, then there is a relation $\preceq$ on Q such that for all $p, q \in P$, $[p]_\equiv \preceq [q]_\equiv$ iff $p \leq q$. Finally, $(Q, \preceq)$ is a *partial order, i.e., $\preceq$ is reflexive on Q, transitive, and antisymmetric ($q_1 \preceq q_2 \preceq q_1$ implies that $q_1 = q_2$); moreover, $q \leq [1]$ for all $q \in Q$.*

Proof. Since $\leq$ is reflexive on P, clearly also $\equiv$ is reflexive on P. Clearly $\equiv$ is symmetric. Now suppose that $p \equiv q \equiv r$. Thus $p \leq q$, $q \leq p$, $q \leq r$, and $r \leq q$. Then $p \leq r$ and $r \leq p$, so $p \equiv r$. So $\equiv$ is an equivalence relation on P.

Let

$$\preceq = \{(a,b) : \exists p, q \in P[p \leq q, \ a = [p], \text{ and } b = [q]]\}.$$

Obviously then $p \leq q$ implies that $[p] \preceq [q]$. Now suppose that $[p] \preceq [q]$. Choose $p', q' \in P$ such that $p' \leq q'$, $[p] = [p']$, and $[q] = [q']$. Then $p \leq p'$ and $q' \leq q$, so $p \leq q$.

To show that $\preceq$ is a partial order on Q, first suppose that $a \in Q$. Write $a = [p]$. Then $p \leq p$, so $a \preceq a$. Thus $\preceq$ is reflexive on Q. Now suppose that $a \preceq b \preceq c$. Then there exist p, q, q', r such that $p \leq q$, $a = [p]$, $b = [q]$, $q' \leq r$, $b = [q']$, and $c = [r]$. Then $q \leq q'$ since $[q] = [q']$. So $p \leq q \leq q' \leq r$, hence $p \leq r$. So $a = [q] \preceq [r] = c$. This shows that $\preceq$ is transitive. Finally, suppose that $a \preceq b \preceq a$. Then there exist p, q, q', r such that $p \leq q$, $a = [p]$, $b = [q]$, $q' \leq r$, $b = [q']$, and $a = [r]$. Then $q \leq q'$ since $[q] = [q']$. Also $r \leq p$ since $[p] = [r]$, so $q \leq q' \leq r \leq p$, hence $q \leq p$. But also $p \leq q$, so $a = [p] = [q] = b$. So $\preceq$ is a partial order. Clearly $a \leq [1]$ for all $a \in Q$. $\square$

Proposition 27.37. *We say that $(P, <)$ is a partial order in the second sense iff $<$ is transitive and irreflexive. (Irreflexive means that for all $p \in P$, $p \not< p$.) Then if $(P, <)$ is a partial order in the second sense and if we define $\preceq$ by $p \preceq q$ iff ($p, q \in P$ and $p < q$ or $p = q$), then $\mathscr{A} \stackrel{\text{def}}{=} (P, \preceq)$ is a partial order. Furthermore, if $(P, \leq)$ is a partial order, and we define $p \prec q$ by $p \prec q$ iff ($p, q \in P$, $p \leq q$, and $p \neq q$), then $\mathscr{B} \stackrel{\text{def}}{=} (P, \prec)$ is a partial order in the second sense.*

Also, $\mathscr{A}$ and $\mathscr{B}$ are inverses of one another.

Proof. Clearly $\preceq$ is reflexive on P. Now suppose that $x \preceq y \preceq z$. If $x = y$ or $y = z$, then $x \preceq z$ by supposition. If $x < y < z$, then $x < z$, and so $x \preceq z$. Thus $\preceq$ is transitive. Suppose that $x \preceq y \preceq x$, but $x \neq y$. Then $x < y < x$, hence $x < x$, contradiction. So $\preceq$ is antisymmetric. Hence $(P, \preceq)$ is a partial order.

Now suppose that $(P, \leq)$ is a partial order, and define $p \prec q$ by $p \prec q$ iff ($p, q \in P$, $p \leq q$, and $p \neq q$). Clearly $\prec$ is irreflexive. Suppose that $p \prec q \prec r$. Then $p \leq q \leq r$, so $p \leq r$. Suppose that $p = r$. Then $p \leq q \leq p$, so $p = q$ by antisymmetry, contradiction. Thus $p \neq r$, and so $p \prec r$. So $(P, \prec)$ is a partial order in the second sense.

Next, suppose that $(P, <)$ is a partial order in the second sense, and let $\mathscr{A}(P, <) = (P, \preceq)$. Furthermore, let $\mathscr{B}(\mathscr{A}(P, <)) = (P, <')$. Then

$$p <' q \text{ iff } (p \preceq q \text{ and } p \neq q) \text{ iff } ((p < q \text{ or } p = q) \text{ and } p \neq q) \text{ iff } p < q.$$

Thus $\mathscr{B}(\mathscr{A}(P, <)) = (P, <)$.

Finally, suppose that $(P, \leq)$ is a partial order. Let $\mathscr{B}(P, \leq) = (P, \prec)$, and let $\mathscr{A}(\mathscr{B}(P, \leq)) = (P, \leq')$. Then

$$p \leq' q \text{ iff } (p \prec q \text{ or } p = q) \text{ iff } ((p \leq q \text{ and } p \neq q) \text{ or } p = q) \text{ iff } p \leq q. \qquad \square$$

Proposition 27.38. *If $(P, \leq, 1)$ is a forcing order and we define $\prec$ by $p \prec q$ iff ($p, q \in P$, $p \leq q$ and $q \not\leq p$), then $(P, \prec)$ is a partial order in the second sense. Moreover, there is an example where this partial order is not isomorphic to the one derived from $(P, \leq, 1)$ by the procedure of Proposition 27.36.*

Proof. $\prec$ is irreflexive, since $x \prec x$ would imply that $x \not\leq x$, a contradiction. For transitivity, suppose that $x \prec y \prec z$. Then $x \leq y$ and $y \leq z$, so $x \leq z$. Also, $y \not\leq x$ and $z \not\leq y$. Suppose that $z \leq x$. Then $y \leq z \leq x$ and hence $y \leq x$, contradiction. Hence $z \not\leq x$, and so $x \prec z$. Thus $(P, \prec)$ is a partial order in the second sense.

For the example, let X be any infinite set, and let $\leq$ be $X \times X$. Fix $1 \in X$. So $(X, \leq, 1)$ is a quasiorder. The partial order constructed in exercise E13.14 has only one element, while the partial order of the present exercise has X, an infinite set, as its underlying set. Note that $\prec$ is empty.

Proposition 27.39. *If $(P, \preceq, 1)$ is a forcing order such that the mapping e from P into* RO(P) *is one-one, then $(P, \preceq)$ is a partial order. There is an example of a forcing order such that e is not one-one. There is an example of an infinite forcing order Q such that e is not one-one, while for any $p, q \in Q$, $p \leq q$ iff $e(p) \subseteq e(q)$.*

Proof. Suppose that P is a forcing order such that e is one-one, and $p \leq q \leq p$. Then $P \downarrow p = P \downarrow q$, and hence $e(p) = e(q)$. So $p = q$. Hence $(P, \leq)$ is a partial order.

For an example of a forcing order such that e is not one-one, take any simple ordering with greatest element; see the remarks preceding Proposition 13.21.

For the final example, take any infinite set Q, and take the forcing order $(Q, Q \times Q, q)$ for any element $q \in Q$. So e is the constant function with value Q. For any $p, q \in Q$ we have $p \leq q$ and $e(p) \subseteq e(q)$, so these statements are equivalent trivially. $\square$

Proposition 27.40. (Continuing Proposition 27.36) *Let $\mathbb{P} = (P, \leq, 1)$ be a forcing order, and let $\mathbb{Q} = (Q, \preceq, [1])$ be as in Proposition 27.36 Then there is an isomorphism f of RO$(\mathbb{P})$ onto RO$(\mathbb{Q})$ such that $f \circ e_{\mathbb{P}} = e_{\mathbb{Q}} \circ \pi$, where $\pi : P \to Q$ is defined by $\pi(p) = [p]$ for all $p \in P$.*

Proof. We will apply Theorem 27.22. For any $p \in P$ let $j(p) = e_{\mathfrak{Q}}(\pi(p))$. Thus $j : P \to \mathrm{RO}(\mathfrak{Q})$.

Suppose that $0 \neq X \in \mathrm{RO}(\mathfrak{Q})$. By Theorem 27.20(i), choose $q \in Q$ such that $e_{\mathfrak{Q}}(q) \leq X$. Say $q = [p]$. Then $j(p) = e_{\mathfrak{Q}}(\pi(p)) \leq X$. So $j[P]$ is dense in RO$(\mathfrak{Q})$.

Suppose that $p, q \in P$ and $p \leq q$. Then $[p] \preceq [q]$, and so $j(p) \leq j(q)$ by Theorem 27.20(ii).

Suppose that $p, q \in P$. If $p \not\perp q$, choose $r \leq p, q$. Then $j(r) \leq j(p), j(q)$, so $j(p) \cap j(q) \neq \emptyset$. If $j(p) \cap j(q) \neq \emptyset$, then by Theorem 27.20(iii), $\pi(p) \not\perp \pi(q)$. So there is an $r \in Q$ such that $r \leq \pi(p), \pi(q)$. Say $r = \pi(s)$. Then $s \leq p, q$, so $p \not\perp q$.

This verifies the hypotheses of Theorem 27.22, and the desired conclusion follows. $\square$

Theorem 27.41. *The mapping e defined just before Theorem 27.20 is a dense embedding of $\mathbb{P}$ into RO$(\mathbb{P})\backslash\{0\}$.*

Proof. We need to check conditions (i)–(iii) and (v) given just before Proposition 25.67. (i) is clear. (ii), (iii), and (v) are given in Theorem 27.20 (ii), (iii), and (i). □

Theorem 27.42. *Let A be a Boolean algebra. For each $a \in A$ let $\mathcal{S}(a) = \{F \in \mathrm{Ult}(A) : a \in F\}$. Then $\mathrm{rng}(\mathcal{S})$ forms a base for a topology on $\mathrm{Ult}(A)$ under which $\mathrm{Ult}(A)$ becomes a compact zero-dimensional Hausdorff space, with $\mathrm{rng}(\mathcal{S})$ the set of all clopen subsets of $\mathrm{Ult}(A)$.*

Proof. If $F \in \mathcal{S}(a) \cap \mathcal{S}(b)$, then $a, b \in F$, hence $a \cdot b \in F$, so $F \in \mathcal{S}(a \cdot b) \subseteq \mathcal{S}(a) \cap \mathcal{S}(b)$. For any $F \in \mathrm{Ult}(A)$, choose $a \in F$; then $F \in \mathcal{S}(a)$. Hence $\mathrm{rng}(f)$ is a base for a topology on $\mathrm{Ult}(A)$.

To see that $\mathrm{Ult}(A)$ is compact, let $X \subseteq \mathrm{rng}(f)$ with fip. Say $X = \{\mathcal{S}(a) : a \in Y\}$. Then Y has the fip in A. Say $Y \subseteq F \in \mathrm{Ult}(A)$. Then $F \in \bigcap X$. This shows that $\mathrm{Ult}(A)$ is compact.

To see that, $\mathrm{Ult}(A)$ is Hausdorff, suppose that $F, G \in \mathrm{Ult}(A)$ with $F \neq G$. Say $a \in F \backslash G$. then $F \in \mathcal{S}(a)$, $G \in \mathcal{S}(-a)$, and $\mathcal{S}(a) \cap \mathcal{S}(-a) = \emptyset$. So $\mathrm{Ult}(A)$ is Hausdorff.

For each $a \in A$, $\mathcal{S}(a) = \mathrm{Ult}(A) \backslash \mathcal{S}(-a)$; so $\mathcal{S}(a)$ is clopen.

Let $U \subseteq \mathrm{Ult}(A)$ be clopen. For each $F \in U$ choose a_F such that $F \in \mathcal{S}(a_F) \subseteq U$. Then $U \subseteq \bigcup_{F \in U} \mathcal{S}(a_F)$. Hence there is a finite subset V of U such that $U \subseteq \bigcup_{F \in V} \mathcal{S}(a_F) = \mathcal{S}(\sum_{F \in V} a_F) \subseteq U$. So $U \in \mathrm{rng}(f)$.

Thus $\mathrm{Ult}(A)$ has a base consisting of clopen sets, so it is zero-dimensional. □

Proposition 27.43. *For any topological space X, let $\mathrm{clop}(X)$ be the set of all clopen subsets of X. Then $\mathrm{clop}(X)$ is a field of subsets of X.* □

Theorem 27.44. *If A is a BA, then A is isomorphic to $\mathrm{clop}(\mathrm{Ult}(A))$.*

Proof. The function $\mathcal{S}$ defined in the proof of Theorem 27.43 is the desired isomorphism. □

Theorem 27.45. *If X is a compact zero-dimensional Hausdorff space, then X is homeomorphic to $\mathrm{Ult}(\mathrm{clop}(X))$.*

Proof. For each $x \in X$ let $g(x)$ be the collection of all clopen subsets U of X such that $x \in U$. Clearly $g(x)$ is an ultrafilter on $\mathrm{clop}(X)$. We claim that g is the desired homeomorphism.

Suppose that V is a member of the base of $\mathrm{Ult}(\mathrm{clop}(X))$ and $x \in g^{-1}[V]$; we want to find an open W in X such that $x \in W \subseteq g^{-1}[V]$. Say $V = \mathcal{S}(a)$ with $a \in \mathrm{clop}(X)$. Since $x \in g^{-1}[V]$, we have $g(x) \in \mathcal{S}(a)$, hence $a \in g(x)$, hence $x \in a$. We claim that $a \subseteq g^{-1}[V]$. For, if $y \in a$ then $a \in g(y)$, hence $g(y) \in \mathcal{S}(a) = V$ hence $y \in g^{-1}[V]$. This shows that g is continuous.

g maps onto $\mathrm{Ult}(\mathrm{clop}(X))$. For, suppose that $F \in \mathrm{Ult}(\mathrm{clop}(X))$. Then $\bigcap F \neq \emptyset$, since F is a family of nonempty clopen sets closed under binary intersection, and X is compact. Take any $x \in \bigcap F$. If $a \in F$, then $x \in a$, so $a \in g(x)$. Thus $F \subseteq g(x)$, so $F = g(x)$.

g is one-one. For, suppose that $x, y \in X$ and $x \neq y$. Let a be clopen with $x \in a$ and $y \notin a$. Then $a \in g(x) \backslash g(y)$.

Now the theorem follows. $\square$

Lemma 27.46. (III.4.5) *For any BA A, $MA_{A\backslash\{0\}}(\kappa)$ holds iff* $\mathrm{Ult}(A)$ *is not the union of $\leq \kappa$ closed nowhere dense sets.*

Proof. $\Rightarrow$: Let $H_\alpha \subseteq \mathrm{Ult}(A)$ be a closed nowhere dense set for each $\alpha < \kappa$. We want to show that $\bigcup_{\alpha<\kappa} H_\alpha \neq \mathrm{Ult}(A)$. For each $\alpha < \kappa$ let $D_\alpha = \{q \in \mathrm{clop}(\mathrm{Ult}(A)) : q \cap H_\alpha = \emptyset\}$. Then D_α is dense; for suppose p is a nonempty clopen set. Then $p\backslash H_\alpha \neq \emptyset$ because H_α is nowhere dense, and $p\backslash H_\alpha$ is open since H_α is closed. Now we apply $\mathrm{MA}_{\mathrm{clop}(\mathrm{ult}(A))\backslash\{\emptyset\}}(\kappa)$ to all these dense sets to obtain a filter G on $\mathrm{clop}(\mathrm{Ult}(A))$ such that $G \cap D_\alpha \neq \emptyset$ for all $\alpha < \kappa$. Then $\bigcap G \cap H_\alpha = \emptyset$ for all $\alpha < \kappa$, so with $F \in \bigcap G$ we have $F \in \mathrm{Ult}(A)\backslash \bigcup_{\alpha<\kappa} H_\alpha$.

$\Leftarrow$: To prove $\mathrm{MA}_{A\backslash\{0\}}(\kappa)$, let $D_\alpha \subseteq A\backslash\{0\}$ be dense for all $\alpha < \kappa$. For each $\alpha < \kappa$ let $U_\alpha = \bigcup_{b\in D_\alpha} \mathcal{S}(b)$. So obviously U_α is open. It is also dense, for take any $\mathcal{S}(b)$ with $b \neq 0$. Choose $c \in D_\alpha$ such that $c \leq b$. Then $\mathcal{S}(c) \subseteq \mathcal{S}(b)$, showing that U_α is dense. Let $H_\alpha = \mathrm{Ult}(A)\backslash U_\alpha$ for each $\alpha < \kappa$. So H_α is closed and nowhere dense. Hence there is an ultrfilter $F \in \mathrm{Ult}(A)\backslash \bigcup_{\alpha<\kappa} H_\alpha$. Hence $F \in \bigcap_{\alpha<\kappa} U_\alpha$. For each $\alpha < \kappa$ there is a $b \in D_\alpha$ such that $F \in \mathcal{S}(b)$, so that $b \in F$. Thus F is a filter intersecting each D_α. $\square$

A topological space X is *extremally disconnected* iff for every open set $U \subseteq X$, also $\overline{U}$ is open.

Proposition 27.47. *If A is a complete BA, then* $\mathrm{Ult}(A)$ *is extremally disconnected.*

Proof. Let $U \subseteq \mathrm{Ult}(A)$ be open. Let $M = \{b \in A : \mathcal{S}(b) \subseteq U\}$. Thus $U = \bigcup_{b\in M} \mathcal{S}(b)$. Let $a = \sum M$. For any $b \in M$ we have $b \leq a$ and so $\mathcal{S}(b) \subseteq \mathcal{S}(a)$. Thus $U = \bigcup_{b\in M} \mathcal{S}(b) \subseteq \mathcal{S}(a)$. We claim that $\mathcal{S}(a) \subseteq \overline{U}$; hence $\mathcal{S}(a) = \overline{U}$ and $\overline{U}$ is open. To prove the claim, suppose to the contrary that $\mathcal{S}(a)\backslash\overline{U} \neq \emptyset$. Then there is a $c \neq 0$ such that $\mathcal{S}(c) \subseteq (\mathcal{S}(a)\backslash\overline{U})$. So $c \leq a$, and hence there is a $b \in M$ such that $c \cap b \neq 0$. Then $\mathcal{S}(c \cap b) \subseteq U$ but also $\mathcal{S}(c \cap b) \cap U = \emptyset$, contradiction. $\square$

Theorem 27.48. (III.4.7) *For any infinite cardinal κ the following are equivalent:*

(i) $MA(\kappa)$.

(ii) $MA_{A\backslash\{0\}}(\kappa)$ holds for any ccc BA A.

(iii) $MA_{A\backslash\{0\}}(\kappa)$ holds for any ccc complete BA A.

(iv) No ccc compact Hausdorff space is the union of $\leq \kappa$ closed nowhere dense sets.

(v) No ccc compact zero-dimensional Hausdorff space is the union of $\leq \kappa$ closed nowhere dense sets.

(vi) No ccc compact extremally disconnected Hausdorff space is the union of $\leq \kappa$ closed nowhere dense sets.

Proof. Obviously (i)$\Rightarrow$(ii)$\Rightarrow$(iii) and (iv)$\Rightarrow$(v)$\Rightarrow$(vi). (i)$\Rightarrow$(iv) by Lemma 25.13. (vi)$\Rightarrow$(iii): assume (vi), and suppose that A is a ccc complete BA. By Proposition 27.47, $\mathrm{Ult}(A)$ is extremally disconnected; and it is obviously ccc. By (vi), $\mathrm{Ult}(A)$ is not the union of $\leq \kappa$ closed nowhere dense sets. Now (iii) follows by Lemma 27.46. So it remains only to show that (iii)$\Rightarrow$(i). So assume (iii), and suppose that $\mathbb{P}$ is a ccc forcing poset. By Theorem 27.19, $\mathrm{RO}(\mathbb{P})$ is a complete BA. By Theorem 27.20(iii), $\mathrm{RO}(\mathbb{P})$ has ccc. Hence by (iii), $\mathrm{MA}_{\mathrm{RO}(\mathbb{P})}(\kappa)$ holds. Hence by Lemma 25.73 $\mathrm{MA}_{\mathbb{P}}(\kappa)$ holds. $\square$

Proposition 27.49. (III.4.11) *With e as defined before Theorem 27.20, $\forall p,q\in\mathbb{P}[e(p)\subseteq e(q)$ iff $\neg\exists r[r\le p$ and $r\perp q]]$.*

Proof. $\Rightarrow$: Suppose that $e(p)\subseteq e(q)$, $r\le p$, and $r\perp q$. By Theorem 27.20(v) there is an $s\le r$ such that $s\le q$, contradiction.

$\Leftarrow$: We apply Theorem 27.20(v). Suppose that $r\le p$. Then by our assumption we have $r\not\perp q$, so there is an $s\le r,q$, as desired. $\square$

Proposition 27.50. (III.4.12) *If x is a non-isolated point in a Hausdorff space X, p is an open set with $x\in p$, and $q=p\backslash\{x\}$, then $e(p)=e(q)$, where e is as as above for the poset $\mathbb{O}_X$.*

Proof. We will apply Proposition 27.49. Suppose that $r\subseteq p$, then $r\backslash\{x\}\subseteq r,q$, so $r\not\perp q$. So $e(p)\subseteq e(q)$. Suppose that $r\subseteq q$. Then also $r\subseteq p$, so $r\not\perp p$. Hence $e(q)\subseteq e(p)$. $\square$

Proposition 27.51. (III.4.13) *Consider the poset $\mathbb{P}$ used in the proof of Theorem 25.34. There are $p,q\in\mathbb{P}$ such that $q\le p$, $p\not\le q$, and $e(p)=e(q)$, where e is as above for the poset $\mathbb{P}$.*

Proof. Take any $Z,W\in\mathscr{E}$ with $Z\subset W$. Let $s_p=s_q=\emptyset$ and $W_p=\{Z\}$, $W_q=\{Z,W\}$. Thus $q\le p$ and $p\not\le q$. To show that $e(p)=e(q)$ we apply Proposition 27.49. Suppose that $r\le p$ and $r\perp q$. Then $s_r\backslash s_p\subseteq Z\subseteq W$. Let $s_t=s_r$ and $W_t=W_r\cup\{W\}$. Then $t\le r$ and $s_t\backslash s_q=s_r\backslash s_p\subseteq W$, so $t\le q$, contradiction.

Suppose that $r\le q$ and $r\perp p$. Since $q\le p$, this is a contradiction. $\square$

Proposition 27.52. (III.4.14) *If $|J|>1$, then $Fn(I,J,\omega)$ is separative.*

Proof. Suppose that $p\le q\le p$. Then $p\supseteq q\supseteq p$, so $p=q$. Suppose that $p\not\le q$. Thus $q\not\subseteq p$. So there is a pair $(i,j)\in q\backslash p$. If $i\in\mathrm{dmn}(p)$, then $p\perp q$, as desired. Suppose that $i\notin\mathrm{dmn}(p)$. Choose $k\in J\backslash\{j\}$. Then $p\cup\{(i,k)\}\le p$ and $(p\cup\{(i,k)\})\perp q$. $\square$

Proposition 27.53. (III.4.14) $\mathrm{RO}(\mathrm{Fn}(I,J,\omega))$ *is isomorphic to* $\mathrm{RO}({}^IJ)$.

Proof. We will use Theorem 27.22. For any $p\in\mathrm{Fn}(I,J,\omega)$ let $F(p)=\{f\in{}^IJ:p\subseteq f\}$. Note that $\{f\in{}^IJ:p\subseteq f\}$ is clopen, hence regular open. In fact, it is obviously open. Its complement is

$$\bigcup_{i\in\mathrm{dmn}(p)}\{f\in{}^IJ:f(i)=1-p(i)\},$$

which is open. Clearly $\mathrm{rng}(F)$ is dense in $\mathrm{RO}({}^IJ)$. If $p\le q$, then $F(q)\subseteq F(p)$. If $p\perp q$, clearly $F(p)\cap F(q)=\emptyset$. Conversely, if $p\not\perp q$, say $r\le p,q$. Then $p,q\subseteq r$, so $F(r)\subseteq F(p)\cap F(q)$, so that $F(p)\cap F(q)\neq\emptyset$. $\square$

Proposition 27.54. (III.4.15) *$\mathbb{P}$ is separative iff $\le$ is antisymmetric and $\forall p\in\mathbb{P}[p\!\downarrow'$ is regular open$]$.*

Proof. $\Rightarrow$: Assume that $\mathbb{P}$ is separative. Then by definition, $\le$ is antisymmetric. To show that $p\!\downarrow'$ is regular open it suffices to show that $p\!\downarrow'=e(p)$, with e as on page 489 for the poset $\mathbb{P}$. We will use Proposition 27.21(ii). Suppose that $q \in e(p)$ but $q \not\leq p$. Then there is an $r \leq q$ such that $r \perp p$, contradiction.

$\Leftarrow$: Assume the indicated condition, and suppose that $q \not\leq p$. Then $q \notin e(p)$, so by Proposition 27.21(ii), there is an $r \leq q$ such that $r \perp p$. $\square$

Proposition 27.55. (III.4.15) *If $\mathbb{P}$ is separative, then $e(p) = p\downarrow$, where e is as above for the poset $\mathbb{P}$.* $\square$

If A and B are complete BAs and A is a subalgebra of B, we say that it is a *complete subalgebra* iff $\forall X \subseteq A[\sum^A X = \sum^B X]$.

Proposition 27.56. (III.4.16) *Suppose that A and B are complete BAs, with $A \leq B$. Then $A \subseteq_c B$ iff A is a complete subalgebra of B.*

Proof. $\Rightarrow$: Assume that $A \subseteq_c B$. Take any $X \subseteq A$. Note that $\sum^B X \leq \sum^A X$. Let Y be an antichain in A maximal subject to the condition that $\forall y \in Y \exists x \in X[y \leq x]$. Then obviously $\sum^A Y \leq \sum^A X$. Actually $\sum^A Y = \sum^A X$. For, suppose that $\sum^A Y < \sum^A X$. Then

$$0 \neq \left(\sum^A X \cdot -\sum^A Y\right) = \left(\sum^A X \cdot -\sum^A Y\right) \cdot \sum^A X,$$

and so there is an $x \in X$ such that $(\sum^A X \cdot - \sum^A Y) \cdot x \neq 0$. Then $Y \cup \{(\sum^A X \cdot - \sum^A Y) \cdot x\}$ contradicts the maximality of Y. Now $Y \cup \{-\sum^A Y\}$ is a maximal antichain in A. For, suppose that $a \in A^+$ and $a \cdot z = 0$ for all $z \in Y \cup \{-\sum^A Y\}$. Then $y \leq -a$ for all $y \in Y$, so $\sum^A Y \leq -a$, hence $a \cdot \sum^A Y = 0$. But also $a \cdot -\sum^A Y = 0$, so $a = 0$, contradiction. It follows that $Y \cup \{-\sum^A Y\}$ is a maximal antichain in B. If $\sum^B X < \sum^A X$, then there is a $y \in Y$ such that $y \cdot \sum^A X \cdot -\sum^B X \neq 0$. Then there is an $x \in X$ such that $y \leq x$. Then $x \cdot \sum^A X \cdot -\sum^B X \neq 0$, contradicting $x \leq \sum^B X$. This shows that $\sum^B X = \sum^A X$. Hence $\prod^B X = -\sum^B\{y : -y \in X\} = -\sum^A\{y : -y \in X\} = \prod^A X$.

$\Leftarrow$: suppose that A is a complete subalgebra of B. Obviously $A \subseteq_{\mathrm{ctr}} B$. If X is a maximal antichain of A, then $\sum^A X = 1$; hence $\sum^B X = 1$ and X is a maximal antichain of B. $\square$

Lemma 27.57. (III.8.1) *If $\theta > \omega_1$ and $M \preceq H(\theta)$, then $\forall R \in M[|R| \leq \omega \to R \subseteq M]$.*

Proof. By Lemma 23.62. $\square$

Lemma 27.58. (III.8.1a) *If $\theta > \omega_1$ and $M \preceq H(\theta)$, then $\forall R \subseteq M[|R| < \omega \to R \in M]$.*

Proof. $H(\theta) \models \exists x \forall y \left[y \in x \leftrightarrow \bigvee_{a \in R}(y = a)\right]$, so $M \models \exists x \forall y \left[y \in x \leftrightarrow \bigvee_{a \in R}(y = a)\right]$, and this gives $R \in M$. $\square$

M is *countably closed* iff $[M]^\omega \subseteq M$.

Proposition 27.59. (233) *If θ is uncountable, $M \preceq H(\theta)$, and M is countably closed, then ${}^\omega M \subseteq M$.*

Proof. If $f : \omega \to M$, then f is a countable subset of $\omega \times M$. Now $\omega \in M$ by Lemma 23.57, and hence $\omega \subseteq M$ by Lemma 23.62. Hence $\omega \times M \subseteq M$ by Lemma 23.58. Hence $f \in [M]^\omega \subseteq M$. $\square$

Lemma 27.60. (III.8.4) *Let θ be uncountable and regular and $S \subseteq H(\theta)$ with $|S| \leq 2^\omega$. Then there is a countably closed $M \preceq H(\theta)$ such that $|M| = 2^\omega$ and $S \subseteq M$.*

Proof. Let $\mathscr{F}$ be the standard countable collection of Skolem functions, and let $g : {}^\omega H(\theta) \to \mathscr{P}(H(\theta))$ be defined by $g(a) = \{a_i : i \in \omega\}$. Let M be the closure of S under $\mathscr{F} \cup \{g\}$. $\square$

Theorem 27.61. (III.8.5) *If X is a first countable compact Hausdorff space, then $|X| \leq 2^\omega$.*

Proof. Let T be the collection of all open subsets of X. Let θ be a large cardinal such that $X, T \in H(\theta)$. Let M be countably closed with $M \preceq H(\theta)$, $X, T \in M$, and $|M| = 2^\omega$, using Lemma 27.60.

(1) $X \cap M$ is closed in X.

In fact, let $a \in X$ be a limit point of $X \cap M$. Let $f : \omega \to (X \cap M)$ converge to a. Then $f \in M$ since M is countably closed. Now $H(\theta) \models$ "f has a limit point", so $M \models$ "f has a limit point". It follows that $a \in M$, proving (1). Now take any $y \in X \cap M$. Then

$$\begin{aligned} H(\theta) \models &\exists \mathscr{U} \exists f[\mathscr{U} \subseteq T \text{ and } f : \omega \to \mathscr{U} \text{ is a surjection} \\ &\text{and } \forall U \in \mathscr{U}[y \in U \text{ and } \forall V \in T[y \in V \to \exists U \in \mathscr{U}[U \subseteq V]]]. \end{aligned}$$

Now $T, \omega, y \in M$, using Lemma 23.57. Hence we get $\mathscr{U}'_y, f'_y \in M$ such that

$$\begin{aligned} H(\theta) \models &[\mathscr{U}'_y \subseteq T \text{ and } f'_y : \omega \to \mathscr{U}'_y \text{ is a surjection} \\ &\text{and } \forall U \in \mathscr{U}'_y[y \in U \text{ and } \forall V \in T[y \in V \to \exists U \in \mathscr{U}'_y[U \subseteq V]]]. \end{aligned}$$

Hence $\mathscr{U}'_y \in M$ is a countable open neighborhood base for y, and $\mathscr{U}'_y \subseteq M$.

Suppose that $X \not\subseteq M$. Pick any $b \in X \backslash M$. For each $y \in X \cap M$ pick $V_y \in \mathscr{U}'_y$ such that $b \notin V_y$. By compactness there exist $n \in \omega$ and $y_i \in X \cap M$ for all $i < n$ such that $X \cap M \subseteq \bigcup_{i<n} V_{y_i}$. Now $b \notin V_{y_i}$ for all $i < n$. Hence

$$H(\theta) \models \exists x \in X[x \notin V_{y_0} \wedge \ldots \wedge x \notin V_{y_{n-1}}] \quad \text{but} \quad M \models \neg\exists x \in X[x \notin V_{y_0} \wedge \ldots \wedge x \notin V_{y_{n-1}}],$$

contradicting $M \preceq H(\theta)$. $\square$

Let θ be an uncountable regular cardinal. A sequence $\langle M_\xi : \xi < \omega_1 \rangle$ is a *nice chain of elementary submodels of* $H(\theta)$ provided that the following conditions hold:

(1) $M_0 = \emptyset$.
(2) $\forall \xi \in (0, \omega_1)[M_\xi \preceq H(\theta)]$.
(3) $\forall \xi < \omega_1[M_\xi$ is countable$]$.
(4) $\forall \xi, \eta < \omega_1[\xi < \eta \to M_\xi \in M_\eta$ and $M_\xi \subseteq M_\eta]$.

(5) $\forall$ limit $\eta < \omega_1 \left[M_\eta = \bigcup_{\xi<\eta} M_\xi\right]$.

For $x \in \bigcup_{\xi<\omega_1} M_\xi$, let $\mathrm{ht}(x)$ be the $\xi < \omega_1$ such that $x \in M_{\xi+1}\backslash M_\xi$.

Lemma 27.62. (III.8.15) *For any regular uncountable cardinal θ a nice chain of elementary substructures of $H(\theta)$ exists.*

Proof. By recursion, let $M_0 = \emptyset$, $M_{\xi+1}$ be a countable elementary substructure of $H(\theta)$ such that $M_\xi \cup \{M_\xi\} \subseteq M_{\xi+1}$, and for limit η let $M_\eta = \bigcup_{\xi<\eta} M_\xi$. $\square$

Lemma 27.63. (III.8.15a) *If $\langle M_\xi : \xi < \omega_1\rangle$ is a nice chain of elementary submodels of $H(\theta)$, then $M_\xi \cap \omega_1$ is an ordinal for each $\xi < \omega_1$, $\forall \xi, \eta[x < \eta < \omega_1 \to M_\xi \cap \omega_1 < M_\eta \cap \omega_1]$, and $\{M_\xi \cap \omega_1 : \xi < \omega_1\}$ is a club in ω_1.*

Proof. $M_\xi \cap \omega_1$ is an ordinal by Lemma 23.65. Now suppose that $\xi < \eta < \omega_1$. Then

$$\begin{aligned} H(\theta) \models &\exists\alpha[\alpha \text{ is an ordinal and } \forall\beta \in M_\xi[\beta \text{ is an ordinal } \to \beta < \alpha] \\ &\quad \text{and } \forall\gamma[\gamma \text{ is an ordinal and } \forall\beta \in M_\xi[\beta \text{ is an ordinal and } \beta < \gamma] \to \alpha \le \gamma] \end{aligned}$$

Since $M_\eta \preceq H(\theta)$ and $M_\xi \in M_\eta$, it follows that there is an $\alpha \in M_\eta$ such that

$$\begin{aligned} H(\theta) \models &[\alpha \text{ is an ordinal and } \forall\beta \in M_\xi[\beta \text{ is an ordinal } \to \beta < \alpha] \\ &\quad \text{and } \forall\gamma[\gamma \text{ is an ordinal and } \forall\beta \in M_\xi[\beta \text{ is an ordinal and } \beta < \gamma] \to \alpha \le \gamma] \end{aligned}$$

Thus α is the union of all ordinals in M_ξ, i.e., $\alpha = M_\xi \cap \omega_1$. Hence $M_\xi \cap \omega_1 < M_\eta \cap \omega_1$. Clearly then $\{M_\xi \cap \omega_1 : \xi < \omega_1\}$ is a club in ω_1. $\square$

Lemma 27.64. (III.8.15b) *If $\langle M_\xi : \xi < \omega_1\rangle$ is a nice chain of elementary submodels of $H(\theta)$, then $\forall \xi < \omega_1[\xi \le M_\xi \cap \omega_1]$.*

Proof. By induction on ξ, using Lemma 27.63. $\square$

Suppose that $f : [I]^2 \to 2$. We define forcing orders $\mathbb{P}_\mu = \mathbb{P}_\mu^f$ for $\mu \in 2$ as follows:

$$\mathbb{P}_\mu = \{p \in [I]^{<\omega} : \forall i, j \in p[i \ne j \to f(\{i, j\}) = \mu\}$$

The order is $\supseteq$. Note that trivially $\emptyset \in \mathbb{P}_\mu$ and $\{i\} \in \mathbb{P}_\mu$ for all $i \in I$.

Proposition 27.65. (239) *If I is arbitrary and $f : [I]^2 \to 2$ has constant value 0, then $\{\{i\} : i \in I\}$ is an antichain in $\mathbb{P}_1$.* $\square$

Proposition 27.66. (III.8.21) *Let $I \in [\mathbb{R}]^{\omega_1}$. Let $<$ be the usual order on $\mathbb{R}$, and $\lhd$ a well-order of $\mathbb{R}$. For $x, y \in I$ with $x \ne y$ define*

$$f(\{x, y\}) = \begin{cases} 0 & \textit{if } < \textit{ and } \lhd \textit{ agree on } \{x, y\}, \\ 1 & \textit{otherwise.} \end{cases}$$

Then neither $\mathbb{P}_0$ nor $\mathbb{P}_1$ has the ccc.

Proof. Let $\langle l_\alpha : \alpha < \omega_1 \rangle$ be a strictly $\triangleleft$-increasing sequence of elements of I. Define $\alpha \ll \beta$ iff $\alpha < \beta$ and $l_\alpha < l_\beta$. So $\ll$ is a well-founded relation. Suppose that $A \in [\omega_1]^\omega$ is pairwise incomparable under $\ll$. Then $\forall \alpha, \beta \in A[\alpha < \beta \to l_\beta < l_\alpha]$. This gives a strictly decreasing ω_1-sequence in $\mathbb{R}$, contradiction. So every pairwise incomparable set is finite. Thus $\mathbb{P}_0$ has ccc.

Suppose that $A \in [\omega_1]^{\omega_1}$ is pairwise incomparable in $\mathbb{P}_1$. Then $\forall \alpha, \beta \in A[\alpha < \beta \to l_\alpha < l_\beta]$. This gives a strictly increasing ω_1-sequence in $\mathbb{R}$, contradiction. $\square$

Lemma 27.67. (III.8.22) *If $f : [I]^2 \to 2$ and I is uncountable, then $\mathbb{P}_0 \times \mathbb{P}_1$ is not ccc.*

Proof. We claim that $\{(\{i\}, \{i\}) : i \in I\}$ is an antichain in $\mathbb{P}_0 \times \mathbb{P}_1$. For, suppose that $i, j \in I$ with $i \neq j$ and $(a, b) \leq (\{i\}, \{i\}), (\{j\}, \{j\})$. Thus $\{i, j\} \subseteq a, b$. Since $\{i, j\} \subseteq a$, we have $f(\{i, j\}) = 0$. Since $\{i, j\} \subseteq b$, we have $f(\{i, j\}) = 1$, contradiction. $\square$

Proposition 27.68. (III.8.23) *Let T be a Suslin tree, and let $f(\{i, j\}) = 0$ iff i and j are incomparable. Then $\mathbb{P}_1$ is ccc.*

Proof. Suppose that $\langle a_\alpha : \alpha < \omega_1 \rangle$ is an antichain in $\mathbb{P}_1$. Thus each a_α is a finite chain, and if $\alpha \neq \beta$ then $\exists i \in a_\alpha \exists j \in a_\beta[i$ and j are incomparable]. For each $\alpha < \omega_1$ let i_α be the top element of a_α. Then i_α and i_β are incomparable for $\alpha \neq \beta$, contradiction. $\square$

Lemma 27.69. (III.8.24) *Assume CH, let θ be a regular cardinal $> \omega_1$, and suppose that $\langle M_\xi : \xi < \omega_1 \rangle$ is a nice chain of elementary submodels of $H(\theta)$. Then $H(\omega_1) \subseteq \bigcup_{\xi < \omega_1} M_\xi$.*

Proof. $|H(\omega_1)| = 2^\omega = \omega_1$, so $H(\omega_1) \in H(\theta)$. If f is a surjection of ω_1 onto $H(\omega_1)$, then $|f| = \omega_1$ and so $f \in H(\theta)$. So $H(\theta) \models \exists f[f$ is a surjection from ω_1 onto $H(\omega_1)$. Since $M_1 \preceq H(\theta)$, it follows that there is a surjection $\Phi \in M_1$ from ω_1 onto $H(\omega_1)$. Hence $\Phi(\zeta) \in M_\xi$ whenever $\zeta < \xi$, by Lemma 27.64. The conclusion of the lemma follows. $\square$

Lemma 27.70. (III.8.25) *With $f : [I]^2 \to 2$, consider the forcing order $\mathbb{P}_\mu$ defined above. If $\mathbb{P}_\mu$ is not ccc, then it has an uncountable antichain whose members are pairwise disjoint.*

Proof. Let $\langle a_\xi : \xi < \omega_1 \rangle$ be an antichain. By the delta system lemma let $A' \in [\omega_1]^{\omega_1}$ be such that $\langle a_\xi : \xi \in A' \rangle$ forms a delta-system, say with kernel b. If $\xi, \eta \in A'$ and $\xi \neq \eta$, then there are $i \in a_\xi$ and $j \in a_\eta$ such that $f(\{i, j\}) = 1 - \mu$. Clearly $i, j \notin b$. Hence $\langle a_\xi \backslash b : \xi \in A' \rangle$ is as desired. $\square$

Theorem 27.71. (III.8.19) *CH implies that there are ccc forcing posets $\mathbb{P}, \mathbb{Q}$ such that $\mathbb{P} \times \mathbb{Q}$ is not ccc.*

Proof. We will define $f : [\omega_1]^2 \to 2$ so that $\mathbb{P}_\mu = \mathbb{P}_\mu^f$ is ccc for $\mu = 0, 1$. Then the result follows by Lemma 27.67.

Let θ be a large cardinal, and let $\langle M_\xi : \xi < \omega_1 \rangle$ be a nice family of elementary submodels of $H(\theta)$. We claim that for each $\xi < \omega_1$ there is a function $g_\xi : \xi \to 2$ such that the following conditions hold:

(1) If $E \in M_\xi$ and E is an infinite disjoint family of nonempty finite subsets of ξ, then for each $\mu \in 2$ there are infinitely many $q \in E$ such that $\forall \zeta \in q[g_\xi(\zeta) = \mu]$.

(2) $g_\xi \in M_{\xi+1}$.

The claim is vacuous if ξ is finite. Suppose that ξ is infinite. Let $\langle E_m : m \in \omega\rangle$ list all members of M_ξ which are infinite disjoint families of nonempty finite subsets of ξ, such that for each $F \in M_\xi$ which is an infinite disjoint family of nonempty finite subsets of ξ, the set $\{m \in \omega : E_m = F\}$ contains infinitely many even numbers and infinitely many odd numbers. Now we define $\langle q_l : l \in \omega\rangle$ by recursion; each $q_l \in E_l$. Suppose that q_l has been defined for all $l < m$. Then $\bigcup_{l<m} q_l$ is a finite subset of ξ. Since E_m is an infinite disjoint family of nonempty finite subsets of ξ, choose $q_m \in E_m$ so that $q_m \cap \bigcup_{l<m} q_l = \emptyset$. Thus $q_l \cap q_m = \emptyset$ for $l < m$. Now we define $g_\xi : \xi \to 2$ as follows:

$$g_\xi(\zeta) = \begin{cases} 0 & \text{if } \zeta \in q_m \text{ and } m \text{ is even,} \\ 1 & \text{if } \zeta \in q_m \text{ and } m \text{ is odd,} \\ 0 & \text{if } \zeta \in \xi \backslash \bigcup_{m\in\omega} q_m. \end{cases}$$

To check (1), suppose that $F \in M_\xi$ is an infinite disjoint family of nonempty finite subsets of ζ. For m even and $E_m = F$ we have $\forall \zeta \in q_m[q_\xi(\zeta) = 0]$. For m odd and $E_m = F$ we have $\forall \zeta \in q_m[q_\xi(\zeta) = 1]$. Then (1) follows. To check (2), note that $M(\theta) \models$"there is a function g_ξ satisfying (1)". Since $\xi, M_\xi \in M_{\xi+1}$ and $M_{\xi+1} \preceq H(\theta)$, we can choose $g_\xi \in M_{\xi+1}$.

Now for $\zeta < \xi < \omega_1$ we define $f(\{\zeta, \xi\}) = g_\xi(\zeta)$. Suppose that $\mathbb{P}^f_\mu$ is not ccc. Then by Lemma 27.70 let A be an uncountable antichain whose members are pairwise disjoint. Let E be any countably infinite subset of A. By Lemma 27.69 there is a $\beta < \omega_1$ such that $E \in M_\beta$. Then $\sup \bigcup E < M_\beta \cap \omega_1$. Take $p \in A$ such that $\min(p) > \beta$. Let $p = \{\xi_0, \ldots, \xi_{n-1}\}$ in increasing order. Let $\xi_n = \xi_{n-1} + 1$. Let $E_0 = E$, and for $l < n$ let $E_{l+1} = \{q \in E_l : \forall \zeta \in q[f(\{\zeta, \xi_l\}) = \mu]$.

(3) $\forall l \le n[E_l \in M_{\xi_l}$ and E_l is infinite].

We prove (3) by induction. It is given for $l = 0$. Assume that $E_l \in M_{\xi_l}$ and E_l is infinite, with $l < n$. We apply (1) and (2) to E_l and ξ_l. This gives that E_{l+1} is infinite and $g_{\xi_l} \in M_{\xi_l+1}$, hence $E_{l+1} \in M_{\xi_l+1} \subseteq M_{\xi_{l+1}}$. So (3) holds.

In particular, $E_n \neq \emptyset$. If $q \in E_n$, then $f(\{\zeta, \xi_l\}) = \mu$ for all $l < n$ and $\zeta \in q$. Hence q and p are compatible, contradicting $q, p \in A$, which is an antichain. □

Proposition 27.72. (III.8.26) *Assume* MA(ω_1) *and assume that* $|I| = \omega_1$. *Suppose that* $f : [I]^2 \to 2$ *and* $\mu \in 2$. *Assume that* $\mathbb{P}^f_\mu$ *is ccc. Then there is an uncountable* $Z \subseteq I$ *such that* $\forall\{i, j\} \in [Z]^2[f(\{i, j\}) = \mu]$.

Proof. Let $W = \{\{i\} : i \in I\}$. By Lemma 25.14, $\mathbb{P}^f_\mu$ has ω_1 as a pre-caliber. Hence there is a $B \in [I]^{\omega_1}$ such that $\{\{i\} : i \in B\}$ is centered. Thus if $i, j \in B$ with $i \neq j$ then there is an $a \in \mathbb{P}^f_\mu$ such that $\{i\}, \{j\} \subseteq a$. It follows that $f(\{i, j\}) = \mu$. □

Proposition 27.73. (III.8.27) *Assume* MA(ω_1) *and assume that* $|I| = \omega_1$. *Suppose that* $f : [I]^2 \to 2$ *and* $\mu \in 2$. *Assume that* $\mathbb{P}^f_\mu$ *is ccc. Then there are subsets* $Z_n \subseteq I$ *for* $n \in \omega$ *such that* $I = \bigcup_{n\in\omega} Z_n$ *and for each* n, $\forall\{i, j\} \in [Z_n]^2[f(\{i, j\}) = \mu$.

Proof. Let $\mathbb{Q} \subseteq {}^\omega\mathbb{P}_\mu$ be the finite support product of countably many copies of $\mathbb{P}_\mu$. By Theorem 25.50, $\mathbb{Q}$ is ccc. For each $i \in I$ let $D_i = \{p \in \mathbb{Q} : \exists n \in \omega[p_n = \{i\}]\}$. Clearly D_i is dense in $\mathbb{Q}$. By MA(ω_1) let G be a filter over $\mathbb{Q}$ intersecting each set D_i. For each $n \in \omega$ let $Z_n = \{i \in I : \exists p \in G[p_n = \{i\}]\}$. Then $I = \bigcup_{n\in\omega} Z_n$. Suppose that $n \in \omega$ and $\{i, j\} \in [Z_n]^2$. Say $p \in G$ with $p_n = \{i\}$ and $q \in G$ with $q_n = \{j\}$. Choose $r \in G$ with $r \le p, q$. Then $r_n \subseteq p_n, q_n$, so $i, j \in r_n$, hence $f(\{i, j\} = \mu$. □

Proposition 27.74. (Proposition III.8.28) *Assume CH, and let f be as in the proof of Theorem 27.71, and let $\mathbb{Q}_\mu$ be the finite support product of ω many copies of $\mathbb{P}^f_\mu$. Then each $\mathbb{Q}_\mu$ has ccc.*

Proof. We assume the notation in the proof of Theorem 27.71, through the definition of f. We claim

$$\forall E \in [[\omega_1]^{<\omega}]^\omega[E \text{ disjoint } \Rightarrow \exists\beta \in (\sup(\bigcup E), \omega_1)$$
$$\forall\mu \in 2\forall p \in [\omega_1\backslash\beta]^{<\omega}\exists q \in E\forall\zeta \in q\forall\xi \in p[f(\zeta, \xi) = \mu]]$$

In fact, suppose that $E \in [[\omega_1]^{<\omega}]^\omega$ and E is disjoint. By Lemma 27.69 there is a $\gamma < \omega_1$ such that $\sup\bigcup E < M_\gamma \cap \omega_1$. Let $\beta = M_\gamma \cap \omega_1$. Suppose that $\mu \in 2$ and $p \in [\omega_1\backslash\beta]^{<\omega}$. Let $p = \{\xi_0, \ldots, \xi_{n-1}\}$ in increasing order. Let $\xi_n = \xi_{n-1} + 1$. Let $E_0 = E$, and for $l < n$ let $E_{l+1} = \{q \in E_l : \forall\zeta \in q[f(\{\zeta, \xi_l\}) = \mu]$.

$(*)$ $\forall l \le n[E_l \in M_{\xi_l}$ and E_l is infinite].

We prove $(*)$ by induction. It is given for $l = 0$. Assume that $E_l \in M_{\xi_l}$ and E_l is infinite, with $l < n$. We apply (1) and (2) to E_l and ξ_l. This gives that E_{l+1} is infinite and $g_{\xi_l} \in M_{\xi_l+1}$, hence $E_{l+1} \in M_{\xi_l+1} \subseteq M_{\xi_{l+1}}$. So $(*)$ holds.

In particular, $E_n \ne \emptyset$. If $q \in E_n$, then $f(\{\zeta, \xi_l\}) = \mu$ for all $l < n$ and $\zeta \in q$. This proves the claim.

Now suppose that $\langle p^\xi : \xi < \omega_1\rangle$ is a system of elements of $\mathbb{Q}_\mu$; we want to show that there are distinct $\xi, \eta < \omega_1$ such that p^ξ and p^η are compatable. For each $\xi < \omega_1$, the *support* of p^ξ, support(p^ξ), is the set of $n \in \omega$ such that $p^\xi_n \ne \emptyset$. Since

$$\omega_1 = \bigcup_{F\in[\omega]^{<\omega}} \{\xi < \omega_1 : \text{support}(p^\xi) = F\},$$

there is an $F \in [\omega]^{<\omega}$ and an $A \in [\omega_1]^{\omega_1}$ such that $\forall\xi \in A[\text{support}(p^\xi) = F]$. For each $n \in F$ define $\xi \equiv_{nA} \eta$ iff $\xi, \eta \in A$ and $p^\xi_n = p^\eta_n$. If some $\equiv_{nA}$-class B has ω_1 elements, we can work with B instead of A. So after finitely many steps we arrive at a subset C of A of size ω_1 such that we can write $F = F' \cup F''$ where $F' \cap F'' = \emptyset$, $\forall n \in F'\exists a \in \mathbb{P}_\mu\forall\xi \in C[p^\xi_n = a]$, and $\forall n \in F''[|\{p^\xi_n : \xi \in C\}| = \omega_1]$. Now we can get a $D \in [C]^{\omega_1}$ such that for all $n \in F''$, $\langle p^\xi_n : \xi \in D\rangle$ is a Δ-system, say with kernel K_n. Write $F'' = \{n_i : i < k\}$ with $n_0 < \cdots < n_{k-1}$. Take a countable subset E of D, and apply the claim to $\{p^\xi_{n_0}\backslash K_0 : \xi \in E\}$, obtaining an ordinal β. Now choose η so that for all $\xi \in D$ with $\eta < \xi$ we have β less than each member of $p^\xi_{n_0}$ and $p^\xi_{n_1}$. Taking a countable subset E' of D with each member of E' greater than η, the claim implies that for each $\rho \in E'$

the element $p^\rho_{n_0}$ is compatible with some member of $\{p^\xi_{n_0}\backslash K_0 : \xi \in E\}$. Since $K_0 \subseteq p^\rho_{n_0}$, actually $p^\rho_{n_0}$ is compatible with some member of $\{p^\xi_{n_0} : \xi \in E\}$. Now we apply the claim to $\{p^\xi_{n_1}\backslash K_1 : \xi \in E'\}$, obtaining an ordinal β'. Now choose η' so that for all $\xi \in D$ with $\eta' < \xi$ we have β' less than each member of $p^\xi_{n_0}$ and $p^\xi_{n_1}$. Taking a countable subset E'' of D with each member of E'' greater than η', the claim implies that for each $\rho \in E''$ the element $p^\rho_{n_0}$ is compatible with some member of $\{p^\xi_{n_0} : \xi \in E\}$, and the element $p^\rho_{n_1}$ is compatible with some member of $\{p^\xi_{n_1} : \xi \in E'\}$. Continuing in this way, we finally see that there are distinct $\xi, \eta < \omega_1$ such that p^ξ and p^η are compatable. $\square$

28. Generic extensions and forcing

In this chapter we give the basic definitions and facts about generic extensions and forcing. Uses of these things will occupy much of remainder of these notes. We use "c.t.m." for "countable transitive model"; see Theorem 15.11. Let M be a c.t.m. of ZFC and let $\mathbb{P} = (P, \leq, 1) \in M$ be a forcing order. We say that G is $\mathbb{P}$-*generic over* M provided that the following conditions hold:

(1) G is a filter on $\mathbb{P}$.

(2) For every dense $D \subseteq P$ such that $D \in M$ we have $G \cap D \neq \emptyset$.

The definition of generic filter just given embodies a choice between two intuitive options. The option chosen corresponds to thinking of stronger conditions—those containing more information—as smaller in the forcing order. This may seen counter-intuitive, but it fits nicely with the embedding of forcing orders into Boolean algebras, as we will see. Many authors take the opposite approach, considering stronger conditions as the greater ones. Of course this requires a corresponding change in the definition of generic filter (and denseness).

The following is the basic existence lemma for generic filters.

Lemma 28.1. *If M is a c.t.m. of* ZFC*, $\mathbb{P} = (P, \leq, 1) \in M$ is a forcing order, and $p \in P$, then there is a G which is $\mathbb{P}$-generic over M and $p \in G$.*

Proof. Let $\langle D_n : n \in \omega \rangle$ enumerate all of the dense subsets of P which are in M. We now define a sequence $\langle q_n : n \in \omega \rangle$ by recursion. Let $q_0 = p$. If $q_n \in P$ has been defined, choose $q_{n+1} \in D_n$ with $q_{n+1} \leq q_n$. Thus $p = q_0 \geq q_1 \geq \cdots$. Now we define

$$G = \{r \in P : q_n \leq r \text{ for some } n \in \omega\}.$$

We check that G is as desired. For (1), suppose that $r, s \in G$. Say $m, n \in \omega$ with $q_m \leq r$ and $q_n \leq s$. By symmetry, say $m \leq n$. Then $q_n \leq r, s$, and $q_n \in G$, as desired.

Condition (2) is clear. Hence (3) holds.

For (4), let $n \in \omega$. Then $q_{n+1} \in G \cap D_n$, as desired. □

It is important to realize that usually generic filters are not in the ground model M; this is expressed in the following lemma.

Lemma 28.2. *Suppose that M is a c.t.m. of ZFC and $\mathbb{P} = (P, \leq, 1) \in M$ is a forcing order. Assume the following:*

(1) For every $p \in P$ there are $q, r \in P$ such that $q \leq p$, $r \leq p$, and $q \perp r$.

Also suppose that G is $\mathbb{P}$-generic over M.

Then $G \notin M$.

Proof. Suppose to the contrary that $G \in M$. Then also $P \backslash G \in M$, since M is a model of ZFC and by absoluteness. We claim that $P \backslash G$ is dense. In fact, given $p \in P$, choose q, r as in (1). Then q, r cannot both be in G, by the definition of filter. So one at least is in $P \backslash G$, as desired. Since $P \backslash G$ is dense and in M, we contradict G being generic. $\square$

Most forcing orders used in forcing arguments satisfy the condition of Lemma 28.2; for more details on this lemma, see later in this chapter.

If $\mathbb{P}$ is a forcing order, a subset E of $\mathbb{P}$ is *predense* iff every $p \in \mathbb{P}$ is compatible with some member of E.

The following elementary proposition gives six equivalent ways to define generic filters.

Proposition 28.3. *Suppose that M is a c.t.m. of ZFC and $\mathbb{P}$ is a forcing order in M. Suppose that $G \subseteq P$ satisfies condition (2), i.e., if $p \in G$ and $p \leq q$, then $q \in G$. Then the following conditions are equivalent:*

(i) $G \cap D \neq \emptyset$ whenever $D \in M$ and D is dense in $\mathbb{P}$.

(ii) $G \cap A \neq \emptyset$ whenever $A \in M$ and A is a maximal antichain of $\mathbb{P}$.

(iii) $G \cap E \neq \emptyset$ whenever $E \in M$ and E is predense in $\mathbb{P}$.

Moreover, suppose that G satisfies (2) and one, hence all, of the conditions (i)–(iii). Then G is $\mathbb{P}$-generic over M iff the following condition holds:

(iv) For all $p, q \in G$, p and q are compatible.

Proof. (i)$\Rightarrow$(ii): Assume (i), and suppose that $A \in M$ is a maximal antichain of $\mathbb{P}$. Let $D = \{p \in P : p \leq q \text{ for some } q \in A\}$. We claim that D is dense. Suppose that r is arbitrary. Choose $q \in A$ such that r and q are compatible. Say $p \leq r, q$. Thus $p \in D$. So, indeed, D is dense. Clearly $D \in M$, since $A \in M$. By (i), choose $p \in D \cap G$. Say $p \leq q \in A$. Then $q \in G \cap A$, as desired.

(ii)$\Rightarrow$(iii): Assume (ii), and suppose that E is as in (iii). By Zorn's lemma, let A be a maximal member of

(1) $\{B \subseteq P : B \text{ is an antichain, and for every } p \in B \text{ there is a } q \in E \text{ such that } p \leq q\}$.

We claim that A is a maximal antichain. For, suppose that $p \perp q$ for all $q \in A$. Choose $s \in E$ such that p and s are compatible. Say $r \leq p, s$. Hence $r \perp q$ for all $q \in A$, so $r \notin A$. Thus $A \cup \{r\}$ is a member of (1), contradiction.

Clearly $A \in M$, since $E \in M$. So, since A is a maximal antichain, choose $p \in A \cap G$. Then choose $q \in E$ such that $p \leq q$. So $q \in E \cap G$, as desired.

(iii)$\Rightarrow$(i): Obvious.

Now we assume (2) in the definition, and (i)–(iii).

If G is $\mathbb{P}$-generic over M, clearly (iv) holds.

Now asume that (i)–(iv) hold, and suppose that $p, q \in G$; we want to find $r \in G$ such that $r \leq p, q$. Let

$$D = \{r : r \perp p \text{ or } r \perp q \text{ or } r \leq p, q\}.$$

We claim that D is dense in $\mathbb{P}$. For, let $s \in P$ be arbitrary. If $s \perp p$, then $s \leq s$ and $s \in D$, as desired. So suppose that s and p are compatible; say $t \leq s, p$. If $t \perp q$, then $t \leq s$ and $t \in D$, as desired. So suppose that t and q are compatible. Say $r \leq t, q$. Then $r \leq t \leq p$ and $r \leq t \leq s$, so $r \leq s$ and $r \leq p, q$, hence $r \in D$, as desired. This proves that D is dense.

Now by (i) choose $r \in D \cap G$. By (iv), r is compatible with p and r is compatible with q. So $r \leq p, q$, as desired. $\square$

We are going to define the generic extension $M[G]$ by first defining *names* in M, and then producing the elements of $M[G]$ by using those names. The notion of a name is defined by recursion, using the following theorem.

Theorem 28.4. *Let P be any set. Then there is a function $\mathbf{F} : \mathbf{V} \to \mathbf{V}$ such that for any set τ,*

$$\mathbf{F}(\tau) = \begin{cases} 1 & \text{if } \tau \text{ is a relation and for all } \sigma, p \\ & \text{if } (\sigma, p) \in \tau \text{ then } p \in P \text{ and } \mathbf{F}(\sigma) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $\mathbf{R} = \{(\sigma, \tau) : \exists p \in P[(\sigma, p) \in \tau]\}$. Then $\mathbf{R}$ is well-founded on $\mathbf{V}$. In fact, let X be any nonempty set, and choose $\tau \in X$ of smallest rank. If $\sigma \mathbf{R} \tau$, then there is a $p \in P$ such that $(\sigma, p) \in \tau$, and then $\sigma \in \{\sigma\} \in \{\{\sigma\}, \{\sigma, p\}\} = (\sigma, p) \in \tau$, and hence $\text{rank}(\sigma) < \text{rank}(\tau)$. It follows that $\sigma \notin X$, as desired.

Also, $\mathbf{R}$ is set-like on $\mathbf{V}$. In fact, for any set τ we have

$$\text{pred}_{\mathbf{VR}}(\tau) = \{\sigma : \exists p \in P[(\sigma, p) \in \tau]\} = \left\{\sigma \in \bigcup\bigcup \tau : \exists p \in P[(\sigma, p) \in \tau]\right\}.$$

Now we define $\mathbf{G} : \mathbf{V} \times \mathbf{V} \to \mathbf{V}$ by setting

$$\mathbf{G}(\tau, f) = \begin{cases} 1 & \text{if } \tau \text{ is a relation, } f \text{ is a function with domain} \\ & \text{pred}_{\mathbf{VR}}(\tau), \text{ and } f(\sigma) = 1 \text{ for all } \sigma \in \text{pred}_{\mathbf{VR}}(\tau) \\ 0 & \text{otherwise.} \end{cases}$$

Now we obtain $\mathbf{F}$ by Theorem 5.7: for any set τ,

$$\begin{aligned} \mathbf{F}(\tau) &= \mathbf{G}(\tau, \mathbf{F} \restriction \text{pred}_{\mathbf{VR}}(\tau)) \\ &= \begin{cases} 1 & \text{if } \tau \text{ is a relation and } \mathbf{F}(\sigma) = 1 \\ & \text{for all } \sigma \in \text{pred}_{\mathbf{VR}}(\tau) \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} 1 & \text{if } \tau \text{ is a relation and for all } \sigma \text{ and } p \in P, \text{ if} \\ & (\sigma, p) \in \tau \text{ then } \mathbf{F}(\sigma) = 1, \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

□

Now with $\mathbf{F}$ as in this theorem, a P name is a set τ such that $\mathbf{F}(\tau) = 1$.

Corollary 28.5. *Let P be any set. Then τ is a P-name iff τ is a relation and for all $(\sigma, p) \in \tau[\sigma$ is a P-name and $p \in P]$.*

Proof. $\Rightarrow$: suppose that τ is a P-name. Thus $\mathbf{F}(\tau) = 1$, so τ is a relation, and for all $(\sigma, p) \in \tau$, $\mathbf{F}(\sigma) = 1$ and $p \in P$. Hence for all $(\sigma, p) \in \tau[\sigma$ is a P-name and $p \in P]$.

Conversely, suppose that τ is a relation and for all $(\sigma, p) \in \tau[\sigma$ is a P-name and $p \in P]$. Then τ is a relation and for all $(\sigma, p) \in \tau[\mathbf{F}(\sigma) = 1$ and $p \in P]$. Hence $\mathbf{F}(\tau) = 1$, so τ is a P-name. □

Note that "τ is a P-name" is absolute.

For any set P, we denote by $\mathbf{V}^P$ the (proper) class of all P-names. If M is a c.t.m. of ZFC, then we let $M^P = \mathbf{V}^P \cap M$. Note by absoluteness that

$$M^P = \{\tau \in M : (\tau \text{ is a } P\text{-name})^M\}.$$

If $G \subseteq P$, we define $\mathrm{val}(\tau, G)$ by recursion.

Theorem 28.6. *If $G \subseteq P$ then there is a function* val *such that for any set τ,*

$$\mathrm{val}(\tau, G) = \{\mathrm{val}(\sigma, G) : \exists p \in G[(\sigma, p) \in \tau]\}.$$

Proof. Let $\mathbf{R}$ be as in the proof of Theorem 28.4. Now we define $\mathbf{G} : \mathbf{V} \times \mathbf{V} \to \mathbf{V}$ by setting

$$\mathbf{G}(\tau, f) = \begin{cases} \{f(\sigma) : \exists p \in G[(\sigma, p) \in \tau]\} & \text{if } f \text{ is a function} \\ & \text{with domain } \mathrm{pred}_{\mathbf{VR}}(\tau), \\ 0 & \text{otherwise.} \end{cases}$$

Now we obtain $\mathbf{F}$ by Theorem 5.7; for any set τ,

$$\begin{aligned} \mathbf{F}(\tau) &= \mathbf{G}(\tau, \mathbf{F} \restriction \mathrm{pred}_{\mathbf{VR}}(\tau)) \\ &= \{\mathbf{F}(\sigma) : \exists p \in G[(\sigma, p) \in \tau]\} \end{aligned}$$ □

We also write τ_G in place of $\mathrm{val}(\tau, G)$. Notice that val is absolute for c.t.m. of ZFC.

Finally, if M is a c.t.m. of ZFC and $G \subseteq P \in M$, we define

$$M[G] = \{\tau_G : \tau \in M^P\}.$$

Note that $M^P \subseteq M$, and hence M^P is countable. Hence by the replacement axiom, $M[G]$ is also countable.

Now we can sketch the goals of chapters 28 and 29. We start with a c.t.m. M, and take $\kappa \in M$ such that κ is regular and greater than ω_1 (in the sense of M). Then we let $\mathbb{P}$ be the forcing order $(P, \supseteq, \emptyset)$, where P is the set of all finite functions contained in $\kappa \times 2$. Let G be any generic filter over $\mathbb{P}$. Then we show that $M[G]$ is a model of ZFC, in $M[G]$ we have $2^\omega = \kappa$, and cardinals in M and in $M[G]$ are the same. This shows the consistency of $\neg$CH.

On the other hand, starting with a c.t.m. M, we let $\mathbb{P}$ be the forcing order $(P, \supseteq, \emptyset)$ with P the set of all countable functions contained in $\omega_1 \times 2$. Then we show that $M[G]$ is a model of ZFC, in $M[G]$ we have $2^\omega = \omega_1$, and ω_1 is the same in M and $M[G]$. This proves the consistency of CH.

Lemma 28.7. *If M is a c.t.m. of ZFC, $\mathbb{P} \in M$ is a forcing order, and G is a filter on $\mathbb{P}$, then $M[G]$ is transitive.*

Proof. Suppose that $x \in y \in M[G]$. Then there is a $\tau \in M^P$ such that $y = \tau_G$. Since $x \in \tau_G$, there is a $\sigma \in M^P$ such that $x = \sigma_G$. So $x \in M[G]$. □

The following Lemma says that $M[G]$ is the smallest c.t.m. of ZFC which contains M as a subset and G as a member, once we show that it really is a model of ZFC. This lemma will be extremely useful in what follows.

Lemma 28.8. (IV.2.19) *Suppose that M is a c.t.m. of ZFC, $\mathbb{P} \in M$ is a forcing order, G is a filter on $\mathbb{P}$, N is a c.t.m. of ZFC, $M \subseteq N$, and $G \in N$. Then $M[G] \subseteq N$.*

Proof. Take any $x \in M[G]$. Say $x = \mathrm{val}(\sigma, G)$ with $\sigma \in M^{\mathbb{P}}$. Then $\sigma, G \in N$, so by absoluteness, $x = (\mathrm{val}(\sigma, G))^N \in N$. $\square$

To show that M is a subset of $M[G]$, we need a function $\check{}$ mapping M into the collection of all P-names. Again the definition is by recursion.

Theorem 28.9. *Suppose that $(P, \leq, 1)$ is a forcing order. Then there is a function $\mathbf{F} : \mathbf{V} \to \mathbf{V}$ such that for every set x, $\mathbf{F}(x) = \{(\mathbf{F}(y), 1) : y \in x\}$.*

Proof. Let $\mathbf{R} = \{(y, x) : y \in x\}$. Clearly $\mathbf{R}$ is well-founded and set-like on $\mathbf{V}$. Define $\mathbf{G} : \mathbf{V} \times \mathbf{V} \to \mathbf{V}$ by

$$\mathbf{G}(x, f) = \begin{cases} \{(f(y), 1) : y \in x\} & \text{if } f \text{ is a function with domain } x, \\ \emptyset & \text{otherwise.} \end{cases}$$

Let $\mathbf{F}$ be obtained from $\mathbf{G}$ by Theorem 5.7. Then for any set x,

$$\mathbf{F}(x) = \mathbf{G}(x, \mathbf{F} \restriction \mathrm{pred}_{\mathbf{VR}}(x)) = \{(\mathbf{F}(y), 1) : y \in x\}. \qquad \square$$

We denote $\mathbf{F}(x)$ by $\check{x}$. Thus for any set x,

$$\check{x} = \{(\check{y}, 1) : y \in x\}.$$

Note that this depends on $\mathbb{P}$; we could denote it by $\mathrm{check}(\mathbb{P}, x)$ to bring this out, if necessary. Again this function is absolute for transitive models of ZFC.

Lemma 28.10. *Suppose that M is a transitive model of ZFC, $\mathbb{P} \in M$ is a forcing order, and G is a non-empty filter on P. Then*

(i) For all $x \in M$, $\check{x} \in M^P$ and $\mathrm{val}(\check{x}, G) = x$.

(ii) $M \subseteq M[G]$.

Proof. Absoluteness implies that $\check{x} \in M^P$ for all $x \in M$. To prove $\mathrm{val}(\check{x}, G) = x$ for all x, suppose that this is not true, and by the foundation axiom take x such that $\mathrm{val}(\check{x}, G) = x$ while $\mathrm{val}(\check{y}, G) = y$ for all $y \in x$. (See Theorem 5.5.) Then

$$\begin{aligned} \mathrm{val}(\check{x}, G) &= \{\mathrm{val}(\sigma, G) : (\sigma, 1) \in \check{x}\} \\ &= \{\mathrm{val}(\check{y}, G) : y \in x\} \\ &= \{y : y \in x\} \\ &= x, \end{aligned}$$

contradiction.

Finally (ii) is immediate from (i). □

Next, for any partial order $\mathbb{P}$ we define a P-name Γ. It depends on $\mathbb{P}$ and could be defined as $\Gamma_{\mathbb{P}}$ to bring this out.

$$\Gamma = \{(\check{p}, p) : p \in P\}.$$

Lemma 28.11. *Suppose that M is a transitive model of ZFC, $\mathbb{P} \in M$ is a forcing order, and G is a non-empty filter on P. Then $\Gamma_G = G$. Hence $G \in M[G]$.*

Proof. $\Gamma_G = \{\check{p}_G : p \in G\} = \{p : p \in G\} = G$. □

We also introduce a name for M. Let $\text{dmn}(\check{M}) = \{\check{a} : a \in M\}$, and for any $a \in M$, $\check{M}(\check{a}) = 1$.

Lemma 28.12. *(i) $p \Vdash \dot{a} \in \check{M}$ iff $\forall q \le p \exists b \in M \exists r \le q[r \Vdash \dot{a} = \check{b}]$.*

(ii) If G is generic, then $\check{M}^G = \{\check{a}^G : a \in M\} = \{a : a \in M\} = M$.

(ii): clear. (i):

$$\begin{aligned} p \Vdash \dot{a} \in \check{M} \quad &\text{iff} \quad e(p) \le ||\dot{a} \in \check{M}|| \\ &\text{iff} \quad e(p) \le \sum_{b \in M} ||\dot{a} = \check{b}|| \\ &\text{iff} \quad \forall q \le p \exists b \in M[e(q) \cdot ||\dot{a} = \check{b}|| \ne 0] \\ &\text{iff} \quad \forall q \le p \exists b \in M \exists r \le q[e(r) \le ||\dot{a} = \check{b}|| \\ &\text{iff} \quad \forall q \le p \exists b \in M \exists r \le q[r \Vdash \dot{a} = \check{b}] \end{aligned}$$ □

Lemma 28.13. *Suppose that M is a transitive model of ZFC, $\mathbb{P} \in M$ is a forcing order, and G is a non-empty filter on P. Then $\text{rank}(\tau_G) \le \text{rank}(\tau)$ for all $\tau \in M^P$.*

Proof. We prove this by induction on τ. Suppose that it is true for all $\sigma \in \text{dmn}(\tau)$. If $x \in \tau_G$, then there is a $(\sigma, p) \in \tau$ such that $p \in G$ and $x = \sigma_G$. Hence by the inductive assumption, $\text{rank}(x) \le \text{rank}(\sigma)$. Hence

$$\text{rank}(\tau_G) = \sup_{x \in \tau_G} (\text{rank}(x) + 1) \le \text{rank}(\tau).$$ □

Note by absoluteness of the rank function that $\text{rank}(\tau)$ is the same within M or $M[G]$.

Lemma 28.14. *Suppose that M is a transitive model of ZFC, $\mathbb{P} \in M$ is a forcing order, and G is a non-empty filter on $\mathbb{P}$. Then M and $M[G]$ have the same ordinals.*

Proof. Since $M \subseteq M[G]$, every ordinal of M is an ordinal of $M[G]$. Now suppose that α is any ordinal of $M[G]$. Write $\alpha = \tau_G$, where $\tau \in M^P$. Now $\text{rank}(\tau) = \text{rank}^M(\tau) \in M$. So by Lemma 28.12, $\alpha = \text{rank}(\alpha) = \text{rank}(\tau_G) \le \text{rank}(\tau) \in M$, so $\alpha \in M$. □

The following lemma will be used often.

Lemma 28.15. *Suppose that M is a transitive model of ZFC, $\mathbb{P} \in M$ is a forcing order, $E \subseteq P$, $E \in M$, and G is a $\mathbb{P}$-generic filter over M. Then:*

(i) Either $G \cap E \neq \emptyset$, or there is a $q \in G$ such that $r \perp q$ for all $r \in E$.

(ii) If E is dense below p and $q \leq p$, then E is dense below q.

(iii) If $p \in G$ and E is dense below p, then $G \cap E \neq \emptyset$.

Proof. Let

$$D = \{p : p \leq r \text{ for some } r \in E\} \cup \{q : q \perp r \text{ for all } r \in E\}.$$

We claim that D is dense. For, suppose that $q \in P$. We may assume that $q \notin D$. So q is not in the second set defining D, and so there is an $r \in E$ which is compatible with q. Take p with $p \leq q, r$. then $p \in D$ and $p \leq q$, as desired.

Since D is dense, we can choose $s \in G \cap D$. Now to prove (i), suppose that $G \cap E = \emptyset$. Then s is not in the first set defining D, so it is in the second set, as desired.

(ii) is clear.

For (iii), suppose that $G \cap E = \emptyset$, and by (i) choose $q \in G$ such that $q \perp r$ for every $r \in E$. By the definition of filter, there is a $t \in G$ with $t \leq p, q$. Since E is dense below p, there is then a $u \in E$ with $u \leq t$. Thus $u \leq q$, so it is not the case that $u \perp q$, contradiction. $\square$

Proposition 28.16. *Suppose that M is a transitive model of ZFC, $\mathbb{P} \in M$ is a forcing order, and G is a $\mathbb{P}$-generic filter over M. Suppose that $p \in P$ and p is compatible with each member of G. Then $p \in G$.*

Proof. The set $\{q \in P : q \leq p \text{ or } q \perp p\}$ is clearly dense in $\mathbb{P}$. $\square$

Now we introduce the idea of forcing. Recall that the logical primitive notions are $\neg$, $\rightarrow$, $\forall$, and $=$.

With each formula $\varphi(v_0, \ldots, v_{m-1})$ of the language of set theory we define another formula

$$p \Vdash_{\mathbb{P},M} \varphi(\sigma_0, \ldots, \sigma_{m-1}),$$

which we read as "p forces $\varphi(\sigma_0, \ldots, \sigma_{m-1})$ with respect to $\mathbb{P}$ and M"; it is the statement

$\mathbb{P}$ is a forcing order, $\mathbb{P} \in M$, $\sigma_0, \ldots, \sigma_{m-1} \in M^P$, $p \in P$, and for every G which is $\mathbb{P}$-generic over M, if $p \in G$, then the relativization of φ to $M[G]$ holds for the elements $\sigma_{0G}, \ldots, \sigma_{(m-1)G}$.

Note that since G in general is not in the model M, this definition cannot be given in M. The main aim of the next part of this chapter is to show that the definition is equivalent to one which is definable in any countable transitive model of ZFC. We do this by defining a notion $\Vdash^*$ in M, and then proving the equivalence of $\Vdash^*$ with $\Vdash$. For the definition we first define a function to take care of atomic φ.

To understand the following theorem, see the definition and corollary following its proof.

Theorem 28.17. *Let $\mathbb{P}$ be a forcing order and e the embedding of $\mathbb{P}$ into* $\mathrm{RO}(\mathbb{P})$*. Then there is a class function F mapping $2\times\mathbf{V}^P\times\mathbf{V}^P$ into* $\mathrm{RO}(\mathbb{P})$ *such that for any $\sigma,\tau\in\mathbf{V}^P$,*

$$\begin{aligned}F(0,\sigma,\tau) &= \prod_{(\xi,p)\in\tau}[-e(p)+F(1,\xi,\sigma)]\cdot\prod_{(\eta,q)\in\sigma}[-e(q)+F(1,\eta,\tau)]\\ F(1,\sigma,\tau) &= \sum_{(\xi,p)\in\tau}[e(p)\cdot F(0,\sigma,\xi)].\end{aligned}$$

Proof. We are going to apply the recursion theorem 5.7. Let $\mathbf{A}=2\times\mathbf{V}^P\times\mathbf{V}^P$. Let

$$\begin{aligned}(\delta',\sigma',\tau')\mathbf{R}(\delta,\sigma,\tau)\quad\text{iff}\quad &(\delta',\sigma',\tau'),(\delta,\sigma,\tau)\in\mathbf{A},\text{ and}\\ &[\delta'=1,\delta=0,\tau'=\sigma\text{ and }\mathrm{rank}(\sigma')<\mathrm{rank}(\tau)]\text{ or}\\ &[\delta'=1,\delta=0,\tau'=\tau\text{ and }\mathrm{rank}(\sigma')<\mathrm{rank}(\sigma)]\text{ or}\\ &[\delta'=0,\delta=1,\sigma'=\sigma\text{ and }\mathrm{rank}(\tau')<\mathrm{rank}(\tau)]\end{aligned}$$

We claim that $\mathbf{R}$ is well-founded on $\mathbf{A}$. In fact, note that if $(0,\sigma'',\tau'')\mathbf{R}(1,\sigma',\tau')\mathbf{R}(0,\sigma,\tau)$, then

$$\begin{aligned}&\sigma''=\sigma',\ \mathrm{rank}(\tau'')<\mathrm{rank}(\tau')\text{ and}\\ &\quad[(\tau'=\sigma\text{ and }\mathrm{rank}(\sigma')<\mathrm{rank}(\tau))\\ &\quad\text{or }(\tau'=\tau\text{ and }\mathrm{rank}(\sigma')<\mathrm{rank}(\sigma))].\end{aligned}$$

Hence one of the following two conditions holds:

(1) $\tau'=\sigma$, $\mathrm{rank}(\sigma')<\mathrm{rank}(\tau)$, $\sigma''=\sigma'$, and $\mathrm{rank}(\tau'')<\mathrm{rank}(\tau')$.

(2) $\tau'=\tau$, $\mathrm{rank}(\sigma')<\mathrm{rank}(\tau)$, $\sigma''=\sigma'$, and $\mathrm{rank}(\tau'')<\mathrm{rank}(\tau')$.

In either case we clearly have $\max(\mathrm{rank}(\sigma''),\mathrm{rank}(\tau''))<\max(\mathrm{rank}(\sigma),\mathrm{rank}(\tau)$. Hence there does not exist a sequence $\cdots a_2\mathbf{R}a_1\mathbf{R}a_0$. Hence $\mathbf{R}$ is well-founded on $\mathbf{A}$.

Next we claim that $\mathbf{R}$ is set-like on $\mathbf{A}$. For, let $(\delta,\sigma,\tau)\in\mathbf{A}$. Say $\sigma\in V_\alpha$ and $\tau\in V_\beta$. Then if $\delta=0$ we have

$$\begin{aligned}\mathrm{pred}_{\mathbf{AR}}(0,\sigma,\tau)&=\{(1,\sigma',\tau')\in 2\times V^P\times V^P:[\tau'=\sigma\text{ and }\mathrm{rank}(\sigma')<\mathrm{rank}(\tau)]\text{ or}\\ &\qquad[\tau'=\tau\text{ and }\mathrm{rank}(\sigma')<\mathrm{rank}(\sigma)]\}\\ &=\{(\delta',\sigma',\tau')\in 2\times V_\beta\times\{\sigma\}:\delta'=1\text{ and }\sigma'\in V^P\\ &\qquad\text{and }\mathrm{rank}(\sigma')<\mathrm{rank}(\tau)\}\cup\\ &\qquad\{(\delta',\sigma',\tau')\in 2\times V_\alpha\times\{\tau\}:\delta'=1\text{ and }\sigma'\in V^P\\ &\qquad\text{and }\mathrm{rank}(\sigma')<\mathrm{rank}(\sigma)\}.\end{aligned}$$

If $\delta=1$, then

$$\begin{aligned}\mathrm{pred}_{\mathbf{AR}}(1,\sigma,\tau)&=\{(0,\sigma',\tau'):\sigma'=\sigma\text{ and }\mathrm{rank}(\tau')<\mathrm{rank}(\tau)\}\\ &=\{(\delta',\sigma',\tau')\in 2\times\{\sigma\}\times V_\beta:\delta'=0\text{ and }\tau'\in V^P\\ &\qquad\text{and }\mathrm{rank}(\tau')<\mathrm{rank}(\sigma)\}.\end{aligned}$$

This proves the claim.

Now we define $\mathbf{G} : 2 \times \mathbf{V}^P \times \mathbf{V}^P \to \mathbf{V}$. Let $\delta \in 2$, $\sigma \in \mathbf{V}^P$, $\tau \in \mathbf{V}^P$, and suppose that f is a function mapping $\mathrm{pred}_{\mathbf{AR}}(\delta, \sigma, \tau)$ into $\mathrm{RO}(\mathbb{P})$. Then we set

$$\text{for } \delta = 0: \quad \mathbf{G}(0, \sigma, \tau, f) = \prod_{(\xi,p)\in\tau} [-e(p) + f(1, \xi, \sigma)] \cdot \prod_{(\eta,q)\in\sigma} [-e(q) + f(1, \eta, \tau)];$$

$$\text{for } \delta = 1: \quad \mathbf{G}(1, \sigma, \tau, f) = \sum_{(\xi,p)\in\tau} [e(p) \cdot f(0, \sigma, \xi)].$$

Note that this makes sense, since $(\xi, p) \in \tau$ implies that $(1, \xi, \sigma)\mathbf{R}(0, \sigma, \tau)$, $(1, \eta, \tau)\mathbf{R}(0, \sigma, \tau)$ and $(0, \sigma, \xi)\mathbf{R}(1, \sigma, , \tau)$.

For any other $f \in \mathbf{V}$ let $\mathbf{G}(\delta, \sigma, \tau, f) = \emptyset$.

Now let $\mathbf{F}$ be obtained by Theorem 5.7: $\mathbf{F}(\delta, \sigma, \tau) = \mathbf{G}(\delta, \sigma, \tau, \mathbf{F} \restriction \mathrm{pred}_{\mathbf{AR}}(\delta, \sigma, \tau))$. Then we have

$$\begin{aligned}
\mathbf{F}(0, \sigma, \tau) &= \mathbf{G}(0, \sigma, \tau, \mathbf{F} \restriction \mathrm{pred}_{\mathbf{AR}}(0, \sigma, \tau)) \\
&= \prod_{(\xi,p)\in\tau} [-e(p) + \mathbf{F}(1, \xi, \sigma)] \cdot \prod_{(\eta,q)\in\sigma} [-e(q) + \mathbf{F}(1, \eta, \tau)]; \\
\mathbf{F}(1, \sigma, \tau) &= \mathbf{G}(1, \sigma, \tau, \mathbf{F} \restriction \mathrm{pred}_{\mathbf{AR}}(1, \sigma, \tau)) \\
&= \sum_{(\xi,p)\in\tau} [e(p) \cdot \mathbf{F}(0, \sigma, \xi)]. \qquad \square
\end{aligned}$$

Now with $\mathbf{F}$ as in this theorem, we define $[\![\sigma = \tau]\!] = \mathbf{F}(0, \sigma, \tau)$ and $[\![\sigma \in \tau]\!] = \mathbf{F}(1, \sigma, \tau)$.

Corollary 28.18. *With $\mathbb{P}$ a forcing order and $\sigma, \tau \in \mathbf{V}^P$ we have*

$$\begin{aligned}
[\![\sigma = \tau]\!] &= \prod_{(\xi,p)\in\tau} [-e(p) + [\![\xi \in \sigma]\!]] \cdot \prod_{(\eta,q)\in\sigma} [-e(q) + [\![\eta \in \tau]\!]]; \\
[\![\sigma \in \tau]\!] &= \sum_{(\xi,p)\in\tau} [e(p) \cdot [\![\sigma = \xi]\!]]. \qquad \square
\end{aligned}$$

Thus we are defining $[\![\sigma = \tau]\!]$ to mean, in a sense, that every element of σ is an element of τ and every element of τ is an element of σ. And we define $[\![\sigma \in \tau]\!]$ to mean, in a sense, that there is some element of τ to which σ is equal. We now extend the definition of Boolean values to arbitrary formulas.

$$\begin{aligned}
[\![\neg\varphi(\sigma_0, \ldots, \sigma_{m-1})]\!] &= -[\![\varphi(\sigma_0, \ldots, \sigma_{m-1})]\!]; \\
[\![\varphi(\sigma_0, \ldots, \sigma_{m-1}) \to \psi(\sigma_0, \ldots, \sigma_{m-1})]\!] &= -[\![\varphi(\sigma_0, \ldots, \sigma_{m-1})]\!] + [\![\psi(\sigma_0, \ldots, \sigma_{m-1})]\!]; \\
[\![\forall x\varphi(\sigma_0, \ldots, \sigma_{m-1}, x)]\!] &= \prod_{\tau\in\mathbf{V}^P} [\![\varphi(\sigma_0, \ldots, \sigma_{m-1}, \tau)]\!].
\end{aligned}$$

Note that the last big product has index set which is a proper class in general. But the values are all in the Boolean algebra $\mathrm{RO}(\mathbb{P})$, so this makes sense. Namely, this part of the definition can be rewritten as follows:

$$[\![\forall x\varphi(\sigma_0, \ldots, \sigma_{m-1}, x)]\!] = \prod\{a \in \mathrm{RO}(\mathbb{P}) : \exists\tau \in \mathbf{V}^P(a = [\![\varphi(\sigma_0, \ldots, \sigma_{m-1}, \tau)]\!])\}.$$

Lemma 28.19.

$$\begin{aligned}
[\![\varphi(\sigma_0,\ldots,\sigma_{m-1})\vee\psi(\sigma_0,\ldots,\sigma_{m-1})]\!] &= [\![\varphi(\sigma_0,\ldots,\sigma_{m-1})]\!]+[\![\varphi(\sigma_0,\ldots,\sigma_{m-1})]\!];\\
[\![\varphi(\sigma_0,\ldots,\sigma_{m-1})\wedge\psi(\sigma_0,\ldots,\sigma_{m-1})]\!] &= [\![\varphi(\sigma_0,\ldots,\sigma_{m-1})]\!]\cdot[\![\varphi(\sigma_0,\ldots,\sigma_{m-1})]\!];\\
[\![\exists x\varphi(\sigma_0,\ldots,\sigma_{m-1},x)]\!] &= \sum_{\tau\in\mathbf{V}^P}[\![\varphi(\sigma_0,\ldots,\sigma_{m-1},\tau)]\!].
\end{aligned}$$

Proof. Recall from Chapter 2 the definitions of $\vee$, $\wedge$, $\exists$. We omit the parameters $\sigma_0,\ldots,\sigma_{m-1}$.

$$\begin{aligned}
[\![\varphi\vee\psi]\!] &= [\![\neg\varphi\to\psi)]\!]\\
&= --[\![\varphi]\!]+[\![\psi]\!])\\
&= [\![\varphi]\!]+[\![\psi]\!];\\
[\![\varphi\wedge\psi) &= [\![\neg(\varphi\to\neg\psi)]\!]\\
&= -(-[\![\varphi]\!]+-[\![\psi]\!])\\
&= [\![\varphi]\!]\cdot[\![\psi]\!];\\
[\![\exists v_i\varphi]\!] &= [\![\neg\forall v_i\neg\varphi]\!]\\
&= -\prod_{\tau\in\mathbf{V}^{\mathbb{P}}}-[\![\varphi(\tau)]\!]\\
&= \sum_{\tau\in\mathbf{V}^{\mathbb{P}}}[\![\varphi(\tau)]\!].
\end{aligned}$$

□

Now we can give our alternate definition of forcing:

$$p\Vdash^*\varphi(\sigma_0,\ldots,\sigma_{m-1})\quad\text{iff}\quad e(p)\le[\![\varphi(\sigma_0,\ldots,\sigma_{m-1})]\!].$$

It is important that Boolean values and $\Vdash^*$ are definable in a c.t.m. M of ZFC. Note that the discussion of Boolean values and of $\Vdash^*$ has taken place in our usual framework of set theory. The complete BA $\mathrm{RO}(\mathbb{P})$ is in general uncountable. Given a c.t.m. M of ZFC, the definitions can take place within M, and while M may be a model of "$\mathrm{RO}(\mathbb{P})$ is uncountable", actually $\mathrm{RO}(\mathbb{P})^M$ is countable. Thus even if $\sigma_0,\ldots,\sigma_{m-1}$ are members of M^P, the statements $p\Vdash^*\varphi(\sigma_0,\ldots,\sigma_{m-1})$ and $(p\Vdash^*\varphi(\sigma_0,\ldots,\sigma_{m-1}))^M$ are much different, since the products and sums involved in the definition of the former range over a possibly uncountable complete BA, while those in the latter range only over a countable BA (which is actually incomplete if it is infinite).

Now we prove the fundamental theorem connecting the notion $\Vdash^*$ in a c.t.m. M with the notion $\Vdash$, whose definition takes place outside M.

Theorem 28.20. (The Forcing Theorem) *Suppose that M is a c.t.m. of ZFC, $\mathbb{P}\in M$ is a forcing order, and G is $\mathbb{P}$-generic over M. Then the following conditions are equivalent:*

(i) There is a $p\in G$ such that $(p\Vdash^*\varphi(\sigma_0,\ldots,\sigma_{m-1}))^M$.

(ii) $\varphi(\sigma_{0G},\ldots,\sigma_{(m-1)G})$ holds in $M[G]$.

Proof. First we prove the equivalence for $\sigma = \tau$ and $\sigma \in \tau$ by induction on the well-founded relation $\mathbf{R}$ given in the proof of Theorem 28.16. For (i)⇒(ii), suppose that $p \in G$ and $(p \Vdash^* \sigma = \tau)^M$. We want to show that $\sigma_G = \tau_G$. Suppose that $a \in \sigma_G$. Then there is an $(\eta, q) \in \sigma$ such that $q \in G$ and $a = \eta_G$. Now $e(p) \leq -e(q) + [\![\eta \in \tau]\!]$, so $e(p) \cdot e(q) \leq [\![\eta \in \tau]\!]$. Since $p, q \in G$, choose $r \in G$ with $r \leq p, q$. Then $e(r) \leq [\![\eta \in \tau]\!]$. So $(r \Vdash^* \eta \in \tau)^M$, so by the inductive hypothesis $a = \eta_G \in \tau_G$. So we have shown that $\sigma_G \subseteq \tau_G$. Similarly, $\tau_G \subseteq \sigma_G$. So we have shown that (i)⇒(ii) for $\sigma = \tau$.

Now suppose that $p \in G$ and $(p \Vdash^* \sigma \in \tau)^M$. Thus $e(p) \leq \sum_{(\xi,s)\in\tau}[e(s) \cdot [\![\sigma = \xi]\!]$. Now we claim

(1) $\{q : \exists(\xi, s) \in \tau[q \leq s \text{ and } e(q) \leq [\![\sigma = \xi]\!]]\}$ is dense below p.

For, suppose that $r \leq p$. Then $e(r) \leq \sum_{(\xi,s)\in\tau}[e(s) \cdot [\![\sigma = \xi]\!]$, and hence

$$e(r) = e(r) \cdot \sum_{(\xi,s)\in\tau}[e(s) \cdot [\![\sigma = \xi]\!] = \sum_{(\xi,s)\in\tau}[e(r) \cdot e(s) \cdot [\![\sigma = \xi]\!].$$

It follows that there is a $(\xi, s) \in \tau$ such that $e(r) \cdot e(s) \cdot [\![\sigma = \xi]\!] \neq 0$. By Theorem 13.20(i) choose t so that $e(t) \leq e(r) \cdot e(s) \cdot [\![\sigma = \xi]\!]$. By Theorem 13.20(iii), t and r are compatible. Say $u \leq t, r$. Also, $e(u) \leq e(t) \leq e(s)$, so u and s are compatible. Say $v \leq u, s$. Then $e(v) \leq e(u) \leq e(t) \leq [\![\sigma = \xi]\!]$, so v is in the set of (1). So (1) holds.

Now by (1) and Theorem 28.14(iii), there exist a $q \in G$ with $q \leq p$ and $(\xi, s) \in \tau$ such that $e(q) \leq [\![\sigma = \xi]\!]$ and $q \leq s$. So $(q \Vdash^* \sigma = \xi)^M$, and by the inductive hypothesis we have $\sigma_G = \xi_G$. Now $q \leq s$ implies that $s \in G$, and so $(\xi, s) \in \tau$ yields $\xi_G \in \tau_G$ (by the definition of val). This proves (i)⇒(ii) for $\sigma \in \tau$.

Now for (ii)⇒(i), suppose that $\sigma_G = \tau_G$. Let

$$\begin{aligned} D = &\{r : (r \Vdash^* \sigma = \tau)^M \text{ or } \exists(\xi, p) \in \tau[r \leq p \text{ and } e(r) \leq -[\![\xi \in \sigma]\!]] \text{ or} \\ &\quad \exists(\eta, q) \in \sigma[r \leq q \text{ and } e(r) \leq -[\![\eta \in \tau]\!]]\}. \end{aligned}$$

We claim that D is dense. For, suppose that $s \in P$. Assume that $(s \not\Vdash^* \sigma = \tau)^M$. Thus $e(s) \not\leq [\![\sigma = \tau]\!]$, so

$$\begin{aligned} 0 &\neq e(s) \cdot -[\![\sigma = \tau]\!] \\ &= e(s) \cdot \left(\sum_{(\xi,p)\in\tau} (e(p) \cdot -[\![\xi \in \sigma]\!]) + \sum_{(\eta,q)\in\sigma} (e(q) \cdot -[\![\eta \in \tau]\!]) \right). \end{aligned}$$

It follows that one of the following conditions holds:

(2) There is a $(\xi, p) \in \tau$ such that $e(s) \cdot e(p) \cdot -[\![\xi \in \sigma]\!] \neq 0$.

(3) There is a $(\eta, q) \in \sigma$ such that $e(s) \cdot e(q) \cdot -[\![\xi \in \tau]\!] \neq 0$.

Suppose that (2) holds, with (ξ, p) as indicated. By Theorem 13.20(i) choose t such that $e(t) \leq e(s) \cdot e(p) \cdot -[\![\xi \in \sigma]\!]$. Since $e(t) \leq e(p)$, by Theorem 13.20(iv) we get $u \leq t, p$. Then $e(u) \leq e(t) \leq e(s)$, so again by Theorem 13.20(iv) we get v such that $v \leq u, s$. Then $v \leq u \leq p$ and $e(v) \leq e(u) \leq e(t) \leq -[\![\xi \in \sigma]\!]$. Thus $v \in D$, as desired.

By a similar argument, (3) gives an element of D below s. Hence D is dense.

Choose $r \in G \cap D$. We claim that $(r \Vdash^* \sigma = \tau)^M$. Otherwise one of the following conditions holds:

(4) $\exists(\xi, p) \in \tau[r \leq p$ and $e(r) \leq -[\![\xi \in \sigma]\!]]$.

(5) $\exists(\eta, q) \in \sigma[r \leq q$ and $e(r) \leq -[\![\eta \in \tau]\!]]$.

Suppose that (4) holds, with (ξ, p) as indicated. Now $e(r) \neq 0$, so $e(r) \not\leq [\![\xi \in \sigma]\!]$. Thus $(r \not\Vdash^* \xi \in \sigma)^M$. Hence by the inductive hypothesis, $\xi_G \notin \sigma_G$. But $r \leq p$, so $p \in G$, and hence $\xi_G \in \tau_G$. This contradicts our assumption that $\sigma_G = \tau_G$.

(5) leads to a contradiction similarly. Hence our claim holds, and we have proved (ii)⇒(i) for $\sigma = \tau$.

For (ii)⇒(i) for $\sigma \in \tau$, assume that $\sigma_G \in \tau_G$. Then there is a $(\xi, p) \in \tau$ such that $p \in G$ and $\sigma_G = \xi_G$. By the inductive hypothesis there is a $q \in G$ such that $(q \Vdash^* \sigma = \xi)^M$. Choose $r \in G$ with $r \leq p, q$. Then $e(r) \leq e(p) \cdot [\![\sigma = \xi]\!]$, and so $e(r) \leq [\![\sigma \in \tau]\!]$. Thus $(r \Vdash^* \sigma \in \tau)^M$.

Thus now the atomic cases are finished.

In the inductive steps we omit the parameters $\sigma_0, \ldots, \sigma_{m-1}$. Suppose that the equivalence holds for φ; we prove it for $\neg\varphi$. For (i)⇒(ii), suppose that $p \in G$ and $(p \Vdash^* \neg\varphi)^M$. We want to show that $\neg\varphi$ holds in $M[G]$. Suppose to the contrary that φ holds in $M[G]$. Then by the equivalence for φ, choose $q \in G$ such that $(q \Vdash^* \varphi)^M$. Choose $r \in G$ with $r \leq p, q$. Then $(r \Vdash^* \neg\varphi)^M$ and $(r \Vdash^* \varphi)^M$, contradiction.

For (ii)⇒(i), suppose that $\neg\varphi$ holds in $M[G]$. We claim that $D \stackrel{\text{def}}{=} \{p : (p \Vdash^* \varphi)^M$ or $(p \Vdash^* \neg\varphi)^M\}$ is dense. For, suppose that q is arbitrary. If $(q \Vdash^* \varphi)^M$, then $q \in D$. Suppose that $(q \not\Vdash^* \varphi)^M$. Then $e(q) \not\leq [\![\varphi]\!]$, so $e(q) \cdot -[\![\varphi]\!] \neq 0$. By Theorem 13.20(i) choose p so that $e(p) \leq e(q) \cdot -[\![\varphi]\!] \neq 0$. By Theorem 13.20(iv) choose $r \leq p, q$. Then $r \leq q$ and $e(r) \leq e(p) \leq -[\![\varphi]\!] = [\![\neg\varphi]\!]$. Hence $(r \Vdash \neg\varphi)^M$. This shows that $r \in D$. Thus D is dense. Choose $p \in D \cap G$. If $(p \Vdash^* \varphi)^M$, then $\varphi^{M[G]}$, contradiction. Hence $(p \Vdash^* \neg\varphi)^M$.

For $\rightarrow$, suppose that $p \in G$, $(p \Vdash^* \varphi \rightarrow \psi)^M$, and φ holds in $M[G]$. By the inductive hypothesis, choose $q \in G$ so that $(q \Vdash^* \varphi)^M$. Choose $r \in G$ with $r \leq p, q$. Then

$$e(r) \leq e(p) \cdot e(q) \leq [\![\varphi \rightarrow \psi]\!] \cdot [\![\varphi]\!] = (-[\![\varphi]\!] + [\![\psi]\!]) \cdot [\![\varphi]\!] \leq [\![\psi]\!].$$

Thus $(r \Vdash^* \psi)^M$, so by the inductive hypothesis, ψ holds in $M[G]$. So we have shown that $\varphi \rightarrow \psi$ holds in $M[G]$.

Conversely, suppose that $\varphi \rightarrow \psi$ holds in $M[G]$.

Case 1. φ holds in $M[G]$. Then also ψ holds in $M[G]$. By the inductive hypothesis we get $p \in G$ such that and $(p \Vdash^* \psi)^M$. Thus $e(p) \leq [\![\psi]\!]$, so $e(p) \leq -[\![\varphi]\!] + [\![\psi]\!] = [\![\varphi \rightarrow \psi]\!]$, hence $(p \Vdash^* \varphi \rightarrow \psi)^M$.

Case 2. φ does not hold in $M[G]$. By the case for $\neg$, there is a $p \in G$ such that $(p \Vdash^* \neg\varphi)^M$. Hence $e(p) \leq -[\![\varphi]\!] \leq -[\![\varphi]\!] + [\![\psi]\!] = [\![\varphi \rightarrow \psi]\!]$, hence $(p \Vdash^* \varphi \rightarrow \psi)^M$.

Finally, we deal with the formula $\forall x\varphi(x)$. For (i)⇒(ii), suppose that $\forall x\varphi(x)$ does not hold in $M[G]$. Then there is a name σ such that $\varphi(\sigma_G)$ does not hold. By the case for $\neg$ it follows that there is a $p \in G$ such that $(p \Vdash^* \neg\varphi(\sigma))^M$, so that $e(p) \leq -[\![\varphi(\sigma)]\!]$. Thus $e(p) \leq -\prod_{\tau \in V^{\mathbb{P}}}[\![\varphi(\tau)]\!]$ and so, since $e(p) \neq 0$, $e(p) \not\leq \prod_{\tau \in V^{\mathbb{P}}}[\![\varphi(\tau)]\!]$, so that it is not true that $(p \Vdash^* \forall x\varphi(x))^M$.

For (ii)⇒(i), suppose that $(p \Vdash^* \forall x\varphi(x))^M$, and suppose that σ is any name. Then $e(p) \leq [\![\varphi(\sigma)]\!]$, hence $(p \Vdash^* \varphi(\sigma))^M$, so $\varphi(\sigma_G)$ holds in $M[G]$, as desired. □

Corollary 28.21. *If M is a c.t.m. of ZFC, $\mathbb{P}$ is a forcing order in M, $p \in P$, and $\varphi(\tau_1, \ldots, \tau_m)$ is a formula, then*

$$p \Vdash \varphi(\tau_1, \ldots, \tau_m) \text{ iff } (p \Vdash^* \varphi(\tau_1, \ldots, \tau_m))^M.$$

Proof. Again we omit the parameters $\tau_1, \ldots, \tau_m$. ⇒: Assume that $p \Vdash \varphi$, but suppose that $(p \not\Vdash^* \varphi)^M$. Thus $e(p) \not\leq [\![\varphi]\!]$, so $e(p) \cdot -[\![\varphi]\!] \neq 0$. By Theorem 13.20(i) choose q such that $e(q) \leq e(p) \cdot -[\![\varphi]\!]$. By Theorem 13.20(iv) choose $r \leq p, q$. Then $e(r) \leq e(q) \leq -[\![\varphi]\!] = [\![\neg\varphi]\!]$. Hence $(r \Vdash \neg\varphi)^M$. Let G be $\mathbb{P}$-generic over M with $r \in G$. Then by Theorem 28.19, $\neg\varphi^{M[G]}$. But $r \leq p$, so by the definition of $\Vdash$, $\varphi^{M[G]}$, contradiction.

⇐: Assume that $(p \Vdash^* \varphi)^M$. Suppose that G is $\mathbb{P}$-generic over M and $p \in G$. Then by Theorem 28.19. $\varphi^{M[G]}$, as desired. □

Corollary 28.22. *Let M be a c.t.m. of ZFC, $\mathbb{P} \in M$ a forcing order, and $G \subseteq M$ a $\mathbb{P}$-generic filter over M. Then*

$$\varphi(\tau_{1G}, \ldots, \tau_{mG})^{M[G]} \text{ iff } \exists p \in G[p \Vdash \varphi(\tau_1, \ldots, \tau_m)].$$

Proof. ⇒: Assume $\varphi(\tau_{1G}, \ldots, \tau_{mG})^{M[G]}$. By Theorem 28.19, choose $p \in G$ such that $(p \Vdash^* \varphi(\tau_1, \ldots, \tau_m))^M$. By Corollary 28.20 we have $p \Vdash \varphi(\tau_1, \ldots, \tau_m)$.

⇐: by the definition of $\Vdash$. □

Now if σ and τ are names, we define

$$\begin{aligned} \mathrm{up}(\sigma, \tau) &= \{(\sigma, 1), (\tau, 1)\}; \\ \mathrm{op}(\sigma, \tau) &= \mathrm{up}(\mathrm{up}(\sigma, \sigma), \mathrm{up}(\sigma, \tau)). \end{aligned}$$

Lemma 28.23. *(i)* $(\mathrm{up}(\sigma, \tau))_G = \{\sigma_G, \tau_G\}$.

(ii) $(\mathrm{op}(\sigma, \tau))_G = (\sigma_G, \tau_G)$. □

Theorem 28.24. *Let M be a c.t.m. of ZFC, $\mathbb{P} \in M$ a forcing order, $G \subseteq P$, and G $\mathbb{P}$-generic over M. Then $M[G]$ is a model of ZFC.*

Proof. We will apply theorems from Chapter 14. Recall from Lemma 28.7 that $M[G]$ is transitive. Hence extensionality and foundation hold in $M[G]$ by Theorems 14.1 and 14.7. For pairing, suppose that $x, y \in M[G]$. Say $x = \sigma_G$ and $y = \tau_G$. By Lemma 28.22 and Theorem 14.3, pairing holds. For union, suppose that $x \in M[G]$. Choose σ such that $x = \sigma_G$. Note that $\mathrm{dmn}(\sigma)$ is a set of $\mathbb{P}$-names, and hence so is $\tau \stackrel{\mathrm{def}}{=} \bigcup \mathrm{dmn}(\sigma)$. We claim that $\bigcup x \subseteq \tau_G$; by Theorem 14.4 this will prove the union axiom. Let $y \in \bigcup x$. Say

$y \in z \in x$. Then there exist $(\rho, r), (\xi, s)$ such that $y = \rho_G$, $r \in G$, $(\rho, r) \in \xi$, $z = \xi_G$, $s \in G$, and $(\xi, s) \in \sigma$. So $\xi \in \mathrm{dmn}(\sigma)$, and hence $(\rho, r) \in \bigcup \mathrm{dmn}(\sigma) = \tau$. It follows that $y = \rho_G \in \tau_G$, as desired.

To check comprehension, we apply Theorem 14.2. So, suppose $\varphi(x, z, w_1, \ldots, w_n)$ is a formula with the indicated free variables, and $\sigma, \tau_1, \ldots, \tau_n$ are $\mathbb{P}$ names. Let

$$y = \{x \in \sigma_G : \varphi^{M[G]}(x, \sigma_G, \tau_{1G}, \ldots, \tau_{nG})\};$$

we want to show that $y \in M[G]$. Let

$$\rho = \{(\pi, p) \in \mathrm{dmn}(\sigma) \times P : (p \Vdash^* (\pi \in \sigma \wedge \varphi(\pi, \sigma, \tau_1, \ldots, \tau_n))^M\}.$$

Thus $\rho \in M^P$. We claim that $\rho_G = y$, as desired. Suppose that $x \in \rho_G$. Then there is a $(\pi, p) \in \mathrm{dmn}(\sigma) \times \mathbb{P}$ such that $p \in G$, $x = \pi_G$, and $(p \Vdash^* (\pi \in \sigma \wedge \varphi(\pi, \sigma, \tau_1, \ldots, \tau_n))^M$. By Corollary 28.20, $p \Vdash (\pi \in \sigma \wedge \varphi(\pi, \sigma, \tau_1, \ldots, \tau_n))$. Hence by definition of $\Vdash$, $\pi_G \in \sigma_G$ and $\varphi^{M[G]}(\pi_G, \sigma_G, \tau_{1G}, \ldots, \tau_{nG})$. Thus $x \in y$. Conversely, suppose that $x \in y$. Thus $x \in \sigma_G$ and $\varphi^{M[G]}(x, \sigma_G, \tau_{1G}, \ldots, \tau_{nG})$. Choose $(\pi, p) \in \sigma$ such that $x = \pi_G$ and $p \in G$. Thus $(\pi_G \in \sigma_G \wedge \varphi^{M[G]}(\pi_G, \sigma_G, \tau_{1G}, \ldots, \tau_{nG}))^{M[G]}$, so by Corollary 28.21 there is a $q \in G$ such that $q \Vdash \pi \in \sigma \wedge \varphi(\pi, \sigma, \tau_1, \ldots, \tau_n)$. Thus by Theorem 28.20 again we have $(\pi, q) \in \rho$, hence $x = \pi_G \in \rho_G$, as desired.

For the power set axiom, we will apply Theorem 14.5. Let σ be a P-name. It suffices to find another P-name ρ such that $\mathscr{P}(\sigma_G) \cap M[G] \subseteq \rho_G$. Let $\rho = S \times \{1\}$, where

$$S = \{\tau \in M^P : \mathrm{dmn}(\tau) \subseteq \mathrm{dmn}(\sigma)\}.$$

Suppose that $\mu \in M^P$ and $\mu_G \subseteq \sigma_G$; we want to show that $\mu_G \in \rho_G$. Let

$$\tau = \{(\pi, p) : \pi \in \mathrm{dmn}(\sigma) \text{ and } p \Vdash \pi \in \mu\}.$$

Thus $\mathrm{dmn}(\tau) \subseteq \mathrm{dmn}(\sigma)$, so $\tau_G \in \rho_G$. It suffices now to show that $\tau_G = \mu_G$. First suppose that $x \in \mu_G$. Since $\mu_G \subseteq \sigma_G$, there is a $(\pi, q) \in \sigma$ such that $q \in G$ and $x = \pi_G$. Thus $\pi_G \in \sigma_G$, so by Theorem 28.21 there is a $p \in G$ such that $p \Vdash \pi \in \sigma$. Hence $(\pi, p) \in \tau$, and so $x = \pi_G \in \tau_G$. Second, suppose that $x \in \tau_G$. Choose $(\pi, p) \in \tau$ such that $p \in G$ and $x = \pi_G$. By definition of τ we have $\pi \in \mathrm{dmn}(\sigma)$ and $p \Vdash \pi \in \mu$. By definition of $\Vdash$, $x = \pi_G \in \mu_G$. Hence we have shown that $\tau_G = \mu_G$, as desired.

For replacement, we apply Theorem 14.6. Let φ be a formula with free variables among $x, y, A, w_1, \ldots, w_n$, suppose that $\sigma, \tau_1, \ldots, \tau_n \in M^P$ and the following holds:

(1) $$(\forall x \in \sigma_G \exists! y[\varphi(x, y, \sigma_G, \tau_{1G}, \ldots, \tau_{1G})])^{M[G]}.$$

We want to find $\rho \in M^P$ such that

$$(\forall y \exists x \in \sigma_G \varphi(x, y, \sigma_G, \tau_{1G}, \ldots, \tau_{nG}) \to y \in \rho_G)^{M[G]}.$$

In view of the uniqueness condition in (1) it suffices to find $\rho \in M^P$ such that

(2) $$\forall x \in \sigma_G \exists y \in \rho_G(\varphi(x, y, \sigma_G, \tau_{1G}, \ldots, \tau_{nG})^{M[G]}.$$

In fact, if (2) holds, $y \in M[G]$, $x \in \sigma_G$, and $(\varphi(x, y, \sigma_G, \tau_{1G}, \ldots, \tau_{nG}))^{M[G]}$. by (2) choose $z \in \rho_G$ such that $(\varphi(x, z, \sigma_G, \tau_{1G}, \ldots, \tau_{nG})^{M[G]}$. Then by (1) we have $y = z$, and so $y \in \rho_G$.

Now we claim

(3) There is an $S \in M$ with $S \subseteq M^P$ such that

$$\forall \pi \in \mathrm{dmn}(\sigma) \forall p \in P[\exists \mu \in M^P[(p \Vdash^* \varphi(\pi, \mu, \sigma, \tau_1, \ldots, \tau_n))^M] \\ \rightarrow \exists \mu \in S[(p \Vdash^* \varphi(\pi, \mu, \sigma, \tau_1, \ldots, \tau_n))^M]].$$

To prove the claim, we make the following argument in M. For each $\pi \in \mathrm{dmn}(\sigma)$ and $p \in P$, if there is a $\mu \in M^P$ such that $(p \Vdash^* \varphi(\pi, \mu, \sigma, \tau_1, \ldots, \tau_n))M$, let $\alpha(\pi, p)$ be the least ordinal such that such a μ is in $V_{\alpha(\pi,p)}$, while $\alpha(\pi, p) = 0$ if there is no such ordinal. Let $\beta = \sup\{\alpha(\pi, p) : \pi \in \mathrm{dmn}(\alpha), p \in P\}$. Then

$$S = \{\mu \in V_\beta : \exists \pi \in \mathrm{dmn}(\sigma) \exists p \in P[(p \Vdash^* \varphi(\pi, \mu, \sigma, \tau_1, \ldots, \tau_n))^M]\}.$$

Clearly S is as desired in the claim.

Let $\rho = S \times \{1\}$. To show that ρ satisfies (2), let $x \in \sigma_G$. say $x = \pi_G$ with $(\pi, p) \in \rho$ and $p \in G$. Then by (1) there is a $\mu \in M^P$ such that $\varphi^{M[G]}(\pi_G, \mu_G, \sigma_G, \tau_{1G}, \ldots, \tau_{mG})$. By Corollary 28.21 choose $q \in G$ such that $q \Vdash \varphi(\pi, \mu, \sigma, \tau_1, \ldots, \tau_m)$. By Corollary 28.20, $(q \Vdash^* \varphi(\pi, \mu, \sigma, \tau_1, \ldots, \tau_m))^M$. Hence by (3) we may assume that $\mu \in S$. Hence $\mu_G \in \rho_G$, as desired.

For the infinity axiom, note that $\omega = \check{\omega}_G$ by Lemma 28.10. Hence the infinity axiom holds by Theorem 14.8.

Finally we consider the axiom of choice. We show that there is a choice function for any family A of nonempty sets, where $A \in M[G]$. By absoluteness, $\bigcup A \in M[G]$. Say $\bigcup A = \sigma_G$. Let f be a bijection from some cardinal κ onto $\mathrm{dmn}(\sigma)$ (in M). Define $\tau = \{\mathrm{op}(\check{\alpha}, f(\alpha)) : \alpha < \kappa\} \times \{1\}$. Thus $\tau_G = \{(\mathrm{op}(\check{\alpha}, f(\alpha)))_G = \{(\alpha, (f(\alpha))_G) : \alpha < \kappa\}$. So τ_G is a function with domain κ. Each $x \in A$ is nonempty, and if $y \in x$ then $y \in \bigcup A$, and hence we can write $y = \tau_G$ with $(\tau, p) \in \sigma$ and $p \in G$. So there is an $\alpha < \kappa$ such that $f(\alpha) = \tau$; so $\tau_G(\alpha) = (f(\alpha))_G = \tau_g = y$. This shows that for each $x \in A$ there is an ordinal $\alpha < \kappa$ such that $\tau_G(\alpha) \in x$; we let α_x be the least such ordinal. Define $g(x) = \tau_G(\alpha_x)$ for all $x \in A$. Then $g(x) \in x$. □

Parts of the following theorem will be used later without reference.

Theorem 28.25. *(i)* $[\![\sigma = \tau]\!] = [\![\tau = \sigma]\!]$.

(ii)

$$[\![\sigma = \tau]\!] = \left(\prod_{(\xi,p)\in\tau} (-e(p) + [\![\xi \in \sigma]\!]) \right) \cdot \left(\prod_{(\rho,q)\in\sigma} (-e(q) + [\![\rho \in \tau]\!]) \right).$$

(iii) $[\![\sigma = \sigma]\!] = 1$.

(iv) If $(\rho, r) \in \sigma$, *then* $e(r) \leq [\![\rho \in \sigma]\!]$, *and hence* $r \Vdash^* \rho \in \sigma$.

(v) $p \Vdash^* \sigma \in \tau$ *iff the set*

$$\{q : \exists(\pi, s) \in \tau(q \leq s \text{ and } q \Vdash^* \sigma = \pi)\}$$

is dense below p.

(vi) $p \Vdash^* \sigma = \tau$ *iff the following two conditions hold:*

(a) For all $(\pi, s) \in \sigma$, *the set*

$$\{q \leq p : \text{if } q \leq s, \text{ then there is a } (\rho, u) \in \tau \text{ such that } q \leq u \text{ and } q \Vdash^* \pi = \rho\}$$

is dense below p;

(b) For all $(\rho, u) \in \tau$, *the set*

$$\{q \leq p : \text{if } q \leq u, \text{ then there is a } (\pi, s) \in \sigma \text{ such that } q \leq s \text{ and } q \Vdash^* \pi = \rho\}$$

is dense below p.

(vii)

$$p \Vdash^* \varphi(\sigma_0, \ldots, \sigma_{n-1}) \wedge \psi(\sigma_0, \ldots, \sigma_{n-1}) \quad \text{iff} \quad p \Vdash^* \varphi(\sigma_0, \ldots, \sigma_{n-1}) \text{ and } p \Vdash^* \psi(\sigma_0, \ldots, \sigma_{n-1}).$$

(viii) $p \Vdash^* \varphi(\sigma_0, \ldots, \sigma_{n-1}) \vee \psi(\sigma_0, \ldots, \sigma_{n-1})$ *iff for all* $q \leq p$ *there is an* $r \leq q$ *such that* $r \Vdash^* \varphi(\sigma_0, \ldots, \sigma_{n-1})$ *or* $r \Vdash^* \psi(\sigma_0, \ldots, \sigma_{n-1})$.

(ix) $p \Vdash^* \neg\varphi(\sigma_0, \ldots, \sigma_{n-1})$ *iff for all* $q \leq p$, $q \not\Vdash^* \varphi(\sigma_0, \ldots, \sigma_{n-1})$.

(x) $\{p : p \Vdash^* \varphi(\sigma_0, \ldots, \sigma_{n-1})$ *or* $p \Vdash^* \neg\varphi(\sigma_0, \ldots, \sigma_{n-1})\}$ *is dense.*

(xi) $p \Vdash^* \exists x\varphi(x, \sigma_0, \ldots, \sigma_{n-1})$ *iff the set*

$$\{r \leq p : \text{there is a } \tau \in \mathbf{V}^P \text{ such that } r \Vdash^* \varphi(\tau, \sigma_0, \ldots, \sigma_{n-1})\}$$

is dense below p.

(xii) $p \Vdash^* \forall x\varphi(x, \sigma_0, \ldots, \sigma_{m-1})$ *iff for all* $\tau \in \mathbf{V}^P$, $p \Vdash^* \varphi(\tau, \sigma_0, \ldots, \sigma_{m-1})$.

(xiiii) The following are equivalent:

(a) $p \Vdash^* \varphi(\sigma_0, \ldots, \sigma_{m-1})$.

(b) For every $r \leq p$, $r \Vdash^* \varphi(\sigma_0, \ldots, \sigma_{m-1})$.

(c) $\{r : r \Vdash^* \varphi(\sigma_0, \ldots, \sigma_{m-1})\}$ *is dense below* p.

(xiv) $p \Vdash^* \varphi(\sigma_0, \ldots, \sigma_{m-1}) \rightarrow \psi(\sigma_0, \ldots, \sigma_{m-1})$ *iff the set*

$$\{q : q \Vdash^* \neg\varphi(\sigma_0, \ldots, \sigma_{m-1}) \text{ or } q \Vdash^* \psi(\sigma_0, \ldots, \sigma_{m-1})\}$$

is dense below p.

(xv) If $p \Vdash^* \neg\forall x\varphi(x, \sigma_0, \ldots, \sigma_{m-1})$, *then the set*

$$\{q : \text{there is a } \tau \in \mathbf{V}^P \text{ such that } q \Vdash \neg\varphi(\tau, \sigma_0, \ldots, \sigma_{m-1})\}$$

is dense below p.

(xvi) If $p \Vdash^* \varphi(\sigma_0, \ldots, \sigma_{m-1})$ and $p \Vdash^* \varphi(\sigma_0, \ldots, \sigma_{m-1}) \to \psi(\sigma_0, \ldots, \sigma_{m-1})$, then $p \Vdash^* \psi(\sigma_0, \ldots, \sigma_{m-1})$.

Proof.
(i): By induction:

$$\begin{aligned}
[\![\sigma = \tau]\!] = & \prod_{(\xi,p)\in\tau} \left[-e(p) + \sum_{(\rho,q)\in\sigma} (e(q) \cdot [\![\rho = \xi]\!])\right] \\
& \cdot \prod_{(\rho,q)\in\sigma} \left[-e(q) + \sum_{(\xi,p)\in\tau} (e(p) \cdot [\![\rho = \xi]\!])\right] \\
& = \prod_{(\rho,q)\in\sigma} \left[-e(q) + \sum_{(\xi,p)\in\tau} (e(p) \cdot [\![\rho = \xi]\!])\right] \\
& \cdot \prod_{(\xi,p)\in\tau} \left[-e(p) + \sum_{(\rho,q)\in\sigma} (e(q) \cdot [\![\rho = \xi]\!])\right] \\
& = \prod_{(\rho,q)\in\sigma} \left[-e(q) + \sum_{(\xi,p)\in\tau} (e(p) \cdot [\![\xi = \rho]\!])\right] \\
& \cdot \prod_{(\xi,p)\in\tau} \left[-e(p) + \sum_{(\rho,q)\in\sigma} (e(q) \cdot [\![\xi = \rho]\!])\right] \\
& = [\![\tau = \sigma]\!].
\end{aligned}$$

(ii):

$$\begin{aligned}
[\![\sigma = \tau]\!] = & \prod_{(\xi,p)\in\tau} \left[-e(p) + \sum_{(\rho,q)\in\sigma} (e(q) \cdot [\![\rho = \xi]\!])\right] \\
& \cdot \prod_{(\rho,q)\in\sigma} \left[-e(q) + \sum_{(\xi,p)\in\tau} (e(p) \cdot [\![\rho = \xi]\!])\right] \\
& = \prod_{(\xi,p)\in\tau} \left[-e(p) + \sum_{(\rho,q)\in\sigma} (e(q) \cdot [\![\xi = \rho]\!])\right] \\
& \cdot \prod_{(\rho,q)\in\sigma} \left[-e(q) + \sum_{(\xi,p)\in\tau} (e(p) \cdot [\![\rho = \xi]\!])\right] \\
& = \left(\prod_{(\xi,p)\in\tau} (-e(p) + [\![\xi \in \sigma]\!])\right) \cdot \left(\prod_{(\rho,q)\in\sigma} (-e(q) + [\![\rho \in \tau]\!])\right).
\end{aligned}$$

We prove (iii) and (iv) simultaneously by induction on the rank of σ; so suppose that they hold for all σ' of rank less than σ. Assume that $(\rho, r) \in \sigma$. Then by the definition of $[\![\rho \in \sigma]\!]$,

$$[\![\rho \in \sigma]\!] = \sum_{(\mu,s)\in\sigma} (e(s) \cdot [\![\rho = \mu]\!]) \geq e(r) \cdot [\![\rho = \rho]\!] = e(r),$$

as desired in (iv). Using this and (ii),

$$[\![\sigma = \sigma]\!] = \prod_{(\rho,r)\in\sigma} (-e(r) + [\![\rho \in \sigma]\!]) = 1,$$

as desired in (iii).

We now use Theorem 13.20(vii) in several of our arguments.

(v):

$$\begin{aligned} p \Vdash^* \sigma \in \tau \quad &\text{iff} \quad e(p) \leq [\![\sigma \in \tau]\!] \\ &\text{iff} \quad e(p) \leq \sum_{(\pi,s)\in\tau} (e(s) \cdot [\![\sigma = \pi]\!]) \\ &\text{iff} \quad \{q : \exists(\pi, s) \in \tau[e(q) \leq e(s) \cdot [\![\sigma = \pi]\!]]\} \text{ is dense below } p. \end{aligned}$$

We claim that the last statement here is equivalent to

$$(*) \qquad \{q : \exists(\pi, s) \in \tau[q \leq s \text{ and } e(s) \leq [\![\sigma = \pi]\!]]\} \text{ is dense below } p.$$

In fact clearly $(*)$ implies the above statement. Now suppose that

$$\{q : \exists(\pi, s) \in \tau[e(q) \leq e(s) \cdot [\![\sigma = \pi]\!]]\} \text{ is dense below } p.$$

Take any $r \leq p$, and choose $q \leq r$ and $(\pi, x) \in \tau$ such that $e(q) \leq e(s) \cdot [\![\sigma = \pi]\!]$. Then q and s are compatible; say $t \leq q, s$. Then $t \leq q \leq r$ and $e(t) \leq e(q) \leq e(s) \cdot [\![\sigma = \pi]\!]$. Thus $(*)$ holds.

Now $(*)$ is clearly equivalent to

$$\{q : \exists(\pi, s) \in \tau[q \leq s \text{ and } s \Vdash^* \sigma = \pi]\} \text{ is dense below } p.$$

(vi): Assume that $p \Vdash^* \sigma = \tau$.

For (a), suppose that $(\pi, s) \in \sigma$ and $r \leq p$. If $r \not\leq s$, then r itself is in the desired set; so suppose that $r \leq s$. Then

$$e(r) \leq e(s) \cdot e(p) \leq e(s) \cdot \left(-e(s) + \sum_{(\rho,u)\in\tau} (e(u) \cdot [\![\pi = \rho]\!]) \right) = e(s) \cdot \sum_{(\rho,u)\in\tau} (e(u) \cdot [\![\pi = \rho]\!]).$$

Hence there is a $(\rho, u) \in \tau$ such that $e(r) \cdot e(s) \cdot e(u) \cdot [\![\pi = \rho]\!] \neq 0$. Hence there exists a $v \leq r, s$ such that $e(v) \leq e(u) \cdot [\![\pi = \rho]\!]$. (See the argument for (v)). It follows that there is a $q \leq v, u$ with $e(q) \leq [\![\pi = \rho]\!]$. So $q \Vdash^* \pi = \rho$, and q is in the desired set.

(b) is treated similarly.

Now assume that (a) and (b) hold. We want to show that $p \Vdash^* \sigma = \tau$, i.e., that $e(p) \leq [\![\sigma = \tau]\!]$. To show that $e(p)$ is below the first big product in the definition of $[\![\sigma = \tau]\!]$, take any $(\xi, q) \in \tau$; we want to show that

$$e(p) \leq -e(q) + \sum_{(\rho,r)\in\sigma} (e(r) \cdot [\![\rho = \xi]\!]),$$

i.e., that

$$e(p) \cdot e(q) \leq \sum_{(\rho,r)\in\sigma} (e(r) \cdot [\![\rho = \xi]\!]).$$

Suppose that this is not the case. Then there is an s such that

$$e(s) \leq e(p) \cdot e(q) \cdot - \sum_{(\rho,r)\in\sigma} (e(r) \cdot [\![\rho = \xi]\!]) = e(p) \cdot e(q) \cdot \prod_{(\rho,r)\in\sigma} (-e(r) + -[\![\rho = \xi]\!]).$$

Hence there is a $u \leq s, p, q$. By (b) choose $v \leq u$ and $(\rho, r) \in \sigma$ such that $v \leq r$ and $v \Vdash^* \rho = \xi$. Then $e(v) \leq e(r) \cdot [\![\rho = \xi]\!]$, and also $e(v) \leq -e(r) + -[\![\rho = \xi]\!])$, contradiction.

Similarly, $e(p)$ is below the second big product in the definition of $[\![\sigma = \tau]\!]$.

(vii): Clear.

(viii): Since

$$p \Vdash^* \varphi(\sigma_0, \ldots, \sigma_{n-1}) \vee \psi(\sigma_0, \ldots, \sigma_{n-1}) \quad \text{iff} \quad e(p) \leq [\![\varphi(\sigma_0, \ldots, \sigma_{n-1}) \vee \psi(\sigma_0, \ldots, \sigma_{n-1})]\!],$$

this is immediate from Theorem 16.20(vii).

(ix) $\Rightarrow$: if $p \Vdash^* \neg\varphi(\sigma_0, \ldots, \sigma_{n-1})$ and $q \leq p$, then

$$e(q) \leq e(p) \leq [\![\neg\varphi(\sigma_0, \ldots, \sigma_{n-1})]\!] = -[\![\varphi(\sigma_0, \ldots, \sigma_{n-1})]\!],$$

and hence $e(q) \not\leq [\![\varphi(\sigma_0, \ldots, \sigma_{n-1})]\!]$, since $e(q) \neq 0$. Thus $q \not\Vdash^* \varphi(\sigma_0, \ldots, \sigma_{n-1})$.

$\Leftarrow$: suppose that $p \not\Vdash^* \neg\varphi(\sigma_0, \ldots, \sigma_{n-1})$. Then

$$e(p) \not\leq [\![\neg\varphi(\sigma_0, \ldots, \sigma_{n-1})]\!] = -[\![\varphi(\sigma_0, \ldots, \sigma_{n-1})]\!],$$

and hence

$$e(p) \cdot [\![\varphi(\sigma_0, \ldots, \sigma_{n-1})]\!] \neq 0,$$

so we can choose r such that

$$e(r) \leq e(p) \cdot [\![\varphi(\sigma_0, \ldots, \sigma_{n-1})]\!],$$

hence there is a $q \leq p, r$, and so $q \Vdash^* \varphi(\sigma_0, \ldots, \sigma_{n-1})$.

(x): Let q be given. If $e(q) \cdot [\![\varphi(\sigma_0, \ldots, \sigma_{n-1})]\!] \neq 0$, choose r such that $e(r) \leq e(q) \cdot [\![\varphi(\sigma_0, \ldots, \sigma_{n-1})]\!]$, and then choose $p \leq q, r$. Thus $p \Vdash^* \varphi(\sigma_0, \ldots, \sigma_{n-1})$, as desired. If $e(q) \cdot [\![\varphi(\sigma_0, \ldots, \sigma_{n-1})]\!] = 0$, then $q \Vdash^* \neg\varphi(\sigma_0, \ldots, \sigma_{n-1})$.

(xi): Suppose that $p \Vdash^* \exists x\varphi(x, \sigma_0, \ldots, \sigma_{n-1})$, and suppose that $q \leq p$. Then $e(q) \leq \sum_{\tau\in M^P} [\![\varphi(\tau, \sigma_0, \ldots, \sigma_{n-1})]\!]$, and so there is a $\tau \in M^P$ such that $e(q)\cdot[\![\varphi(\tau, \sigma_0, \ldots, \sigma_{n-1})]\!] \neq 0$; hence we easily get $r \leq q$ such that $e(r) \leq [\![\varphi(\tau, \sigma_0, \ldots, \sigma_{n-1})]\!]$. This implies that $r \Vdash^* \varphi(\tau, \sigma_0, \ldots, \sigma_{n-1})$, as desired.

Conversely, suppose that the set

$$\{r \leq p : \text{there is a } \tau \in \mathbf{V}^P \text{ such that } r \Vdash^* \varphi(\tau, \sigma_0, \ldots, \sigma_{n-1})\}$$

is dense below p, while $p \not\Vdash^* \exists x\varphi(x, \sigma_0, \ldots, \sigma_{n-1})$. Thus $e(p) \not\leq [\![\exists x\varphi(x, \sigma_0, \ldots, \sigma_{n-1})]\!]$, so

$$e(p) \cdot \prod_{\tau\in M^P} -[\![\varphi(\tau, \sigma_0, \ldots, \sigma_{n-1})]\!] \neq 0.$$

Then we easily get $q \leq p$ such that

$$e(q) \leq \prod_{\tau\in M^P} -[\![\varphi(\tau, \sigma_0, \ldots, \sigma_{n-1})]\!]. \tag{4}$$

By assumption, choose $r \leq q$ and $\tau \in M^P$ such that $r \Vdash^* \varphi(\tau, \sigma_0, \ldots, \sigma_{n-1})$. Thus $e(r) \leq [\![\varphi(\tau, \sigma_0, \ldots, \sigma_{n-1})]\!]$. This clearly contradicts (4).

(xii): Clear.

(xiii): Clearly (a)$\Rightarrow$(b)$\Rightarrow$(c). Now assume (c). Suppose that $p \not\Vdash^* \varphi(\sigma_0, \ldots, \sigma_{m-1})$. Thus $e(p) \not\leq [\![\varphi(\sigma_0, \ldots, \sigma_{m-1})]\!]$, so we easily get $q \leq p$ such that

$$e(q) \leq -[\![\varphi(\sigma_0, \ldots, \sigma_{m-1})]\!]. \tag{5}$$

By (c), choose $r \leq q$ such that $r \Vdash^* \varphi(\sigma_0, \ldots, \sigma_{m-1})$. Clearly this contradicts (5).

(xiv): First suppose that $p \Vdash^* \varphi(\sigma_0, \ldots, \sigma_{m-1}) \to \psi(\sigma_0, \ldots, \sigma_{m-1})$. Thus by the definition of $\to$ we have $p \Vdash^* \neg\varphi(\sigma_0, \ldots, \sigma_{m-1}) \vee \psi(\sigma_0, \ldots, \sigma_{m-1})$. Hence the desired conclusion follows by (viii). The converse follows by reversing these steps.

(xv): This is very similar to part of the proof of (xi), but we give it anyway. We have

$$\begin{aligned} i(p) &\leq [\![\neg\forall x\varphi(x, \sigma_0, \ldots, \sigma_{m-1})]\!] \\ &= \sum_{\tau\in M^P} -[\![\varphi(\tau, \sigma_0, \ldots, \sigma_{m-1})]\!]. \end{aligned}$$

Now suppose that $q \leq p$. Then $e(q)$ is $\leq$ the sum here, so we easily get $r \leq q$ and $\tau \in M^P$ such that $e(r) \leq [\![\neg\varphi(\tau, \sigma_0, \ldots, \sigma_{m-1})]\!]$. Hence $r \Vdash^* \neg\varphi(\tau, \sigma_0, \ldots, \sigma_{m-1})$, as desired.

(xvi): The hypotheses yield

$$\begin{aligned} &e(p) \leq [\![\varphi(\sigma_0, \ldots, \sigma_{m-1})]\!] \quad \text{and} \\ &e(p) \leq -[\![\varphi(\sigma_0, \ldots, \sigma_{m-1})]\!] + [\![\psi(\sigma_0, \ldots, \sigma_{m-1})]\!], \end{aligned}$$

so $e(p) \leq [\![\psi(\sigma_0, \ldots, \sigma_{m-1})]\!]$ and hence $p \Vdash^* \psi(\sigma_0, \ldots, \sigma_{m-1})]\!]$. $\square$

Proposition 28.26. *Let I and J be sets with I infinite and $|J| > 1$, and let $\mathbb{P} = (P, \leq, \emptyset)$, where P is the collection of all finite functions contained in $I \times J$ and $\leq$ is $\supseteq$ restricted to P. Then $\mathbb{P}$ satisfies the condition of Lemma 28.2.*

Proof. Suppose that $p \in P$. Pick any $i \in I\backslash \mathrm{dmn}(p)$, and let j, k be distinct elements of J. Then $p \subseteq p \cup \{(i, j)\}$, $p \subseteq p \cup \{(i, k)\}$, and these two extensions of p are incompatible. $\square$

Proposition 28.27. *If the condition in the hypothesis of Lemma 28.2 fails, then there is a $\mathbb{P}$-generic filter G over M such that $G \in M$, and G intersects every dense subset of P (not only those in M).*

Proof. Let p be such that for all q, r, if q and r are $\leq p$ then they are compatible. Define

$$G = \{q : \exists r[r \leq q \text{ and } (r \leq p \text{ or } p \leq r)].$$

We claim that G is as desired. Clearly $G \in M$.

Note that $p \in G$, by taking $r = p$.

To check (1), suppose that $q, r \in G$; we want to find $s \in G$ with $s \leq q, r$. Choose $t \leq q$ such that $t \leq p$ or $p \leq t$, and choose $u \leq r$ such that $u \leq p$ or $p \leq u$. If $t \leq p$ and $u \leq p$, then by the choice of p there is a $v \leq t, u$. Then $v \leq u \leq p$, and $v \leq t \leq q$, hence $v \in G$, and $v \leq q$ and $v \leq u \leq r$, as desired.

If $t \leq p$ and $p \leq u$, then $t \leq u \leq r$, so $t \in G$ and $t \leq q, r$, as desired.

If $p \leq t$ and $u \leq p$, then $u \leq r$ implies that $u \in G$, and $u \leq p \leq t \leq q$, as desired.

Finally, if $p \leq t, u$, then $p \leq q, r$ and $p \in G$, as desired. So (1) holds.

(2) is obvious.

Now suppose that D is dense. Choose $q \in D$ such that $q \leq p$. Then $q \in G$, as desired. $\square$

Proposition 28.28. *Assume the hypothesis of Lemma 28.2. Then there does not exist a $\mathbb{P}$-generic filter over M which intersects every dense subset of P (not only those which are in M).*

Proof. Take G generic; we show that $\{p \in P : p \notin G\}$ is dense. Let D be the set indicated in the hint. Let p be any element of P, and choose incompatible $q, r \leq p$ by the hypothesis of Lemma 28.2. Then it is not true that both $q, r \in G$, as desired; this checks that D is dense. Obviously $G \cap D = \emptyset$, proving the assertion of the proposition. $\square$

Proposition 28.29. *Show that if $\mathbb{P}$ satisfies the condition of Lemma 28.2, then it has uncountably many dense subsets.*

Proof. (Proof due to Josh Sanders) For each $p \in P$ let $p(0), p(1)$ be members of P such that $p(0), p(1) \leq p$ and $p(0) \perp p(1)$; thus $p(0), p(1) < p$. We now define an element p_f for each finite sequence f of 0s and 1s, by recursion on the domain of f. Let $p_\emptyset$ be any element of P. Having defined p_f, let $p_{f0} = p_f(0)$ and $p_{f1} = p_f(1)$. For each $f \in {}^\omega 2$ let $K_f = \{p_{f\restriction m} : m \in \omega\}$. Now if $f, g \in {}^\omega 2$ and $f \neq g$, choose m minimum such that $f(m) \neq g(m)$. Clearly then $p_{f\restriction(m+1)} \in K_f \backslash K_g$. Thus $K_f \neq K_g$ for distinct $f, g \in {}^\omega 2$. For each $f \in {}^\omega 2$ let $D_f = P\backslash K_f$. So $D_f \neq D_g$ for $f \neq g$.

We claim that each D_f is dense; this will prove the statement of the proposition. To see this, take any $q \in P$. If $q \in D_f$, then there is nothing to prove. Suppose that $q \notin D_f$. Thus $q \in K_f$. Say $q = p_{f\restriction m}$. Let $\varepsilon = 1 - f(m)$. Then $p_{(f\restriction m)\varepsilon} \in D_f$ and $p_{(f\restriction m)\varepsilon} \leq q$, as desired. $\square$

Proposition 28.30. *Assume the hypothesis of Lemma 28.2. Then there are 2^ω filters which are $\mathbb{P}$-generic over M.*

Proof. Let $D_0, D_1, \ldots$ list all of the dense subsets of $\mathbb{P}$ which are in M. We now define elements r_f and p_f in $\mathbb{P}$ for f a finite sequence of 0's and 1's, by recursion on the length of f. Let $p_\emptyset = 1$ and choose $r_\emptyset \in D_0$ so that $r_\emptyset \leq p_\emptyset$. Now suppose that p_f and r_f have been defined, with f having domain $n \in \omega$. Choose p_{f0} and p_{f1} both $\leq r_f$ so that $p_{f0} \perp p_{f1}$. Then choose $r_{f0} \leq p_{f0}$ with $r_{f0} \in D_{n+1}$, and choose $r_{f1} \leq p_{f1}$ with $r_{f1} \in D_{n+1}$.

For any $f \in {}^\omega 2$ let

$$G_f = \{q \in \mathbb{P} : p_{f\restriction n} \leq q \text{ for some } n \in \omega\}.$$

Clearly G_f is $\mathbb{P}$-generic; and there are 2^ω such filters. $\square$

Proposition 28.31. *Let $\mathbb{P} = (\{1\}, \leq, 1)$. The collection of all $\mathbb{P}$-names is a proper class.*

Proof. Clearly the following two facts hold with no assumption about P.

(1) If σ is a P-name, then so is $\{(\sigma, 1)\}$.

(2) If A is a set of P-names, then $\{(\sigma, 1) : \sigma \in A\}$ is a P-name.

Now let A be the collection of all P-names, and suppose that A is a set. By (2), also $\tau \stackrel{\text{def}}{=} \{(\sigma, 1) : \sigma \in A\}$ is a P-name, and by (1), $\{(\tau, 1)\}$ is a P-name. So $\sigma \stackrel{\text{def}}{=} \{(\tau, 1)\} \in A$. Thus

$$\tau \in \{\tau\} \in \{\{\tau\}, \{\tau, 1\}\} = (\tau, 1) \in \{(\tau, 1)\} = \sigma \in \{\sigma\} \in \{\{\sigma\}, \{\sigma, 1\}\} = (\sigma, 1) \in \tau,$$

contradiction. $\square$

Proof.

Proposition 28.32. *$p \Vdash \sigma = \tau$ iff the following two conditions hold.*
(i) For every $(\xi, q) \in \sigma$ and every $r \leq p, q$ one has $r \Vdash \xi \in \tau$.
(ii) For every $(\xi, q) \in \tau$ and every $r \leq p, q$ one has $r \Vdash \xi \in \sigma$.

Proof. First assume that $p \Vdash \sigma = \tau$. For (i), suppose that $(\xi, q) \in \sigma$ and $r \leq p, q$. Let G be $\mathbb{P}$-generic over M with $r \in G$. Then also $p, q \in G$, so $\xi_G \in \sigma_G$ and $\sigma_G = \tau_G$. Hence $\xi_G \in \tau_G$, as desired. (ii) is similar.

Second assume the two conditions. To show that $p \Vdash \sigma = \tau$, let G be $\mathbb{P}$-generic over M with $p \in G$. Suppose that $x \in \sigma_G$. Then there is a $(\xi, q) \in \sigma$ such that $q \in G$ and $x = \xi_G$. Choose $r \in G$ such that $r \leq p, q$. By (i) we have $\xi_G \in \tau_G$. This shows that $\sigma_G \subseteq \tau_G$. The other inclusion is similar. $\square$

Proof.

Proposition 28.33. *Assume that $\mathbb{P} \in M$, $p, q \in P$, and $p \perp q$. Then $\{\tau \in M^P : p \Vdash \tau = \check{\emptyset}\}$ is a proper class in M.*

Proof. In M we define members τ_α of $M^{\mathbb{P}}$ by recursion, such that $p \Vdash \tau_\alpha = \check{0}$, and such that the ranks increase. let $\tau_0 = \check{0}$. Having defined τ_α, let $\tau_{\alpha+1} = \{(\tau_\alpha, q)\}$. Note that if G is generic and $p \in G$, then $q \notin G$, and so $(\tau_{\alpha+1})_G = \emptyset$; so $p \Vdash \tau_{\alpha+1} = \emptyset$. For λ a limit ordinal, let $\tau_\lambda = \{(\tau_\alpha, q) : \alpha < \lambda\}$. Clearly $p \Vdash \tau_\lambda = \check{0}$. □

Proposition 28.34. *The forcing order of Proposition 28.25 is separative.*

Proof. If $p \supseteq q \supseteq p$, then $p = q$. Now suppose that $p \not\supseteq q$. Choose $(i, j) \in q \backslash p$. If $i \in \text{dmn}(p)$, then $p \perp q$, as desired. If $i \notin \text{dmn}(p)$, let $k \in J \backslash \{j\}$ and define $r = p \cup \{(i, k)\}$. Then $r \supseteq p$ and $r \perp q$. □

Proposition 28.35. *Assume that $\mathbb{P} \in M$ is separative and $p, q, r \in P$. Then the following two conditions are equivalent:*

(i) $p \Vdash \{(\{(\emptyset, q)\}, r)\} = \check{1}$.

(ii) $p \leq r$ and $p \perp q$.

Proof. $\Rightarrow$: Assume that

$$p \Vdash \{(\{(0, q)\}, r)\} = \check{1}. \tag{1}$$

Suppose that $p \not\leq r$. By the definition of separative, choose s such that $s \leq p$ and $s \perp r$. Let G be $\mathbb{P}$-generic over M with $s \in G$. Then $\{(\{(0, q)\}, r)\}_G = 0$, contradiction. Thus $p \leq r$. Suppose that p and q are compatible; say $t \leq p, q$. Let G be $\mathbb{P}$-generic over M with $t \in G$. Then $q \in G$, so $\{(0, q)\}_G = \{0\}$. Also $r \in G$, so $\{(\{(0, q)\}, r)\}_G = \{\{0\}\} \neq 1$, contradiction.

$\Leftarrow$: Suppose that $p \leq r$ and $p \perp q$. Suppose that G is $\mathbb{P}$-generic over M and $p \in G$. Then $q \notin G$, so $\{(0, q)\}_G = 0$. Also, $r \in G$, so $\{(\{(0, q)\}, r)\}_G = \{\{(0, q)\}_G\} = \{0\} = 1$, as desired. □

Proposition 28.36. *Suppose that $f : A \to M$ with $f \in M[G]$. Then there is a $B \in M$ such that $f : A \to B$.*

Proof. Let $f = \tau_G$ and define $B = \{b : \exists p \in P[p \Vdash [\check{b} \in \text{rng}(\tau)]]\}$. The definition of B takes place in M; so $B \in M$. Suppose that b is in the range of f. Thus $\check{b}_G = b \in \text{rng}(\tau_G)$, so we can choose $p \in B$ such that $p \Vdash \check{b} \in \text{rng}(\tau)$. So $b \in B$, as desired. □

Proposition 28.37. *Assume that $\mathbb{P} \in M$ and α is a cardinal of M. Then for any $\mathbb{P}$-generic G over M the following conditions are equivalent:*

(1) For all $B \in M$, ${}^\alpha B \cap M = {}^\alpha B \cap M[G]$.

(2) ${}^\alpha M \cap M = {}^\alpha M \cap M[G]$.

Proof. (1)$\Rightarrow$(2): Assume (1). Obviously $\subseteq$ in (2) holds. Now suppose that $f \in {}^\alpha M \cap M[G]$. By proposition 28.35 choose $B \in M$ such that $f : \alpha \to B$. So by (1), $f \in M$, as desired.

(2)⇒(1): Assume (2). Then $\subseteq$ in (1) is clear. Suppose that $f \in {}^{\alpha}B \cap M[G]$. Then $f \in {}^{\alpha}M \cap M[G]$ since M is transitive, so $f \in M$ by (2). $\square$

Proposition 28.38. *Suppose that $\mathbb{P} \in M$ is a forcing order satisfying the condition of Lemma 28.2. Assume that*

$$M = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \subseteq \cdots \quad (n \in \omega),$$

where $M_{n+1} = M_n[G_n]$ for some G_n which is $\mathbb{P}$-generic over M_n, for each $n \in \omega$. Then the power set axiom fails in $\bigcup_{n\in\omega} M_n$.

Proof. Assume that $R = \bigcup_{n\in\omega} M_n$ does satisfy the power set axiom. Then $R \models \exists y \forall z(z \subseteq P \to z \in y)$. Choose $y \in R$ so that $R \models \forall z(z \subseteq P \to z \in y)$. Say $y \in M_n$. Then $R \models G_n \subseteq P \to z \in y$. By absoluteness, $R \models G_n \subseteq P$. So $R \models G_n \in y$, hence $G_n \in y \in M_n$. This contradicts Lemma 28.2.

Proposition 28.39. *The following conditions are equivalent:*

$$[\![\varphi(\sigma_0, \ldots, \sigma_{m-1}) \leftrightarrow \psi(\sigma_0, \ldots, \sigma_{m-1})]\!] = 1$$
$$[\![\varphi(\sigma_0, \ldots, \sigma_{m-1})]\!] = [\![\psi(\sigma_0, \ldots, \sigma_{m-1})]\!].$$

Proof. We omit the parameters $\sigma_0, \ldots, \sigma_{m-1}$. First assume that $[\![\varphi \leftrightarrow \psi]\!] = 1$. Then

$$\begin{aligned}
0 &= -[\![\varphi \leftrightarrow \psi]\!] \\
&= -[\![(\varphi \to \psi) \wedge (\psi \to \varphi)]\!] \\
&= -[\![\neg(\varphi \wedge \neg\psi) \wedge \neg(\psi \wedge \neg\varphi)]\!] \\
&= -([\![\neg(\varphi \wedge \neg\psi)]\!] \cdot [\![\neg(\psi \wedge \neg\varphi)]\!] \\
&= -(-[\![\varphi \wedge \neg\psi) \cdot -[\![\psi \wedge \neg\varphi]\!]) \\
&= [\![\varphi \wedge \neg\psi]\!] + [\![\psi \wedge \neg\varphi]\!] \\
&= ([\![\varphi]\!] \cdot -[\![\psi]\!]) + ([\![\psi]\!] \cdot -[\![\psi]\!].
\end{aligned}$$

It follows that $[\![\varphi]\!] = [\![\psi]\!]$.

Conversely, if $[\![\varphi]\!] = [\![\psi]\!]$, then we can reverse the above equations to get $-[\![\varphi \leftrightarrow \psi]\!] = 0$, so that $[\![\varphi \leftrightarrow \psi]\!] = 1$. $\square$

Proposition 28.40. $[\![\sigma = \tau]\!] \cdot [\![\tau = \rho]\!] \leq [\![\sigma = \rho]\!]$.

Proof. Suppose not. Then $[\![\sigma = \tau]\!] \cdot [\![\tau = \rho]\!] \cdot -[\![\sigma = \rho]\!] \neq 0$. By Theorem 9.20(i) choose p so that $e(p) \leq [\![\sigma = \tau]\!] \cdot [\![\tau = \rho]\!] \cdot -[\![\sigma = \rho]\!]$. Then $(p \Vdash^* \sigma = \tau)^M$, $(p \Vdash^* \tau = \rho)^M$, and $p \Vdash^* (\neg\sigma = \rho)^M$. Let G be $\mathbb{P}$-generic over M. Then by Theorem 28.19, $\sigma_G = \tau_G$, $\tau_G = \rho_G$, and $\sigma_G \neq \rho_G$, contradiction. $\square$

Proposition 28.41. *If ZFC $\models \varphi$ then $[\![\varphi]\!] = 1$, for any sentence φ.*

Proof. Suppose that $[\![\varphi]\!] \neq 1$. Thus $-[\![\varphi]\!] \neq 0$, so by Theorem 13.20(i) choose p so that $e(p) \leq -[\![\varphi]\!] = [\![\neg\varphi]\!]$. Thus $(p \Vdash^* \neg\varphi)^M$. Let G be $\mathbb{P}$-generic over M. Then by Theorem 28.19 we have $\neg\varphi^{M[G]}$. $\square$

29. Forcing and cardinal arithmetic

The main forcing orders used in this chapter are special cases of the following. For sets I, J and for λ an infinite cardinal,

$$\mathrm{Fn}(I, J, \lambda) = (\{f : f \text{ is a function contained in } I \times J \text{ and } |f| < \lambda\}, \supseteq, \emptyset).$$

We first show that $\neg$CH is consistent. The main part of the proof is given in the following theorem.

Theorem 29.1. (Cohen) *Let M be a c.t.m. of ZFC. Suppose that κ is any infinite cardinal of M. Let G be $\mathrm{Fn}(\kappa, 2, \omega)$-generic over M. Then $2^\omega \geq \kappa$ in $M[G]$.*

Proof. Let $g = \bigcup G$. Since any two members of G are compatible, g is a function.

(1) For each $\alpha \in \kappa$, the set $\{f \in \mathrm{Fn}(\kappa, 2, \omega) : \alpha \in \mathrm{dmn}(f)\}$ is dense in $\mathrm{Fn}(\kappa, 2, \omega)$ (and it is a member of M).

In fact, given $f \in \mathrm{Fn}(\kappa, 2, \omega)$, either f is already in the above set, or else $\alpha \notin \mathrm{dmn}(f)$ and then $f \cup \{(\alpha, 0)\}$ is an extension of f which is in that set. So (1) holds.

Since G intersects each set (1), it follows that g maps κ into 2. Let (in M) $h : \kappa \times \omega \to \kappa$ be a bijection. For each $\alpha < \kappa$ let $a_\alpha = \{m \in \omega : g(h(\alpha, m)) = 1\}$. We claim that $a_\alpha \neq a_\beta$ for distinct α, β; this will give our result. In fact, for distinct $\alpha, \beta < \kappa$, the set

$$\begin{aligned}\{f \in \mathrm{Fn}(\kappa, 2, \omega) &: \text{there is an } m \in \omega \text{ such that} \\ &h(\alpha, m), h(\beta, m) \in \mathrm{dmn}(f) \text{ and } f(h(\alpha, m)) \neq f(h(\alpha, m))\}\end{aligned}$$

is dense in $\mathrm{Fn}(\kappa, 2, \omega)$ (and it is in M). In fact, let distinct α and β be given, and suppose that $f \in \mathrm{Fn}(\kappa, 2, \omega)$. Now $\{m : h(\alpha, m) \in f \text{ or } h(\beta, m) \in f\}$ is finite, so choose $m \in \omega$ not in this set. Thus $h(\alpha, m), h(\beta, m) \notin f$. Let $h = f \cup \{(h(\alpha, m), 0), (h(\beta, m), 1)\}$. Then h extends f and is in the above set, as desired.

It follows that G contains a member of this set. Hence $a_\alpha \neq a_\beta$. □

By taking $\kappa > \omega_1$ in M, it would appear that we have shown the consistency of $\neg$CH. But there is a major detail that we have to take care of. Possibly ω_1 means something different in $M[G]$ than it does in M; maybe we have accidentally introduced a bijection from the ω_1 of M onto ω. Since M is countable, this is conceivable.

To illustrate this problem, let $\mathbb{P}$ be the forcing order consisting of all finite functions mapping a subset of ω into ω_1, ordered by $\supseteq$, with $\emptyset$ as "largest" element. Suppose that G is $\mathbb{P}$-generic over M. Now the following sets are dense:

$$\begin{aligned} A_m &\stackrel{\text{def}}{=} \{f \in P : m \in \mathrm{dmn}(f)\} \quad \text{for each } m \in \omega, \\ B_\alpha &\stackrel{\text{def}}{=} \{f \in P : \alpha \in \mathrm{rng}(f)\} \quad \text{for each } \alpha \in \omega_1^M. \end{aligned}$$

In fact, given any $g \in P$, if $m \in \omega \backslash \mathrm{dmn}(g)$, then $g \cup \{(m, 0)\}$ is a member of P which in A_m and contains g; and given any $g \in P$ and $\alpha < \omega_1^M$, choose $m \in \omega \backslash \mathrm{dmn}(g)$; then $g \cup \{(m, \alpha)\}$ is a member of P which in B_α and contains g. Now if G is $\mathbb{P}$-generic over M

and intersects all of the sets A_m and B_α, then clearly $\bigcup G$, which is a member of $M[G]$, is a function mapping ω onto ω_1^M. So ω_1^M gets "collapsed" to a countable ordinal in $M[G]$. Note that $\omega^M = \omega^{M[G]}$ by absoluteness.

Thus to finish the proof of consistency of ¬CH we need to study the preservation of cardinals in the passage from M to $M[G]$.

$\mathbb{P}$ *preserves cardinals* $\geq \kappa$ iff for every G which is $\mathbb{P}$-generic over M and every ordinal $\alpha \geq \kappa$ in M, α is a cardinal in M iff α is a cardinal in $M[G]$.

$\mathbb{P}$ *preserves cofinalities* $\geq \kappa$ iff for every G which is $\mathbb{P}$-generic over M and every limit ordinal α in M such that $(\mathrm{cf}(\alpha))^M \geq \kappa$, $(\mathrm{cf}(\alpha))^M = (\mathrm{cf}(\alpha))^{M[G]}$.

$\mathbb{P}$ *preserves regular cardinals* $\geq \kappa$ iff for every G which is $\mathbb{P}$-generic over M and every ordinal $\alpha \geq \kappa$ which is a regular cardinal of M, α is also a regular cardinal of $M[G]$.

If $\kappa = \omega$, we say simply that $\mathbb{P}$ preserves cardinals, cofinalities, or regular cardinals.

In these definitions, if we replace "$\geq$" by "$\leq$" we obtain new definitions which will be used below also.

The relationship between these notions that we want to give uses the following fact.

Lemma 29.2. *Suppose that α is a limit ordinal, κ and λ are regular cardinals, $f : \kappa \to \alpha$ is strictly increasing with* $\mathrm{rng}(f)$ *cofinal in α, and $g : \lambda \to \alpha$ is strictly increasing with* $\mathrm{rng}(g)$ *cofinal in α. Then $\kappa = \lambda$.*

Proof. Suppose not; say by symmetry $\kappa < \lambda$. For each $\xi < \kappa$ choose $\eta_\xi < \lambda$ such that $f(\xi) < g(\eta_\xi)$. Let $\rho = \sup_{\xi<\kappa} \eta_\xi$. Thus $\rho < \lambda$ by the regularity of λ. But then $f(\xi) < g(\rho) < \alpha$ for all $\xi < \kappa$, contradiction. □

Proposition 29.3. *Let M be a c.t.m. of ZFC, $\mathbb{P} \in M$ be a forcing order, and κ be a cardinal of M.*

(i) If $\mathbb{P}$ preserves regular cardinals $\geq \kappa$, then it preserves cofinalities $\geq \kappa$.

(ii) If $\mathbb{P}$ preserves cofinalities $\geq \kappa$ and κ is regular, then $\mathbb{P}$ preserves cardinals $\geq \kappa$.

(iii) If $\mathbb{P}$ preserves cofinalities, then $\mathbb{P}$ preserves cardinals.

Proof. (i): Let α be a limit ordinal of M with $(\mathrm{cf}(\alpha))^M \geq \kappa$. Then $(\mathrm{cf}(\alpha))^M$ is a regular cardinal of M which is $\geq \kappa$ and hence is also a regular cardinal of $M[G]$. Now we can apply Lemma 29.2 within $M[G]$ to $\kappa = (\mathrm{cf}(\alpha))^M$ and $\lambda = (\mathrm{cf}(\alpha))^{M[G]}$ to infer that $(\mathrm{cf}(\alpha))^M = (\mathrm{cf}(\alpha))^{M[G]}$.

(ii): Suppose that cardinals $\geq \kappa$ are not preserved, and let λ be the least cardinal of M which is $\geq \kappa$ but which is not a cardinal of $M[G]$. If λ is regular in M, then

$$\lambda = (\mathrm{cf}(\lambda))^M = (\mathrm{cf}(\lambda))^{M[G]},$$

and so λ is a regular cardinal in $M[G]$, contradiction. If λ is singular in M, then $\lambda > \kappa$ since κ is regular and $\lambda \geq \kappa$. So λ is the supremum of a set S of cardinals of M which are regular and $\geq \kappa$, so each member of S is a cardinal of $M[G]$ by the minimality of λ, so λ is a cardinal of $M[G]$.

(iii): follows from (ii), with $\kappa = \omega$. □

We can replace "$\geq$" by "$\leq$" in this proposition and its proof; call this new statement *Proposition 29.3′*. The very last part of the proof of 29.3 can be simplified for $\leq$, and actually one does not need to assume that κ is regular in this case.

A forcing order $\mathbb{P}$ satisfies the *κ-chain condition*, abbreviated κ-c.c., iff every antichain in $\mathbb{P}$ has size less than κ.

The following theorem is very useful in forcing arguments.

Theorem 29.4. *Let M be a c.t.m. of ZFC, $\mathbb{P} \in M$ be a forcing order, κ be a cardinal of M, G be $\mathbb{P}$-generic over M, and suppose that $\mathbb{P}$ satisfies the κ-c.c. Suppose that $f \in M[G]$, $A, B \in M$, and $f : A \to B$. Then there is an $F : A \to \mathscr{P}(B)$ with $F \in M$ such that:*

(i) $f(a) \in F(a)$ for all $a \in A$.

(ii) $(|F(a)| < \kappa)^M$ for all $a \in A$.

Proof. Let $\tau \in M^P$ be such that $\tau_G = f$. Thus the statement "$\tau_G : A \to B$" holds in $M[G]$. Hence by Corollary 28.21 there is a $p \in G$ such that

$$p \Vdash \tau : \check{A} \to \check{B}.$$

Now for each $a \in A$ let

$$F(a) = \{b \in B : \text{there is a } q \leq p \text{ such that } q \Vdash \mathrm{op}(\check{a}, \check{b}) \in \tau\}.$$

To prove (i), suppose that $a \in A$. Let $b = f(a)$. Thus $(a, b) \in f$, so by Theorem 15.21 there is an $r \in G$ such that $r \Vdash \mathrm{op}(\check{a}, \check{b}) \in \tau$. Let $q \in G$ with $q \leq p, r$. Then q shows that $b \in F(a)$.

To prove (ii), again suppose that $a \in A$. By the axiom of choice in M, there is a function $Q : F(a) \to P$ such that for any $b \in F(a)$, $Q(b) \leq p$ and $Q(b) \Vdash \mathrm{op}(\check{a}, \check{b}) \in \tau$.

(1) If $b, b' \in F(a)$ and $b \neq b'$, then $Q(b) \perp Q(b')$.

In fact, suppose that $r \leq Q(b), Q(b')$. Then

$$r \Vdash \mathrm{op}(\check{a}, \check{b}) \in \tau \wedge \mathrm{op}(\check{a}, \check{b'}) \in \tau; \tag{2}$$

but also $r \leq Q(b) \leq p$, so $r \Vdash \tau : \check{A} \to \check{B}$, hence

$$r \Vdash \forall x, y, z[\mathrm{op}(x, y) \wedge \mathrm{op}(x, z) \to y = z]$$

and hence

$$r \Vdash \mathrm{op}(\check{a}, \check{b}) \in \tau \wedge \mathrm{op}(\check{a}, \check{b'}) \in \tau \to \check{b} = \check{b'}. \tag{3}$$

Now let H be $\mathbb{P}$-generic over M with $r \in H$. By the definition of forcing and (2) we have $(a, b) = (\mathrm{op}(\check{a}, \check{b}))_G \in \tau_G$ and $(a, b') = (\mathrm{op}(\check{a}, \check{b'}) \in \tau_G$. By (3) and the definition of forcing it follows that $b = b'$. Thus (1) holds.

By (1), $\langle Q(b) : b \in F(a)\rangle$ is a one-one function onto an antichain of P. Hence $(|F(a)| < \kappa)^M$ by the κ-cc. $\square$

Proposition 29.5. *If M is a c.t.m. of ZFC, κ is a cardinal of M, and $\mathbb{P} \in M$ satisfies κ-cc in M, then $\mathbb{P}$ preserves regular cardinals $\geq \kappa$, and also preserves cofinalities $\geq \kappa$. If also κ is regular in M, then $\mathbb{P}$ preserves cardinals $\geq \kappa$.*

Proof. First we want to show that if $\lambda \geq \kappa$ is regular in M then also λ is regular in $M[G]$ (and hence is a cardinal of $M[G]$). Suppose that this is not the case. Hence in $M[G]$ there is an $\alpha < \lambda$ and a function $f : \alpha \to \lambda$ such that the range of f is cofinal in λ. Recall from Lemma 28.13 that M and $M[G]$ have the same ordinals. Thus $\alpha \in M$. By Theorem 29.4, let $F : \alpha \to \mathscr{P}(\lambda)$ be such that $f(\xi) \in F(\xi)$ and $(|F(\xi)| < \lambda)^M$ for all $\xi < \alpha$. Let $S = \bigcup_{\xi<\alpha} F(\xi)$. Then S is a subset of λ which is cofinal in λ and has size less than λ, contradiction.

The rest of the proposition follows from Proposition 29.3. □

By the *countable chain condition*, abbreviated ccc, we mean the ω_1-chain condition.

Lemma 29.6. *If κ is an infinite cardinal, then $\mathrm{Fn}(\kappa, 2, \omega)$ satisfies ccc.*

Proof. Suppose that $\mathscr{F} \subseteq \mathrm{Fn}(\kappa, 2, \omega)$ is uncountable. Since for each finite $F \subseteq \kappa$ there are only finitely many members of $\mathscr{F}$ with domain F, it is clear that $\{\mathrm{dmn}(f) : f \in \mathscr{F}\}$ is an uncountable collection of finite sets. By the Δ-system lemma, let $\mathscr{G}$ be an uncountable subset of this collection which forms a Δ-system, say with root R. Then

$$\mathscr{G} = \bigcup_{k \in {}^R 2} \{f \in \mathscr{G} : f \restriction R = k\};$$

since ${}^R 2$ is finite, there is a $k \in {}^R 2$ such that

$$\mathscr{H} \stackrel{\text{def}}{=} \{f \in \mathscr{G} : f \restriction R = k\}$$

is uncountable. Clearly f and g are compatible for any $f, g \in \mathscr{H}$. □

Theorem 29.7. (Cohen) *Let M be a c.t.m. of ZFC. Suppose that κ is any cardinal of M. Let G be $\mathrm{Fn}(\kappa, 2, \omega)$-generic over M. Then $M[G]$ has the same cofinalities and cardinals as M, and $2^\omega \geq \kappa$ in $M[G]$.*

Proof. By Theorems 29.1, 29.5, and 29.6, also using the fact that ω is absolute. □

The method of proof of Theorem 29.7 is called *Cohen forcing.*

Theorem 29.8. (Cohen) *If* ZFC *is consistent, then so is* ZFC + ¬CH.

Proof. Apply Theorem 29.7 with κ a cardinal of M greater than ω_1^M. □

We now turn to the proof of consistency of CH. This depends on a new notion which is important in its own right.

Let λ be an infinite cardinal. A forcing order $\mathbb{P} = (P, \leq, 1)$ is λ-*closed* iff for all $\gamma < \lambda$ and every system $\langle p_\xi : \xi < \gamma \rangle$ of elements of P such that $p_\eta \leq p_\xi$ whenever $\xi < \eta < \gamma$, there is a $q \in P$ such that $q \leq p_\xi$ for all $\xi < \gamma$.

The importance of this notion for generic extensions comes about because of the following theorem, which is similar to Theorem 29.4.

Theorem 29.9. *Suppose that M is a c.t.m. of ZFC, $\mathbb{P} \in M$ is a forcing order, λ is a cardinal of M, $\mathbb{P}$ is λ-closed, $A, B \in M$, and $|A| < \lambda$. Suppose that G is $\mathbb{P}$-generic over M and $f \in M[G]$ with $f : A \to B$. Then $f \in M$.*

Proof. It suffices to prove this when A is an ordinal. For, suppose that this special case has been shown, and now suppose that A is arbitrary. In M, let j be a bijection from $\alpha \overset{\text{def}}{=} |A|^M$ onto A. Then $f \circ j : \alpha \to B$, so $f \circ j \in M$ by the special case. Hence $f \in M$.

So now we assume that $A = \alpha$, an ordinal less than λ. Let $K = ({}^{\alpha}B)^M$. Let $f = \tau_G$. We want to show that $f \in K$, for then $f \in M$. Suppose not. Now $\tau_G : \alpha \to B$ and $\tau_G \in K$. Hence by Theorem 28.21 there is a $p \in G$ such that

$$p \Vdash \tau : \check{\alpha} \to \check{B} \wedge \tau \notin \check{K}. \tag{1}$$

For a while we work entirely in M. We will define sequences $\langle p_\eta : \eta \leq \alpha \rangle$ of elements of P and $\langle z_\eta : \eta < \alpha \rangle$ of elements of B by recursion, so that the following conditions hold:

(2) $p_0 = p$.
(3) $p_\eta \leq p_\xi$ if $\xi < \eta$.
(4) $p_{\eta+1} \Vdash \tau(\check{\eta}) = \check{z}_\eta$.

Of course we start out by defining $p_0 = p$, so that (2) holds. Now suppose that p_η has been defined so that (2)–(4) hold; we define $p_{\eta+1}$. In fact, we claim that there exist a $p_{\eta+1} \leq p_\eta$ and a $z_\eta \in B$ such that $p_{\eta+1} \leq p_\eta$ and $p_{\eta+1} \Vdash \tau(\check{\eta}) = \check{z}_\eta$. To prove this claim, suppose that $p_\eta \in H$ where H is $\mathbb{P}$-generic over M. Then by (1), $\tau_H : \alpha \to B$ and $\tau_H \notin K$. Hence $\tau_H(\eta) \in B$; say $\tau_H(\eta) = z_\eta$. By Theorem 28.21, there is a $q \in H$ such that $q \Vdash \tau(\check{\eta}) = \check{z}_\eta$. Let $p_{\eta+1} \in H$ with $p_{\eta+1} \leq p_\eta, q$. This proves the claim. Thus (2)–(4) holds.

For η limit, p_η is given by the definition of λ-closed.

Note that the function z defined in this way is in K.

This finishes our argument within M. Now let H be $\mathbb{P}$-generic over M with $p_\alpha \in H$. Then $\tau_H(\eta) = z_\eta$ for each $\eta < \alpha$ by (4), so that $\tau_H = z \in K$. This contradicts (1), since $p_\alpha \leq p$. □

Proposition 29.10. *Suppose that M is a c.t.m. of ZFC. $\mathbb{P} \in M$ is a forcing order, λ is a regular cardinal of M, and $\mathbb{P}$ is λ-closed. Then $\mathbb{P}$ preserves cofinalities and cardinals $\leq \lambda$.*

Proof. Otherwise, by Proposition 29.3′ there is a regular cardinal $\kappa \leq \lambda$ of M which is not regular in $M[G]$. Thus there exist in $M[G]$ an ordinal $\alpha < \kappa$ and a function $f : \alpha \to \kappa$ such that $\text{rng}(f)$ is cofinal in κ. By Theorem 29.9, $f \in M$, contradiction. □

Theoerem 29.11. *Let M be a c.t.m. of ZFC, and let G be $\mathbb{F}(\omega_1, 2, \omega_1)$-generic over M. Then CH holds in $M[G]$, and $\omega_1^M = \omega_1^{M[G]}$.*

First we show that $\mathbb{F}(\omega_1, 2, \omega_1)$ is ω_1-closed. Let $\langle f_\xi : \xi < \alpha \rangle$ be a sequence of members of $\mathbb{F}(\omega_1, 2, \omega_1)$ such that $\alpha < \omega_1$ and $\forall \xi, \eta[\xi < \eta < \alpha \to f_\eta \supseteq f_\xi]$. Then clearly $\bigcup_{\xi<\alpha} f_\xi \in$

$\mathbb{F}(\omega_1, 2, \omega_1)$ and $\bigcup_{\xi<\alpha} f_\xi \supseteq f_\eta$ for each $\eta < \xi$. Now it follows from Proposition 29.10 that $\omega_1^M = \omega_1^{M[G]}$.

By Theorem 29.9 we have $({}^\omega 2)^{M[G]} \subseteq M$. Let $F : \omega_1 \times \omega \to \omega_1$ be a bijection. For each $f \in {}^\omega 2$ let

$$D_f = \{g \in \mathrm{Fn}(\omega_1, 2, \omega_1) : \exists \alpha < \omega_1 \forall n \in \omega[F(\alpha, n) \in \mathrm{dmn}(g) \text{ and } g(F(\alpha, n)) = f(n)]\}\}.$$

Clearly D_f is dense in $\mathrm{Fn}(\omega_1, 2, \omega_1)$. Define $h : \omega_1^M \to {}^\omega 2$ in $M[G]$ by: $(h(\alpha))(n) = (\bigcup G)(F(\alpha, n))$. Then h maps onto ${}^\omega 2$ by the denseness of the D_f's, as desired. □

Theorem 29.12. (Gödel) *If ZFC is consistent, then so is ZFC + CH.* □

Gödel also showed that if ZF is consistent, then so is ZFC + GCH. For this he introduced the notion of constructible sets.

We give some elementary facts about forcing which will be used later.

Theorem 29.13. $p \Vdash \check{a} \in \check{b}$ *iff* $a \in b$. □

Theorem 29.14. $p \Vdash \exists x\varphi(x, \sigma_0, \ldots, \sigma_{m-1})$ *iff the set*

$$\{r \leq p : \textit{there is a } \tau \in M^P[r \Vdash \varphi(\tau, \sigma_0, \ldots, \sigma_{m-1})]\}$$

is dense below p.

Proof. $\Rightarrow$: Assume that $p \Vdash \exists x\varphi(x, \sigma_0, \ldots, \sigma_{m-1})$, and $q \leq p$. Let G be $\mathbb{P}$-generic over M with $q \in G$. Then also $p \in G$, so $(\exists x\varphi(\sigma_0, \ldots, \sigma_{m-1}))^{M[G]}$. Hence there is a $\tau \in M^P$ such that $\varphi^{M[G]}(\tau_G, \sigma_{0G}, \ldots, \sigma_{(m-1)G})$ holds. Choose $s \in G$ such that $s \Vdash \varphi(\tau, \sigma_0, \ldots, \sigma_{m-1})$. Then choose $r \in G$ with $r \leq q, s$. Thus r is in the indicated set, as desired.

$\Leftarrow$: Assume the indicated condition, and suppose that $p \in G$ with G $\mathbb{P}$-generic over M. Then there is an $r \in G$ with r in the indicated set. Hence $\varphi^{M[G]}(\tau_G, \sigma_{0G}, \ldots, \sigma_{(m-1)G})$, and so also $\exists x\varphi^{M[G]}(x, \sigma_{0G}, \ldots, \sigma_{(m-1)G})$, as desired. □

Proposition 29.15. *If* $p \Vdash \exists x \in \sigma\varphi(x)$, *then there exist a* $q \leq p$ *and a* $(\tau, r) \in \sigma$ *such that* $q \leq r$ *and* $q \Vdash \varphi(\tau)$.

Proof. Let G be generic such that $p \in G$. Then there is an $x \in \sigma_G$ such that $M[G] \models \varphi(x)$. Say $(\tau, r) \in \sigma$ with $r \in G$ and $x = \tau_G$. Thus $M[G] \models \varphi(\tau_G)$, so there is an $s \in G$ such that $s \Vdash \varphi(\tau)$. Choose $q \in G$ such that $q \leq p, r, s$ □

Proposition 29.16. $\mathrm{Fn}(\omega_1, 2, \omega_1)$ *preserves cardinals* $\geq \omega_2$.

Proof. Clearly $|\mathrm{Fn}(\omega_1, 2, \omega_1)| = \omega_1$, so $\mathrm{Fn}(\omega_1, 2, \omega_1)$ is ω_2-cc. Hence the result follows by Proposition 29.5. □

Proposition 29.17. *Work only in ZFC (or in a fixed model of it). Suppose that* $(X, <)$ *is a linear order. Let* P *be the set of all pairs* (p, n) *such that* $n \in \omega$ *and* $p \subseteq X \times n$ *is a finite* *function. Define* $(p,n) \le (q,m)$ *iff* $m \le n$, $\mathrm{dmn}(q) \subseteq \mathrm{dmn}(p)$, $\forall x \in \mathrm{dmn}(q)[p(x) \cap m = q(x)$, *and*

$$\forall x, y \in \mathrm{dmn}(q),\ \textit{if } x < y \textit{ then } p(x) \backslash p(y) \subseteq m.$$

Then $\mathbb{P}$ *has ccc.*

Proof. Suppose that $\mathscr{A}$ is an uncountable subset of P. By the Δ-system theorem, we may assume that $\langle \mathrm{dmn}(p) : (p,n) \in \mathscr{A} \rangle$ is a Δ-system, say with root r. We may also assume that $p \restriction r = q \restriction r$ whenever $(p,n),(q,m) \in \mathscr{A}$. Now suppose that $(p,n),(q,m) \in \mathscr{A}$. Let s be the maximum of m and n. Clearly $p \cup q$ is a function, and so $(p \cup q, s) \in P$. We claim that $(p \cup q, s) \le (p,n),(q,m)$, as desired. By symmetry it suffices to show that $(p \cup q, s) \le (q,m)$. Suppose that $x \in \mathrm{dmn}(q)$. Then $(p \cup q)(x) \cap m = q(x) \cap m = q(x)$. If $x, y \in \mathrm{dmn}(q)$ and $x < y$, then $(p \cup q)(x) \backslash (p \cup q)(y) = q(x) \backslash q(y) \subseteq m$. □

Proposition 29.18. *Continuing Proposition 29.17, suppose that we are working in a c.t.m. M of ZFC. Let G be* $\mathbb{P}$*-generic over M. For each* $x \in X$ *let*

$$a_x = \bigcup \{p(x) : (p,n) \in G \textit{ for some } n \in \omega, \textit{ with } x \in \mathrm{dmn}(p)\}.$$

Thus $a_x \subseteq \omega$. *Then if* $x < y$, *then* $a_x \backslash a_y$ *is finite.*

Proof. For each $z \in X$ let $D_z = \{(p,n) : z \in \mathrm{dmn}(p)\}$. Given any $(q,m) \in P$, if $z \notin \mathrm{dmn}(q)$ clearly $(q \cup \{(z,0)\}, m) \in P$, $(q \cup \{(z,0)\}, m) \in D_z$, and $(q \cup \{(z,0)\}, m) \le (q,m)$. So D_z is dense.

Choose $(p,n) \in D_x \cap G$ and $(q,m) \in D_y \cap G$. Say $(p,n),(q,m) \ge (r,s) \in G$. We claim then that $a_x \backslash a_y \subseteq s$. Let $i \in a_x \backslash a_y$. Say $i \in u(x)$ with $(u,t) \in G$ and $x \in \mathrm{dmn} u$. Say $(u,t),(r,s) \ge (v,z) \in G$. Thus $v(x) \backslash v(y) \subseteq s$. Now $i \in u(x)$, so $i \in v(x)$. Also, $i \notin v(y)$ since $i \notin a_y$. So $i \in s$, as desired.

Proposition 29.19. *Continuing propositions 29.17 and 29.18, if* $x < y$, *then* $a_y \backslash a_x$ *is infinite.*

Proof. For each $i < \omega$ let

$$E^i = \{(p,n) : x, y \in \mathrm{dmn}(p) \text{ and } |p(y) \backslash p(x)| \ge i\}.$$

We claim that E^i is dense. Let (q,n) be given. Wlog $x, y \in \mathrm{dmn}(q)$. Say $\mathrm{dmn}(q)$ is

$$u_0 < \cdots < u_j = x < \cdots < u_{m-1}.$$

Let $\mathrm{dmn}(r) = \mathrm{dmn}(q)$, $r(u_t) = q(u_t)$ for $t \le j$; choose $w > n$ with $|w - n| = i$, and let $r(u_t) = q(u_t) \cup (w \backslash n)$ for $j < t$. Then $(q,n) \ge (r,w) \in E^i$, as desired.

Now for any $i \in \omega$ we show that $|a_y \backslash a_x| \ge i$. Choose $(p,n) \in E^i \cap G$. We claim that $p(y) \backslash p(x) \subseteq a_y \backslash a_x$ (as desired). Let $j \in p(y) \backslash p(x)$. So $j \in a_y$, and $j < n$ since $p(y) \subseteq n$. Suppose that $j \in a_x$. Say $y \in q(x)$, with $(q,v) \in G$ and $x \in \mathrm{dmn}(q)$. Say $(p,n),(q,v) \ge (r,s) \in G$. Then $j \in r(x)$ since $j \in q(x)$. Hence $j \in r(x) = p(x) \cap n$, so $j \in p(x)$, contradiction. □

If $\tau \in V^{\mathbb{P}}$, a *nice name* for a subset of τ is a name of the form $\bigcup\{\{\sigma\} \times A_\sigma : \sigma \in \mathrm{dmn}(\tau)\}$, where each A_σ is an antichain in $\mathbb{P}$.

Lemma 29.20. *Assume that $\mathbb{P}$ is ccc, $\kappa = |\mathbb{P}|$, $\lambda = |\mathrm{dmn}(\tau)|$, and κ and λ are infinite. Then there are at most κ^λ nice $\mathbb{P}$-names for subsets of τ.*

Proof. By ccc, there are $\le \kappa^\omega$ antichains in $\mathbb{P}$. Let K be the set of all functions g mapping $\mathrm{dmn}(\tau)$ into the set of all antichains of $\mathbb{P}$. For each $g \in K$ let $f(g) = \bigcup\{\{\sigma\}\times g(\sigma) : \sigma \in \mathrm{dmn}(\tau)\}$. Now $|K| \le \kappa^\lambda$, and f maps K onto the set of all nice names for subsets of τ. □

Proposition 29.21. *Let M be a c.t.m. of ZFC, $\mathbb{P} \in M$ a forcing order, and $\sigma \in M^P$.*

(i) For any $\mu \in M^P$ there is a nice name $\tau \in M^{\mathbb{P}}$ for a subset of σ such that

$$(*) \qquad 1 \Vdash \tau = \mu \cap \sigma.$$

(ii) If G is $\mathbb{P}$-generic over M and $a \subseteq \sigma_G$ in $M[G]$, then $a = \tau_G$ for some nice name τ for a subset of σ.

Proof. Assume the hypotheses of the proposition.

(i): Assume also that $\mu \in M^P$. For each $\pi \in \mathrm{dmn}(\sigma)$ let $A_\pi \subseteq P$ be such that

(1) $p \Vdash (\pi \in \mu \wedge \pi \in \sigma)$ for all $p \in A_\pi$.

(2) A_π is an antichain of $\mathbb{P}$.

(3) A_π is maximal with respect to (1) and (2).

Moreover, we do this definition inside M, so that $\langle A_\pi : \pi \in \mathrm{dmn}(\sigma)\rangle \in M$. Now let

$$\tau = \bigcup_{\pi \in \mathrm{dmn}(\sigma)} (\{\pi\} \times A_\pi).$$

To prove $(*)$, suppose that G is $\mathbb{P}$-generic over M; we want to show that $\tau_G = \mu_G \cap \sigma_G$.

First suppose that $a \in \mu_G \cap \sigma_G$. Choose $(\pi, p) \in \sigma$ such that $p \in G$ and $a = \pi_G$. Clearly $p \Vdash \pi \in \sigma$.

(4) $A_\pi \cap G \neq \emptyset$.

For, suppose that $A_\pi \cap G = \emptyset$. By Lemma 28.14(i), there is a $q \in G$ such that $q \perp r$ for all $r \in A_\pi$. Now since $\pi_G \in \mu_G$, by Corollary 28.21 there is a $q' \in G$ such that $q' \Vdash \pi \in \mu$. Let $r \in G$ with $r \le q, q'$. Then $r \Vdash (\pi \in \mu \wedge \pi \in \sigma)$. It follows that $A_\pi \cup \{r\}$ is an antichain, contradicting (3). Thus (4) holds.

By (4), take $q \in A_\pi \cap G$. Then $(\pi, q) \in \tau$ and $q \in G$, so $a = \pi_G \in \tau_G$. Thus we have shown that $\mu_G \cap \sigma_G \subseteq \tau_G$.

Now suppose that $a \in \tau_G$. Choose $(\pi, p) \in \tau$ such that $p \in G$ and $a = \pi_G$. Thus $p \in A_\pi$, so by (1), $p \Vdash (\pi \in \mu \wedge \pi \in \sigma)$. By the definition of forcing, $a = \pi_G \in \mu_G \cap \sigma_G$. This shows that $\tau_G \subseteq \mu_G \cap \sigma_G$. Hence $\tau_G = \mu_G \cap \sigma_G$.

(ii): Assume the hypotheses of (ii). Write $a = \mu_G$. Taking τ as in (i), we have $a = \mu_G = \mu_G \cap \sigma_G = \tau_G$, as desired. $\square$

Proposition 29.22. *Suppose that M is a c.t.m. of ZFC, and in M, $\mathbb{P}$ is a forcing order, $|P| = \kappa \geq \omega$, $\mathbb{P}$ has the λ-cc, and μ is an infinite cardinal. Suppose that G is $\mathbb{P}$-generic over M. Then there is a function in $M[G]$ mapping $((\kappa^{<\lambda})^\mu)^M$ onto a set containing $\mathscr{P}(\mu)^{M[G]}$.*

Proof. We do some calculations in M. Each antichain in $\mathbb{P}$ has size at most $\kappa^{<\lambda}$. Since $|\mathrm{dmn}(\check{\mu})|$ has size μ, we thus have at most $\nu \stackrel{\text{def}}{=} (\kappa^{<\lambda})^\mu$ nice names for subsets of $\check{\mu}$. Let $\langle \tau_\alpha : \alpha < \nu \rangle$ enumerate all of these names. Define

$$\pi = \{(\mathrm{op}(\check{\alpha}, \tau_\alpha), 1) : \alpha < \nu\}.$$

Now π_G is a function. For, if $x \in \pi_G$, then there is an $\alpha < \nu$ such that $x = (\alpha, (\tau_\alpha)_G)$, by Lemma 28.22. Thus π_G is a relation. Now suppose that $(x, y), (x, z) \in \pi_G$. Then there exist $\alpha, \beta < \nu$ such that $(x, y) = (\alpha, (\tau_\alpha)_G)$ and $(x, z) = (\beta, (\tau_\beta)_)$. Hence $\alpha = \beta$ and $y = z$. Clearly the domain of π_G is ν. By Proposition 29.21, $\mathscr{P}(\mu) \subseteq \mathrm{rng}(\pi_G)$ in $M[G]$, as desired. $\square$

Now we can prove a more precise version of Theorem 29.7.

Theorem 29.23. (Solovay) *Let M be a c.t.m. of ZFC. Suppose that κ is a cardinal of M such that $\kappa^\omega = \kappa$. Let $\mathbb{P}$ be the partial order* $\mathrm{Fn}(\kappa, 2, \omega)$ *ordered by $\supseteq$, and let G be $\mathbb{P}$-generic over M. Then $M[G]$ has the same cofinalities and cardinals as M, and $2^\omega = \kappa$ in $M[G]$.*

Moreover, if λ is any infinite cardinal in M, then $\kappa \leq (2^\lambda)^{M[G]} \leq (\kappa^\lambda)^M$.

Proof. By Theorem 29.7, $M[G]$ has the same cofinalities and cardinals as M and $\kappa \leq 2^\omega$.

Note that $|\mathrm{Fn}(\kappa, 2, \omega)| = \kappa$ in M. Hence by Proposition 29.22, for any infinite cardinal λ of M we have

$$\kappa \leq (2^\omega)^{M[G]} \leq (2^\lambda)^{M[G]} \leq ((\kappa^{<\lambda})^\lambda)^M = (\kappa^\lambda)^M.$$

Applying this to $\lambda = \omega$ we get $2^\omega = \kappa$ in $M[G]$. $\square$

By assuming that the ground model satisfies GCH, which is consistent by the theory of constructible sets, we can obtain a sharper result.

Corollary 29.24. *Suppose that M is a c.t.m. of $ZFC + GCH$. Suppose that κ is an uncountable regular cardinal of M. Let $\mathbb{P}$ be the partial order* $\mathrm{Fn}(\kappa, 2, \omega)$ *ordered by $\supseteq$, and let G be $\mathbb{P}$-generic over M. Then $M[G]$ has the same cofinalities and cardinals as M, and $2^\omega = \kappa$ in $M[G]$.*

Moreover, for any infinite cardinal λ of M we have

$$(2^\lambda)^{M[G]} = \begin{cases} \kappa & \text{if } \lambda < \kappa, \\ \lambda^+ & \text{if } \kappa \leq \lambda. \end{cases}$$

Proof. By GCH we have $\kappa^\omega = \kappa$. Hence the hypothesis of Theorem 29.23 holds, and the conclusion follows using GCH in M. $\square$

We give several more specific corollaries.

Corollary 29.25. *If ZFC is consistent, then so is each of the following:*
(i) $ZFC + 2^{\aleph_0} = \aleph_2$.
(ii) $ZFC + 2^{\aleph_0} = \aleph_{203}$.
(iii) $ZFC + 2^{\aleph_0} = \aleph_{\omega_1}$.
(iv) $ZFC + 2^{\aleph_0} = \aleph_{\omega_4}$. $\square$

Corollary 29.26. *If ZFC+ "there is an uncountable regular limit cardinal" is consistent, so is ZFC+ "2^ω is a regular limit cardinal".* $\square$

Corollary 29.27. *Suppose that M is a c.t.m. of ZFC. Then there is a generic extension $M[G]$ such that in it, $2^\omega = ((2^\omega)^+)^M$.*

Since clearly $((2^\omega)^+)^\omega = (2^\omega)^+$ in M, this is immediate from Theorem 29.23. $\square$

Proposition 29.28. (IV.3.12a) *There are σ, τ, G such that σ is a nice name for a subset of τ and $\sigma_G \not\subseteq \tau_G$.*

Proof. Assume that $p \perp q$. Let $\tau = \{(\emptyset, p)\}$ and $\sigma = \{(\emptyset, q)\}$. Let G be such that $q \in G$. Then $\tau_G = \emptyset$ and $\sigma_G = \{\emptyset\}$. $\square$

Proposition 29.29. (IV.3.18) *Assume that $\mathbb{P}, J \in M$, $\mathbb{P}$ is countable, and J is a set of size ω_1 in the sense of M. In $M[G]$ let E be an uncountable subset of J. Then there is an $E' \in M$ such that $E' \subseteq E$ and E' is uncountable in the sense of M.*

Proof. Let $p \in G$ be such that $p \Vdash (\sigma$ is an injection of ω_1 into $\dot{E})$, where $\dot{E}$ is a name for E. Take any $\alpha < \omega_1$. Then $\sigma_G(\alpha) \in J = \check{J}_G$, so there is a $(\hat{j_\alpha}, \mathbb{1}) \in \check{J}$ and a $q_\alpha \leq p$ such that $q_\alpha \Vdash \sigma(\alpha) = \hat{j_\alpha}$. Since $\mathbb{P}$ is countable, there is a $M \in [\omega_1]^{\omega_1}$ such that $q_\alpha = q_\beta$ for all $\alpha, \beta \in M$. Fix $\alpha \in M$. Let $E' = \{j \in J : q_\alpha \Vdash \hat{j} \in \dot{E}\}$. If $\alpha \neq \beta$ with $\alpha, \beta \in M$, then $p \Vdash \sigma(\alpha) \neq \sigma(\beta)$, hence $q_\alpha \Vdash j_\alpha \neq j_\beta$, hence $j_\alpha \neq j_\beta$. $\square$

Proposition 29.30. (IV.3.18a) *If $\mathbb{P} = \mathrm{Fn}(J, 2, \omega)$ with J uncountable, then there is an uncountable $E \subseteq J$ in $M[G]$ such that there is no infinite $E' \in M$ such that $E' \subseteq E$.*

Proof. Assume that $J = \kappa$ with $\kappa \geq \omega_1$. For each $\alpha < \omega_1$ let $D_\alpha = \{p : \exists \beta \in (\alpha, \omega_1)[\beta \in \mathrm{dmn}(p)$ and $p(\beta) = 1]\}$. Clearly D_α is dense. Let $E = \{\alpha < \omega_1 : (\bigcup G)(\alpha) = 1\}$. So E is uncountable, $E \subseteq J$, $E \in M[G]$. Suppose that $E' \subseteq E$ with E' infinite and $E' \in M$. Let $F = \{p : \exists \alpha \in E'[\alpha \in \mathrm{dmn}(p)$ and $p(\alpha) = 0]\}$. Clearly F is dense. Choose $\alpha \in F \cap G$. then $\alpha \in E' \backslash E$, contradiction. $\square$

We now need some elementary facts about cardinals. For cardinals κ, λ, we define

$$\kappa^{<\lambda} = \sup_{\alpha<\lambda} |{}^\alpha\kappa|.$$

Note here that the supremum is over all ordinals less than λ, not only cardinals.

Proposition 29.31. *Let κ and λ be cardinals with $\kappa \geq 2$ and λ infinite and regular. Then $(\kappa^{<\lambda})^{<\lambda} = \kappa^{<\lambda}$.*

Proof. Clearly $\geq$ holds. For $\leq$, by the fact that $\lambda \cdot \lambda = \lambda$ it suffices to find an injection from

$$\bigcup_{\alpha<\lambda} {}^{\alpha}\left(\bigcup_{\beta<\lambda} {}^{\beta}\kappa\right) \tag{1}$$

into

$$\bigcup_{\alpha,\beta<\lambda} {}^{\alpha\times\beta}(\kappa+1). \tag{2}$$

Let x be a member of (1), and choose $\alpha < \lambda$ accordingly. Then for each $\xi < \alpha$ there is a $\beta_{x,\xi} < \lambda$ such that $x(\xi) \in {}^{\beta_{x,\xi}}\kappa$. Let $\gamma_x = \sup_{\xi<\alpha} \beta_{x,\xi}$. Then $\gamma_x < \lambda$ by the regularity of λ. We now define $f(x)$ with domain $\alpha \times \gamma_x$ by setting, for any $\xi < \alpha$ and $\eta < \gamma_x$

$$(f(x))(\xi,\eta) = \begin{cases} (x(\xi))(\eta) & \text{if } \eta < \beta_{x,\xi}, \\ \kappa & \text{otherwise.} \end{cases}$$

Then f is one-one. In fact, suppose that $f(x) = f(y)$. Let the domain of $f(x)$ be $\alpha \times \gamma_x$ as above. Suppose that $\xi < \alpha$. If $\beta_{x,\xi} \neq \beta_{y,\xi}$, say $\beta_{x,\xi} < \beta_{y,\xi}$. Then $\gamma_x = \gamma_y \geq \beta_{y,\xi} > \beta_{x,\xi}$, and $(f(x))(\xi, \beta_{x,\xi}) = \kappa$ while $(f(y))(\xi, \beta_{x,\xi}) < \kappa$, contradiction. Hence $\beta_{x,\xi} = \beta_{y,\xi}$. Finally, take any $\eta < \beta_{x\xi}$. Then

$$(x(\xi))(\eta) = (f(x))(\xi,\eta) = (f(y))(\xi,\eta) = (y(\xi))(\eta);$$

it follows that $x = y$.

Now the direction $\leq$ follows.

Proposition 29.32. *For any cardinals κ, λ, $|[\kappa]^{<\lambda}| \leq \kappa^{<\lambda}$.*

Proof. For each cardinal $\mu < \lambda$ define $f : {}^{\mu}\kappa \to [\kappa]^{\leq\mu}\backslash\{\emptyset\}$ by setting $f(x) = \text{rng}(x)$ for any $x \in {}^{\mu}\kappa$. Clearly f is an onto map. It follows that $|[\kappa]^{\leq\mu}| \leq |{}^{\mu}\kappa| \leq \kappa^{<\lambda}$. Hence

$$\begin{aligned}
|[\kappa]^{<\lambda}| &= \left| \bigcup_{\substack{\mu<\lambda, \\ \mu \text{ a cardinal}}} [\kappa]^{\leq\mu} \right| \\
&\leq \sum_{\substack{\mu<\lambda, \\ \mu \text{ a cardinal}}} |[\kappa]^{\leq\mu}| \\
&\leq \sum_{\substack{\mu<\lambda, \\ \mu \text{ a cardinal}}} \kappa^{<\lambda} \\
&\leq \lambda \cdot \kappa^{<\lambda} \\
&= \kappa^{<\lambda}.
\end{aligned}$$

$\square$

Lemma 29.33. *If I, J are sets and λ is an infinite cardinal, then* $\mathrm{Fn}(I, J, \lambda)$ *has the* $(|J|^{<\lambda})^+$*-cc.*

Proof. Let $\theta = (|J|^{<\lambda})^+$, and suppose that $\{p_\xi : \xi < \theta\}$ is a collection of elements of $\mathrm{Fn}(I, J, \lambda)$; we want to show that there are distinct $\xi, \eta < \theta$ such that p_ξ and p_η are compatible. We want to apply the general indexed Δ-system theorem 24.4, with κ, λ, $\langle A_i : i \in I\rangle$ replaced by λ, θ, $\langle \mathrm{dmn}(p_\xi) : \xi < \theta\rangle$ respectively. Obviously θ is regular. If $\alpha < \theta$, then $|[\alpha]^{<\lambda}| \le |\alpha|^{<\lambda}$ (by Proposition 29.33) $\le (|J|^{<\lambda})^{<\lambda} = |J|^{<\lambda}$ (by Proposition 29.32) $< \theta$. Thus we can apply 24.4, and we get $J \in [\theta]^\theta$ such that $\langle \mathrm{dmn}(p_\xi) : \xi \in J\rangle$ is an indexed Δ-system, say with root r. Now $|{}^rJ| \le |J|^{<\lambda} < \theta$, so there exist a $K \in [J]^\theta$ and an $f \in {}^rJ$ such that $p_\xi \restriction r = f$ for all $\xi \in K$. Clearly p_ξ and p_η are compatible for any two $\xi, \eta \in K$. $\square$

Lemma 29.34. *If I, J are sets and λ is a regular cardinal, then* $\mathrm{Fn}(I, J, \lambda)$ *is λ-closed.*

Proof. Suppose that $\gamma < \lambda$ and $\langle p_\xi : \xi < \gamma\rangle$ is a system of elements of $\mathrm{Fn}(I, J, \lambda)$ such that $p_\eta \supseteq p_\xi$ whenever $\xi < \eta < \gamma$. Let $q = \bigcup_{\xi<\gamma} p_\xi$. Clearly $q \in \mathrm{Fn}(I, J, \lambda)$ and $q \supseteq p_\xi$ for each $\xi < \gamma$. $\square$

We now need another little fact about cardinal arithmetic.

Lemma 29.35. *If λ is regular, then $\lambda^{<\lambda} = 2^{<\lambda}$.*

Proof. Note that if $\alpha < \lambda$, then by the regularity of λ,

$$|{}^\alpha\lambda| = \left|\bigcup_{\beta<\lambda} {}^\alpha\beta\right| \le \sum_{\beta<\lambda} |\beta|^{|\alpha|} \le \sum_{\beta<\lambda} |\max(\alpha, \beta)|^{|\max(\alpha,\beta)|} \le \sum_{\beta<\lambda} 2^{|\max(\alpha,\beta)|} \le 2^{<\lambda} \le \lambda^{<\lambda};$$

hence the lemma follows. $\square$

Lemma 29.36. *Suppose that M is a c.t.m. of ZFC, $I, J, \lambda \in M$, and in M, λ is a regular cardinal, $2^{<\lambda} = \lambda$ and $|J| \le \lambda$. Then $Fn(I, J, \lambda)^M$ preserves cofinalities and cardinalities.*

Proof. By Lemma 29.34, the set $\mathrm{Fn}(I, J, \lambda)$ is λ-closed, and so by Proposition 29.10, $\mathrm{Fn}(I, J, \lambda)$ preserves cofinalities and cardinalities $\le \lambda$. Now $|J|^{<\lambda} \le \lambda^{<\lambda} = 2^{<\lambda} = \lambda$ by Lemma 29.35, Hence by Lemma 29.33, $\mathrm{Fn}(I, J, \lambda)$ has the λ^+-cc. By Proposition 29.5 $\mathrm{Fn}(I, J, \lambda)$ preserves cofinalities and cardinals $\ge \lambda^+$. $\square$

Now we can give our main theorem concerning making 2^λ as large as we want, for any regular λ given in advance.

Theorem 29.37. *Suppose that M is a c.t.m. of ZFC and in M we have cardinals κ, λ such that $\lambda < \kappa$, λ is regular, $2^{<\lambda} = \lambda$, and $\kappa^\lambda = \kappa$. Let $P = \mathrm{Fn}(\kappa, 2, \lambda)$ ordered by $\supseteq$. Then P preserves cofinalities and cardinalities. Let G be $\mathbb{P}$-generic over M. Then*

(i) $(2^\lambda = \kappa)^{M[G]}$.

(ii) If μ and ν are cardinals of M and $\omega \le \mu < \lambda$, then $(\nu^\mu)^M = (\nu^\mu)^{M[G]}$.

(iii) For any cardinal μ of M, if $\mu \ge \lambda$ then $(2^\mu)^{M[G]} = (\kappa^\mu)^M$.

Proof. Preservation of cofinalities and cardinalities follows from Lemma 29.36. Now we turn to (i). To show that $\kappa \le (2^\lambda)^{M[G]}$, we proceed as in the proof of Theorem 29.1 Let $g = \bigcup G$. So g is a function mapping a subset of κ into 2.

(1) For each $\alpha \in \kappa$, the set $\{f \in \mathrm{Fn}(\kappa, 2, \lambda) : \alpha \in \mathrm{dmn}(f)\}$ is dense in $\mathbb{P}$ (and it is a member of M).

In fact, given $f \in \mathrm{Fn}(\kappa, 2, \lambda)$, either f is already in the above set, or else $\alpha \notin \mathrm{dmn}(f)$ and then $f \cup \{(\alpha, 0)\}$ is an extension of f which is in that set. So (1) holds.

Since G intersects each set (1), it follows that g maps κ into 2. Let (in M) $h : \kappa \times \lambda \to \kappa$ be a bijection. For each $\alpha < \kappa$ let $a_\alpha = \{\xi \in \lambda : g(h(\alpha, \xi)) = 1\}$. We claim that $a_\alpha \ne a_\beta$ for distinct α, β; this will give $\kappa \le (2^\lambda)^{M[G]}$. The set

$$\begin{aligned}\{f \in \mathrm{Fn}(\kappa, 2, \lambda) : &\text{ there is a } \xi \in \lambda \text{ such that} \\ &h(\alpha, \xi), h(\beta, \xi) \in \mathrm{dmn}(f) \text{ and } f(h(\alpha, \xi)) \ne f(h(\alpha, \xi))\}\end{aligned}$$

is dense in $\mathbb{P}$ (and it is in M). In fact, let distinct α and β be given, and suppose that $f \in \mathrm{Fn}(\kappa, 2, \lambda)$. Now $\{\xi : h(\alpha, \xi) \in f \text{ or } h(\beta, \xi) \in f\}$ has size less than λ, so choose $\xi \in \lambda$ not in this set. Thus $h(\alpha, \xi), h(\beta, \xi) \notin f$. Let $h = f \cup \{(h(\alpha, \xi), 0), (h(\beta, \xi), 1)\}$. Then h extends f and is in the above set, as desired.

It follows that G contains a member of this set. Hence $a_\alpha \ne a_\beta$. Thus we have now shown that $\kappa \le (2^\lambda)^{M[G]}$.

For the other inequality, note by Lemma 29.33 that $\mathbb{P}$ has the $(2^{<\lambda})^+$-cc, and by hypothesis $(2^{<\lambda})^+ = \lambda^+$. By the assumption that $\kappa^\lambda = \kappa$ we also have $|P| = \kappa$. Hence by Proposition 29.22 the other inequality follows. Thus we have finished the proof of (i).

For (ii), assume the hypothesis. If $f \in M[G]$ and $f : \mu \to \nu$, then $f \in M$ by Theorem 29.9. Hence (ii) follows.

Finally, for (iii), suppose that μ is a cardinal of M such that $\mu \ge \lambda$. By Proposition 29.22 with λ replaced by λ^+ we have $(2^\mu)^{M[G]} \le (\kappa^\mu)^M$. Now $(\kappa^\mu)^M \le (\kappa^\mu)^{M[G]} = ((2^\lambda)^\mu)^{M[G]} = (2^\mu)^{M[G]}$, so (iii) holds. □

Corollary 29.38. *Suppose that M is a c.t.m. of ZFC+ GCH, and in M we have cardinals κ, λ, both regular, with $\lambda < \kappa$. Let $P = \mathrm{Fn}(\kappa, 2, \lambda)$ ordered by $\supseteq$. Then P preserves cofinalities and cardinalities. Let G be $\mathbb{P}$-generic over M. Then for any infinite cardinal μ,*

$$(2^\mu)^{M[G]} = \begin{cases} \mu^+ & \text{if } \mu < \lambda, \\ \kappa & \text{if } \lambda \le \mu < \kappa, \\ \mu^+ & \text{if } \kappa \le \mu. \end{cases}$$

Proof. Immediate from Theorem 29.37. □

Theorem 29.37 gives quite a bit of control over what can happen to powers 2^κ for κ regular. We can apply this theorem to obtain a considerable generalization of it.

Theorem 29.39. *Suppose that $n \in \omega$ and M is a c.t.m. of ZFC. Also assume the following:*

(i) $\lambda_1 < \cdots < \lambda_n$ are regular cardinals in M.

(ii) $\kappa_1 \le \cdots \le \kappa_n$ *are cardinals in* M*.*
(iii) $(\mathrm{cf}(\kappa_i) > \lambda_i)^M$ *for each* $i = 1, \ldots, n$*.*
(iv) $(2^{<\lambda_i} = \lambda_i)^M$ *for each* $i = 1, \ldots, n$*.*
(v) $(\kappa_i^{\lambda_i})^M = \kappa_i$ *for each* $i = 1, \ldots, n$

Then there is a c.t.m. $N \supseteq M$ *with the same cofinalities and cardinals such that:*

(vi) $(2^{\lambda_i} = \kappa_i)^N$ *for each* $i = 1, \ldots, n$*.*
(vii) $(2^\mu)^N = (\kappa_n^\mu)^M$ *for all* $\mu > \lambda_n$*.*

Proof. The statement vacuously holds for $n = 0$. Suppose that it holds for $n-1$, and the hypothesis holds for n, where n is a positive integer. Let $\mathbb{P}_n = \mathrm{Fn}(\kappa_n, 2, \lambda_n)$. Then by Lemma 29.33, $\mathbb{P}_n$ has the $(2^{<\lambda_n})^+$-cc, i.e., by (iv) it has the λ_n^+-cc. By Lemma 29.34 it is λ_n-closed. So by Proposition 29.36, $\mathbb{P}_n$ preserves all cofinalities and cardinalities. Let G be $\mathbb{P}_n$-generic over M. By Theorem 29.37, $(2^{\lambda_n} = \kappa_n)^{M[G]}$, $(2^\mu)^{M[G]} = (\kappa_n^\mu)^M$ for all $\mu > \lambda_n$, and also conditions (i)-(v) hold for $M[G]$ for $i = 1, \ldots, n-1$. Hence by the inductive hypothesis, there is a c.t.m. N with $M[G] \subseteq N$ such that

(1) $(2^{\lambda_i} = \kappa_i)^N$ for each $i = 1, \ldots, n-1$.

(2) $(2^\mu)^N = (\kappa_{n-1}^\mu)^{M[G]}$ for all $\mu > \lambda_{n-1}$.

In particular,

$$\begin{aligned}(2^{\lambda_n})^N = (\kappa_{n-1}^{\lambda_n})^{M[G]} &\le (\kappa_n^{\lambda_n})^{M[G]} = ((2^{\lambda_n})^{\lambda_n})^{M[G]} = (2^{\lambda_n})^{M[G]}\\ &= \kappa_n = (2^{\lambda_n})^{M[G]} \le (2^{\lambda_n})^N.\end{aligned}$$

Thus $(2^{\lambda_n})^N = \kappa_n$. Furthermore, if $\mu > \lambda_n$ then

$$\begin{aligned}(2^\mu)^N = (\kappa_{n-1}^\mu)^{M[G]} &\le (\kappa_n^\mu)^{M[G]} = ((2^{\lambda_n})^\mu)^{M[G]} = (2^\mu)^{M[G]}\\ &= (\kappa_n^\mu)^M \le (\kappa_n^\mu)^N = ((2^{\lambda_n})^\mu)^N = (2^\mu)^N.\end{aligned}$$

It follows that $(2^\mu)^N = (\kappa_n^\mu)^M$. This completes the inductive proof. $\square$

Corollary 29.40. *Suppose that* $n \in \omega$ *and* M *is a c.t.m. of ZFC + GCH. Also assume the following:*

(i) $\lambda_1 < \cdots < \lambda_n$ *are regular cardinals in* M*.*
(ii) $\kappa_1 \le \cdots \le \kappa_n$ *are cardinals in* M*.*
(iii) $(\mathrm{cf}(\kappa_i) > \lambda_i)^M$ *for each* $i = 1, \ldots, n$*.*

Then there is a c.t.m. $N \supseteq M$ *with the same cofinalities and cardinals such that:*

(iv) $(2^{\lambda_i} = \kappa_i)^N$ *for each* $i = 1, \ldots, n$*.*
(v) $(2^\mu)^N = (\kappa_n^\mu)^M$ *for all* $\mu > \lambda_n$*.* $\square$

Corollary 29.41. *If ZFC is consistent, then so are each of the following:*

(i) ZFC $+ 2^{\aleph_0} = 2^{\aleph_1} = \aleph_3$*.*
(ii) ZFC $\cup \{2^{\aleph_n} = \aleph_{n+2} : n < 100\}$*.*

(iii) $ZFC \cup \{2^{\aleph_n} = \aleph_{\omega+1} : n < 300\}$.
(iv) $ZFC \cup \{2^{\aleph_n} = \aleph_{\omega+n} : n < 33\}$. $\square$

Corollary 29.42. *If it is consistent with ZFC that there is an uncountable regular limit cardinal, then the following is consistent:*
$ZFC \cup \{2^{\aleph_n}$ *is the first regular limit cardinal:* $n < 1000\}$. $\square$

30. General theory of forcing

Theorem 30.1. (IV.4.1) *Let M be a transitive model of ZFC, let $\mathbb{P} \in M$ be a forcing poset, and fix $G \subseteq \mathbb{P}$. Consider the eight statements we get by letting φ be one of*

(1) dense set,
(2) dense open set.
(3) maximal antichain.
(4) predense set.

and letting ψ be one of

(a) filter
(b) upward closed linked family.

and asserting that G has property ψ, and $G \cap E \neq \emptyset$ for all $E \subseteq \mathbb{P}$ such that $E \in M$ and E has property φ.

Then these eight statements are equivalent.

Proof. (A) (xa)$\Rightarrow$(xb). For, Assume (xa). Then G clearly has property b and the other property for x holds.

(B) (4y)$\Rightarrow$(3y). For, Assume (4y), and suppose that $E \subseteq \mathbb{P}$ with $E \in M$ and E is a maximal antichain. Then E is predense, so $G \cap E \neq \emptyset$.

(C) (3y)$\Rightarrow$(1y). For, Assume (3y). Suppose that $E \subseteq \mathbb{P}$ with $E \in M$ and E dense. Let F be a maximal antichain consisting of elements of E. Then $G \cap F \neq \emptyset$, so $G \cap E \neq \emptyset$.

(D) (1y)$\Rightarrow$(2y). This is clear since 2 implies 1; see (B).

(E) (2a)$\Rightarrow$(4a). Assume (2a), and suppose that G is a filter, $E \subseteq \mathbb{P}$, $E \in M$, and E is predense. By Lemma 25.63, $E\downarrow'$ is dense open. So $G \cap (E\downarrow') \neq \emptyset$. Hence $G \cap E \neq \emptyset$.

(F) (1a), (2a), (3a), (4a) are all equivalent. This is true by (B)–(E).

(G) (2b)$\Rightarrow$(2a). Assume (2b). We want to show that G is a filter. So, suppose that $p, q \in G$; we want to get $r \in G$ such that $r \leq p, q$. Let $D = \{r \in \mathbb{P} : r \perp p$ or $r \perp q$ or $r \leq p, q\}$. So $D \in M$. It is clearly open. To show that it is dense, let $s \in \mathbb{P}$. If $s \perp p$, then $s \in D$. Suppose that $s \not\perp p$; say $t \leq s, p$. If $t \perp q$ then $t \in D$; and $t \leq s$. Suppose that $t \not\perp q$. Say $r \leq t, q$. Then $r \in D$ and $r \leq p$, as desired. Choose $r \in D \cap G$. Clearly $r \leq p, q$.

Now all are equivalent. For, (1a)–(4a) are equivalent by (F), (4a)$\Rightarrow$(4b) by (A), (4b)$\Rightarrow$(3b)$\Rightarrow$(1b)$\Rightarrow$(2b) by (B), (C), (D), and (2b)$\Rightarrow$(2a) by (G). $\square$

Lemma 30.2. (IV.4.2) *Let M be a transitive model of ZFC, with forcing posets $\mathbb{P}, \mathbb{Q} \in M$, and suppose that $i : \mathbb{Q} \to \mathbb{P}$ is a complete embedding, with $i \in M$. Let $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over M. Then $i^{-1}[G]$ is $\mathbb{Q}$-generic over M.*

Proof. First, $i^{-1}[G]$ is linked. For, suppose that $p, q \in i^{-1}[G]$. Thus $i(p), i(q) \in G$, so they are compatible. By (iii) in the definition of complete embedding, also p and q are compatible.

Next, $i^{-1}[G]$ is upwards closed. For, suppose that $p \in i^{-1}[G]$ and $p \leq q$. Then $i(p) \in G$ and $i(p) \leq i(q)$ by (ii) in the definition of complete embedding, so $i(q) \in G$ and hence $q \in i^{-1}[G]$.

Now by Theorem 30.1 it suffices to show that $i^{-1}[G] \cap A \neq \emptyset$ whenever $A \in M$ and A is a maximal antichain in $\mathbb{Q}$. By definition of complete embedding, $i[A]$ is a maximal antichain in $\mathbb{P}$. Hence $G \cap i[A] \neq \emptyset$; say $p \in G \cap i[A]$. Say $p = i(q)$ with $q \in A$. Then $q \in i^{-1}[G] \cap A$. $\square$

If $\mathbb{P}$ and $\mathbb{Q}$ are forcing posets and $i : \mathbb{Q} \to \mathbb{P}$ we define i_* with domain $V^{\mathbb{Q}}$ by

$$i_*(\tau) = \{(i_*(\sigma), i(q)) : (\sigma, q) \in \tau\}.$$

If $i : \mathbb{Q} \to \mathbb{P}$ and $H \subseteq \mathbb{Q}$, let $\tilde{i}(H) = \{p \in \mathbb{P} : \exists q \in H[i(q) \leq p]\}$.

Lemma 30.3. (IV.4.4) *Let M be a transitive model of ZFC, with forcing posets $\mathbb{P}$, $\mathbb{Q} \in M$. Suppose that $i : \mathbb{Q} \to \mathbb{P}$ is a complete embedding, with $i \in M$. Let $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over M, and let $H = i^{-1}[G]$. Then*

(i) $\forall \tau \in M^{\mathbb{Q}}[i_*(\tau) \in M^{\mathbb{P}}$ and $\mathrm{val}(i_*(\tau), G) = \mathrm{val}(\tau, H)$.

(ii) $M[H] \subseteq M[G]$.

Proof. (i): By induction.

$$i_*(\tau) = \{(i_*(\sigma), i(q)) : (\sigma, q) \in \tau\};$$

the inductive hypothesis is that each $i_*(\sigma) \in M^{\mathbb{P}}$ for $\sigma \in \mathrm{dmn}(\tau)$, so $i_*(\tau) \in M^{\mathbb{P}}$. Also,

$$\begin{aligned}
\mathrm{val}(i_*(\tau), G) &= \{\mathrm{val}(i_*(\sigma), G) : \exists p \in G[(i_*(\sigma), p) \in i_*(\tau)]\} \\
&= \{\mathrm{val}(\sigma, H) : \exists p \in G \exists q[(\sigma, q) \in \tau \text{ and } (i_*(\sigma), p) = (i_*(\sigma), i(q))]\} \\
&= \{\mathrm{val}(\sigma, H) : \exists p \in G \exists q[(\sigma, q) \in \tau \text{ and } p = i(q)]\} \\
&= \{\mathrm{val}(\sigma, H) : \exists q \exists p \in G[(\sigma, q) \in \tau \text{ and } p = i(q)]\} \\
&= \{\mathrm{val}(\sigma, H) : \exists q[\exists p \in G[p = i(q)] \text{ and } (\sigma, q) \in \tau]\} \\
&= \{\mathrm{val}(\sigma, H) : \exists q[q \in i^{-1}[G] \text{ and } (\sigma, q) \in \tau]\} \\
&= \{\mathrm{val}(\sigma, H) : \exists q \in H[(\sigma, q) \in \tau]\} \\
&= \mathrm{val}(\tau, H).
\end{aligned}$$

(ii): If $x \in M[H]$, then there is a $\tau \in M^{\mathbb{Q}}$ such that $x = \mathrm{val}(\sigma, H)$. Then by (i), $x = \mathrm{val}(i_*(\sigma), G) \in M[G]$. $\square$

Lemma 30.4. (IV.4.6) *If G_1, G_2 are both $\mathbb{P}$-generic over M and $G_1 \subseteq G_2$, then $G_1 = G_2$.*

Proof. Take any $p \in G_2$. Let $D = \{r \in \mathbb{P} : r \leq p \text{ or } r \perp p\}$. Then D is dense and in M. Pick $r \in G_1 \cap D$. Then $r \perp p$ would contradict $r, p \in G_2$. so $r \leq p$, hence $p \in G_1$. $\square$

Lemma 30.5. (IV.4.7) *Let M be a transitive model of ZFC, with $\mathbb{Q}, \mathbb{P}, i \in M$ and assume that $i : \mathbb{Q} \to \mathbb{P}$ is a dense embedding. Then*

1. If $H \subseteq \mathbb{Q}$ is $\mathbb{Q}$-generic over M and $G = \tilde{i}(H)$, then G is $\mathbb{P}$-generic over M, and $H = i^{-1}[G]$.

2. If $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over M and $H = i^{-1}[G]$, then H is $\mathbb{Q}$-generic over M, and $G = \tilde{i}(H)$.

3. In (1) and (2), $M[H] = M[G]$.

4. $q \Vdash \varphi(\tau_1,\ldots,\tau_n)$ *iff* $i(q) \Vdash \varphi(i_*(\tau_1),\ldots,i_*(\tau_n))$.

Proof. 1: G is a filter: suppose that $p \in G$ and $p \le p' \in \mathbb{P}$. Then there is a $q \in H$ such that $i(q) \le p$. So $i(q) \le p'$, hence $p' \in G$. Suppose that $p, p' \in G$. Choose $q, q' \in H$ such that $i(q) \le p$ and $i(q') \le p'$. Choose $r \in H$ such that $r \le q, q'$. Then $i(r) \le i(q) \le p$ and $i(r) \le i(q') \le p'$, as desired.

Now suppose that D is dense open in $\mathbb{P}$ and $D \in M$. Then $i^{-1}[D]$ is dense by Proposition 25.71. Choose $q \in H \cap i^{-1}[D]$. Then $i(q) \in G \cap D$. So G is $\mathbb{P}$-generic.

Now suppose that $q \in H$. Then $i(q) \in \tilde{\imath}(H) = G$, so $q \in i^{-1}[G]$. Thus $H \subseteq i^{-1}[G]$. By Lemma 30.2,, $i^{-1}[G]$ is $\mathbb{Q}$-generic over M. Hence $H = i^{-1}[G]$ by Lemma 30.4.

2: H is $\mathbb{Q}$-generic over M by Lemma 30.2. If $p \in \tilde{i}(H)$, choose $q \in H$ such that $i(q) \le p$. Thus $i(q) \in G$, so also $p \in G$. Thus $\tilde{i}(H) \subseteq G$. By Lemma 30.4, $\tilde{i}(H) = G$.

3: Assume 1. Then $M[H] \subseteq M[G]$ by Lemma 30.3. Clearly $G \in M[H]$, so $M[G] \subseteq M[H]$ by Lemma 28.8.

Same argument assuming 2.

4: For $\Rightarrow$, assume that $q \Vdash_{\mathbb{Q}} \varphi(\tau_1,\ldots,\tau_n)$ and $i(q) \in G$ with G $\mathbb{P}$-generic over M. Let $H = i^{-1}[G]$. Then H is $\mathbb{Q}$-generic over M by 2, and $q \in H$. Hence $M[H] \models \varphi(\tau_{1H},\ldots,\tau_{nH})$. Now by Lemma 30.3 we have $\tau_{iH} = i_*(\tau_i)_G$ for each i, and $M[H] = M[G]$ by 3. Hence $M[G] \models \varphi(i_*(\tau_1)_G,\ldots,i_*(\tau_n)_G)$. Hence $i(q) \Vdash \varphi(i_*(\tau_1),\ldots,i_*(\tau_n))$.

$\Leftarrow$: Assume that $i(q) \Vdash \varphi(i_*(\tau_1),\ldots,i_*(\tau_n))$, and suppose that $q \in H$ with H $\mathbb{Q}$-generic over M. Let $G = \tilde{\imath}(H)$. Then $i(q) \in G$, and G is $\mathbb{P}$-generic over M by 1. Hence $M[G] \models \varphi(i_*(\tau_1)_G,\ldots,i_*(\tau_n)_G)$. By 1, $H = i^{-1}[G]$, and by Lemma 30.3, $\tau_{iH} = i_*(\tau_i)_G$ for each i. By 3, $M[H] = M[G]$. Hence $M[H] \models \varphi(\tau_{1H},\ldots,\tau_{nH})$. This shows that $q \Vdash \varphi(\tau_1,\ldots,\tau_n)$. $\square$

Proposition 30.6. (IV.4.8) *If I is infinite, then* $\mathrm{Fn}(I,\omega,\omega) \not\cong \mathrm{Fn}(I,2,\omega)$.

Proof. Suppose that f is an isomorphism from $\mathrm{Fn}(I,\omega,\omega)$ onto $\mathrm{Fn}(I,2,\omega)$. Clearly $f(\emptyset) = \emptyset$. Fix $i \in I$. Then every non-empty member of $\mathrm{Fn}(I,2,\omega)$ extends $k_0 \stackrel{\text{def}}{=} \{(i,0)\}$ or $k_1 \stackrel{\text{def}}{=} \{(i,1)\}$. Say $f(x_0) = k_0$ and $f(x_1) = k_1$. Choose $r \in \omega$ with $r \notin \mathrm{rng}(x_0) \cup \mathrm{rng}(x_1)$. Then $\{(i,r)\}$ does not extend x_0 or x_1, so $f(\{(i,r)\})$ does not extend k_0 or k_1, contradiction. $\square$

Proposition 30.7. (IV.4.8a) *For I infinite, $\prod^{\mathrm{fin}}_{i\in I}({}^{<\omega}\omega)$ densely embeds in $Fn(I,\omega,\omega)$.*

Proof. Write $I = \bigcup_{i\in I} J_i$ with the J_i's pairwise disjoint and each of size ω. For each $i \in I$ let g_i be a bijection from ω to J_i. Now take any $x \in \prod^{\mathrm{fin}}_{i\in I}({}^{<\omega}\omega)$. Let

$$\mathrm{dmn}(f(x)) = \{k \in I : \exists i \in I[x_i \ne \emptyset \wedge \exists l \in \mathrm{dmn}(x_i)[k = g_i(l)]]\}.$$

Clearly for each $k \in \mathrm{dmn}(f(x))$ there are unique i and l as above. We define $(f(x))(k) = x_i(l)$. Note that $f(x) = \emptyset$ if $\forall i \in I[x_i = \emptyset]$.

To check (ii) in the definition of dense embedding, suppose that $x, y \in \prod^{\mathrm{fin}}_{i\in I}({}^{<\omega}\omega)$ and $y \le x$. Thus $\forall i \in I[x_i \subseteq y_i]$. Take any $k \in \mathrm{dmn}(f(x))$. Choose i and l as above. Then $y_i \ne \emptyset$, $l \in \mathrm{dmn}(y_i)$, and $k = g_i(l)$. Hence $k \in \mathrm{dmn}(f(y))$. Furthermore, $(f(y))(k) = y_i(l) = x_i(l) = (f(x))(k)$. Hence $f(x) \subseteq f(y)$.

To check (iii) in the definition of dense embedding, first note that $\Leftarrow$ holds by (ii) in the definition of dense embedding. Now suppose that $x, y \in \prod_{i\in I}^{\mathrm{fin}}({}^{<\omega}\omega)$ and $f(x)$ and $f(y)$ are compatible; say $f(x), f(y) \subseteq h \in \mathrm{Fn}(I, \omega, \omega)$. Suppose that $i \in I$ and $l \in \mathrm{dmn}(x_i) \cap \mathrm{dmn}(y_i)$. Let $k = g_i(l)$. Then

$$x_i(l) = (f(x))(k) = h(k) = (f(y))(k) = y_i(l).$$

It follows that if we let $z_i = x_i \cup y_i$ for all $i \in \omega$ then $z \in \prod_{i\in I}^{\mathrm{fin}}({}^{<\omega}\omega)$ and $z \le x, y$.

Now to show that $\mathrm{rng}(f)$ is dense in $\mathrm{Fn}(I, \omega, \omega)$, let $s \in \mathrm{Fn}(I, \omega, \omega)$ with $s_i \neq \emptyset$ for some i. We define, for any $i \in I$,

$$\mathrm{dmn}(x_i) = \{l \in \omega : \exists k \in \mathrm{dmn}(s)[s(k) \in J_i \wedge l = g_i^{-1}(s(k))]\}.$$

Then we define $x_i(l) = s(k)$. To show that this does not depend on the choice of k, suppose tha also $k' \in \mathrm{dmn}(s)$, $s(k') \in J_i$, and $l = g_i^{-1}(s(k'))$. Then $g_i^{-1}(s(k)) = g_i^{-1}(s(k'))$, so $s(k) = s(k')$.

Now we claim that $s \subseteq f(x)$. For, suppose that $k \in \mathrm{dmn}(s)$. Say $s(k) \in J_i$. Let $l = g_i^{-1}(s(k))$. So $l \in \mathrm{dmn}(x_i)$ and $x_i(l) = s(k)$. Hence $k \in \mathrm{dmn}(f(x))$ and $(f(x))(k) = x_i(l) = s(k)$. □

Proposition 30.8. (IV.4.8b) $\prod_{i\in I}^{\mathrm{fin}}({}^{<\omega}\omega)$ *densely embeds in* $\mathit{Fn}(I, 2, \omega)$.

Proof. Claim: there is a dense embedding of ${}^{<\omega}\omega$ into $\mathrm{Fn}(\omega, 2, \omega)$.

For, if $x = \langle m_0, \ldots, m_{k-1}\rangle \in ({}^{<\omega}\omega)$, define

$$f(x) = 0^{m_0}10^{m_1}1\cdots 0^{m_{k-1}}1.$$

Clearly $y \subseteq x$ iff $f(y) \subseteq f(x)$. To show that f is dense, let $z \in \mathrm{Fn}(\omega, 2, \omega)$. Let $m \in \omega$ and $w \in {}^m2$ be such that $z \subseteq w$, with $w(\mathrm{dmn}(w) - 1) = 1$. We can write

$$w = 0^{n_0}10^{n_1}1\cdots 0^{n_{k-1}}1.$$

Then $f(\langle n_0, n_1, \ldots, n_{k-1}\rangle) = w$, as desired. Now see Proposition 25.68.

Fix such an f.

Now write $I = \bigcup_{i\in I} J_i$ with the J_i's pairwise disjoint and each of size ω. For each $i \in I$ let g_i be a bijection of ω onto J_i. Now for each $x \in \prod_{i\in I}^{\mathrm{fin}}({}^{<\omega}\omega)$ define $h(x) \in \mathrm{Fn}(I, 2, \omega)$ by setting

$$\begin{aligned} &\mathrm{dmn}(h(x)) = \{j \in I : \exists i \in I[j \in J_i \wedge \exists k \in \mathrm{dmn}(f(x_i))[g_i(k) = j]]\} \\ &\text{and } (h(x))(j) = (f(x_i))(k). \end{aligned}$$

Suppose that $x \le y$. Thus $\forall i \in I[x_i \supseteq y_i]$. Suppose that $j \in \mathrm{dmn}(h(y))$, and choose i, k correspondingly. In particular, $k \in \mathrm{dmn}(f(y_i))$. Since $x_i \supseteq y_i$ we have $f(x_i) \supseteq f(y_i)$, so $j \in \mathrm{dmn}(h(x))$. Then $(h(y))(j) = f(y_i)(k) = f(x_i)(k(h(x))(j)$. This shows that $h(x) \supseteq h(y)$, hence $h(x) \le h(y)$. Conversely, suppose that $h(x) \le h(y)$. Take any $i \in I$;

we want to show that $x_i \supseteq y_i$. So, take any $k \in \mathrm{dmn}(y_i)$. Let $j = g_i(k)$. Then $j \in J_i$; so $j \in \mathrm{dmn}(h(y))$. So $j \in \mathrm{dmn}(h(x))$, and $(f(x_i))(k) = (h(x))(j) = (h(y))(j) = (f(y_i))(k)$. This shows that $f(y_i) \subseteq f(x_i)$; hence $y_i \subseteq x_i$.

Finally, to show that $\mathrm{rng}(h)$ is dense in $\mathrm{Fn}(I, 2, \omega)$, let $z \in \mathrm{Fn}(I, 2, \omega)$. Let $i \in I$ and set $w_i = z \restriction J_i$. Then $w_i \circ g_i \in \mathrm{Fn}(\omega, 2, \omega)$. Choose $v_i \in ({}^{<\omega}\omega)$ such that $w_i \circ g_i \subseteq f(v_i)$. We claim now that $z \subseteq h(v)$. For, take any $j \in \mathrm{dmn}(z)$. Choose $i \in I$ such that $j \in J_i$. Thus $j \in \mathrm{dmn}(w_i)$. Let $k = g_i^{-1}(j)$. Then $k \in \mathrm{dmn}(w_i \circ g_i)$, so also $k \in \mathrm{dmn}(f(v_i))$. Hence $j \in \mathrm{dmn}(h(v))$, and

$$(h(v))(j) = (f(v_i))(k) = w_i(g_i(k)) = w_i(j) = z(j). \qquad \square$$

Lemma 30.9. (IV.4.9) *Let M be a ctm of ZFC and let $\mathbb{Q} \in M$ such that $(|\mathbb{Q}| \leq \omega)^M$. Let G be $\mathbb{Q}$-generic over M. Then there is no $h \in ({}^\omega\omega) \cap M[G]$ such that $\forall f \in ({}^\omega\omega) \cap M[f \leq^* h]$.*

Proof. First we note:

(1) If $p \Vdash \tau \in \omega$, then $X \stackrel{\text{def}}{=} \{l \in \omega : p \Vdash \check{l} \leq \tau\}$ is finite.

In fact, suppose that X is infinite. Let $p \in G$ generic. Let $m = \tau_G$, and choose distinct $l_0, \ldots, l_m \in X$. Then $l_0, \ldots, l_m < m$, contradiction.

Now for the lemma, suppose that there is such an h. Take $\dot{h}$ such that $\dot{h}_G = h$. Let $W = ({}^\omega\omega) \cap M$. Then $M[G] \models \forall x \in \check{W}[x \leq^* \dot{h}]$, so there is a $p \in G$ such that $p \Vdash (\dot{h} : \omega \to \omega$ and $\forall x \in \check{W}[x \leq^* \dot{h}])$.

Now we work in M. List $\{s \in \mathbb{Q} : s \leq p\}$ as $\{r_j : j \in \omega\}$. For each $n \in \omega$ let $E_n = \{l \in \omega : \exists j < n[r_j \Vdash \check{l} \leq \dot{h}(n)\}$. By (1), for each $j < n$ the set $\{l \in \omega : r_j \Vdash \check{l} \leq \dot{h}(n)\}$ is finite, and so E_n is finite. For each $n \in \omega$ let $f(n) = \max(E_n) + 1$; $f(n) = 1$ if $E_n = \emptyset$. Thus $f \in ({}^\omega\omega)$. Hence $f \in W$.

Since $p \Vdash \forall x \in \check{W}[x \leq^* \dot{h}]$ and $\mathbb{1} \Vdash \check{f} \in \check{W}$, it follows that $p \Vdash \exists m \forall n \geq m[\check{f}(n) \leq \dot{h}(n)]$. Hence there exist a $q \leq p$ and an m such that $q \Vdash \forall n \geq m[\check{f}(n) \leq \dot{h}(n)]$. Say $q = r_j$. Take n with $n > j$ and $n > m$, and let $l = f(n)$. Then $r_j \Vdash \check{l} \leq \dot{h}(n)$, so $l \in E_n$ and hence $f(n) > l$ by the definition of f, contradiction. $\square$

Lemma 30.10. (IV.4.10) *Let M be a ctm of ZFC and let $\mathbb{P} = \mathrm{Fn}(I, J, \omega)$, where $(J \leq \omega)^M$. Let G be $\mathbb{P}$-generic over M.*

Then there is no $h \in ({}^\omega\omega) \cap M[G]$ such that $\forall f \in ({}^\omega\omega) \cap M[f \leq^* h]$.

Proof. Suppose there is such an h. Thus $h \subseteq \omega \times \omega$. Say $h = \sigma_G$. Let $\dot{h}$ be a nice $\mathbb{P}$-name for a subset of $(\omega \times \omega)^v$ such that $\mathbb{1} \Vdash (\sigma \subseteq (\omega \times \omega)^v \to \sigma = \dot{h})$. Say $p \Vdash \sigma \subseteq (\omega \times \omega)^v$. Then $p \Vdash \sigma = \dot{h}$, so $\dot{h}_G = h$. Note that $(\omega \times \omega)^v = \{((m, n)^v, \mathbb{1}) : m, n \in \omega\}$, so that $\hat{h}$ has the form

$$\bigcup_{m,n \in \omega} \{\{(m, n)^v\} \times A_{mn}\}.$$

Here each A_{mn} is an antichain in $\mathbb{P}$, and hence is countable. Let $S = \bigcup_{m,n \in \omega} A_{mn}$. Then S is a countable subset of $\mathrm{Fn}(I, J, \omega)$. Let $K = \bigcup_{p \in S} \mathrm{dmn}(p)$ and $\mathbb{Q} = \mathrm{Fn}(K, J, \omega)$. Then K is a countable subset of I, and $\dot{h}$ is a $\mathbb{Q}$-name.

Now $\mathbb{Q} \subseteq_c \mathbb{P}$ by Proposition 25.66, so Lemma 30.3 applies to the inclusion $\mathbb{Q} \subseteq_c \mathbb{P}$. So with $H = G \cap \mathbb{Q}$ we have $M \subseteq M[H] \subseteq M[G]$ and $h = \dot{h}_G = \dot{h}_H$. Moreover, $\forall f \in ({}^\omega\omega) \cap M[f \leq^* h$. This contradicts Lemma 30.9. $\square$

Lemma 30.11. (IV.4.10a) *Under the conditions of Lemma 30.9, if $M \models CH$, then $M[G] \models \mathfrak{b} = \omega_1$.*

Proof. $({}^\omega\omega) \cap M$ is an unbounded family of size ω_1 in $M[G]$. $\square$

Let $\mathscr{E} \subseteq ({}^\omega\omega)$ be infinite. We define the *dominating function order* (dfo) $(\mathbb{P}(\mathscr{E}), \leq)$ as follows. $\mathbb{P}(\mathscr{E})$ consists of all pairs $p = (s_p, Y_p)$ such that $s_p \in \mathrm{Fn}(\omega, \omega, \omega)$ and $Y_p \in [\mathscr{E}]^{<\omega}$. We define $q \leq p$ iff $s_q \supseteq s_p$, $Y_q \supseteq Y_p$, and $\forall f \in Y_p \forall n \in \mathrm{dmn}(s_q) \backslash \mathrm{dmn}(s_p)[s_q(n) > f(n)]$.

Proposition 30.12. (IV.4.11) *dfo $\mathbb{P}$ is transitive. $\mathbb{P}$ is σ-centered, and there is a family of $|\mathscr{E}|$ dense sets such that whenever G is a filter meeting all of them and $h = \bigcup_{p \in G} s_p$, then $h \in ({}^\omega\omega)$ and $f \leq^* h$ for all $f \in \mathscr{E}$.*

Proof. First we check transitivity. Suppose that $r \leq q \leq p$, $f \in \mathcal{Y}_p$, and $n \in \mathrm{dmn}(s_r) \backslash \mathrm{dmn}(s_p)$. Note that $f \in \mathcal{Y}_q$. If $n \notin \mathrm{dmn}(s_q)$, then $s_r(n) > f(n)$. If $n \in \mathrm{dmn}(s_q)$, then $s_r(n) = s_q(n) > f(n)$.

$\mathbb{P}$ is σ-centered, since for any $t \in \mathrm{Fn}(\omega, \omega, \omega)$ the set of $p \in \mathbb{P}$ with $s_p = t$ is centered.

For each $n \in \omega$ let $D_n = \{p \in \mathbb{P} : n \in \mathrm{dmn}(s_p)\}$. Then D_n is dense. For, suppose that $p \in \mathbb{P}$. If $n \in \mathrm{dmn}(s_p)$, then $p \in D_n$. Suppose that $n \notin \mathrm{dmn}(s_p)$. Let m be greater than $f(n)$ for each $f \in \mathcal{Y}_p$, and let $s_q = s_p \cup \{(n, m)\}$ and $\mathcal{Y}_p = \mathcal{Y}_p$. Then $q \leq p$ and $q \in D_n$. So D_n is dense. Also, for each $f \in \mathcal{E}$ let $E_f = \{p \in \mathbb{P} : f \in \mathcal{Y}_p\}$. Clearly E_f is dense. Let $\mathscr{A} = \{D_n : n \in \omega\} \cup \{E_f : f \in \mathcal{E}\}$. So $|\mathscr{A}| \leq |\mathcal{E}|$.

Suppose that G is a filter meeting all members of $\mathscr{A}$. Because of the D_n's, we have $h \in {}^\omega\omega$. Take any $f \in \mathcal{E}$. Choose $p \in G$ such that $f \in \mathcal{Y}_p$. We claim that $h(n) > f(n)$ for all n greater than each member of s_p. For, take such an n, and choose $q \in G$ so that $n \in \mathrm{dmn}(s_q)$. Choose $r \in G$ with $r \leq p, q$. Then $n \in \mathrm{dmn}(s_r)$ since $r \leq q$, and $r(n) > f(n)$ since $r \leq p$ and $f \in \mathcal{Y}_p$. Hence $h(n) = r(n) > f(n)$. $\square$

Proposition 30.13. *For any ctm M there is a forcing order $\mathbb{P} \in M$ such that in $M[G]$ there is an $h \in {}^\omega\omega$ which almost dominates each $f \in ({}^\omega\omega) \cap M$.*

Proof. Let $\mathscr{E} = ({}^\omega\omega) \cap M$ and apply Proposition 30.12. $\square$

Proposition 30.14. (IV.4.12) *Let M be a ctm of ZFC + CH. Then there is an $A \in M$ such that $A \subseteq [\omega]^\omega$, ($A$ is a mad family)^M, $(|A| = \omega_1)^M$, and A is still mad in any extension $M[G]$ with G $\mathrm{Fn}(I, J, \omega)$-generic, where $I, J \in M$ and $(2 \leq |J| \leq \omega)^M$.*

Proof. First we work in M with the poset $\mathbb{T} \stackrel{\text{def}}{=} {}^{<\omega}\omega$. Note that $\mathbb{T}$ is countable. Hence by CH there are at most ω_1 nice names for subsets of ω. Let $\langle(\tau_\xi, p_\xi) : \omega \leq \xi < \omega_1\rangle$ list all pairs (τ, p) such that τ is a nice name for a subset of ω, $p \in \mathbb{T}$, and $p \Vdash |\tau| = \omega$. We define $\langle A_\xi : \xi < \omega_1\rangle \in {}^{\omega_1}([\omega]^\omega)$ by recursion. Let $\langle A_n : n \in \omega\rangle$ be disjoint infinite subsets of ω. Now suppose that A_η has been defined for all $\eta < \xi$, where $\omega \leq \xi < \omega_1$, so that $A_\eta \cap A_\rho$ is finite for $\eta \neq \rho$.

Case 1. There is an $\eta < \xi$ such that $p_\xi \not\Vdash \tau_\xi \perp \check{A}_\eta$. Let $\langle B_n : n \in \omega\rangle$ enumerate $\{A_\eta : \eta < \xi\}$ without repetitions. Note that for each $n \in \omega$ the set $\omega\backslash\bigcup_{m<n} B_m$ is infinite, since B_n is infinite and $B_n \cap \bigcup_{m<n} B_m = \bigcup_{m<n}(B_n \cap B_m)$ is finite. By recursion choose

$$a_n \in \left(\omega\backslash\bigcup_{m<n} B_m\right)\backslash\{a_m : m < n\}.$$

Let $A_\xi = \{a_n : n \in \omega\}$. Then $A_\xi \cap A_\eta$ is finite for all $\eta < \xi$.

Case 2. $\forall\eta < \xi[p_\xi \Vdash \tau_\xi \perp \check{A}_\eta]$. Let $\langle B_n : n \in \omega\rangle$ enumerate $\{A_\eta : \eta < \xi\}$ without repetitions. Note that for each $n \in \omega$ the set $\omega\backslash\bigcup_{m<n} B_m$ is infinite. Let $\langle(m_i, q_i) : i \in \omega\rangle$ enumerate all pairs (m, q) such that $m \in \omega$ and $q \leq p_\xi$. We now define $\langle a_i : i \in \omega\rangle$ by recursion. Suppose that a_j has been defined for all $j < i$. Then

$$q \Vdash \exists n[n > \check{m}_i \wedge n \in \tau_\xi \wedge \forall j < i[n \notin \check{B}_j \wedge n \neq \check{a}_j]].$$

It follows that there is an $r \leq q$ and an $a_i \in \omega$ such that $a_i > m_i$, $a_i \notin \bigcup_{j<i} B_j$, $a_i \neq a_j$ for all $j < i$, and $r \Vdash \check{a}_i \in \tau_\xi$. Let $A_\xi = \{a_i : i \in \omega\}$. Then $A_\xi \cap A_\eta$ is finite for all $\eta < \xi$, A_ξ is infinite, and

$$(*) \qquad \forall m \in \omega\forall q \leq p_\xi\exists r \leq q\exists n > m[n \in A_\xi \wedge r \Vdash \check{n} \in \tau_\xi].$$

Now suppose that H is $\mathbb{T}$-generic over M, and in $M[H]$ we have a set $B \in [\omega]^\omega$. We want to show that $B \cap A_\xi$ is infinite for some $\xi < \omega_1$. Say $B = \mu_H$. Let ν be a nice name for a subset of ω such that $\mathbb{1} \Vdash \mu \subseteq \omega \rightarrow \mu = \nu$. Say $p \in H$ and $p \Vdash \mu \subseteq \omega \wedge |\mu| = \omega$. Then $p \Vdash |\nu| = \omega$. Choose $\xi < \omega_1$ so that $(p, \nu) = (p_\xi, \tau_\xi)$. Take any $m \in \omega$; we will find $n > m$ such that $n \in B \cap A_\xi$. Now by $(*)$, the set $\{r : \exists n > m[r \Vdash \check{n} \in \tau_\xi]\}$ is dense below p_ξ. Hence there is an $n > m$ with $n \in A_\xi$ and an $r \in H$ such that $r \Vdash \check{n} \in \tau_\xi$, hence $n \in \tau_{\xi H} = \nu_H = \mu_H = B$, as desired.

Claim 1. *$\mathbb{T}$ densely embeds in* $\mathrm{Fn}(\omega, \omega, \omega)$.

Proof. Let f be the inclusion map. It suffices to show that $\mathbb{T}$ is dense in $\mathrm{Fn}(\omega, \omega, \omega)$, and this is clear. □

Claim 2. *If $2 \leq m \in \omega$, then $\mathbb{T}$ densely embeds in* $\mathrm{Fn}(\omega, m, \omega)$.

Proof. For $m = 2$, see the proof of Proposition 30.8. Now suppose that $m > 2$. For each natural number n, let g_n be the representation of n with base $m - 1$. Thus g_n is a sequence of the integers $0, \ldots, m - 2$. For any $\langle a_0, \ldots, a_{p-1}\rangle \in {}^p\omega$ let

$$f(\langle a_0, \ldots, a_{p-1}\rangle) = g_{a_0}(m-1)g_{a_1}(m-1)\cdots g_{a_{p-1}}(m-1).$$

Clearly f is the required dense embedding. □

Now for the exercise itself, let $\mathscr{A} = \{A_\xi : \xi < \omega_1\}$. So $(|\mathscr{A}| = \aleph_1)^M$. Suppose that $I, J \in M$ and $(2 \leq |J| \leq \omega)^M$, and let $\mathbb{P} = \mathrm{Fn}(I, J, \omega)$. Let G be $\mathbb{P}$-generic over M. Suppose that $B \in [\omega]^\omega$ with $B \in M[G]$; we want to find $\xi < \omega_1$ such that $B \cap A_\xi$ is infinite. Say $B = \sigma_G$, where σ is a nice name for a subset of ω; say $\sigma = \bigcup_{n\in\omega}(\{\check{n}\} \times A_n)$ with each A_n an antichain in $\mathbb{P}$. So each A_n is countable. Let $K = \bigcup_{n\in\omega, f\in A_n} \mathrm{dmn}(f)$. Hence K is countable. We may assume that $K = \omega$. Then σ is a $\mathrm{Fn}(K, J, \omega)$-name. Then by c;ao,s 1 and 2, there is an H in $\mathbb{T}$ over M such that $M[G] = M[H]$. Hence the existence of the desired $\xi < \omega_1$ follows. $\square$

$\mathbb{P}$ is *weakly homogeneous* iff $\forall p \in \mathbb{P}[\{f(p) : p \in \mathrm{Aut}(\mathbb{P})\}$ is predense in $\mathbb{P}]$.

Theorem 30.15. (IV.4.15) *If $\mathbb{P}$ is weakly homogeneous and φ is a sentence, then $\mathbb{1} \Vdash \varphi$ or $\mathbb{1} \Vdash \neg\varphi$.*

Proof. Assume that $\mathbb{1} \not\Vdash \varphi$ and $\mathbb{1} \not\Vdash \neg\varphi$. By Theorem 28.24(ix) there are p, q such that $p \Vdash \neg\varphi$ and $q \Vdash \varphi$. Since $\mathbb{P}$ is weakly homogeneous, there is an $f \in \mathrm{Aut}(\mathbb{P})$ such that q and $f(p)$ are compatible. But $f(p) \Vdash \neg\varphi$ by Lemma 30.5, so q and $f(p)$ are not compatible, contradiction. $\square$

Proposition 30.16. (IV.4.16) *Theorem 30.15 can fail if the condition that $\mathbb{P}$ is weakly homogeneous is omitted.*

Proof. Let $\mathbb{Q}$ and $\mathbb{R}$ be such that $\mathbb{Q} \Vdash 2^\omega = \omega_2$ and $\mathbb{R} \Vdash 2^\omega = \omega_3$ Form $\mathbb{P}$ by putting disjoint copies of $\mathbb{Q}$ and $\mathbb{R}$ together side-by-side, with a new $\mathbb{1}$ above them. Then $\mathbb{P} \not\Vdash 2^\omega = \omega_2$ and $\mathbb{P} \not\Vdash \neg(2^\omega = \omega_2)$. $\square$

Proposition 30.17. (IV.4.17) *Let $\mathscr{E} \subseteq ({}^\omega\omega)$ be such that $\forall f, f'[f =^* f' \Rightarrow (f \in \mathscr{E}$ iff $f' \in \mathscr{E})]$. Then the dominating function order is weakly homogeneous.*

Proof. Let $p, q \in \mathbb{P}$. Choose $m \in \omega$ so that $\mathrm{dmn}(s_q) \cup \mathrm{dmn}(s_p) \subseteq m$. Define $h : \omega \to \omega$ by setting, for any $i \in \omega$,

$$h(i) = \begin{cases} m + i & \text{if } i < m, \\ i - m & \text{if } m \le i < 2m, \\ i & \text{otherwise.} \end{cases}$$

Now define k with domain $\mathbb{P}$ by setting, for any $r \in \mathbb{P}$,

$$s_{k(r)} = s_r \circ h \quad \text{and} \quad \mathcal{Y}_{k(r)} = \{f \circ h : f \in \mathcal{Y}_r\}.$$

If $r \le r'$, then $s_{r'} \subseteq s_r$ and hence $s_{k(r')}) \subseteq s_{k(r)}$. Also, $\mathcal{Y}_{r'} \subseteq \mathcal{Y}_r$. Clearly $f =^* (f \circ h)$ for each $f \in \mathscr{E}$. Suppose that $f \in \mathcal{S}_{r'}$ and $n \in \mathrm{dmn}(s_{k(r)} \backslash \mathrm{dmn}(s_{k(r')}))$. Thus $h(n) \in \mathrm{dmn}(s_r) \backslash \mathrm{dmn}(s_{r'})$, so $s_r(h(n)) > f(h(n))$. Thus $(s_{k(r)})(n) > (f \circ h)(n)$. This shows that $k(r) \le k(r')$. Clearly $k \circ k$ is the identity on $\mathbb{P}$. So k is an automorphism of $\mathbb{P}$.

Now we define $r \in \mathbb{P}$. Note that $\mathrm{dmn}(s_{k(p)}) = h^{-1}[\mathrm{dmn}(p)]$, which is disjoint from $\mathrm{dmn}(s_q)$. Let $\mathrm{dmn}(s_r) = \mathrm{dmn}(s_{k(p)} \cup \mathrm{dmn}(s_q)$. For any $i \in \mathrm{dmn}(s_r)$ let

$$s_r(i) = \begin{cases} \max\{f(i) : f \in \mathcal{Y}_{k(p)}\} + 1 & \text{if } i \in \mathrm{dmn}(s_q), \\ \max\{f(i) : f \in \mathcal{Y}_q\} + 1 & \text{if } i \in \mathrm{dmn}(s_{k(p)}). \end{cases}$$

Let $\mathcal{Y}_r = \mathcal{Y}_{k(p)} \cup \mathcal{Y}_q$. Thus $r \in \mathbb{P}$.

To check that $r \leq k(p)$, suppose that $f \in \mathcal{Y}_{k(p)}$. Suppose $n \in \text{dmn}(s_r) \backslash \text{dmn}(s_{k(p)})$. Hence $n \in \text{dmn}(s_q)$. and $s_r(n) > f(n)$.

To check that $r \leq q$, suppose that $f \in \mathcal{Y}_q$. Suppose that $n \in \text{dmn}(s_r) \backslash \text{dmn}(q)$. Then $n \in \text{dmn}(s_{k(p)})$. and $s_r(n) > f(n)$.

Thus $r \leq k(p), q$, as desired. □

Let A be a complete BA. Set $\mathbb{P} = A \backslash \{0\}$. Define $j(p) = p$ for all $p \in \mathbb{P}$. By Theorem 27.22 there is an isomorphism f from $\text{RO}(\mathbb{P})$ onto A such that $f \circ e = j$.

Proposition 30.18.

$$f([\![\varphi(\sigma_1, \ldots, \sigma_n)]\!] = \sum^A \{p \in \mathbb{P} : p \Vdash \varphi(\sigma_1, \ldots, \sigma_n)\}.$$

Proof. If $p \Vdash \varphi(\sigma_1, \ldots, \sigma_n)$, then $e(p) \leq [\![\varphi(\sigma_1, \ldots, \sigma_n)]\!]$, and hence $p = f(e(p)) \leq f([\![\varphi(\sigma_1, \ldots, \sigma_n)]\!])$. Thus $f([\![\varphi(\sigma_1, \ldots, \sigma_n)]\!])$ is an upper bound for $\{p : p \Vdash \varphi(\sigma_1, \ldots, \sigma_n)\}$. Suppose that q is any upper bound, but $f([\![\varphi(\sigma_1, \ldots, \sigma_n)]\!]) \not\leq q$. Then $[\![\varphi(\sigma_1, \ldots, \sigma_n)]\!] \not\leq f^{-1}(q)$, so there is a $p \in \mathbb{P}$ such that $e(p) \leq [\![\varphi(\sigma_1, \ldots, \sigma_n)]\!] \cdot -f^{-1}(q)$. So $p \Vdash \varphi(\sigma_1, \ldots, \sigma_n)$ and $p = f(e(p)) \leq -q$. Thus $p \cdot q = 0$; but also $p \leq q$, contradiction. □

We write $\mathbb{P} \rightarrow_d \mathbb{Q}$ iff there is a dense embedding of $\mathbb{P}$ into $\mathbb{Q}$. We write $\mathbb{P} \approx_d \mathbb{Q}$ iff there is an $\mathbb{R}$ such that $\mathbb{P} \rightarrow_d \mathbb{R}$ and $\mathbb{Q} \rightarrow_d \mathbb{R}$.

Proposition 30.19. *Let $\mathbb{P}$ be a forcing poset, and let $e : \mathbb{P} \rightarrow \text{RO}(\mathbb{P})$ be given on page 494. Then e is a dense embedding.* □

Proposition 30.20. *If $\mathbb{P} \rightarrow_d \mathbb{Q}$, then $\mathbb{P}$ and $\mathbb{Q}$ have isomorphic completions.*

Proof. Let $i : \mathbb{P} \rightarrow \mathbb{Q}$ be a dense embedding and let A and B be the completions of $\mathbb{P}$, $\mathbb{Q}$ respectively. Let $j : \mathbb{P} \rightarrow A \backslash \{0\}$ and $k : \mathbb{Q} \rightarrow B \backslash \{0\}$ be dense embeddings given on page 494. Then $k \circ i : \mathbb{P} \rightarrow B \backslash \{0\}$ is a dense embedding. By Theorem 27.22 there is an isomorphism $h : B \rightarrow A$ such that $h \circ k \circ i = j$. □

Lemma 30.21. (IV.4.26) *For posets $\mathbb{P}, \mathbb{Q}$, (1)$\leftrightarrow$(2) and (3)$\rightarrow$* (1).

(1) $\mathbb{P} \approx_d \mathbb{Q}$.

(2) $\mathbb{P}$ and $\mathbb{Q}$ have isomorphic completions.

(3) There is an $\mathbb{S}$ such that $\mathbb{S} \rightarrow_d \mathbb{P}$ and $\mathbb{S} \rightarrow_d \mathbb{Q}$.

Proof. From Proposition 30.20, (1)$\Rightarrow$(2).

Now assume (2); say $j : \mathbb{P} \rightarrow A \backslash \{0\}$ and $k : \mathbb{Q} \rightarrow B \backslash \{0\}$ are dense embeddings given on page 494, and $h : A \rightarrow B$ is an isomorphism. Then $h \circ j : \mathbb{P} \rightarrow B \backslash \{0\}$ is a dense embedding. By symmetry, (1) follows.

Assume (3). Then by Proposition 30.20 the completion of $\mathbb{S}$ is isomorphic to the completions of $\mathbb{P}$ and $\mathbb{Q}$, so the latter two have isomorphic completions. □

Lemma 30.22. (IV.6.1) *If $r \Vdash \exists x \in \pi \varphi(x)$, then there exist a $q \leq r$ and $\sigma \in \text{dmn}(\pi)$ such that $q \Vdash [\sigma \in \pi$ and $\varphi(\sigma)]$.*

Proof. Let G be generic with $r \in G$. Then there is an $x \in \pi_G$ such that $M[G] \models \varphi(x)$. Say $x = \sigma_G$. Then $M[G] \models [\sigma \in \pi$ and $\varphi(\sigma)]$. Hence there is a $q \in G$ such that $q \Vdash [\sigma \in \pi$ and $\varphi(\sigma)]$. □

Lemma 30.23. (IV.6.4) *Let T be a well-pruned Suslin tree in M, and force with $(T, \geq)$. Then $(T, \geq)$ is ccc, and in $M[G]$, G is a path through T.*

Proof. p and q are compatible iff there is an r such that $r \geq p$ and $r \geq q$, thus iff p and q are comparable. So $p \perp q$ iff p and q are incomparable. So ccc holds. If $p, q \in G$, then p and q are compatible, hence comparable. So G is a chain. For any $p \in G$ and any $\alpha < \omega_1$, the set $\{q : p \leq q$ and q has height $\geq \alpha\}$ is dense above p, and hence there is a $q \in G$ with height $\geq \alpha$. Since G is upwards closed under $\geq$, G meets each level of T and so is a path through T. □

Theorem 30.24. *If φ is a sentence which holds in every ctm M, then $ZFC \vdash \varphi$.*

Proof. If $ZFC+\{\neg\varphi\}$ is consistent, apply Theorem 15.10 to get a ctm of $ZFC+\{\neg\varphi\}$. □

Lemma 30.25. (IV.6.5) *If T is a Suslin tree, then there is no order preserving map $\varphi : T \to \mathbb{R}$.*

Proof. Working in a ctm M, suppose such a map exists. Let T' be a well-pruned subtree of T. Applying Lemma 30.23, we get a ccc generic extension $M[G]$ with G a path through T. Applying φ gives a strictly increasing function mappling ω_1 into $\mathbb{R}$, contradiction. This shows that the statement of the lemma holds in every ctm M. Hence the lemma follows by Theorem 30.24. □

Lemma 30.26. (IV.6.6) *Let T be a well-pruned Suslin tree. Then in the poset topology for $(T, \geq)$, every countable intersection of dense open sets is dense open.*

Proof. Note that U is open iff $\forall s \in U[(s \uparrow) \subseteq U$. Now suppose that each U_n is dense open, and let $V = \bigcap_{n \in \omega} U_n$. Clearly V is open. For each $n \in \omega$ let A_n be an antichain, maximal with respect to $A_n \subseteq U_n$. Since U_n is dense, A_n is actually a maximal antichain. Now let $p \in T$ be given; we want to find $q \in V$ such that $p \leq q$. Since each A_n is countable, there is an $\alpha > \operatorname{ht}(x)$ for all $x \in \bigcup_{n \in \omega} A_n$. Since T is well-pruned, there is a q of height α such that $p \leq q$. For each $n \in \omega$ there is an $s_n \in A_n$ such that $s_n < q$. Hence $s_n \in U_n$ and since U_n is open, also $q \in U_n$. So $p \leq q \in \bigcap_{n \in \omega} U_n = V$. □

A topological space X is *Baire* iff in X every countable intersection of dense open sets is dense.

Lemma 30.27. (IV.6.9) *Working in a ctm M, for a forcing poset $\mathbb{P}$, (1)⇒(2) and (2)↔(3):*

(1) $\mathbb{P}$ is Baire in the poset topology.

(2) For each set $E \in M$ and each generic G, M and $M[G]$ have the same elements of ${}^{\omega}E$.

(3) For each $E \in M$, each name τ, and each $p \in \mathbb{P}$, if $p \Vdash \tau : \omega \to \check{E}$, then there exist a function $h : \omega \to E$ and a $q \leq p$ such that $h \in M$ and $q \Vdash \tau = \check{h}$.

Claim 1. *For any forcing poset $\mathscr{P}$, $D \subseteq \mathbb{P}$ is dense iff it is dense in the poset topology.*

Proof. $\Rightarrow$: Assume that D is dense and U is nonempty open in the poset topology. Fix $s \in U$. Then $s\downarrow \subseteq U$. Take $t \in D$ with $t \leq s$. Then $t \in D \cap U$.

$\Leftarrow$: Assume that D is dense in the poset topology. Take any $s \in \mathbb{P}$. Choose $t \in D \cap s\downarrow$. Thus $t \in D$ and $t \leq s$. $\square$

Proof of 30.27 (2)$\Rightarrow$(3): Assume (2), and suppose that $E \in M$, τ is a name, $p \in \mathbb{P}$, and $p \Vdash \tau : \omega \to \check{E}$. Let G be generic with $p \in G$. Then $h = \tau_G : \omega \to E$. By (2), $h \in M$. Thus $M[G] \models \tau = h$. Hence there is a $q \in G$ such that $q \Vdash \tau = \check{h}$ and $q \leq p$.

(3)$\Rightarrow$(2): Assume (3), and suppose that $E \in M$ and G is generic. We want to show that $({}^{\omega}E)^{M[G]} = ({}^{\omega}E)^M$. Take any $h \in ({}^{\omega}E)^{M[G]}$. Say $h = \tau_G$. Then there is an $r \in G$ such that $r \Vdash \tau : \omega \to \check{E}$. Now

$$D \stackrel{\text{def}}{=} \{q : q \perp r \text{ or } (q \leq r \text{ and } \exists k \in ({}^{\omega}E)^M[q \Vdash \tau = \check{h}])\}$$

is dense, by (3). Take any $q \in G \cap D$. Then $q \leq r$ and there is an $k \in {}^{\omega}E$ such that $q \Vdash \tau = \check{k}$, hence $h = \tau_G = k$.

(1)$\Rightarrow$(3): Assume (1), and suppose that $E \in M$, τ is a name, $p \in \mathbb{P}$, and $p \Vdash \tau : \omega \to \check{E}$. For each $n \in \omega$ let

$$U_n = \{q : q \perp p \text{ or } \exists e \in E[q \Vdash \tau(\check{n}) = \check{e}]\}$$

Clearly U_n is open. To show that U_n is dense, suppose that $s \in \mathbb{P}$. If $s \perp p$ then $s \in U_n$. If $s \not\perp p$, take $t \leq s, p$. Now $t \Vdash \exists x \in \check{E}[(\check{n}, x) \in \tau]$, so by Proposition 29.15 there exist a $q \leq t$ and an $e \in E$ such that $q \Vdash (\check{n}, \check{e}) \in \tau$. This shows that U_n is dense. By claim 1, U_n is dense in the poset topology.

It follows now from (1) and claim 1 that there is a $q \leq p$ such that $q \in \bigcap_{n \in \omega} U_n$. Note that $q \not\perp p$. For each $n \in \omega$ let $h(n)$ be the e such that $q \Vdash \tau(\check{n}) = \check{e}$ Then $q \Vdash \tau = \check{h}$. In fact, if G is generic and $q \in G$, then for each $n \in \omega$, $\tau_G(n) = e = h(n)$; so $\tau_G = h$. $\square$

Proposition 30.28. *Let $\mathbb{P}$ be ω with the order $>$. For each $n \in \omega$ let $U_n = \{n, n+1, n+2, \ldots\}$. Then:*

(i) Each U_n is dense open.

(ii) $\bigcap_{n \in \omega} U_n = \emptyset$.

(iii) For every filter G, $M[G] = M$.

(iv) (2) and (3) in Lemma 30.27 hold.

Proof. (i): Clearly each U_n is open. For any $n, k \in \omega$, $\max(k, n) + 1$ is a member of U_n greater than k, so U_n is dense.

(ii): Obvious.

(iii): If G is a filter, then $G = \omega$, or there is an $n \in \omega$ such that $G = \{0, 1, \ldots, n\}$. Thus $G \in M$ and so $M[G] = M$.

(iv): (2) is obvious, and (3) follows by Lemma IV.6.9. □

Proposition 30.29. (IV.6.10 *For a separative poset $\mathbb{P}$ in a ctm M, conditions (1)–(3) of Lemma 30.27 are equivalent.*

Proof. By Lemma 30.27 it suffices to prove that (3)⇒(1). So assume (3) and suppose that U_n is dense open for each $n \in \omega$. We want to show that $\bigcap_{n\in\omega} U_n$ is dense. Let A_n be maximal such that

(1) $A_n \subseteq U_n$.

(2) A_n is pairwise incompatible.

Since U_n is dense open, A_n is a maximal antichain. Let

$$\tau = \{(\mathrm{op}(\check{n}, \check{p}), \mathbb{1}) : n \in \omega, p \in A_n\}.$$

Then for any generic G, $\tau_G = \{(n,p) : n \in \omega \text{ and } p \in A_n \cap G\}$, and so $\tau_G : \omega \to \mathbb{P}$. Thus $\mathbb{1} \Vdash \tau : \omega \to \check{\mathbb{P}}$. and for any generic G and any $n \in \omega$, $G \cap A_n = \{\tau_G(n)\}$.

By (3) of Lemma 30.27 there is a $h \in M$ and a q such that $q \Vdash \tau = \check{h}$. To show that $q \in U_n$, note that $h(n) \in A_n \subseteq U_n$. Now we claim

(3) There is no $r \leq q$ such that $r \perp h(n)$.

For, suppose that such an r exists. Then r is compatible with some $t \in A_n\backslash\{h(n)\}$. Say $u \leq r, t$. Let H be generic with $u \in H$. Since $u \leq q$ and $q \Vdash \tau = \check{h}$, it follows that $\tau_H(n) = h(n)$. Now $t, h(n) \in H$, contradiction.

Now by (3) and separativity, $q \leq h(n)$. Since $h(n) \in U_n$ and U_n is open, it follows that $q \in U_n$. □

Proposition 30.30. (IV.6.11) *There is an atomless ccc Baire poset iff there is a Suslin tree.*

Proof. ⇒: Assume that $\mathbb{P}$ is atomless, ccc, Baire. We define $\langle U_\alpha : \alpha < \omega_1\rangle$ by recursion. Let $U_0 = \{1\}$. Suppose that U_α has been defined so that it is maximal incompatible. For each $w \in U_\alpha$ let $S_{\alpha,w}$ be a maximal pairwise incompatible collection of elements $\leq w$. Then $2 \leq |S_{\alpha,w}|$ by atomlessness, and $|S_{\alpha,w}| \leq \omega$ by ccc. Let $U_{\alpha+1} = \bigcup_{w\in U_\alpha} S_{\alpha,w}$. Clearly $U_{\alpha+1}$ is maximal incompatible. Now suppose that $\alpha < \omega_1$ is limit and U_β has been defined for all $\beta < \alpha$. For each $\beta < \alpha$ let $D_\beta = \{s : \exists t \in U_\beta[s \leq t]\}$. Clearly D_β is open. It is dense; for let $t \in \mathbb{P}$. Choose $w \in U_\beta$ such that w and t are compatible. Say $x \leq w, t$. Then $x \in D_\beta$ and $x \leq t$. Let $V = \bigcap_{\beta<\alpha} D_\beta$. Then V is dense by Baireness. Let U_α be maximal pairwise incompatible $\subseteq V$.

Let $T = \bigcup_{\alpha<\omega_1} U_\alpha$. Then T is a Souslin tree by Proposition 22.8.

⇐: Assume that there is a Suslin tree T. By Lemma 22.26 we may assume that T is well-pruned. Consider the poset $\mathbb{P} \stackrel{\text{def}}{=} (T, \geq)$. Since T is well-pruned, $\mathbb{P}$ is atomless; otherwise we would get an uncountable chain. Clearly $\mathbb{P}$ is ccc. Now suppose that $\langle D_n : n \in \omega\rangle$ is a system of dense open sets in $\mathbb{P}$, and $t \in \mathbb{P}$; we want to find an element of $\bigcap_{n\in\omega} D_n$ which is below t in $\mathbb{P}$, i.e., above t in T. For each $n \in \omega$ let A_n be maximal pairwise incompatible $\subseteq D_n$. Let $\alpha < \omega_1$ be greater than the height of any element of $\bigcup_{n\in\omega} A_n$. Choose $s \in T$ of height α. For each $n \in \omega$ choose $r_n \in A_n$ which is compatible with s. Thus $r_n < s$ in T. It follows that $s \in \bigcap_{n\in\omega} D_n$.

Proposition 30.31. (IV.6.13) *If $\mathscr{P}$ is ccc and does not add new real numbers ($({}^\omega\omega)^M = ({}^\omega\omega)^{M[G]}$ for any generic G), then $\mathbb{P}$ does not add ω-sequences (for any $A \in M$, $({}^\omega A)^M = ({}^\omega A)^{M[G]}$).*

Proof. Suppose that $A \in M$, $f : \omega \to A$, $f \in M[G]$. By Theorem 29.4 there is a $F : \omega \to \mathscr{P}(A)$ such that $F \in M$, $\forall n \in \omega[f(n) \in F(n)]$, and $(|F(n)| \leq \omega)^M$ for all $n \in \omega$. Let $g_n : F(n) \to \omega$ be an injection. Define $h(n) = g_n(f(n))$ for all $n \in \omega$. Then $h : \omega \to \omega$, so $h \in M$ by assumption. We have $f(n) = g_n^{-1}(h(n))$ for all $n \in \omega$, so $f \in M$. □

Proposition 30.32. *If $\lambda \geq \aleph_1$, I is infinite, and $|J| \geq 2$, then there is an antichain in* $\mathrm{Fn}(I, J, \lambda)$ *of size $2^{\aleph_0}$.*

Proof. Let $K \subseteq I$ with $|K| = \aleph_0$ and let $a, b \in J$ with $a \neq b$. Then members of ${}^K\{a, b\}$ are in $\mathrm{Fn}(I, J, \lambda)$, and any two members of ${}^K\{a, b\}$ are incompatible. □

Proposition 30.33. *Under GCH we have $(2^{<\lambda})^{<\lambda} > 2^{<\lambda}$ when $\lambda = \aleph_\omega$*

Proof. $2^{<\aleph_\omega} = \aleph_\omega$ and $(2^{<\aleph_\omega})^{<\aleph_\omega} \geq {\aleph_\omega}^{\aleph_0} > \aleph_\omega$. □

Lemma 30.34. (IV.7.2) *Suppose that M is a c.t.m. of ZFC and in M we have a forcing order $\mathbb{P}$, an antichain A of $\mathbb{P}$, and a system $\langle \sigma_q : q \in A\rangle$ of members of $M^{\mathbb{P}}$. Then there is a name $\pi \in M^{\mathbb{P}}$ such that $q \Vdash \pi = \sigma_q$ for every $q \in A$.*

Proof. We define

$$(\tau, r) \in \pi \quad \text{iff} \quad \begin{array}{l}(\tau, r) \in M^P \text{ and there is a } q \in A \text{ such that } r \leq q\\ \text{and } r \Vdash \tau \in \sigma_q \text{ and } \tau \in \mathrm{dmn}(\sigma_q).\end{array}$$

Fix $q \in A$ and fix a generic G for $\mathbb{P}$ over M such that $q \in G$; we want to show that $\pi_G = (\sigma_q)_G$.

First suppose that $x \in \pi_G$. Choose $(\tau, r) \in \pi$ such that $r \in G$ and $x = \tau_G$. By the definition of π, there is a $q' \in A$ such that $r \leq q'$, $r \Vdash \tau \in \sigma_{q'}$, and $\tau \in \mathrm{dmn}(\sigma_{q'})$. Since $r \in G$, also $q' \in G$. But A is an antichain, $q, q' \in A$, and $q \in G$, so $q = q'$. So $r \Vdash \tau \in \sigma_q$, and since $r \in G$ it follows that $\tau_G \in (\sigma_q)_G$.

Second, suppose that $y \in (\sigma_q)_G$. Choose $(\tau, r) \in \sigma_q$ such that $r \in G$ and $y = \tau_G$. Since $\tau_G \in (\sigma_q)_G$, there is a $p \in G$ such that $p \Vdash \tau \in \sigma_q$. Also $q \in G$, so let $s \in G$ be such that $s \leq p, q$. Then $(\tau, s) \in \pi$, and so $y = \tau_G \in \pi_G$. □

Theorem 30.35. (IV.7.1) (maximal principle) *Suppose that M is a c.t.m. of ZFC, $\mathbb{P} \in M$ is a forcing order, $\tau_1, \ldots, \tau_n \in M^{\mathbb{P}}$, $p \in P$, and $p \Vdash \exists x\varphi(x, \tau_1, \ldots, \tau_n)$. Then there is a $\pi \in M^P$ such that $p \Vdash \varphi(\pi, \tau_1, \ldots, \tau_n)$.*

Proof. This argument takes place in M, unless otherwise indicated. By Zorn's lemma, let A be an antichain, maximal with respect to the property

(1) For all $q \in A$, $q \leq p$ and $q \Vdash \varphi(\sigma, \pi_1, \ldots, \pi_n)$ for some $\sigma \in M^P$.

By the axiom of choice, for each $q \in A$ let $\sigma_q \in M^{\mathbb{P}}$ be such that $q \Vdash \varphi(\sigma_q, \pi_1, \ldots, \pi_n)$. By Lemma 26.17, let $\pi \in M^P$ be such that $q \Vdash \pi = \sigma_q$ for every $q \in A$. Since also $q \Vdash \varphi(\sigma_q, \tau_1, \ldots, \tau_n)$, an easy argument using the definition of forcing, thus external to M, shows that $q \Vdash \varphi(\pi, \tau_1, \ldots, \tau_n)$.

Now we show that $p \Vdash \varphi(\pi, \tau_1, \ldots, \tau_n)$. To this end we argue outside M. Suppose that G is $\mathbb{P}$-generic over M. We claim that $G \cap A \neq \emptyset$. In fact the set

$$\{r \leq p : \text{there is a } \sigma \in M^{\mathbb{P}} \text{ such that } r \Vdash \varphi(\sigma, \tau_1, \ldots, \tau_n)\} \tag{2}$$

is dense below p, and hence there is an $r \in G$ which is also in (2). If $G \cap A = \emptyset$, then there is an element $q \in G$ incompatible with each member of A; in this case, choose $s \in G$ with $s \leq r, q$. Then s is in (2) and s is incompatible with each element of A, contradicting the maximality of A. So $G \cap A \neq \emptyset$.

Say $q \in G \cap A$. Choose $r \in G$ such that $r \leq p, q$. Since $q \Vdash \varphi(\pi, \tau_1, \ldots, \tau_n)$, also $r \Vdash \varphi(\sigma, \tau_1, \ldots, \tau_n)$, and hence $\varphi(\tau_G, (\tau_1)_G, \ldots, (\tau_n)_G)$ holds in $M[G]$, as desired. □

Proposition 30.36. (IV.7.10) *In M, let $\mathbb{P} = \mathrm{Fn}(\kappa, \lambda, \omega)$, where $\omega \leq \kappa < \lambda$. Then λ is countable in $M[G]$.*

Proof. Clearly $f \stackrel{\text{def}}{=} \bigcup G$ is a function with domain κ. For each $\alpha < \lambda$ let $D_\alpha = \{p : \exists m \in \omega[m \in \mathrm{dmn}(p) \text{ and } p(m) = \alpha\}$. Clearly D_α is dense. It follows that $f \restriction \omega$ maps onto λ. □

Proposition 30.37. (IV.7.10a) *In M, let $\mathbb{P} = \mathrm{Fn}(\kappa, \lambda, \omega)$, where $\omega \leq \kappa < \lambda$. Then for every cardinal $\mu > \lambda$ in M, μ is a cardinal in $M[G]$.*

Proof. By Lemma 29.33, $\mathbb{P}$ has the λ^+-cc. Hence the result follows by Theorem 29.5. □

Proposition 30.38. (IV.7.10b) *In M, let $\mathbb{P} = \mathrm{Fn}(\kappa, \lambda, \omega)$, where $\omega \leq \kappa < \lambda$. Assume that $M \models$GCH. Then $M[G] \models$GCH.*

Proof. Let $\lambda = \aleph_\alpha^M$. We claim

$$\forall \beta \left[\aleph_{\alpha+\beta}^M = \aleph_\beta^{M[G]}\right]. \tag{1}$$

We prove (1) by induction on β. It holds for $\beta = 0$ by Proposition 30.36. Now assume it for β. Write bij for "bijective". Then

$$\aleph_{\alpha+\beta+1}^M \text{ bij } \mathscr{P}(\aleph_{\alpha+\beta}^M) \text{ bij } \mathscr{P}(\aleph_\beta^{M[G]}).$$

Since $\aleph_{\alpha+\beta+1}^M$ is a cardinal in $M[G]$ by Proposition 30.37, it follows that $\aleph_{\alpha+\beta+1}^M = \aleph_{\beta+1}^{M[G]}$. Next suppose that γ is a limit ordinal, and $\aleph_{\alpha+\beta}^M = \aleph_\beta^{M[G]}$ for all $\beta < \gamma$. Then

$$\aleph_{\alpha+\gamma}^M = \bigcup_{\beta<\gamma} \aleph_{\alpha+\beta}^M = \bigcup_{\beta<\gamma} \aleph_\beta^{M[G]} = \aleph_\gamma^{M[G]}.$$

Thus (1) holds.

Now take any ordinal β. Then

$$\mathscr{P}(\aleph_\beta^{M[G]}) \text{ bij } \mathscr{P}(\aleph_{\alpha+\beta}^M) \text{ bij } \aleph_{\alpha+\beta+1}^M = \aleph_{\beta+1}^{M[G]};$$

so $2^{\aleph_\beta^{M[G]}} = \aleph_{\beta+1}^{M[G]}$. □

Lemma 30.39. (IV.7.12) *Assume that $\omega < \lambda < o(M)$ and that for all ordinals $\delta < \lambda$, $({}^\delta\lambda)^M = ({}^\delta\lambda)^{M[G]}$. Then*

(i) For all limit $\gamma \le \lambda\big[\mathrm{cf}^M(\gamma) = \mathrm{cf}^{M[G]}(\gamma)\big]$.

(ii) For all $\beta \le \lambda\big[(\beta$ is a cardinal$)^M$ iff $(\beta$ is a cardinal$)^{M[G]}\big]$.

Proof. (i): Let γ be limit $\le \lambda$, and let $\delta = \mathrm{cf}^M(\gamma)$. Then there is a function $f : \delta \to \gamma$ with $\sup(\mathrm{rng}(f)) = \gamma$, with $f \in M$. So $f \in M[G]$, and hence $\mathrm{cf}^{M[G]}(\gamma) \le \mathrm{cf}^M(\gamma)$. Suppose that $\mathrm{cf}^{M[G]}(\gamma) < \mathrm{cf}^M(\gamma)$. Say $\mathrm{cf}^{M[G]}(\gamma) = \varepsilon < \delta \le \lambda$. Let $g : \varepsilon \to \gamma$ be such that $g \in M[G]$ and $\sup(\mathrm{rng}(g)) = \gamma$. Then $g \in ({}^\varepsilon\lambda)^{M[G]}$, so $g \in ({}^\varepsilon\lambda)^M$, and hence $\mathrm{cf}^M(\gamma) \le \varepsilon < \delta$, contradiction.

(ii): Assume that $\beta \le \lambda$. Clearly $(\beta$ is a cardinal$)^{M[G]} \to (\beta$ is a cardinal$)^M$. Now suppose that not$(\beta$ is a cardinal$)^{M[G]}$. Then there exist a $\delta < \beta$ and a bijection $f : \delta \to \beta$. So $\delta < \lambda$ and $f \in ({}^\delta\lambda)^{M[G]}$. So $f \in ({}^\delta\lambda)^M$. Hence not$(\beta$ is a cardinal$)^M$. □

Proposition 30.40. *If $\lambda \ge \omega_1$, then* $\mathrm{Fn}(\omega_1, 2, \omega_1) \subseteq_c \mathrm{Fn}(\lambda, 2, \omega_1)$. □

Proposition 30.41. *There is a dense embedding of* $\mathrm{Fn}(\omega_1, 2^\omega, \omega_1)$ *into* $\mathrm{Fn}(\omega_1, 2, \omega_1)$.

Proof. Let $g : {}^\omega 2 \to 2^\omega$ be a bijection, and let $h : \omega_1 \times \omega \to \omega_1$ be a bijection. If $\alpha < \omega_1$, $p \in \mathrm{Fn}(\omega_1, 2^\omega, \omega_1)$, and $1^{st}(h^{-1}(\alpha)) \in \mathrm{dmn}(p)$, let

$$(f(p))(\alpha) = (g^{-1}(p(1^{st}(h^{-1}(\alpha))))(2^{nd}(h^{-1}(\alpha)).$$

Note that

$$\mathrm{dmn}(f(p)) = \{\alpha < \omega_1 : 1^{st}(h^{-1}(\alpha)) \in \mathrm{dmn}(p)\} = \{h(\beta, \gamma) : \beta \in \mathrm{dmn}(p), \gamma < \omega\},$$

and so $\mathrm{dmn}(f(p))$ is countable. Thus $f(p) \in \mathrm{Fn}(\omega_1, 2, \omega_1)$.

(1) If $p_1 \le p_2$ then $f(p_1) \le f(p_2)$.

In fact, suppose that $p_1 \le p_2$. Thus $p_1 \supseteq p_2$. Take any $\alpha \in \mathrm{dmn}(f(p_2))$. So $1^{st}(h^{-1}(\alpha)) \in \mathrm{dmn}(p_2)$, hence $1^{st}(h^{-1}(\alpha)) \in \mathrm{dmn}(p_1)$. Moreover,

$$\begin{aligned}
p_2(1^{st}(h^{-1}(\alpha))) &= p_1(1^{st}(h^{-1}(\alpha)));\\
(f(p_2))(\alpha) &= (g^{-1}(p_2(1^{st}(h^{-1}(\alpha))))(2^{nd}(h^{-1}(\alpha)))\\
&= (g^{-1}(p_1(1^{st}(h^{-1}(\alpha))))(2^{nd}(h^{-1}(\alpha)))\\
&= (f(p_1))(\alpha).
\end{aligned}$$

This proves (1).

(2) $\text{rng}(f)$ is dense in $\text{Fn}(\omega_1, 2, \omega_1)$.

For, let $q \in \text{Fn}(\omega_1, 2, \omega_1)$. We want to find $p \in \text{Fn}(\omega_1, 2^\omega, \omega_1)$ such that $q \subseteq f(p)$. For each $\beta \in \omega_1$ let $p_\beta \in 2^\omega$ be defined by setting, for $\gamma \in \omega$,

$$k_\beta(\gamma) = \begin{cases} q(h(\beta,\gamma)) & \text{if } h(\beta,\gamma) \in \text{dmn}(q), \\ 0 & \text{otherwise.} \end{cases}$$

Then we set $p_\beta = g(k_\beta)$. Now suppose that $\alpha \in \text{dmn}(q)$. Say $h^{-1}(\alpha) = (\beta, \gamma)$. Then

$$(f(p))(\alpha) = (g^{-1}(p_\beta))(\gamma) = k_\beta(\gamma) = q(\alpha).$$

(3) If $f(p_1) \le f(p_2)$, then $p_1 \le p_2$.

Indeed, assume that $f(p_1) \le f(p_2)$. So $f(p_2) \subseteq f(p_1)$. Suppose that $\beta \in \text{dmn}(p_2)$. Let $\gamma \in \omega$ be arbitrary, and let $\alpha = h(\beta, \gamma)$. Then $\alpha \in \text{dmn}(f(p_2))$, so $\alpha \in \text{dmn}(f(p_1))$ and $(g^{-1}(p_2(\beta)))(\gamma) = (f(p_2))(\alpha) = (f(p_1))(\alpha) = (g^{-1}(p_1(\beta)))(\gamma)$. Hence $g^{-1}(p_2(\beta)) = g^{-1}(p_1(\beta))$, so $p_2(\beta) = p_1(\beta)$.

Now see Proposition 25.68. □

Proposition 30.42. (IV.7.19) *Assume that $M \models \neg CH$, and let $\mathbb{P} = (\text{Fn}(I, 2, \omega_1))^M$, where $(|I| > \omega)^M$. Then $M[G] \models CH$.*

Proof. We may assume that I is a cardinal $\lambda > \omega$ in M. In M, $\mathbb{P}$ is countably closed, so by Lemma 29.9, $({}^\omega 2)^M = ({}^\omega 2)^{M[G]}$. Also, in $M[G]$ there is no bijection from $\alpha < \omega_1^M$ onto ω_1^M, again by Lemma 29.9. So $(2^\omega)^M = (2^\omega)^{M[G]}$ and $\omega_1^M = \omega_1^{M[G]}$. Now by Propositions 30.38 and 30.39 we have $\text{Fn}(\omega_1, 2^\omega, \omega_1) \subseteq_c \text{Fn}(\lambda, 2, \omega_1)$. Hence by Lemma 30.3, in $M[G]$ there is a function mapping ω_1^M onto $(2^\omega)^M$. Thus CH holds in $M[G]$. □

Proposition 30.43. (IV.7.19) *Assume that $M \models \neg CH$, and let $\mathbb{P} = (\text{Fn}(I, 2, \omega_1))^M$, where $(|I| > \omega)^M$. Suppose that $(\omega < \kappa \le 2^\omega)^M$ Then $(\kappa = \omega_1)^{M[G]}$.* □

Lemma 30.44. (IV.7.23) *If $f \in M[G]$ and f is a function, then* $\text{rng}(f) \subseteq M$ *iff there is an $E \in M$ such that* $\text{rng}(f) \subseteq E$.

Proof. $\Leftarrow$: holds since M is transitive. For $\Rightarrow$, say $\alpha = \text{rank}(f)$. Thus $f \in V_{\alpha+1}$, so $\text{rng}(f) \subseteq V_\alpha$. By absoluteness of rank, $\alpha < o(M[G]) = o(M)$, so $\text{rng}(f) \subseteq V_\alpha \cap M \in M$; see Theorem 13.13.. □

Proposition 30.44. *If $\mathbb{P}$ is λ-closed and $\delta < \lambda$, then $\mathbb{1} \Vdash \forall f \in {}^\delta\check{M}[f \in M]$. That is, for all $f \in M[G]$, if f is a function with domain δ and range included in M, then $f \in M$.*

Proof. By Lemma 30.44, choose $E \in M$ so that $\text{rng}(f) \subseteq E$. Then Lemma 29.9 applies. □

Lemma 30.46. (IV.7.25) $\mathbb{1} \Vdash \forall x[x \cap \check{M}$ *exists*]. *That is,* $\forall x \in M[G][x \cap M \in M[G]]$.

Proof. Let $\alpha = \mathrm{rank}(x)$, so that $x \subseteq V_\alpha$. Then $\alpha < o(M[G]) = o(M)$, so $E \stackrel{\mathrm{def}}{=} V_\alpha \cap M \in M$. Then $x \cap M = x \cap E$, and $x, E \in M[G]$, so $x \cap E \in M[G]$. $\square$

Proposition 30.47. (IV.7.24) *Let $\mathbb{P}$ be an atomless poset. Then $\mathbb{1} \Vdash \forall x \exists y \supseteq x[|y \cap \check{M}| = |y \backslash \check{M}|]$. That is, if G is generic, then*

$$\forall x \in M[G] \exists y \in M[G][y \supseteq x \textit{ and } (|y \cap M|^{M[G]} = |y \backslash M|^{M[G]})].$$

Proof. By Lemma 30.46, $x \cap M \in M[G]$ and $x \backslash M = x \backslash (x \cap M) \in M[G]$.

Case 1. $|x \backslash M| < |x \cap M|$. Let $y = x \cup (\{G\} \times |x \cap M|)$. Then $|y \cap M| = |x \cap M|$ and $|y \backslash M| = \max(|x \backslash M|, |x \cap M|) = |x \cap M|$.

Case 2. $|x \backslash M| = |x \cap M|$. Let $y = x$.

Case 3. $|x \backslash M| > |x \cap M|$. Let $y = x \cup |x \backslash M|$. Then $|y \cap M| = |x \backslash M|$ and $|y \backslash M| = |x \backslash M|$, $\square$

Proposition 30.48. (IV.7.27) *In M let $\theta = |\delta|$. Then*

$$\mathbb{1} \Vdash \forall f \in {}^{\delta}\check{M}[f \in \check{M}] \textit{ iff } \mathbb{1} \Vdash \forall f \in {}^{\theta}\check{M}[f \in \check{M}],$$

that is,

For every generic G,

(i) for every $f \in M[G]$((f is a function with domain δ and range $\subseteq M$) implies that $f \in M$)

iff

(ii) for every $f \in M[G]$((f is a function with domain θ and range $\subseteq M$) implies that $f \in M$).

Proof. Let g be a bijection from θ onto δ. Assume (i), and suppose that $f \in M[G]$ is a function with domain θ and range $\subseteq M$. Then $f \circ g^{-1}$ is in $M[G]$ and is a function with domain δ and range $\subseteq M$. Hence by (i), $f \circ g^{-1} \in M$. Hence $f \in M$. So (ii) holds.

(ii)⇒(i) is similar. $\square$

Proposition 30.49. (IV.7.28) *Let θ be a cardinal. Then (1)⇒(2):*

(1) In $\mathbb{P}$, every intersection of θ dense open sets is dense.

(2) $\mathbb{P}$ does not add θ-sequences. That is, for every $f \in M[G]$((f is a function with domain θ and range $\subseteq M$) implies that $f \in M$).

Proof. (1)⇒(2): assume (1), and suppose that $f \in M[G]$ with f a function with domain θ and with $\mathrm{rng}(f) \subseteq M$. By Lemma 30.44 let $E \in M$ be such that $\mathrm{rng}(f) \subseteq E$. By Lemma 29.9, $f \in M$. $\square$

Proposition 30.50. (IV.7.29) *Assume that κ is uncountable and regular, and T is a well-pruned κ-Suslin tree. Then in the poset topology, every intersection of fewer than κ dense open sets is dense.*

Proof. Let $\alpha < \kappa$ and let $\langle D_\xi : \xi < \alpha \rangle$ be a system of dense open sets in T. For each $\xi < \alpha$ let $A_\xi = \{q \in D_\xi : (q\downarrow') \cap D_\xi = \emptyset\}$. Then A_ξ is an antichain, so $|A_\xi| < \kappa$. Let β be a level above the levels of all members of $\bigcup_{\xi<\alpha} A_\xi$. Take any q of level β, and take any $\xi < \alpha$; we claim that $q \in D_\xi$. For, choose $r \in D_\xi$ such that $q \leq r$. Then choose $p \in A_\xi$ such that $p \leq r$. Then p and q are comparable, so $p < q$. Hence $q \in D_\xi$. □

Proposition 30.51. (IV.7.29a) *Assume that κ is uncountable and regular, and T is a well-pruned κ-Suslin tree. Then forcing with T preserves cofinalities and cardinals.*

Proof. Since T is $\kappa-cc$, cofinalities and cardinals $\geq \kappa$ are preserved by Theorem 29.5. By Propositions 30.49 and 30.50, cofinalities and cardinals less than κ are preserved. □

Proposition 30.52. (IV.7.29b) *Assume that κ is uncountable and regular, and T is a well-pruned κ-Suslin tree. Then in $M[G]$, G is a path through T.*

Proof. Clearly any two elements of G are comparable, and G is closed downwards. For any $\alpha < \kappa$ the set $D = \{p \in T : \text{level}(p) > \alpha\}$ is dense; the intersection of G with D shows that G has an element of level greater than α. □

If μ is a probability measure on X, then $\mathbb{MB} = \mathbb{MB}(X, \mu)$ is the Boolean algebra of measurable sets modulo the ideal of measure zero sets.

Lemma 30.53. (IV.7.34) $\mathbb{MB}(X, \mu)$*, μ a probability measure, is ccc.*

Proof. If A is an antichain, then $|\{p \in A : \mu(p) \geq 1/n\}| \leq n$ for all $n \in \omega$. Now $A = \bigcup_{n\in\omega}\{p \in A : \mu(p) \geq 1/n\}$, so A is countable. □

Lemma 30.54. (IV.7.34a) *Let μ be a probability measure, and consider* $\mathbb{MB}(X, \mu)$*. Then*

$$\mathbb{1} \Vdash \forall f \in {}^\omega\omega \exists h \in {}^\omega\omega \cap \check{M} \forall n \in \omega[f(n) \leq h(n)],$$

That is: given a ctm M, suppose that G is generic and $f \in {}^\omega\omega \cap M[G]$. Then there is an $h \in {}^\omega\omega \cap M$ such that $\forall n \in \omega[f(n) \leq h(n)]$.

Proof. Let $\dot{f}$ be a name such that $\dot{f}_G = f$. Choose $p \in G$ such that $p \Vdash \dot{f} : \omega \to \omega$.

(1) There exist an $h \in {}^\omega\omega \cap M$ and a $q \leq p$ such that $q \Vdash \forall n[\dot{f}(n) \leq \check{h}(n)]$.

For, for each $n \in \omega$, $p \leq [\![\dot{f}(n) \in \omega]\!] = \sum_{m\in\omega}[\![\dot{f}(n) = m]\!]$, and so $p = \sum_{m\in\omega}(p \cdot [\![\dot{f}(n) = m]\!])$. Now for each $n \in \omega$ let $h(n) \in \omega$ be such that

$$\mu\left(\sum_{m\leq h(n)}(p \cdot [\![\dot{f}(n) = m]\!])\right) \geq \mu(p)(1 - 2^{-n-2}).$$

Now let $r_{mn} = p \cdot [\![\dot{f}(n) = m]\!]$ and $q = \prod_{n\in\omega}\sum_{m\leq h(n)} r_{mn}$. Now for any n,

$$\begin{aligned}\mu(p) &= \mu\left(p \cdot \prod_{m\leq h(n)} -r_{mn} + p \cdot \sum_{m\leq h(n)} r_{mn}\right)\\ &= \mu\left(p \cdot \prod_{m\leq h(n)} -r_{mn}\right) + \mu\left(p \cdot \sum_{m\leq h(n)} r_{mn}\right),\end{aligned}$$

so

$$\mu\left(p\cdot\prod_{m\le h(n)}-r_{mn}\right)=\mu(p)-\mu\left(p\cdot\sum_{m\le h(n)}r_{mn}\right)$$
$$\le\mu(p)-\mu(p)(1-2^{-n-2})$$
$$=\mu(p)\cdot 2^{-n-2}.$$

Now $\mu(p)=\mu(q)+\mu(p\cdot -q)$, so

$$\begin{aligned}\mu(q)&=\mu(p)-\mu(p\cdot -q)\\&=\mu(p)-\mu\left(p\cdot\sum_{n\in\omega}\prod_{m\le h(n)}-r_{mn}\right)\\&=\mu(p)-\mu\left(\sum_{n\in\omega}\left(p\cdot\prod_{m\le h(n)}-r_{mn}\right)\right)\\&\ge\mu(p)-\sum_{n\in\omega}\mu\left(p\cdot\prod_{m\le h(n)}-r_{mn}\right)\\&\ge\mu(p)-\sum_{n\in\omega}(\mu(p)\cdot 2^{-n-2})\\&=\mu(p)\left(1-\sum_{n\in\omega}2^{-n-2}\right)\\&=\mu(p)/2.\end{aligned}$$

It follows that $\mu(q)>0$.

Now for all $n\in\omega$, $q\le\sum_{m\le h(n)}[\![\dot f(n)=m]\!]$, so $q\Vdash\exists m\le\check h(n)[\dot f(n)=m]$. Thus $q\Vdash\forall n\in\omega[\dot f(n)\le\check h(n)]$. This proves (1).

Let $D=\{q:p\perp q$ or $(q\le p$ and $\exists h\in\check M[q\Vdash\forall n\in\omega[\dot f(n)\le\check h(n)]\}$. By (1), D is dense, and this gives the desired h. $\square$

Lemma 30.55. (IV.7.35) *Let G be $\mathrm{Fn}(\omega,\omega,\omega)$-generic, and in $M[G]$ let $f=\bigcup G:\omega\to\omega$. Then there is no $h\in({}^\omega\omega)\cap M$ such that $f\le^* h$.*

Proof. Let $h\in({}^\omega\omega)\cap M$. For each $m\in\omega$, the set $\{p\in\mathrm{Fn}(\omega,\omega,\omega):\exists n\in\mathrm{dmn}(p)[n>m$ and $p(n)>h(n)]\}$ is dense, and so $M[G]\models\forall m\exists n>m[f(n)>h(n)]$. $\square$

Lemma 30.56. (IV.7.36) *Suppose that M is a model for $\mathrm{ZFC}+\mathrm{CH}$, $\mathbb{MB}$ is a measure algebra in M, and G is generic. Then $M[G]\models\mathfrak{b}=\mathfrak{d}=\omega_1$.*

Proof. By Lemma 30.54, $({}^\omega\omega)\cap M$ is a dominating family of size ω_1. $\square$

Lemma 30.57. (IV.7.37) *Suppose that M is a ctm for ZFC, κ is an infinite cardinal in M, and $\mathbb{MB} = \mathbb{MB}(2^{\kappa\times\omega}, \mu)$. Let G be generic.*

Then M and $M[G]$ have the same cardinals and cofinalities. Assume that $k^\omega = \kappa$ in M. Then $2^\omega = \kappa$ in $M[G]$.

Proof. M and $M[G]$ have the same cardinals and cofinalities by Lemma 30.53. Now for $\alpha < \kappa$, $n \in \omega$ and $i \in 2$ let $p_i^{\alpha n} = \{x \in {}^{\kappa\times\omega}2 : x(\alpha, n) = i\}$. Then $p_i^{\alpha n}$ is measurable and $\mu(p_i^{\alpha n}) = \frac{1}{2}$, by Proposition 18.85. $\{p_0^{\alpha n}, p_1^{\alpha n}\}$ is a maximal antichain, so by Theorem 30.1 exactly one of them is in G. Define $F(\alpha, n)$ to be the $i \in 2$ such that $p_i^{\alpha n} \in G$. Let $h_\alpha(n) = F(\alpha, n)$ for all $\alpha < \kappa$ and $n \in \omega$. Now suppose that $\alpha, \beta \in \kappa$ and $\alpha \neq \beta$. Then

$$\prod_{n\in\omega} \{x \in {}^{\kappa\times\omega}2 : x(\alpha, n) = x(\beta, n)\} = 0 \tag{1}$$

In fact, take any $m \in \omega\backslash 1$. With $U_f = \{x \in {}^{\kappa\times\omega}2 : f \subseteq x\}$ for each finite function $f \subseteq (\kappa \times \omega) \times 2$, we have

$$\prod_{n<m} \{x \in {}^{\kappa\times\omega}2 : x(\alpha, n) = x(\beta, n)\} = \bigcup_{f\in {}^m 2} \{x \in {}^{\kappa\times\omega}2 : \forall n < m[x(\alpha, n) = x(\beta, n) = f(n)]\}.$$

Now by Proposition 18.86, for each $f \in {}^m 2$ we have

$$\mu(\{x \in {}^{\kappa\times\omega}2 : \forall n < m[x(\alpha, n) = x(\beta, n) = f(n)]\}) = \frac{1}{2^{2m}}.$$

Hence

$$\mu\left(\prod_{n<m} \{x \in {}^{\kappa\times\omega}2 : x(\alpha, n) = x(\beta, n)\}\right) = \frac{2^m}{2^{2m}} = \frac{1}{2^m}.$$

Hence (1) follows.

By (1), $\prod_{n\in\omega}\{x \in {}^{\kappa\times\omega}2 : x(\alpha, n) = x(\beta, n)\} \notin G$, so

$$-\prod_{n\in\omega} \{x \in {}^{\kappa\times\omega}2 : x(\alpha, n) = x(\beta, n)\} = \sum_{n\in\omega} \{x \in {}^{\kappa\times\omega}2 : x(\alpha, n) \neq x(\beta, n)\} \in G.$$

Now $D \stackrel{\text{def}}{=} \{y \in \mathbb{MB}^+ : \exists n \in \omega[y \leq \{x \in {}^{\kappa\times\omega}2 : x(\alpha, n) \neq x(\beta, n)\}]\}$ is dense below $\sum_{n\in\omega}\{x \in {}^{\kappa\times\omega}2 : x(\alpha, n) \neq x(\beta, n)\}$. It follows that there is an $n \in \omega$ such that $\{x \in {}^{\kappa\times\omega}2 : x(\alpha, n) \neq x(\beta, n)\}]\} \in G$. Clearly then $F(\alpha, n) \neq F(\beta, n)$, so $h_\alpha(n) \neq h_\beta(n)$. Thus $2^\omega \geq \kappa$ in $M[G]$.

Now note that $|\mathbb{MB}| = \kappa^\omega$ by Proposition 18.88. It follows that there are exactly κ^ω nice names for subsets of κ. Hence by the argument in the proof of Lemma 29.22, $2^\omega \leq \kappa^\omega$. □

Lemma 30.58. (IV.7.37a) *Suppose that M is a ctm for GCH, κ is an infinite cardinal in M, and $\mathbb{MB} = \mathbb{MB}(2^{\kappa\times\omega}, \mu)$. Let G be generic.*

Then M and $M[G]$ have the same cardinals and cofinalities. Assume that $\mathrm{cf}(\kappa) = \omega$. *Then* $2^\omega = \kappa^+$ *in* $M[G]$.

Proof. $\kappa^\omega = \kappa^+$ if $\mathrm{cf}(\kappa) = \omega$. □

Lemma 30.59. (IV.7.37b) *Suppose that M is a ctm for GCH, κ is an infinite cardinal in M, and $\mathbb{MB} = \mathbb{MB}(2^{\kappa\times\omega}, \mu)$. Let G be generic.*

Then M and $M[G]$ have the same cardinals and cofinalities. Assume that μ is a cardinal, and $\omega \le \mu < \mathrm{cf}(\kappa)$. Then $2^\mu = \kappa$ in $M[G]$.

Proof. By Proposition 29.22 we have $(2^\mu)^{M[G]} \le \kappa$. Obviously $\kappa = 2^\omega \le 2^\mu$ in $M[G]$. □

Lemma 30.60. (IV.7.37c) *Suppose that M is a ctm for GCH, κ is an infinite cardinal in M, and $\mathbb{MB} = \mathbb{MB}(2^{\kappa\times\omega}, \mu)$. Let G be generic.*

Then M and $M[G]$ have the same cardinals and cofinalities. Assume that μ is a cardinal, and $\kappa \le \mu$. Then $2^\mu = \mu^+$ in $M[G]$.

Proof. By Proposition 29.22 we have $(2^\mu)^{M[G]} \le \mu^+$. □

Proposition 30.61. (IV.7.38) *Let $\mathbb{BO}$ be the σ-subalgebra of $\mathscr{P}({}^I2)$ generated by the open sets, and let $\mathbb{P}_{\mathbb{BO}} = \{p \in \mathbb{BO} : \mu(p) > 0\}$. Then $\mathbb{P}_{\mathbb{BO}}$ is densely embedded in $\mathbb{MB}({}^I2, \mu)$.*

Proof. Clearly $\mathbb{P}_{\mathbb{BO}}$ is a subposet of $\mathbb{MB}({}^I2, \mu)$, so by Proposition 188a it suffices to show that $\mathbb{P}_{\mathbb{BO}}$ is dense in $\mathbb{MB}({}^I2, \mu)$. So let $q \in \mathbb{MB}({}^I2, \mu)$. Thus $\mu(q) > 0$, so $\mu(-q) < 1$. By the definition of μ (see page 346), there is an open set U such that $-q \subseteq U$ and $\mu(U) \le \mu(-q) + (1 - \mu(-q))/2 < 1$. Then $-U \subseteq q$ and $\mu(-U) > 0$. $-U$ is closed, hence in $\mathbb{P}_{\mathbb{BO}}$. □

Proposition 30.62. (IV.7.38a) *Let $\mathbb{BA}$ be the σ-subalgebra of $\mathscr{P}({}^I2)$ generated by the clopen sets, and let $\mathbb{P}_{\mathbb{BA}} = \{p \in \mathbb{BA} : \mu(p) > 0\}$. Then $\mathbb{P}_{\mathbb{BA}}$ is densely embedded in $\mathbb{MB}({}^I2, \mu)$.*

Proof. $\mathbb{P}_{\mathbb{BO}} \subseteq \mathbb{P}_{\mathbb{BA}}$, so this follows from Proposition 30.61. □

Proposition 30.63. (IV.7.38b) *Let $\mathscr{F}$ be the set of all closed G_δ's, and let $\mathbb{P}_{\mathscr{F}} = \{p \in \mathscr{F} : \mu(p) > 0\}$. Then $\mathbb{P}_{\mathscr{F}}$ is densely embedded in $\mathbb{MB}({}^I2, \mu)$.*

Proof. The set $-U$ in the proof of Proposition 30.61 is a closed G_δ. □

Proposition 30.64. (IV.7.41) *Let $\kappa > \omega$ be regular. Let $\mathbb{P}$ consist of the empty tree $\mathbb{1}$ together with all subtrees p of ${}^{<\kappa}2$ such that $|p| < \kappa$, $\mathrm{height}(p) = \alpha+1$ for some limit $\alpha < \kappa$, for each $s \in p$ of height less than α we have $s^\frown\langle 0\rangle \in p$ and $s^\frown\langle 1\rangle \in p$ and $\exists t \in \mathcal{L}_\alpha[s < t]$. Define $q \le p$ iff q is an end extension of p. Then $\mathbb{P}$ is ω_1-closed but not ω_2-closed.*

Proof. Suppose that α is a countable limit ordinal and $p_0 > p_1 > \cdots > p_\xi > \cdots$ with $\xi < \alpha$. Let $\langle \beta_n : n \in \omega\rangle$ be strictly increasing with supremum α. Define q as follows. For each $n \in \omega$ let $\mathrm{height}(p_{\beta_n}) = \alpha_n + 1$. Let $\gamma = \bigcup_{n\in\omega} \alpha_n$. Define

$$q = \bigcup_{n\in\omega} p_{\beta_n} \cup \{f \in {}^\gamma 2 : \forall n \in \omega[f \restriction \beta_n \in p_{\beta_n}]\}.$$

Clearly $q < p_\xi$ for each $\xi < \alpha$.

For "not ω_2-closed", let T be a well-pruned ω_1 Aronszajn tree such that each element has two immediate successors. This gives rise to a decreasing sequence of length ω_1 with no element below it. $\square$

Proposition 30.65. (IV.7.42) *Let κ and $\mathbb{P}$ be as in Proposition IV.7.41. In $\mathbb{P}$ every intersection of fewer than κ dense open sets is dense.*

Proof. Suppose that $\alpha < \kappa$ and D_ξ is dense open for each $\xi < \alpha$. Let $p_0 \in \mathbb{P}$ be arbitrary. Let $\mathrm{height}(p_0) = \alpha_0 + 1$. For $s \in p_0$ let $f(s, 0) \in \mathcal{L}_{\alpha_0}$ be such that $s \leq f(s, 0)$.

Now suppose that p_η and $f(s, \eta)$ have been defined for all $\eta < \xi$. First suppose that $\xi = \eta + 1$ for some η. Choose $p_\xi \in D_\eta$ so that $p_\xi < p_\eta$. Say $\mathrm{height}(p_\xi) = \alpha_\xi + 1$. For each $s \in p_\xi \backslash p_\eta$ let $f(s, \xi) \in \mathcal{L}_{\alpha_\xi}$ be such that $s \leq f(s, \xi)$. For $s \in p_\eta$ let $f(s, \xi)$ be such that $f(s, \eta) \leq f(s, \xi)$. Thus in this case $s \leq f(s, \xi)$.

Second suppose that ξ is limit, and for each $\eta < \xi$ and each $s \in p_\eta$ we have $f(s, \eta) \leq f(s, \rho)$ for all ρ such that $\eta < \rho < \xi$. Let $q = \bigcup_{\eta<\xi} p_\eta$. for each $s \in p_\eta$ with $\eta < \xi$ the sequence $\langle f(s, \rho) : \eta \leq \rho < \xi\rangle$ is a chain, and we let $f(s, \xi) = \bigcup_{\eta\leq\rho<\xi} f(s, \rho)$. Then p_ξ is $q \cup \{f(s, \xi) : s \in q\}$. $\square$

Proposition 30.66. (IV.7.43) *Let $\mathbb{Q}$ be the set of all $p \in \mathrm{Fn}(\omega_1, 2^\omega, \omega_1)$ such that* $\mathrm{dmn}(p)$ *is a successor ordinal or 0. Then there is a dense embedding of $\mathbb{Q}$ into* $\mathrm{Fn}(\omega_1, 2, \omega_1)$.

Proof. This is clear from Proposition 30.41. $\square$

Proposition 30.67. (IV.7.45) *Let κ be regular and let T be a well-pruned Aronszajn tree. As a forcing poset, T is not κ-closed.*

Proof. Suppose that T is κ-closed. Then we can define a strictly increasing sequence $\langle p_\xi : \xi < \kappa\rangle$ in T. $\square$

Proposition 30.68. (IV.7.46) *Assume in M that κ is uncountable and regular and $\mathbb{P}$ is a κ-closed poset. Then every κ-Aronszajn tree in M remains κ-Aronszajn in $M[G]$.*

Proof. Assume the hypothesis, but suppose that p forces that $\dot{C}$ is a path through T. Define $\langle p_\xi : \xi < \kappa\rangle$ and $\langle x_\xi : \xi < \kappa\rangle$ as follows. Let $p_0 = p$ and $p_0 \Vdash [\check{x_0} \in \dot{C}$. If p_ξ and x_ξ have been defined for $\xi < \kappa$, since $\sup\{\mathrm{height}(x_\eta) : \eta \leq \xi\} < \kappa$, there exist $p_{\xi+1}$ and $x_{\xi+1}$ such that $p_{\xi+1} \Vdash x_{\xi+1} \in \dot{C}$ and $x_{\xi+1} \neq x_\eta$ for all $\eta \leq \xi$. At limit steps, use κ-closure. Then $\{x_\xi : \xi < \kappa\}$ is a chain in T, in M, contradiction. $\square$

Proposition 30.69. (IV.7.46a) *Assume in M that κ is uncountable and regular and $\mathbb{P}$ is a κ-closed poset. Then every κ-Suslin tree in M remains κ-Suslin in $M[G]$.*

Proof. Similar to the proof of Proposition 30.68, starting with a well-pruned κ-Suslin tree and working with an antichain of size κ. $\square$

Proposition 30.70. (IV.7.46b) *Assume in M that κ is uncountable and regular and $\mathbb{P}$ is a κ-closed poset. If S is stationary in κ in M, then it is stationary in κ in $M[G]$.*

Proof. Suppose not; say C is club in κ in $M[G]$ and $C \cap S = \emptyset$. Let $f : \kappa \to C$ be the strictly increasing enumeration of C. We now define $g : \kappa \to \kappa$ by recursion. Say $p \Vdash [\dot{f} : \kappa \to \kappa$ and $\dot{f}$ is strictly increasing and continuous]. Now $p \Vdash \exists\alpha[\dot{f}(0) = \alpha]$, so there is a $q_0 \leq p$ and an ordinal $g(0)$ such that $q_0 \Vdash [\dot{f}(0) = g(0)\check{}]$. Now suppose that q_ξ and $g(\xi)$ have been defined, with $q_\xi \leq p$ and $g(\xi)$ an ordinal so that $q_\xi \Vdash [\dot{f}(\xi) = g(\xi)\check{}$. Then $q_\xi \Vdash \exists\alpha[\dot{f}(\xi+1) = \alpha$ and $f(\xi) < f(\xi+1)]$. Hence there is a $q_{\xi+1} \leq q_\xi$ and an ordinal $g(\xi+1)$ such that $q_{\xi+1} \Vdash \dot{f}(\xi+1) = g(\xi+1)\check{}$ and $g(\xi) < g(\xi+1)$. Now suppose that η is a limit ordinal, and q_ξ and $g(\xi)$ have been defined for all $\xi < \eta$ so that $g(\xi)$ is an ordinal and $q_\xi \Vdash \dot{f}(\xi) = g(\xi)\check{}$. By κ closure let $r \leq q_\xi$ for all $\xi < \eta$. Then

$$r \Vdash \exists\alpha[\dot{f}(\eta) = \alpha \text{ and } \forall\sigma < \eta[\dot{f}(\sigma) < \alpha] \text{ and } \forall\tau[\forall\sigma < \eta[\dot{f}(\sigma) < \tau] \to [\alpha < \tau \text{ or } \alpha = \tau]]].$$

Hence there is a $q_\eta \leq r$ and an ordinal $g(\eta)$ such that $q_\eta \Vdash [\dot{f}(\eta) = g(\eta)\check{}]$ and also

$$q_\eta \Vdash \forall\sigma < \eta[\dot{f}(\sigma) < g(\eta)] \text{ and } \forall\tau[\forall\sigma < \eta[\dot{f}(\sigma) < \tau] \to [g(\eta) < \tau \text{ or } g(\eta) = \tau]]].$$

It follows that $g(\eta) = \bigcup_{\xi<\eta} g(\xi)$.

Hence g is strictly increasing and continuous. Hence there is a $\xi < \kappa$ such that $g(\xi) \in S$. But $q_\xi \Vdash [\dot{f}(\xi) = g(\xi)\check{}$ and $\dot{f}(\xi) \notin \check{S}]$, contradiction. □

Proposition 30.71. (IV.7.48) *Assume that κ be weakly inaccessible in M, and define $\mathbb{P} = \prod^{\mathrm{fin}}_{\alpha<\kappa} \mathrm{Fn}(\omega, \alpha, \omega)$. Let G be generic. Then $\omega_1^{M[G]} = \kappa$.*

Proof. For $\alpha < \kappa$ and $n \in \omega$ let $D_{n\alpha} = \{p \in \mathbb{P} : n \in \mathrm{dmn}(p_\alpha)\}$. Clearly $D_{n\alpha}$ is dense in $\mathbb{P}$. Hence for any $\alpha < \kappa$ and $n \in \omega$ there is a $p \in G$ such that $n \in \mathrm{dmn}(p_\alpha)$. For $\alpha < \kappa$ and $n \in \omega$ let $g_\alpha(n) = p_\alpha(n)$, where $p \in G$ and $n \in \mathrm{dmn}(p_\alpha)$. For each $\alpha < \kappa$ and $\xi < \alpha$ let $E_{\alpha\xi} = \{p \in \mathbb{P} : \exists n \in \omega[n \in \mathrm{dmn}(p_\alpha)$ and $p_\alpha(n) = \xi\}$. Clearly $E_{\alpha\xi}$ is dense in $\mathbb{P}$. It follows that g_α maps ω onto α. Now by Proposition 25.100, $\mathbb{P}$ has the κ-ccc. Hence by Proposition 29.5, all cardinals $\geq \kappa$ are still cardinals in $M[G]$. Hence $\omega_1^{M[G]} = \kappa$. □

Proposition 30.72. (IV.7.48a) *Assume that κ be weakly inaccessible in M, and define $\mathbb{P} = \prod^{\mathrm{fin}}_{\alpha<\kappa} \mathrm{Fn}(\omega, \alpha, \omega)$. Let G be generic. Then $\forall\alpha[\aleph_{1+\alpha}^{M[G]} = \aleph_{\kappa+\alpha}^M]$.*

Proof. Since $\kappa = \aleph_\kappa$ in M, this is clear from Propositon 30.71 and its proof. □

Proposition 30.73. (IV.7.50) *Let $\mathbb{B}$ be a complete BA. Then the following are equivalent:*

(i) $\mathbb{B}$ collapses ω_1, i.e., $\mathbb{1} \Vdash \exists f[f$ maps ω onto $\omega_1]$.

(ii) There are $b_n^\alpha \in \mathbb{B}$ for $\alpha < \omega_1$ and $n \in \omega$ such that $\forall n \in \omega[\langle b_n^\alpha : \alpha < \omega_1\rangle$ is an antichain] and $\forall\alpha < \omega_1 \left[\sum_{n\in\omega} b_n^\alpha = 1\right]$.

Proof. (i)⇒(ii): By the maximal principle let $\dot{f}$ be such that $\mathbb{1} \Vdash \dot{f} : \omega \to \omega_1$ is surjective. Let $b_n^\alpha = [\![\dot{f}(n) = \alpha]\!]$.

(ii)⇒(i): Assume (ii). Let G be generic. Then for all $\alpha < \omega_1$ the set $\{b_n^\alpha : n \in \omega\}$ is predense, since $\sum_{n\in\omega} b_n^\alpha = 1$; hence there is an $n \in \omega$ such that $b_n^\alpha \in G$, since for all $\alpha < \omega_1$, $\sum_{n\in\omega} b_n^\alpha = 1$. For each $n \in \omega$ there is only one $\alpha < \omega_1$ such that $b_n^\alpha \in G$, since

$\langle b^\alpha_n : \alpha < \omega_1\rangle$ is an antichain. So we can define $f(n)$ to be the $\alpha < \omega_1$ such that $b^\alpha_n \in G$. Clearly f is as desired. $\square$

Proposition 30.74. (IV.7.51) *Assume that M is a model of GCH, and $\mathbb{P}$ is a poset in M with ccc. Then in $M[G]$. $|D| \le \aleph_2$ for every almost disjoint family $D \subseteq [\omega_1]^{\omega_1}$.*

Proof. Suppose to the contrary that $\langle A_\xi : \xi < \omega_3\rangle$ is a system of elements of $[\omega_1]^{\omega_1}$ which is pairwise almost disjoint, in $M[G]$. For distinct $\alpha, \beta \in \omega_3$ let $f(\alpha, \beta)$ be the least $\gamma < \omega_1$ such that $A_\alpha \cap A_\beta \subseteq \gamma$. Thus $f : \omega_3 \times \omega_3 \to \omega_1$. By Theorem 29.4 there is a function F in M such that $F : \omega_3 \times \omega_3 \to \mathscr{P}(\omega_1)$, and for all $(\alpha, \beta) \in \omega_3 \times \omega_3$ we have $f(\alpha, \beta) \in F(\alpha, \beta)$ and $|F(\alpha, \beta)| \le \omega$. let $g : \omega_3 \times \omega_3 \to \omega_1$ be a member of M such that $\forall(\alpha, \beta) \in \omega_3 \times \omega_3[F(\alpha, \beta) \subseteq g(\alpha, \beta)]$. Thus for all $\alpha, \beta \in \omega_3$ with $\alpha \ne \beta$ we have $A_\alpha \cap A_\beta \subseteq g(\alpha, \beta)$. Then there is a $p \in G$ such that for all distinct $\alpha, \beta < \omega_3$, $p \Vdash \dot{A}_\alpha \cap \dot{A}_\beta \subseteq g(\alpha, \beta)\check{}$. By the Erdös-Rado theorem $(2^{\omega_1})^+ \to (\omega_2)^2_{\omega_1}$ there exist a $\Gamma \in [\omega_3]^{\omega_2}$ and a $\gamma < \omega_1$ such that for all distinct $\alpha, \beta \in \Gamma$, $p \Vdash \dot{A}_\alpha \cap \dot{A}_\beta \subseteq \gamma$. Thus for all distinct $\alpha, \beta \in \Gamma$, $A_\alpha \cap A_\beta \subseteq \gamma$. Hence $\langle A_\alpha \backslash \gamma : \alpha \in \Gamma\rangle$ is a system of ω_2 pairwise disjoint uncountable subsets of ω_1, contradiction. $\square$

Proposition 30.75. (IV.7.52) *The existence of an almost disjoint $D \subseteq [\omega_1]^{\omega_1}$ of size $\aleph_3$ is not provable from* $\mathrm{ZFC} + 2^{\aleph_0} = 2^{\aleph_1} = 2^{\aleph_2} = \aleph_3$.

Proof. Start with $M \models$ GCH. Let $\mathbb{P} = \mathrm{Fn}(\omega_3, 2, \omega)$. Then by Theorem 29.23, in $M[G]$ we have $2^\omega = \omega_3$. Also, $\omega_3 \le 2^{\omega_1} \le \omega_3$, so $2^{\omega_1} = \omega_3$. Similarly, $2^{\omega_2} = \omega_3$. Finally, by Proposition 30.74 there is no almost disjoint $D \subseteq [\omega_1]^{\omega_1}$ of size $\aleph_3$. $\square$

Proposition 30.76. (IV.7.52a) *The existence of an almost disjoint $D \subseteq [\omega_1]^{\omega_1}$ of size $\aleph_3$ is not disprovable from* $\mathrm{ZFC} + 2^{\aleph_0} = 2^{\aleph_1} = 2^{\aleph_2} = \aleph_3$.

Proof. We start with $M \models$ GCH. Apply Corollary 29.39 with $\lambda_1 = \aleph_0$, $\lambda_2 = \aleph_1$, $\lambda_3 = \aleph_2$ and $\kappa_1 = \aleph_1$, $\kappa_2 = \kappa_3 = \aleph_3$, to get an extension N of M in which $2^\omega = \omega_1$ and $2^{\omega_1} = 2^{\omega_2} = \omega_3$. In N take $\mathrm{Fn}(\omega_3, 2, \omega)$. Hence by Theorem 29.23 we have $N[G] \models 2^\omega = \omega_3$, $\omega_3 \le (2^{\omega_1})^{N[G]} \le (\omega_3^{\omega_1})^N = \omega_3$, so that $2^{\omega_1} = \omega_3$ in $N[G]$. Similarly $2^{\omega_2} = \omega_3$ in $N[G]$. Now in N, the set ${}^{<\omega_1}\omega_1$ has size ω_1, and if $f \in {}^{\omega_1}\omega_1$ then $P_f = \{f \restriction \alpha : \alpha < \omega_1\}$ has size ω_1 and for $f \ne g$, $M_f \cap M_g$ is countable. So in N there is an almost disjoint $D \subseteq [\omega_1]^{\omega_1}$ of size ω_3. It is still such a set in $N[G]$. $\square$

Recall the definition of i_* from before Lemma 30.3.

Proposition 30.77. *$i_*(\hat{x}) = \hat{x}$ for any x.*

Proof. By induction: $i_*(\hat{x}) = i_*(\{(\hat{y}, 1) : y \in x\} = \{(i_*(\hat{y}), 1) : y \in x\} = \{(\hat{y}, 1) : y \in x\} = \hat{x}$. $\square$

Proposition 30.78. *If σ is a nice name for a subset of ω, then so is $i_*(\sigma)$.*

Proof. Say $\sigma = \bigcup_{n\in\omega}(\{\hat{n}\} \times A_n)$ with A_n an antichain. Thus $\sigma = \{(\hat{n}, p) : n \in \omega, p \in A_n\}$. So $i_*(\sigma) = \{(i_*(\hat{n}), i(p)) : n \in \omega, p \in A_n\} = \{(\hat{n}, p) : n \in \omega, p \in i[A_n]\}$. $\square$

Proposition 30.79. (IV.7.53) *Suppose that $M \models$ GCH and $\mathbb{P} = \mathrm{Fn}(\kappa, 2, \omega)$. Then in $M[G]$, ω_2 is not embeddable in $\mathscr{P}(\omega)/\mathrm{fin}$.*

Proof. Suppose to the contrary that

$$\begin{aligned}\mathbb{1} \Vdash \exists\sigma[&\sigma \text{ is a function with domain } \omega_2 \text{ and } \forall\alpha < \omega_2[\dot{\sigma}_\alpha \in [\omega]^\omega]\\ &\text{and } \forall\alpha, \beta < \omega_2[\alpha < \beta \to \dot{\sigma}_\alpha \subseteq^* \dot{\sigma}_\beta \text{ and } \dot{\sigma}_\beta \backslash \dot{\sigma}_\alpha \text{ is infinite}]].\end{aligned}$$

By the maximal principle we get a name $\dot{E}$ such that

$$\begin{aligned}\mathbb{1} \Vdash &\dot{E} \text{ is a function with domain } \omega_2 \text{ and } \forall\alpha < \omega_2[\dot{E}_\alpha \in [\omega]^\omega]\\ &\text{and } \forall\alpha, \beta < \omega_2[\alpha < \beta \to \dot{E}_\alpha \subseteq^* \dot{E}_\beta \text{ and } \dot{E}_\beta \backslash \dot{E}_\alpha \text{ is infinite}].\end{aligned}$$

For each $\alpha < \omega_2$ let $\dot{F}_\alpha$ be a nice name for a subset of ω such that $\mathbb{1} \Vdash [\dot{E}_\alpha \subseteq \omega \to \dot{E} = \dot{F}]$. It follows that

$$\begin{aligned}\mathbb{1} \Vdash &\dot{F} \text{ is a function with domain } \omega_2 \text{ and } \forall\alpha < \omega_2[\dot{F}_\alpha \in [\omega]^\omega]\\ &\text{and } \forall\alpha, \beta < \omega_2[\alpha < \beta \to \dot{F}_\alpha \subseteq^* \dot{F}_\beta \text{ and } \dot{F}_\beta \backslash \dot{F}_\alpha \text{ is infinite}].\end{aligned}$$

By the definition of nice name, for each $\alpha < \omega_2$ there is a $J_\alpha \in [\kappa]^\omega$ such that $\dot{F}_\alpha$ is a $\mathrm{Fn}(J_\alpha, 2, \omega)$-name. We now apply Lemma 24.4 with $\lambda = \omega_1$ and $\kappa = \omega_2$ to obtain a Δ-system $\langle J_\alpha : \alpha \in \Gamma\rangle$ with $|\Gamma| = \omega_2$, say with kernel Q. Now for each $\alpha \in \Gamma$, $J_\alpha \backslash Q$ is a countable subset of κ. Hence there exist a $\Delta \in [\Gamma]^{\omega_2}$ and a countable ordinal γ such that $J_\alpha \backslash Q$ has order type γ, for each $\alpha \in \Delta$.

Now take any $\alpha, \beta \in \Delta$. Let $j_{\alpha\beta}$ be the permutation of κ which is the identity on Q, is the order preserving map between $J_\alpha \backslash Q$ and $J_\beta \backslash Q$, and is the identity outside $J_\alpha \cup J_\beta$. $j_{\alpha\beta}$ induces an automorphism $j'_{\alpha\beta}$ of $\mathbb{P}$. Namely, if $r \in \mathbb{P}$, then $\mathrm{dmn}(j'_{\alpha\beta}(r)) = j_{\alpha\beta}[\mathrm{dmn}(r)]$, and for any $\xi \in \mathrm{dmn}(r)$, $((j'_{\alpha\beta}(r))(j_{\alpha\beta}(\xi)) = r(\xi)$.

Now fix $\alpha \in \Delta$. For each $\beta \in \Delta$, $(j'_{\beta\alpha})_*(F_\beta)$ is a nice name for a subset of ω. Morover, $(j'_{\beta\alpha})_*(F_\beta)$ is a $\mathrm{Fn}(J_\alpha, 2, \omega)$-name. Now $\mathrm{Fn}(J_\alpha, 2, \omega)$ is countable, and there are only ω_1 nice names for a subset of ω in $\mathrm{Fn}(J_\alpha, 2, \omega)$ So there are distinct $\beta, \delta \in \Delta$ such that $(j'_{\beta\alpha})_*(F_\beta) = (j'_{\delta\alpha})_*(F_\delta)$. By Lemma 30.5.4, for any $n \in \omega$ we have

$$\begin{aligned}p \Vdash \check{n} \in \dot{F}_\beta \quad &\text{iff} \quad i(p) \Vdash \check{n} \in (j'_{\beta\alpha})_*(F_\beta)\\ &\text{iff} \quad i(p) \Vdash \check{n} \in (j'_{\delta\alpha})_*(F_\delta)\\ &\text{iff} \quad p \Vdash \check{n} \in \dot{F}_\delta.\end{aligned}$$

So $F_\beta = F_\delta$, contradiction. □

Proposition 30.80. (IV.7.55) *Assume that in M, κ is strongly inaccessible, and $|\mathbb{P}| < \kappa$. Then κ is strongly inaccessible in $M[G]$.*

Proof. κ is regular in $M[G]$: suppose that $\alpha < \kappa$ and $f : \alpha \to \kappa$ has cofinal range. By Theorem 29.4 there is a function $F : \alpha \to \mathscr{P}(\kappa)$ in M such that $\forall\xi < \alpha[f(\xi) \in F(\xi)$

and $|F(\xi)| < \kappa]$ in M. Define, in M, $g(\xi) = \sup(F(\xi)) + 1$ for all $\xi < \alpha$. Then $g : \alpha \to \kappa$ has cofinal range, contradiction.

For the strong limit property, suppose that $\mu < \kappa \leq 2^\mu$. Then there is a system $\langle E_\alpha : \alpha < \kappa\rangle$ of nice names for a subset of μ such that $\langle E_{\alpha G} : \alpha < \kappa\rangle$ is one-one. But there are less than κ nice names for a subset of μ, contradiction. $\square$

Proposition 30.81. (IV.7.56) *Let G be $\mathbb{P}$-generic over M, $\mathbb{P}$ is ccc in M. Let $\kappa > \omega$ be regular in M. Suppose that $C \subseteq \kappa$ is club in $M[G]$. Then there is a club $C' \subseteq \kappa$ in M such that $C' \subseteq C$.*

Proof. In $M[G]$ define $f : \kappa \to \kappa$ so that $\forall\alpha < \kappa[\alpha < f(\alpha) \in C]$. By Theorem 29.4 there is a function $F : \kappa \to \mathscr{P}(\kappa)$ in M such that $f(\alpha) \in F(\alpha)$ and $|F(\alpha)| < \kappa$ for all $\alpha < \kappa$. Now we define $\langle \beta_\alpha : \alpha < \kappa\rangle$ by recursion: $\beta_0 = 0$,

$$\beta_{\alpha+1} \in \kappa \text{ and } \beta_{\alpha+1} > \text{ all members of } \bigcup_{\gamma\leq\beta_\alpha} F(\gamma),$$

and $\beta_\gamma = \bigcup_{\alpha<\gamma} \beta_\alpha$ for γ limit. Clearly β is strictly increasing, and $\beta_\alpha < f(\beta_\alpha) < \beta_{\alpha+1}$ for all $\alpha < \kappa$. Clearly then $C' \stackrel{\text{def}}{=} \{\beta_\gamma : \gamma \text{ limit}\}$ is a club in κ, $C' \in M$, and $C' \subseteq C$. $\square$

Proposition 30.82. (IV.7.57) *If $\mathbb{P}$ is ccc in M and $M[G] \models \diamondsuit$, then $M \models \diamondsuit$.*

Proof. Let $\langle A_\alpha : \alpha \in \omega_1\rangle$ be a $\diamondsuit$-sequence in $M[G]$. Thus $A \in \prod_{\alpha<\omega_1} \mathscr{P}(\alpha)$. Let $B = \bigcup_{\alpha<\omega_1} \mathscr{P}(\alpha)$. So $A : \omega_1 \to B$. By Theorem 29.4 let $\mathscr{B} : \omega_1 \to \mathscr{P}(B)$ be such that $\mathscr{B} \in M$ and $\forall\alpha \in \omega_1[A_\alpha \in \mathscr{B}_\alpha$ and $|\mathscr{B}_\alpha| \leq \omega]$. Let $\mathscr{A}_\alpha = \mathscr{B}_\alpha \cap \mathscr{P}(\alpha)$. Clearly $\langle \mathscr{A}_\alpha : \alpha < \omega_1\rangle$ is a $\diamondsuit^-$-sequence. $\square$

Proposition 30.83. (IV.7.58) *If $\mathbb{P}$ is ccc in M, $\diamondsuit$ holds in M, and $|\mathbb{P}| \leq \omega_1$, then $\diamondsuit$ holds in $M[G]$.*

Proof. Let f be a bijection from $\{(\check{\xi}, p) : \xi < \omega_1, p \in \mathbb{P}\}$ onto ω_1. Let $A'_\alpha = \{\xi < \alpha : \exists p \in G[f(\check{\xi}, p) \in A_\alpha]\}$. Given $B \subseteq \omega_1$ in $M[G]$, let $B = \dot{B}_G$ where $\dot{B}$ is a nice name for a subset of ω_1. Then $S \stackrel{\text{def}}{=} \{\alpha < \kappa : f[\dot{B}] \cap \alpha = A_\alpha\}$ is stationary. We claim that $S' \stackrel{\text{def}}{=} \{\alpha : B \cap \alpha = A'_\alpha\}$ is stationary. Let $C = \{\alpha < \omega_1 : \forall\xi < \alpha\forall p[(\check{\xi}, p) \in \dot{B} \to f(\check{\xi}, p) < \alpha]\}$. Then

(1) C is club in ω_1.

In fact, to show that C is closed, suppose that $\gamma < \omega_1$ is limit and $C \cap \gamma$ is unbounded in γ. Suppose that $\xi < \gamma$, $p \in \mathbb{P}$, and $(\check{\xi}, p) \in \dot{B}$. Choose $\eta \in (\xi, \gamma)$ with $\eta \in C$. Then $f(\check{\xi}, p) < \eta < \gamma$. So C is closed.

To show that C is unbounded, let $\beta < \omega_1$. Define $\gamma_0 = \beta$. If γ_n has been defined, let

$$\gamma_{n+1} = (\gamma_n + 1) \cup \sup\{f(\check{\xi}, p) : \xi < \gamma_n, (\check{\xi}, p) \in \dot{B}\}.$$

(Note that for any $\xi < \gamma_n$, $\{p : (\check{\xi}, p) \in \dot{B}\}$ is countable, by ccc.) Let $\gamma_\omega = \sup_{n\in\omega} \gamma_n$. Then $\gamma_\omega \in C$.

Now to show that S' is stationary, suppose also that D is club (in $M[G]$). By Proposition 30.81 let $D' \subseteq D$ be club, with $D' \in M$. Choose $\alpha \in S \cap C \cap D'$. We claim that $B \cap \alpha = A'_\alpha$. First suppose that $\xi \in B \cap \alpha$. Say $p \in G$ and $p \Vdash \check{\xi} \in \dot{B}$. Thus $\xi \in \dot{B}_G$, so there is a $q \in G$ such that $(\xi, q) \in \dot{B}$. Then $f(\check{\xi}, q) \in f[\dot{B}]$. Also $\alpha \in C$, so $f(\check{\xi}, q) < \alpha$. Hence $f(\check{\xi}, q) \in A_\alpha$, so $\xi \in A'_\alpha$. Conversely, suppose that $\xi \in A'_\alpha$. Choose $p \in G$ so that $f(\check{\xi}, p) \in A_\alpha$. Then $(\check{\xi}, p) \in \dot{B}$, so $p \Vdash \check{\xi} \in \dot{B}$ and hence $\xi \in B$. $\square$

Let A be uncountable. Let $\mathbf{K}_A$ be the collection of all algebras with universe A and countably many operations. We allow 0-ary operations, i.e., elements of the universe. For each $M \in \mathbf{K}_A$ let $Sm(M)$ be the set of all countable subuniverses of M. Clearly $Sm(M)$ is a club of $[A]^{\leq\omega}$.

Theorem 30.84. *For A uncountable, for every club W of $[A]^{\leq\omega}$ there is an algebra $M \in \mathbf{K}_A$ such that $Sm(M) \subseteq W$.*

Proof. We define a function $s : {}^{<\omega}A \to W$. Let $s(\emptyset)$ be any member of W. Suppose that $s(a) \in W$ has been defined for any $a \in {}^mA$, where $m \in \omega$. Take any $a \in {}^{m+1}A$. Then we let $s(a)$ be any member of W containing the set $s(a \restriction m) \cup \{a_m\}$. This is possible since W is unbounded. Note that $\mathrm{rng}(a) \subseteq s(a)$ for any $a \in {}^{<\omega}A$. Now for any positive m and any $a \in {}^mA$, let x_a be a function mapping ω onto $s(a)$. We now define for each positive integer m and each $i \in \omega$ an m-ary operation F_i^m on A by setting $F_i^m(a) = x_a(i)$. Let $M = (A, F_i^m)_{m,i\in\omega,m>0}$. We claim that $Sm(M) \subseteq W$.

To prove this, let $C \in Sm(M)$. Write $C = \{a_i : i \in \omega\}$. For each positive integer m let $t_m = s(a_0, \ldots, a_{m-1})$. Now by construction, $m < n$ implies that $t_m \subseteq t_n$. Moreover, t_m is the range of $x_{\langle a_0,\ldots,a_{m-1}\rangle}$, which is $\{F_i^m(a_0, \ldots, a_{m-1}) : i \in \omega\}$. Thus $t_m \subseteq C$. It follows that $C = \bigcup_{m>0} t_m \in W$. $\square$

Theorem 30.85. *Suppose that P is a ccc forcing order in a c.t.m. M. Let G be P-generic over M. Let λ be an uncountable cardinal in the sense of M, and let C be club in $[\lambda]^{\leq\omega}$ in the sense of $M[G]$. Then C includes a club of $[\lambda]^{\leq\omega}$ in the sense of M.*

Proof. In $M[G]$ let $N = (\lambda, F_i^n)_{i,n\in\omega,0<n}$ be an algebra in $M[G]$ such that $Sm(N) \subseteq C$. Fix $n > 0$ and $a \stackrel{\text{def}}{=} \langle a_0, \ldots, a_{n-1}\rangle \in {}^n\lambda$. Define $f^{na} : \omega \to \lambda$ by setting $f^{na}(i) = F_i^n(a_0, \ldots, a_{n-1})$ for all $i \in \omega$. Thus $f^{na} : \omega \to \lambda$ and $f^{na} \in M[G]$. By Theorem 29.4 let $g^{na} : \omega \to \mathscr{P}(\lambda)$ be such that $\forall i \in \omega[f^{na}(i) \in g^{na}(i)]$ and $\forall i \in \omega[|g^{na}(i)| \leq \omega]$, with $g^{na} \in M$. In M, for each $i \in \omega$ let $h_i^{na} : \omega \to g^{na}(i)$ be a surjection. Now we define $H_{ik}^n(a_0, \ldots, a_{n-1}) = h_i^{na}(k)$ for all $i, k \in \omega$. Let $P = (\lambda, H_{ik}^n)_{i,k,n<\omega,n>0}$. We claim that $Sm(P) \subseteq Sm(N)$. For, let $s \in Sm(P)$. Suppose that $n, i \in \omega$ with $n > 0$ and $a \stackrel{\text{def}}{=} \langle a_0, \ldots, a_{n-1}\rangle \in {}^ns$. Then $F_i^n(a_0, \ldots, a_{n-1}) = f^{na}(i) \in g^{na}(i)$, so there is a k such that $h_i^{na}(k) = f^{na}(i)$. Hence $F_i^n(a_0, \ldots, a_{n-1}) = f^{na}(i) = h_i^{na}(k) = H_{ik}^n(a_0, \ldots, a_{n-1}) \in s$. Thus $s \in Sm(N)$. Now since $Sm(N) \subseteq C$ we have $Sm(P) \subseteq C$. $\square$

Corollary 30.86. (IV.7.59) *Suppose that P is a ccc forcing order in a c.t.m. M. Let G be P-generic over M. Let λ be an uncountable cardinal in the sense of M, and let S be a stationary subset of $[\lambda]^{\leq\omega}$ in M. Then S is also stationary in $M[G]$.* $\square$

Proposition 30.87. (IV.7.60) *In the ground model M let $\mathbb{P} = \mathrm{Fn}(\omega_1, 2, \omega_1)$. Let G be generic. Then*

(i) $\bigcup G : \omega_1 \to 2$.

(ii) $X \stackrel{\text{def}}{=} (\bigcup G)^{-1}[\{1\}]$ is an unbounded subset of ω_1.

(iii) If $E \in M$ and $E \subseteq$ the closure of X, then E is countable.

Proof. (i) is clear. For (ii), suppose that $\alpha < \omega_1$. Let $D = \{p \in \mathbb{P} : \exists\beta > \alpha[p(\beta) = 1]\}$. Then D is dense. For suppose that $p \in \mathbb{P}$. Choose $\beta > \alpha$ so that $\beta \notin \mathrm{dmn}(p)$. Then extend p by taking 1 as the value of β. So D is dense, and (ii) follows.

For (iii), suppose that $E \in M$, $E \subseteq \omega_1$, E uncountable. For any $\alpha < \omega_1$ let $D_\alpha = \{p : \exists\beta > \alpha[\beta \in E,\ \beta \in \mathrm{dmn}(p), \text{ and } p(\beta) = 0]\}$. Then D_α is dense, and (iii) follows. $\square$

Proposition 30.88. (IV.8.15) *Let $f : \omega \to \mathbb{Q}$ and set*

$$W_f = \bigcap_{n\in\omega} \bigcup_{m>n} (f(m) - 2^{-m}, f(m) + 2^{-m}).$$

Then W_f is a null G_δ in $\mathbb{R}$.

Proof. For any $m \in \omega$ we have $\mu(f(m) - 2^{-m}, f(m) + 2^{-m}) = 2^{-m+1}$, and so for any $n \in \omega\backslash 1$,

$$\mu\left(\bigcup_{m>n} (f(m) - 2^{-m}, f(m) + 2^{-m})\right) \le \sum_{m>n} 2^{-m+1} = 2^{-n+1}.$$

It follows that W_f has measure 0. Clearly W_f is a G_δ. $\square$

Proposition 30.89. (IV.8.15) *Let $f : \omega \to \mathbb{Q}$ and set*

$$W_f = \bigcap_{n\in\omega} \bigcup_{m>n} (f(m) - 2^{-m}, f(m) + 2^{-m}).$$

Let $\mathbb{P} = \mathrm{Fn}(I, 2, \omega)$ and let G be $\mathbb{P}$-generic over M. Then for every real number x in $M[G]$ there is an $f \in {}^\omega\mathbb{Q}$ such that $f \in M$ and $x \in W_f$.

Proof. For each $m \in \omega$ let $h(m) \in \mathbb{Q}$ be such that $x - 2^{-m} < h(m) < x + 2^{-m}$. Thus $h \subseteq \omega \times \mathbb{Q}$. Let $\dot{h}$ be a nice name for a subset of $(\omega \times \mathbb{Q})^v$ such that $\dot{h}_G = h$. By the argument in the proof of Lemma 30.10 there is a countable $K \subseteq I$ such that $\dot{h}$ is a $\mathrm{Fn}(K, 2, \omega)$-name. Now $\mathrm{Fn}(K, 2, \omega) \subseteq_c \mathbb{P}$ by Example 25.64, so Lemma 30.3 applies to the inclusion $\mathrm{Fn}(K, 2, \omega) \subseteq_c \mathbb{P}$. So with $H = G \cap \mathrm{Fn}(K, 2, \omega)$ we have $M \subseteq M[H] \subseteq M[G]$ and $h = \dot{h}_G = \dot{h}_H$.

Now let $\dot{x}$ be a $\mathbb{P}$-name such that $\dot{x}_G = x$. Choose $p \in H$ such that $p \Vdash \forall m \in \omega[\dot{h}(m) \in \check{\mathbb{Q}}]$. Let $\{q_i : i \in \omega\}$ enumerate all the members of $\mathrm{Fn}(K, 2, \omega)$ which are $\le p$. We now define two sequences $i \in {}^\omega\omega$ and $s \in {}^\omega\mathbb{Q}$ by recursion. Let $i(0)$ and s_0 be such that $q_{i(0)} \le p$, $q_{i(0)} \in H$, and $q_{i(0)} \Vdash [\dot{h}(0) = \check{s}_0]$. Let $i(m+1)$ and s_{m+1} be such that $q_{i(m+1)} \le q_{i(m)}$, $q_{i(m+1)} \in H$, and $q_{i(m+1)} \Vdash [\dot{h}(m+1) = \check{s}_{m+1}]$.

Take $t \in G$ such that $t \Vdash \forall m \in \omega[\dot{x} - 2^{-m} < \dot{h}(m) < \dot{x} + 2^{-m}]$. Now we claim

(1) $\forall m \in \omega[x - 2^{-m} < s_m < x + 2^{-m}]$.

In fact, take any $m \in \omega$. Let $u \in G$ such that $u \leq t, q_{i(m)}$. Then $u \Vdash \dot{x} - 2^{-m} < \check{s}_m < \dot{x} + 2^{-m}$, and hence $x - 2^{-m} < s_m < x + 2^{-m}$, proving (1).

From (1) we get $\forall m \in \omega[s_m - 2^{-m} < x < s + 2^{-m}]$. Hence $x \in W_s$. $\square$

31. Iterated forcing

Lemma 31.1. (V.1.1) *Let $\mathbb{P},\mathbb{Q}$ be posets in the ground model M. Define $i : \mathbb{P} \to \mathbb{P}\times\mathbb{Q}$ by $i(p) = (p, \mathbb{1})$ and $j : \mathbb{Q} \to \mathbb{P}\times\mathbb{Q}$ by $i(q) = (\mathbb{1}, q))$. Let K be $\mathbb{P}\times\mathbb{Q}$-generic over M. In $M[K]$ let $G = i^{-1}[K]$ and $H = j^{-1}[K]$. Then*

(i) G is $\mathbb{P}$-generic over M.

(ii) H is $\mathbb{Q}$-generic over M.

(iii) $K = G \times H$.

Proof. By Lemma 25.81, i and j are complete embeddings. Hence G and H are generic by Lemma 30.2. To prove that $K \subseteq G \times H$, suppose that $(p, q) \in K$. Then $(p, \mathbb{1}), (\mathbb{1}, q) \in K$, so $p \in G$ and $q \in H$. Thus $K \subseteq G \times H$. For $G \times H \subseteq K$, suppose that $p \in G$ and $q \in H$. Thus $(p, \mathbb{1}), (\mathbb{1}, q) \in K$. Since K is a filter, there is a $(p', q') \in K$ such that $p', q') \leq (p, \mathbb{1}), (\mathbb{1}, q)$. Thus $(p', q') \leq (p, q)$, so $(p, q) \in K$. □

Theorem 31.2. (V.1.2) *Let $\mathbb{P},\mathbb{Q}, i, j$ be as in Lemma 31.1. Outside of M let $G \subseteq \mathbb{P}$ and $H \subseteq \mathbb{Q}$. Then the following are equivalent:*

(i) $G \times H$ is $\mathbb{P}\times\mathbb{Q}$-generic over M.

(ii) G is $\mathbb{P}$-generic over M and H is $\mathbb{Q}$-generic over $M[G]$.

(iii) H is $\mathbb{Q}$-generic over M and G is $\mathbb{P}$-generic over $M[H]$.

Moreover, if (i)-(iii) hold, then $M[G \times H] = M[G][H] = M[H][G]$.

Proof. (i)⇒(ii): Assume (i). Then $i^{-1}[G\times H] = G$, so by Lemma 31.1, G is $\mathbb{P}$-generic over M. Also, H is $\mathbb{Q}$-generic over M, and hence H is a filter on $\mathbb{Q}$. Suppose that $D \subseteq \mathbb{Q}$ is dense and $D \in M[G]$. Then there is a $\mathbb{P}$-name $\dot{D}$ such that $D = \dot{D}_G$. Take $p \in G$ such that $p \Vdash [\dot{D}$ is dense in $\check{\mathbb{Q}}]$. Let

$$D' = \{(p_1, q_1) \in \mathbb{P}\times\mathbb{Q} : p_1 \leq p \text{ and } p_1 \Vdash [\check{q}_1 \in \dot{D}]\}.$$

Then D' is dense below $(p, \mathbb{1})$. In fact, suppose that $(p_2, q_2) \leq (p, \mathbb{1})$. Then $p_2 \leq p$, so $p_2 \Vdash [\dot{D}$ is dense in $\check{\mathbb{Q}}]$. Hence $p_2 \Vdash \exists y \in \check{\mathbb{Q}}[y \in \dot{D}$ and $y \leq \check{q}_2]$. By Proposition 29.15 there exist $p_3 \leq p_2$ and $q_3 \in \mathbb{Q}$ such that $p_3 \Vdash [\check{q}_3 \in \dot{D}$ and $\check{q}_3 \leq \check{q}_2]$. Thus $(p_3, q_3) \in D'$ and $(p_3, q_3) \leq (p_2, q_2)$ as desired. So D' is dense below $(p, \mathbb{1})$.

Now $p \in G$, so $(p, \mathbb{1}) \in G \times H$. Choose $(p_4, q_4) \in D' \cap (G \times H)$. Then $p_4 \Vdash \check{q}_4 \in \dot{D}$, so $q_4 \in \dot{D}_G = D$ and $q_4 \in H$. This proves that H is $\mathbb{Q}$-generic over $M[G]$. Hence (ii) holds.

(ii)⇒(i): Assume (ii). Since G and H are filters, clearly $G \times H$ is a filter. To prove that it is generic, suppose that $D \subseteq \mathbb{P}\times\mathbb{Q}$ is dense and $D \in M$. Let

$$D^* = \{q \in \mathbb{Q} : \exists p \in G[(p, q) \in D]\}.$$

Thus $D^* \in M[G]$. We claim that D^* is dense in $\mathbb{Q}$. To prove this, suppose that $q_0 \in \mathbb{Q}$. Let

$$D' = \{p_1 : \exists q_1 \leq q_0[(p_1, q_1) \in D]\}.$$

Clearly D' is dense in $\mathbb{P}$. Choose $p_1 \in G \cap D'$. Then choose $q_1 \leq q_0$ such that $(p_1, q_1) \in D$. Then $q_1 \in D^*$ and $q_1 \leq q_0$. So D^* is dense in $\mathbb{Q}$.

Choose $q \in D^* \cap H$. Say $p \in G$ with $(p,q) \in D$. Thus $(p,q) \in D \cap (G \times H)$, as desired; (i) holds.

Similarly, (i)⇔(iii).

For the "moreover" statement, we use Lemma 28.8. We have $G \times H \in M[G][H]$ and $M \subseteq M[G][H]$, so $M[G \times H] \subseteq M[G][H]$.

Also, $G \in M[G\times H]$ and $M \subseteq M[G\times H]$, so $M[G] \subseteq M[G\times H]$. Also, $H \in M[G\times H]$, so $M[G][H] \subseteq M[G \times H]$.

By symmetry, $M[G \times H] = M[H][G]$. □

An *index function* is a function E such that $\mathrm{dmn}(E)$ is a set of regular cardinals. An *Easton index function* is an index function E such that:

(1) $\forall \kappa \in \mathrm{dmn}(E)[E(\kappa)$ is an infinite cardinal such that $\mathrm{cf}(E(\kappa)) > \kappa]$.

(2) $\forall \kappa, \lambda \in \mathrm{dmn}(E)[\kappa < \lambda \to E(\kappa) \le E(\lambda)]$.

If E is an Easton index function with domain I and $\mathbb{R} = \prod_{\kappa \in I} \mathrm{Fn}(E(\kappa), 2, \kappa)$, then the *Easton poset* $\mathbb{P}(E)$ is defined by

$$p \in \mathbb{P}(E) \quad \text{iff} \quad p \in \mathbb{R} \text{ and } \forall \lambda[\lambda \text{ regular } \to |\{\kappa \in \lambda \cap I : p(\kappa) \neq \mathbb{1}\}| < \lambda].$$

Note that $\mathbb{1} = \emptyset$.

Proposition 31.3. *Let E be an Easton index function such that there is no regular limit cardinal λ such that there is a $p \in \mathbb{R}$ such that $|\{\kappa \in \lambda \cap \mathrm{dmn}(E) : p(\kappa) \neq \mathbb{1}\}| = \lambda$. Then $\mathbb{P}(E) = \mathbb{R}$, with $\mathbb{R}$ as above.*

Proof. Assume the hypothesis, but suppose that λ is regular and there is a $p \in \mathbb{R}$ such that $|\{\kappa \in \lambda \cap \mathrm{dmn}(E) : p(\kappa) \neq \mathbb{1}\}| = \lambda$. Then λ is a successor cardinal $\aleph_{\alpha+1}$. But then $|\{\kappa \in \lambda \cap \mathrm{dmn}(E) : p(\kappa) \neq \mathbb{1}\}| \le \max(\omega, |\alpha|) < \lambda$, contradiction. □

Lemma 31.4. (V.2.3) *Suppose that E is an Easton index function such that* $\mathrm{dmn}(E) \subseteq \lambda^+$*, where λ is a regular cardinal such that $2^{<\lambda} = \lambda$. Then $\mathbb{P}(E)$ has the λ^+-cc.*

Proof. Say $\mathrm{dmn}(E) = I$. Let $W = \{p_\alpha : \alpha < \lambda^+\} \subseteq \mathbb{P}(E)$; we want to show that W is not an antichain. Thus each p_α is a function with domain I, with $p_\alpha(\kappa) \in \mathrm{Fn}(E(\kappa), 2, \kappa)$ for each $\kappa \in I$. For each $\alpha < \lambda^+$ let $D_\alpha = \{(\kappa, x) : \kappa \in I, x \in \mathrm{dmn}(p_\alpha(\kappa))\}$.

(1) $|D_\alpha| < \lambda$ for each $\alpha < \lambda^+$.

In fact, let $X = \{\kappa \in \lambda \cap I : p_\alpha(\kappa) \neq \mathbb{1}\}$. Then $|X| < \lambda$. If $\lambda \notin I$, then

$$|D_\alpha| = \sum_{\kappa \in I} |\mathrm{dmn}(p_\alpha(\kappa))| = \sum_{\kappa \in X} |\mathrm{dmn}(p_\alpha(\kappa))| < \lambda,$$

since each $|\mathrm{dmn}(p_\alpha(\kappa))| < \kappa < \lambda$. If $\lambda \in I$, then

$$|D_\alpha| = \sum_{\kappa \in I} |\mathrm{dmn}(p_\alpha(\kappa))| = \sum_{\kappa \in X} |\mathrm{dmn}(p_\alpha(\kappa))| + |\mathrm{dmn}(p_\alpha(\lambda))| < \lambda.$$

Note by Proposition 29.32 and Lemma 29.35 that for $\alpha < \lambda^+$ we have $|[\alpha]^{<\lambda} \leq \lambda^{<\lambda} = 2^{<\lambda} = \lambda$. Hence we can apply Theorem 24.4 with κ, λ replaced by λ, λ^+ to obtain $B \in [\lambda^+]^{\lambda^+}$ and R such that $D_\alpha \cap D_\beta = R$ for all distinct $\alpha, \beta \in B$. Now $2^{|R|} \leq 2^{<\lambda} = \lambda$ and

$$B = \bigcup_{h\in Q} \{\alpha \in B : \forall(\kappa, s) \in R[(p_\alpha(\kappa))(s) = h(\kappa, s)]\},$$

where $Q = {}^R 2$, so there exist distinct $\alpha, \beta \in B$ such that $\forall(\kappa, s) \in R[(p_\alpha(\kappa))(s) = (p_\beta(\kappa))(s)]$. Thus p_α and p_β are compatible. □

If E is an Easton index function and λ is an ordinal, then $E^+_\lambda = E \upharpoonright \{\kappa : \kappa > \lambda\}$ and $E^-_\lambda = E \upharpoonright \{\kappa : \kappa \leq \lambda\}$.

Lemma 31.5. (V.2.5) $\mathbb{P}(E) \cong \mathbb{P}(E^-_\lambda) \times \mathbb{P}(E^+_\lambda)$.

Proof. For any $x \in \mathbb{P}(E)$ let $f(x) = (x \upharpoonright \{\kappa : \kappa \leq \lambda\}, x \upharpoonright \{\kappa : \kappa > \lambda\})$. □

Lemma 31.6. (V.2.6) *Assuming GCH, if E is any Easton index function, then $\mathbb{P}(E)$ preserves cofinalities and cardinals.*

Proof. By Proposition 29.3 it suffices to show that every uncountable regular cardinal in M remains regular in $M[K]$ whenever K is $\mathbb{P}(E)$-generic over M. Suppose not; say θ is uncountable and regular in M while $\lambda \stackrel{\text{def}}{=} (\mathrm{cf}(\theta))^{M[K]} < \theta$. Thus λ is regular in $M[K]$. Let $f \in M[K]$, $f : \lambda \to \theta$ with $\sup(\mathrm{rng}(f)) = \theta$.

By Lemmas 30.5 and 31.5 and Theorem 31.2 we can write $M[K] = M[H][G]$ with H $(\mathbb{P}(E^+_\lambda))^M$-generic over M and G $(\mathbb{P}(E^-_\lambda))^M$-generic over $M[H]$.

Now $(\mathbb{P}(E^+_\lambda))^M$ is λ-closed in M. For, if $\alpha < \lambda$ and $\langle p_\xi : \xi < \alpha\rangle$ is decreasing in $(\mathbb{P}(E^+_\lambda))^M$, recall that $(\mathbb{P}(E^+_\lambda))^M \subseteq \prod_{\kappa\in I,\lambda<\kappa} \mathrm{Fn}(E(\kappa), 2, \kappa)$; hence we can define $q(\kappa) = \bigcup_{\xi<\alpha} p_\xi(\kappa)$ for all $\kappa \in I$ with $\kappa > \lambda$ and we get an extension of $\langle p_\xi : \xi < \alpha\rangle$. It follows from Lemma 29.9 that $(\mathbb{P}(E^+_\lambda))^M$ does not add λ-sequences. Hence $2^{<\lambda} = \lambda$ in $M[H]$ and $(\mathbb{P}(E^-_\lambda))^{M[H]} = (\mathbb{P}(E^-_\lambda))^M$. Now by Lemma 31.4 applied in $M[H]$, $(\mathbb{P}(E^-_\lambda))^M$ is λ^+-cc in $M[H]$. Now by Theorem 29.4 there is an $F : \lambda \to \mathscr{P}(\theta)$ such that $\forall\xi < \lambda[f(\xi) \in F(\xi)$ and $(|F(\xi)| \leq \lambda)^{M[H]}$. Now again $(\mathbb{P}(E^+_\lambda))^M$ is λ-closed in M, so by Theorem 29.9 we get $F \in M$ and $\forall\xi < \lambda[(|F(\xi)| \leq \lambda)^M]$. Now in M, $\bigcup_{\xi<\lambda} F(\xi)$ is of size $\leq \lambda$ and is cofinal in θ, contradiction. □

Proposition 31.7. *Assume GCH, and let E be an Easton index function with domain I. For any infinite cardinal θ, $|\mathbb{P}(E^-_\theta)| \leq \prod_{\kappa\in I,\kappa\leq\theta} E(\kappa)$.*

Proof. In fact, $\mathbb{P}(E^-_\theta) \subseteq \prod_{\kappa\in I,\kappa\leq\theta} \mathrm{Fn}(E(\kappa), 2, \kappa)$. Now if $\kappa \in I$ and $\kappa \leq \theta$, then $|[E(\kappa)]^{<\kappa}| = E(\kappa)$ by (1). Hence

$$\begin{aligned}
|\mathrm{Fn}(E(\kappa), 2, \kappa)| &= |\{f : f \text{ is a function, } \mathrm{dmn}(f) \in [E(\kappa)]^{<\kappa}, \mathrm{rng}(f) \subseteq 2\}| \\
&= |\{f : \exists X \in [E(\kappa)]^{<\kappa}[f \in {}^X 2]\}| \\
&= \left|\bigcup_{X\in[E(\kappa)]^{<\kappa}} {}^X 2\right|
\end{aligned}$$

$$\leq \sum_{X \in [E(\kappa)]^{<\kappa}} 2^{|X|}$$
$$\leq |E(\kappa)|.$$

It follows that $|\mathbb{P}(E_\theta^-)| \leq \prod_{\kappa \in I, \kappa \leq \theta} E(\kappa)$. □

Theorem 31.8. (V.2.7) *Let $M \models$ GCH. In M let E be an Easton index function and let $\mathbb{P} = \mathbb{P}(E)$. Let K be $\mathbb{P}$-generic over M. Then $\mathbb{P}$ preserves cofinalities and cardinals, and $M[K] \models \forall \kappa \in \operatorname{dmn}(E)[2^\kappa = E(\kappa)]$.*

Proof. Preservation of cofinalities and cardinals is given by Lemma 31.6.

Now let $\kappa \in \operatorname{dmn}(E)$. We define $F_\kappa : E(\kappa) \to 2$ by saying for $\delta < E(\kappa)$ that $F_\kappa(\delta) = i$ iff there is a $p \in K$ such that $\delta \in \operatorname{dmn}(p(\kappa))$ and $(p(\kappa))(\delta) = i$. Then we define for $\alpha < E(\kappa)$ $h_\alpha \in {}^\kappa 2$ by defining $h_\alpha(\xi) = F_\kappa(\kappa \cdot \alpha + \xi)$. Clearly for any $\delta < \kappa$ the set $D_\delta \overset{\text{def}}{=} \{p \in \mathbb{P} : \delta \in \operatorname{dmn}(p(\kappa))\}$ is dense, so $F_\kappa(\delta)$ is defined. If $\alpha, \beta \in \kappa$ and $\alpha \neq \beta$, then the set

$$N_{\alpha\beta} \overset{\text{def}}{=} \{p \in \mathbb{P} : \exists \xi < \kappa[\kappa{\cdot}\alpha{+}\xi, \kappa{\cdot}\beta{+}\xi \in \operatorname{dmn}(p(\kappa)) \text{ and } (p(\kappa))(\kappa{\cdot}\alpha{+}\xi) \neq (p(\kappa))(\kappa{\cdot}\beta{+}\xi)]\}$$

is dense. It follows that $h_\alpha \neq h_\beta$ for $\alpha \neq \beta$.

(1) $|\mathbb{P}(E_\kappa^-)| \leq E(\kappa)$.

In fact, by Proposition 31.7, $|\mathbb{P}(E_\kappa^-)| \leq \prod_{\mu \in I, \mu \leq \kappa} |E(\mu)| \leq (E(\kappa))^{<\kappa} = E(\kappa)$.

Now by Lemma 31.4, $\mathbb{P}(E_\kappa^-)$ has the κ^+-cc. It follows that there are at most $E(\kappa)$ nice $\mathbb{P}(E_\kappa^-)$-names for subsets of $\check{\kappa}$. Now by Lemma 29.9, forcing with $\mathbb{P}(E_\kappa^+)$ does not add any new κ-sequences. Now a nice $\mathbb{P}(E_\kappa^-)$-name for a subset of $\check{\kappa}$ can be considered as a κ-sequence. Hence in $M[H]$ there are at most $E(\kappa)$ nice $\mathbb{P}(E_\kappa^-)$-names for subsets of $\check{\kappa}$. Now by the proof of Proposition 29.22 we get $(2^\kappa \leq E(\kappa))^{M[H][G]}$. □

Corollary 31.9. *If GCH holds in the ground model, then $\forall n \in \omega[2^{\aleph_n} = \aleph_{3+2n}]$ in the generic extension by $\prod_{n \in \omega} \operatorname{Fn}(\omega_{3+2n}, 2, \omega_n)$.*

Proof. Let $\operatorname{dmn}(E) = \{\omega_n : n \in \omega\}$ and $E(\omega_n) = \omega_{3+2n}$ for all $n \in \omega$. Then E is an Easton index function and $\prod_{\omega_n \in \operatorname{dmn}(E)} \operatorname{Fn}(\omega_{3+2n}, 2, \omega_n)$ and $\prod_{n \in \omega} \operatorname{Fn}(\omega_{3+2n}, 2, \omega_n)$ are isomorphic. □

Proposition 31.10. *If κ is a limit cardinal, $\langle \lambda_\xi : \xi < \operatorname{cf}(\kappa) \rangle$ is a strictly increasing sequence of cardinals with supremum κ, then $2^\kappa = (\sum_{\xi < \operatorname{cf}(\kappa)} 2^{\lambda_\xi})^{\operatorname{cf}(\kappa)}$.*

Proof. See the proof of Proposition 11.80. □

Theorem 31.11. (Bukovsky, Hechler) *If κ is a singular cardinal ν is a cardinal, and $\exists \alpha < \kappa \forall \mu \in (\alpha, \kappa)[2^\mu = \nu]$, then $2^\kappa = \nu$.*

Proof. Let $\langle \lambda_\xi : \xi < \operatorname{cf}(\kappa) \rangle$ be a strictly increasing sequence of cardinals with supremum κ, and let $\mu < \kappa$ be such that $2^\mu = \nu$. Then $\sum_{\xi < \operatorname{cf}(\kappa)} 2^{\lambda_\xi} = (2^\mu)^{\operatorname{cf}(\kappa)}$, so by Proposition 31.10, $2^\kappa = (2^\mu)^{\operatorname{cf}(\kappa)} = \nu$. □

Corollary 31.12. *It is relatively consistent that $2^{\aleph_\omega}$ is singular.*

Proof. Let $\mathrm{dmn}(E) = \{\omega_n : n \in \omega\}$ and $E(\omega_n) = \aleph_{\aleph_{\omega+1}}$ for all $n \in \omega$. Thus $\mathrm{cf}(E(\omega_n)) = \aleph_{\omega+1} > \omega_n$ for all n. So E is an Easton function. In the generic extension, $2^{\omega_n} = \aleph_{\aleph_{\omega+1}}$ for all n. Hence by Theorem 31.11, $2^{\aleph_\omega} = \aleph_{\aleph_{\omega+1}}$. $\square$

Proposition 31.13. (V.2.9) *Working in ZFC, let E be an Easton index function, and assume that $\forall\kappa \in \mathrm{dmn}(E)[2^\kappa = E(\kappa)]$. For any infinite cardinal θ let*

$$E'(\theta) = \max(\theta^+, \sup\{E(\kappa) : \kappa \in \theta^+ \cap \mathrm{dmn}(E)\}).$$

Then let

$$E^*(\theta) = \begin{cases} E'(\theta) & \text{if } \mathrm{cf}(E'(\theta)) > \theta, \\ (E'(\theta))^+ & \text{if } \mathrm{cf}(E'(\theta)) \le \theta. \end{cases}$$

Then $2^\theta \ge E^*(\theta)$.

Proof. Case 1. $\mathrm{cf}(E'(\theta) > \theta$. Then $E^*(\theta) = E'(\theta) = \max(\theta^+, \sup\{E(\kappa) : \kappa \in \theta^+ \cap \mathrm{dmn}(E)\}) = \max(\theta^+, \sup\{2^\kappa : \kappa \in \theta^+ \cap \mathrm{dmn}(E)\})$.

Subcase 1.1. $\theta^+ \le \sup\{2^\kappa : \kappa \in \theta^+ \cap \mathrm{dmn}(E)\}$. Then $E^*(\theta) = \sup\{2^\kappa : \kappa \in \theta^+ \cap \mathrm{dmn}(E)\} \le 2^\theta$.

Subcase 1.2. $\sup\{2^\kappa : \kappa \in \theta^+ \cap \mathrm{dmn}(E)\} < \theta^+$. Then $E^*(\theta) = \theta^+ \le 2^\theta$.

Case 2. $\mathrm{cf}(E'(\theta)) \le \theta$. Then $E'(\theta) = \sup\{2^\kappa : \kappa \in \theta^+ \cap \mathrm{dmn}(E)\}$. Thus $E'(\theta) \le 2^\theta$. Now $\mathrm{cf}(2^\theta) > \theta$ by Corollary 11.55, so by the case condition, $(E'(\theta))^+ \le 2^\theta$. $\square$

Proposition 31.14. (V.2.10) *In the model of Theorem 31.8, $2^\theta = E^*(\theta)$ for all infinite θ, where E^* is as in Proposition 31.13*

Proof. Recall that $M[K]$ is obtained from $\mathbb{P} = \mathbb{P}(E) \subseteq \prod_{\kappa\in I} \mathrm{Fn}(E(\kappa), 2, \kappa)$, where $I = \mathrm{dmn}(E)$. Let θ be any infinite cardinal.

Case 1. $\theta \in I$. Then $2^\theta = E(\theta) = E'(\theta)$ and $\mathrm{cf}(E'(\theta)) > \theta$, so $E^*(\theta) = E'(\theta) = 2^\theta$.

Case 2. $\theta \notin I$, $\mathrm{cf}(E'(\theta)) > \theta$. So $E^*(\theta) = E'(\theta)$. By Proposition 31.13, $2^\theta \ge E'(\theta)$.

Subcase 2.1. $\sup\{E(\kappa) : \kappa \in \theta^+ \cap \mathrm{dmn}(E)\} \le \theta^+$. Thus $\sup\{2^\kappa : \kappa \in \theta^+ \cap \mathrm{dmn}(E)\} \le \theta^+$. Now by Proposition 31.7, $|\mathbb{P}(E^-_\theta)| \le \prod_{\kappa\in I,\kappa\le\theta} E(\kappa) = \prod_{k\in I,\kappa\le\theta} 2^\kappa \le (\theta^+)^\theta = \theta^+$. It follows that there are at most θ^+ nice names for a subset of $\check\theta$. Hence $M[H] \models 2^\theta = \theta^+$. Now $\mathbb{P}(E^+_\theta)$ is θ^+-closed, So no further θ-seqences are introduced. So $M[K] = M[H][G] \models 2^\theta = \theta^+ = E'(\theta) = E^*(\theta)$.

Subcase 2.2. $\sup\{E(\kappa) : \kappa \in \theta^+ \cap \mathrm{dmn}(E)\} > \theta^+$. Then $E'(\theta) = \sup\{E(\kappa) : \kappa \in \theta^+ \cap \mathrm{dmn}(E)\}$. Now by Proposition 31.7, $|\mathbb{P}(E^-_\theta)| \le \prod_{\kappa\in I,\kappa\le\theta} E(\kappa) \le (\sup\{E(\kappa) : \kappa \in \theta^+ \cap \mathrm{dmn}(E)\})^\theta = \sup\{E(\kappa) : \kappa \in \theta^+ \cap \mathrm{dmn}(E)\}$ since $\mathrm{cf}(E'(\theta)) > \theta$. So there are at most $E'(\theta) = \sup\{E(\kappa) : \kappa \in \theta^+ \cap \mathrm{dmn}(E)\}$ nice names for a subset of $\check\theta$. Now $E'(\theta) \le 2^\theta$ by Propositionn 31.13, and so $2^\theta = \sup\{E(\kappa) : \kappa \in \theta^+ \cap \mathrm{dmn}(E)\} = E'(\theta) = E^*(\theta)$ in $M[H]$, hence also in $M[K]$.

Case 3. $\theta \notin I$, $\mathrm{cf}(E'(\theta)) \le \theta$. So $E^*(\theta) = (E'(\theta)^+$. By Proposition 31.13, $2^\theta \ge E^*(\theta)$. Now by Proposition 31.7, $|\mathbb{P}(E^-_\theta)| \le \prod_{\kappa\in I,\kappa\le\theta} E(\kappa) \le (\sup\{E(\kappa) : \kappa \in \theta^+ \cap \mathrm{dmn}(E)\})^\theta = (E'(\theta))^\theta = (E'(\theta))^+$. So there are at most $(E'(\theta))^+$ nice names for subsets of θ in M, so

$(2^\theta)^{M[H]} \le (E'(\theta))^+ = E^*(\theta)$. No new subsets of θ are introduced by G, so $2^\theta = E^*(\theta)$ in $M[K]$. $\square$

Proposition 31.15. *Let $E(\omega_n) = \omega_{n+2}$ for all $n \in \omega$ and $E(\omega_{\omega+1}) = \omega_{\omega+3}$ and assume GCH in M. Force with $\mathbb{P}(E)$. Then in $M[G]$ we have:*

(i) $2^{\aleph_n} = \aleph_{n+2}$ for all $n \in \omega$.

(ii) $2^{\aleph_\omega} = \aleph_{\omega+1}$

(iii) $2^{\aleph_{\omega+1}} = \aleph_{\omega+3}$.

(iv) $2^{\aleph_{\omega+\beta}} = \aleph_{\alpha+\beta+1}$ for $\beta \ge 2$.

Proof. (i) and (iii) are clear from Theorem 31.8. For (ii), $E'(\aleph_\omega) = \aleph_{\omega+1} = E^*(\aleph_\omega)$, and (ii) follows. For (iv), $E'(\aleph_{\omega+\beta}) = \max(\aleph_{\omega+\beta+1}, \aleph_{\omega+3}) = \aleph_{\omega+\beta+1}$, and (iv) follows. $\square$

Proposition 31.16. *Let $E(\omega_n) = \omega_{\omega+n+1}$ for all $n \in \omega$ and $E(\omega_{\omega+1}) = \omega_{\omega+\omega+1}$ and assume GCH in M. Force with $\mathbb{P}(E)$. Then in $M[G]$ we have:*

(i) $2^{\aleph_n} = \aleph_{\omega+n+1}$ for all $n \in \omega$.

(ii) $2^{\aleph_\omega} = \aleph_{\omega+\omega+1}$.

(iii) $2^{\aleph_{\omega+1}} = \aleph_{\omega+\omega+1}$.

(iv) $2^{\aleph_{\omega+\beta}} = \aleph_{\omega+\omega+1}$ for $\omega > \beta \ge 2$.

(v) $2^{\aleph_{\omega+\omega}} = \aleph_{\omega+\omega+1}$.

(vi) $2^{\aleph_{\omega+\omega+\beta}} = \aleph_{\omega+\omega+\beta+1}$ for $\beta \ge 1$.

Proof. (i) and (iii) are clear from Theorem 31.8. For (ii),

$$E'(\aleph_\omega) = \max(\aleph_{\omega+1}, \aleph_{\omega+\omega}) = \aleph_{\omega+\omega}$$

and $E^*(\aleph_\omega) = \aleph_{\omega+\omega+1}$; (ii) follows. For (iv), $E'(\aleph_{\omega+\beta}) = \max(\aleph_{\omega+\beta+1}, \aleph_{\omega+\omega+1}) = \aleph_{\omega+\omega+1} = E^*(\aleph_{\omega+\beta})$, and (iv) follows. For (v), $E'(\aleph_{\omega+\omega}) = \max(\aleph_{\omega+\omega+1}, \aleph_{\omega+\omega+1}) = \aleph_{\omega+\omega+1} = E^*(\aleph_{\omega+\omega})$, and (v) follows. For (vi), $E'(\aleph_{\omega+\omega+\beta}) = \max(\aleph_{\omega+\omega+\beta+1}, \aleph_{\omega+\omega+1}) = \aleph_{\omega+\omega+\beta+1} = E^*(\aleph_{\omega+\omega+\beta})$, and (vi) follows. $\square$

We work out the class version of Easton's theorem.

A *class Easton function* is a class function E such that $\mathrm{dmn}(E)$ is a class and conditions (1) and (2) preceding Proposition 31.3 hold. $\mathbb{R}(E)$ is the class of all p such that $\mathrm{dmn}(p)$ is a subset of $\mathrm{dmn}(E)$ and for each $\kappa \in \mathrm{dmn}(p)$, $p(\kappa) \in \mathrm{Fn}(E(\kappa), 2, \kappa)\backslash\{\emptyset\}$. For $p, q \in \mathbb{R}$ we write $p \le q$ iff $\mathrm{dmn}(q) \subseteq \mathrm{dmn}(p)$ and for all $\kappa \in \mathrm{dmn}(q)[p(\kappa) \subseteq q(\kappa)]$. $\mathbb{P}(E)$ is the class of all $p \in \mathbb{R}(E)$ such that

$$\forall \text{regular } \lambda[|\{\kappa \in \mathrm{dmn}(p) : \kappa < \lambda\}| < \lambda].$$

We now go through the basic results of forcing, for proper classes.

Forcing posets are allowed to be classes.
Generic filters can be classes.

Lemma 28.1–28.6 hold.
$M[G] = \{\tau_G : \tau \in M^{\mathbb{P}}\} \cup \{G\}$; it is countable.
For each $\alpha \in \mathbf{On}$ let $\Gamma_\alpha = \{(\check{p}, p) : p \in \mathbb{P} \cap V_\alpha\}$.

Proposition 31.17. *If G is $\mathbb{P}$-generic over M, then $G = \bigcup_{\alpha \in \mathbf{On}} \Gamma_{\alpha G}$.*

Proof. If $\alpha \in \mathbf{On}$ and $x \in \Gamma_{\alpha G}$, then there is a $p \in G \cap V_\alpha$ such that $x = \check{p}_G = p$. So $\supseteq$ holds.

If $x \in G$, say $x \in V_\alpha$. Then $x = \check{x}$ and $(\check{x}, x) \in \Gamma_\alpha$, so $\subseteq$ holds. □

Lemma 28.7 requires an addition to the proof given. Suppose that $x \in G$; we want to show that $x \in M[G]$. By Proposition 31.17, choose α so that $x \in \Gamma_{\alpha G}$. Thus $x = \check{p}_G$ with $(\check{p}, p) \in \mathbb{P} \cap V_\alpha$, so $x \in M[G]$. □

Now Lemmas 28-9–28.10 go through
Lemma 28.11 does not go through.
Lemmas 28.12–28.22 go though.
The proof of Theorem 28.23 goes through except for the power set axiom; the set S defined in that proof is a proper class.

Proposition 31.18. (V.2.14) *Let M be a c.t.m. and let $\mathbb{P} = \mathrm{Fn}(\omega, o(M))$. Note that $\mathbb{P} \notin M$, as otherwise $o(M) \in M$, contradiction. Let G be $\mathbb{P}$-generic over M. Then $M[G]$ is not a model of ZFC.*

Proof. Suppose it is. For each $\alpha < o(M)$ let $D_\alpha = \{p \in \mathbb{P} : \exists n \in \omega[n \in \mathrm{dmn}(p)$ and $p(n) = \alpha$. Then D_α is dense. It follows that $\bigcup G : \alpha \to o(M)$ is surjective, and so $o(M) \in M[G]$. But $o(M) = o(M[G])$ by Lemma 28.13, contradiction. □

Theorem 28.24 holds.
Propositions 28.30–28.31 hold.
Propositions 28.38–28.40 hold.
The results 29.2–29.5 hold.
The results 29.9–29.10 hold.
The results 29.13–29.15 hold.
The results 29.20–29.21 hold.
The results 30.1–30.5 hold.
The results 31.1–31.2 hold.
In the definition preceding Lemma 31.5, note that $\mathbb{P}(E^+_\lambda)$ can be a proper class.

Proposition 31.19. (V.2.15) *If E is a class Easton function, then forcing with $\mathbb{P}(E)$ produces a model of ZFC.*

Proof. By the above, only the power set axiom remains. So we want to show that $M[K] \models \forall x \exists y \forall z[z \subseteq x \to z \in y]$. So, suppose that $x \in M[K]$.

Now for each infinite cardinal κ, let $\mathbb{P}(E^-_\kappa)'$ be the set of all functions p such that $\mathrm{dmn}(p) \subseteq E^-_\kappa$, $p \in \prod_{\kappa \in \mathrm{dmn}(p)} \mathrm{Fn}(E(\kappa), 2, \kappa)$ and

$$\forall \text{regular } \lambda[|\{\kappa \in \mathrm{dmn}(p) : \kappa < \lambda\}| < \lambda].$$

(1) $\mathbb{P}(E_\kappa^-) \cong \mathbb{P}(E_\kappa^-)'$

In fact, for each $p \in \mathbb{P}(E_\kappa^-)$ let $f(p) = p \restriction \{\lambda \in \mathrm{dmn}(p) : p_\lambda \neq \emptyset\}$. Clearly f is the desired isomorphism.

Now let f_κ be the isomorphism of $\mathbb{P}(E)$ onto $\mathbb{P}(E_\kappa^-)' \times \mathbb{P}(E_\kappa^+)$ given as follows: $f_\kappa(p) = (f_\kappa \restriction E_\kappa^-), f_\kappa \restriction E_\kappa^+)$. For each $p \in \mathbb{P}(E_\kappa^-)'$ let $i_\kappa(p) = f_\kappa^{-1}(p, \mathbb{1})$, and for each $q \in \mathbb{P}(E_\kappa^+)$ let $j_\kappa(q) = f_\kappa^{-1}(\mathbb{1}, q)$. Let $G_\kappa = i_\kappa^{-1}[K]$ and $H_\kappa = j_\kappa^{-1}[K]$.

$$\mathbb{P}(E) = \bigcup_{\kappa \text{ an infinite cardinal}} \mathbb{P}(E_\kappa^-)'. \tag{1}$$

In fact, suppose that $p \in \mathbb{P}(E)$. Then p is a set, and so there is an infinite cardinal κ such that $\mathrm{dmn}(p) \subseteq \kappa$. It follows that $p \in \mathbb{P}(E_\kappa^-)'$.

Now for each infinite cardinal κ, let $N_\kappa = \{\tau : \tau \text{ is a } \mathbb{P}(E_\kappa^-)'\text{-name}\}$. Let Q be the class of all $\mathbb{P}(E)$ names.

$$Q = \bigcup_{\kappa \text{ an infinite cardinal}} N_\kappa. \tag{2}$$

We prove this by induction on the name $\tau \in Q$. For $(\sigma, p) \in \tau$, say $\sigma \in N_{\kappa(\sigma,p)}$ and $p \in \mathbb{P}(E_{\lambda(\sigma,p)}^-)$. Let μ be such that $\kappa(\sigma,p), \lambda(\sigma,p) < \mu$ for all $(\sigma,p) \in \tau$. Then $\tau \in N_\mu$. This proves (2).

(3) There is an infinite cardinal κ such that $x \in M[H_\kappa]$.

In fact, say $x = \tau_K$. By (2) choose κ so that $\tau \in N_\kappa$. Then $x = \tau_{H_\kappa} \in M[H_\kappa]$. Now $M[H_\kappa]$ is a model of ZFC. Hence there is a y as called for in the power set axiom. $\square$

Theorem 31.20. (V.2.15a) *If E is a class Easton function, then forcing with $\mathbb{P}(E)$ produces a model $M[K]$ of ZFC, and $M[K] \models \forall \kappa \in \mathrm{dmn}(E)[2^\kappa = E(\kappa)]$.*

Proof. By Proposition 31.19 and the proof of Theorem 31.8. $\square$

Corollary 31.21. *There is a model of $\forall$regular $\kappa[2^\kappa = \kappa^{++}]$.* $\square$

This completes the exposition of class forcing.

Proposition 31.22. *If $W \subseteq I$, then* $\mathrm{Fn}(I, J, \omega) \cong \mathrm{Fn}(W, J, \omega) \times \mathrm{Fn}(I \backslash W, J, \omega)$. $\square$

Lemma 31.23. (V.2.18) *In M, assume that $W \subseteq I$ and $\mathbb{P} = \mathrm{Fn}(I, J, \omega)$. Let K be $\mathbb{P}$-generic over M. Let $G = K \cap \mathrm{Fn}(W, J, \omega)$ and let $H = K \cap \mathrm{Fn}(I \backslash W, J, \omega)$.*

Then G is $\mathrm{Fn}(W, J, \omega)$-generic over M and H is $\mathrm{Fn}(I \backslash W, J, \omega)$-generic over $M[G]$. Moreover, $M[K] = M[G][H]$.

Proof. In M define $\zeta : \mathrm{Fn}(W, J, \omega) \times \mathrm{Fn}(I \backslash W, J, \omega) \to \mathrm{Fn}(I, J, \omega)$ by $\zeta(p, q) = p \cup q$. Then ζ is an isomorphism from $\mathrm{Fn}(W, J, \omega) \times \mathrm{Fn}(I \backslash W, J, \omega)$ onto $\mathrm{Fn}(I, J, \omega)$. Note that $\zeta^{-1}(p) = (p \restriction W, p \restriction (I \backslash W))$. Let $\tilde{K} = \zeta^{-1}[K]$. So $\tilde{K}$ is $\mathrm{Fn}(W, J, \omega) \times \mathrm{Fn}(I \backslash W, J, \omega)$-generic over M, and $M[K] = M[\tilde{K}]$. Now let $i(p) = (p, \mathbb{1})$ and $j(p) = (\mathbb{1}, p)$, $G = i^{-1}[\tilde{K}]$, and

$H = j^{-1}[\tilde{K}]$. Then by Lemma 31.1, G is $\mathrm{Fn}(W, J, \omega)$-generic over M, H is $\mathrm{Fn}(I\backslash W, J, \omega)$-generic over $M[G]$. By Theorem 31.2, $M[\tilde{K}] = M[G][H]$. Now

$$\begin{aligned} G &= \{p \in \mathrm{Fn}(W, J, \omega) : (p, \mathbb{1}) \in \tilde{K}\} = \{p \in \mathrm{Fn}(W, J, \omega) : p \cup \emptyset \in K\} \\ &= K \cap \mathrm{Fn}(W, J, \omega); \\ H &= \{p \in \mathrm{Fn}(I\backslash W, J, \omega) : (\mathbb{1}, p) \in \tilde{K}\} = \{p \in \mathrm{Fn}(I\backslash W, J, \omega) : \emptyset \cup p \in K\} \\ &= K \cap \mathrm{Fn}(I\backslash W, J, \omega). \end{aligned}$$

□

Lemma 31.24. (V.2.19) *In M, let $\mathbb{P} = \mathrm{Fn}(\kappa, \omega, \omega)$. Let K be $\mathbb{P}$-generic over M. Then $M[K] \models \mathfrak{d} \geq \kappa$.*

Proof. Suppose not; then there is a dominating family $\{h_\alpha : \alpha < \theta\}$ with $\theta < \kappa$. Let $k : \theta \times \omega \to \omega$ be defined by $k(\alpha, n) = h_\alpha(n)$. Let τ be a nice name for a subset of $(\theta \times \omega) \times \omega$ such that $\tau_K = k$. Note that there is a $W_0 \in [\kappa]^{\leq\theta}$ such that τ is a $\mathrm{Fn}(W_0, \omega)$-name. Let W be such that $W_0 \subseteq W \subseteq \kappa$ and $|\kappa\backslash W| = \omega$. Then τ is a $\mathrm{Fn}(W, \omega)$-name, and $\mathrm{Fn}(\kappa\backslash W, \omega) \cong \mathrm{Fn}(\omega, \omega)$. Let $G = K \cap \mathrm{Fn}(W, \omega)$ and $H = K \cap \mathrm{Fn}(\kappa\backslash W, \omega)$. Then by Lemma 31.23, G is $\mathrm{Fn}(W, \omega)$-generic over M, H is $\mathrm{Fn}(\kappa\backslash W, \omega)$-generic over $M[G]$, and $M[K] = M[G][H]$. Now $\mathrm{Fn}(\kappa\backslash W, \omega) \cong \mathrm{Fn}(\omega, \omega)$; let k be an isomorphism. Let $f = \bigcup k[H] : \omega \to \omega$. Applying Lemma 30.35 with $k[H], M[G]$ in place of G, M, we infer that there is no $l \in ({}^\omega\omega) \cap M[G]$ such that $f \leq^* l$. But $h_\alpha \in M[G]$ for each $\alpha < \theta$, contradiction. □

For $c \subseteq \mathbb{Q} \times \mathbb{Q}$ define $U_c = \bigcup_{(x,y)\in c}(x, y) \subseteq \mathbb{R}$.

Lemma 31.25. (V.2.21) *Let $\mathbb{P} = \mathrm{Fn}(\omega, 2, \omega)$ and let H be $\mathbb{P}$-generic over M. In $M[H]$ let $h = \bigcup H : \omega \to 2$, and let $r = \sum_{j\in\omega}(h(j) \cdot 2^{-j})$. Thus $r \in [0, 2] \subseteq \mathbb{R}$. Then $r \in U_c$ for all $c \in \mathbb{P}(\mathbb{Q} \times \mathbb{Q}) \cap M$ such that U_c is dense in $\mathbb{R}$.*

Proof. For any $c \subseteq \mathbb{Q} \times \mathbb{Q}$ let

$$D_c = \left\{ p \in \mathbb{P} : \exists n \in \omega \exists (x, y) \in c \left[\mathrm{dmn}(p) = n \text{ and } x < \sum_{j<n}(p(j) \cdot 2^{-j}) < y - 2^{1-n} \right] \right\}.$$

(*) If U_c is dense in $\mathbb{R}$, then D_c is dense in $\mathbb{P}$.

In fact, suppose that U_c is dense in $\mathbb{R}$, and suppose that $p \in \mathbb{P}$. Let $m \in \omega\backslash 1$ be such that $m - 1$ is greater than all members of $\mathrm{rng}(p)$. Take $(x, y) \in c$ such that $(s + 2^{-m-1}, s + 2^{-m}) \cap (x, y) \neq \emptyset$. Say $z \in (s + 2^{-m-1}, s + 2^{-m}) \cap (x, y)$. Thus $s + 2^{-m-1} < z < y$ and $x < z < s + 2^{-m}$. So $x < s + 2^{-m}$ and $s + 2^{-m-1} < y$. Now we claim:

(1) There are $q(0) < q(1) < \cdot < q(k)$ with $m - 1 \leq q(0)$ and $k > 0$ such that $x < s + 2^{-q(0)} + 2^{-q(1)} + \cdots + 2^{-q(k-1)} < y$ and $s + 2^{-q(0)} + 2^{-q(1)} + \cdots + 2^{-q(k)} < y$.

Case 1. $s + 2^{-m} < y$. Take $r > m$ with $s + 2^{-m} + 2^{-r} < y$. Then (1) holds with $k = 1$, $q(0) = m$, $q(1) = r$.

Case 2. $x < s + 2^{-m-1}$. Take $r > m+1$ with $s + 2^{-m-1} + 2^{-r} < y$. Then (1) holds with $k = 1$, $q(0) = m+1$, $q(1) = r$.

Case 3. $s + 2^{-m-1} \le x$ and $y \le s + 2^{-m}$. Thus $y - x \le 2^{-m-1}$. Note that $s + 2^{-m} - s - 2^{-m-1} = 2^{-m-1}$.

Subcase 3.1. $s + 2^{-m-1} = x$. Take $u > m+2$ so that $2^{-u} < (y-x)/3$. Then $2^{1-u} = 2 \cdot 2^{-u} < 2(y-x)/3 < y - x$, hence $x < s + 2^{-m-1} + 2^{1-u} < y$ and $2^{1-u} + 2^{-u} = 2 \cdot 2^{-u} + 2^{-u} = 3 \cdot 2^{-u} < y - x$, hence $s + 2^{-m-1} + 2^{1-u} + 2^{-u} < y$, giving (1).

Subcase 3.2. $s + 2^{m-1} < x$. We now define $n(0) < n(1) < \cdots$ by recursion. Let $n(0) = m+1$. If $n(i)$ has been defined, $n(0) < \cdots < n(i)$, $s + 2^{-n(0)} + \cdots + 2^{-n(i)} < x$, and let $n(i+1) > n(i)$ be minimum such that $s + 2^{-n(0)} + \cdots + 2^{-n(i)} + 2^{-n(i+1)} \le x$. If $s + 2^{-n(0)} + \cdots + 2^{-n(i)} + 2^{-n(i+1)} = x$, then (1) can be satisfied as in Subcase 3.1. Otherwise the construction continues.

So suppose the construction continues forever. We say that i is a *gap* iff $n(i+1) > n(i) + 1$.

(1) There is a gap.

Otherwise we have $x \ge s + 2^{-n(0)} + 2^{-n(1)} + \cdots = s + 2^{-n(0)+1} = s + 2^{-m} \ge y > x$, contradiction.

(2) There are arbitrarily large gaps.

In fact, suppose not, and let i be the largest gap. Then

$$
\begin{aligned}
x &\ge s + 2^{-n(0)} + \cdots + 2^{-n(i)} + 2^{-n(i+1)} + 2^{-n(i+1)-1} + \cdots \\
&= s + 2^{-n(0)} + \cdots + 2^{-n(i)} + 2^{-n(i+1)+1} > x,
\end{aligned}
$$

contradiction.

Now let i be a gap such that $2^{-n(i)-1} < (y-x)/2$. Then $s + 2^{-n(0)} + \cdots + 2^{-n(i)} < x$, while $s + 2^{-n(0)} + \cdots + 2^{-n(i)} + 2^{-n(i)-1} > x$. So $x < s + 2^{-n(0)} + \cdots + 2^{-n(i)} + 2^{-n(i)-1} < y$. Also $n(i+1) > n(i) + 1$, so $2^{-n(i+1)} < (y-x)/2$. hence $s + 2^{-n(0)} + \cdots + 2^{-n(i)} + 2^{-n(i)-1} + 2^{-n(i+1)} < y$. This gives (1).

Now we take $q(0) < \cdots < q(k)$ as in (1). Then

$$
\begin{aligned}
s &+ 2^{-q(0)} + \cdots + 2^{-q(k-1)} + 2^{-q(k)-1} + 2^{1-q(k)-2} \\
&= s + 2^{-q(0)} + \cdots + 2^{-q(k-1)} + 2^{-q(k)-1} + 2^{-q(k)-1} \\
&= s + 2^{-q(0)} + \cdots + 2^{-q(k)},
\end{aligned}
$$

and hence $s + 2^{-q(0)} + \cdots + 2^{-q(k)-1} < y - 2^{1-q(k)-2}$. Define r with domain $q(k)+2$ by setting, for any $i < q(k)+2$,

$$
r(i) = \begin{cases} p(i) & \text{if } i \in \operatorname{dmn}(p), \\ 1 & \text{if } i = q(j) \text{ for some } j < k, \\ 1 & \text{if } i = q(k)+1, \\ 0 & \text{otherwise.} \end{cases}
$$

Then $p \subseteq r \in D_c$. So D_c is dense.

Now with G generic, let $g = \bigcup G$. Suppose that $c \in \mathscr{P}(\mathbb{Q} \times \mathbb{Q})$ and U_c is dense in $\mathbb{R}$. By the above, D_c is dense. Take $p \in G \cap D_c$. Take any $q \in G$, and take $r \leq p, q$. Then

$$x < \sum_j (g(j) \cdot 2^{-j}) \leq \sum_{j<n} (r(j) 2^{-j}) + \sum_{j=n}^{\infty} 2^{-j} = \sum_{j<n} (r(j) 2^{-j}) + 2^{1-n} < y. \qquad \square$$

Lemma 31.26. (V.2.22) *Let* $\mathbb{P} = \mathrm{Fn}(\kappa \times \omega, 2, \omega)$ *where* κ *is any cardinal. Let* K *be* $\mathbb{P}$*-generic over* M*. Then* $M[K] \models [\mathrm{cov(meag)} \geq \kappa]$.

Proof. This is clear if $\kappa = \omega$. So assume that $\kappa > \omega$ and the lemma is false. Then there is a $\theta < \kappa$ and a collection $\mathscr{A}$ of meager sets with $|\mathscr{A}| = \theta$ such that $\bigcup \mathscr{A} = \mathbb{R}$. We may assume that each member of $\mathscr{A}$ is closed nowhere dense, by Lemma 18.25. Then for any $Y \in \mathscr{A}$, the set $\mathbb{R} \backslash Y$ is open dense. So we may assume that we have given a collection $\mathscr{B}$ of size θ, with each member of $\mathscr{B}$ of the form U_c, with $c \in \mathscr{P}(\mathbb{Q} \times \mathbb{Q})$, each U_c open dense and in M (by absoluteness). Now one can proceed as in the proof of Lemma 31.24, and obtain a generic $h : \omega \to \omega$. Then Lemma 31.25 gives a contradiction. $\square$

Proposition 31.27. (V.2.23) *Assume in* M *that CH holds,* $\kappa > \omega$*, and* $\kappa^\omega = \kappa$*. Let* K *be* $\mathrm{Fn}(\kappa, 2, \omega)$*-generic over* M*. Then in* $M[K]$ *we have* $\mathfrak{p} = \mathfrak{b} = \mathfrak{a} = \omega_1$*.*

Proof. By chapter 20 we have $\omega_1 \leq \mathfrak{p} \leq \mathfrak{b} \leq \mathfrak{a}$, and by Proposition 30.14 we have $\mathfrak{a} \leq \omega_1$. $\square$

Proposition 31.28. (V.2.23a) *Assume in* M *that CH holds,* $\kappa > \omega$*, and* $\kappa^\omega = \kappa$*. Let* K *be* $\mathrm{Fn}(\kappa, 2, \omega)$*-generic over* M*. Then in* $M[K]$ *we have* $\mathfrak{d} = 2^\omega = \kappa$*.*

Proof. By Theorem 29.23 we have $2^\omega = \kappa$, and $\mathfrak{d} = \kappa$ by Lemma 31.24. $\square$

Proposition 31.29. (V.2.23b) *Assume in* M *that CH holds,* $\kappa > \omega$*, and* $\kappa^\omega = \kappa$*. Let* K *be* $\mathrm{Fn}(\kappa, 2, \omega)$*-generic over* M*. Then in* $M[K]$ *we have* $\mathrm{cov(meag)} = \mathrm{non(null)} = 2^\omega$*.*

Proof. $\mathrm{cov(meag)} = 2^\omega$ by Lemma 31.26 $\mathrm{cov(meag)} \leq \mathrm{non(null)}$ by Chapter 19. $\square$

Lemma 31.30. (V.2.26) *In* M*, let* $(T, \sqsubset)$ *be an* ω_1*-tree and let* $\mathbb{P}$ *be a countably closed forcing poset. Then if* $C \in M[G]$ *is a path through* T*, it follows that* $C \in M$*.*

Proof. Assume not. Let $\mathscr{C}$ be the set of all paths through T in M. Then there exisst a name $\dot{C}$ and a $p \in \mathbb{P}$ such that $p \Vdash [\dot{C} \notin \check{\mathscr{C}}$ and $\dot{C}$ is a path through $T]$. Now we will define p_s and x_s for $s \in {}^{<\omega}2$ and α_n for $n \in \omega$ so that the following conditions hold:

(1) $0 = \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \omega_1$.

(2) $p_s \in \mathbb{P}$ and $p_s \leq p$.

(3) If $\mathrm{dmn}(s) = n$, then $x_s \in \mathcal{L}_{\alpha_n}(T)$.

(4) $p_s \Vdash x_s \in \dot{C}$.

(5) $p_{s \frown \langle 0 \rangle} \leq p_s$ and $p_{s \frown \langle 1 \rangle} \leq p_s$.

(6) $x_{s\frown\langle 0\rangle} \neq x_{s\frown\langle 1\rangle}$.

We start with any $p_\emptyset \leq p$ and $x_\emptyset \in \mathcal{L}_0$ such that $p_\emptyset \Vdash \check{x}_\emptyset \in \dot{C}$; and we set $\alpha_0 = 0$. Clearly (2)–(4) hold, and (1), (5), (6) are vacuously true.

Now suppose that n is given such that p_s and x_s have been constructed for all $s \in {}^n2$. For each $s \in {}^n2$ let $E_s = \{y \in T : \exists q \leq p_s[q \Vdash y \in \dot{C}]\}$. Then E_s is a subtree of T and meets every level. Moreover, $E_s \cap T_{\alpha_n+1} = x_s\!\downarrow'$.

(7) E_s is well-pruned.

In fact, suppose that $y \in E_s$; say $q \leq p_s$ and $q \Vdash y \in \dot{C}$. Also suppose that height$(y) < \beta < \omega_1$. Then since q forces that $\dot{C}$ is a path, there exist an $r \leq q$ and a $z \in \mathcal{L}_\beta \cap y\uparrow$ such that $r \Vdash z \in \dot{C}$.

(8) E_s is not a path.

For, suppose it is a path. Then $E_s \in \mathscr{C}$. But $p_s \Vdash [\dot{C} \notin \check{\mathscr{C}}]$, so $p_s \Vdash [\dot{C} \neq \check{E}_s]$. Since $p_s \Vdash [\dot{C}$ is a path and $\check{E}_s$ is a path$]$, it follows that $p_s \Vdash [\dot{C} \not\subseteq \check{E}_s]$. Hence there exist $y \in T\backslash E_s$ and $q \leq p_s$ such that $q \Vdash [\check{y} \in \dot{C}]$, contradicting the definition of E_s.

Since the E_s are well-pruned and not paths, there is an $\alpha_{n+1} > \alpha_n$ such that for each $s \in {}^n2$ there are distinct. $x_{s\frown\langle 0\rangle}, x_{s\frown\langle 1\rangle}$ in $E_s \cap \mathcal{L}_{\alpha_{n=1}}$. Then there are $p_{s\frown\langle\varepsilon\rangle}$ for $\varepsilon = 0, 1$ such that $p_{s\frown\langle\varepsilon\rangle} \leq p_s$ and $p_{s\frown\langle\varepsilon\rangle} \Vdash [x_{s\frown\langle\varepsilon\rangle} \in \dot{C}]$. Thus (1)–(6) hold.

Now let $\gamma = \sup\{\alpha_n : n \in \omega\}$. So $\gamma < \omega_1$. For each $f \in {}^\omega2$, by countable closure and (5) there is a $p_f \leq p_{f\restriction n}$ for all $n \in \omega$. By (2), $p_f \Vdash [\dot{C}$ is a path through $T]$. Hence $p_f \Vdash \exists y \in \mathcal{L}_\gamma[y \in \dot{C}]$. Hence there exist a $q_f \leq p_f$ and an $x_f \in \mathcal{L}_\gamma$ such that $q_f \Vdash [x_f \in \dot{C}]$. For each n we have $q_f \Vdash [x_{f\restriction n} \in \dot{C}]$, and height$(x_n) = \alpha_n < \gamma$, so $x_{f\restriction n} \sqsubset x_f$. Hence by (6), the x_f are all different, so $|\mathcal{L}_\gamma| \geq 2^\omega$, contradiction. $\square$

Proposition 31.31. (V.2.26a) *In M let $\theta \geq 2^{\omega_1}$ and $\mathbb{P} = \mathrm{Fn}(\omega_1, \theta, \omega_1)$, and let $(T, \sqsubset)$ be an ω_1-tree. Then T is not a Kurepa tree in $M[G]$.*

Proof. Clearly $\bigcup G$ is a mapping from $\omega_1^{M[G]}$ onto ω_1^M. In M let $\mathscr{C}$ be the set of all paths through T. Thus $(|\mathscr{C}| \leq \theta)^M$. By Lemma 31.30, $\mathscr{C}$ is also the set of all paths through T in $M[G]$. So in $M[G]$, $|\mathscr{C}| \leq \theta| = \omega_1$; so T is not Kurepa. $\square$

Theorem 31.32. (V.2.25) *Suppose that M is a c.t.m. of ZFC and (κ is strongly inaccessible)^M. Then there is a generic extension $M[G]$ such that $\kappa = \omega_2^{M[G]}$ and there is no Kurepa tree in $M[G]$.*

Proof. In M, for each $S \subseteq \kappa$ let

$$\mathbb{P}_S = \left\{p \in \prod_{\alpha\in S} \mathrm{Fn}(\omega_1, \alpha, \omega_1) : |\{\alpha \in S : p_\alpha \neq \mathbb{1}\}| \leq \omega\right\}.$$

Let $\mathbb{P} = \mathbb{P}_\kappa$.

(1) $\forall S \subseteq \kappa[\mathbb{P} \cong \mathbb{P}_S \times \mathbb{P}_{\kappa\backslash S}]$.

In fact, for each $x \in \mathbb{P}$ let $f(x) = (x \restriction S, x \restriction (\kappa \backslash S)$. Clearly f is the desired isomorphism.

(2) $\mathbb{P}$ has the κ-cc.

In fact, let $\langle p_\alpha : \alpha < \kappa \rangle$ be a system of elements of $\mathbb{P}$. For each $\alpha < \kappa$ let $S_\alpha = \{\beta < \kappa : p_\alpha(\beta) \neq \mathbb{1}\}$. Thus each S_α is countable. Hence by Theorem 24.4 there is a $T \in [\kappa]^\kappa$ such that $\langle S_\alpha : \alpha \in T \rangle$ is a Δ-system, say with kernel K. Let $\gamma < \kappa$ be greater than $\sup(K)$. Then

$$|\{p_\alpha \restriction K : \alpha \in T\}| \leq \left| \prod_{\alpha \in K} \mathrm{Fn}(\omega_1, \alpha, \omega_1) \right| \leq \prod_{\alpha \in K} |\gamma|^{\omega_1} \leq (|\gamma|^{\omega_1})^{|K|} < \kappa.$$

Hence there is a $k \in \prod_{\alpha \in K} \mathrm{Fn}(\omega_1, \alpha, \omega_1)$ such that $\{\alpha \in T : p_\alpha \restriction K = k\}$ has size κ. For any two $\beta, \gamma \in \{\alpha \in T : p_\alpha \restriction K = k\}$ the elements p_β and p_γ are compatible. So (2) holds.

By (2), $\mathbb{P}$ preserves cofinalities and cardinals $\geq \kappa$. Also, $\mathbb{P}$ is clearly countably closed, so it does not add new ω-sequences. Hence κ is still a cardinal in $M[G]$, and $\omega_1^M = \omega_1^{M[G]}$.

(3) For each $\alpha < \kappa$ there is a complete embedding from $\mathrm{Fn}(\omega_1, \alpha, \omega_1)^M$ into $\mathbb{P}$.

In fact, for any $x \in \mathrm{Fn}(\omega_1, \alpha, \omega_1)$ define $f(x) \in \mathbb{P}$ by setting, for any $\beta < \kappa$,

$$f(x)_\beta = \begin{cases} x & \text{if } \alpha = \beta, \\ \mathbb{1} & \text{otherwise.} \end{cases}$$

We check the conditions. (i) and (ii) are clear, as is $\Leftarrow$ in (iii). For $\Rightarrow$, suppose that $f(x)$ and $f(y)$ are compatible. So there is a $z \in \mathbb{P}$ with $z \leq f(x), f(y)$. Then $z_\alpha \leq x, y$, as desired. For (iv), suppose that A is a maximal antichain in $\mathrm{Fn}(\omega_1, \alpha, \omega_1)$. For any $x \in \mathbb{P}$, x_α is compatible with some $y \in A$. Hence x is compatible with $f(y)$, as desired. So (3) holds.

Let f be as in (3). Then by Lemma 30.3, $H \stackrel{\text{def}}{=} f^{-1}[G]$ is $\mathrm{Fn}(\omega_1, \alpha, \omega_1)$-generic over M, and $M[H] \subseteq M[G]$. We have a surjection $\bigcup H : \omega_1 \to \alpha$, and $\bigcup H \in M[G]$. It follows that $\omega_2^{M[G]} = \kappa$.

Now suppose that in $M[G]$ there is a Kurepa tree $(T, \sqsubset)$. We may assume that $T = \omega_1$. Thus $\sqsubset \subseteq (\omega_1 \times \omega_1)$. We may assume that $\sqsubset = \dot{\sqsubset}_G$ with $\dot{\sqsubset}$ a nice name for a subset of $\omega_1 \times \omega_1$. Since $\mathbb{P}$ has the $\kappa - cc$, there is an $A \in [\mathbb{P}]^{<\kappa}$ such that $\dot{\sqsubset} \subseteq ((\omega_1 \times \omega_1) \times A$. Let $S = \bigcup_{p \in A} \mathrm{support}(p)$. Thus $|S| < \kappa$. Let f be the isomorphism from $\mathbb{P}$ onto $\mathbb{P}_S \times \mathbb{P}_{\kappa \backslash S}$ given by (1). Then $M[G] = M[f[G]]$. By Lemma 31.1 and Theorem 31.2 we obtain generic G^- over $\mathbb{P}_S$ and G^+ over $\mathbb{P}_{\kappa \backslash S}$ such that $M[f[G]] = M[G^-][G^+]$. By Lemma 30.3 we have $(f_*(\dot{\sqsubset}))_{f[G]} = (\dot{\sqsubset})_G = \sqsubset$. Now write $\dot{\sqsubset} = \bigcup_{(m,n) \in \omega \times \omega} (\{(m.n)\} \times B_{mn})$ with each B_{mn} an antichain in $\mathbb{P}$. Then $f_*(\dot{\sqsubset}) = \bigcup_{(m,n) \in \omega \times \omega} (\{(m.n)\} \times \{f(p) : p \in B_{mn}\})$. For each $p \in A$ write $f(p) = (k(p), \mathbb{1})$. For $p \in \mathbb{P}_S$ let $h(p) = (p, \mathbb{1})$. Let $\dot{\prec} = \bigcup_{(m,n) \in \omega \times \omega} (\{(m.n)\} \times \{k(p) : p \in B_{mn}\})$. Then

$$\begin{aligned}
\dot{\prec}_{G^-} &= \{(m,n) : \exists q \in G^-[((m,n),q) \in \dot{\prec}]\} \\
&= \{(m,n) : \exists q \in h^{-1}[f[G]][((m,n),q) \in \dot{\prec}]\} \\
&= \{(m,n) : \exists q[h(q) \in f[G] \text{ and } ((m,n),q) \in \dot{\prec}]\} \\
&= \{(m,n) : \exists q \exists r[r \in G \text{ and } (q, \mathbb{1}) = f(r) \text{ and } ((m,n),q) \in \dot{\prec}]\}
\end{aligned}$$

$$= \{(m,n) : \exists q \exists r[r \in G \text{ and } q = k(r) \text{ and } ((m,n),q) \in \dot{\prec}]\}$$
$$= \{(m,n) : \exists r[r \in G \text{ and } ((m,n), f(r)) \in f_*(\dot{\sqsubset})]$$
$$= f_*(\sqsubset)_{f[G]} = \sqsubset .$$

Thus $\sqsubset \in M[G^-]$.

Now $\mathbb{P}_S$ is countably closed, so it does not add any new ω-sequences. Hence ($\mathbb{P}_{\kappa\backslash S}$ is countably closed)$^{M[G^-]}$. By Lemma 31.30 it follows that every path through the tree $(\omega_1, \sqsubset)$ that is in $M[G]$ is already in $M[G^-]$. Let $\mathscr{C}$ be the collection of all paths through $(\omega_1, \sqsubset)$ which are in $M[G]$.

(4) $|\mathbb{P}_S| < \kappa$ in M.

In fact, Let $\gamma < \kappa$ be greater than each member of S. Then

$$|\mathbb{P}_S| \leq \prod_{\alpha \in S} |\mathrm{Fn}(\omega_1, \alpha, \omega_1)| \leq \prod_{\alpha \in S} |\gamma|^{\omega_1} = (|\gamma|^{\omega_1})^{|S|} < \kappa.$$

So (4) holds. Now by Proposition 30.80, κ is strongly inaccessible in $\mathbb{P}[G^-]$. We have $|\mathscr{C}| \leq 2^{\omega_1} < \kappa$ in $M[G^-]$, so there is an $\alpha < \kappa$ such that $\alpha \notin S$ and $|\mathscr{C}| \leq \alpha$. In $M[G]$ we have $|\mathscr{C}| \leq |\alpha| \leq \omega_1$, so that $(\omega_1, \sqsubset)$ is not a Kurepa tree in $M[G]$. □

Proposition 31.33. (V.2.28) *Let $S \subseteq \omega_1$ be stationary. Let $\mathbb{P}_S$ be the set of all countable $p \subseteq S$ such that p is closed in ω_1, i.e., such that if γ is a limit ordinal and $p \cap \gamma$ is unbounded in γ, then $\gamma \in p$.*

(i) $\forall p \in \mathbb{P}_S[p = \emptyset$ or $\sup(p) = \max(p) \in p]$.

For $p, q \in \mathbb{P}_S$, define $q \leq p$ iff ($p = \emptyset$ or $q \cap (\max(p) + 1) = p$).

Then $\mathbb{P}_S$ is Baire.

Proof. Suppose that $\langle D_n : n \in \omega\rangle$ is a system of dense open subsets of $\mathbb{P}_S$, and $p \in \mathbb{P}_S$; we want to find a member of $\bigcap_{n\in\omega} D_n$ which is below p. For each $n \in \omega$ define $f_n : \mathbb{P}_S \times S \to \mathbb{P}_S$ by setting, for each $q \in \mathbb{P}_S$ and each $\alpha \in S$, $f_n(q, \alpha) =$ some $p \in D_n$ such that $p \leq q \cup \{\alpha\}$. Let θ be such that $\omega_1, \mathbb{P}_S \in H(\theta)$. Let $\langle M_\alpha : \alpha < \omega_1\rangle$ be a nice chain of elementary submodels of $H(\theta)$ such that $\omega_1, p, S \in M_0$ and each $f_n \in M_0$. By Lemma 27.63, $C \stackrel{\text{def}}{=} \{M_\xi \cap \omega_1 : \xi < \omega_1\}$ is club in ω_1. Take any $\xi < \omega_1$. Then $M(\theta) \models \forall \alpha < \omega_1 \exists \beta \in S[\alpha < \beta]$, so $M_\xi \models \forall \alpha < \omega_1 \exists \beta \in S[\alpha < \beta]$. Thus $M_\xi \cap \omega_1$ is a limit ordinal, and is the supremum of members of S.

Now choose $\xi < \omega_1$ such that $(M_\xi \cap \omega_1) \in S$. Let $\langle \alpha_n : n \in \omega\rangle$ be a strictly increasing sequence of members of S with supremum $M_\xi \cap \omega_1$. Now we define a sequence $\langle q_n : n \in \omega\rangle$. Let $q_0 = p$. If q_n has been defined, let $q_{n+1} = f_n(q_n, \alpha_n)$. Then $p = q_0 \geq q_1 \geq \cdots$, and $\sup_{n\in\omega} q_n = M_\xi \cap \omega_1$. Since $M_\xi \cap \omega_1 \in S$, we have $r \stackrel{\text{def}}{=} \bigcup_{n\in\omega} q_n \cup \{M_\xi \cap \omega_1\} \in \mathbb{P}_S$ and $r \leq q_n$ for all n, so $r \in \bigcap_{n\in\omega} D_n$. □

Proposition 31.34. (V.2.28a) *Let $S \subseteq \omega_1$ be stationary. Let $\mathbb{P}_S$ be the set of all countable $p \subseteq S$ such that p is closed in ω_1, i.e., such that if γ is a limit ordinal and $p \cap \gamma$ is unbounded in γ, then $\gamma \in p$.*

For $p, q \in \mathbb{P}_S$, define $q \leq p$ iff ($p = \emptyset$ or $q \cap (\max(p) + 1) = p$).

Let G be $\mathbb{P}_S$-generic over M. Then $\mathbb{P}_S$ does not add ω-sequences, and hence $\omega_1^M = \omega_1^{M[G]}$. Also, $\bigcup G$ is a club which is a subset of S.

Proof. $\mathbb{P}_S$ does not add ω-sequences by Proposition 31.33 and Proposition 30.49. Clearly $\bigcup G$ is club and is contained in S. $\square$

Proposition 31.35. (V.2.28b) *If S and T are disjoint stationary subsets of ω_1, then $\mathbb{P}_S \times \mathbb{P}_T$ collapses ω_1.*

Proof. Let G be $\mathbb{P}_S \times \mathbb{P}_T$-generic over M. By Lemma 31.1 there exist h which is $\mathbb{P}_S$-generic over M and K which is $\mathbb{P}_T$-generic over M. By Proposition 31.34, $\bigcup H$ is club in $\omega_1^H = \omega_1^M$ and is contained in S, and $\bigcup K$ is club in $\omega_1^K = \omega_1^M$ and is contained in T. By Theorem 31.2 we have $\bigcup H, \bigcup K \in M[G]$. Since S and T are disjoint, so are $\bigcup H$ and $\bigcup K$. Since the intersection of two clubs in ω_1 is a club, it follows that $\omega_1^{M[G]} = \omega$. $\square$

Let H denote the assertion that there are ccc forcing posets $\mathbb{P}$, $\mathbb{Q}$ such that $\mathbb{P} \times \mathbb{Q}$ is not ccc.

Proposition 31.36. (V.2.29) $\mathrm{MA}(\omega_1) \Rightarrow \neg H$.

Proof. By Theorem 25.50. $\square$

Proposition 31.37. (V.2.29a) *If there is a Suslin line, then H holds.*

Proof. By Lemma 21.37, taking $\mathbb{P} = \mathbb{Q} = \mathbb{O}(X)$. $\square$

Proposition 31.38. (V.2.29b) CH $\Rightarrow H$.

Proof. By Theorem 27.71. $\square$

Let H^+ denote the assertion that there are ccc forcing posets $\mathbb{P}$, $\mathbb{Q}$ such that $\mathbb{P}\times\mathbb{Q}$ collapses ω_1.

Proposition 31.39. (V.2.29c) $H^+ \Rightarrow H$.

Proof. Assume $\neg H$. Thus for all ccc poset $\mathbb{P}$ and $\mathbb{Q}$, also $\mathbb{P} \times \mathbb{Q}$ is ccc and hence it preserves cardinals, so that H^+ fails. $\square$

Proposition 31.40. (V.2.29d) *If there is a Suslin tree, then H^+ holds.*

Proof. Let T be a well-pruned Suslin tree. Then by Proposition 30.50, T is ccc and adds a path through T. Let $\mathbb{Q}$ be the forcing poset given in the proof of Theorem 25.101. By that proof, $\mathbb{Q}$ is ccc. Let G be $(T \times \mathbb{Q})$-generic over M. Let $i(p) = (p, \mathbb{1})$ for $p \in T$, and $j(p) = (\mathbb{1}, p)$ for $p \in \mathbb{Q}$. Let $H = i^{-1}[G]$ and $K = j^{-1}[G]$. Then by Lemma 31.1 and Theorem 31.2, H is T-generic over M, K is $\mathbb{Q}$-generic over M, $M[H] \subseteq M[G]$, and $M[K] \subseteq M[G]$. Hence there is a path C through T in $M[G]$ and there is a continuous order preserving map from T into the rationals. Hence C is countable in $M[G]$, so that $\omega_1^{M[G]}$ is countable. $\square$

Proposition 31.41. (V.2.29e) *CH implies* H^+.

Proof. Let $\mathbb{Q}_\mu$ be as in Proposition 27.74 for $\mu \in 2$. Let G be $(\mathbb{Q}_0 \times \mathbb{Q}_1)$-generic over M. Let $i(p) = (p, \mathbb{1})$ and $j(p) = (\mathbb{1}, p)$. Let $H = i^{-1}[G]$ and $K = j^{-1}[G]$. So by Lemma 31.1 and Theorem 31.2, H is $\mathbb{Q}_0$-generic over M, K is $\mathbb{Q}_1$-generic over M, $M[H] \subseteq M[G]$, and $M[K] \subseteq M[G]$. For each $i \in \omega_1$ the set $D_i \stackrel{\text{def}}{=} \{p \in \mathbb{Q}_0 : \exists n \in \omega[p_n = \{i\}]\}$ is dense in $\mathbb{Q}_0$. Hence $D_i \cap H \neq \emptyset$, for each $i \in I$. Let $Z_n = \{i \in \omega_1 : \exists p \in H[p_n = \{i\}\}$. Then $\omega_1 = \bigcup_{n\in\omega} Z_n$. Similarly, for each $i \in \omega_1$ the set $D'_i \stackrel{\text{def}}{=} \{p \in \mathbb{Q}_1 : \exists n \in \omega[p_n = \{i\}]\}$ is dense in $\mathbb{Q}_1$. Hence $D'_i \cap K \neq \emptyset$, for each $i \in I$. Let $Y_n = \{i \in \omega_1 : \exists p \in K[p_n = \{i\}\}$. Then $\omega_1 = \bigcup_{n\in\omega} Y_n$. Now suppose that $i, j \in \omega_1$, $i \neq j$, and $i, j \in Z_m \cap Y_n$. Say $p, q \in H$ with $p_m = \{i\}$ and $q_m = \{j\}$, and $r, s \in K$ with $r_n = \{i\}$ and $s_n = \{j\}$. Choose $t \in G$ such that $t \leq (p, \mathbb{1}), (q, \mathbb{1}), (\mathbb{1}, r), (\mathbb{1} s)$. Say $t = (u, v)$. Then $u \leq p, q$, so $i, j \in u_m$, hence $f(\{i, j\}) = 0$. But also $v \leq r, s$, so $i, j \in v_n$, hence $f(\{i, j\}) = 1$, contradiction. It follows that $|Z_m \cap Y_n| \leq 1$. If $\omega_1^M = \omega_1^{M[G]}$, then

$$\omega_1^M = \left(\bigcup_{m\in\omega} Z_m\right) \cap \left(\bigcap_{n\in\omega} Y_n\right) = \bigcup_{m,n\in\omega} (Z_m \cap Y_n),$$

and this last set is countable, contradiction. □

Proposition 31.42. (V.2.31) *For ccc forcing posets* $\mathbb{P}, \mathbb{Q}$ *the following are equivalent:*

(i) $\mathbb{P} \times \mathbb{Q}$ *is ccc.*

(ii) $\mathbb{1}_\mathbb{P} \Vdash_\mathbb{P} [\check{\mathbb{Q}}$ *is ccc*$]$.

(iii) $\mathbb{1}_\mathbb{Q} \Vdash_\mathbb{Q} [\check{\mathbb{P}}$ *is ccc*$]$.

Proof. For (i)⇒(ii), assume ¬(ii). Then there exist $p \in \mathbb{P}$ and a $\mathbb{P}$-name $\dot{f}$ such that $p \Vdash [\dot{f} : \omega_1 \to \check{\mathbb{Q}}$ and $\dot{f}$ is one-one and $\operatorname{rng}\dot{f}$ is an antichain$]$. For each $\xi < \omega_1$ choose $p_\xi \leq p$ and $q_\xi \in \mathbb{Q}$ such that $p_\xi \Vdash [\dot{f}(\xi) = \check{q_\xi}]$. Then $(p_\xi, q_\xi) : \xi < \omega_1\}$ is an antichain. In fact, suppose that $\xi, \eta < \omega_1$ and $(r, s) \leq (p_\xi, q_\xi), (p_\eta, q_\eta)$. Then $r \Vdash [\dot{f}(\xi) = \check{q_\xi}]$ and $r \Vdash [\dot{f}(\eta) = \check{q_\eta}]$ Hence $r \Vdash \neg\exists s[s \leq \check{q_\xi}$ and $s \leq \check{q_\eta}]$. But $s \leq q_\xi, q_\eta$, contradiction.

For (ii)⇒(i), assume (ii) and ¬(i). Let $\{(p_\xi, q_\xi) : \xi < \omega_1\}$ be an antichain. Let $\dot{D}$ be the $\mathbb{P}$-name $\{(\check{\xi}, p_\xi) : \xi < \omega_1\}$. Now if G is $\mathbb{P}$-generic over M, then $\dot{D}_G = \{\xi < \omega_1 : p_\xi \in G\}$. If $\xi, \eta \in D_G$, then p_ξ and p_η are compatible, hence q_ξ and q_η are incompatible, since $\{(p_\xi, q_\xi) : \xi < \omega_1\}$ is an antichain. It follows from (ii) that $\dot{D}_G$ is countable. Thus $\mathbb{1} \Vdash [\dot{D}$ is countable$]$.

Let $E = \{\xi < \omega_1 : \exists r \in \mathbb{P}[r \Vdash [\check{\xi} = \sup \dot{D}]]\}$. For each $\xi \in E$ let $r_\xi \in \mathbb{P}$be such that $r_\xi \Vdash [\check{\xi} = \sup \dot{D}]$. Then $\{r_\xi : \xi \in E\}$ is an antichain. So E is countable. Say $E \subseteq \beta \in \omega_1$. If G is $\mathbb{P}$-generic and $\alpha = \sup \dot{D}_G$, then there is an $r \in G$ such that $r \Vdash [\check{\alpha} = \sup \dot{D}]$, so $\alpha \in E$ and hence $\alpha < \beta$. Thus $\mathbb{1} \Vdash [\dot{D} \subseteq \check{\beta}]$. Let G be $\mathbb{P}$-generic with $p_\beta \in G$. Then $\beta \in \dot{D}_G \subseteq \beta$, contradiction. □

Let $\mathbb{P}$ be a forcing poset. A $\mathbb{P}$-*name for a forcing poset* is a triple $(\dot{\mathbb{Q}}, \dot{\leq}_\mathbb{Q}, \dot{\mathbb{1}}_\mathbb{Q})$ of $\mathbb{P}$-names such that $\mathbb{1}_\mathbb{Q} \in \operatorname{dmn}(\dot{\mathbb{Q}})$ and

$$\mathbb{1}_\mathbb{P} \Vdash_\mathbb{P} [\dot{\mathbb{1}}_\mathbb{Q} \in \dot{\mathbb{Q}} \text{ and } \dot{\leq}_\mathbb{Q} \text{ is a forcing order with largest element } \dot{\mathbb{1}}_\mathbb{Q}].$$

Lemma 31.43. (V.3.2) *Let M be a c.t.m. for ZFC, $\mathbb{P} \in M$ a forcing poset, and let G be $\mathbb{P}$-generic over M. In $M[G]$ let $(X, \leq_X, \mathbb{1}_X)$ be a forcing poset.*

Then there is a $\mathbb{P}$-name $(\dot{\mathbb{Q}}, \dot{\leq}_{\mathbb{Q}}, \dot{\mathbb{1}}_{\mathbb{Q}})$ for a forcing poset such that, in $M[G]$, the forcing poset $(\dot{\mathbb{Q}}_G, (\dot{\leq}_{\mathbb{Q}})_G, (\dot{\mathbb{1}}_{\mathbb{Q}})_G)$ is isomorphic to $(X, \leq_X, \mathbb{1}_X)$.

Moreover, we may take $\dot{\mathbb{Q}}$ to be $\check{\alpha}$ for some ordinal α, and $\mathbb{1}_{\mathbf{Q}}$ to be $\check{\emptyset}$, and $\dot{\leq}_{\mathbb{Q}}$ to be a nice name for a subset of $(\alpha \times \alpha)\check{}$.

Proof. In $M[G]$ let $\alpha = |X|$. Let f be a bijection from X onto α which takes $\mathbb{1}_X$ to 0. Define $\xi \sqsubseteq \eta$ iff $\xi, \eta < \alpha$ and $f^{-1}(\xi) \leq_X f^{-1}(\eta)$. Thus f is an isomorphism from $(X, \leq_X, \mathbb{1}_X)$ onto $(\alpha, \sqsubseteq, 0)$. Let $\varphi(u, v, w)$ abbreviate the statement "$u \subseteq v \times v$ is a pre-order of the set v with largest element $w \in v$". Thus $\varphi(\sqsubseteq, \alpha, 0)$ holds.

Fix a $\mathbb{P}$-name τ over M with $\tau_G = \sqsubseteq$. Let $\dot{\mathbb{Q}} = \check{\alpha}$ and $\dot{\mathbb{1}}_{\mathbb{Q}} = \check{0}$. Now

$$\mathbb{1} \Vdash \exists y[\varphi(y, \check{\alpha}, \check{0}) \wedge [\varphi(\tau, \check{\alpha}, \check{0}) \rightarrow y = \tau]]$$

In fact, let H be $\mathbb{P}$-generic over M. If $\varphi(\tau_H, \alpha, 0)$, then we can take $y = \tau_H$. If $\neg\varphi(\tau_H, \alpha, 0)$, then we can take $y = \alpha \times \alpha$. By the maximal principle, Theorem 30.35, there is a name σ such that

$$\mathbb{1} \Vdash [\varphi(\sigma, \check{\alpha}, \check{0}) \wedge [\varphi(\tau, \check{\alpha}, \check{0}) \rightarrow \sigma = \tau]]. \tag{1}$$

By Proposition 29.21 let $\dot{\sqsubseteq}_{\mathbb{Q}}$ be a nice name for a subset of $(\alpha \times \alpha)\check{}$ such that

$$\mathbb{1} \Vdash [\sigma \subseteq (\alpha \times \alpha)\check{} \rightarrow \sigma = \dot{\sqsubseteq}_{\mathbb{Q}}]. \tag{2}$$

Now $\varphi(\tau_G, \alpha, 0)$, so by (1), $\sigma_G = \tau_G = \sqsubseteq \subseteq (\alpha \times \alpha)$. Hence by (2), $\sqsubseteq = \sigma_G = (\dot{\sqsubseteq}_{\mathbb{Q}})_G$. $\square$

If $\mathbb{P}$ is a forcing poset and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a forcing poset, then $\mathbb{P} * \dot{\mathbb{Q}}$ is the triple $(\mathbb{R}, \leq, \mathbb{1})$, where

$$\begin{aligned} \mathbb{R} &= \{(p, \dot{q}) \in \mathbb{P} \times \operatorname{dmn}(\dot{\mathbb{Q}}) : p \Vdash [\dot{q} \in \dot{\mathbb{Q}}]\}; \\ \mathbb{1} &= (\mathbb{1}_{\mathbb{Q}}, \dot{\mathbb{1}}_{\mathbb{Q}}); \\ (p_1, \dot{q}_1) \leq (p_2, \dot{q}_2) \quad &\text{iff} \quad p_1 \leq_{\mathbb{P}} p_2 \text{ and } p_1 \Vdash \dot{q}_1 \dot{\leq}_{\mathbb{Q}} \dot{q}_2. \end{aligned}$$

Moreover, $i(p) = (p, \dot{\mathbb{1}}_{\mathbb{Q}})$ for any $p \in \mathbb{P}$.

Lemma 31.44. *With the above notation,*
(i) $\mathbb{P} * \dot{\mathbb{Q}}$ is a forcing poset.
(ii) $p_0 \leq p_1$ iff $i(p_1) \leq i(p_2)$.
(iii) $i(\mathbb{1}_{\mathbb{P}}) = \mathbb{1}_{\mathbb{P} * \dot{\mathbb{Q}}}$.
(iv) If $(p_0, \dot{q}_0), (p_1, \dot{q}_1) \in \mathbb{P} * \dot{\mathbb{Q}}$ and $p_0 \perp p_1$, then $(p_0, \dot{q}_0) \perp (p_1, \dot{q}_1)$.
(v) If $p_0 \perp p_1$ and $(p_1, \dot{q}_1) \in \mathbb{P} * \dot{\mathbb{Q}}$, then $(p_0, \dot{\mathbb{1}}_{\mathbb{Q}}) \perp (p_1, \dot{q}_1)$.
(vi) $p_0 \perp p_1$ iff $i(p_1) \perp i(p_2)$.
(vii) i is a complete embedding.

Proof. All except (vii) are clear. For (vii), we claim that p is a reduction of $(p,\dot q)$ to $\mathbb{P}$; see Lemma 25.78. In fact, suppose that $i(r)\perp(p,\dot q)$. Thus $(r,\dot{\mathbb{1}}_{\mathbb{Q}})\perp(p,\dot q)$. Clearly then $p\perp r$. $\square$

Proposition 31.45. *If $\mathbb{P},\mathbb{Q}\in M$, then $\mathbb{P}\times\mathbb{Q}$ is isomorphic to $\mathbb{P}*(\check{\mathbb{Q}},\check{\leq}_{\mathbb{Q}},\check{\mathbb{1}}_{\mathbb{Q}})$.*

Proof. Define $f(p,q)=(p,\check q)$. Clearly $p\Vdash[\check q\in\check{\mathbb{Q}}]$, so $f(p,q)\in\mathbb{R}$. Clearly f is a bijection. $f(\mathbb{1}_{\mathbb{P}},\mathbb{1}_{\mathbb{Q}})=(\mathbb{1}_{\mathbb{P}},\check{\mathbb{1}}_{\mathbb{Q}})=\mathbb{1}$. For $p_i\in\mathbb{P}$ and $q_i\in\mathbb{Q}$ with $i\in 2$ we have

$$(p_1,q_1)\le(p_2,q_2)\quad\text{iff}\quad p_1\le p_2\text{ and }q_1\le q_2\quad\text{iff}\quad p_1\le p_2\text{ and }p_1\Vdash\check q_1\check{\le}_{\mathbb{Q}}\check q_2. \qquad\square$$

With $\mathbb{P}$ and $\dot{\mathbb{Q}}$ as above, suppose that G is $\mathbb{P}$-generic over M and H is a subset of $\dot{\mathbb{Q}}_G$. We define $G*H=\{(p,\dot q)\in\mathbb{P}*\dot{\mathbb{Q}}:p\in G\text{ and }\dot q_G\in H\}$.

Theorem 31.46. (V.3.6) *Assume given a product as above in M. Let K be $\mathbb{P}*\dot{\mathbb{Q}}$-generic over M. Let $G=i^{-1}[K]$ and $H=\{\dot q_G:\dot q\in\operatorname{dmn}(\dot{\mathbb{Q}})$ and $\exists p[(p,\dot q)\in K]\}$.*

Then G is $\mathbb{P}$-generic over M and H is $\dot{\mathbb{Q}}_G$-generic over $M[G]$. Also $K=G*H$ and $M[K]=M[G][H]$.

Proof. G is generic because i is a complete embedding; see Lemma 30.2.

(1) $H\subseteq\dot{\mathbb{Q}}_G$.

For, suppose that $x\in H$. Say $x=\dot q_G$ with $\dot q\in\operatorname{dmn}(\dot{\mathbb{Q}})$ and $(p,\dot q)\in K$. Then $p\in G$ and $p\Vdash\dot q\in\dot{\mathbb{Q}}$, so $x=\dot q_G\in\dot{\mathbb{Q}}_G$, proving (1).

(2) $\mathbb{1}_{\dot{\mathbb{Q}}_G}\in H$.

For, $\mathbb{1}_{\dot{\mathbb{Q}}_G}=(\dot{\mathbb{1}}_{\mathbb{Q}})_G$, $\dot{\mathbb{1}}_{\mathbb{Q}}\in\operatorname{dmn}(\mathbb{Q})$. and $(\mathbb{1}_{\mathbb{P}},\dot{\mathbb{1}}_{\mathbb{Q}})\in K$, so (2) follows.

(3) If $x\in H$ and $x\le y$, then $y\in H$.

For, suppose that $x\in H$ and $x\le y$. Say $x=\dot q_G$ and $(p,\dot q)\in K$. Since $y\in\dot{\mathbb{Q}}_G$, there is an $(\dot r,p')\in\mathbb{Q}$ such that $p'\in G$ and $y=\dot r_G$. So $\dot q_G\le\dot r_G$. Let $p''\in G$ be such that $p''\Vdash\dot q\le\dot r$. Hence $(p,\dot q),(p',\dot{\mathbb{1}}),(p'',\dot{\mathbb{1}})\in K$. Let $(p''',\dot s)\in K$ be such that $(p''',\dot s)\le(p,\dot q),(p',\dot{\mathbb{1}}),(p'',\dot{\mathbb{1}})$. Now $(p''',\dot s)\le(p,\dot q)$, so $p'''\Vdash\dot s\le\dot q$. Also, $(p''',\dot s)\le(p'',\dot{\mathbb{1}})$, so $p'''\le p''$ and hence $p'''\Vdash\dot q\le\dot r$. So $p'''\Vdash\dot s\le\dot r$. Thus $(p''',\dot s)\le(p''',\dot r)$, so $(p''',\dot r)\in K$. It follows that $y=\dot r_G\in H$, proving (3).

(4) If $x,y\in H$, then there is a $z\in H$ such that $z\le x,y$.

For, suppose that $x,y\in H$. Say $x=\dot q_G$ with $\dot q\in\dot{\mathbb{Q}}$, $(p,\dot q)\in K$, and $y=\dot r_G$ with $\dot r\in\dot{\mathbb{Q}}$. $(s,\dot r)\in K$, Choose $(t,\dot u)\in K$ such that $(t,\dot u)\le(p,\dot q),(s,\dot r)$. Then $\dot u_G\in H$ and $t\Vdash\dot u\le\dot q$, and $t\in G$, so $\dot u_G\le\dot q_G$. Similarly $\dot u_G\le\dot r_G$. So (4) holds.

(5) H is $\dot{\mathbb{Q}}_G$-generic over $M[G]$.

In fact, it remains only to take any $D\subseteq\dot{\mathbb{Q}}_G$ which is dense in $\dot{\mathbb{Q}}_G$ with $D\in M[G]$ and show that $D\cap H\neq\emptyset$. There is a $\mathbb{P}$-name $\dot D$ such that $D=\dot D_G$. Take $p\in G$ such that $p\Vdash[\dot D$ is dense in $\check{\mathbb{Q}}]$. Let

$$D'=\{(p_1,\dot q_1)\in\mathbb{P}*\dot{\mathbb{Q}}:p_1\le p\text{ and }p_1\Vdash[\check q_1\in\dot D]\}.$$

Then D' is dense below $(p, \dot{\mathbb{1}})$. In fact, suppose that $(p_2, \dot{q}_2) \leq (p, \dot{\mathbb{1}})$. Then $p_2 \leq p$, so $p_2 \Vdash [\dot{D}$ is dense in $\dot{\mathbb{Q}}]$. Hence $p_2 \Vdash \exists y \in \dot{\mathbb{Q}}[y \in \dot{D}$ and $y \leq \dot{q}_2]$. Hence by Proposition 29.15 there exist $p_3 \leq p_2$ and $\dot{q}_3 \in \mathrm{dmn}(\dot{\mathbb{Q}})$ such that $p_3 \Vdash [\dot{q}_3 \in \dot{D}$ and $\dot{q}_3 \leq \dot{q}_2]$. Thus $(p_3, \dot{q}_3) \in D'$ and $(p_3, \dot{q}_3) \leq (p_2, \dot{q}_2)$ as desired. So D' is dense below $(p, \dot{\mathbb{1}})$.

Choose $(p_4, \dot{q}_4) \in D' \cap K$. Then $p_4 \Vdash \dot{q}_4 \in \dot{D}$, so $\dot{q}_{4G} \in \dot{D}_G = D$ and $\dot{q}_{4G} \in H$. This proves that H is $\mathbb{Q}$-generic over $M[G]$. Hence (5) holds.

(6) $K \subseteq G * H$.

For, suppose that $(p, \dot{q}) \in K$. Then $p \in G$ and $\dot{q}_G \in H$, so $(p, \dot{q}) \in G * H$.

(7) $G * H \subseteq K$.

In fact, suppose that $x \in G * H$. Say $x = (p, \dot{q})$ with $(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}}$, $p \in G$, and $\dot{q}_G \in H$. Say $\dot{q}_G = \dot{q}'_G$ with $\dot{q}' \in \mathrm{dmn}(\dot{\mathbb{Q}})$ and $(p', \dot{q}') \in K$. Choose $p'' \in G$ such that $p'' \Vdash \dot{q} = \dot{q}'$. Take $(p''', \dot{r}) \in K$ such that $(p''', \dot{r}) \leq (p, \dot{\mathbb{1}}), (p', \dot{q}'), (p'', \dot{\mathbb{1}})$. Then $p''' \Vdash \dot{r} \leq \dot{q}'$. Also, $p''' \leq p''$, so $p''' \Vdash \dot{q}' \leq \dot{q}$. Hence $(p''', r) \leq (p, \dot{q})$ and so $x = (p, \dot{q}) \in K$, proving (7).

(8) $M[K] \subseteq M[G][H]$.

This holds since $K = G * H \in M[G][H]$, so Lemma 28.8 applies.

(9) $M[G][H] \subseteq M[K]$

For, $M[G] \subseteq M[K]$ since $G \in M[K]$, using Lemma 28.8, and $H \in M[K]$, so $M[G][H] \subseteq M[K]$ by Lemma 28.8 again. □

Lemma 31.47. (V.3.8) *In M suppose that κ is uncountable and regular, and $\mathbb{P}$ is ccc. Suppose that $\dot{S}$ is a name, and $\mathbb{1} \Vdash [\dot{S} \subseteq \kappa \wedge |\dot{S}| < \kappa]$. Then there is a $\beta < \kappa$ such that $\mathbb{1} \Vdash [\dot{S} \subseteq \check{\beta}]$.*

Proof. Let $E = \{\alpha < \kappa : \exists p[p \Vdash [\check{\alpha} = \sup(\dot{S})]]\}$. For each $\alpha \in E$ let p_α be such that $p_\alpha \Vdash [\check{\alpha} = \sup(\dot{S})]$. Clearly $\{p_\alpha : \alpha \in E\}$ is an antichain, so E is countable. Hence there is a $\beta < \kappa$ such that $E \subseteq \beta$. We claim that $\mathbb{1} \Vdash [\dot{S} \subseteq \check{\beta}]$.

Suppose not. Then there is a p such that $p \Vdash \neg[\dot{S} \subseteq \check{\beta}]$. So $p \Vdash [\sup(\dot{S}) \geq \beta]$, hence $p \Vdash \exists x \in \kappa[x \geq \beta \wedge \sup(\dot{S}) = x]$. Hence there exist a $q \leq p$ and an $\alpha < \kappa$ such that $q \Vdash [\check{\alpha} \geq \check{\beta} \wedge \sup(\dot{S}) = \check{\alpha}]$. Hence $\alpha \geq \beta$ and $q \Vdash \sup(\dot{S}) = \alpha$, so $\alpha \in E$, contradicting $E \subseteq \beta$. □

Lemma 31.48. (V.3.9) *If κ is uncountable and regular, $\mathbb{P}$ is κ-cc, $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a poset, and $\mathbb{1} \Vdash [\dot{\mathbb{Q}}$ is κ-cc$]$, then $\mathbb{P} * \dot{\mathbb{Q}}$ is κ-cc.*

Proof. Assume the hypotheses, but suppose that $\mathbb{P} * \dot{\mathbb{Q}}$ is not κ-cc. Let $\langle (p_\xi, \dot{q}_\xi) : \xi < \kappa \rangle$ be an antichain. Let $\dot{S}$ be the $\mathbb{P}$-name $\{(\check{\xi}, p_\xi) : \xi < \kappa\}$. Clearly $\mathbb{1} \Vdash [\dot{S} \subseteq \kappa]$.

(1) There is no $\beta < \kappa$ such that $\mathbb{1} \Vdash [\dot{S} \subseteq \check{\beta}]$.

In fact, suppose that there is such a β. Let G be $\mathbb{P}$-generic such that $p_\beta \in G$. Then $\beta \in \dot{S}_G \subseteq \beta$, contradiction.

But now we will show that $\mathbb{1} \Vdash [|\dot{S}| < \check{\kappa}]$; then Lemma 31.47 gives a contradiction. Let G be $\mathbb{P}$-generic over M. Since $\dot{\mathbb{Q}}_G$ has the κ-cc, $|\dot{S}_G| < \kappa$ will follow if we prove that

$(\dot{q}_\xi)_G \perp (\dot{q}_\eta)_G$ for distinct $\xi,\eta \in \dot{S}_G$. So suppose that $\xi,\eta \in \dot{S}_G$ but $(\dot{q}_\xi)_G$ and $(\dot{q}_\eta)_G$ are compatible; say $\dot{q} \in \mathrm{dmn}(\dot{\mathbb{Q}})$ and $\dot{q}_G \dot{\leq}_G (\dot{q}_\xi)_G, (\dot{q}_\eta)_G$. Then there is a $p \in G$ such that $p \Vdash [\dot{q} \in \dot{\mathbb{Q}}]$, $p \Vdash [\dot{q} \dot{\leq} \dot{q}_\xi]$, and $p \Vdash [\dot{q} \dot{\leq} \dot{q}_\eta]$. Now $\xi,\eta \in \dot{S}_G$, so $p_\xi, p_\eta \in G$. Let $p' \in G$ be such that $p' \leq p, p_\xi, p_\eta$. Then $(p', \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}}$, $(p', \dot{q}) \leq (p_\xi, \dot{q}_\xi)$, and $(p', \dot{q}) \leq (p_\eta, q_\eta)$. This contradicts $(p_\xi, \dot{q}_\xi) \perp (p_\eta, q_\eta)$. □

Let κ be an infinite cardinal and α an ordinal. A *κ-supported α-stage iterated forcing construction* is a pair

$$(\langle(\mathbb{P}_\xi, \leq_\xi, \mathbb{1}_\xi) : \xi \leq \alpha\rangle, \langle(\dot{\mathbb{Q}}_\xi, \dot{\leq}_{\dot{\mathbb{Q}}_\xi}, \dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\xi}) : \xi < \alpha\rangle)$$

satisfying the following conditions:

(1) Each $(\mathbb{P}_\xi, \leq_\xi, \mathbb{1}_\xi)$ is a forcing poset.

(2) Each $(\dot{\mathbb{Q}}_\xi, \dot{\leq}_{\dot{\mathbb{Q}}_\xi}, \dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\xi})$ is a $(\mathbb{P}_\xi, \leq_\xi, \mathbb{1}_\xi)$-name for a forcing poset.

(3) Each $p \in \mathbb{P}_\xi$ is a sequence of length ξ, with $p_\mu \in \mathrm{dmn}(\dot{\mathbb{Q}}_\mu)$ for each $\mu < \xi$.

(4) If $\xi < \eta$ and $p \in \mathbb{P}_\eta$, then $(p \restriction \xi) \in \mathbb{P}_\xi$.

(5) If $\xi < \eta$ and $p \in \mathbb{P}_\xi$, then $i^\eta_\xi(p) \in \mathbb{P}_\eta$, where $i^\eta_\xi(p) = p \cup \{(\mu, \dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\mu}) : \xi \leq \mu < \eta\}$.

(6) $\mathbb{1}_\xi = \langle \dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\eta} : \eta < \xi\rangle$.

(7) If $p, p' \in \mathbb{P}_\xi$, then $p \leq_\xi p'$ iff $\forall \mu < \xi[p \restriction \mu \Vdash_{\mathbb{P}_\mu} (p_\mu \leq p'_\mu)]$.

(8) If $\xi + 1 \leq \alpha$, then $\mathbb{P}_{\xi+1} = \{p^\frown\langle \dot{q}\rangle : p \in \mathbb{P}_\xi \text{ and } \dot{q} \in \mathrm{dmn}(\dot{\mathbb{Q}}_\xi) \text{ and } p \Vdash_{\mathbb{P}_\xi} [\dot{q} \in \dot{\mathbb{Q}}_\xi]\}$.

(9) If η is limit $\leq \alpha$, then $p \in P_\eta$ iff p is a sequence of length η, $\forall \xi < \eta[(p \restriction \xi) \in \mathbb{P}_\xi]$, and $|\mathrm{supp}(p_\eta)| < \kappa$, where $\mathrm{supp}(p_\eta) = \{\xi < \eta : p_\sigma \neq \dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\xi}\}$.

Proposition 31.49. *With the above notation, $\mathbb{P}_0 = \{\emptyset\}$, $\emptyset \leq_0 \emptyset$, $\mathbb{1}_0 = \emptyset$, and the $\mathbb{P}_0$-names are*

$$\begin{aligned}&\emptyset, \{(\emptyset,\emptyset)\}, \{(\{(\emptyset,\emptyset)\},\emptyset)\}, \{(\{(\{(\emptyset,\emptyset)\},\emptyset)\},\emptyset)\},\\ &\{(\emptyset,\emptyset), (\{(\emptyset,\emptyset)\},\emptyset)\}\ldots\end{aligned}$$ □

Proposition 31.50. *With the above notation, G is $\mathbb{P}_0$-generic over M iff $G = \mathbb{P}_0$.* □

Proposition 31.51. *With the above notation, there is a $\mathbb{P}_0$-name $\dot{\mathbb{Q}}$ for a forcing poset such that if G is $\mathbb{P}_0$-generic over M then $\mathbb{Q}_G = 2$ with its natural order.*

Proof. We use Lemma 28.22.

$$\begin{aligned}\dot{\mathbb{Q}} &= \mathrm{up}(\emptyset, \mathrm{up}(\emptyset,\emptyset))\\ \dot{\mathbb{1}}_{\dot{\mathbb{Q}}} &= \emptyset;\\ \leq_{\dot{\mathbb{Q}}} &= \mathrm{up}(\mathrm{up}(\mathrm{op}(\emptyset,\emptyset), \mathrm{op}(\emptyset, \mathrm{up}(\emptyset,\emptyset))), \mathrm{op}(\mathrm{up}(\emptyset,\emptyset), \mathrm{up}(\emptyset,\emptyset)))\end{aligned}$$ □

Proposition 31.52. *With the above notation, $|\mathrm{supp}(p)| < \kappa$ for all p in any $\mathbb{P}_\xi$.*

Proof. By induction on ξ. $\square$

Proposition 31.53. *With the above notation, if $\xi \le \eta \le \alpha$, $p \in \mathbb{P}_\eta$, $r \in \mathbb{P}_\xi$, and $r \le (p \restriction \xi)$, then $r \cup (p \restriction (\eta \backslash \xi)) \in \mathbb{P}_\eta$.*

Proof. With ξ fixed we prove this by induction on η. It is trivial for $\eta = \xi$. Now assume that it is true for $\eta \ge \xi$, and we have $\xi \le \eta + 1$, $p \in \mathbb{P}_{\eta+1}$, $r \in \mathbb{P}_\xi$, and $r \le p \restriction \xi$. The case $\xi = \eta + 1$ is trivial, so suppose that $\xi < \eta + 1$.

Case 1. $\xi = \eta$. By Definition (8) we have$(p \restriction \eta) \Vdash [p_\eta \in \dot{\mathbb{Q}}_\eta]$, hence $r \Vdash [p_\eta \in \dot{\mathbb{Q}}_\eta]$, hence by Definition (8) we get $r \cup (p \restriction ((\eta+1) \backslash \eta)) \in \mathbb{P}_{\eta+1}$.

Case 2. $\xi < \eta$. By the inductive hypothesis applied to $p \restriction \eta$ we get $r \cup (p \restriction (\eta \backslash \xi)) \in \mathbb{P}_\eta$. Now by Definition (7) we have $r \cup (p \restriction (\eta \backslash \xi)) \le (p \restriction \eta)$, and by Definition (8) we have $(p \restriction \eta) \Vdash [p_\eta \in \dot{\mathbb{Q}}_\eta]$, so $r \cup (p \restriction (\eta \backslash \xi)) \Vdash [p_\eta \in \dot{\mathbb{Q}}_\eta]$. Hence by Definition (8) $r \cup (p \restriction ((\eta+1) \backslash \xi)) \in \mathbb{P}_{\eta+1}$.

The limit case follows from the Definition (9). $\square$

Proposition 31.54. *With the notation of the above Definition, for any $\xi < \alpha$ the poset $\mathbb{P}_{\xi+1}$ is isomorphic to $\mathbb{P}_\xi * \dot{\mathbb{Q}}_\xi$.*

Proof. For any $p \in \mathbb{P}_{\xi+1}$ let $f(p) = (p \restriction \xi, p_\xi)$. By Definition (4), $p \restriction \xi \in \mathbb{P}_\xi$. By Definition (3), $p_\xi \in \mathrm{dmn}(\dot{\mathbb{Q}}_\xi)$. By Definition (8), $(p \restriction \xi) \Vdash [p_\xi \in \dot{\mathbb{Q}}_\xi]$. Hence $f(p) \in \mathbb{P}_\xi * \dot{\mathbb{Q}}_\xi$. Clearly f is a bijection. Suppose that $p^1, p^2 \in \mathbb{P}_{\xi+1}$. Then

$$\begin{aligned}
p^1 \le p^2 \quad &\text{iff} \quad \forall \mu \le \xi [p \restriction \mu \Vdash [p^1_\mu \le p^2_\mu]] \quad \text{by Definition (7)}\\
&\text{iff} \quad [p^1 \restriction \xi \le p^2 \restriction \xi] \text{ and } p \restriction \xi \Vdash [p^1_\xi \le p^2_\xi]\\
&\text{iff} \quad (p^1 \restriction \xi, p^1_\xi) \le (p^2 \restriction \xi, p^2_\xi)\\
&\text{iff} \quad f(p^1) \le f(p^2).
\end{aligned}$$

$\square$

Lemma 31.55. (V.3.12) *With the notation of the above Definition, $i^\eta_\xi : \mathbb{P}_\xi \to \mathbb{P}_\eta$ is a complete embedding.*

Proof. See the definition of complete embedding on page 444. For (i), we have

$$i^\eta_\xi(\mathbb{1}_\xi) = \mathbb{1}_\xi \cup \{(\mu, \mathbb{1}_{\dot{\mathbb{Q}}_\mu}) : \mu \in [\xi, \eta)\} = \langle \mathbb{1}_{\dot{\mathbb{Q}}_\mu} : \mu < \eta \rangle = \mathbb{1}_\eta.$$

For (ii), suppose that $p_1, p_2 \in \mathbb{P}_\xi$ and $p_1 \le_\xi p_2$. Then by Definition (7), $\forall \mu < \xi [p_1 \restriction \mu \Vdash_{\mathbb{P}_\mu} [p_{1\mu} \le p_{2\mu}]$. If $\xi \le \mu < \eta$, then by Definition (2), $(\dot{\mathbb{Q}}_\mu, \dot{\le}_{\dot{\mathbb{Q}}_\mu}, \dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\mu})$ is a $(\mathbb{P}_\mu, \le_\mu, \mathbb{1}_\mu)$-name for a forcing poset. Hence by Definition (1), $\mathbb{1}_\mu \Vdash_{\mathbb{P}_\mu} [\dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\mu} \in \dot{\mathbb{Q}}_\mu \wedge \mathbb{1}_{\dot{\mathbb{Q}}_\mu} \le_{\dot{\mathbb{Q}}_\mu} \mathbb{1}_{\dot{\mathbb{Q}}_\mu}]$. It follows that for all $\mu < \eta$, $i^\eta_\xi(p_1) \restriction \mu \Vdash_{\mathbb{P}_\mu} [(i^\eta_\xi(p_1))_\mu \le i^\eta_\xi(p_2)_\mu]$. Hence by Definition, $i^\eta_\xi(p_1) \le i^\eta_\xi(p_2)$.

$\Leftarrow$ in (iii) follows from (ii). Now suppose that $p_1, p_2 \in \mathbb{P}_\xi$ and $i^\eta_\xi(p_1)$ and $i^\eta_\xi(p_2)$ are compatible. Say $r \le i^\eta_\xi(p_1), i^\eta_\xi(p_2)$. By Definition (4), $r \restriction \xi \in \mathbb{P}_\xi$, and by Definition (7), $r \restriction \xi \le p_1, p_2$.

Finally, for the notion of reduction, see page 447. Suppose that $p \in \mathbb{P}_\eta$, $r \in \mathbb{P}_\xi$, and r and $p \upharpoonright \xi$ are compatible; say $s \le r, p \upharpoonright \xi$. Let $s' = s \cup (p \upharpoonright [\xi, \eta))$. By Proposition 31.53, $s' \in \mathbb{P}_\eta$. Clearly $s' \le i^\eta_\xi(r), p$. $\square$

In a forcing poset, $\not\perp (p_1, \ldots, p_n)$ means that there is a $q \le p_i$ for all i.

Lemma 31.56. (V.3.15) *In a $<\kappa$-supported iteration, using the above Definition, take $p^1, \ldots, p^n \in \mathbb{P}_\alpha$ and suppose that $\xi < \alpha$ and $\mathrm{supp}(p^i) \cap \mathrm{support}(p^j) \subseteq \xi$ whenever $i \neq j$. Then $\not\perp (p^1, \ldots, p^n)$ iff $\not\perp (p^1 \upharpoonright \xi, \ldots, p^n \upharpoonright \xi)$.*

Proof. $\Rightarrow$: assume that $\not\perp (p^1, \ldots, p^n)$. Say $q \le p^i$ for all i. Then by Definition (7), $(q \upharpoonright \xi) \le (p^i \upharpoonright \xi)$ for all i.

$\Leftarrow$: assume that $\not\perp (p^1 \upharpoonright \xi, \ldots, p^n \upharpoonright \xi)$. Say $q \le p^i \upharpoonright \xi$ for all i. Define r by

$$r_\eta = \begin{cases} q_\eta & \text{if } \eta < \xi, \\ p^i_\eta & \text{if } \xi \le \eta \in \mathrm{supp}(p^i) \text{ for some } i, \\ \mathbb{1} & \text{otherwise.} \end{cases}$$

Then $r \in \mathbb{P}_\xi$ by induction as in the proof of Proposition 31.53. Clearly $r \le p^i$ for all i. $\square$

Lemma 31.57. (V.3.16) *In a finite support iteration, using the above Definition, assume that α is a limit ordinal and $p^\mu \in \mathbb{P}_\alpha$ for all $\mu < \omega_1$. Then*

$$\exists \xi < \alpha \exists I \in [\omega_1]^{\omega_1} \forall n \in \omega \backslash 2 \forall \mu \in {}^n I [\not\perp (p^{\mu_0}, \ldots, p^{\mu_{n-1}}) \textit{ iff } \not\perp (p^{\mu_0} \upharpoonright \xi, \ldots, p^{\mu_{n-1}} \upharpoonright \xi)].$$

Proof. By the Δ-system lemma, choose $I \in [\omega_1]^{\omega_1}$ such that $\{\mathrm{supp}(p^\mu) : \mu \in I\}$ forms a Δ-system, say with kernel R. Take $\xi < \alpha$ such that $R \subseteq \xi$. Now Lemma 31.56 applies. $\square$

Lemma 31.58. (V.3.17) *In a finite support iteration, using the above definition, if $\alpha \ge 1$ and $\forall \xi < \alpha [\mathbb{1}_\xi \Vdash [\dot{\mathbb{Q}}_\xi$ is ccc$]]$, then $\mathbb{P}_\alpha$ is ccc.*

Proof. We proceed by induction on α. For $\alpha = 0$, note that $\mathbb{P}_0 = \{\emptyset\}$, which is ccc. The successor case follows from Proposition 31.54 and Lemma 31.48. The limit case follows from Lemma 31.57. $\square$

$\mathrm{tbc}(\alpha, \sqsubseteq)$ abbreviates the statement that α is a nonzero ordinal, $\sqsubseteq$ is a subset of $\alpha \times \alpha$, and $(\alpha, \sqsubseteq, 0)$ is a ccc forcing poset.

Lemma 31.59. (V.4.3) *For any infinite cardinal θ, $MA(\theta)$ holds iff $MA_{\mathbb{Q}}(\theta)$ holds for every poset $\mathbb{Q}$ of the form $(\alpha, \sqsubseteq, 0)$, where $\mathrm{tbc}(\alpha, \sqsubseteq)$ and $\alpha \le \theta$.*

Proof. $\Rightarrow$: trivial. $\Leftarrow$: Assume the indicated condition, and suppose that $MA_{\mathbb{P}}(\theta)$ is false. By Lemma 25.57, there is a $\mathbb{Q} \subseteq_{\mathrm{ctr}} \mathbb{P}$ such that $MA_{\mathbb{Q}}(\theta)$ is false and $|\mathbb{Q}| \le \theta$. let f be a bijection of $\mathbb{Q}$ onto $|\mathbb{Q}|$ such that $f(\mathbb{1}_{\mathbb{Q}}) = 0$. Define $\xi \sqsubseteq \eta$ iff $\xi, \eta < |\mathbb{Q}|$ and

$f^{-1}(\xi) \leq_{\mathbb{Q}} f^{-1}(\eta)$. Then $\mathbb{R} \stackrel{\text{def}}{=} (|\mathbb{Q}|, \sqsubseteq, 0)$ is a forcing poset, $\text{tbc}(|\mathbb{Q}|, \sqsubseteq)$, and $MA_{\mathbb{R}}(\theta)$ is false, contradiction. $\square$

Proposition 31.60. *Let a finite support α-stage interation be given and suppose that $\xi < \alpha$. Let G be $\mathbb{P}_\alpha$-generic over M. Then:*

(i) Let $G_\xi = (i^\alpha_\xi)^{-1}[G]$; then G_ξ is $\mathbb{P}_\xi$-generic over M, and $M[G_\xi] \subseteq M[G]$.

(ii) Let $\mathbb{R}_\xi = (\mathbb{Q}_\xi)_{G_\xi}$, by Proposition 31.54 let f be an isomorphism from $\mathbb{P}_{\xi+1}$ onto $\mathbb{P}_\xi * \dot{\mathbb{Q}}_\xi$, and let $G' = f[(i^\alpha_{\xi+1})^{-1}[G]]$. Then G' is $\mathbb{P}_\xi * \dot{\mathbb{Q}}_\xi$-generic over M. Now let

$$H_\xi = \{\rho_{G_\xi} : \rho \in \mathbb{Q}_\xi \wedge \exists p \in \mathbb{P}_\xi[(p, \rho) \in G']\}.$$

Then H_ξ is $\mathbb{R}_\xi$-generic over $M[G_\xi]$, and $M[G'] = M[G_\xi][H_\xi]$.

Proof. (i) holds by Lemma 30.3 and Lemma 31.55. For (ii), clearly G' is $\mathbb{P}_\xi * \dot{\mathbb{Q}}_\xi$-generic over M. The rest follows from Theorem 31.46. $\square$

Proposition 31.61. *If $\mathbb{P}$ is a subposet of $\mathbb{Q}$, then every $\mathbb{P}$-name is a $\mathbb{Q}$-name.*

Proof. An easy induction, using Corollary 28.5. $\square$

For a forcing poset $\mathbb{P}$, $\text{Ntbc}(\alpha, \dot{\sqsubseteq}, \mathbb{P})$ holds iff $\dot{\sqsubseteq}$ is a nice name for a subset of $\alpha \times \alpha$ and $\mathbb{1} \Vdash_{\mathbb{P}} \text{tbc}(\alpha, \dot{\sqsubseteq})$.

Lemma 31.62. (V.4.5) *Suppose that $\mathbb{P}_1$ is ccc, $\mathbb{P}_0 \subseteq_c \mathbb{P}_1$, and $\dot{\sqsubseteq}$ is a $\mathbb{P}_0$-name; then* $\text{Ntbc}(\alpha, \dot{\sqsubseteq}, \mathbb{P}_1)$ *implies* $\text{Ntbc}(\alpha, \dot{\sqsubseteq}, \mathbb{P}_0)$.

Proof. Assume that $\text{Ntbc}(\alpha, \dot{\sqsubseteq}, \mathbb{P}_1)$ and $\neg\text{Ntbc}(\alpha, \dot{\sqsubseteq}, \mathbb{P}_0)$ hold in M. Then there is a $p \in \mathbb{P}_0$ such that $p \Vdash_{\mathbb{P}_0} \neg\text{tbc}(\check{\alpha}, \dot{\sqsubseteq})$. Let G be $\mathbb{P}_1$-generic over M with $p \in G$. By Lemmas 30.2 and 30.3, $G \cap \mathbb{P}_0$ is $\mathbb{P}_0$-generic over M, and $M[G \cap \mathbb{P}_0] \subseteq M[G]$. Now by Lemma 30.3 $\dot{\sqsubseteq}_G = \dot{\sqsubseteq}_{G\cap\mathbb{P}_0}$. But $\text{tbc}(\alpha, \dot{\sqsubseteq}_G)$ holds in $M[G]$ because $\text{Ntbc}(\alpha, \dot{\sqsubseteq}, \mathbb{P}_1)$, while $\text{tbc}(\alpha, \dot{\sqsubseteq}_{G\cap\mathbb{P}_0})$ does not hold in $M[G \cap \mathbb{P}_0]$ since $p \Vdash_{\mathbb{P}_0} \neg\text{tbc}(\check{\alpha}, \dot{\sqsubseteq})$. This contradicts the absoluteness of the formula $\text{tbc}(x, y)$. $\square$

Lemma 31.63. (V.4.6) *Let M be a c.t.m. for ZFC. In M let θ be an infinite cardinal. Let G be $\mathbb{P}$-generic over M. In $M[G]$ let $(X, \leq_X, \mathbb{1}_X)$ be a ccc forcing poset with $|X| \leq \theta$.*

Then there is a name $\dot{\sqsubseteq}$ in M and an $\alpha \leq \theta$ such that $\text{Ntbc}(\alpha, \dot{\sqsubseteq}.\mathbb{P})$ *holds in M and such that in $M[G]$ the poset $(\alpha, \dot{\sqsubseteq}_G, 0)$ is isomorphic to $(X, \leq_X, \mathbb{1}_X)$.*

Proof. In $M[G]$ let $\alpha = |X|$. Let f be a bijection from X onto α which takes $\mathbb{1}_X$ to 0. Define $\xi \sqsubseteq \eta$ iff $\xi, \eta < \alpha$ and $f^{-1}(\xi) \leq_X f^{-1}(\eta)$. Thus f is an isomorphism from $(X, \leq_X, \mathbb{1}_X)$ onto $(\alpha, \sqsubseteq, 0)$.

Fix a $\mathbb{P}$-name τ over M with $\tau_G = \sqsubseteq$. Let $\dot{\mathbb{Q}} = \check{\alpha}$ and $\dot{\mathbb{1}}_{\mathbb{Q}} = \check{0}$. Now

$$\mathbb{1} \Vdash \exists y[\text{tbc}(\alpha, y) \wedge [\text{tbc}(\alpha, \tau) \rightarrow y = \tau]].$$

In fact, let H be $\mathbb{P}$-generic over M. If $\text{tbc}(\alpha, \tau_H)$, then we can take $y = \tau_H$. If $\neg\text{tbc}(\alpha, \tau_H)$, then we can take $y = \alpha \times \alpha$. By the maximal principle, Theorem 30.35, there is a name σ such that

$$\mathbb{1} \Vdash [\text{tbc}(\alpha.\sigma) \wedge [\text{tbc}(\alpha, \tau) \rightarrow \sigma = \tau]]. \tag{1}$$

By Proposition 29.21 let $\dot{\sqsubseteq}_{\mathbb{Q}}$ be a nice name for a subset of $(\alpha \times \alpha)\check{}$ such that

$$\mathbb{1} \Vdash [\sigma \subseteq (\alpha \times \alpha)\check{} \rightarrow \sigma = \dot{\sqsubseteq}_{\mathbb{Q}}]. \tag{2}$$

Now $\text{tbc}(\alpha, \tau_G)$, so by (1), $\sigma_G = \tau_G = \sqsubseteq \subseteq (\alpha \times \alpha)$. Hence by (2), $\sqsubseteq = \sigma_G = (\dot{\sqsubseteq}_{\mathbb{Q}})_G$. $\square$

Proposition 31.64. *Let κ be an infinite cardinal, and let $f : \kappa \to \kappa \times \kappa$ be the bijection given by the proof of Theorem 11.32. Then $\zeta \leq f^{-1}(\zeta, \mu)$ for any $\zeta, \mu < \kappa$.*

Proof. We prove this by induction on ζ. It is trivial for $\zeta = 0$. Assume that it is true for ζ. Now $(\zeta, \mu) \prec (\zeta + 1, \mu)$, so $\zeta \leq f^{-1}(\zeta, \mu) < f^{-1}(\zeta + 1, \mu)$ and so $\zeta + 1 \leq f^{-1}(\zeta + 1, \mu)$. Assume that ζ is a limit ordinal and $\rho < f^{-1}(\rho, \mu)$ for all $\rho < \zeta$ and all μ. Then for any $\rho < \zeta$ we have $\rho \leq f^{-1}(\rho, \mu) < f^{-1}(\zeta, \mu)$, and hence $\zeta \leq f^{-1}(\zeta, \mu)$. $\square$

Proposition 31.65. *Suppose that $i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding.*

(i) For any $x \in M$, $i_* \check{x} = \check{x}$.

(ii) For any $x, y \in M$, $i_* \text{up}(\check{x}, \check{y}) = \text{up}(\check{x}, \check{y})$.

(iii) For any $x, y \in M$, $i_* \text{op}(\check{x}, \check{y}) = \text{op}(\check{x}, \check{y})$.

(iv) If σ is a nice $\mathbb{P}$-name for a subset of $\check{\alpha} \times \check{\alpha}$, then $i_* \sigma$ is a nice $\mathbb{Q}$-name for a subset of $\check{\alpha} \times \check{\alpha}$.

Proof. (i): By induction:

$$i_* \check{x} = i_* \{(\check{y}, \mathbb{1}) : y \in x\} = \{(i_* \check{y}, \mathbb{1}) : y \in x\} = \{(\check{y}, \mathbb{1}) : y \in x\} = \check{x}.$$

(ii):

$$i_* \text{up}(\check{x}, \check{y}) = i_* \{(\check{x}, \mathbb{1}), (\check{y}, \mathbb{1})\} = \{(i_* \check{x}, \mathbb{1}), (i_* \check{y}, \mathbb{1})\} = \{(\check{x}, \mathbb{1}), (\check{y}, \mathbb{1})\} = \text{up}(\check{x}, \check{y}).$$

(iii):

$$\begin{aligned} i_* \text{op}(\check{x}, \check{y}) &= i_* \text{up}(\text{up}(\check{x}, \check{x}), \text{up}(\check{x}, \check{y})) \\ &= i_* \{(\text{up}(\check{x}, \check{x}), \mathbb{1}), (\text{up}(\check{x}, \check{y}), \mathbb{1})\} \\ &= \{(i_* \text{up}(\check{x}, \check{x}), \mathbb{1}), (i_* \text{up}(\check{x}, \check{y}), \mathbb{1})\} \\ &= \{(\text{up}(\check{x}, \check{x}), \mathbb{1}), (\text{up}(\check{x}, \check{y}), \mathbb{1})\} \\ &= \text{op}(\check{x}, \check{y}). \end{aligned}$$

(iv): Say $\sigma = \bigcup \{\{\text{op}(\check{\rho}, \check{\sigma})\} \times A_{\rho\sigma} : \rho, \sigma \in \alpha\}$, with each $A_{\rho\sigma}$ an antichain in $\mathbb{P}$. Then

$$i_* \sigma = \bigcup \{\{i_* \text{op}(\check{\rho}, \check{\sigma})\} \times i[A_{\rho\sigma}] : \rho, \sigma \in \alpha\} = \bigcup \{\{\text{op}(\check{\rho}, \check{\sigma})\} \times i[A_{\rho\sigma}] : \rho, \sigma \in \alpha\},$$

with each $i[A_{\rho\sigma}]$ an antichain in $\mathbb{Q}$; this is a nice $\mathbb{Q}$-name for a subset of $\check{\alpha} \times \check{\alpha}$. $\square$

Theorem 31.66. (V.4.1) *Assume that κ is an uncountable regular cardinal and $2^{<\kappa} = \kappa$. Then there is a ccc forcing poset $\mathbb{P}$ such that $\mathbb{1}_{\mathbb{P}} \Vdash [MA \wedge 2^\omega = \kappa]$.*

Proof. Let $f : \kappa \to \kappa \times \kappa$ be the bijection given by the proof of Theorem 11.32; see Proposition 31.64.

We are going to define by recursion a finite support κ-stage iteration. The starting stage is trivial; $\mathbb{P} = \{\emptyset\}$. The limit stage is determined by the previous stages. Now we make the step from $\xi < \kappa$ to $\xi + 1$. We assume that for each $\zeta \leq \xi$ we have specified a sequence $\langle(\alpha^\mu_\zeta, \dot{\sqsubseteq}^\mu_\zeta) : \mu < \kappa\rangle$ listing all pairs $(\alpha, \dot{\sqsubseteq})$ such that $0 < \alpha < \kappa$ and $\dot{\sqsubseteq}$ is a nice $\mathbb{P}_\zeta$-name for a subset of $\alpha \times \alpha$. As the inductive hypothesis we assume that $|\mathrm{dmn}(\mathbb{Q}_\zeta)|, |\mathbb{P}_\zeta| < \kappa$ for each $\zeta < \xi$, and

$$(\langle(\mathbb{P}_\zeta, \leq_\zeta, \mathbb{1}_\zeta) : \zeta \leq \xi\rangle, \langle(\dot{\mathbb{Q}}_\zeta, \dot{\sqsubseteq}_{\dot{\mathbb{Q}}_\zeta}, \mathbb{1}_{\dot{\mathbb{Q}}_\zeta}) : \zeta < \xi\rangle)$$

is a finite support ξ-stage iterated forcing construction.

Let $f(\xi) = (\zeta, \mu)$. By Proposition 31.64, $\zeta \leq \xi$. Then $\dot{\sqsubseteq}^\mu_\zeta$ is a nice $\mathbb{P}_\zeta$-name for a subset of $\alpha^\mu_\zeta \times \alpha^\mu_\zeta$. Hence by Proposition 31.65, $(i^\xi_\zeta)_*(\dot{\sqsubseteq}^\mu_\zeta)$ is a nice $\mathbb{P}_\xi$-name for a subset of $\alpha^\mu_\zeta \times \alpha^\mu_\zeta$. If $\mathrm{Ntbc}(\alpha^\mu_\zeta, (i^\xi_\zeta)_*(\dot{\sqsubseteq}^\mu_\zeta), \mathbb{P}_\xi)$, then we let $(\dot{\mathbb{Q}}_\xi, \dot{\leq}_{\dot{\mathbb{Q}}_\xi}, \dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\xi})$ be $(\check{\alpha}^\mu_\zeta, (i^\xi_\zeta)_*(\dot{\sqsubseteq}^\mu_\zeta), \check{0})$. If $\neg\mathrm{Ntbc}(\alpha^\mu_\zeta, (i^\xi_\zeta)_*(\dot{\sqsubseteq}^\mu_\zeta), \mathbb{P}_\xi)$, then we let $\dot{\mathbb{Q}}_\xi = \{(\emptyset, \mathbb{1}_{\mathbb{P}_\xi})\}$, $\dot{\leq}_{\dot{\mathbb{Q}}_\xi} = \emptyset$, and $\dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\xi} = \emptyset$.

This completes the construction of our finite support κ-stage iteration.

(1) $\forall \xi < \kappa[|\mathbb{P}_\xi| < \kappa \wedge |\dot{\mathbb{Q}}_\xi| < \kappa]$.

This is clear by induction, using the regularity of κ at the limit stages.

(2) $\forall \xi < \kappa[\mathbb{1}_{\mathbb{P}_\xi} \Vdash [\dot{\mathbb{Q}}_\xi \text{ is ccc}]]$.

This holds by definition of Ntbc.

Next, note that

(3) $\kappa^\omega = \kappa$.

In fact,

$$\kappa^\omega = |{}^\omega\kappa| \leq \sum_{\lambda<\kappa} \lambda^\omega \leq \sum_{\lambda<\kappa} 2^\lambda = 2^{<\kappa} = \kappa.$$

Let $\mathbb{P} = \mathbb{P}_\kappa$, and let G be $\mathbb{P}$-generic over M. Note that Proposition 29.22 holds if we replace $|\mathbb{P}| = \kappa$ by $|\mathbb{P}| \leq \kappa$. Hence using (3) we have

(4) $M[G] \models [2^\omega \leq \kappa]$.

Now if we prove that $MA(\theta)$ for every $\theta < \kappa$, then by Theorem 25.3 we will have $\kappa \leq 2^\omega$, so $2^\omega = \kappa$ by (4). Also, MA follows.

Now for each $\xi < \kappa$ let $\mathbb{P}'_\xi = i^\kappa_\xi[\mathbb{P}_\xi]$. Then for $\xi < \eta < \kappa$ we have $\mathbb{P}'_\xi \subseteq \mathbb{P}'_\eta \subseteq \mathbb{P}$.

Suppose that $\theta < \kappa$. Take a ccc forcing poset $\mathbb{Q}$ and a family $\mathscr{D}$ of dense subsets of $\mathbb{Q}$ with $|\mathscr{D}| \leq \theta$. By Lemma 25.57 we may assume that $|\mathbb{Q}| \leq \theta$. Then by Lemma 31.63 we get a $\mathbb{P}$-name $\dot{\sqsubseteq}$ and an $\alpha \leq \theta$ such that $\mathrm{Ntbc}(\alpha, \dot{\sqsubseteq}, \mathbb{P})$ holds and $(\alpha, \dot{\sqsubseteq}_G, \mathbb{P})$ is isomorphic to $\mathbb{Q}$. So we may assume that $\mathbb{Q} = (\alpha, \dot{\sqsubseteq}_G, 0)$. Let $\langle D^\nu : \nu < \theta\rangle$ enumerate $\mathscr{D}$. Thus $D^\nu \subseteq \alpha$ for each $\nu < \theta$. Let $\dot{D}^\nu$ be a nice $\mathbb{P}$-name for a subset of α such that $D^\nu = \dot{D}^\nu_G$. The names $\dot{\sqsubseteq}$ and $\dot{D}^\nu$ for $\nu < \theta$ altogether involve fewer than κ members of $\mathbb{P}$. Hence there exists a $\zeta < \kappa$ such that all of these names are $\mathbb{P}'_\zeta$-names. Let $\dot{\sqsubseteq}'$ be a $\mathbb{P}_\zeta$-name such that $\dot{\sqsubseteq} = (i^\kappa_\zeta)_*(\dot{\sqsubseteq}')$. Then there is a $\mu < \kappa$ such that $(\alpha^\mu_\zeta, \dot{\sqsubseteq}^\mu_\zeta)$ is $(\alpha, \dot{\sqsubseteq}')$. Let $\xi = f^{-1}(\zeta, \mu)$. Now by Lemma 31.62 we get $\text{Ntbc}(\alpha, \dot{\sqsubseteq}, \mathbb{P}'_\xi)$. Hence $\text{Ntbc}(\alpha, (i^\xi_\zeta)_*(\dot{\sqsubseteq}'), \mathbb{P}_\xi)$. That is, $\text{Ntbc}(\alpha^\mu_\zeta, (i^\xi_\zeta)_*(\dot{\sqsubseteq}^\mu_\zeta), \mathbb{P}_\xi)$. Hence $(\dot{\mathbb{Q}}_\xi, \dot{\leq}_{\dot{\mathbb{Q}}_\xi}, \dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\xi}) = (\check{\alpha}^\mu_\zeta, (i^\xi_\zeta)_*(\dot{\sqsubseteq}^\mu_\zeta), \check{0})$ by construction. Note that

$$\dot{\sqsubseteq}_G = ((i^\kappa_\zeta)_*(\dot{\sqsubseteq}'))_G = (i^\kappa_\xi)_*((i^\xi_\zeta)_*(\dot{\subseteq}')) = ((i^\xi_\zeta)_*(\dot{\subseteq}'))_{G_\xi}.$$

Now we apply Proposition 31.60. Let $G_{\xi+1} = (i^\kappa_{\xi+1})^{-1}[G]$, f an isomorphism of $\mathbb{P}_{\xi+1}$ onto $\mathbb{P}_\xi * \dot{\mathbb{Q}}_\xi$, $G' = f[G_{\xi+1}]$, $H_\xi = \{\rho_{G_\xi} : \rho \in \mathbb{Q}_\xi \wedge \exists p \in \mathbb{P}_\xi[(p, \rho) \in G']\}$. Now by Proposition 31.60, H_ξ is $(\dot{\mathbb{Q}}_\xi)_{G_\xi}$-generic over $M[G_\xi]$ and $M[G'] = M[G_\xi][H_\xi]$. Let $\dot{D}^{\nu\prime}$ be a $\mathbb{P}_\xi$-name such that $\dot{D}^\nu = (i^\kappa_\xi)_*(\dot{D}^{\nu\prime})$. Then $D^\nu = \dot{D}^\nu_G = (\dot{D}^{\nu\prime})_{G_\xi} \in \mathbb{P}_\xi[G_\xi]$. Each D^ν is dense in $(\dot{\mathbb{Q}}_\xi)_{G_\xi}$, so $H_\xi \cap D^\nu \neq \emptyset$ for all $\nu < \theta$. □

A forcing poset $\mathbb{P}$ *has property* K iff $\forall E \in [\mathbb{P}]^{\omega_1} \exists L \in [E]^{\omega_1}[L$ is linked$]$.

Lemma 31.67. (V.4.9) *Assume that $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a forcing poset. Suppose that $\mathbb{P}$ has property* K *and $\mathbb{1} \Vdash [\dot{\mathbb{Q}}$ has property* K$]$. *Then $\mathbb{P} * \dot{\mathbb{Q}}$ has property K.*

Proof. Assume the hypotheses. Let $\langle (p_\xi, \dot{q}_\xi) : \xi < \omega_1 \rangle$ be a system of elements of $\mathbb{P} * \dot{\mathbb{Q}}$. Let $\dot{S}$ be the $\mathbb{P}$-name $\{(\check{\xi}, p_\xi) : \xi < \omega_1\}$. Let $\dot{F} = \{(\text{op}(\check{\xi}, \dot{q}_\xi), p_\xi) : \xi < \omega_1\}$. Thus for any generic G, $\dot{F}_G$ is the function with domain $\{\xi : p_\xi \in G\}$ such that $\dot{F}_G(\xi) = \dot{q}_{\xi G}$ for any ξ with $p_\xi \in G$.

(1) There is no $\beta < \omega_1$ such that $\mathbb{1} \Vdash [\dot{S} \subseteq \beta]$.

In fact, otherwise let G be generic with $p_\beta \in G$. Then $\beta \in \dot{S}_G \subseteq \beta$, contradiction.

Now clearly $\mathbb{1} \Vdash [\dot{S} \subseteq \omega_1]$. Hence by Lemma 31.47 we have $\mathbb{1} \not\Vdash [|\dot{S}| < \omega_1]$. So there is a p such that $p \Vdash [|\dot{S}| = \omega_1]$. Let G be generic with $p \in G$. Then

(2) $M[G] \models [|\dot{S}_G| = \omega_1]$.

Now $\dot{F}_G$ maps $\dot{S}_G$ into $\dot{\mathbb{Q}}_G$ and $\dot{\mathbb{Q}}_G$ has property K, so in $M[G]$ there is a set $B \in [\dot{S}_G]^{\omega_1}$ such that $\{\dot{F}_G(\xi) : \xi \in B\}$ is linked. Say $B = \dot{B}_G$. Take $p^* \in G$ with $p^* \leq p$ and

$$p^* \Vdash [\dot{B} \subseteq \dot{S} \text{ and } \{\dot{F}(\xi) : \xi \in \dot{B}\} \text{ is linked}].$$

In M let

$$A = \{\xi < \omega_1 : \exists q[(q \leq p^*) \wedge (q \leq p_\xi) \wedge q \Vdash [\xi \in \dot{B}]\}.$$

(3) $B \subseteq A$.

In fact, suppose that $\xi \in B$. Since $B \subseteq \dot{S}_G$, we have $\xi \in \dot{S}_G$ and hence $p_\xi \in G$. Also, there is an $r \in G$ such that $r \Vdash [\xi \in \dot{B}]$. Choose $q \in G$ so that $q \leq r, p^*, p_\xi$. Thus $\xi \in A$.

It follows that A is uncountable, as otherwise $A \subseteq \beta$ for some $\beta < \omega_1$, hence by (3) $B \subseteq \beta$, contradicting $|B| = \omega_1$. Now in M, for any $\xi \in A$ choose p'_ξ such that $p'_\xi \leq p^*, p_\xi$ and $p'_\xi \Vdash \xi \in \dot{B}$. Since A is uncountable and $\mathbb{P}$ has property K, choose $L \in [A]^{\omega_1}$ such that $\{p'_\xi : \xi \in L\}$ is linked.

Finally, to show that $\{(p_\xi, \dot{q}_\xi) : \xi \in L\}$ is linked, take any $\xi, \eta \in L$. Take any $p'' \leq p'_\xi, p'_\eta$. Then $p'' \Vdash [\xi \in \dot{B} \wedge \eta \in \dot{B}]$. Then $p'' \Vdash [\dot{q}_\xi \not\perp \dot{q}_\eta]$, so $p'' \Vdash \exists q'[(q' \leq$

$\dot{q}_\xi) \wedge (q' \le \dot{q}_\eta)]$. Hence by Lemma 30.22, there is a $p''' \le p''$ and a $q'' \in \mathrm{dmn}(\dot{\mathbb{Q}})$ such that $p''' \Vdash [(q'' \le \dot{q}_\xi) \wedge (q'' \le \dot{q}_\eta)]$. By the definition of the order on $\mathbb{P} * \dot{\mathbb{Q}}$, this shows that $(p''', q'') \le (p_\xi, \dot{q}_\xi), (p_\eta, \dot{q}_\eta)$. □

Lemma 31.68. (V.4.9) *Assume that $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a forcing poset. Suppose that $\mathbb{P}$ has pre-caliber ω_1 and $\mathbb{1} \Vdash [\dot{\mathbb{Q}}$ has pre-caliber $\omega_1]$. Then $\mathbb{P} * \dot{\mathbb{Q}}$ has pre-caliber ω_1.*

Proof. Assume the hypotheses. Let $\langle (p_\xi, \dot{q}_\xi) : \xi < \omega_1\rangle$ be a system of elements of $\mathbb{P} * \dot{\mathbb{Q}}$. Let $\dot{S}$ be the $\mathbb{P}$-name $\{(\check{\xi}, p_\xi) : \xi < \omega_1\}$. Let $\dot{F} = \{(\mathrm{op}(\check{\xi}, \dot{q}_\xi), p_\xi) : \xi < \omega_1\}$. Thus for any generic G, $\dot{F}_G$ is the function with domain $\{\xi : p_\xi \in G\}$ such that $\dot{F}_G(\xi) = \dot{q}_{\xi G}$ for any ξ with $p_\xi \in G$.

(1) There is no $\beta < \omega_1$ such that $\mathbb{1} \Vdash [\dot{S} \subseteq \beta]$.

In fact, otherwise let G be generic with $p_\beta \in G$. Then $\beta \in \dot{S}_G \subseteq \beta$, contradiction.

Now clearly $\mathbb{1} \Vdash [\dot{S} \subseteq \omega_1]$. Hence by Lemma 31.47 we have $\mathbb{1} \not\Vdash [|\dot{S}| < \omega_1]$. So there is a p such that $p \Vdash [|\dot{S}| = \omega_1]$. Let G be generic with $p \in G$. Then

(2) $M[G] \models [|\dot{S}_G| = \omega_1]$.

Now $\dot{F}_G$ maps $\dot{S}_G$ into $\dot{\mathbb{Q}}_G$ and $\dot{\mathbb{Q}}_G$ has pre-caliber ω_1, so in $M[G]$ there is a set $B \in [\dot{S}_G]^{\omega_1}$ such that $\{\dot{F}_G(\xi) : \xi \in B\}$ is centered. Say $B = \dot{B}_G$. Take $p^* \in G$ with $p^* \le p$ and

$$p^* \Vdash [\dot{B} \subseteq \dot{S} \text{ and } \{\dot{F}(\xi) : \xi \in \dot{B}\} \text{ is centered}].$$

In M let

$$A = \{\xi < \omega_1 : \exists q[(q \le p^*) \wedge (q \le p_\xi) \wedge q \Vdash [\xi \in \dot{B}]]\}.$$

(3) $B \subseteq A$.

In fact, suppose that $\xi \in B$. Since $B \subseteq \dot{S}_G$, we have $\xi \in \dot{S}_G$ and hence $p_\xi \in G$. Also, there is an $r \in G$ such that $r \Vdash [\xi \in \dot{B}]$. Choose $q \in G$ so that $q \le r, p^*, p_\xi$. Thus $\xi \in A$.

It follows that A is uncountable, as otherwise $A \subseteq \beta$ for some $\beta < \omega_1$, hence by (3) $B \subseteq \beta$, contradicting $|B| = \omega_1$. Now in M, for any $\xi \in A$ choose p'_ξ such that $p'_\xi \le p^*, p_\xi$ and $p'_\xi \Vdash \xi \in \dot{B}$. Since A is uncountable and $\mathbb{P}$ has pre-caliber ω_1, choose $L \in [A]^{\omega_1}$ such that $\{p'_\xi : \xi \in L\}$ is centered.

Finally, to show that $\{(p_\xi, \dot{q}_\xi) : \xi \in L\}$ is centered, take any finite subset F of L. Take any $p'' \le p'_\xi$ for all $\xi \in L$. Then $p'' \Vdash \forall \xi \in F[\xi \in \dot{B}]$. Then $p'' \Vdash \exists q' \forall \xi \in F[(q' \le \dot{q}_\xi)]$. Hence by Lemma 30.22, there is a $p''' \le p''$ and a $q'' \in \mathrm{dmn}(\dot{\mathbb{Q}})$ such that $p''' \Vdash \forall \xi \in F[(q'' \le \dot{q}_\xi)]$. By the definition of the order on $\mathbb{P} * \dot{\mathbb{Q}}$, this shows that $(p''', q'') \le (p_\xi, \dot{q}_\xi)$ for all $\xi \in F$. □

Proposition 31.69. *Suppose that a κ-supported α-stage iterated forcing construction is given. For each $\xi \le \alpha$ let $\mathbb{P}'_\xi = \{i^\alpha_\xi(p) : p \in \mathbb{P}_\xi\}$. Then*

(i) If $\eta < \xi \le \alpha$ then $\mathbb{P}'_\eta \subseteq_c \mathbb{P}'_\xi$.

(ii) If $\xi \le \alpha$ is a limit ordinal, κ is regular, and $\kappa \le \mathrm{cf}(\xi)$, then $\mathbb{P}'_\xi = \bigcup_{\eta<\xi} \mathbb{P}'_\eta$.

(iii) If $\xi \le \alpha$ is a limit ordinal and $\kappa = \omega$, then $\mathbb{P}'_\xi = \bigcup_{\eta<\xi} \mathbb{P}'_\eta$.

Proof. (i): Assume that $\eta < \xi \le \alpha$. Clearly $\mathbb{P}'_\eta$ is a subposet of $\mathbb{P}'_\xi$. To show that $\mathbb{P}'_\eta \subseteq_{\rm ctr} \mathbb{P}'_\xi$, suppose that $F \in [\mathbb{P}'_\eta]^{<\omega}$, $p \in \mathbb{P}'_\xi$, and $\forall q \in F[p \le q]$; see the definition of $\subseteq_{\rm ctr}$ following Proposition 25.54. Say $\forall q \in F[q = i^\alpha_\eta(q')]$, and $p = i^\alpha_\xi(p')$. Now $\forall u, v \in \mathbb{P}_\eta[u \le v$ iff $i^\xi_\eta(u) \le i^\xi_\eta(v)]$. It follows from Proposition 25.70 that

(1) $i^\xi_\eta[\mathbb{P}_\eta] \subseteq_{\rm c} \mathbb{P}_\xi$.

Now $\forall q \in F[p' \le i^\xi_\eta(q')]$, so by (1) there is an $r \in \mathbb{P}_\eta$ such that $\forall q \in F[r \le i^\xi_\eta(q')]$. Hence $\forall q \in F[i^\alpha_\eta(r) \le q]$. This shows that $\mathbb{P}'_\eta \subseteq_{\rm ctr} \mathbb{P}'_\xi$.

Now suppose that $A \subseteq \mathbb{P}_\eta$ and $i^\alpha_\eta[A]$ is a maximal antichain in $\mathbb{P}'_\eta$. Then clearly A is a maximal antichain in $\mathbb{P}_\eta$. Since i^ξ_η is an isomorphism from $\mathbb{P}_\eta$ onto $i^\xi_\eta[\mathbb{P}_\eta]$, it follows that $i^\xi_\eta[A]$ is a maximal antichain in $i^\xi_\eta[\mathbb{P}_\eta]$. Hence by (1), $i^\xi_\eta[A]$ is a maximal antichain in $\mathbb{P}_\xi$. So $i^\alpha_\eta[A] = i^\alpha_\xi[i^\xi_\eta[A]]$ is a maximal antichain in $\mathbb{P}'_\xi$. This proves (i).

(ii) and (iii) are clear. □

MAK is the assertion that $MA_{\mathbb{P}}(\kappa)$ holds for all $\kappa < 2^\omega$ and all forcing posets $\mathbb{P}$ which have property K. MAP is the assertion that $MA_{\mathbb{P}}(\kappa)$ holds for all $\kappa < 2^\omega$ and all forcing posets $\mathbb{P}$ which have pre-caliber ω_1.

Proposition 31.70. *MA implies MAK.* □

Proposition 31.71. *MA implies MAP.* □

Proposition 31.72. *MAP implies that* $\mathfrak{p} = 2^\omega$.

Proof. Assume MAP, and suppose that $\mathbb{P}$ is σ-centered. Then by Proposition 25.41, $\mathbb{P}$ has ω_1 as a pre-caliber. Hence $MA_{\mathbb{P}}(\kappa)$ holds for all $\kappa < 2^\omega$ and all σ-centered forcing posets $\mathbb{P}$. Hence by Theorem 25.65, $\mathfrak{p} = 2^\omega$. □

Proposition 31.73. *If $\mathbb{Q} \subseteq_{\rm ctr} \mathbb{P}$ and $\mathbb{P}$ has ω_1 as a pre-caliber, then $\mathbb{Q}$ has ω_1 as a pre-caliber.*

Proof. Assume the hypotheses, and suppose that $\langle q_\xi : \xi < \omega_1\rangle$ is a system of elements of $\mathbb{Q}$. Let $A \in [\omega_1]^{\omega_1}$ be such that $\{q_\xi : \xi \in A\}$ is centered, in $\mathbb{P}$. By the definition of $\subseteq_{\rm ctr}$, $\{q_\xi : \xi \in A\}$ is centered, in $\mathbb{Q}$. □

$\mathrm{tbc}'(\alpha, \sqsubseteq)$ abbreviates the statement that α is a nonzero ordinal, $\sqsubseteq$ is a subset of $\alpha \times \alpha$, and $(\alpha, \sqsubseteq, 0)$ is a forcing poset having property K.

Proposition 31.74. *For any infinite cardinal θ, $MAK(\theta)$ holds iff $MA_{\mathbb{Q}}(\theta)$ holds for every poset $\mathbb{Q}$ of the form $(\alpha, \sqsubseteq, 0)$, where $\mathrm{tbc}'(\alpha, \sqsubseteq)$ and $\alpha \le \theta$.*

Proof. $\Rightarrow$: trivial. $\Leftarrow$: Assume the indicated condition, and suppose that $MA_{\mathbb{P}}(\theta)$ is false, where $\mathbb{P}$ has property K. By Lemma 25.57, there is a $\mathbb{Q} \subseteq_{\rm ctr} \mathbb{P}$ such that $MA_{\mathbb{Q}}(\theta)$ is false and $|\mathbb{Q}| \le \theta$. let f be a bijection of $\mathbb{Q}$ onto $|\mathbb{Q}|$ such that $f(\mathbb{1}_{\mathbb{Q}}) = 0$. Define $\xi \sqsubseteq \eta$ iff $\xi, \eta < |\mathbb{Q}|$ and $f^{-1}(\xi) \le_{\mathbb{Q}} f^{-1}(\eta)$. Then $\mathbb{R} \stackrel{\rm def}{=} (|\mathbb{Q}|, \sqsubseteq, 0)$ is a forcing poset, $\mathrm{tbc}'(|\mathbb{Q}|, \sqsubseteq)$, and $MA_{\mathbb{R}}(\theta)$ is false, contradiction. □

For a forcing poset $\mathbb{P}$, $\mathrm{Ntbc}'(\alpha,\dot{\sqsubseteq},\mathbb{P})$ holds iff $\dot{\sqsubseteq}$ is a nice name for a subset of $\alpha\times\alpha$ and $\mathbb{1}\Vdash_{\mathbb{P}}\mathrm{tbc}'(\alpha,\dot{\sqsubseteq})$.

Proposition 31.75. *Suppose that $\mathbb{P}_1$ has property K, $\mathbb{P}_0\subseteq_c\mathbb{P}_1$, and $\dot{\sqsubseteq}$ is a $\mathbb{P}_0$-name; then* $\mathrm{Ntbc}'(\alpha,\dot{\sqsubseteq},\mathbb{P}_1)$ *implies* $\mathrm{Ntbc}'(\alpha,\dot{\sqsubseteq},\mathbb{P}_0)$.

Proof. Assume that $\mathrm{Ntbc}'(\alpha,\dot{\sqsubseteq},\mathbb{P}_1)$ and $\neg\mathrm{Ntbc}'(\alpha,\dot{\sqsubseteq},\mathbb{P}_0)$ hold in M. Then there is a $p\in\mathbb{P}_0$ such that $p\Vdash_{\mathbb{P}_0}\neg\mathrm{tbc}'(\check\alpha,\dot{\sqsubseteq})$. Let G be $\mathbb{P}_1$-generic over M with $p\in G$. By Lemmas 30.2 and 30.3, $G\cap\mathbb{P}_0$ is $\mathbb{P}_0$-generic over M, and $M[G\cap\mathbb{P}_0]\subseteq M[G]$. Now by Lemma 30.3, $\dot{\sqsubseteq}_G=\dot{\sqsubseteq}_{G\cap\mathbb{P}_0}$. But $\mathrm{tbc}'(\alpha,\dot{\sqsubseteq}_G)$ holds in $M[G]$ because $\mathrm{Ntbc}'(\alpha,\dot{\sqsubseteq},\mathbb{P}_1)$, while $\mathrm{tbc}'(\alpha,\dot{\sqsubseteq}_{G\cap\mathbb{P}_0})$ does not hold in $M[G\cap\mathbb{P}_0]$ since $p\Vdash_{\mathbb{P}_0}\neg\mathrm{tbc}'(\check\alpha,\dot{\sqsubseteq})$. This contradicts the absoluteness of the formula $\mathrm{tbc}'(x,y)$. $\square$

Proposition 31.76. *Let M be a c.t.m. for ZFC. In M let $\mathbb{P}$ be a forcing poset with property K, and let θ be an infinite cardinal. Let G be $\mathbb{P}$-generic over M. In $M[G]$ let $(X,\le_X,\mathbb{1}_X)$ be a ccc forcing poset with $|X|\le\theta$.*

Then there is a name $\dot{\sqsubseteq}$ in M and an $\alpha\le\theta$ such that $\mathrm{Ntbc}'(\alpha,\dot{\sqsubseteq}.\mathbb{P})$ *holds in M and such that in $M[G]$ the poset $(\alpha,\dot{\sqsubseteq}_G,0)$ is isomorphic to $(X,\le_X,\mathbb{1}_X)$.*

Proof. In $M[G]$ let $\alpha=|X|$. Let f be a bijection from X onto α which takes $\mathbb{1}_X$ to 0. Define $\xi\sqsubseteq\eta$ iff $\xi,\eta<\alpha$ and $f^{-1}(\xi)\le_X f^{-1}(\eta)$. Thus f is an isomorphism from $(X,\le_X,\mathbb{1}_X)$ onto $(\alpha,\sqsubseteq,0)$.

Fix a $\mathbb{P}$-name τ over M with $\tau_G=\sqsubseteq$. Let $\dot{\mathbb{Q}}=\check\alpha$ and $\dot{\mathbb{1}}_{\mathbb{Q}}=\check 0$. Now

$$\mathbb{1}\Vdash\exists y[\mathrm{tbc}'(\alpha.y)\wedge[\mathrm{tbc}'(\alpha,\tau)\to y=\tau]].$$

In fact, let H be $\mathbb{P}$-generic over M. If $\mathrm{tbc}'(\alpha,\tau_H)$, then we can take $y=\tau_H$. If $\neg\mathrm{tbc}'(\alpha,\tau_H)$, then we can take $y=\alpha\times\alpha$. By the maximal principle, Theorem 30.35, there is a name σ such that

$$\mathbb{1}\Vdash[\mathrm{tbc}'(\alpha.\sigma)\wedge[\mathrm{tbc}'(\alpha,\tau)\to\sigma=\tau]]. \tag{1}$$

By Lemma 29.21 let $\dot{\sqsubseteq}_{\mathbb{Q}}$ be a nice name for a subset of $(\alpha\times\alpha)\check{}$ such that

$$\mathbb{1}\Vdash[\sigma\subseteq(\alpha\times\alpha)\check{}\to\sigma=\dot{\sqsubseteq}_{\mathbb{Q}}]. \tag{2}$$

Now $\mathrm{tbc}'(\alpha,\tau_G)$, so by (1), $\sigma_G=\tau_G=\sqsubseteq\subseteq(\alpha\times\alpha)$. Hence by (2), $\sqsubseteq=\sigma_G=(\dot{\sqsubseteq}_{\mathbb{Q}})_G$. $\square$

Theorem 31.77. (V.4.12) *Assume that $\kappa>\omega$ is regular and $2^{<\kappa}=\kappa$. Then there is a forcing poset $\mathbb{P}_K$ of size κ such that $\mathbb{1}_{\mathbb{P}_\kappa}\Vdash[MAK\wedge 2^\omega=\kappa]$; and $\mathbb{P}_\kappa$ has property K.*

Proof. Let $f:\kappa\to\kappa\times\kappa$ be the bijection given by the standard proof that $\kappa=\kappa\cdot\kappa$; see the proof of Theorem 11.32 on page 143; and see Proposition 31.64.

We are going to define by recursion a finite support κ-stage iteration. The starting stage is trivial; $\mathbb{P}=\{\emptyset\}$. The limit stage is determined by the previous stages. Now we make the step from $\xi<\kappa$ to $\xi+1$. We assume that for each $\zeta\le\xi$ we have specified a sequence $\langle(\alpha^\mu_\zeta, \dot{\sqsubseteq}^\mu_\zeta) : \mu < \kappa\rangle$ listing all pairs $(\alpha, \dot{\sqsubseteq})$ such that $0 < \alpha < \kappa$ and $\dot{\sqsubseteq}$ is a nice $\mathbb{P}_\zeta$-name for a subset of $\alpha \times \alpha$. As the inductive hypothesis we assume that $|\mathrm{dmn}(\mathbb{Q}_\zeta|, |\mathbb{P}_\zeta| < \kappa$ for each $\zeta < \xi$, and

$$(\langle(\mathbb{P}_\zeta, \leq_\zeta, \mathbb{1}_\zeta) : \zeta \leq \xi\rangle, \langle(\dot{\mathbb{Q}}_\zeta, \dot{\sqsubseteq}_{\dot{\mathbb{Q}}_\zeta}, \mathbb{1}_{\dot{\mathbb{Q}}_\zeta}) : z < \xi\rangle)$$

is a finite support ξ-stage iterated forcing construction.

Let $f(\xi) = (\zeta, \mu)$. By Proposition 31.64, $\zeta \leq \xi$. Then $\dot{\sqsubseteq}^\mu_\zeta$ is a nice $\mathbb{P}_\zeta$-name for a subset of $\alpha^\mu_\zeta \times \alpha^\mu_\zeta$. Hence by Lemma 30.3, and Lemma 31.55, $(i^\xi_\zeta)_*(\dot{\sqsubseteq}^\mu_\zeta)$ is a nice $\mathbb{P}_\xi$-name for a subset of $\alpha^\mu_\zeta \times \alpha^\mu_\zeta$. If $\mathrm{Ntbc}'(\alpha^\mu_\zeta, (i^\xi_\zeta)_*(\dot{\sqsubseteq}^\mu_\zeta), \mathbb{P}_\xi)$, then we let $(\dot{\mathbb{Q}}_\xi, \dot{\leq}_{\dot{\mathbb{Q}}_\xi}, \dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\xi})$ be $(\check{\alpha}^\mu_\zeta, (i^\xi_\zeta)_*(\dot{\sqsubseteq}^\mu_\zeta), \check{0})$. If $\neg\mathrm{Ntbc}'(\alpha^\mu_\zeta, (i^\xi_\zeta)_*(\dot{\sqsubseteq}^\mu_\zeta), \mathbb{P}_\xi)$, then we let $\dot{\mathbb{Q}}_\xi = \{(\emptyset, \mathbb{1}_{\mathbb{P}_\xi})\}$, $\dot{\leq}_{\dot{\mathbb{Q}}_\xi} = \emptyset$, and $\dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\xi} = \emptyset$.

This completes the construction of our finite support κ-stage iteration.

(1) $\forall\xi < \kappa[|\mathbb{P}_\xi| < \kappa \wedge |\dot{\mathbb{Q}}_\xi| < \kappa]$.

This is clear by induction, using the regularity of κ at the limit stages.

(2) $\forall\xi \leq \kappa[\mathbb{1}_{\mathbb{P}_\xi} \Vdash [\dot{\mathbb{Q}}_\xi$ has property K$]]$.

This holds by definition of Ntbc$'$.

Next, note that

(3) $\kappa^\omega = \kappa$.

In fact,

$$\kappa^\omega = |{}^\omega\kappa| \leq \sum_{\lambda<\kappa} \lambda^\omega \leq \sum_{\lambda<\kappa} 2^\lambda = 2^{<\kappa} = \kappa.$$

Let $\mathbb{P} = \mathbb{P}_\kappa$, and let G be $\mathbb{P}$-generic over M. Note that Lemma 29.22 holds if we replace $|\mathbb{P}| = \kappa$ by $|\mathbb{P}| \leq \kappa$. Hence using (3) we have

(4) $M[G] \models [2^\omega \leq \kappa]$.

Now if we prove that $MA_{\mathbb{R}}(\theta)$ for every $\theta < \kappa$ and every forcing poset $\mathbb{R}$ having property K, then (1) $2^\omega = \kappa$ by the proof of Theorem 25.3; (2) MAK holds.

Now for each $\xi < \kappa$ let $\mathbb{P}'_\xi = i^\kappa_\xi[\mathbb{P}_\xi]$. Then for $\xi < \eta < \kappa$ we have $\mathbb{P}'_\xi \subseteq \mathbb{P}'_\eta \subseteq \mathbb{P}$.

Suppose that $\theta < \kappa$. Take a forcing poset $\mathbb{Q}$ with property K, and a family $\mathscr{D}$ of dense subsets of $\mathbb{Q}$ with $|\mathscr{D}| \leq \theta$. By Proposition 31.74 we may assume that $|\mathbb{Q}| \leq \theta$. Then by Proposition 31.76 we get a $\mathbb{P}$-name $\dot{\sqsubseteq}$ and an $\alpha \leq \theta$ such that $\mathrm{Ntbc}'(\alpha, \dot{\sqsubseteq}, \mathbb{P})$ holds and $(\alpha, \dot{\sqsubseteq}_G, \mathbb{P})$ is isomorphic to $\mathbb{Q}$. So we may assume that $\mathbb{Q} = (\alpha, \dot{\sqsubseteq}_G, 0)$. Let $\langle D^\nu : \nu < \theta\rangle$ enumerate $\mathscr{D}$. Thus $D^\nu \subseteq \alpha$ for each $\nu < \theta$. Let $\dot{D}^\nu$ be a nice $\mathbb{P}$-name for a subset of α such that $D^\nu = \dot{D}^\nu_G$. The names $\dot{\sqsubseteq}$ and $\dot{D}^\nu$ for $\nu < \theta$ altogether involve fewer than κ members of $\mathbb{P}$. Hence there exists a $\zeta < \kappa$ such that all of these names are $\mathbb{P}'_\zeta$-names. Let $\dot{\sqsubseteq}'$ be a $\mathbb{P}_\zeta$-name such that $\dot{\sqsubseteq} = (i^\kappa_\zeta)_*(\dot{\sqsubseteq}')$. Then there is a $\mu < \kappa$ such that $(\alpha^\mu_\zeta, \dot{\sqsubseteq}^\mu_\zeta)$ is $(\alpha, \dot{\sqsubseteq}')$. Let $\xi = f^{-1}(\zeta, \mu)$. Now by Proposition 31.75 we get

$\mathrm{Ntbc}'(\alpha,\dot{\sqsubseteq},\mathbb{P}'_\xi)$. Hence $\mathrm{Ntbc}'(\alpha,(i^\xi_\zeta)_*(\dot{\sqsubseteq}'),\mathbb{P}_\xi)$. That is, $\mathrm{Ntbc}'(\alpha^\mu_\zeta,(i^\xi_\zeta)_*(\dot{\sqsubseteq}^\mu_\zeta),\mathbb{P}_\xi)$. Hence $(\dot{\mathbb{Q}}_\xi,\dot{\preceq}_{\dot{\mathbb{Q}}_\xi},\dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\xi})=(\check{\alpha}^\mu_\zeta,(i^\xi_\zeta)_*(\dot{\sqsubseteq}^\mu_\zeta),\check{0})$ by construction. Note that

$$\dot{\sqsubseteq}_G=((i^\kappa_\zeta)_*(\dot{\sqsubseteq}'))_G=(i^\kappa_\xi)_*((i^\xi_\zeta)_*(\dot{\subseteq}'))=((i^\xi_\zeta)_*(\dot{\subseteq}'))_{G_\xi}.$$

Now we apply Proposition 31.60. Let $G_{\xi+1}=(i^\kappa_{\xi+1})^{-1}[G]$, f an isomorphism of $\mathbb{P}_{\xi+1}$ onto $\mathbb{P}_\xi*\dot{\mathbb{Q}}_\xi$, $G'=f[G_{\xi+1}]$, $H_\xi=\{\rho_{G_\xi}:\rho\in\mathbb{Q}_\xi\wedge\exists p\in\mathbb{P}_\xi[(p,\rho)\in G']\}$. Now by Proposition 31.60, H_ξ is $(\dot{\mathbb{Q}}_\xi)_{G_\xi}$-generic over $M[G_\xi]$ and $M[G']=M[G_\xi][H_\xi]$. Let $\dot{D}^{\nu\prime}$ be a $\mathbb{P}_\xi$-name such that $\dot{D}^\nu=(i^\kappa_\xi)_*(\dot{D}^{\nu\prime})$. Then $D^\nu=\dot{D}^\nu_G=(\dot{D}^{\nu\prime})_{G_\xi}\in\mathbb{P}_\xi[G_\xi]$. Each D^ν is dense in $(\dot{\mathbb{Q}}_\xi)_{G_\xi}$, so $H_\xi\cap D^\nu\neq\emptyset$ for all $\nu<\theta$. $\square$

$\mathrm{tbc}''(\alpha,\sqsubseteq)$ abbreviates the statement that α is a nonzero ordinal, $\sqsubseteq$ is a subset of $\alpha\times\alpha$, and $(\alpha,\sqsubseteq,0)$ is a forcing poset with pre-caliber ω_1.

Proposition 31.78. *For any infinite cardinal θ, $MAP(\theta)$ holds iff $MA_{\mathbb{Q}}(\theta)$ holds for every poset $\mathbb{Q}$ of the form $(\alpha,\sqsubseteq,0)$, where* $\mathrm{tbc}''(\alpha,\sqsubseteq)$ *and $\alpha\le\theta$.*

Proof. $\Rightarrow$: trivial. $\Leftarrow$: Assume the indicated condition, and suppose that $MA_{\mathbb{P}}(\theta)$ is false, where $\mathbb{P}$ has pre-caliber ω_1. By Lemma 25.57, there is a $\mathbb{Q}\subseteq_{\mathrm{ctr}}\mathbb{P}$ such that $MA_{\mathbb{Q}}(\theta)$ is false and $|\mathbb{Q}|\le\theta$. let f be a bijection of $\mathbb{Q}$ onto $|\mathbb{Q}|$ such that $f(\mathbb{1}_{\mathbb{Q}})=0$. Define $\xi\sqsubseteq\eta$ iff $\xi,\eta<|\mathbb{Q}|$ and $f^{-1}(\xi)\le_{\mathbb{Q}}f^{-1}(\eta)$. Then $\mathbb{R}\stackrel{\mathrm{def}}{=}(|\mathbb{Q}|,\sqsubseteq,0)$ is a forcing poset, $\mathrm{tbc}''(|\mathbb{Q}|,\sqsubseteq)$, and $MA_{\mathbb{R}}(\theta)$ is false, contradiction. $\square$

For a forcing poset $\mathbb{P}$, $\mathrm{Ntbc}''(\alpha,\dot{\sqsubseteq},\mathbb{P})$ holds iff $\dot{\sqsubseteq}$ is a nice name for a subset of $\alpha\times\alpha$ and $\mathbb{1}\Vdash_{\mathbb{P}}\mathrm{tbc}''(\alpha,\dot{\sqsubseteq})$.

Proposition 31.79. *Suppose that $\mathbb{P}_1$ has pre-caliber ω_1, $\mathbb{P}_0\subseteq_c\mathbb{P}_1$, and $\dot{\sqsubseteq}$ is a $\mathbb{P}_0$-name; then* $\mathrm{Ntbc}''(\alpha,\dot{\sqsubseteq},\mathbb{P}_1)$ *implies* $\mathrm{Ntbc}''(\alpha,\dot{\sqsubseteq},\mathbb{P}_0)$.

Proof. Assume that $\mathrm{Ntbc}''(\alpha,\dot{\sqsubseteq},\mathbb{P}_1)$ and $\neg\mathrm{Ntbc}''(\alpha,\dot{\sqsubseteq},\mathbb{P}_0)$ hold in M. Then there is a $p\in\mathbb{P}_0$ such that $p\Vdash_{\mathbb{P}_0}\neg\mathrm{tbc}''(\check{\alpha},\dot{\sqsubseteq})$. Let G be $\mathbb{P}_1$-generic over M with $p\in G$. By Lemmas 30.2 and 30.3, $G\cap\mathbb{P}_0$ is $\mathbb{P}_0$-generic over M, and $M[G\cap\mathbb{P}_0]\subseteq M[G]$. Now by Lemma 30.3, $\dot{\sqsubseteq}_G=\dot{\sqsubseteq}_{G\cap\mathbb{P}_0}$. But $\mathrm{tbc}''(\alpha,\dot{\sqsubseteq}_G)$ holds in $M[G]$ because $\mathrm{Ntbc}''(\alpha,\dot{\sqsubseteq},\mathbb{P}_1)$, while $\mathrm{tbc}''(\alpha,\dot{\sqsubseteq}_{G\cap\mathbb{P}_0})$ does not hold in $M[G\cap\mathbb{P}_0]$ since $p\Vdash_{\mathbb{P}_0}\neg\mathrm{tbc}''(\check{\alpha},\dot{\sqsubseteq})$. This contradicts the absoluteness of the formula $\mathrm{tbc}''(x,y)$. $\square$

Proposition 31.80. *Let M be a c.t.m. for ZFC. In M let $\mathbb{P}$ be a forcing poset with pre-caliber ω_1 and let θ be an infinite cardinal. Let G be $\mathbb{P}$-generic over M. In $M[G]$ let $(X,\le_X,\mathbb{1}_X)$ be a forcing poset with pre-caliber ω_1 and with $|X|\le\theta$.*

Then there is a name $\dot{\sqsubseteq}$ in M and an $\alpha\le\theta$ such that $\mathrm{Ntbc}''(\alpha,\dot{\sqsubseteq}.\mathbb{P})$ *holds in M and such that in $M[G]$ the poset $(\alpha,\dot{\sqsubseteq}_G,0)$ is isomorphic to $(X,\le_X,\mathbb{1}_X)$.*

Proof. In $M[G]$ let $\alpha=|X|$. Let f be a bijection from X onto α which takes $\mathbb{1}_X$ to 0. Define $\xi\sqsubseteq\eta$ iff $\xi,\eta<\alpha$ and $f^{-1}(\xi)\le_X f^{-1}(\eta)$. Thus f is an isomorphism from $(X,\le_X,\mathbb{1}_X)$ onto $(\alpha,\sqsubseteq,0)$.

Fix a $\mathbb{P}$-name τ over M with $\tau_G = \sqsubseteq$. Let $\dot{\mathbb{Q}} = \check{\alpha}$ and $\dot{1\!\!1}_{\mathbb{Q}} = \check{0}$. Now

$$1\!\!1 \Vdash \exists y[\mathrm{tbc}''(\alpha.y) \wedge [\mathrm{tbc}''(\alpha, \tau) \to y = \tau]].$$

In fact, let H be $\mathbb{P}$-generic over M. If $\mathrm{tbc}'(\alpha, \tau_H)$, then we can take $y = \tau_H$. If $\neg\mathrm{tbc}'(\alpha, \tau_H)$, then we can take $y = \alpha \times \alpha$. By the maximal principle, Theorem 30.35, there is a name σ such that

$$(1) \qquad 1\!\!1 \Vdash [\mathrm{tbc}''(\alpha.\sigma) \wedge [\mathrm{tbc}''(\alpha, \tau) \to \sigma = \tau]].$$

By Lemma 29.21 let $\dot{\sqsubseteq}_{\mathbb{Q}}$ be a nice name for a subset of $(\alpha \times \alpha)\check{}$ such that

$$(2) \qquad 1\!\!1 \Vdash [\sigma \subseteq (\alpha \times \alpha)\check{} \to \sigma = \dot{\sqsubseteq}_{\mathbb{Q}}].$$

Now $\mathrm{tbc}''(\alpha, \tau_G)$, so by (1), $\sigma_G = \tau_G = \sqsubseteq \subseteq (\alpha \times \alpha)$. Hence by (2), $\sqsubseteq = \sigma_G = (\dot{\sqsubseteq}_{\mathbb{Q}})_G$. $\square$

Theorem 31.81. (V.4.12) *Assume that $\kappa > \omega$ is regular and $2^{<\kappa} = \kappa$. Then there is a forcing poset $\mathbb{P}_P$ of size κ such that $1\!\!1_{\mathbb{P}_P} \Vdash [MAP \wedge 2^\omega = \kappa]$ and $\mathbb{P}_P$ has pre-caliber ω_1.*

Proof. Let $f : \kappa \to \kappa \times \kappa$ be the bijection given by the standard proof that $\kappa = \kappa \cdot \kappa$; see the proof of Theorem 11.32; and see Proposition 31.64.

We are going to define by recursion a finite support κ-stage iteration. The starting stage is trivial; $\mathbb{P} = \{\emptyset\}$. The limit stage is determined by the previous stages. Now we make the step from $\xi < \kappa$ to $\xi + 1$. We assume that for each $\zeta \leq \xi$ we have specified a sequence $\langle(\alpha^\mu_\zeta, \dot{\sqsubseteq}^\mu_\zeta) : \mu < \kappa\rangle$ listing all pairs $(\alpha, \dot{\sqsubseteq})$ such that $0 < \alpha < \kappa$ and $\dot{\sqsubseteq}$ is a nice $\mathbb{P}_\zeta$-name for a subset of $\alpha \times \alpha$. As the inductive hypothesis we assume that $|\mathrm{dmn}(\mathbb{Q}_\zeta|, |\mathbb{P}_\zeta| < \kappa$ for each $\zeta < \xi$, and

$$(\langle(\mathbb{P}_\zeta, \leq_\zeta, 1\!\!1_\zeta) : \zeta \leq \xi\rangle, \langle(\dot{\mathbb{Q}}_\zeta, \dot{\sqsubseteq}_{\dot{\mathbb{Q}}_\zeta}, 1\!\!1_{\dot{\mathbb{Q}}_\zeta}) : z < \xi\rangle)$$

is a finite support ξ-stage iterated forcing construction.

Let $f(\xi) = (\zeta, \mu)$. By Proposition 31.64, $\zeta \leq \xi$. Then $\dot{\sqsubseteq}^\mu_\zeta$ is a nice $\mathbb{P}_\zeta$-name for a subset of $\alpha^\mu_\zeta \times \alpha^\mu_\zeta$. Hence by Lemma 30.3 and Lemma 31.55, $(i^\xi_\zeta)_*(\dot{\sqsubseteq}^\mu_\zeta)$ is a nice $\mathbb{P}_\xi$-name for a subset of $\alpha^\mu_\zeta \times \alpha^\mu_\zeta$. If $\mathrm{Ntbc}''(\alpha^\mu_\zeta, (i^\xi_\zeta)_*(\dot{\sqsubseteq}^\mu_\zeta), \mathbb{P}_\xi)$, then we let $(\dot{\mathbb{Q}}_\xi, \dot{\preceq}_{\dot{\mathbb{Q}}_\xi}, \dot{1\!\!1}_{\dot{\mathbb{Q}}_\xi})$ be $(\check{\alpha}^\mu_\zeta, (i^\xi_\zeta)_*(\dot{\sqsubseteq}^\mu_\zeta), \check{0})$. If $\neg\mathrm{Ntbc}''(\alpha^\mu_\zeta, (i^\xi_\zeta)_*(\dot{\sqsubseteq}^\mu_\zeta), \mathbb{P}_\xi)$, then we let $\dot{\mathbb{Q}}_\xi = \{(\emptyset, 1\!\!1_{\mathbb{P}_\xi})\}$, $\dot{\preceq}_{\dot{\mathbb{Q}}_\xi} = \emptyset$, and $\dot{1\!\!1}_{\dot{\mathbb{Q}}_\xi} = \emptyset$.

This completes the construction of our finite support κ-stage iteration.

(1) $\forall \xi < \kappa[|\mathbb{P}_\xi| < \kappa \wedge |\dot{\mathbb{Q}}_\xi| < \kappa]$.

This is clear by induction, using the regularity of κ at the limit stages.

(2) $\forall \xi \leq \kappa[1\!\!1_{\mathbb{P}_\xi} \Vdash [\dot{\mathbb{Q}}_\xi$ has pre-caliber $\omega_1]]$.

This holds by definition of Ntbc''.

Next, note that

(3) $\kappa^\omega = \kappa$.

In fact,

$$\kappa^\omega = |{}^\omega\kappa| \le \sum_{\lambda<\kappa} \lambda^\omega \le \sum_{\lambda<\kappa} 2^\lambda = 2^{<\kappa} = \kappa.$$

Let $\mathbb{P} = \mathbb{P}_\kappa$, and let G be $\mathbb{P}$-generic over M. Note that Proposition 29.22 holds if we replace $|\mathbb{P}| = \kappa$ by $|\mathbb{P}| \le \kappa$. Hence using (3) we have

(4) $M[G] \models [2^\omega \le \kappa]$.

Now if we prove that $MA_{\mathbb{R}}(\theta)$ for every $\theta < \kappa$ and every forcing poset $\mathbb{R}$ having pre-caliber ω_1, then (1) $2^\omega = \kappa$ by the proof of Theorem 25.3; (2) MAP holds.

Now for each $\xi < \kappa$ let $\mathbb{P}'_\xi = i^\kappa_\xi[\mathbb{P}_\xi]$. Then for $\xi < \eta < \kappa$ we have $\mathbb{P}'_\xi \subseteq \mathbb{P}'_\eta \subseteq \mathbb{P}$.

Suppose that $\theta < \kappa$. Take a forcing poset $\mathbb{Q}$ with pre-caliber ω_1. and a family $\mathscr{D}$ of dense subsets of $\mathbb{Q}$ with $|\mathscr{D}| \le \theta$. By Proposition 31.78 we may assume that $|\mathbb{Q}| \le \theta$. Then by Proposition 31.80 we get a $\mathbb{P}$-name $\dot{\sqsubseteq}$ and an $\alpha \le \theta$ such that $\mathrm{Ntbc}''(\alpha, \dot{\sqsubseteq}, \mathbb{P})$ holds and $(\alpha, \dot{\sqsubseteq}_G, \mathbb{P})$ is isomorphic to $\mathbb{Q}$. So we may assume that $\mathbb{Q} = (\alpha, \dot{\sqsubseteq}_G, 0)$. Let $\langle D^\nu : \nu < \theta\rangle$ enumerate $\mathscr{D}$. Thus $D^\nu \subseteq \alpha$ for each $\nu < \theta$. Let $\dot{D}^\nu$ be a nice $\mathbb{P}$-name for a subset of α such that $D^\nu = \dot{D}^\nu_G$. The names $\dot{\sqsubseteq}$ and $\dot{D}^\nu$ for $\nu < \theta$ altogether involve fewer than κ members of $\mathbb{P}$. Hence there exists a $\zeta < \kappa$ such that all of these names are $\mathbb{P}'_\zeta$-names. Let $\dot{\sqsubseteq}'$ be a $\mathbb{P}_\zeta$-name such that $\dot{\sqsubseteq} = (i^\kappa_\zeta)_*(\dot{\sqsubseteq}')$. Then there is a $\mu < \kappa$ such that $(\alpha^\mu_\zeta, \dot{\sqsubseteq}^\mu_\zeta)$ is $(\alpha, \dot{\sqsubseteq}')$. Let $\xi = f^{-1}(\zeta, \mu)$. Now by Proposition 31.79 we get $\mathrm{Ntbc}''(\alpha, \dot{\sqsubseteq}, \mathbb{P}'_\xi)$. Hence $\mathrm{Ntbc}''(\alpha, (i^\xi_\zeta)_*(\dot{\sqsubseteq}'), \mathbb{P}_\xi)$. That is, $\mathrm{Ntbc}''(\alpha^\mu_\zeta, (i^\xi_\zeta)_*(\dot{\sqsubseteq}^\mu_\zeta), \mathbb{P}_\xi)$. Hence $(\dot{\mathbb{Q}}_\xi, \dot{\le}_{\dot{\mathbb{Q}}_\xi}, \dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\xi}) = (\check{\alpha}^\mu_\zeta, (i^\xi_\zeta)_*(\dot{\sqsubseteq}^\mu_\zeta), \check{0})$ by construction. Note that

$$\dot{\sqsubseteq}_G = ((i^\kappa_\zeta)_*(\dot{\sqsubseteq}'))_G = (i^\kappa_\xi)_*((i^\xi_\zeta)_*(\dot{\subseteq}')) = ((i^\xi_\zeta)_*(\dot{\subseteq}'))_{G_\xi}.$$

Now we apply Proposition 31.60. Let $G_{\xi+1} = (i^\kappa_{\xi+1})^{-1}[G]$, f an isomorphism of $\mathbb{P}_{\xi+1}$ onto $\mathbb{P}_\xi * \dot{\mathbb{Q}}_\xi$, $G' = f[G_{\xi+1}]$, $H_\xi = \{\rho_{G_\xi} : \rho \in \mathbb{Q}_\xi \wedge \exists p \in \mathbb{P}_\xi[(p, \rho) \in G']\}$. Now by Proposition 31.60, H_ξ is $(\dot{\mathbb{Q}}_\xi)_{G_\xi}$-generic over $M[G_\xi]$ and $M[G'] = M[G_\xi][H_\xi]$. Let $\dot{D}^{\nu\prime}$ be a $\mathbb{P}_\xi$-name such that $\dot{D}^\nu = (i^\kappa_\xi)_*(\dot{D}^{\nu\prime})$. Then $D^\nu = \dot{D}^\nu_G = (\dot{D}^{\nu\prime})_{G_\xi} \in \mathbb{P}_\xi[G_\xi]$. Each D^ν is dense in $(\dot{\mathbb{Q}}_\xi)_{G_\xi}$, so $H_\xi \cap D^\nu \ne \emptyset$ for all $\nu < \theta$. $\square$

Lemma 31.82. (V.4.13) *If T is a Suslin tree and $\mathbb{P}$ has property K, then $\mathbb{1} \Vdash_{\mathbb{P}} [T$ is Suslin$]$.*

Proof. Assume the hypotheses, but suppose that $\mathbb{1} \nVdash_{\mathbb{P}} [T$ is Suslin$]$. Then there is a $p \in \mathbb{P}$ such that p forces T to have an uncountable chain or an uncountable antichain.

Case 1. p forces T to have an uncountable antichain. Thus $p \Vdash \exists f[f : \omega_1 \to T$ and f is one-one and $\mathrm{rng}(f)$ is an antichain$]$. By the maximal principle we get a name $\dot{f}$ such that $p \Vdash [\dot{f} : \omega_1 \to T$ and $\dot{f}$ is one-one and $\mathrm{rng}(\dot{f})$ is an antichain$]$. For each $\xi < \omega_1$ choose $q_\xi \le p$ and $x_\xi \in T$ such that $q_\xi \Vdash [\dot{f}(\xi) = \check{x}_\xi]$. By Property K, there is an $L \in [\omega_1]^{\omega_1}$ such that $\{q_\xi : \xi \in L\}$ is linked. So for distinct $\xi, \eta \in L$ we get $r \le q_\xi, q_\eta$ and $r \Vdash \check{x}_\xi \perp \check{x}_\eta$, hence $x_\xi \perp x_\eta$. This contradicts T being Suslin.

Case 2. p forces T to have an uncountable chain. This is treated similarly. $\square$

Corollary 31.83. (V.4.14) *Assume that there is a c.t.m. of ZFC. Then there is a generic extension $M[G]$ satisfying $\neg$SH plus MAK.*

Proof. Start with a model M of $V = L$. Then $M \models$GCH and in M there is a Suslin tree. Then apply Theorem 31.77 and Lemma 31.82. $\square$

Lemma 31.84. (V.4.15) *The amoeba order used in the proof of Theorem 25.12 is σ-linked.*

Proof. If $F \in [\mathbb{Q} \times \mathbb{Q}]^{<\omega}$ and δ is a positive rational, let

$$A_{F\delta} = \left\{ p \in \mathbb{P} : \mu\left(p \triangle \sum_{(a,b)\in F} (a,b) \right) \le \delta \wedge \mu(p) < \varepsilon - 5\delta \right\}.$$

We claim that $A_{F\delta}$ is linked. For, suppose that $p, q \in A_{F\delta}$ with $p \neq q$. Let $W = \sum_{(a,b)\in F}(a,b)$. Then

$$\begin{aligned}\mu(p \cup q) &= \mu((p \cup q) \cap W + \mu((p \cup q)\backslash W)\\ &\le \mu((p\backslash q) \cap W) + \mu((q\backslash p) \cap W) + \mu(p \cap q \cap W) + \mu(p\backslash W) + \mu(q\backslash W)\\ &\le \varepsilon - 5\delta + 4\delta < \varepsilon.\end{aligned}$$

So p and q are compatible by (1) in the proof of Theorem 25.12.

To show that $\mathbb{P}$ is the union of all of the sets $A_{F\delta}$. let $p \in \mathbb{P}$. Then $\mu(p) < \varepsilon$, so there is a positive δ such that $\mu(p) < \varepsilon - 5\delta$. By Lemma 18.98, there is an $F \in [\mathbb{Q}]^{<\omega}$ such that $\mu(p \triangle \sum_{(a,b)\in F}(a,b) < \delta$. Thus $p \in A_{F\delta}$. $\square$

Lemma 31.85. (V.4.15) *The amoeba order used in the proof of Theorem 25.1 has property K.* $\square$

Proposition 31.86. *If T is a Suslin tree and $\mathbb{P}$ has pre-caliber ω_1, then $\mathbb{1} \Vdash_{\mathbb{P}} [T$ is Suslin$]$.*

Proof. See the proof of Lemma 31.82. $\square$

Corollary 31.87. (V.4.16) *Assume CH in M, and suppose that $\kappa > \omega$ is regular and $2^{<\kappa} = \kappa$. Take the forcing poset $\mathbb{P}_P$ given by Theorem 31.81. Let G be $\mathbb{P}_P$-generic over M. Then $M[G] \models [MAP \wedge 2^\omega = \kappa]$, in $M[G]$ there is a Suslin tree, and* $\mathrm{cov}(\mathrm{null}) = \omega_1$.

Proof. $M[G] \models [MAP \wedge 2^\omega = \kappa]$ and $\mathbb{P}_P$ has pre-caliber ω_1 by Theorem 31.81. Hence in $M[G]$ there is a Suslin tree by Proposition 31.86. Recall that if $c \subseteq \mathbb{Q} \times \mathbb{Q}$, then $U_c = \bigcup_{(a,b)\in c}(a,b)$. By absoluteness, $(U_c)^M = (U_c)^{M[G]} \cap M$. and $\mu((U_c)^{M[G]}) = \mu((U_c)^M)$.

If $\vec{c} \subseteq \omega \times (\mathbb{Q} \times \mathbb{Q})$, for each $n \in \omega$ let $\vec{c}_n = \{(q, r) : (n, q, r) \in \vec{c}\}$. Let $B_{\vec{c}} = \bigcap_{n\in\omega} U_{\vec{c}_n}$. Again, $\mu(B_{\vec{c}}^M) = \mu(B_{\vec{c}}^{M[H]})$.

Now in M let $\mathbb{R} = \{r_\xi : \xi < \omega_1\}$, and for each $\alpha < \omega_1$ let $E_\alpha = \{r_\xi : \xi < \alpha\}$. Then E_α is countable, and hence is a null set. Now by Lemma 18.98 there is a $(\vec{c})^\alpha \subseteq \omega \times (\mathbb{Q} \times \mathbb{Q})$ such that $E_\alpha \subseteq B_{(\vec{c})^\alpha}$ and $\mu(B_{(\vec{c})^\alpha}) = 0$.

Now we work in $M[G]$. Now the proof will be finished when we prove

(*) $\mathbb{R} = \bigcup\{B_{(\vec{c})^\alpha} : \alpha < \omega_1\}$.

Suppose that (*) fails; say $x \in \mathbb{R}\backslash \bigcup\{B_{(\vec{c})^\alpha} : \alpha < \omega_1\}$. Fix $a \in \omega$ such that $-a \le x \le a$. Let $f(\alpha) = (\vec{c})^\alpha$ for all $\alpha < \omega_1$. Thus $f : \omega_1 \to \mathscr{P}(\omega \times (\mathbb{Q} \times \mathbb{Q}))$. Let $\dot{x}$ be a name such that $\dot{x}_G = x$. Fix $p \in \mathbb{P}_P$ such that

$$p \Vdash [\dot{x} \text{ is a real number and } -a \le \dot{x} \le a \text{ and } \forall\alpha < \omega_1[x \notin B_{f(\alpha)}]].$$

Now in M, for each $\alpha < \omega_1$ we have $p \Vdash [\dot{x} \notin B_{(\vec{c})^\alpha}]$, so $p \Vdash \exists n \in \omega[\dot{x} \notin U_{(c_n)^\alpha}]$. Hence there exist an $n(\alpha) \in \omega$ and a $p_\alpha \le p$ such that $p_\alpha \Vdash [\dot{x} \notin U_{(c_n)^\alpha}]$. Since $\mathbb{P}_P$ has pre-caliber ω_1, there is a $J \in [\omega_1]^{\omega_1}$ such that $\{p_\alpha : \alpha \in J\}$ is centered. Now $[-a, a] \subseteq \mathbb{R} = \bigcup_{\alpha\in J} E_\alpha \subseteq \bigcup_{\alpha\in J} U_{(\vec{c}_{n(\alpha)})^\alpha}$. Since $[-a, a]$ is compact, there is a finite $S \subseteq J$ such that $[-a, a] \subseteq \bigcup_{\alpha\in S} U_{(\vec{c}_{n(\alpha)})^\alpha}$. This last statement is absolute, so $\mathbb{1} \Vdash [[-a, a] \subseteq \bigcup_{\alpha\in S} U_{(\vec{c}_{n(\alpha)})^\alpha}]$. Now there is a $q \le p_\alpha$ for all $\alpha \in S$. Then $q \Vdash \dot{x} \notin U_{(\vec{c}_{n(\alpha)})^\alpha}$ for all $\alpha \in S$, so that $q \Vdash \dot{x} \notin [-a, a]$, contradicting $p \Vdash [-a \le \dot{x} \le a]$. □

Corollary 31.88. (V.4.16) *Assume CH in M and that $\kappa > \omega$ is regular and $2^{<\kappa} = \kappa$. Take the forcing poset $\mathbb{P}_P$ given by Theorem 31.81. Let G be $\mathbb{P}_P$-generic over M. Then $M[G] \models [MAP \wedge 2^\omega = \kappa]$, in $M[G]$ there is a Suslin tree, and* $\operatorname{add}(\text{null}) = \omega_1$.

Proof. See Corollary 31.87 and Lemma 18.9. □

Corollary 31.89. (V.4.16) *Assume CH in M and that $\kappa > \omega$ is regular and $2^{<\kappa} = \kappa$. Take the forcing poset $\mathbb{P}_P$ given by Theorem 31.81. Let G be $\mathbb{P}_P$-generic over M. Then $M[G] \models [MAP \wedge 2^\omega = \kappa]$, in $M[G]$ there is a Suslin tree, and $\mathfrak{p} = 2^\omega$.*

Proof. See Corollary 31.87 and Proposition 31.72. □

Proposition 31.90. *If $\mathfrak{p} = 2^\omega$, then* $\operatorname{non}(\text{null}) = 2^\omega$.

Proof. By Chapter 20 we have $\mathfrak{p} \le \mathfrak{t} \le \mathfrak{s} \le \operatorname{non}(\text{null})$. □

Proof.

Proposition 31.91. *If X is Lebesgue measurable with finite measure, then there is a G_δ Y such that $X \subseteq Y$ and $\mu(X) = \mu(Y)$.*

Proof. This is immediate from Lemma 18.98. □

Let M be a c.t.m. of ZFC, and fix $x \in \mathbb{R}$ (not necessarily in M). Then x is *random over* M iff $\forall\vec{c} \subseteq (\omega \times \mathbb{Q} \times \mathbb{Q})$ with $\vec{c} \in M$ and $\mu(B_{\vec{c}}) = 0$ we have $x \notin B^M_{\vec{c}}$.

Proposition 31.92. (V.4.19) *Let M be a c.t.m. of ZFC, and let $\mathbb{P} = \mathbb{MB}([0, 1], \mu)$ with μ Lebesgue measure. Let G be $\mathbb{P}$-generic over M. Then in $M[G]$ there is an $x \in [0, 1]$ such that $\forall a, b \in \mathbb{Q}[0 \le a < b \le 1$ and $[a, b] \in G$ imply that $x \in [a, b]]$.*

Proof. Recall that $\mathbb{P}$ consists of equivalence classes $[S]$ of measurable sets of positive measure modulo the ideal of measure 0 sets. $[S]$ and $[T]$ are compatible iff $\mu(S \cap T) > 0$. Hence G is centered. Moreover, $\{[a, b] : 0 \le a < b \le 1$ and $[[a, b]] \in G\}$ is centered.

This is a collection of closed sets in the compact space $[0,1]$, so there is a point x in the intersection. $\square$

Proposition 31.93. (V.4.19) *Continuing Proposition 31.92, the point x is unique.*

Proof. Suppose that also $\forall a, b \in \mathbb{Q}[0 \le a < b \le 1$ and $[a, b] \in G$ imply that $y \in [a, b]]$, with $x \neq y$. Say $x < y$. Let $D = \{[[a, b]] : a < b$ and $b - a < (y - x)/2\}$. Clearly D is dense in $\mathbb{P}$. So, choose $[[a, b]] \in D \cap G$. Then $x, y \in [a, b]$, but $y - x > b - a$, contradiction. $\square$

Proposition 31.94. (V.4.19) *Continuing Propositions 31.92, 31.93, the point x is random over M.*

Proof. Suppose that $\vec{c} \subseteq (\omega \times \mathbb{Q} \times \mathbb{Q})$, $\vec{c} \in M$, and $\mu(B_{\vec{c}}) = 0$. Let $D = \{[X] \in \mathbb{P} : \exists m[X \cap U_{\vec{c}_m} = \emptyset]\}$. We claim that D is dense in $\mathbb{P}$. For, let $p \in \mathbb{P}$. Say $p = [X]$. Then $\mu(X) = \mu(X \cap B_{\vec{c}}) + \mu(X \backslash B_{\vec{c}}) = \mu(X \backslash B_{\vec{c}})$. Now

$$\mu(X \backslash B_{\vec{c}}) = \mu\left(X \cap \bigcup_{m \in \omega} ([0,1] \backslash U_{\vec{c}_m})\right) \le \sum_{m \in \omega} \mu(X \cap ([0,1] \backslash U_{\vec{c}_m}));$$

it follows that there is an $m \in \omega$ such that $\mu(X \cap ([0,1] \backslash U_{\vec{c}_m})) \neq 0$. Hence $X \cap ([0,1] \backslash U_{\vec{c}_m}) \in D$, as desired.

Now choose $[X] \in D \cap G$. Say $X \cap U_{\vec{c}_m} = \emptyset$. It follows that $X \cap B_{\vec{c}} = \emptyset$, and so $x \notin B_{\vec{c}}$. $\square$

The x given in Propositions 31.92, 31.93 is denoted by $x(G)$.

Proposition 31.95. (V.4.20) *Using the notation of Proposition 31.92, suppose that $y \in [0,1]$ is random over M. Then there is a $\mathbb{P}$-generic filter G over M such that $y = x(G)$.*

Proof. Let $G = \{B^M_{\vec{c}} : \mu(B^M_{\vec{c}}) > 0$ and $y \in B^M_{\vec{c}}\}$. G is closed under intersection, since if $B^M_{\vec{c}}, B^M_{\vec{d}} \in G$ then $y \in B^M_{\vec{c}} \cap B^M_{\vec{d}}$, while if $\mu(B^M_{\vec{c}} \cap B^M_{\vec{d}}) = 0$, say $B^M_{\vec{c}} \cap B^M_{\vec{d}} = B^M_{\vec{e}}$; then $y \in B^M_{\vec{e}}$, contradiction. Clearly G is closed upwards. Suppose that D is dense. Then $\mu([0,1] \backslash \bigcup D) = 0$, so $y \in \bigcup D$. Say $y \in a \in D$. Then also $a \in G$.

Clearly $x(G) = y$. $\square$

Proposition 31.96. (V.4.21) *Let $\mathbb{P}$ be a ccc forcing poset and let κ be regular. Let T be a κ-tree.*

If T is an ω_1-Aronszajn tree and no subtree of T is a Suslin tree, then $\mathbb{1} \Vdash [\check{T}$ is Aronszajn$]$.

Proof. Suppose not. Then there is a $p \in \mathbb{P}$ such that $p \Vdash [\check{T}$ is not Aronszajn$]$. If $p \in G$, then T is still an ω_1-tree in $M[G]$. Hence there must be an uncountable chain C in $M[G]$. Let $\dot{C}$ be a name and $q \le p$ such that $q \Vdash [\dot{C}$ is a chain through $T]$. Let $S = \{x \in T : \exists r \le q[r \Vdash \check{x} \in \dot{C}]\}$. Clearly S is a subtree of T. Hence S has height κ and there is no chain through S. Suppose that K is an uncountable antichain in S. For each $x \in K$ let $r_x \le q$ be such that $r_x \Vdash [\check{x} \in \dot{C}]$. Since $\mathbb{P}$ is ccc, there exist distinct $x, y \in K$ such that r_x and r_y are compatible. Say $s \le r_x, r_y$. Then $s \Vdash [\check{x}, \check{y} \in \dot{C}]$, hence

$s \Vdash [\check{x} \leq \check{y} \text{ or } \check{y} \leq \check{x}]$, hence x and y are comparable, contradiction. Thus S does not have an uncountable antichain, so it is Suslin, contradiction. $\square$

Proposition 31.97. (V.4.21) *Let $\mathbb{P}$ be a ccc forcing poset and let κ be regular. Let T be a κ-tree.*

If T is a Hausdorff κ-Aronszajn tree and $\kappa > \omega_1$, then $\mathbb{1} \Vdash [\check{T}$ is κ-Aronszajn$]$.

Proof. Suppose not, and let G be $\mathbb{P}$-generic over M such that T is not κ-Aronszajn. Clearly T is still a κ-tree, so there is a chain through T; say $f : \kappa \to T$ with $f(\alpha) \in \mathrm{Lev}_\alpha(T)$ for each $\alpha < \kappa$ and $f(\alpha) < f(\beta)$ for $\alpha < \beta$. For each $b \in T$ and each $\beta < \mathrm{height}(b)$ let b_β be the element of height β which is less than b. Thus $f \in \prod_{\alpha<\kappa} L_\alpha(T)$. Let $\dot{f}$ be a name and $p \in G$ such that $p \Vdash \dot{f} \in \prod_{\alpha<\kappa} L_\alpha(T)$. For each $\alpha < \kappa$ let $F(\alpha) = \{b \in L_\alpha(T) : \exists q \leq p[q \Vdash \dot{f}(\alpha) = \check{b}]\}$. By the argument for Theorem 29.4, $F(\alpha)$ is countable and $f(\alpha) \in F(\alpha)$ for all $\alpha < \kappa$. If $b \in X_\alpha$, $q \leq p$, and $q \Vdash [\dot{f}(\alpha) = \check{b}]$, and $\beta < \alpha$, then $q \Vdash [\dot{f}(\beta) = \check{b}_\beta]$. Let $\Delta = \{\alpha < \kappa : \mathrm{cf}(\alpha) = \omega_1\}$.

(1) $\forall \alpha \in \Delta \forall s \in F(\alpha) \exists \beta < \alpha[\{t_\gamma : \beta \leq \gamma < \alpha, s_\beta < t \in F(\alpha)\}$ is a chain$]$.

For, suppose not; say $\alpha \in \Delta$, $s \in F(\alpha)$, and for all $\beta < \alpha$ the set $\{t_\gamma : \beta \leq \gamma < \alpha, s_\beta < t \in F(\alpha)\}$ is not a chain.

(2) $\forall \beta < \alpha \exists \gamma \in [\beta, \alpha) \exists t \in F(\alpha)[s_\gamma < t \neq s \wedge s_{\gamma+1} \not\leq t]$.

In fact, say $t, t' \in F(\alpha)$, $\gamma, \gamma' \in [\beta, \alpha)$, $s_\beta < t, t'$, $t_\gamma, t'_{\gamma'}$ incomparable. So $t \neq t'$. Say $t \neq s$. Choose $\delta \in (\beta, \alpha)$ such that $s_\delta < t$ and $s_{\delta+1} \not\leq t$. So (2) holds.

Now we use (2) to construct by recursion two sequences $\langle \gamma_\xi : \xi < \omega_1 \rangle$ and $\langle t_\xi : \xi < \omega_1 \rangle$. Suppose that these have been defined for all $\xi < \eta$, where $\eta < \omega_1$, so that each $\gamma_\xi < \alpha$. Let $\delta = \bigcup_{\xi<\eta} \gamma_\xi$. So $\delta < \alpha$ since $\mathrm{cf}(\alpha) = \omega_1$. By (2), choose $\gamma_\eta \in [\delta + 1, \alpha)$ and $t_\eta \in \mathrm{Lev}_\alpha(T)$ such that $s_{\gamma_\eta} < t_\eta \neq s$ and $s_{\gamma_\eta+1} \not\leq t_\eta$. Since $F(\alpha)$ has size less than ω_1, there exist ξ, η with $\xi < \eta$ and $t_\xi = t_\eta$. Then $s_{\gamma_\xi+1} \leq s_{\gamma_\eta} < t_\eta = t_\xi$, contradiction. Hence (1) holds.

(3) For every $\alpha \in \Delta$ there is a $\beta < \alpha$ such that for each $s \in F(\alpha)$ the set $\{t_\gamma : s_\beta \leq t \in F(\alpha),\ \beta \leq \gamma < \alpha\}$ is a chain.

To prove this, let $\alpha \in \Delta$. By (1), for each $s \in F(\alpha)$ choose $\gamma_s < \alpha$ such that the set $\{t_\delta : s_{\gamma_s} \leq t \in F(\alpha),\ \gamma_s \leq \delta < \alpha\}$ is a chain. Let $\beta = \sup_{\mathrm{ht}(s)=\alpha} \gamma_s$. Clearly β is as desired in (3).

Now for each $\alpha \in \Delta$ choose $h(\alpha)$ to be a β as in (3). So h is a regressive function defined on the stationary set Δ. Hence there is a $\beta < \alpha$ such that $h^{-1}[\{\beta\}]$ is stationary, and hence of size κ.

(4) If $\alpha, \delta \in h^{-1}[\{\beta\}]$, $\alpha < \delta$, $s \in F(\alpha)$, then there is a $t \in F(\delta)$ such that $t_\alpha = s$.

In fact, say $q \leq p$ and $q \Vdash [\dot{f}(\alpha) = s]$. Also $q \Vdash \exists t \in \check{L}_\delta(T)[\dot{f}(\delta) = t]$, so there exist an $r \leq q$ and a $t \in L_\alpha(T)$ such that $r \Vdash [\dot{f}(\delta) = \check{t}]$. Thus $t \in F(\delta)$ and $t_\alpha = s$.

Now let α be the least member of $h^{-1}[\{\beta]$. Suppose that $s \in F(\alpha)$ and $s < t, u$ with $t \neq u$ and $t, u \in F(\gamma)$ with $\gamma \in h^{-1}[\{\beta\}]$. Then there are $t' < t$ and $u' < u$ with t' and u' incomparable. Choose $s' \in F(\gamma)$ such that $s < s'$, by (4). This contradicts (3).

By (4) there are κ many elements of T in some $F(\gamma)$ with $\alpha < \gamma \in h^{-1}[\{\beta\}]$, and they are all comparable. This gives a chain in T of size κ, contradiction. $\square$

Proposition 31.98. *If T is a κ-Suslin tree with $\omega_1 < \kappa$, then any subset of T of size κ contains an uncountable chain.*

Proof. Let $X \in [T]^\kappa$. We define a sequence $S_0, S_1, \ldots$ of subsets of X. Let S_0 be a subset of X which is a maximal tree with a single root under the order of T. If S_ξ has been defined for all $\xi < \eta$, and $X \setminus \bigcup_{\xi<\eta} S_\xi$ is nonempty, let S_η be a subset of $X \setminus \bigcup_{\xi<\eta} S_\xi$ which is a maximal tree with a single root under the order of T. This construction stops at some ordinal α, and $X = \bigcup_{\xi<\alpha} S_\xi$. By the maximality condition, the roots of the trees S_ξ form an incomparable set, and hence $\alpha < \kappa$. Hence some S_ξ has height κ, and this gives an uncountable chain. $\square$

Proposition 31.99. (V.4.21) *Let $\mathbb{P}$ be a ccc forcing poset and let κ be regular. Let T be a κ-tree.*

If T is a κ-Suslin tree and $\kappa > \omega_1$, then $\mathbb{1} \Vdash [\check{T}$ is κ-Suslin$]$.

Proof. Suppose to the contrary that in $M[G]$ there is a one-one function $f : \kappa \to T$ whose range is an antichain. Say $p \in \mathbb{P}$ and $\dot{f}_G = f$ and $p \Vdash [\dot{f}$ is a one-one function whose range is an antichain$]$. For each $\xi < \kappa$ choose $x_\xi \in T$ and $q_\xi \le p$ such that $q_\xi \Vdash [\dot{f}(\xi) = \check{x}_\xi]$. By Proposition 31.98 let $A \in [\kappa]^{\omega_1}$ be such that $\{x_\xi : \xi \in A\}$ is a chain. Choose ξ, η distinct members of A such that q_ξ and q_η are compatible. Say $r \le q_\xi, q_\eta$. Then $r \Vdash [\dot{f}(\xi)$ is incomparable with $\dot{f}(\eta)]$, hence x_ξ and x_η are incomparable, contradiction. $\square$

Proposition 31.100. (V.4.24) *Suppose that $\langle \mathbb{P}_\xi : \xi \le \omega \rangle$ is a finite support iterated forcing construction, $\lambda \ge 2$ is a cardinal, and $\forall n[\mathbb{1} \Vdash_{\mathbb{P}_n} [\dot{Q}_n$ has an antichain of size $\lambda]$. Then there is a $\mathbb{P}_\omega$-generic filter G such that in $M[G]$ there is a surjection from ω onto λ.*

Proof. For each $n \in \omega$ we have $\mathbb{1}_{\mathbb{P}_n} \Vdash [\dot{Q}_n$ has a maximal antichain of size at least $\lambda]$. Hence by the maximal principle, for each n there is a $\mathbb{P}_n$-name $\dot{A}_n$ such that $\mathbb{1}_{\mathbb{P}_n} \Vdash [\dot{A}_n$ is a maximal antichain of size at least λ in $\dot{Q}_n]$. Again by the maximal principle, for each $n \in \omega$ there is a $\mathbb{P}_n$-name B such that

$$\begin{aligned}\mathbb{1}_{\mathbb{P}_n} \Vdash [&\dot{B} \text{ is a function with domain } \lambda \times \omega \text{ such that for each } n,\\ &\langle \dot{B}(\xi,\nu) : \xi < \lambda \rangle \text{ is a partition of } \dot{A}_n \text{ into nonempty subsets}].\end{aligned}$$

Now for each $\alpha < \lambda$ let $D_\alpha = \{p \in \mathbb{P}_\omega : \exists n \in \omega[p \restriction n \Vdash \exists q \in \dot{B}(\alpha, n) \exists r[r \le q, p_n]]\}$. We claim that D_α is dense in $\mathbb{P}_\omega$. For, let $s \in \mathbb{P}_\omega$. Choose n so that $\forall m \ge n[s_m = \mathbb{1}]$. Now $\mathbb{1}_{\mathbb{P}_n} \Vdash \exists q \in \dot{B}(\alpha, n)$. It follows that there is a $p \le s$ such that $\mathbb{P}_n \Vdash [p_n \in \dot{B}(\alpha, n)]$. Clearly $p \in D_\alpha$, as desired.

Now let G be $\mathbb{P}_\omega$-generic intersecting each set D_α. Take any $n \in \omega$. Let $G_n = (i_n^\omega)^{-1}[G]$. Then G_n is $\mathbb{P}_n$-generic over M, and hence $\langle \dot{B}_{G_n}(\xi, \nu) : \xi < \lambda \rangle$ is a partition of $\dot{A}_{nG_n}$ into nonempty subsets. Hence there is an $\alpha < \lambda$ such that $|G_n \cap B_{G_n}(\alpha, n)| = 1$. Clearly α is unique, and we let $f(n)$ be this α.

For any $\alpha < \lambda$ let $p \in G \cap D_\alpha$. Choose $n \in \omega$ so that $p \upharpoonright n \Vdash \exists q \in \dot{B}(\alpha, n)\exists r[r \leq q, p_n]$. Then $G_n \cap B_{G_n}(\alpha, n) \neq \emptyset$, so $f(n) = \alpha$. $\square$

Proposition 31.101. *Suppose that in M we are given an α-stage finite support iterated forcing construction*

$$(\langle(\mathbb{P}_\xi, \leq_\xi, \mathbb{1}_\xi) : \xi \leq \omega_1\rangle, \langle(\dot{\mathbb{Q}}_\xi, \dot{\preceq}_{\dot{\mathbb{Q}}_\xi}, \dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\xi}) : \xi < \alpha\rangle)$$

with α a limit ordinal. Suppose that G is $\mathbb{P}_\alpha$-generic over M, $S \in M$, $X \in M[G]$, $X \subseteq S$, and $(|S| < \mathrm{cf}(\alpha))^{M[G]}$. Then there is an $\eta < \alpha$ such that $X \in M[(i^\alpha_\eta)^{-1}[G]]$.

Proof. Clearly $\forall s \in S[s \in X \leftrightarrow \exists p \in G[p \Vdash_{\mathbb{P}_\alpha} \check{s} \in \dot{X}]$, where $\dot{X}$ is a $\mathbb{P}_\alpha$-name such that $\dot{X}_G = X$. Now $\mathbb{P}_\alpha = \bigcup_{\xi<\alpha} i^\alpha_\xi[\mathbb{P}_\xi]$ and $G = \bigcup_{\xi<\alpha} i^\alpha_\xi[(i^\alpha_\xi)^{-1}[G]]$. Let $G_\xi = (i^\alpha_\xi)^{-1}[G]$.

In $M[G]$, for each $s \in X$ there is a $\xi = \xi_s < \alpha$ such that $\exists p \in G_\xi[i^\alpha_\xi(p) \Vdash_{\mathbb{P}_\alpha} \check{s} \in \dot{X}]$. Let $\eta = \sup_{s\in X} \xi_s$. So $\eta < \alpha$ since $(|S| < \mathrm{cf}(\alpha))^{M[G]}$. Then $X = \{s \in S : \exists p \in G_\eta[i^\alpha_\eta(p) \Vdash \check{s} \in \dot{X}]\}$. Hence $X \in M[G_\eta]$. $\square$

Lemma 31.102. (V.4.27) *Let M be a c.t.m. of ZFC. In M let $\kappa = 2^\omega$. Then there is in M a ccc forcing poset $\mathbb{P}$ of size κ such that whenever G is $\mathbb{P}$-generic over M, in $M[G]$ we have $2^\omega = \kappa$ and there is a non-principal ultrafilter U on ω with character ω_1.*

Proof. We construct a finite support ω_1-stage iterated forcing construction

$$(\langle(\mathbb{P}_\xi, \leq_\xi, \mathbb{1}_\xi) : \xi \leq \omega_1\rangle, \langle(\dot{\mathbb{Q}}_\xi, \dot{\preceq}_{\dot{\mathbb{Q}}_\xi}, \dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\xi}) : \xi < \omega_1\rangle)$$

along with a sequences of $\mathbb{P}_\xi$-names $\langle \dot{U}_\xi : \xi < \omega_1\rangle$, and $\mathbb{P}_{\xi+1}$-names $\langle \dot{K}_\xi : \xi < \omega_1\rangle$, by recursion. Let $\mathbb{P}_0 = \{0\}$. Let V be a non-principal ultrafilter on ω in M, and let $\dot{U}_0 = \check{V}$. The construction of $\mathbb{P}_\xi$ for ξ limit is given by the finite support property. Also for ξ limit we have

$$\mathbb{1}_\xi \Vdash \exists U[U \text{ is an ultrafilter and } \forall x \forall \eta < \xi[x \in (i^\xi_\eta)_* \dot{U}_\eta \to x \in U]],$$

so by the maximal principle there is a $\mathbb{P}_\xi$-name $\dot{U}_\xi$ such that

$$\mathbb{1}_\xi \Vdash [\dot{U}_\xi \text{ is an ultrafilter and } \forall x \forall \eta < \xi[x \in (i^\xi_\eta)_* \dot{U}_\eta \to x \in \dot{U}_\xi]],$$

Now suppose that $\mathbb{P}_\xi$ and $\dot{U}_\xi$ have been defined. We now define $\dot{Q}_\xi$, $\dot{U}_{\xi+1}$, $\mathbb{P}_{\xi+1}$, and $\dot{K}_\xi$. We have

$$\mathbb{1}_{\mathbb{P}_\xi} \Vdash_\xi \exists Q \forall u[u \in Q \leftrightarrow \exists x, y[u = \mathrm{op}(x, y) \wedge x \in [\omega]^{<\omega} \wedge y \in [\dot{U}_\xi]^{<\omega}]].$$

By the maximal principle there is a name $\dot{Q}_\xi$ such that

$$\mathbb{1}_{\mathbb{P}_\xi} \Vdash_\xi \forall u[u \in \dot{Q}_\xi \leftrightarrow \exists x, y[u = \mathrm{op}(x, y) \wedge x \in [\omega]^{<\omega} \wedge y \in [\dot{U}_\xi]^{<\omega}]].$$

Also,

$$\mathbb{1}_{\mathbb{P}_\xi} \Vdash_\xi \exists R\forall u[u \in R \leftrightarrow \exists x, y, x', y'[u = \mathrm{op}(\mathrm{op}(x, y), \mathrm{op}(x', y'))\wedge \\ \mathrm{op}(x, y) \in \dot{Q}_\xi \wedge \mathrm{op}(x', y') \in \dot{Q}_\xi \wedge x' \subseteq x \wedge y' \subseteq y\wedge \\ \forall Z \in y'[x\backslash x' \subseteq Z]]].$$

Hence by the maximal principle there is a $\mathbb{P}_\xi$-name $\dot{\preceq}_{\dot{\mathbb{Q}}_\xi}$ such that

$$\mathbb{1}_{\mathbb{P}_\xi} \Vdash_\xi \forall u[u \in \dot{\preceq}_{\dot{\mathbb{Q}}_\xi} \leftrightarrow \exists x, y, x', y'[u = \mathrm{op}(\mathrm{op}(x, y), \mathrm{op}(x', y'))\wedge \\ \mathrm{op}(x, y) \in \dot{Q}_\xi \wedge \mathrm{op}(x', y') \in \dot{Q}_\xi \wedge x' \subseteq x \wedge y' \subseteq y\wedge \\ \forall Z \in y'[x\backslash x' \subseteq Z]]].$$

Let $\mathbb{1}_{\dot{\mathbb{Q}}_\xi} = \mathrm{op}(\emptyset, \emptyset)$. Now $(\dot{Q}_\xi, \dot{\preceq}_{\dot{\mathbb{Q}}_\xi}, \mathbb{1}_{\dot{Q}_\xi})$ is a $\mathbb{P}_\xi$-name for a forcing poset. This also defines $\mathbb{P}_{\xi+1}$. Next, let $\Gamma = \{(\check{p}, p) : p \in \mathbb{P}_{\xi+1}\}$ and $\Gamma' = \{(\check{p}, p) : p \in \mathbb{P}_\xi\}$. Then

$$\mathbb{1}_{\mathbb{P}_{\xi+1}} \Vdash_{\xi+1} \exists K\forall x[x \in K \leftrightarrow \exists p \in \Gamma\exists u, v[(i_\xi^{\xi+1})_* p_\xi \in (i_\xi^{\xi+1})_*\Gamma' \text{ and } p_\xi = \mathrm{op}(u, v) \wedge x \in u]].$$

Hence by the maximal principle there is a $\mathbb{P}_{\xi+1}$-name $\dot{K}_\xi$ such that

$$\mathbb{1}_{\mathbb{P}_{\xi+1}} \Vdash_{\xi+1} \forall x[x \in \dot{K}_\xi \leftrightarrow \exists p \in \Gamma\exists u, v[(i_\xi^{\xi+1})_* p_\xi \in (i_\xi^{\xi+1})_*\Gamma' \text{ and } p_\xi = \mathrm{op}(u, v) \wedge x \in u]].$$

Now

$$\mathbb{1}_{\mathbb{P}_{\xi+1}} \Vdash_{\xi+1} \exists U[U \text{ is an ultrafilter, and } ((i_\xi^{\xi+1})_*\dot{U}_\xi) \subseteq U \text{ and } \dot{K}_\xi \in U]$$

Hence by the maximal principle there is a $\mathbb{P}_{\xi+1}$-name $\dot{U}_{\xi+1}$ such that

$$\mathbb{1}_{\mathbb{P}_{\xi+1}} \Vdash_{\xi+1} [\dot{U}_{\xi+1} \text{ is an ultrafilter, and } ((i_\xi^{\xi+1})_*\dot{U}_\xi) \subseteq \dot{U}_{\xi+1} \text{ and } \dot{K}_\xi \in \dot{U}_{\xi+1}].$$

This finishes the construction.

(1) For each $\xi \leq \omega_1[\mathbb{1}_\xi \Vdash [\dot{Q}_\xi \text{ is ccc}]]$.

In fact, let G be $\mathbb{P}_\xi$-generic over M. Then $\dot{Q}_{\xi G}$ is the poset described in the proof of Lemma 25.34, and so it is ccc.

(2) $\mathbb{P}_{\omega_1}$ is ccc.

This follows from (1) and Lemma 31.58.

Now let G be $\mathbb{P}_{\omega_1}$-generic over M, and for each $\xi \leq \omega_1$ let $G_\xi = (i_\xi^{\omega_1})^{-1}[G]$. So by Lemma 30.3, G_ξ is $\mathbb{P}_\xi$-generic over M.

(3) $\forall\xi \leq \omega_1[\dot{U}_{\xi G_\xi}$ is a nonprincipal ultrafilter on $\omega]$.

We prove (3) by induction on ξ. It is given for $\xi = 0$. Assume that it is true for $\xi < \omega_1$. By the definition of $\dot{U}_{\xi+1}$ it follows that $\dot{U}_{\xi+1,G}$ is an ultrafilter on ω, and it is non principal since $V \subseteq \dot{U}_{\xi+1,G}$. The case of limit ξ is clear.

(4) $\forall\xi<\omega_1\forall x\in\dot U_{\xi G_\xi}[\dot K_{\xi G_{\xi+1}}\subseteq^* x]$.

In fact, let $\xi<\omega_1$. By Lemma 30.3, if $p\in\mathbb{P}_\xi$, then $((i_\xi^{\xi+1})_*p_\xi)_{G_{\xi+1}}=p_{\xi G_\xi}$ and $((i_\xi^{\xi+1})_*\Gamma')_{G_{\xi+1}}=\Gamma'_{G_\xi}=G_\xi$. Hence by the definition of $\dot K_\xi$ we have $\forall x[x\in\dot K_{\xi G_{\xi+1}}$ iff $\exists p\in G_{\xi+1}\exists u,v[p_{\xi G_\xi}\in G_\xi$ and $p_\xi=(u,v)$ and $x\in u]]$. Now let $f(p)=(p\restriction\xi,p_\xi)$ for all $p\in\mathbb{P}_{\xi+1}$. Then $f[G_{\xi+1}]$ is $(\mathbb{P}_\xi*\dot Q_\xi)$-generic over M. Let $j(p)=(p,\mathbb{1}_{\dot Q_\xi})$ for any $p\in\mathbb{P}_\xi$. Then j is a complete embedding of $\mathbb{P}_\xi$ into $(\mathbb{P}_\xi*\dot Q_\xi)$. Moreover, $f^{-1}(j(p))=i_\xi^{\xi+1}(p)$. We have $j^{-1}[f[G_{\xi+1}]]=(j^{-1}\circ f)[G_{\xi+1}]=(i_\xi^{\xi+1})^{-1}[G_{\xi+1}]=G_\xi$. Let $H=\{\dot q_{G_\xi}:\exists p[(p,\dot q)\in f[G_{\xi+1}]]\}$. By Proposition 31.60, H is $\dot Q_{\xi G_\xi}$-generic over $M[G_\xi]$. Moreover, $H=\{p_{\xi G_\xi}:p\in G_{\xi+1}\}$. Hence $x\in\dot K_{\xi G_{\xi+1}}$ iff $p_{\xi G_\xi}\in H$ and there are u,v such that $p_\xi=(u,v)$ and $x\in u$. By the proof of Lemma 25.34 it now follows that $\forall x\in\dot U_{\xi G_\xi}[\dot K_{\xi G_{\xi+1}}\subseteq^* x]$.

(5) If $\xi<\eta\le\omega_1$, then $\dot U_{\xi G_\xi}\subseteq\dot U_{\eta G_\eta}$.

We prove this by induction on η, with ξ fixed. It is trivial for $\xi=\eta$. Assume that it is true for η. Then

$$\dot U_{\xi G_\xi}\subseteq\dot U_{\eta G_\eta}=((i_\eta^{\eta+1})_*\dot U_\eta)_{G_{\eta+1}}\subseteq\dot G_{\eta+1,G_{\eta+1}}.$$

For η limit the conclusion is clear.

(6) $\dot U_{\omega_1 G_{\omega_1}}$ is generated by $\{X\subseteq\omega:\omega\backslash X\text{ is finite}\}\cup\{\dot K_{\xi G_\xi}:\xi<\omega_1\}$.

In fact, suppose that $X\in\dot U_{\omega_1 G_{\omega_1}}$. Then by Proposition 31.101 there is an $\eta<\omega_1$ such that $X\in M[G_\eta]$, and hence $X\in\dot U_{\eta G_\eta}$. Then $\dot K_{\eta G_\eta}\subseteq^* X$.

If follows that the character of $\dot U_{\omega_1 G_{\omega_1}}$ is ω_1; it is uncountable by Chapter 20.

By induction, $|\mathbb{P}_\xi|\le\kappa$ for all $\xi\le\omega_1$. Hence $2^\omega=\kappa$ in $M[G]$ by Proposition 29.22. $\square$

Proposition 31.103. (V.4.32) *MA implies that* cov(meager) = cov(null) = 2^ω.

Proof. See Proposition 25.40 and Chapter 19. $\square$

Proposition 31.104. (V.4.32) *It is consistent that* 2^ω *is large and* cov(meager) = 2^ω *and* cov(null) = ω_1

Proof. Use $\mathrm{Fn}(\kappa,2,\omega)$ in a model of CH. By Propositions 30.88 and 30.89, cov(null) = ω_1 in the extension. By Lemma 31.26, cov(meager) = 2^ω in the extension. $\square$

Proposition 31.105. (V.4.32) *It is consistent that* 2^ω *is large and* cov(meager) = cov(null) = ω_1

Proof. First get the continuum large, and then use Lemma 31.102; see Chapter 20. $\square$

Proposition 31.106. (V.4.37) *Let M be a ctm for $ZFC+GCH$. Then there is a cardinal preserving extension $M[G]$ of M satisfying $ZFC+MA+(2^\omega=\omega_2)+(2^{\omega_2}=\omega_3)$.*

Proof. First force with $\mathrm{Fn}(\omega_3, 2, \omega_2)$, obtaining M'. By Theorem 29.38, in M' we have $2^\omega = \omega_1$, $2^{\omega_1} = \omega_2$, and $2^{\omega_2} = \omega_3$. In particular, in M', $2^{<\omega_2} = \omega_2$. So we can apply Theorem 31.66 to get an extension $M[G]$ such that $M[G] \models [MA + (2^\omega = \omega_2)]$. In $M[G]$ we have $2^{\omega_2} \geq \omega_3$. Since $|\mathbb{P}| = \omega_2$ in M', there are $\omega_2^\omega = \omega_2$ nice names for subsets of ω_2. So $M[G]$ adds at most ω_2 new subsets of ω_2, so $2^{\omega_2} = \omega_3$ in $M[G]$. $\square$

For an ordinal α, a *normal α-chain of posets* is a sequence $\langle \mathbb{P}_\xi : \xi < \alpha \rangle$ of posets such that $\mathbb{P}_\xi \subseteq_c \mathbb{P}_\eta$ whenever $\xi \leq \eta$ and $\mathbb{P}_\eta = \bigcup_{\xi<\eta} \mathbb{P}_\xi$ whenever $\eta < \alpha$ is a limit ordinal.

Proposition 31.107. (V.4.40) *Let $\langle \mathbb{P}_\xi : \xi < \gamma \rangle$ be a normal γ-chain of posets, where γ is a limit ordinal. Let $\mathbb{P}_\gamma = \bigcup_{\xi<\gamma} \mathbb{P}_\xi$. Then $\langle \mathbb{P}_\xi : \xi \leq \gamma \rangle$ is a normal $(\gamma + 1)$-chain.*

Proof. First we check that each $\mathbb{P}_\xi$ is a subposet of $\mathbb{P}_\gamma$. We have

$$\leq_{\mathbb{P}_\gamma} \cap (\mathbb{P}_\xi \times \mathbb{P}_\xi) = (\mathbb{P}_\xi \times \mathbb{P}_\xi) \cap \bigcup_{\eta<\gamma} (\mathbb{P}_\eta \times \mathbb{P}_\eta) = (\mathbb{P}_\xi \times \mathbb{P}_\xi).$$

Next we check that $\mathbb{P}_\xi \subseteq_{\mathrm{ctr}} \mathbb{P}_\gamma$. Suppose that $n \in \omega$, $q_1, \ldots, q_n \in \mathbb{P}_\xi$, $p \in \mathbb{P}_\gamma$, and $\forall i[p \leq q_i]$. Choose $\eta < \gamma$ such that $\xi \leq \eta$ and $p \in \mathbb{P}_\eta$. Then $\exists q \in \mathbb{P}_\xi \forall i[q \leq q_i]$ since $\mathbb{P}_\xi \subseteq_{\mathrm{ctr}} \mathbb{P}_\eta$.

Finally, to show that $\mathbb{P}_\xi \subseteq_{\mathrm{c}} \mathbb{P}_\gamma$, we apply Lemma 25.79, with i the inclusion embedding. First note

$\mathbb{1}_{\mathbb{P}_\xi} = \mathbb{1}_{\mathbb{P}_\gamma}$.
$\forall q_1, q_2 \in \mathbb{P}_\xi[q_1 \leq_{\mathbb{P}_\xi} q_2 \rightarrow q_1 \leq_{\mathbb{P}_\gamma} q_2]$.
$\forall q_1, q_2 \in \mathbb{P}_\xi[q_1 \perp_{\mathbb{P}_\xi} q_2 \leftrightarrow q_1 \perp_{\mathbb{P}_\gamma} q_2]$.

Now let $p \in \mathbb{P}_\gamma$. Choose $\eta < \gamma$ such that $\xi < \eta$ and $p \in \mathbb{P}_\eta$. Let $p' \in \mathbb{P}_\xi$ be a reduction of p to $\mathbb{P}_\xi$. Then for any $q \in \mathbb{P}_\xi$, $q \perp_{\mathbb{P}_\gamma} p$ iff $q \perp_{\mathbb{P}_\eta} p$ iff $q \perp_{\mathbb{P}_\xi} p'$. $\square$

Proposition 31.108. (V.4.43) *There are posets $\mathbb{P}_\xi$ for $\xi \leq \omega_1$ such that $\forall \xi, \eta \leq \omega_1[\xi, \eta \rightarrow \mathbb{P}_\xi \subseteq_c \mathbb{P}_\eta]$ and $\mathbb{P}_{\omega_1} = \bigcup_{\xi<\omega_1} \mathbb{P}_\xi$, each $\mathbb{P}_\xi$ is countable, and $\mathbb{P}_{\omega_1}$ is not ccc.*

Proof. Let T be a tree of height ω_1 consisting of a branch $\langle b_\xi : \xi < \omega_1 \rangle$ together with elements x_ξ with $b_\xi < x_\xi$ for all $\xi < \omega_1$. Let $\mathbb{P}_\xi = T_{\xi+1}$ for each $\xi < \omega_1$ and $\mathbb{P}_{\omega_1} = T$. If $\xi < \eta \leq \omega_1$ then $\mathbb{P}_\xi \subseteq_c \mathbb{P}_\eta$. In fact, suppose that $u \in \mathbb{P}_\eta$. If $\mathrm{height}(u) \leq \xi$, then u is a reduction of u to $\mathbb{P}_\xi$. If $\mathrm{height}(u) > \xi$, let p be of height ξ with $p < u$. Then p is a reduction of u to $\mathbb{P}_\xi$, since if $q \in \mathbb{P}_\xi$ and $q \not\perp p$, then q and p are comparable, hence $q \leq p \leq u$, so that $q \not\perp u$. Clearly T has an antichain of size ω_1, so $\mathbb{P}_{\omega_1}$ is not ccc. $\square$

Lemma 31.109. (V.4.44) *If $i : \mathbb{P} \rightarrow \mathbb{R}$ is a complete embedding and G is a filter on $\mathbb{P}$, let $\mathbb{R}/G = \{r \in \mathbb{R} : \forall p \in G[r \not\perp i(p)]\}$. Then $\mathbb{R}/G$ is an upward closed subset of $\mathbb{R}$, and $i[G] \subseteq \mathbb{R}/G$.*

Proof. Suppose that $r \in \mathbb{R}/G$ and $r \leq s$. Take any $p \in G$. Say $t \leq r, i(p)$. Then $t \leq s, i(p)$, so $s \in \mathbb{R}/G$.

Next, suppose that $q \in G$. Take any $p \in G$. Then $p \not\perp q$, so $i(p) \not\perp i(q)$, hence $i(q) \in \mathbb{R}/G$. $\square$

Lemma 31.110. (V.4.44) *If $i : \mathbb{P} \to \mathbb{R}$ is a complete embedding, G is a filter on $\mathbb{P}$, $\mathbb{P}$, $\mathbb{R}$, i are in M, and Γ is the standard $\mathbb{P}$-name for a generic filter, then for any $p \in \mathbb{P}$ and $r \in \mathbb{R}$, $p \Vdash_{\mathbb{P}} [r \in \mathbb{R}/\Gamma]$ iff p is a reduction of r to $\mathbb{P}$.*

Proof. $\Rightarrow$: Suppose that $p \Vdash_{\mathbb{P}} [r \in \mathbb{R}/\Gamma]$. Suppose that $q \in \mathbb{P}$ and $q \not\perp p$; we want to show that $i(q) \not\perp r$. Let G be generic such that $p, q \in G$. Then $r \in \mathbb{R}/G$ and $q \in G$, so $r \not\perp i(q)$.

$\Leftarrow$: Suppose that p is a reduction of r to $\mathbb{P}$. Let G be any generic filter such that $p \in G$; we want to show that $r \in \mathbb{R}/G$. So suppose that $q \in G$. Then $p \not\perp q$, so $i(q) \not\perp r$, as desired. $\square$

Let $i : \mathbb{P} \to \mathbb{R}$ is a complete embedding. Then $\mathrm{red}(p, r)$ is an abbreviation for "$p \in \mathbb{P}$ and $r \in \mathbb{R}$ and p is a reduction of r to $\mathbb{P}$". In M we define

$$\dot{Q} = \{(\check{r}, p) : r \in \mathbb{R} \text{ and } p \in \mathbb{P} \text{ and } \mathrm{red}(p, r)\}.$$

Proposition 31.111. *With the above definitions, if G is $\mathbb{P}$-generic over M then $\dot{Q}_G = \{r \in \mathbb{R} : \exists p \in G[p \Vdash [r \in \mathbb{R}/\Gamma]]\}$.*

Proof. By definition, $\dot{Q}_G = \{r \in \mathbb{R} : \exists p \in G[\mathrm{red}(p, r)]\}$. By Lemma 31.110, this is $\{r \in \mathbb{R} : \exists p \in G[p \Vdash [r \in \mathbb{R}/\Gamma]]\}$. $\square$

Proposition 31.112. *With the above definitions, if G is $\mathbb{P}$-generic over M then $\dot{Q}_G = \mathbb{R}/G$.*

Proof. First suppose that $r \in \dot{Q}_G$. Then by Proposition 31.111, choose $p \in G$ such that $p \Vdash [r \in \mathbb{R}/\Gamma]$. so $r \in \mathbb{R}/G$.

Second, suppose that $r \in \mathbb{R}/G$. Then there is a $p \in G$ such that $p \Vdash [r \in \mathbb{R}/\Gamma]$. By Proposition 31.111, $r \in \dot{Q}_G$. $\square$

Proposition 31.113. *With the above definitions, $\mathbb{1} \Vdash [\dot{Q} = \mathbb{R}/\Gamma]$.* $\square$

Proposition 31.114. *With the above definitions, if $p \in \mathbb{P}$ and $r \in \mathbb{R}$, then $p \Vdash [\check{r} \in \dot{Q}]$ iff* $\mathrm{red}(p, r)$.

Proof. $p \Vdash [\check{r} \in \dot{Q}]$ iff ($p \Vdash [\check{r} \in \dot{Q}]$ and $p \Vdash [\dot{Q} = \mathbb{R}/\Gamma]$) iff $p \Vdash [\check{r} \in \mathbb{R}/\Gamma]$ iff $\mathrm{red}(p, r)$, using Proposition 31.113 and Lemma 31.110. $\square$

Proposition 31.115. *Suppose that $i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding. Then the following are equivalent:*

(i) $\mathrm{red}(p, r)$*,*

(ii) $\forall q \in \mathbb{P}[q \not\perp p \to i(q) \not\perp r]$.

(iii) $\forall q \in \mathbb{P}[q \leq p \to i(q) \not\perp r]$.

Proof. By definition, $\mathrm{red}(p, r) \leftrightarrow \forall q \in \mathbb{P}[i(q) \perp r \to q \perp p]$. Hence (i) and (ii) are equivalent. Now suppose that $\mathrm{red}(p, r)$, $q \in \mathbb{P}$, and $q \leq p$. Then $q \not\perp p$, so $i(q) \not\perp r$ by (ii). Finally, assume (iii), and suppose that $q \in \mathbb{P}$ and $q \not\perp p$. Say $q' \leq q, p$. Then by (iii), $i(q') \not\perp r$. Say $s \leq i(q'), r$. Then $s \leq i(q), r$, so $i(q) \not\perp r$, as desired. $\square$

Proposition 31.116. *Suppose that $i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding and* $\mathrm{red}(p, r)$*. Then $i(p) \not\perp r$.*

Proof. Obviously $p \not\perp p$, fo $i(p) \not\perp r$ by Proposition 31.115(ii). $\square$

Proposition 31.117. *Suppose that $i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding,* $\mathrm{red}(p, r)$*, and $p' \leq p$. Then* $\mathrm{red}(p', r)$*.*

Proof. By Proposition 31.115(iii), $\forall q[q \leq p \to i(q) \not\perp r]$, so $\forall q[q \leq p' \to i(q) \not\perp r]$. Hence $\mathrm{red}(p', r)$ by Proposition 31.115(iii). $\square$

Proposition 31.118. *Suppose that $i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding,* $\mathrm{red}(p, r)$*, and $r \leq i(p_1)$. Then $\exists p_2[p_2 \leq p_1$ and $p_2 \leq p$ and* $\mathrm{red}(p_2, r)]$*.*

Proof. By Proposition 31.115(iii) we have $\forall q \in P[q \leq p \to i(q) \not\perp r]$. Hence $\forall q \in P[q \leq p \to i(q) \not\perp i(p_1)]$, hence $\forall q \in P[q \leq p \to q \not\perp p_1]$. Since $p \leq p$, it follows that $p \not\perp p_1$. Choose $p_2 \leq p, p_1$. By Proposition 31.117, $\mathrm{red}(p_2, r)$. $\square$

Proposition 31.119. *Suppose that $i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding and $r_3 \leq i(p_1), i(p_2)$. Then $\exists p_3 \leq p_1, p_2[\mathrm{red}(p_3, r_3)]$.*

Proof. Let p' be a reduction of r_3 to $\mathbb{P}$. Thus $\mathrm{red}(p', r_3)$ and $r_3 \leq i(p_1)$. Hence by Proposition 31.118 there is a $p'' \leq p', p_1$ such that $\mathrm{red}(p'', r_3)$. So $\mathrm{red}(p'', r_3)$ and $r_3 \leq p_2$. Hence by Proposition 31.118 there is a $p_3 \leq p'', p_2$ such that $\mathrm{red}(p_3, r_3)$. Note that $p_3 \leq p'' \leq p_1$. $\square$

Proposition 31.120. *Let $i : \mathbb{P} \to \mathbb{Q}$ be a dense embedding. Suppose that $n \in \omega$, $p_1, \ldots, p_n \in \mathbb{P}$, and $\not\perp (i(p_1), \ldots, i(p_n))$. Then $\not\perp (p_1, \ldots, p_n)$.*

Proof. Say $r \leq i(p_1), \ldots, i(p_n)$. Choose $s \in \mathbb{P}$ so that $i(s) \leq r$. Then $i(s) \leq i(p_1)$, so $i(s) \not\perp i(p_1)$ and hence $s \not\perp p_1$. Say $t_1 \leq s, p_1$. Suppose that we have found t_k so that $t_k \leq s, p_1, \ldots, p_k$, with $1 \leq k < n$. Then $i(t_k) \leq i(s) \leq i(p_{k+1})$, so $i(t_k) \not\perp i(p_{k+1})$. Hence $t_k \not\perp p_{k+1}$. Say $t_{k+1} \leq t_k, p_{k+1}$. Then $t_{k+1} \leq s, p_1, \ldots, p_{k+1}$.

Now $t_n \leq p_1, \ldots, p_n$, as desired. $\square$

Lemma 31.121. (V.4.45) *Suppose that $i : \mathbb{P} \to \mathbb{R}$ is a complete embedding. Let*

$$\dot{Q} = \{(\check{r}, p) : r \in \mathbb{R} \text{ and } p \in \mathbb{P} \text{ and } p \text{ is a reduction of } r \text{ to } \mathbb{P}\}.$$

Then $\mathbb{P} * \dot{Q} \approx_d \mathbb{R}$. (See the definition of $\approx_d$ on page 567.)

Proof. By definition,

$$\mathbb{P} * \dot{Q} = \{(p, \check{r}) \in \mathbb{P} \times \mathrm{dmn}(\dot{Q}) : r \in \mathbb{R} \text{ and } p \Vdash [\check{r} \in \dot{Q}]\};$$
$$(p_1, \check{r}_1) \leq (p_2, \check{r}_2) \quad \text{iff} \quad p_1 \leq p_2 \text{ and } p_1 \Vdash [\check{r}_1 \leq \check{r}_2].$$

Using Proposition 31.114 we get

$$\mathbb{P} * \dot{Q} = \{(p, \check{r}) \in \mathbb{P} \times \mathrm{dmn}(\dot{Q}) : r \in \mathbb{R} \text{ and } \mathrm{red}(p, r)\};$$
$$(p_1, \check{r}_1) \leq (p_2, \check{r}_2) \quad \text{iff} \quad p_1 \leq p_2 \text{ and } r_1 \leq r_2].$$

Hence we get $\mathbb{P} * \dot{Q} \cong \{(p,r) \in \mathbb{P} \times \mathbb{R} : \mathrm{red}(p,r)\}$.

Now let $\mathbb{A}$ be the completion of $\mathbb{R}$ with the dense embedding $j : \mathbb{R} \to \mathbb{A}$. To prove the lemma it suffices to find a dense embedding k of $\{(p,r) \in \mathbb{P} \times \mathbb{R} : \mathrm{red}(p,r)\}$ into $\mathbb{A}$. If $\mathrm{red}(p,r)$, then by Proposition 31.113 we have $r \not\perp i(p)$ since $p \not\perp p$. We define $k(p,r) = j(r) \cdot j(i(p))$. Then $\mathrm{rng}(k)$ is dense in $\mathbb{A}$ since $k(p,r) \le j(r)$. To show that k is a dense embedding, we check conditions 1-3 of the definition on page 445. We have $k(\mathbb{1},\mathbb{1}) = j(\mathbb{1}) \cdot j(i(\mathbb{1})) = \mathbb{1}$. If $(p_1,r_1) \le (p_2,r_2)$, then $k(p_1,r_1) = j(r_1) \cdot j(i(p_1)) \le j(r_2) \cdot j(i(p_2)) = k(p_2,r_2)$. For the $\to$ direction of 3, suppose that $k(p_1,r_1) \not\perp k(p_2,r_2)$. Thus $j(r_1) \cdot j(i(p_1)) \cdot j(r_2) \cdot j(p_2)) \neq 0$. By Proposition 31.120 choose $r_3 \le r_1, i(p_1), r_2, i(p_2)$. By Proposition 31.119 there is a $p_3 \le p_1, p_2$ such that $\mathrm{red}(p_3,r_3)$. Since also $r_3 \le r_1, r_2$, we have $(p_3,r_3) \le (p_1,r_1), (p_2,r_2)$.

For the $\leftarrow$ direction of 3, suppose $(p_1,r_1) \not\perp (p_2,r_2)$; say $(p_3,r_3) \le (p_1,r_1), (p_2,r_2)$. Then by 2, $k(p_3,r_3) \le k(p_1,r_1) \cdot k(p_2,r_2)$, and hence $k(p_1,r_1) \not\perp k(p_2,r_2)$. □

Let $\langle \mathbb{P}_\xi : \xi < \alpha\rangle$ and $\langle \mathbb{X}_\xi : \xi < \alpha\rangle$ be two normal α-chains of posets. Then these two chains are *strongly forcing equivalent* iff there is another normal α chain $\langle \mathbb{A}_\xi : \xi < \alpha\rangle$ of posets and dense embeddings $i_\xi : \mathbb{P}_\xi \to \mathbb{A}_\xi$ and $j_\xi : \mathbb{X}_\xi \to \mathbb{A}_\xi$ such that $i_\xi = i_\eta \restriction \mathbb{P}_\xi$ and $j_\xi = j_\eta \restriction \mathbb{X}_\xi$ whenever $\xi < \eta < \alpha$.

Proposition 31.122. *Let*

$$(\langle(\mathbb{P}_\xi, \le_\xi, \mathbb{1}_\xi) : \xi \le \alpha\rangle, \langle(\dot{\mathbb{Q}}_\xi, \dot{\preceq}_{\dot{\mathbb{Q}}_\xi}, \dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\xi}) : \xi < \alpha\rangle)$$

be an α-stage finite support iterated forcing construction. For each $\xi \le \alpha$ let $\overline{\mathbb{P}}_\xi = i^\alpha_\xi[\mathbb{P}_\xi]$. $\overline{\mathbb{P}}_\xi$ is considered as a subposet of $\mathbb{P}_\alpha$.

Then $\langle \overline{\mathbb{P}}_\xi : \xi \le \alpha\rangle$ is a normal $(\alpha+1)$-chain of posets.

Proof. Suppose that $\xi < \eta \le \alpha$. Clearly $\overline{\mathbb{P}}_\xi \subseteq \overline{\mathbb{P}}_\eta$ and in fact $\overline{\mathbb{P}}_\xi$ is a subposet of $\overline{\mathbb{P}}_\eta$. To show that $\overline{\mathbb{P}}_\xi \subseteq_{\mathrm{ctr}} \overline{\mathbb{P}}_\eta$, suppose that $n \in \omega$, $q_1, \ldots, q_n \in \overline{\mathbb{P}}_\xi$, $p \in \overline{\mathbb{P}}_\eta$, and $\forall i[p \le q_i]$. Say $q_i = i^\alpha_\xi(q'_i)$ and $p = i^\alpha_\eta(p')$. Let $p'' = p' \restriction \xi$. Then $p'' \in \mathbb{P}_\xi$ by 4 in the definition of iterated forcing. We have $\forall i[p'' \le q'_i]$ by 7 in the definition, so $\forall i[i^\alpha_\xi(p'') \le q_i]$. Thus $\overline{\mathbb{P}}_\xi \subseteq_{\mathrm{ctr}} \overline{\mathbb{P}}_\eta$. To show that $\overline{\mathbb{P}}_\xi \subseteq_{\mathrm{c}} \overline{\mathbb{P}}_\eta$, let $p \in \overline{\mathbb{P}}_\eta$. Let $p' = i^\alpha_\xi(p \restriction \xi)$. We claim that p' is a reduction of p to $\overline{\mathbb{P}}_\xi$. For, suppose that $q \in \overline{\mathbb{P}}_\xi$ and $q \le p'$. (We are going to apply Proposition 31.115(iii).) Let $r = (q \restriction \xi) \cup (p \restriction (\alpha\backslash\xi))$. Then $r \in \overline{\mathbb{P}}_\eta$ and $r \le q, p$ as desired.

Finally, if $\eta \le \alpha$ is a limit ordinal, then

$$\overline{\mathbb{P}}_\eta = i^\alpha_\eta[\mathbb{P}_\eta] = \bigcup_{\xi<\eta} i^\alpha_\xi[\mathbb{P}_\xi] = \bigcup_{\xi<\eta} \overline{\mathbb{P}}_\xi.$$ □

Proposition 31.123. *Suppose that $\mathbb{P} \subseteq_c \mathbb{Q}$, $\mathbb{A}$ is the completion of $\mathbb{P}$ with dense embedding $i : \mathbb{P} \to \mathbb{A}$, and $\mathbb{B}$ is the completion of $\mathbb{Q}$ with dense embedding $j : \mathbb{Q} \to \mathbb{B}$.*

Then there is a complete embedding $k : \mathbb{A} \to \mathbb{B}$ such that $k \circ i = j \restriction \mathbb{P}$.

A diagram for this:

$$\begin{tikzcd}
\mathbb{P} \arrow[rr, "\subseteq_c"] \arrow[d, "i"'] & & \mathbb{Q} \arrow[d, "j"] \\
\mathbb{A} \arrow[rr, "k"'] & & \mathbb{B}
\end{tikzcd}$$

Proof. First we claim

(1) If $X.Y \subseteq \mathbb{P}$ and $\sum^{\mathbb{A}} i[X] = \sum^{\mathbb{A}} i[Y]$, then $\sum^{\mathbb{B}} j[X] = \sum^{\mathbb{B}} j[Y]$.

In fact, assume not. Say $\sum^{\mathbb{B}} j[X] \not\leq \sum^{\mathbb{B}} j[Y]$. Thus $\sum^{\mathbb{B}} j[X] \cdot -\sum^{\mathbb{B}} j[Y] \neq 0$ so there is an $x \in X$ such that $j(x) \cdot -\sum^{\mathbb{B}} j[Y] \neq 0$. Choose $u \in \mathbb{Q}$ such that $j(u) \leq j(x) \cdot -\sum^{\mathbb{B}} j[Y]$. Thus

(2) $u \leq x$.

(3) $\forall y \in Y[u \perp y]$.

Let v be a reduction of u to $\mathbb{P}$. Then by (3), $\forall y \in Y[v \perp y]$. Also $v \not\perp u$, so by (2), $v \not\perp x$. Say $w \leq v, x$. Then $i(w) \leq \sum^{\mathbb{A}} i[X]$ and $i(w) \cdot \sum^{\mathbb{A}} i[Y] = 0$, contradiction. So (1) holds.

Now for any $a \in \mathbb{A}$ write $a = \sum^{\mathbb{A}} i[X]$ and define $k(a) = \sum^{\mathbb{B}} j[X]$. This definition is unambiguous by (1).

(4) $k(-a) = -k(a)$.

In fact, write $a = \sum^{\mathbb{A}} i[X]$ and $-a = \sum^{\mathbb{A}} i[Y]$. Clearly $\forall x \in X \forall y \in Y[i(x) \cdot i(y) = 0]$, so $\forall x \in X \forall y \in Y[x \perp y]$, hence $\forall x \in X \forall y \in Y[j(x) \cdot j(y) = 0]$, so $\sum^{\mathbb{B}} j[X] \cdot \sum^{\mathbb{B}} j[Y] = 0$. Let Z be a maximal antichain in $\mathbb{P}$. Then $\sum^{\mathbb{A}} i[Z] = 1 = a + -a = \sum^{\mathbb{A}} i[X \cup Y]$, so by (1), $\sum^{\mathbb{B}} j[Z] = \sum^{\mathbb{B}} j[X \cup Y] = \sum^{\mathbb{B}} j[x] + \sum^{\mathbb{B}} j[Y]$. Now Z is a maximal antichain in $\mathbb{Q}$, so $\sum^{\mathbb{B}} j[Z] = 1$. This proves (4).

(5) If $Z \subseteq \mathbb{A}$, then $k(\sum^{\mathbb{A}} Z) = \sum^{\mathbb{B}} k[Z]$.

In fact, for each $z \in Z$ write $z = \sum^{\mathbb{A}} i[X_z]$ with $X_z \subseteq \mathbb{P}$. Then

$$\begin{aligned}
k\left(\sum^{\mathbb{A}} Z\right) &= k\left(\sum_{z \in Z}^{\mathbb{A}} \sum^{\mathbb{A}} i[X_z]\right) \\
&= k\left(\sum^{\mathbb{A}} i\left[\bigcup_{z \in Z} X_z\right]\right) \\
&= \sum^{\mathbb{B}} j\left[\bigcup_{z \in Z} X_z\right]
\end{aligned}$$

$$= \sum_{z\in Z}^{\mathbb{B}} \sum^{\mathbb{B}} j[X_z]$$

$$= \sum^{\mathbb{B}} k[Z]$$

Clearly k is one-one. Finally, if $p \in \mathbb{P}$, then $(k \circ i)(p) = k(i(p)) = j(p)$. $\square$

Proposition 31.124. (V.4.47) *Let $\langle \mathbb{X}_\xi : \xi \leq \alpha \rangle$ be a normal $(\alpha+1)$-chain of posets. Then there is an α-stage finite support iterated forcing construction*

$$(\langle(\mathbb{P}_\xi, \leq_\xi, \mathbb{1}_\xi) : \xi \leq \alpha\rangle, \langle(\dot{\mathbb{Q}}_\xi, \dot{\leq}_{\dot{\mathbb{Q}}_\xi}, \dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\xi}) : \xi < \alpha\rangle)$$

such that, with $\overline{\mathbb{P}}_\xi = i^\alpha_\xi[\mathbb{P}_\xi]$ for each $\xi \leq \alpha$, the chain $\langle \mathbb{X}_\xi : \xi \leq \alpha\rangle$ is strongly forcing equivalent to $\langle \overline{\mathbb{P}}_{1+\xi} : \xi \leq \alpha \rangle$.

Proof. First we define $\mathbb{P}_0$, $\dot{\mathbb{Q}}_0$, $\mathbb{P}_1$, $\mathbb{A}_0$, i_0, and j_0. Let $\mathbb{P}_0 = \{\emptyset\}$. Let $\dot{\mathbb{Q}}_0 = \{(\check{r}, \emptyset) : r \in \mathbb{X}_0\}$. Let $\mathbb{P}_1 = \{\{((\check{r}, \emptyset), 0)\} : r \in \mathbb{X}_0\}$. Let $\mathbb{A}_0$ be the Boolean completion of $\mathbb{X}_0$, with associated dense embedding j_0. Let $i_0(\{((\check{r}, \emptyset), 0)\}) = j_0(r)$ for each $r \in \mathbb{X}_0$.

Now suppose that $\mathbb{P}_{1+\beta}$, $\mathbb{A}_\beta$, i_β, j_β have been defined so that $i_\beta : \mathbb{P}_{1+\beta} \to \mathbb{A}_\beta$ is a dense embedding and $j_\beta : \mathbb{X}_\beta \to \mathbb{A}_\beta$ is a dense embedding. (These statements are true for $\beta = 0$.) We now define $\mathbb{P}_{1+\beta+1}$, $\dot{Q}_{\beta+1}$, $\mathbb{A}_{\beta+1}$, $i_{\beta+1}$, and $j_{\beta+1}$. Let $\mathbb{A}'_{\beta+1}$ be the completion of $\mathbb{X}_{\beta+1}$ with associated dense embedding $j'_{\beta+1}$. We now apply Proposition 31.123 to get a complete embedding k of $\mathbb{A}_\beta$ into $\mathbb{A}'_{\beta+1}$ such that $k \circ j_\beta = j'_{\beta+1} \restriction \mathbb{X}_\beta$:

$$\begin{tikzcd}
\mathbb{X}_\beta \arrow[r, "\subseteq_c"] \arrow[d, "j_\beta"'] & \mathbb{X}_{\beta+1} \arrow[d, "j'_{\beta+1}"] \\
\mathbb{A}_\beta \arrow[r, "k"'] \arrow[dr, "\subseteq_c"'] & \mathbb{A}'_{\beta+1} \arrow[d, "s"] \\
& \mathbb{A}_{\beta+1}
\end{tikzcd}$$

Now there is a complete BA $\mathbb{A}_{\beta+1}$ such that $\mathbb{A}_\beta$ is a subalgebra of $\mathbb{A}_{\beta+1}$ and there is an isomorphism s of $\mathbb{A}'_{\beta+1}$ onto $\mathbb{A}_{\beta+1}$ such that $s \circ k$ is the identity on A_β. We claim that $\mathbb{A}_\beta \subseteq_c \mathbb{A}_{\beta+1}$. Clearly $\mathbb{A}_\beta$ is a subposet of $\mathbb{A}_{\beta+1}$. To check that $\mathbb{A}_\beta \subseteq_{\text{ctr}} \mathbb{A}_{\beta+1}$, suppose that $n \in \omega$, $q_1, \ldots, q_n \in \mathbb{A}_\beta$, $p \in \mathbb{A}_{\beta+1}$, and $\forall i[p \leq q_i]$. Then $\prod_i q_i \neq 0$ and $\forall i[\prod_i q_i \leq q_i]$. Finally, suppose that $p \in \mathbb{A}_{\beta+1}$. Let $q \in \mathbb{A}_\beta$ be a reduction of $s^{-1}(p)$ to $\mathbb{A}_\beta$ using k. Then for any $r \in \mathbb{A}_\beta$ we have $k(r) \perp s^{-1}(p)$ implies that $r \perp q$, so $r \perp p$ implies that $s^{-1}(r) \perp s^{-1}(p)$. Since $s \circ k$ is the inclusion map, we have $s^{-1}(r) = s^{-1}(s(k(r)) = k(r)$. So $r \perp p$ implies that $k(r) \perp s^{-1}(p)$ and hence $r \perp p$, as desired. Let $j_{\beta+1} = s \circ j'_{\beta+1}$. Clearly $j_{\beta+1}$ is a dense embedding of $\mathbb{X}_{\beta+1}$ into $\mathbb{A}_{\beta+1}$. Also, $j_{\beta+1} \restriction \mathbb{X}_\beta = (s \circ j'_{\beta+1}) \restriction \mathbb{X}_\beta = s \circ k \circ j_\beta = j_\beta$.

Note that i_β is a complete embedding of $\mathbb{P}_{1+\beta}$ into $\mathbb{A}_{\beta+1}$. Let $\dot{Q}_{\beta+1} = \{(\check{r}, p) : r \in \mathbb{A}_{\beta+1}$ and $p \in \mathbb{P}_{1+\beta}$ and p is a reduction of r to $\mathbb{P}_{1+\beta}$ using $i_\beta\}$. Then by the proof of V.4.45,

$$\begin{aligned}\mathbb{P}_{1+\beta} * \dot{Q}_{\beta+1} = \{(p, \check{r}) \in \mathbb{P}_{1+\beta} \times \mathrm{dmn}(\dot{Q}_{\beta+1}) :&\\ p \text{ is a reduction of } r \text{ to } \mathbb{P}_{1+\beta} \text{ using } i_\beta\}.&\end{aligned}$$

Hence

$$\begin{aligned}\mathbb{P}_{1+\beta+1} = \{p : p &\text{ is a function with domain } 1+\beta+1 \text{ and}\\ &p \restriction (1+\beta) \in \mathbb{P}_{1+\beta} \text{ and } p(1+\beta) \in \mathbb{A}_{\beta+1} \text{ and}\\ &p \restriction (1+\beta) \text{ is a reduction of } p(1+\beta) \text{ to } \mathbb{P}_{1+\beta} \text{ using } i_\beta\}.\end{aligned}$$

Now if $p \in \mathbb{P}_{1+\beta+1}$, then $(p \restriction (1+\beta)) \not\perp p(1+\beta)$, so $i_\beta(p \restriction (1+\beta)) \not\perp p(1+\beta)$. We define $i_{\beta+1}(p) = i_\beta(p \restriction (1+\beta)) \cdot p(1+\beta)$. Then if $p \in \mathbb{P}_{1+\beta}$ then $i_{\beta+1}(i_\beta^{\beta+1}(p)) = i_{\beta+1}(p^\frown\langle \mathbb{1}\rangle) = i_\beta(p) \cdot \mathbb{1} = i_\beta(p)$. Now $\mathrm{rng}(i_{\beta+1})$ is dense in $\mathbb{A}_{\beta+1}$. For, suppose that $r \in \mathbb{A}^+_{\beta+1}$. Let $p \in \mathbb{P}_{1+\beta}$ be a reduction of r to $\mathbb{P}_{1+\beta}$ using i_β. Then $p^\frown\langle \check{r}\rangle \in \mathbb{P}_{1+\beta+1}$, and $i_{\beta+1}(p^\frown\langle \check{r}\rangle) = i_\beta(p) \cdot r \leq r$. Clearly $p \leq q$ implies that $i_{\beta+1}(p) \leq i_{\beta+1}(q)$, and $p \not\perp q$ implies that $i_{\beta+1}(p) \not\perp i_{\beta+1}(q)$. Now suppose that $i_{\beta+1}(p) \not\perp i_{\beta+1}(q)$. Thus $r \stackrel{\text{def}}{=} i_\beta(p \restriction (1+\beta)) \cdot p(1+\beta) \cdot i_\beta(q \restriction (1+\beta)) \cdot q(1+\beta) \neq 0$. We have $r \leq i_\beta(p \restriction (1+\beta)), i_\beta(q \restriction (1+\beta))$. Hence by Proposition 350h there is an $s \leq p \restriction (1+\beta), q \restriction (1+\beta)$ such that s is a reduction of r to $\mathbb{P}_{1+\beta}$ using i_β. Let $t = s^\frown\langle \check{r}\rangle$. Then $t \in \mathbb{P}_{1+\beta+1}$ and $t \leq p, q$. So $p \not\perp q$. So $i_{\beta+1}$ is a dense embedding of $\mathbb{P}_{1+\beta+1}$ into $\mathbb{A}_{\beta+1}$.

This completes the successor step.

Now suppose that β is a limit ordinal. $\mathbb{P}_\beta$ is given by the definition of finite support iteration. We let $\mathbb{A}_\beta = \bigcup_{\gamma<\beta} \mathbb{A}_\gamma$. Then $\mathbb{A}_\beta$ is not in general complete, but $\mathbb{A}_\gamma \subseteq_c \mathbb{A}_\beta$. Let $j_\beta = \bigcup_{\gamma<\beta} j_\gamma$. Then $j_\beta : \mathbb{X}_\beta \to \mathbb{A}_\beta$ is a dense embedding. For $p \in \mathbb{P}_\beta$, choose $\xi < \beta$ so that $p(\eta) = \mathbb{1}$ for all $\eta \in [\xi, \beta)$ and define $i_\beta(p) = i_\xi(p \restriction \xi)$. Then i_β is a dense embedding of $\mathbb{P}_\beta$ into $\mathbb{A}_\beta$. □

$MA^*(\kappa)$ is the statement that $MA_{\mathbb{P}}(\kappa)$ holds for all posets which are ω_2-cc and countably closed. MA^* is the statement that $\forall \kappa < 2^{\omega_1} MA^*(\kappa)$.

Lemma 31.125. (V.5.2) *If $\mathbb{P}$ is countably closed, then $MA_{\mathbb{P}}(\omega_1)$.*

Proof. Let $\mathbb{P}$ be countably closed, and let $\langle D_\xi : \xi < \omega_1\rangle$ be a system of dense subsets of $\mathbb{P}$. We define $r_\xi \in \mathbb{P}$ for each $\xi < \omega_1$, by recursion. Let $r_0 \in D_0$. Suppose that r_ξ has been defined for all $\xi < \eta$, so that $r_0 \geq r_1 \geq \cdots \geq r_\xi \geq \cdots$. By countable closure, let $r_\eta \leq r_\xi$ for all $\xi < \eta$, with $r_\eta \in D_\eta$. Now let $G = \{q \in \mathbb{P} : \exists \xi < \omega_1[r_\xi \leq q]$. □

Lemma 31.126. (V.5.3) *If $\mathbb{P}$ is countably closed and $\mathbb{1} \Vdash [\dot{\mathbb{Q}}$ is countably closed], then $\mathbb{P} * \dot{\mathbb{Q}}$ is countably closed.*

Proof. Suppose that $\langle (p_\xi, \dot{q}_\xi) : \xi < \alpha \rangle$ is such that α is countable and $\forall \xi, \eta < \alpha[\xi < \eta \to (p_\eta, \dot{q}_\eta) \le (p_\xi, \dot{q}_\zeta)]$. Choose $r \in \mathbb{P}$ such that $r \le p_\xi$ for all $\xi < \alpha$. Then $\forall \xi, \eta < \alpha[\xi < \eta \to r \Vdash [\dot{q}_\eta \le \dot{q}_\xi]]$, so $r \Vdash \exists s \forall \xi < \alpha[s \le \dot{q}_\xi]$. Then there is a $t \le r$ and a $\dot{u}$ such that $t \Vdash \forall \xi < \alpha[\dot{u} \le \dot{q}_\xi]$. Hence $\forall \xi < \alpha[(t, \dot{u}) \le (p_\xi, \dot{q}_\xi)]$. □

Proposition 31.127. *$\{t \in \mathrm{Fn}(\omega_1, \omega, \omega_1) : \mathrm{dmn}(t) \in \omega_1\}$ is a complete subposet of $\mathrm{Fn}(\omega_1, \omega, \omega_1)$.*

Proof. It is obviously a subposet. To show that $\{t \in \mathrm{Fn}(\omega_1, \omega, \omega_1) : \mathrm{dmn}(t) \in \omega_1\} \subseteq_{\mathrm{ctr}} \mathrm{Fn}(\omega_1, \omega, \omega_1)$, suppose that $n \in \omega$, $q_1, \ldots, q_n \in \{t \in \mathrm{Fn}(\omega_1, \omega, \omega_1) : \mathrm{dmn}(t) \in \omega_1\}$, $p \in \mathrm{Fn}(\omega_1, \omega, \omega_1)$, and $\forall i[p \le q_i]$. Let $r \in \{t \in \mathrm{Fn}(\omega_1, \omega, \omega_1) : \mathrm{dmn}(t) \in \omega_1\}$ be such that $p \subseteq r$. Then $\forall i[r \le q_i$. Finally denseness follows in the same way. □

Example 31.128. (V.5.4) *There is an ω-stage countable support iterated forcing construction such that $\forall n[\mathbb{1} \Vdash [\dot{Q}_n$ is countably closed]] and $\forall n[\mathbb{P}_n$ is countably closed], but $\mathbb{P}_\omega$ is not countably closed, and in fact collapses ω_1.*

Proof. In M let $\mathbb{T} = \{t \in \mathrm{Fn}(\omega_1, \omega, \omega_1) : \mathrm{dmn}(t) \in \omega_1\}$. Let $\mathbb{P}_0 = \{\emptyset\}$ and $\dot{\mathbb{Q}}_0 = \check{\mathbb{T}}$. This also defines $\mathbb{P}_1$.

Now suppose that $\mathbb{P}_{n+1}$ has been defined so that $\mathbb{1}_{\mathbb{P}_n} \Vdash [\dot{\mathbb{Q}}_n \subseteq_d \mathbb{T}]$. Let $i : \mathbb{P}_n \to \mathbb{P}_{n+1}$ be the usual complete embedding. Let $\Gamma'' = \{(i_* p_n, p) : p \in \mathbb{P}_{n+1}\}$ using the definition of i_* on page 560. Then

$$\begin{aligned}\mathbb{1}_{\mathbb{P}_{n+1}} \Vdash &\exists f : \omega_1 \to \omega \forall \alpha < \omega_1 \forall k \in \omega \\ &[f(\alpha) = k \leftrightarrow \exists p \in \Gamma''[\alpha \in \mathrm{dmn}(p) \text{ and } p(\alpha) = k]]\end{aligned}$$

Hence there is a $\mathbb{P}_{n+1}$-name $\dot{f}_n$ such that

$$\begin{aligned}\mathbb{1}_{\mathbb{P}_{n+1}} \Vdash &\dot{f}_n : \omega_1 \to \omega \text{ and } \forall \alpha < \omega_1 \forall k \in \omega \\ &[\dot{f}_n(\alpha) = k \leftrightarrow \exists p \in \Gamma''[\alpha \in \mathrm{dmn}(p) \text{ and } p(\alpha) = k]]\end{aligned}$$

Now we claim

(1) For all $p \in \mathbb{P}_{n+1}$, $\alpha < \omega_1$, and $k \in \omega$, $p \Vdash \dot{f}_n(\alpha) = k$ iff for every K which is $\mathbb{P}_{n+1}$-generic over M with $p \in K$, and with $G = \{r \in \mathbb{P}_n : r^\frown\langle \mathbb{1} \rangle \in K\}$, we have $\alpha \in \mathrm{dmn}(p_{nG})$ and $p_{nG}(\alpha) = k$.

For, suppose that $p \in \mathbb{P}_{n+1}$, $\alpha < \omega_1$, and $k \in \omega$. First suppose that $p \Vdash \dot{f}_n(\alpha) = k$. Suppose that K is $\mathbb{P}_{n+1}$-generic over M with $p \in K$, and $G = \{r \in \mathbb{P}_n : r^\frown\langle \mathbb{1} \rangle \in K\}$. But suppose that $\alpha \notin \mathrm{dmn}(p_{nG})$. Then there is a $q \le p$ such that $\alpha \in \mathrm{dmn}(q_{nG})$, $q \restriction n = p \restriction n$, and $q_{nG}(\alpha) \ne k$. Let $q_{nG} \in L$ generic, and $H = G * L$; see the definition on page 605. Now $q \le p$, so $q \Vdash \dot{f}_n(\alpha) = k$. Hence $\dot{f}_{nL}(\alpha) = k$. Now $\Gamma''_L = \{(i_* r_n)_L : r \in L\}$, so $(i_* q_n)_L \in \Gamma''_L$. By Lemma 30.3, $(i_* q)_L = q_{nG}$. Hence $\dot{f}_{nL}(\alpha) \ne k$, contradiction. It follows that $\alpha \in \mathrm{dmn}(p_{nG})$. Suppose that $p_{nG}(\alpha) \ne k$. Now $\Gamma''_K = \{(i_* r_n)_K : r \in K\}$, so $(i_*p_n)_K \in \Gamma''_K$. By Lemma 30.3, $(i_*p)_K = p_{nG}$. Hence $\dot{f}_{nK} \neq k$. But $p \Vdash \dot{f}_n(\alpha) = k$, contradiction.

Second, suppose that the condition in (1) holds. Let K be $\mathbb{P}_{n+1}$-generic over M with $p \in K$, and $G = \{r \in \mathbb{P}_n : r^\frown\langle \mathbb{1}\rangle \in K\}$. Now $\Gamma''_K = \{(i_*r_n)_K : r \in K\}$, so $(i_*p_n)_K \in \Gamma''_K$. By Lemma IV.4.4, $(i_*p)_K = p_{nG}$. Then by the condition in (1), $\alpha \in \mathrm{dmn}(p_{nG})$ and $p_{nG}(\alpha) = k$. Hence $\dot{f}_{nK}(\alpha) = k$, as desired. This completes the proof of (1).

Now

$$\begin{aligned}\mathbb{1}_{\mathbb{P}_{n+1}} \Vdash &\exists Q[Q \text{ is a forcing poset and } Q \subseteq_d \mathbb{T} \text{ and } \forall t \in \mathbb{T}\\ &\quad [t \in Q \leftrightarrow \exists \text{ limit } \gamma \exists k \in \omega[\mathrm{dmn}(t) = \gamma + k \text{ and } k = \dot{f}_n(\gamma+\omega)]]]\end{aligned}$$

Hence there is a $\mathbb{P}_{n+1}$-name $\dot{\mathbb{Q}}_{n+1}$ such that

$$\begin{aligned}\mathbb{1}_{\mathbb{P}_{n+1}} \Vdash &\dot{\mathbb{Q}}_{n+1} \text{ is a forcing poset and } \dot{\mathbb{Q}}_{n+1} \subseteq_d \mathbb{T} \text{ and } \forall t \in \mathbb{T}\\ &\quad [t \in \dot{\mathbb{Q}}_{n+1} \leftrightarrow \exists \text{ limit } \gamma \exists k \in \omega[\mathrm{dmn}(t) = \gamma + k \text{ and } k = \dot{f}_n(\gamma+\omega)]]]\end{aligned}$$

Now by definition, $q \in \mathbb{P}_\omega$ iff q is a function with domain ω and $\forall n \in \omega[q \restriction n \Vdash [q_n \in \dot{\mathbb{Q}}_n]]$. Now let $q \in \mathbb{P}_\omega$, $n \in \omega$, and suppose that $q_{n+1} \neq \mathbb{1}$. Then $q \restriction (n+1) \Vdash [q_{n+1} \in \dot{\mathbb{Q}}_{n+1}]$, so

$$q \restriction (n+1) \Vdash \exists \text{ limit } \gamma \exists k \in \omega[\mathrm{dmn}(q_{n+1}) = \gamma + k \text{ and } k = \dot{f}_n(\gamma+\omega)].$$

By the maximal principle twice, there are $\mathbb{P}_{n+1}$ names $\dot{\gamma}$ and $\dot{k}$ such that

$$q \restriction (n+1) \Vdash [\dot{\gamma} \text{ is a limit ordinal and } \mathrm{dmn}(q_{n+1}) = \dot{\gamma} + \dot{k} \text{ and } \dot{k} = \dot{f}_n(\dot{\gamma}+\omega)].$$

Now let G be $\mathbb{P}_\omega$-generic over M. Let $G_n = (i^\omega_n)^{-1}[G]$ and $G_{n+1} = (i^\omega_{n+1})^{-1}[G]$. Since $q \restriction (n+1) \Vdash \dot{k} = \dot{f}_n(\dot{\gamma}+\omega)$, it follows by (1) that $\dot{\gamma}_{G_{n+1}} + \omega \in \mathrm{dmn}(q_{nG_n})$. But also $\dot{\gamma}_{G_{n+1}} + k = \mathrm{dmn}(q_{n+1,G_{n+1}})$. Thus $\mathrm{dmn}(q_{n+1,G_{n+1}}) < \mathrm{dmn}(q_{nG_n})$. It follows that there is an $m \in \omega$ such that $q_m = \mathbb{1}$. Hence our interated forcing construction has finite support.

Now clearly $\forall n[\mathbb{1} \Vdash [\dot{Q}_n$ has an antichain of size at least $\omega_1]]$, so by Proposition 31.100, $\mathbb{P}_\omega$ collapses ω_1. By Theorem 29.9, $\mathbb{P}_\omega$ is not countably closed. $\square$

If $\mathbb{P}$ is a forcing poset, then a $\mathbb{P}$-name τ is *full* iff for all $\mathbb{P}$-names σ and all $p \in \mathbb{P}$, if $p \Vdash [\sigma \in \tau]$, then there is a $\sigma' \in \mathrm{dmn}(\tau)$ such that $p \Vdash [\sigma' = \sigma]$ and $(\sigma', p) \in \tau$.

Lemma 31.129. (V.5.6) *If τ is a $\mathbb{P}$-name, then there is a full $\mathbb{P}$-name $\tilde{\tau}$ such that $\mathbb{1} \Vdash [\tilde{\tau} = \tau]$.*

Proof. Let $\pi = \bigcup \mathrm{dmn}(\tau)$. Thus π is a P-name. We claim

(1) $\mathbb{1} \Vdash \forall x \forall y[x \in y \in \tau \to x \in \pi]$.

In fact, let G be generic. Suppose that $x \in y \in \tau_G$. Then there is a $(\rho, p) \in \tau$ such that $p \in G$ and $y = \rho_G$; and there is a $(\xi, q) \in \rho$ such that $q \in G$ and $x = \xi_G$. So $\rho \in \mathrm{dmn}(\tau)$ and $(\xi, q) \in \rho$, so $(\xi, q) \in \pi$. It follows that $x = \xi_G \in \pi_G$, as desired.

Let E be the set of all nice names for subsets of π, and let $\tilde{\tau} = \{(\sigma', p) \in E \times P : p \Vdash \sigma' \in \tau\}$. Then

(2) $\mathbb{1} \Vdash \tilde{\tau} \subseteq \tau$.

In fact, suppose that G is generic and $x \in \tilde{\tau}_G$. Then there is a $(\sigma', p) \in \tilde{\tau}$ such that $p \in G$ and $x = \sigma'_G$. Now $p \Vdash \sigma' \in \tau$, so $x = \sigma'_G \in \tau_G$. This proves (2).

(3) If $p \Vdash \sigma \in \tau$, then there is a $\sigma' \in E$ such that $p \Vdash \sigma' = \sigma$; hence $p \Vdash \sigma' \in \tau$, and so $(\sigma', p) \in \tilde{\tau}$.

In fact, let σ' be a nice name for a subset of π such that $\mathbb{1} \Vdash \sigma' = \sigma \cap \pi$. Hence $\mathbb{1} \Vdash \sigma \subseteq \pi \to \sigma = \sigma'$. Now $p \Vdash \sigma \subseteq \pi$ by (1), so $p \Vdash \sigma = \sigma'$, as desired in (3).

A consequence of (3) is that $\mathbb{1} \Vdash \tau \subseteq \tilde{\tau}$. In fact, suppose that G is P-generic over M and $x \in \tau_G$. Say $x = \sigma_G$. Choose $p \in G$ such that $p \Vdash \sigma \in \tau$. Choose σ' by (3). Then $(\sigma', p) \in \tilde{\tau}$ and $p \in G$, so $x = \sigma_G = \sigma'_G \in \tilde{\tau}'_G$, as desired.

Together with (2) this shows that $\mathbb{1} \Vdash \tau = \tilde{\tau}$. Clearly $\tilde{\tau}$ is full. □

Proposition 31.130. (V.5.7) *If $\mathbb{P}$ is atomless and τ is a full $\mathbb{P}$-name such that $\tau_G = 2$ for every $\mathbb{P}$-generic G, then $|\tau| \geq 2^\omega$.*

Proof. Let B be a an infinite maximal antichain in $\mathbb{P}$. Suppose that $A \stackrel{\text{def}}{=} (A_0, A_1)$ is a partition of B. Let $\sigma^A = \{(\emptyset, p) : p \in A_0\}$. Suppose that $p \in G$, generic. If $p \in A_0$, then $\sigma^A_G = \{\emptyset_G\} = \{\emptyset\}$. If $p \notin A_0$, then $\sigma^A_G = \emptyset$. Hence $p \Vdash \sigma^A \in \tau$, so there is a $\sigma'^A \in \mathrm{dmn}(\tau)$ such that $p \Vdash \sigma'^A = \sigma^A$ and $(\sigma'^A, p) \in \tau$. Suppose that also $C \stackrel{\text{def}}{=} (C_0, C_1)$ is a partition of B, and $A_0 \neq C_0$. Say $p \in A_0 \backslash C_0$. Then for $p \in G$ generic we have $\sigma^A_G = \{\emptyset\}$ and $\sigma^C_G = \emptyset$. Since $p \Vdash \sigma'^A = \sigma^A$ and $\sigma'^C = \sigma^C$, we have $\sigma'^A_G \neq \sigma'^C_G$, hence $\sigma'^A \neq \sigma'^C$. So we get at least 2^ω members of τ. □

Lemma 31.131. (V.5.8) *Suppose given an α-stage iterated forcing construction such that for each $\xi < \alpha$, $\dot{Q}_\xi$ is a full name and $\mathbb{1}_{\mathbb{P}_\xi} \Vdash [\dot{Q}_\xi$ is countably closed]. Then $\mathbb{P}_\alpha$ is countably closed.*

Proof. Suppose that $p^n \in \mathbb{P}_\alpha$ for all $n \in \omega$, and $p^0 \geq p^1 \geq \cdots$. We now define $\langle q^\omega_\mu : \mu < \alpha \rangle$ by recursion so that the following conditions hold for each $\xi \leq \alpha$:

(1_ξ) $q^\omega \restriction \xi \in \mathbb{P}_\xi$.

(2_ξ) $\forall n \in \omega[q^\omega \restriction \xi \leq p^n \restriction \xi]$.

(3_ξ) $\forall \mu \notin \bigcup_{n\in\omega} \mathrm{supp}(p^n)[q^\omega_\mu = \mathbb{1}]$.

For the successor step, suppose that $q^\omega \restriction \xi$ has been defined so that (1_ξ)–(3_ξ) hold. We want to define q^ω_ξ. If $\xi \notin \bigcup_{n\in\omega} \mathrm{supp}(p^n)$, let $q^\omega_\xi = \mathbb{1}$. Now by Definition V.3.11.7, $p^{n+1} \restriction \xi \Vdash [p^{n+1}_\xi \leq p^n_\xi]$, and hence $p^\omega \restriction \xi \Vdash [p^{n+1}_\xi \leq p^n_\xi]$. Since $\mathbb{1}_{\mathbb{P}_\xi} \Vdash [\dot{Q}_\xi$ is countably closed], it follows that $p^\omega \restriction \xi \Vdash \exists r \forall n \in \omega[r \leq p^n_\xi]$. Hence by the maximal principle there is a $\mathbb{P}_\xi$-name s such that $p^\omega \restriction \xi \Vdash \forall n \in \omega[s \leq p^n_\xi]$. Now $p^\omega \restriction \xi \Vdash s \in \dot{\mathbb{Q}}_\xi$ and $\dot{Q}_\xi$ is a full name. So there is a $q^\omega_\xi \in \mathrm{dmn}(\mathbb{Q}_\xi)$ such that $p^\omega \restriction \xi \Vdash s = q^\omega_\xi$ and $(q^\omega_\xi, p^\omega \restriction \xi) \in \dot{\mathbb{Q}}_\xi$. Hence ($1_{\xi+1}$)–($3_{\xi+1}$) hold.

The limit step is clear. Finally, we have $q^\omega \leq p^n$ for all $n \in \omega$. □

$\mathbb{P}$ is ω_1-*linked* iff $\mathbb{P}$ is the union of ω_1 linked subfamilies. $\mathbb{P}$ is ω_1-*centered* iff $\mathbb{P}$ is the union of ω_1 centered subfamilies.

Proposition 31.132. *If* $\mathrm{Fn}(\kappa, 2, \lambda)$ *is linked, then it is centered.*

Proof. Assume that $\mathrm{Fn}(\kappa, 2, \lambda)$ is linked, and $p_1, \ldots, p_n \in \mathrm{Fn}(\kappa, 2, \lambda)$. Then $p_1 \cup p_2 \in \mathrm{Fn}(\kappa, 2, \lambda)$. If $\alpha \in \mathrm{dmn}(p_1) \cap \mathrm{dmn}(p_3)$, then $p_1(\alpha) = p_3(\alpha)$. Similarly for p_2 and p_3. So $p_1 \cup p_2 \cup p_3 \in \mathrm{Fn}(\kappa, 2, \lambda)$. Etc. $\square$

Proposition 31.133. (V.5.10) *Let* κ *be an infinite cardinal, and let* $\mathbb{P}_\kappa = \mathrm{Fn}(\kappa, 2, \omega_1)$. *Then* $\mathbb{P}_\kappa$ *is* ω_1*-linked iff it is* ω_1*-centered.*

Proof. $\Leftarrow$ is obvious. Now suppose that $\mathbb{P}_\kappa$ is ω_1-linked; say $\mathbb{P}_\kappa = \bigcup_{\alpha<\omega_1} \mathbb{Q}_\alpha$ with each $\mathbb{Q}_\alpha$ linked.

$(*)$ $\forall \alpha < \omega_1 \forall n \in \omega \forall p_1, \ldots, p_n \in \mathbb{Q}_\alpha[(p_1 \cup \ldots \cup p_n) \in \mathbb{P}_\kappa]$.

This is clear by induction on n. Now for each $\alpha < \omega_1$ let $\mathbb{Q}'_\alpha = \{p_1 \cup \ldots \cup p_n : n \in \omega, p_1, \ldots, p_n \in \mathbb{Q}_\alpha\}$. Then by $(*)$, $\mathbb{Q}'_\alpha \subseteq \mathbb{P}_\kappa$. $\mathbb{Q}'_\alpha$ is clearly centered; and clearly $\mathbb{P}_\kappa = \bigcup_{\alpha<\omega_1} \mathbb{Q}'_\alpha$. $\square$

Proposition 31.134. *Assume CH. There is an* $\mathscr{F} \subseteq {}^{2^{\omega_1}}2$ *such that* $|\mathscr{F}| = 2^{\omega_1}$ *and for all* $j \in {}^\omega 2$ *and one-one* $f \in {}^\omega\mathscr{F}$ *there is an* $\alpha < 2^{\omega_1}$ *such that* $\forall n \in \omega[f_n(\alpha) = j_n]$.

Proof. Let $E = \{(s, p) : s \in [\omega_1]^\omega, p : \mathscr{P}(s) \to 2\}$. Note that $|E| = 2^{\omega_1}$. Let $g : 2^{\omega_1} \to E$ be a bijection. For each $A \subseteq \omega_1$ and each $\alpha < 2^{\omega_1}$, with $g(\alpha) = (s, p)$ let $f_A(\alpha) = p(A \cap s)$. Suppose that $A \in {}^\omega\mathscr{P}(\omega_1)$ is injective and $i \in {}^\omega\omega_1$. For $m \neq n$ let $a_{mn} \in A_m \triangle A_n$. Suppose that $\{a_{mn} : m \neq n\} \subseteq s \in [\omega_1]^\omega$. For each $X \subseteq s$ let

$$p(X) = \begin{cases} i_n & \text{if } X = A_n \cap s, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\forall n \in \omega[f_{A_n}(g^{-1}(s, p)) = p(A_n \cap s) = i_n]$. $\square$

Proposition 31.135. (V.5.10) *Assume CH. Let* κ *be an infinite cardinal, and let* $\mathbb{P}_\kappa = \mathrm{Fn}(\kappa, 2, \omega_1)$. *Then* $\mathbb{P}_\kappa$ *is* ω_1*-centered iff* $\kappa \leq 2^{\omega_1}$.

Proof. First suppose that $\mathbb{P}_\kappa$ is ω_1-centered, but $\kappa > 2^{\omega_1}$. Say $\mathbb{P}_\kappa = \bigcup_{\alpha<\omega_1} \mathbb{Q}_\alpha$ with each $\mathbb{Q}_\alpha$ centered. For each $\alpha < \omega_1$ let $f_\alpha \in {}^\kappa 2$ be such that $\bigcup \mathbb{Q}_\alpha \subseteq f_\alpha$. Then for each $\beta \in \kappa$ we have $\langle f_\alpha(\beta) : \alpha < \omega_1\rangle \in {}^{\omega_1}2$. Hence there exist distinct $\beta, \gamma < \kappa$ such that $\langle f_\alpha(\beta) : \alpha < \omega_1\rangle = \langle f_\alpha(\gamma) : \alpha < \omega_1\rangle$. Let $h = \{(\beta, 0), (\gamma, 1)\}$. Thus $h \in \mathbb{P}_\kappa$. Say $h \in \mathbb{Q}_\alpha$. Then $f_\alpha(\beta) = h(\beta) = 0 \neq 1 = h(\gamma) = f_\alpha(\gamma)$, contradiction.

Second suppose that $\kappa \leq 2^{\omega_1}$. Let $\mathscr{F}$ be as in Proposition 31.134, and let $f : \kappa \to \mathscr{F}$ be an injection. Now for each $\alpha < \omega_1$ we define $x_\alpha \in {}^\kappa 2$ by setting $x_\alpha(\beta) = f_\beta(\alpha)$ for any $\beta < \kappa$. For each $\alpha < \omega_1$ let $K_\alpha = \{h \in \mathrm{Fn}(\kappa, 2, \omega_1) : h \subseteq x_\alpha\}$. Clearly K_α is centered. We claim that $\mathbb{P}_\kappa = \bigcup_{\alpha<\omega_1} K_\alpha$. For, suppose that $h \in \mathbb{P}_\kappa$. Let $\mathrm{dmn}(h) = \{\beta_i : i \in \omega\}$. Define $g_i = f_{\beta_i}$. By the condition of Proposition 31.134, choose $\alpha < 2^{\omega_1}$ such that $\forall i \in \omega[g_i(\alpha) = h(\beta_i)$. Then for any $i \in \omega$, $x_\alpha(\beta_i) = f_{\beta_i}(\alpha) = g_i(\alpha) = h(\beta_i)$. Thus $h \in K_\alpha$. $\square$

A forcing poset $\mathbb{P}$ is *well-met* iff for all $p_1, p_2 \in \mathbb{P}$, if p_1, p_2 are compatible, then there is a $q \leq p_1, p_2$ such that for all $r \leq p_1, p_2[r \leq q]$.

Corollary 31.136. *Assume CH. Let κ be an infinite cardinal, and let $\mathbb{P}_\kappa = \mathrm{Fn}(\kappa, 2, \omega_1)$. Then $\mathbb{P}_\kappa$ is countably closed, well-met, ω_1-centered, ω_1-linked, and ω_2-cc.*

Proof. It is obviously countably closed and well-met. It is ω_1-centered by Proposition 31.135, and it is ω_1-linked by Proposition 31.133. Clearly then it is also ω_2-cc. □

BA is the statement that $MA_{\mathbb{P}}(\kappa)$ holds for all $\kappa < 2^{\omega_1}$ and all countably closed well-met ω_1-linked forcing posets $\mathbb{P}$. BACH is the statement BA+CH.

Proposition 31.137. *$\neg CH$ implies BA.*

Proof. By Lemma 31.125. □

Proposition 31.138. *$CH + 2^{\omega_1} = \omega_2$ implies BACH.*

Proof. By Lemma 31.125. □

Proposition 31.139. *Assume CH. Let κ be an infinite cardinal, and let $\mathbb{P} = \mathrm{Fn}(\kappa, 2, \omega_1)$. Then $\mathrm{MA}_{\mathbb{P}}(2^{\omega_1})$ does not hold.*

Proof. Suppose it does hold. For each $\alpha \in \omega_1$ let

$$D_\alpha = \{f \in \mathbb{P} : \alpha \in \mathrm{dmn}(f)\}.$$

Each such set is dense in $\mathbb{P}$.

For each $h \in {}^{\omega_1}2$ let

$$E_h = \{f \in \mathbb{P} : \text{there is an } \alpha \in \mathrm{dmn}(f) \text{ such that } f(\alpha) \neq h(\alpha)\}.$$

Again, each such set E_h is dense in $\mathbb{P}$.

Now let G be $\mathbb{P}$-generic over M, and set $k = \bigcup G$. Clearly $k \in {}^{\omega_1}2$. Now take any $f \in G \cap E_k$. Choose $\alpha \in \mathrm{dmn}(f)$ such that $f(\alpha) \neq k(a)$. But $f \subseteq k$, contradiction. □

Lemma 31.140. (V.5.14) *Assume CH, and suppose given a countable support α-stage iterated forcing construction such that for each $\xi < \alpha$ $\dot{\mathbb{Q}}_\xi$ is a full name, and $\mathbb{1} \Vdash [\dot{\mathbb{Q}}_\xi$ is countably closed, well-met, and ω_1-linked$]$. Then $\mathbb{P}_\alpha$ has the ω_2-cc.*

Proof. By Lemma 31.131, $\mathbb{P}_\alpha$ is countably closed, and hence ω_1 is preserved. Now for each $\xi < \alpha$ let $\dot{L}_\xi$ be a $\mathbb{P}_\xi$-name such that

$$\mathbb{1}_{\mathbb{P}_\xi} \Vdash [\dot{L}_\xi : \dot{\mathbb{Q}}_\xi \to \omega_1 \text{ and } \forall\lambda < \omega_1[\dot{L}_\xi^{-1}[\{\gamma\}] \text{ is linked] and } \dot{L}_\xi(\mathbb{1}) = 0].$$

Now we claim

$$(*) \qquad \begin{aligned} &\forall p \in \mathbb{P}_\alpha \exists p^* \leq p[\mathrm{supp}(p) \subseteq \mathrm{supp}(p^*) \text{ and } \forall \xi < \alpha \\ &\qquad \exists \gamma(p,\xi) < \omega_1[p^* \restriction \xi \Vdash [\dot{L}_\xi(p_\xi) = \gamma(p,\xi)]]] \end{aligned}$$

In fact, suppose that $p \in \mathbb{P}_\alpha$. Let $\mathrm{supp}(p) = \{\eta_n : n \in \omega\}$. We now define $\langle r_n : n \in \omega\rangle$ by recursion. Let $r_0 = p$. Suppose that $r_n \le p$ has been defined. Then $r_n \restriction \eta_n \Vdash \exists\delta < \omega_1[\dot{L}_{\eta_n}(p_{\eta_n}) \in \omega_1]$, so there exist an $s \le r_n \restriction \eta_n$ and a $\gamma(p, \eta_n) \in \omega_1$ such that $s \Vdash [\dot{L}_{\eta_n}(p_{\eta_n}) = \gamma(p, \eta_n)]$. Let $r_{n+1} \restriction \eta_n = s$, $r_{n+1,\eta_n} = p_{\eta_n}$ if $r_{n\eta_n} = \mathbb{1}$, equal to $r_{n\eta_n}$ otherwise, and $r_{n+1} \restriction (\alpha\backslash(\eta_n + 1) = r_n \restriction (\alpha\backslash(\eta_n + 1))$. Finally, let $p^* \le r_n$ for all n using the fact that $\mathbb{P}_\alpha$ is countably closed. This proves $(*)$.

Now for each $p \in \mathbb{P}_\alpha$ let ${}^0p = p$ and ${}^{n+1}p = ({}^np)^*$. Let $\Sigma(p) = \bigcup_{n\in\omega} \mathrm{supp}({}^np)$.

Now take any $A \in [\mathbb{P}_\alpha]^{\omega_2}$; we show that A is not an antichain. We apply the Δ-system theorem, Theorem 24.4, to the sets $\Sigma(p)$ for $p \in A$, with $\kappa = \omega_1$ and $\lambda = \omega_2$, recalling that CH is assumed. We get a $B \in [A]^{\omega_2}$ such that $\langle\Sigma(p) : p \in B\rangle$ forms a Δ-system, say with root $R \in [\alpha]^{\le\omega}$. Now

$$B = \bigcup\{\{x \in B : \langle\gamma({}^nx, \xi) : n \in \omega, \xi \in R\rangle = f\} : f \in {}^{\omega\times R}\omega_1\}.$$

It follows that there are distinct $x, y \in B$ such that $\langle\gamma({}^nx, \xi) : n \in \omega, \xi \in R\rangle = \langle\gamma({}^ny, \xi) : n \in \omega, \xi \in R\rangle$. We claim that $x \not\perp y$ (as desired).

We define p_ξ for $\xi < \alpha$ by recursion. Assume inductively that for all n, $p \restriction \xi \le ({}^nx) \restriction \xi$ and $p \restriction \xi \le ({}^ny) \restriction \xi$. If $\xi \notin \Sigma(x)\cup\Sigma(y)$. let $p_\xi = \mathbb{1}$. Now suppose that $\xi \in (\Sigma(x)\cup\Sigma(y))\backslash R$. Say $\xi \in \Sigma(x)$. Note that $\xi \notin \Sigma(y)$. For each $n \in \omega$ let $r_n = ({}^nx)_\xi$. Thus $r_0 = x_\xi$. We claim:

$(**)$ $\forall n \in \omega[p \restriction \xi \Vdash [r_{n+1} \le r_n]]$.

In fact, take any $n \in \omega$. Now $r_{n+1} = ({}^{n+1}x)_\xi = (({}^nx)^*)_\xi$ and $({}^nx)^* \le {}^nx$, so $({}^nx)^* \restriction \xi \Vdash [({}^nx)^*_\xi \le ({}^nx)_\xi]$, i.e. $({}^{n+1}x) \restriction \xi \Vdash [r_{n+1} \le r_n]$. Since $p \restriction \xi \le ({}^{n+1}x) \restriction \xi$, $(**)$ follows.

Since $\mathbb{1} \Vdash [\dot{\mathbb{Q}}_\xi$ is countably closed], and $\dot{\mathbb{Q}}_\xi$ is a full name, there is a $\dot{p}_\xi \in \mathrm{dmn}(\dot{\mathbb{Q}}_\xi)$ such that $(p \restriction \xi)^\frown\langle\dot{p}_\xi\rangle \le ({}^nx) \restriction (\xi + 1)$ for all n. Since $\xi \notin \Sigma(y)$, we have $({}^ny)_\xi = \mathbb{1}$ for all n, so also $(p \restriction \xi)^\frown\langle\dot{p}_\xi\rangle \le ({}^ny) \restriction (\xi + 1)$ for all n. Now since $\mathbb{1} \Vdash [\dot{Q}_\xi$ is countably closed, there is a full name $p_\xi \in \mathrm{dmn}(\dot{Q}_\xi)$ such that $p \restriction \xi \Vdash [p_\xi \le ({}^nx)_\xi, ({}^ny)_\xi]$ for all n.

Now suppose that $\xi \in R$. For brevity let ${}^n\gamma = \gamma(x^n, \xi)$ $(= \gamma(y^n, \xi))$. Now for any $n \in \omega$, ${}^{n+1}x = ({}^nx)^*$, and $({}^nx)^* \restriction \xi \Vdash [\dot{L}_\xi(({}^nx)_\xi) = \gamma({}^nx, \xi) = {}^n\gamma]$, so $p \restriction \xi \Vdash [\dot{L}_\xi(({}^nx)_\xi) = {}^n\gamma]$. Similarly, $p \restriction \xi \Vdash [\dot{L}_\xi(({}^ny)_\xi) = {}^n\gamma]$. Thus $p \restriction \xi \Vdash [{}^nx, {}^ny \in \dot{L}_\xi^{-1}[\{{}^n\gamma\}]$. Since $\mathbb{1} \Vdash [\dot{L}_\xi^{-1}[\{{}^n\gamma\}]$ is linked and $\dot{Q}_\xi$ is well-met], it follows that $p \restriction \xi \Vdash [({}^nx)_\xi \wedge ({}^ny)_\xi \in \dot{Q}_\xi]$. Now ${}^{n+1}x = ({}^nx)^* \le {}^nx$, so ${}^{n+1}x \restriction \xi \Vdash [({}^{n+1}x)_\xi \le (nx)_\xi]$, so $p \restriction \xi \Vdash [({}^{n+1}x)_\xi \le (nx)_\xi]$. Similarly, $p \restriction \xi \Vdash [({}^{n+1}y)_\xi \le (ny)_\xi]$. Hence $p \restriction \xi \Vdash [({}^{n+1}x)_\xi \wedge ({}^{n+1}y)_\xi \le ({}^nx)_\xi \wedge ({}^ny)_\xi]$. Again since $\mathbb{1} \Vdash [\dot{Q}_\xi$ is countably closed, there is a full name $p_\xi \in \mathrm{dmn}(\dot{Q}_\xi)$ such that $p \restriction \xi \Vdash [p_\xi \le ({}^nx)_\xi, ({}^ny)_\xi]$ for all n.

This finishes the definition of the p_ξ. Hence $p \le x, y$. $\square$

Proposition 31.141. *Suppose that $\lambda < 2^\omega$, $\mathbb{P}$ is a countably closed well-met ω_1-linked poset such that $\mathrm{MA}_\mathbb{P}(\lambda)$ is false. Then there is a countably closed well-met ω_1-linked poset $\mathbb{Q} \le_{\mathrm{ctr}} \mathbb{P}$ such that $|\mathbb{Q}| \le \lambda^\omega$ and $\mathrm{MA}_\mathbb{Q}(\lambda)$ is false.*

Proof. Fix dense sets D_α in $\mathbb{P}$ for $\alpha < \lambda$ such that there is no filter intersecting each D_α. Consider the structure $(\mathbb{P}, \le, \mathbb{1}, D_\alpha)_{\alpha<\kappa}$. Let A be a set of Skolem functions for this structure. Let F be an ω-ary operation on $\mathbb{P}$ assigning to each decreasing sequence an element below it. Then let $(\mathbb{Q}, \leq, \mathbb{1}, D'_\alpha)_{\alpha<\kappa}$ be the closure of some one-element subset of $\mathbb{P}$ under A and P.

(1) $|\mathbb{Q} \leq \lambda^\omega$.

(2) D'_α is dense in $\mathbb{Q}$ for each $\alpha < \lambda$.

In fact, if $q \in \mathbb{Q}$, then $\mathbb{P} \models \exists x \in D_\alpha[x \leq q]$, so $\mathbb{Q} \models \exists x \in D'_\alpha[x \leq q]$.

(3) $\mathbb{Q} \subseteq_{\text{ctr}} \mathbb{P}$.

For, if $F \in [\mathbb{Q}]^{<\omega}$ and $\mathbb{P} \models \exists p[\bigwedge_{q\in F}[p \leq q]$, then $\mathbb{Q} \models \exists p[\bigwedge_{q\in F}[p \leq q]$.

(4) $\mathbb{Q}$ is countably closed.

(5) $\mathbb{Q}$ is well-met.

In fact, this holds in $\mathbb{P}$ and it is a first-order sentence, so it holds in $\mathbb{Q}$.

(6) $\mathbb{Q}$ is ω_1-linked.

For say $\mathbb{P} = \bigcup_{\alpha<\omega_1} S_\alpha$ with each S_α linked. Suppose that $p, q \in S_\alpha \cap \mathbb{Q}$. Then $\mathbb{P} \models \exists r[r \leq p, q]$, so $\mathbb{Q} \models \exists r[r \leq p, q]$.

(7) $\text{MA}_\mathbb{Q}(\kappa)$ does not hold.

Suppose that $\text{MA}_\mathbb{Q}(\kappa)$ holds. Let G be a filter on $\mathbb{Q}$ which intersects each D'_α. Let $G^+ = \{p \in \mathbb{P} : \exists q \in G[q \leq p]\}$. Clearly G^+ is a filter on $\mathbb{P}$, and it intersects each D_α, contradiction. □

$\text{tbc}^*(\alpha, \sqsubseteq)$ abbreviates the statement that α is a nonzero ordinal, $\sqsubseteq$ is a subset of $\alpha \times \alpha$, and $(\alpha, \sqsubseteq, 0)$ is a countably closed, well-met, ω_1-linked forcing poset.

$\text{ntbc}^*(\alpha, \dot{\sqsubseteq}, \mathbb{P})$ holds iff $\dot{\sqsubseteq}$ is a nice name for a subset of $\alpha \times \alpha$ and $\mathbb{1} \Vdash_\mathbb{P} \text{tbc}^*(\alpha, \dot{\sqsubseteq})$.

Lemma 31.142. *Let M be a c.t.m. for ZFC. In M let θ be an infinite cardinal. Let G be $\mathbb{P}$-generic over M. In $M[G]$ let $(X, \leq_X, \mathbb{1}_X)$ be a countably closed well-met ω_1-linked forcing poset with $|X| \leq \theta$.*

Then there is a name $\dot{\sqsubseteq}$ in M and an $\alpha \leq \theta$ such that $\text{Ntbc}(\alpha, \dot{\sqsubseteq}.\mathbb{P})$ *holds in M and such that in $M[G]$ the poset $(\alpha, \dot{\sqsubseteq}_G, 0)$ is isomorphic to $(X, \leq_X, \mathbb{1}_X)$.*

Proof. See the proof of Lemma 31.63.

Theorem 31.143. *Assume that $\kappa > \omega_1$ is a regular cardinal, $2^{<\kappa} = \kappa$, and $\forall\lambda < \kappa[\lambda^\omega < \kappa]$. Then there is a countably closed ω_2-cc poset $\mathbb{P}$ of size κ such that $\mathbb{1}_\mathbb{P} \Vdash BACH$ and $2^{\omega_1} = \kappa$.*

Proof. We start with a model M of CH. Let $f : \kappa \to \kappa \times \kappa$ be the bijection given by the proof of Theorem 11.32; see Proposition 31.64.

We are going to define by recursion a countable support κ-stage iteration. The starting stage is trivial; $\mathbb{P} = \{\emptyset\}$. The limit stage is determined by the previous stages. Now we make the step from $\xi < \kappa$ to $\xi + 1$. We assume that for each $\zeta \leq \xi$ we have specified a sequence $\langle(\alpha^\mu_\zeta, \dot{\sqsubseteq}^\mu_\zeta) : \mu < \kappa\rangle$ listing all pairs $(\alpha, \dot{\sqsubseteq})$ such that $0 < \alpha < \kappa$ and $\dot{\sqsubseteq}$ is a nice $\mathbb{P}_\zeta$-name for a subset of $\alpha\times\alpha$. As the inductive hypothesis we assume that $|\mathrm{dmn}(\mathbb{Q}_\zeta)|, |\mathbb{P}_\zeta| < \kappa$ for each $\zeta < \xi$, and

$$(\langle(\mathbb{P}_\zeta, \leq_\zeta, \mathbb{1}_\zeta) : \zeta \leq \xi\rangle, \langle(\dot{\mathbb{Q}}_\zeta, \dot{\sqsubseteq}_{\dot{\mathbb{Q}}_\zeta}, \mathbb{1}_{\dot{\mathbb{Q}}_\zeta}) : \zeta < \xi\rangle)$$

is a countable support ξ-stage iterated forcing construction.

Let $f(\xi) = (\zeta, \mu)$. By Proposition 31.64, $\zeta \leq \xi$. Then $\dot{\sqsubseteq}^\mu_\zeta$ is a nice $\mathbb{P}_\zeta$-name for a subset of $\alpha^\mu_\zeta\times\alpha^\mu_\zeta$. Hence by Proposition 31.65, $(i^\xi_\zeta)_*(\dot{\sqsubseteq}^\mu_\zeta)$ is a nice $\mathbb{P}_\xi$-name for a subset of $\alpha^\mu_\zeta\times\alpha^\mu_\zeta$. If $\mathrm{Ntbc}^*(\alpha^\mu_\zeta, (i^\xi_\zeta)_*(\dot{\sqsubseteq}^\mu_\zeta), \mathbb{P}_\xi)$, then we let $(\dot{\mathbb{Q}}_\xi, \dot{\preceq}_{\dot{\mathbb{Q}}_\xi}, \dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\xi})$ be $(\check{\alpha}^\mu_\zeta, \star, \check{0})$, where $\star$ is the full name obtained from $(i^\xi_\zeta)_*(\dot{\sqsubseteq}^\mu_\zeta)$ by the proof of Lemma 31.129. If $\neg\mathrm{Ntbc}^*(\alpha^\mu_\zeta, (i^\xi_\zeta)_*(\dot{\sqsubseteq}^\mu_\zeta), \mathbb{P}_\xi)$, then we let $\dot{\mathbb{Q}}_\xi = \{(\emptyset, \mathbb{1}_{\mathbb{P}_\xi})\}$, $\dot{\preceq}_{\dot{\mathbb{Q}}_\xi} = \emptyset$, and $\dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\xi} = \emptyset$.

This completes the construction of our countable support κ-stage iteration.

(1) $\forall\xi < \kappa[|\mathbb{P}_\xi| < \kappa \wedge |\dot{\mathbb{Q}}_\xi| < \kappa]$.

This is clear by induction, using the regularity of κ at the limit stages.

(2) $\forall\xi < \kappa[\mathbb{1}_{\mathbb{P}_\xi} \Vdash [\dot{\mathbb{Q}}_\xi$ is countably closed, well-met, and ω_1-linked$]]$.

This holds by definition of Ntbc^*.

Next, note that

(3) $\kappa^{\omega_1} = \kappa$.

In fact,

$$\kappa^{\omega_1} = |{}^{\omega_1}\kappa| \leq \sum_{\lambda<\kappa} \lambda^{\omega_1} \leq \sum_{\lambda<\kappa} 2^\lambda = 2^{<\kappa} = \kappa.$$

Let $\mathbb{P} = \mathbb{P}_\kappa$, and let G be $\mathbb{P}$-generic over M.

(4) $\mathbb{P}$ is countably closed, and hence CH holds in $M[G]$.

This is true by Lemma 31.131.

(5) $\mathbb{P}_\kappa$ has the ω_2-cc.

This holds by Lemma 31.138.

Now we apply Proposition 29.22 with $\lambda = \omega_2$ and $\mu = \omega_1$; we get

(6) $M[G] \models [2^{\omega_1} \leq \kappa]$.

Now for each $\xi < \kappa$ let $\mathbb{P}'_\xi = i^\kappa_\xi[\mathbb{P}_\xi]$. Then for $\xi < \eta < \kappa$ we have $\mathbb{P}'_\xi \subseteq \mathbb{P}'_\eta \subseteq \mathbb{P}$.

Suppose that $\theta < \kappa$. Take a countably closed well-met ω_1-linked forcing poset $\mathbb{Q}$; we want to show that $\mathrm{MA}_\mathbb{Q}(\theta)$. By Lemma 31.140 we may assume that $|\mathbb{Q}| \leq \theta$. Take a family $\mathscr{D}$ of dense subsets of $\mathbb{Q}$ with $|\mathscr{D}| \leq \theta$. Then by Lemma 31.142 we get a $\mathbb{P}$-name $\dot{\sqsubseteq}$ and an $\alpha \leq \theta$ such that $\mathrm{Ntbc}^*(\alpha, \dot{\sqsubseteq}, \mathbb{P})$ holds and $(\alpha, \dot{\sqsubseteq}_G, \mathbb{P})$ is isomorphic to $\mathbb{Q}$. So we may assume that $\mathbb{Q} = (\alpha, \dot{\sqsubseteq}_G, 0)$. Let $\langle D^\nu : \nu < \theta\rangle$ enumerate $\mathscr{D}$. Thus $D^\nu \subseteq \alpha$ for each $\nu < \theta$. Let $\dot{D}^\nu$ be a nice $\mathbb{P}$-name for a subset of α such that $D^\nu = \dot{D}^\nu_G$. The names $\dot{\sqsubseteq}$ and $\dot{D}^\nu$ for $\nu < \theta$ altogether involve fewer than κ members of $\mathbb{P}$. Hence there exists a $\zeta < \kappa$ such that all of these names are $\mathbb{P}'_\zeta$-names. Let $\dot{\sqsubseteq}'$ be a $\mathbb{P}_\zeta$-name such that $\dot{\sqsubseteq} = (i^\kappa_\zeta)_*(\dot{\sqsubseteq}')$. Then there is a $\mu < \kappa$ such that $(\alpha^\mu_\zeta, \dot{\sqsubseteq}^\mu_\zeta)$ is $(\alpha, \dot{\sqsubseteq}')$. Let $\xi = f^{-1}(\zeta, \mu)$. Now by Lemma 31.142 we get $\mathrm{Ntbc}^*(\alpha, \dot{\sqsubseteq}, \mathbb{P}'_\xi)$. Hence $\mathrm{Ntbc}^*(\alpha, (i^\xi_\zeta)_*(\dot{\sqsubseteq}'), \mathbb{P}_\xi)$. That is, $\mathrm{Ntbc}^*(\alpha^\mu_\zeta, (i^\xi_\zeta)_*(\dot{\sqsubseteq}^\mu_\zeta), \mathbb{P}_\xi)$. Hence $(\dot{\mathbb{Q}}_\xi, \dot{\preceq}_{\dot{\mathbb{Q}}_\xi}, \dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\xi}) = (\check{\alpha}^\mu_\zeta, (i^\xi_\zeta)_*(\dot{\sqsubseteq}^\mu_\zeta), \check{0})$ by construction. Note that

$$\dot{\sqsubseteq}_G = ((i^\kappa_\zeta)_*(\dot{\sqsubseteq}'))_G = (i^\kappa_\xi)_*((i^\xi_\zeta)_*(\dot{\subseteq}')) = ((i^\xi_\zeta)_*(\dot{\subseteq}'))_{G_\xi}.$$

Now we apply Proposition 31.60. Let $G_{\xi+1} = (i^\kappa_{\xi+1})^{-1}[G]$, f an isomorphism of $\mathbb{P}_{\xi+1}$ onto $\mathbb{P}_\xi * \dot{\mathbb{Q}}_\xi$, $G' = f[G_{\xi+1}]$, $H_\xi = \{\rho_{G_\xi} : \rho \in \mathbb{Q}_\xi \wedge \exists p \in \mathbb{P}_\xi[(p, \rho) \in G']\}$. Now by Proposition 31.60, H_ξ is $(\dot{\mathbb{Q}}_\xi)_{G_\xi}$-generic over $M[G_\xi]$ and $M[G'] = M[G_\xi][H_\xi]$. Let $\dot{D}^{\nu\prime}$ be a $\mathbb{P}_\xi$-name such that $\dot{D}^\nu = (i^\kappa_\xi)_*(\dot{D}^{\nu\prime})$. Then $D^\nu = \dot{D}^\nu_G = (\dot{D}^{\nu\prime})_{G_\xi} \in \mathbb{P}_\xi[G_\xi]$. Each D^ν is dense in $(\dot{\mathbb{Q}}_\xi)_{G_\xi}$, so $H_\xi \cap D^\nu \neq \emptyset$ for all $\nu < \theta$. $\square$

Let θ be a regular cardinal. We define

$$\begin{array}{rcl} f \leq^\theta g & \text{iff} & f, g \in {}^\theta\theta \text{ and } |\{\xi < \theta : f(\xi) > g(\xi)\}| < \theta; \\ \mathscr{D} \subseteq {}^\theta\theta \text{ is almost dominating} & \text{iff} & \forall f \in {}^\theta\theta \exists g \in \mathscr{D}[f \leq^\theta g]; \\ \mathfrak{d}_\theta & = & \min\{|\mathscr{D}| : \mathscr{D} \subseteq {}^\theta\theta \text{ is almost dominating}\}; \\ \mathscr{B} \subseteq {}^\theta\theta \text{ is almost unbounded} & \text{iff} & \neg\exists f \in {}^\theta\theta \forall g \in \mathscr{B}[g \leq^\theta f]; \\ \mathfrak{b}_\theta & = & \min\{|\mathscr{B}| : \mathscr{B} \subseteq {}^\theta\theta \text{ is almost unbounded}\}. \end{array}$$

Also, $\mathscr{E} \subseteq \mathscr{P}(\theta)$ *has the strong θ-intersection property, SθIP*, iff $\forall \mathscr{F} \in [\mathscr{E}]^{<\theta}[|\bigcap \mathscr{F}| = \theta]$.

$$\mathfrak{p}_\theta = \min\{|\mathscr{E}| : \mathscr{E} \text{ has the strong S}\theta\text{IP and } \neg\exists K \in [\theta]^\theta \forall Z \in \mathscr{E}[|K \backslash Z| < \theta]\}.$$

Proposition 31.144. (V.5.17) *For any regular cardinal θ, $\theta^+ \leq \mathfrak{p}_\theta$.*

Proof. Suppose that $\mathscr{F} = \{A_\alpha : \alpha \in \theta\}$ has SθIP. For each $\alpha \in \theta$ choose $k_\alpha \in \left(\bigcap_{\beta \leq \alpha} A_\beta\right) \backslash \{k_\beta : \beta < \alpha\}$. Then $y \stackrel{\text{def}}{=} \{k_\alpha : \alpha \in \theta\}$ has size θ, and for each $\alpha < \theta$, $y \backslash A_\alpha \subseteq \{k_\beta : \beta < \alpha\}$. $\square$

Proposition 31.145. (V.5.17) *For any regular cardinal θ, $\mathfrak{p}_\theta \leq \mathfrak{b}_\theta$.*

Proof. Let $\kappa < \mathfrak{p}_\theta$. Suppose that $\mathscr{B} \subseteq {}^\theta\theta$ and $|\mathscr{B}| \leq \kappa$. For each $\alpha < \theta$ and $f \in \mathscr{B}$ let $Z^\alpha_f = \{\beta < \theta : \beta > f(\alpha)\}$. Then for any $\alpha < \theta$,

$$\bigcap_{f \in \mathscr{B}} Z^\alpha_f = \{\beta < \theta : \forall f \in \mathscr{B}[\beta > f(\alpha)]\} = \left\{\beta < \theta : \beta > \sup_{f \in \mathscr{B}} f(\alpha)\right\},$$

a set of size θ. Hence there is a $K^\alpha \subseteq \theta$ such that for each $f \in \mathscr{B}$, $K^\alpha \subseteq^\theta Z^\alpha_f$, with $|K^\alpha| = \theta$. For each $f \in \mathscr{B}$ choose $\beta_{\alpha f} < \theta$ such that $K^\alpha \backslash Z^\alpha_f \subseteq \beta_{\alpha f}$. Thus $\forall f \in \mathscr{B} \forall \gamma \geq \beta_{\alpha f}[\gamma \in K^\alpha \to \gamma \in Z^\alpha_f]$. Let $\gamma = \sup_{f \in \mathscr{B}} \beta_{\alpha f}$, and let $g(\alpha)$ be a member of K^α which is $\geq \gamma$. Then for all $\alpha < \theta$, $\forall f \in \mathscr{B}[g(\alpha) \in Z^\alpha_f]$, so $g(\alpha) > f(\alpha)$. $\square$

Proposition 31.146. (V.5.17) *For any regular cardinal θ,* $\text{cf}(\mathfrak{b}_\theta) = \mathfrak{b}_\theta$.

Proof. Suppose that $\text{cf}(\mathfrak{b}_\theta) < \mathfrak{b}_\theta$. Let X be almost unbounded with $|X| = \mathfrak{b}_\theta$. Then we can write $X = \bigcup_{\alpha<\text{cf}(\mathfrak{b}_\theta)} Y_\alpha$ with $|Y_\alpha| < \mathfrak{b}_\theta$ for all $\alpha < \text{cf}(\mathfrak{b}_\theta)$. Choose a bound g^α for Y_α for each $\alpha < \text{cf}(\mathfrak{b}_\theta)$, and then by the above argument choose a bound h for $\{g^\alpha : \alpha < \text{cf}(\mathfrak{b}_\theta)\}$. Then h is a bound for X, contradiction. Thus $\text{cf}(\mathfrak{b}_\theta) = \mathfrak{b}_\theta$. $\square$

Proposition 31.147. (V.5.17) *For any regular cardinal θ,* $\mathfrak{b}_\theta \le \text{cf}(\mathfrak{d}_\theta)$.

Proof. Let D be a almost dominating family of size $\mathfrak{d}_\theta$, and write $D = \bigcup_{\alpha<\text{cf}(\mathfrak{d}_\theta)} E_\alpha$, with each E_α of size less than $\mathfrak{d}_\theta$. Since then E_α is not almost dominating, there is an $f^\alpha \in {}^\omega\omega$ such that for all $g \in E_\alpha$ we have $f^\alpha \not\le^\theta g$. Suppose that $\text{cf}(\mathfrak{d}_\theta) < \mathfrak{b}_\theta$, and accordingly let $h \in {}^\theta\theta$ be such that $f^\alpha \le^\theta h$ for all $\alpha < \text{cf}(\mathfrak{d}_\theta)$. Choose $k \in D$ such that $h \le^\theta k$. Say $k \in E_\alpha$. But $f^\alpha \le^\theta h \le^\theta k$, contradiction. $\square$

Proposition 31.148. (V.5.17) *For any regular cardinal θ,* $\mathfrak{d}_\theta \le 2^\theta$. $\square$

Theorem 31.149. (V.5.18) *BACH*$\vdash \mathfrak{p}_{\omega_1} = \mathfrak{b}_{\omega_1} = \mathfrak{d}_{\omega_1} = 2^{\omega_1}$.

Proof. Assume BACH. Suppose that $\kappa < 2^{\omega_1}$ is an infinite cardinal; we show that $\kappa < \mathfrak{p}$. Let $\mathscr{E} \subseteq [\omega_1]^{\omega_1}$ have Sω_1IP, with $|\mathscr{E}| = \kappa$. We want to find a pseudo-intersection of $\mathscr{E}$. Let

$$\mathbb{P} = \{(s_p, W_p) : s_p \in [\omega_1]^{<\omega_1} \text{ and } W_p \in [\mathscr{E}]^{<\omega_1}\}.$$

We define $q \le p$ iff the following hold:

(1) $s_p \subseteq s_q$.
(2) $W_p \subseteq W_q$.
(3) $\forall Z \in W_p[(s_q \backslash s_p) \subseteq Z]$.

This is a forcing order. For transitivity, suppose that $r \le q \le p$. Clearly (1) and (2) for r and p hold. Now suppose that $Z \in W_p$. Then $Z \in W_q$, and $(s_r\backslash s_p) = (s_r\backslash s_q) \cup (s_q\backslash s_p) \subseteq Z$.

(4) $\mathbb{P}$ is countably closed.

For, suppose that $(s_{p(0)}, W_{p(0)}) \ge (s_{p(1)}, W_{p(1)}) \ge \cdots$. Let $s' = \bigcup_{i\in\omega} s_{p(i)}$ and $W' = \bigcup_{i\in\omega} W_{p(i)}$. Clearly $s' \in [\omega_1]^{<\omega_1}$ and $W' \in [\mathscr{E}]^{<\omega_1}$. Take any $i \in \omega$. Then $s_{p(i))} \subseteq s'$ and $W_{p(i)} \subseteq W'$. Suppose that $Z \in W_{p(i)}$. Then

$$s'\backslash s_{p(i)} = \bigcup_{j\in\omega}(s_{p(j)}\backslash s_{p(i)}) = \bigcup_{j>i}(s_{p(j)}\backslash s_{p(i)}) \subseteq Z.$$

Hence $(s', W') \le (s_{p(i)}, W_{p(i)})$. Hence (4) holds

(5) $\mathbb{P}$ is well-met.

For, suppose that $p, q \in \mathscr{P}$ are compatible. Then $(s_p \cup s_q, W_p \cup W_q) \le p, q$ and $r \le (s_p \cup s_q, W_p \cup W_q)$ whenever $r \le p, q$; so (5) holds.

(6) $\mathbb{P}$ is ω_1-linked.

In fact, $\mathbb{P} = \bigcup\{\{p \in \mathscr{P} : s_p = t\} : t \in [\omega_1]^{<\omega_1}\}$, and each $\{\{p \in \mathscr{P} : s_p = t\}$ is linked.

Thus $MA_{\mathbb{P}}(\kappa)$ applies.

Now for each $\alpha \in \omega_1$ let $D_\alpha = \{p \in \mathbb{P} : s_p \cap (\alpha, \omega_1) \neq \emptyset\}$. Then D_α is dense, for if $p \in \mathbb{P}$, then $|\bigcap W_p| = \omega_1$, so we can choose $\beta \in \bigcap W_p$ with $\beta > \alpha$, and then $(s_p \cup \{\beta\}, W_p) \in D_\alpha$ and $(s_p \cup \{\beta\}, W_p) \leq p$.

For any $Z \in \mathscr{E}$ let $E_Z = \{p \in \mathbb{P} : Z \in W_p\}$. Then E_Z is dense, since if $p \in \mathbb{P}$, then $(s_p, W_p \cup \{Z\}) \in E_Z$ and $(s_p, W_p \cup \{Z\}) \leq p$.

Let G be a filter intersecting all of these dense sets. Let $K_G = \bigcup_{p \in G} s_p$. Then G intersecting all sets D_α for $\alpha \in \omega_1$ implies that $|K_G| = \omega_1$.

Given $Z \in \mathscr{E}$, choose $p \in G \cap E_Z$. Suppose that $\alpha \in K_G \backslash Z$. Say $\alpha \in s_q$ with $q \in G$. Choose $r \in G$ such that $r \leq p, q$. Then $\alpha \in s_r$ since $r \leq q$. If $\alpha \notin s_p$, then $\alpha \in Z$ since $r \leq p$. Thus $K_G \backslash Z \subseteq s_p$ and hence $|K_G \backslash Z| < \omega_1$. $\square$

Lemma 31.150. *Let M be a ctm of ZFC and let $\mathbb{Q} \in M$ such that $(|\mathbb{Q}| \leq \omega_1)^M$. Let G be $\mathbb{Q}$-generic over M. Then there is no $h \in ({}^{\omega_1}\omega_1) \cap M[G]$ such that $\forall f \in ({}^{\omega_1}\omega_1) \cap M[f \leq^{\omega_1} h]$.*

Proof. First we note:

(1) If $p \Vdash \tau \in \omega_1$, then $X \stackrel{\text{def}}{=} \{\alpha \in \omega_1 : p \Vdash \check{\alpha} \leq \tau\}$ is countable.

In fact, suppose that $|X| = \omega_1$. Let $p \in G$ generic. Let $\alpha = \tau_G$, and choose distinct $\beta_\xi \in X$ for $\xi < \omega_1$. Then $\beta_\xi \leq \alpha$ for all $\xi < \omega_1$, contradiction.

Now for the lemma, suppose that there is such an h. Take $\dot{h}$ such that $\dot{h}_G = h$. Let $W = ({}^{\omega_1}\omega_1) \cap M$. Then $M[G] \models \forall x \in \check{W}[x \leq^{\omega_1} \dot{h}]$, so there is a $p \in G$ such that $p \Vdash (\dot{h} : \omega_1 \to \omega_1$ and $\forall x \in \check{W}[x \leq^{\omega_1} \dot{h}])$.

Now we work in M. List $\{s \in \mathbb{Q} : s \leq p\}$ as $\{r_\alpha : \alpha \in \omega_1\}$. For each $\alpha \in \omega_1$ let $E_\alpha = \{\beta \in \omega_1 : \exists \gamma < \alpha[r_\gamma \Vdash \check{\beta} \leq \dot{h}(\alpha)]\}$. By (1), for each $\gamma < \omega_1$ the set $\{\beta \in \omega_1 : r_\gamma \Vdash \check{\beta} \leq \dot{h}(\alpha)\}$ is countable, and so E_α is countable. For each $\alpha \in \omega_1$ let $f(\alpha) = \sup(E_\alpha) + 1$. Thus $f \in ({}^{\omega_1}\omega_1)$. Hence $f \in W$.

Since $p \Vdash \forall x \in \check{W}[x \leq^{\omega_1} \dot{h}]$ and $\mathbb{1} \Vdash \check{f} \in \check{W}$, it follows that $p \Vdash \exists \alpha < \omega_1 \forall \beta \geq \alpha[\check{f}(\beta) \leq \dot{h}(\beta)]$. Hence there exist a $q \leq p$ and an α such that $q \Vdash \forall \beta \geq \alpha[\check{f}(\beta) \leq \dot{h}(\beta)]$. Say $q = r_\gamma$. Take β with $\beta > \gamma$ and $\beta > \alpha$, and let $\delta = f(\beta)$. Then $r_\gamma \Vdash \check{\delta} \leq \dot{h}(\beta)$, so $\delta \in E_\beta$ and hence $f(\beta) > \delta$ by the definition of f, contradiction. $\square$

Lemma 31.151. *Let M be a ctm of ZFC and let $\mathbb{P} = \mathrm{Fn}(I, J, \omega_1)$, where $(|J| \leq \omega_1)^M$. Assume that CH holds in M. Let G be $\mathbb{P}$-generic over M.*

Then there is no $h \in ({}^{\omega_1}\omega_1) \cap M[G]$ such that $\forall f \in ({}^{\omega_1}\omega_1) \cap M[f \leq^{\omega_1} h]$.

Proof. Suppose there is such an h. Thus $h \subseteq \omega_1 \times \omega_1$. Say $h = \sigma_G$. Let $\dot{h}$ be a nice $\mathbb{P}$-name for a subset of $(\omega_1 \times \omega_1)^v$ such that $\mathbb{1} \Vdash (\sigma \subseteq (\omega_1 \times \omega_1)^v \to \sigma = \dot{h})$. Say $p \Vdash \sigma \subseteq (\omega_1 \times \omega_1)^v$. Then $p \Vdash \sigma = \dot{h}$, so $\dot{h}_G = h$. Note that $(\omega_1 \times \omega_1)^v = \{((\alpha, \beta)^v, \mathbb{1}) : \alpha, \beta \in \omega_1\}$, so that $\hat{h}$ has the form

$$\bigcup_{\alpha, \beta \in \omega_1} \{\{(\alpha, \beta)^v\} \times A_{\alpha\beta}\}.$$

Here each $A_{\alpha\beta}$ is an antichain in $\mathbb{P}$, and hence has size $\leq \omega_1$ by CH and Lemma 29.33. Let $S = \bigcup_{\alpha,\beta\in\omega_1} A_{\alpha\beta}$. Then S is a subset of $\mathrm{Fn}(I, J, \omega_1)$ of size $\leqq \omega_1$. Let $K = \bigcup_{p\in S} \mathrm{dmn}(p)$ and $\mathbb{Q} = \mathrm{Fn}(K, J, \omega)$. Then K is a subset of I of size $\leq \omega_1$, and $\dot{h}$ is a $\mathbb{Q}$-name.

Now $\mathbb{Q} \subseteq_c \mathbb{P}$ by Proposition 25.66, so Lemma 30.3 applies to the inclusion $\mathbb{Q} \subseteq_c \mathbb{P}$. So with $H = G \cap \mathbb{Q}$ we have $M \subseteq M[H] \subseteq M[G]$ and $h = \dot{h}_G = \dot{h}_H$. Moreover, $\forall f \in ({}^{\omega_1}\omega_1) \cap M[f \leq^* h]$. This contradicts Lemma 31.150. $\square$

Lemma 31.152. *Let G be $\mathrm{Fn}(\omega_1, \omega_1, \omega_1)$-generic, and in $M[G]$ let $f = \bigcup G : \omega_1 \to \omega_1$. Then there is no $h \in ({}^{\omega_1}\omega_1) \cap M$ such that $f \leq^{\omega_1} h$.*

Proof. Assume that G is $\mathrm{Fn}(\omega_1, \omega_1, \omega_1)$-generic, and in $M[G]$ $f = \bigcup G : \omega_1 \to \omega_1$. Suppose that $h \in ({}^{\omega_1}\omega_1) \cap M$. For each $\alpha \in \omega_1$, the set $\{p \in \mathrm{Fn}(\omega_1, \omega_1, \omega_1) : \exists \beta \in \mathrm{dmn}(p)[\beta > \alpha \text{ and } p(\beta) > h(\beta)]\}$ is dense, and so $M[G] \models \forall\alpha\exists\beta > \alpha[f(\alpha) > h(\alpha)]$. Thus $f \not\leq^{\omega_1} h$. $\square$

Lemma 31.153. *In M, assume that $W \subseteq I$ and $\mathbb{P} = \mathrm{Fn}(I, J, \omega_1)$. Let K be $\mathbb{P}$-generic over M. Let $G = K \cap \mathrm{Fn}(W, J, \omega_1)$ and let $H = K \cap \mathrm{Fn}(I\backslash W, J, \omega_1)$.*

Then G is $\mathrm{Fn}(W, J, \omega_1)$-generic over M and H is $\mathrm{Fn}(I\backslash W, J, \omega_1)$-generic over $M[G]$. Moreover, $M[K] = M[G][H]$.

Proof. In M define $\zeta : \mathrm{Fn}(W, J, \omega_1) \times \mathrm{Fn}(I\backslash W, J, \omega_1) \to \mathrm{Fn}(I, J, \omega_1)$ by $\zeta(p, q) = p \cup q$. Then ζ is an isomorphism from $\mathrm{Fn}(W, J, \omega_1) \times \mathrm{Fn}(I\backslash W, J, \omega_1)$ onto $\mathrm{Fn}(I, J, \omega_1)$. Note that $\zeta^{-1}(p) = (p \restriction W, p \restriction (I\backslash W))$. Let $\tilde{K} = \zeta^{-1}[K]$. So $\tilde{K}$ is $\mathrm{Fn}(W, J, \omega_1) \times \mathrm{Fn}(I\backslash W, J, \omega_1)$-generic over M, and $M[K] = M[\tilde{K}]$. Now let $i(p) = (p, \mathbb{1})$ and $j(p) = (\mathbb{1}, p)$, $G = i^{-1}[\tilde{K}]$, and $H = j^{-1}[\tilde{K}]$. Then by Lemma 31.1, G is $\mathrm{Fn}(W, J, \omega_1)$-generic over M, H is $\mathrm{Fn}(I\backslash W, J, \omega_1)$-generic over $M[G]$. By Theorem 31.2, $M[\tilde{K}] = M[G][H]$. Now

$$\begin{aligned} G &= \{p \in \mathrm{Fn}(W, J, \omega_1) : (p, \mathbb{1}) \in \tilde{K}\} = \{p \in \mathrm{Fn}(W, J, \omega_1) : p \cup \emptyset \in K\} \\ &= K \cap \mathrm{Fn}(W, J, \omega_1); \\ H &= \{p \in \mathrm{Fn}(I\backslash W, J, \omega_1) : (\mathbb{1}, p) \in \tilde{K}\} = \{p \in \mathrm{Fn}(I\backslash W, J, \omega_1) : \emptyset \cup p \in K\} \\ &= K \cap \mathrm{Fn}(I\backslash W, J, \omega_1). \end{aligned}$$

$\square$

Lemma 31.154. *In M, let $\mathbb{P} = \mathrm{Fn}(\kappa, \omega_1, \omega_1)$. Let K be $\mathbb{P}$-generic over M. Then $M[K] \models \mathfrak{d}_{\omega_1} \geq \kappa$.*

Proof. Suppose not; then there is a dominating family $\{h_\alpha : \alpha < \theta\}$ with $\theta < \kappa$. Let $k : \theta \times \omega_1 \to \omega_1$ be defined by $k(\alpha, \beta) = h_\alpha(\beta)$. Let τ be a nice name for a subset of $(\theta \times \omega_1) \times \omega_1$ such that $\tau_K = k$. Note that there is a $W_0 \in [\kappa]^{\leq\theta}$ such that τ is a $\mathrm{Fn}(W_0, \omega_1, \omega_1)$-name. Let W be such that $W_0 \subseteq W \subseteq \kappa$ and $|\kappa\backslash W| = \omega_1$. Then τ is a $\mathrm{Fn}(W, \omega_1, \omega_1)$-name, and $\mathrm{Fn}(\kappa\backslash W, \omega_1, \omega_1) \cong \mathrm{Fn}(\omega_1, \omega_1, \omega_1)$. Let $G = K \cap \mathrm{Fn}(W, \omega_1, \omega_1)$ and $H = K \cap \mathrm{Fn}(\kappa\backslash W, \omega_1)$. Then by Lemma 31.153, G is $\mathrm{Fn}(W, \omega_1, \omega_1)$-generic over M, H is $\mathrm{Fn}(\kappa\backslash W, \omega_1, \omega_1)$-generic over $M[G]$, and $M[K] = M[G][H]$. Now $\mathrm{Fn}(\kappa\backslash W, \omega_1, \omega_1) \cong \mathrm{Fn}(\omega_1, \omega_1, \omega_1)$; let l be an isomorphism. Let $f = \bigcup l[H] : \omega_1 \to \omega_1$. Applying Lemma 31.152 with $k[H], M[G]$ in place of G, M, we infer that there is no $r \in ({}^{\omega_1}\omega_1) \cap M[G]$ such that $f \leq^* r$. But $h_\alpha \in M[G]$ for each $\alpha < \theta$, contradiction. $\square$

Theorem 31.155. (V.5.19.2) *Let M model GCH, κ any regular cardinal $\geq \omega_2$, and take $\mathbb{P} = \mathrm{Fn}(\kappa, \omega_1, \omega_1)$. Then for G $\mathbb{P}$-generic over M, in $M[G]$ we have $2^\omega = \omega_1$, $2^{\omega_1} = \kappa$, $\mathfrak{p}_{\omega_1} = \mathfrak{b}_{\omega_1} = \omega_2$, and $\mathfrak{d}_{\omega_1} = \kappa$.*

Proof. $2^\omega = \omega_1$ and $2^{\omega_1} = \kappa$ by Theorem 29.37. $\mathfrak{p}_{\omega_1} = \mathfrak{b}_{\omega_1} = \omega_2$ by Propositions 31.144–31.146 and Lemma 31.151. $\mathfrak{d}_{\omega_1} = \kappa$ by Lemma 31.154. $\square$

Proposition 31.156. *Assume CH. Let $\mathscr{E} \subseteq ({}^{\omega_1}\omega_1)$ be infinite. We define $\mathbb{P} \stackrel{\text{def}}{=} (\mathbb{P}(\mathscr{E}), \leq)$ as follows. $\mathbb{P}(\mathscr{E})$ consists of all pairs $p = (s_p, Y_p)$ such that $s_p \in \mathrm{Fn}(\omega_1, \omega_1, \omega_1)$ and $Y_p \in [\mathscr{E}]^{<\omega_1}$. We define $q \leq p$ iff $s_q \supseteq s_p$, $Y_q \supseteq Y_p$, and $\forall f \in Y_p \forall \alpha \in \mathrm{dmn}(s_q)\backslash\mathrm{dmn}(s_p)[s_q(\alpha) > f(\alpha)]$.*

Then $\mathbb{P}$ is transitive. $\mathbb{P}$ is ω_1-centered, and there is a family of $|\mathscr{E}|$ dense sets such that whenever G is a filter meeting all of them and $h = \bigcup_{p\in G} s_p$, then $h \in ({}^{\omega_1}\omega_1)$ and $f \leq^{\omega_1} h$ for all $f \in \mathscr{E}$.

Proof. First we check transitivity. Suppose that $r \leq q \leq p$, $f \in \mathcal{Y}_p$, and $\alpha \in \mathrm{dmn}(s_r)\backslash\mathrm{dmn}(s_p)$. Note that $f \in \mathcal{Y}_q$. If $\alpha \notin \mathrm{dmn}(s_q)$, then $s_r(\alpha) > f(\alpha)$. If $\alpha \in \mathrm{dmn}(s_q)$, then $s_r(\alpha) = s_q(\alpha) > f(\alpha)$.

$\mathbb{P}$ is ω_1-centered, since for any $t \in \mathrm{Fn}(\omega_1, \omega_1, \omega_1)$ the set of $p \in \mathbb{P}$ with $s_p = t$ is centered.

For each $\alpha \in \omega_1$ let $D_\alpha = \{p \in \mathbb{P} : \alpha \in \mathrm{dmn}(s_p)\}$. Then D_α is dense. For, suppose that $p \in \mathbb{P}$. If $\alpha \in \mathrm{dmn}(s_p)$, then $p \in D_\alpha$. Suppose that $\alpha \notin \mathrm{dmn}(s_p)$. Let β be greater than $f(\alpha)$ for each $f \in \mathcal{Y}_p$, and let $s_q = s_p \cup \{(\alpha, \beta)\}$ and $\mathcal{Y}_p = \mathcal{Y}_p$. Then $q \leq p$ and $q \in D_\alpha$. So D_α is dense. Also, for each $f \in \mathcal{E}$ let $E_f = \{p \in \mathbb{P} : f \in \mathcal{Y}_p\}$. Clearly E_f is dense. Let $\mathscr{A} = \{D_\alpha : \alpha \in \omega_1\} \cup \{E_f : f \in \mathcal{E}\}$. So $|\mathscr{A}| \leq |\mathcal{E}|$.

Suppose that G is a filter meeting all members of $\mathscr{A}$. Because of the D_α's, we have $h \in {}^{\omega_1}\omega_1$. Take any $f \in \mathcal{E}$. Choose $p \in G$ such that $f \in \mathcal{Y}_p$. We claim that $h(\alpha) > f(\alpha)$ for all α greater than each member of s_p. For, take such an α, and choose $q \in G$ so that $\alpha \in \mathrm{dmn}(s_q)$. Choose $r \in G$ with $r \leq p, q$. Then $\alpha \in \mathrm{dmn}(s_r)$ since $r \leq q$, and $r(\alpha) > f(\alpha)$ since $r \leq p$ and $f \in \mathcal{Y}_p$. Hence $h(\alpha) = r(\alpha) > f(\alpha)$. $\square$

Proposition 31.157. *The poset $\mathbb{P}$ in Proposition 31.156 is well-met and countably closed.*

Proof. well-met: suppose that $p, q, r \in \mathbb{P}$ and $r \leq p, q$. Let $t = s_p \cup s_q$ and $X = Y_p \cup Y_q$. It suffices to show that $(t, X) \leq p$. Suppose that $f \in Y_p$ and $\alpha \in \mathrm{dmn}(s_q)\backslash\mathrm{dmn}(s_p)$. So $\alpha \in \mathrm{dmn}(s_r)\backslash\mathrm{dmn}(s_p)$, so $s_q(\alpha) = s_r(\alpha) > f(\alpha)$, as desired.

Countably closed: Suppose that $p_0 \geq p_1 \geq \cdots$. Let $s_q = \bigcup_{n\in\omega} s_{p_n}$ and $X = \bigcup_{n\in\omega} Y_{p_n}$. Suppose that $n \in \omega$, $f \in Y_{p_n}$, and $\alpha \in \mathrm{dmn}(s_q)\backslash\mathrm{dmn}(s_{p_n})$. Say $\alpha \in \mathrm{dmn}(s_{p_m})$. Then $n < m$ and $s_q(\alpha) = s_{p_m}(\alpha) > f(\alpha)$, as desired. $\square$

Proposition 31.158. *Suppose that in M we are given an ω_2-stage countable support iterated forcing construction*

$$(\langle \mathbb{P}_\xi, \leq_\xi, \mathbb{1}_\xi) : \xi \leq \omega_2\rangle, \langle \dot{\mathbb{Q}}_\xi, \dot{\preceq}_{\dot{\mathbb{Q}}_\xi}, \dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\xi}) : \xi < \omega_2\rangle)$$

Also suppose that $\mathbb{P}_{\omega_2}$ has the ω_2-cc. Suppose that G is $\mathbb{P}_{\omega_2}$-generic over M, $S \in M$, $X \in M[G]$, $X \subseteq S$, and $(|S| < \omega_2)^{M[G]}$. Then there is an $\eta < \omega_2$ such that $X \in M[(i_\eta^{\omega_2})^{-1}[G]]$.

Proof. Clearly $\forall s \in S[s \in X \leftrightarrow \exists p \in G[p \Vdash_{\mathbb{P}_{\omega_2}} \check{s} \in \dot{X}]$, where $\dot{X}$ is a $\mathbb{P}_{\omega_2}$-name such that $\dot{X}_G = X$. Now $\mathbb{P}_{\omega_2} = \bigcup_{\xi<\omega_2} i_\xi^{\omega_2}[\mathbb{P}_\xi]$ and $G = \bigcup_{\xi<\omega_2} i_\xi^{\omega_2}[(i_\xi^{\omega_2})^{-1}[G]]$. Let $G_\xi = (i_\xi^{\omega_2})^{-1}[G]$.

In $M[G]$, for each $s \in X$ there is a $\xi = \xi_s < \omega_2$ such that $\exists p \in G_\xi[i_\xi^{\omega_2}(p) \Vdash_{\mathbb{P}_{\omega_2}} \check{s} \in \dot{X}]$. Let $\eta = \sup_{s\in X} \xi_s$. So $\eta < \omega_2$ since $(|S| < \mathrm{cf}(\omega_2))^{M[G]}$. Then $X = \{s \in S : \exists p \in G_\eta[i_\eta^{\omega_2}(p) \Vdash \check{s} \in \dot{X}]\}$. Hence $X \in M[G_\eta]$. $\square$

Theorem 31.159. (V.5.19.1) *Let $\kappa > \omega_1$ be a regular cardinal. Then there is a generic extension N of M such that CH holds in N, $2^{\omega_1} \geq \kappa$ in N, and in N, $\mathfrak{p}_{\omega_1} = \mathfrak{b}_{\omega_1} = \mathfrak{d}_{\omega_1} = \omega_2$.*

Proof. First we extend M to a model S such that CH holds in S and $2^{\omega_1} \geq \kappa$ in S. Then starting with S we form an ω_2-stage countable support iteration $\langle \mathbb{P}_\xi : \xi \leq \omega_2 \rangle$, applying Proposition 31.156 at the successor steps, using full names. Namely, if $\mathbb{P}_\xi$ has been constructed, then

$$\mathbb{1}_{\mathbb{P}_\xi} \Vdash \exists Q \forall x[x \in Q \leftrightarrow \exists s \exists Y[s \in (\mathrm{Fn}(\omega_1, 1, \omega_1))^v \wedge Y \in ((^\omega\omega)^{<\omega_1})^v \wedge x = (s, Y)]]$$

Now by Theorem 30.35 and Lemma 31.129 we get a full name $\dot{\mathbb{Q}}_\xi$ such that

$$\mathbb{1}_{\mathbb{P}_\xi} \Vdash \forall x[x \in \dot{\mathbb{Q}}_\xi \leftrightarrow \exists s \exists Y[s \in (\mathrm{Fn}(\omega_1, 1, \omega_1))^v \wedge Y \in ((^\omega\omega)^{<\omega_1})^v \wedge x = (s, Y)]]$$

Then

$$\begin{aligned}\mathbb{1}_{\mathbb{P}_\xi} \Vdash &\exists R \forall z[z \in R \leftrightarrow \exists s, Y, s', Y'[(s, Y), (s', Y') \in \dot{\mathbb{Q}}_\xi \wedge z = ((s, Y), (s', Y')) \\ &\wedge s' \subseteq s \wedge Y' \subseteq Y \wedge \forall f \in Y' \forall \alpha \in \mathrm{dmn}(s) \backslash \mathrm{dmn}(s')[s(\alpha) > f(\alpha)]]\end{aligned}$$

Again by Theorem 30.35 and Lemma 31.129 we get a full name $\dot{\leq}_\xi$ such that

$$\begin{aligned}\mathbb{1}_{\mathbb{P}_\xi} \Vdash &\forall z[z \in \dot{\leq}_\xi \leftrightarrow \exists s, Y, s', Y'[(s, Y), (s', Y') \in \dot{\mathbb{Q}}_\xi \wedge z = ((s, Y), (s', Y')) \\ &\wedge s' \subseteq s \wedge Y' \subseteq Y \wedge \forall f \in Y' \forall \alpha \in \mathrm{dmn}(s) \backslash \mathrm{dmn}(s')[s(\alpha) > f(\alpha)]]\end{aligned}$$

By Propositions 31.156 and 31.157,

$$\mathbb{1}_{\mathbb{P}_\xi} \Vdash [\dot{\mathbb{Q}}_\xi \text{ is } \omega_1\text{-centered, countably closed, and well-met}]$$

By Lemma 31.131 each $\mathbb{P}_\xi$ is countably closed. By Lemma 31.140, $\mathbb{P}_{\omega_2}$ has the ω_2-cc. Hence cardinals and cofinalities are preserved. Also, CH holds since each $\mathbb{P}_\xi$ is countably closed, and of course $2^{\omega_1} \geq \kappa$ in $\mathbb{P}_{\omega_2}$. By Propositions 31.144–31.148 we have $\omega_2 \leq \mathfrak{p}_{\omega_1} \leq \mathfrak{b}_{\omega_1} \leq \mathfrak{d}_{\omega_1}$.

Now let G be $\mathbb{P}_{\omega_2}$-generic over S. Take any $\xi < \omega_2$. Let $G_\xi = (i_\xi^{\omega_2})^{-1}[G]$; so G_ξ is $\mathbb{P}_\xi$-generic over S, by Lemma 30.2, and $S[G_\xi] \subseteq S[G]$ by Proposition 31.60(i). By Proposition 31.54, $\mathbb{P}_{\xi+1}$ is isomorphic to $\mathbb{P}_\xi * \dot{\mathbb{Q}}_\xi$; say k_ξ is an isomorphism. Then $k_\xi[G_{\xi+1}]$ is $(\mathbb{P}_\xi * \dot{\mathbb{Q}}_\xi)$-generic over S. Now let

$$H_\xi = \{\rho_{G_\xi} : \rho \in \dot{\mathbb{Q}}_\xi \wedge \exists p \in \mathbb{P}_\xi[(p, \rho) \in k_\xi[G_{\xi+1}]]\}.$$

Then by Proposition 31.60, H_ξ is $(\dot{\mathbb{Q}}_\xi)_{G_\xi}$-generic over $S[G_\xi]$, and $S[k_\xi[G_{\xi+1}]] = S[G_\xi][H_\xi]$. Also, $S[k_\xi[G_{\xi+1}]] = S[G_{\xi+1}]$. By the above, in $S[G_\xi]$ we have a poset $(\dot{\mathbb{Q}}_{\xi G_\xi}, \dot{\preceq}_{\xi G_\xi})$ satisfying the conditions of Proposition 31.156. Let $h_\xi = \bigcup_{p\in G_\xi} s_p$. Then $h_\xi \in {}^\omega\omega$ and for all $f \in ({}^\omega\omega)^{\mathbb{P}_\xi}$ we have $f \leq^{\omega_1} h$; and $h_\xi \in S[G_\xi] \subseteq S[G]$. Let $D = \{h_\xi : \xi < \omega_2\}$. We claim that in $S[G]$ the set D is dominating. For, suppose that $f \in ({}^\omega\omega)^{S[G]}$. By Proposition 31.158 there is a $\xi < \omega_2$ such that $f \in S[G_\xi]$. Then $f \leq^{\omega_1} h_\xi$, as desired. $\square$

Proposition 31.160. *Suppose that κ is an infinite cardinal, $\mathscr{F} \subseteq \mathscr{P}(\omega_1)$, $|\mathscr{F}| \geq \kappa$, and $\forall \alpha < \omega_1[|\{X \cap \alpha : X \in \mathscr{F}\}| \leq \omega$.*

Then there is an ω_1-tree T such that there are $\geq \kappa$ paths through T.

Proof. Let $T = \bigcup_{\alpha<\omega_1}\{\chi_{X\cap\alpha} : X \in \mathscr{F}\}$. $\square$

Proposition 31.161. *Assume BACH and $\kappa < 2^{\omega_1}$. Then there is an ω_1-tree with at least κ paths through T.*

Proof. We may assume that $\omega_1 \leq \kappa$. Let $\mathbb{P}$ consist of all pairs $p = (\alpha_p, F_p)$ such that $\alpha_p < \omega_1$ and F_p is a nonempty countable subset of $\mathscr{P}(\omega_1)$. Define $q \leq p$ iff $\alpha_q \geq \alpha_p$, $F_q \supseteq F_p$, and $\{X \cap \alpha_p : X \in F_q\} = \{X \cap \alpha_p : X \in F_p\}$. Let $\mathbb{1}_\mathbb{P} = (0, \{\emptyset\})$.

(1) $\leq$ is transitive.

For, suppose that $r \leq q \leq p$. If $X \in F_r$, choose $Y \in F_q$ such that $X \cap \alpha_q = Y \cap \alpha_q$. Then choose $Z \in F_p$ such that $Y \cap \alpha_p = Z \cap \alpha_p$. Then $X \cap \alpha_p = X \cap \alpha_q \cap \alpha_p = Y \cap \alpha_q \cap \alpha_p = Z \cap \alpha_q \cap \alpha_p = Z \cap \alpha_p$.

Conversely, suppose that $Z \in F_p$. Choose $Y \in F_q$ such that $Y \cap \alpha_p = Z \cap \alpha_p$. Then choose $X \in F_r$ so that $X \cap \alpha_q = Y \cap \alpha_q$. Then $X \cap \alpha_p = X \cap \alpha_q \cap \alpha_p = Y \cap \alpha_q \cap \alpha_p = Z \cap \alpha_q \cap \alpha_p = Z \cap \alpha_p$.

(2) $\mathbb{P}$ is countably closed.

For, suppose that $p_0 \geq p_1 \geq \cdots$. Let $\alpha_q = \sup_{n\in\omega} \alpha_{p_n}$ and $F_q = \bigcup_{n\in\omega} F_{p_n}$. To check that $q \leq p_n$, first suppose that $X \in F_q$; we want to find $Y \in F_{p_n}$ such that $X \cap \alpha_{p_n} = Y \cap \alpha_{p_n}$. Choose $m \in \omega$ such that $X \in F_{p_m}$.

Case 1. $m \leq n$. Then $p_n \leq p_m$. Hence $F_{p_m} \subseteq F_{p_n}$, so $X \in F_{p_n}$. So we can take $Y = X$.

Case 2. $n < m$. Then $p_m \leq p_n$. Choose $Y \in F_{p_n}$ so that $X \cap \alpha_{p_n} = Y \cap \alpha_{p_n}$.

Second suppose that $Y \in F_{p_n}$; we want to find $X \in F_q$ such that $X \cap \alpha_{p_n} = Y \cap \alpha_{p_n}$. We can take $X = Y$.

This proves (2).

(3) If $p, q \in \mathbb{P}$ and $\alpha_q \geq \alpha_p$, then p and q are compatible iff the following conditions hold:

(a) $\{X \cap \alpha_p : X \in F_p\} = \{Y \cap \alpha_p : Y \in F_q\}$.

(b) $\{X \cap \alpha_q : X \in F_p\} \subseteq \{Y \cap \alpha_q : Y \in F_q\}$.

For, first suppose that p and q are compatible. Say $r \leq p, q$. Thus

(4) $\{Z \cap \alpha_p : Z \in F_r\} = \{X \cap \alpha_p : X \in F_p\}$ and

(5) $\{Z \cap \alpha_q : Z \in F_r\} = \{Y \cap \alpha_q : Y \in F_q\}$.

Now suppose that $X \in F_p$; we want to find $Y \in F_q$ such that $X \cap \alpha_p = Y \cap \alpha_p$. By (4) choose $Z \in F_r$ such that $Z \cap \alpha_p = X \cap \alpha_p$. Then by (5) choose $Y \in F_q$ such that $Z \cap \alpha_q = Y \cap \alpha_q$. Then $Y \cap \alpha_p = Y \cap \alpha_q \cap \alpha_p = Z \cap \alpha_q \cap \alpha_p = Z \cap \alpha_p = X \cap \alpha_p$.

Next, suppose that $Y \in F_q$; we want to find $X \in F_p$ such that $X \cap \alpha_p = Y \cap \alpha_p$. By (5) choose $Z \in F_r$ such that $Z \cap \alpha_q = Y \cap \alpha_q$. Then by (4) choose $X \in F_p$ such that $Z \cap \alpha_p = X \cap \alpha_p$. Then $X \cap \alpha_p = Z \cap \alpha_p = Z \cap \alpha_q \cap \alpha_p = Y \cap \alpha_q \cap \alpha_p = Y \cap \alpha_p$.

Finally, suppose that $X \in F_p$; we want to find $Y \in F_q$ so that $X \cap \alpha_q = Y \cap \alpha_q$. Now $X \in F_r$, so such a Y exists by (5).

Second, suppose that (a) and (b) hold. Let $\alpha_r = \alpha_q$ and $F_r = F_p \cup F_q$, To show that $r \le p$, suppose that $Z \in F_r$; we want to find $W \in F_p$ such that $Z \cap \alpha_p = W \cap \alpha_p$.

Case 1. $Z \in F_p$. Take $W = Z$.

Case 2. $Z \in F_q$. Then (a) gives the desired W.

Conversely, suppose that $W \in F_p$; we want $Z \in F_r$ so that $Z \cap \alpha_p = W \cap \alpha_p$. Take $Z = W$.

To show that $r \le q$, suppose that $Z \in F_r$; we want to find $W \in F_q$ such that $Z \cap \alpha_q = W \cap \alpha_q$.

Case 1. $Z \in F_p$. Then W exists by (b).

Case 2. $Z \in F_q$. Take $W = Z$.

On the other hand, suppose that $W \in F_q$; we want to find $Z \in F_r$ such that $Z \cap \alpha_q = W \cap \alpha_q$. Take $Z = W$.

Hence (3) holds.

(6) $\mathbb{P}$ is well-met.

In fact, suppose that p and q are compatible. Say $\alpha_q \ge \alpha_p$. Let r be defined as in the proof of (3). Now suppose that $s \le p, q$; we want to show that $s \le r$.

First suppose that $X \in F_s$; we want to find $Y \in F_r$ such that $X \cap \alpha_r = Y \cap \alpha_r$. Since $s \le q$, choose $y \in F_q$ such that $X \cap \alpha_q = Y \cap \alpha_q$. Then $Y \in F_r$ and $\alpha_q = \alpha_r$, as desired.

Second suppose that $Y \in F_r$; we want to find $X \in F_s$ such that $X \cap \alpha_r = Y \cap \alpha_r$.

Case 1. $Y \in F_q$. Since $s \le q$, choose $X \in F_s$ such that $X \cap \alpha_q = Y \cap \alpha_q$. Since $\alpha_q = \alpha_r$, this is as desired.

Case 2. $Y \in F_p$. Then by (3)(b) choose $Z \in F_q$ such that $Y \cap \alpha_q = Z \cap \alpha_q$. Then since $s \le q$, choose $X \in F_s$ such that $X \cap \alpha_q = Z \cap \alpha_q$. So $X \cap \alpha_r = Y \cap \alpha_r$.

This proves (6).

(7) $|\mathbb{P}| = \omega_1$.

This is clear, by CH.

Thus $MA_{\mathbb{P}}(\kappa)$ holds.

(8) For each $\xi < \omega_1$ the set $D_\xi \stackrel{\text{def}}{=} \{p : \alpha_p \ge \xi\}$ is dense.

To prove this, take any $q \in \mathbb{P}$. If $\xi \le \alpha_q$, this is ok. If $\alpha_q < \xi$, let $\alpha_p = \xi$ and $F_p = F_q$. Then $p \le q$, as desired. So (8) holds.

(9) For any $X \subseteq \omega_1$ the set $E_X \stackrel{\text{def}}{=} \{p \in \mathbb{P} : \exists Y \in F_p[X \triangle Y \text{ is countable}]\}$ is dense.

For, let $q \in \mathbb{P}$. For each $Y \in F_q$ let $Z_Y = (X \backslash \alpha_q) \cup (Y \cap \alpha_q)$. Let $\alpha_p = \alpha_q$ and $F_p = F_q \cup \{Z_Y : y \in F_q\}$. If $Y \in F_q$, then $Y \cap \alpha_q = Z_Y \cap \alpha_q$. Hence $p \leq q$. For any $Y \in F_q$ we have $X \backslash Z_y \subseteq \alpha_q$ and $Z_Y \backslash X \subseteq \alpha_q$. So $X \triangle Z_Y$ is countable, proving (9).

Let$\mathscr{E}$ be a strongly independent subset of $\mathscr{P}(\omega_1)$ of size κ, and let $G \subseteq \mathbb{P}$ be a filter intersecting each set D_ξ for $\xi < \omega_1$ and each set E_X for $X \in \mathscr{E}$. Let $\mathscr{F} = \bigcup_{p \in G} F_p$.

(10) $\forall \xi < \omega_1[|\{Y \cap \xi| : Y \in \mathscr{F}\}| \leq \omega]$.

In fact, pick $p \in G \cap D_\xi$. Thus $\alpha_p \geq \xi$. We claim that

(11) $\forall Y \in \mathscr{F} \exists X \in F_p[X \cap \alpha_p = Y \cap \alpha_p]$.

For, suppose that $Y \in \mathscr{F}$. Say $Y \in F_q$ with $q \in G$. Say $r \in G$ and $r \leq p, q$. Thus $F_q \subseteq F_r$, so $Y \in F_r$. Since $r \leq p$, there is an $X \in F_p$ such that $X \cap \alpha_p = Y \cap \alpha_p$. So (11) holds.

Now it follows from (11) that $\{Y \cap \alpha_p : Y \in \mathscr{F}\}$ is countable. Hence also $\{Y \cap \xi : Y \in \mathscr{F}\}$ is countable, as desired in (10).

(12) $|\mathscr{F}| \geq \kappa$.

In fact, for each $X \in \mathscr{E}$ choose $p \in G \cap E_X$, and choose $Y_X \in F_p$ such that $X \triangle Y_X$ is countable. We claim that $E_X \neq E_Z$ for distinct $X, Z \in \mathscr{E}$; in fact, actually $E_X \triangle E_Z$ is uncountable. This follows from

(13) If $X, Z \in \mathscr{E}$ and $X \neq Z$, then $(X \triangle Z) \subseteq (X \triangle Y_X) \cup (Y_X \triangle Y_Z) \cup (Y_Z \triangle Z)$.

To prove (13), suppose that $X, Z \in \mathscr{E}$ and $X \neq Z$. Take any $\alpha \in X \triangle Z$. By symmetry say $\alpha \in X \backslash Z$. We may assume that $\alpha \in Y_X$ and $\alpha \in Y_Z$. Hence $\alpha \in (Y_Z \backslash Z)$. This proves (13), and hence also (12).

Now the proposition follows from Proposition 31.160. $\square$

Proposition 31.162. (V.5.21) *Suppose that $M \vdash GCH$ and there is a strongly inaccessible cardinal. Then there is a generic extension $M[G]$ in which there is no Kurepa tree, CH holds, and $2^{\omega_1} \geq \aleph_{85}$.*

Proof. Let κ be an inaccessible cardinal in M. By Theorem 31.32 there is a generic extension $M[H]$ in which there is no Kurepa tree, CH holds, and $\kappa = \omega_2$. Note that in $M[H]$ we have $2^\omega = \omega_1$. Now in $M[H]$ we take G $\mathrm{Fn}(\omega_{85}^{\omega_1}, \omega_1, \omega_1)$-generic over $M[H]$. Then by Theorem 29.37 with κ, λ replaced by $\omega_{85}^{\omega_1}$, ω_1 respectively we obtain $M[H][G]$ in which CH holds, and $2^{\omega_1} = \omega_{85}^{\omega_1}$. By Lemma 31.30 there is no Kurepa tree in $M[H][G]$. $\square$

A poset $\mathbb{P}$ is *proper* iff for all uncountable κ and all stationary $S \subseteq [\kappa]^{\leq\omega}$ we have $\mathbb{1} \Vdash_{\mathbb{P}} [S$ is stationary].

Proposition 31.163. *Every ccc poset is proper.*

Proof. By Corollary 30.86. $\square$

Lemma 31.164. (V.7.3) *Assume that $\mathbb{P}$ is proper, κ is uncountable, and $p \Vdash [\dot{A} \in [\kappa]^{\leq\omega}]$. Then there exist a $B \in [\kappa]^{\leq\omega}$ and a $q \leq p$ such that $q \Vdash [\dot{A} \subseteq \check{B}]$.*

Proof. Obviously $([\kappa]^{\leq\omega})^M$ is club in $[\kappa]^{\leq\omega}$ and hence is also stationary in M. Let G be $\mathbb{P}$-generic over M with $p \in G$. Then $([\kappa]^{\leq\omega})^M$ is stationary in $M[G]$. Now obviously $\{X \in [\kappa]^{\leq\omega} : \dot{A}_G \subseteq X\}$ is club. Hence there is a $B \in ([\kappa]^{\leq\omega})^M$ such that $\dot{A}_G \subseteq B$. So there is a $q \leq p$ such that $q \Vdash [\dot{A} \subseteq \check{B}]$. □

Corollary 31.165. *Assume that $\mathbb{P}$ is proper, and G is $\mathbb{P}$-generic over M. Then $\omega_1^M = \omega_1^{M[G]}$.*

Proof. Suppose not. Then there is a $p \in G$ such that $p \Vdash \check{\omega}_1 \in [\omega_1]^{\leq\omega}$. Hence by Lemma 31.164 there exist a $B \in [\omega_1]^{\leq\omega}$ and a $q \leq p$ such that $q \Vdash [\check{\omega}_1 \subseteq \check{B}]$. Let $q \in H$ generic. Then $\omega_1 = \check{\omega}_{1G} \subseteq B$, contradiction. □

Proposition 31.166. *If $\mathbb{P}$ is proper and $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} [\dot{\mathbb{Q}}$ is proper$]$, then $\mathbb{P} * \dot{\mathbb{Q}}$ is proper.*

Proof. Assume the hypothesis, and suppose that κ is uncountable and $S \subseteq [\kappa]^{\leq\omega}$ is stationary. Let K be $(\mathbb{P} * \dot{\mathbb{Q}})$-generic over M. Form G and H as in Theorem 31.46. Then S is stationary in $M[G]$. Also, $\dot{\mathbb{Q}}_G$ is proper, so S is stationary in $M[G][H] = M[K]$. □

Proposition 31.167. (V.7.8) *If $A \subseteq B$, define $p : [B]^{\leq\omega} \to [A]^{\leq\omega}$ by $p(y) = y \cap A$.*
If $C \subseteq [A]^{\leq\omega}$ is club in $[A]^{\leq\omega}$, then $p^{-1}[C]$ is club in $[B]^{\leq\omega}$.

Proof. Assume the hypotheses. Unbounded: Suppose that $x \in [B]^{\leq\omega}$. Choose $y \in C$ such that $x \cap A \subseteq y$. Then $x \cup y \in p^{-1}[C]$ and $x \subseteq x \cup y$.

Closed: suppose that $x_0 \subseteq x_1 \subseteq \cdots$ and each $x_i \in p^{-1}[C]$. Thus $p(x_0) \subseteq p(x_1) \subseteq \cdots$ and each $p(x_i) \in C$. Hence $p(\bigcup_{n\in\omega} x_i) = \bigcup_{n\in\omega} p(x_n) \in C$, so $\bigcup_{n\in\omega} x_n \in p^{-1}[C]$. □

Proposition 31.168. (V.7.8) *If $A \subseteq B$, define $p : [B]^{\leq\omega} \to [A]^{\leq\omega}$ by $p(y) = y \cap A$.*
If $C \subseteq [B]^{\leq\omega}$ is club in $[B]^{\leq\omega}$, then $p[C]$ contains a club subset of $[A]^{\leq\omega}$.

Proof. Assume the hypotheses.

(1) For each finite $s \subseteq A$ there is an $f(s) \in C$ with $s \subseteq f(s)$ such that if $s \subseteq t$ then $f(s) \subseteq f(t)$.

We prove this by induction on $|s|$. It is obvious for $|s| = 0$, i.e., for $s = \emptyset$. Now suppose it is true for $|s| = n$. For $|s| = n + 1$, write $s = \{x_0, \ldots, x_n\}$. Choose $f(s) \in C$ such that

$$\bigcup_{i\leq n} f(s\backslash\{x_i\}) \cup s \subseteq f(s).$$

Then if $t \subset s$ then there is an $i \leq n$ such that $t \subseteq s\backslash\{x_i\}$, hence $f(t) \subseteq f(s\backslash\{x_i\}) \subseteq f(s)$. So (1) holds.

Let $C_0 = \{x \in [A]^{\leq\omega} : \forall s \in [x]^{<\omega}[f(s) \cap A \subseteq x]\}$.

(2) C_0 is club in $[A]^{\leq\omega}$.

To prove (2), first we show that C_0 is unbounded. Suppose that $x \in [A]^{\leq\omega}$. Define $y_0 = x$ and $y_{n+1} = \bigcup\{f(s) \cap A : s \in [y_n]^{<\omega}\}$, and $z = \bigcup_{n\in\omega} y_n$. Then $x \subseteq z \in C_0$.

Closed: suppose that $x_0 \subseteq x_1 \subseteq \cdots$ with each $x_i \in C_0$. Clearly $\bigcup_{n\in\omega} x_n \in C_0$.

(3) $C_0 \subseteq p[C]$.

In fact, suppose that $x \in C_0$. If x is finite, then $f(x) \cap A = x$, i.e., $p(f(x)) = x$, so $x \in p[C]$. Suppose that x is infinite; say $x = \{a_0, a_1, \ldots\}$. Then $f(\{a_0\}) \subseteq f(\{a_0, a_1\}) \subseteq \cdots$, and each term here is in C, so $f(\{a_0\}) \cup f(\{a_0, a_1\}) \cup \cdots \in C$. We have

$$x \subseteq ((f(\{a_0\}) \cup f(\{a_0, a_1\}) \cup \cdots) \cap A) \subseteq x,$$

so $x = (f(\{a_0\}) \cup f(\{a_0, a_1\}) \cup \cdots) \cap A$. □

Proposition 31.169. (V.7.8) *If $A \subseteq B$, define $p : [B]^{\leq\omega} \to [A]^{\leq\omega}$ by $p(y) = y \cap A$.*
If $S \subseteq [A]^{\leq\omega}$ is stationary in $[A]^{\leq\omega}$, then $p^{-1}[S]$ is stationary in $[B]^{\leq\omega}$.

Proof. Suppose that $C \subseteq [B]^{\leq\omega}$ is club in $[B]^{\leq\omega}$. By Proposition 31.168 let $D \subseteq [A]^{\leq\omega}$ be club in $[A]^{\leq\omega}$ such that $D \subseteq p[C]$. Choose $a \in D \cap S$. Say $a = p(b)$ with $b \in C$. Then $b \in p^{-1}[S]$. So $C \cap p^{-1}[S]] \neq \emptyset$. □

Proposition 31.170. (V.7.8) *If $A \subseteq B$, define $p : [B]^{\leq\omega} \to [A]^{\leq\omega}$ by $p(y) = y \cap A$.*
If $S \subseteq [B]^{\leq\omega}$ is stationary in $[B]^{\leq\omega}$, then $p[S]$ is stationary in $[A]^{\leq\omega}$.

Proof. Suppose that $C \subseteq [A]^{\leq\omega}$ is club in $[A]^{\leq\omega}$. By Proposition 31.167, $p^{-1}[C]$ is club in $[B]^{\leq\omega}$. Choose $a \in S \cap p^{-1}[C]$. Then $p(a) \in C \cap p[S]$. □

For $E \subseteq \mathbb{P}$, $p \perp E$ means $\forall q \in E[p \perp q]$. We say that E is *predense below* p, in symbols $p \leq \bigvee E$, iff $\forall q \leq p \exists r \in E[q$ and r are compatible$]$.

Lemma 31.171. (V.7.10) *$q \perp E$ iff $q \Vdash [\check{E} \cap \Gamma = \emptyset]$.*

Proof. $\Rightarrow$: Assume that $q \perp E$ and $q \in G$ generic. Suppose that $p \in E \cap G$. Then p and q are compatible, contradiction.

$\Leftarrow$: Suppose that $q \not\perp E$. Say $p \in E$ with p, q compatible. Say $r \leq p, q$. Let $r \in G$, generic. Then $p \in E \cap G$. Thus $E \cap G \neq \emptyset$. Hence $q \not\Vdash [\check{E} \cap \Gamma = \emptyset]$. □

Lemma 31.172. (V.7.10) *$p \leq \bigvee E$ iff $p \Vdash [\check{E} \cap \Gamma \neq \emptyset]$.*

Proof. $\Rightarrow$: Assume that $p \leq \bigvee E$. Thus $\forall q \leq p[q \not\perp E]$, so by Lemma 31.171,

(1) $\forall q \leq p[q \not\Vdash [\check{E} \cap \Gamma = \emptyset]]$.

Now suppose that $p \not\Vdash [\check{E} \cap \Gamma \neq \emptyset]$. Then there is a generic G with $p \in G$ such that $E \cap G = \emptyset$. Hence there is a $q \in G$ such that $q \Vdash [\check{E} \cap \Gamma = \emptyset]$. Say $r \leq p, q$. Then $r \Vdash [\check{E} \cap \Gamma = \emptyset]$. This contradicts (1).

$\Leftarrow$: Assume that $p \Vdash [\check{E} \cap \Gamma \neq \emptyset]$. Suppose that $p \not\leq \bigvee E$. Then there is a $q \leq p$ such that $\forall r \in E[q \perp r]$; that is, such that $q \perp E$. By Lemma 31.171, $q \Vdash [\check{E} \cap \Gamma = \emptyset]$. But $q \leq p$ so $q \Vdash [\check{E} \cap \Gamma \neq \emptyset]$, contradiction. □

If $\mathbb{P} \in M$, then $p \in \mathbb{P}$ is $(M, \mathbb{P})$-*generic* iff for all dense $D \subseteq \mathbb{P}$ such that $D \in M$ we have $p \leq \bigvee (D \cap M)$.

Proposition 31.173. (V.7.12) *If θ is uncountable and regular, $\mathbb{P} \in M \preceq H(\theta)$ and $p \in \mathbb{P}$, then the following conditions are equivalent:*

(i) p is $(M, \mathbb{P})$-generic.

(ii) For all open dense $D \subseteq \mathbb{P}$ such that $D \in M$ we have $p \leq (D \cap M)$.

(iii) For all predense $D \subseteq \mathbb{P}$ such that $D \in M$ we have $p \leq (D \cap M)$.

Proof. First we claim:

(1) If $D \in M$, then $(D{\downarrow}') \in M$.

In fact, suppose that $D \in M$. Then

$$H(\theta) \models \exists X \forall x[x \in X \leftrightarrow \exists y \in D[x \leq y]]; \quad \text{hence}$$
$$M \models \exists X \forall x[x \in X \leftrightarrow \exists y \in D[x \leq y]];$$

taking $X \in M$ such that $M \models \forall x[x \in X \leftrightarrow \exists y \in D[x \leq y]]$, we have $H(\theta) \models \forall x[x \in X \leftrightarrow \exists y \in D[x \leq y]]$, so $X = (D{\downarrow}')$. So (1) holds.

Now obviously (i)$\Rightarrow$(ii). (ii)$\Rightarrow$(iii): Assume (ii), and suppose that $D \subseteq \mathbb{P}$ is predense and $D \in M$. By Lemma 25.63, $D{\downarrow}'$ is open dense, and by (1) $(D{\downarrow}') \in M$. Hence by (ii), $p \leq \bigvee((D{\downarrow}') \cap M)$. Thus $\forall q \leq p \exists r \in (D{\downarrow}') \cap M[q$ and r are compatible$]$. Suppose that $q \leq p$, and choose $r \in (D{\downarrow}') \cap M$ so that q and r are compatible. Choose $s \in D$ such that $r \leq s$. Then q and s are compatible. This shows that $p \leq \bigvee(D \cap M)$, as desired in (iii).

(iii)$\Rightarrow$(i): similarly. □

Proposition 31.174. *If $D \subseteq \mathbb{P}$ is dense, then there is an $A \subseteq D$ such that A is a maximal antichain.*

Proof. Let $A \subseteq D$ be maximal such that it is an antichain. We claim that A is a maximal antichain in $\mathbb{P}$. For, suppose not; then there is a $b \in \mathbb{P}$ incompatible with each member of A. Choose $d \leq b$ with $d \in D$. Then d is incompatible with each member of A; so $A \cup \{d\}$ is a subset of D which is an antichain, and $d \notin A$ since otherwise b would be incompatible with d. This contradiction proves the proposition. □

Proposition 31.175. (V.7.12) *If θ is uncountable and regular, $\mathbb{P} \in M \preceq H(\theta)$ and $p \in \mathbb{P}$, then the following conditions are equivalent:*

(i) p is $(M, \mathbb{P})$-generic.

(ii) For every maximal antichain $D \subseteq \mathbb{P}$ such that $D \in M$ we have $p \leq (D \cap M)$.

Proof. (i)$\Rightarrow$(ii): Assume (i), and suppose that $D \subseteq \mathbb{P}$ is a maximal antichain such that $D \in M$. Then $(D{\downarrow}') \in M$ by (1) in the proof of Proposition 31.173. We claim that $D{\downarrow}'$ is dense. For, let $q \in \mathbb{P}$. Choose $r \in D$ such that q and r are compatible; say $s \leq q, r$. Thus $s \in (D{\downarrow}')$ and $s \leq q$, as desired. It follows that $p \leq ((D{\downarrow}') \cap M)$. As in the proof of 31.173 this shows that $p \leq (D \cap M)$.

(ii)$\Rightarrow$(i): Assume (ii), and suppose that $D \subseteq M$ with D dense. Then

$$H(\theta) \models \exists X[X \text{ is a maximal antichain and } X \subseteq D], \quad \text{hence}$$
$$M \models \exists X[X \text{ is a maximal antichain and } X \subseteq D].$$

Take $X \in M$ such that $M \models [X \text{ is a maximal antichain and } X \subseteq D]$. Then $H(\theta) \models [X \text{ is a maximal antichain and } X \subseteq D]$; so X is a maximal antichain and $X \subseteq D$. Hence by (ii), $p \leq (X \cap M)$. To show that $p \leq (D \cap M)$, take any $q \leq p$. Since $p \leq (X \cap M)$, choose $r \in X$ so that q and r are compatible. Then $r \in D$, as desired. □

Proposition 31.176. (V.7.13) *If $\mathbb{P} \in M \preceq H(\theta)$ and $\mathbb{P}$ is ccc, then every $p \in \mathbb{P}$ is $(M, \mathbb{P})$-generic.*

Proof. Suppose that $A \subseteq \mathbb{P}$ is a maximal antichain of $\mathbb{P}$ and $A \in M$. By Lemma 23.61, $A \subseteq M$. Hence $A \cap M = A$. Clearly $p \leq \bigvee A$. So $p \in \mathbb{P}$ is $(M, \mathbb{P})$-generic by Proposition 31.175. □

Proposition 31.177. (V.7.14) *If $\mathbb{P} \in M \preceq H(\theta)$, $\mathbb{P}$ is countably closed, $p \in M$, and M is countable, then there is a $q \leq p$ such that q is $(M, \mathbb{P})$-generic.*

Proof. Let $\langle D_n : n \in \omega \rangle$ list all of the dense subsets of $\mathbb{P}$ which are in M.

(1) $\forall q \leq p \forall n \in \omega \exists r \leq q[r \in D_n \cap M]$.

For, assume that $q \leq p$ and $n \in \omega$. Then $H(\theta) \models \exists r \leq q[r \in D_n]$, so $M \models \exists r \leq q[r \in D_n]$; this gives (1).

By (1) we get $p = p_0 \geq p_1 \geq \cdots$ such that each $p_{n+1} \in D_n \cap M$. Choose $q \in \mathbb{P}$ such that $q \leq$ each p_n. Then $q \leq \bigvee (D_n \cap M)$ for all n. In fact, if $r \leq q$ then $r \leq p_{n+1} \in D_n \cap M$, so r is compatible with the member p_{n+1} of $D_n \cap M$. □

Proposition 31.178. (V.7.15) *If $\theta \geq \omega_2$, θ regular, M is countable, $M \preceq H(\theta)$, and $\mathbb{P} = \mathrm{Fn}(\omega, \omega_1, \omega)$, then $\mathbb{P} \in M$ and no q is $(M, \mathbb{P})$-generic.*

Proof.

$$H(\theta) \models \exists X \forall f[f \in X \text{ iff } f \text{ is a finite function with domain} \subseteq \omega \text{ and range} \subseteq \omega_1];$$

as usual, this shows that $\mathbb{P} \in M$. Suppose that q is $(M, \mathbb{P})$-generic. Fix $n \in \omega \backslash \mathrm{dmn}(g)$. Now $A \stackrel{\text{def}}{=} \{\{(n, \xi)\} : \xi < \omega_1\} \in M$ by the usual argument. A is a maximal antichain in $\mathbb{P}$. Hence by Proposition 31.175, $q \leq \bigvee (A \cap M)$. Thus $M \models \forall r \leq q \exists f \in A \cap M[r$ and f are compatible$]$. So $H(\theta) \models \forall r \leq q \exists f \in A \cap M[r$ and f are compatible$]$. Let $r = q \cup \{(n, (\sup_{f \in A \cap M} f(n)) + 1)\}$. Then r is not compatible with any member of $A \cap M$, contradiction. □

Proposition 31.179. (V.7.16) *Let $S \subseteq \omega_1$ be stationary, and let $\mathbb{P}$ be the poset described in Proposition 31.33. Suppose that $M \preceq H(\theta)$, M countable, and $S \in M$. Let $\gamma = \sup(M \cap \omega_1)$, and assume that $\gamma \in S$. Then*

(i) $\mathbb{P} \in M$.

(ii) $\forall p \in \mathbb{P} \cap M \exists q \leq p[q$ is $(M, \mathbb{P})$-generic$]$.

Proof. For (i), note that

$$\begin{aligned} H(\theta) \models \exists X \forall f[f \in X \text{ iff } p \subseteq S \text{ and } |p| \leq \omega \text{ and } \forall \text{limit } \rho < \omega_1 \\ [\forall \delta \in p \cap \rho \exists \varepsilon \in (\delta, \rho)[\varepsilon \in p \cap \rho] \to \rho \in p] \text{ and } \exists Y[Y \subseteq X \times X \\ \text{and } \forall p, q \in X[(q, p) \in Y \text{ iff } [p = \emptyset \text{ or } q \cap (\max(p) + 1) = p]]]] \end{aligned}$$

Hence this statement holds in M, so that there exist $X, Y \in M$ such that

$$\begin{aligned} M \models &\forall f[f \in X \text{ iff } p \subseteq S \text{ and } |p| \leq \omega \text{ and } \forall \text{limit } \rho < \omega_1 \\ &\quad [\forall \delta \in p \cap \rho \exists \varepsilon \in (\delta, \rho)[\varepsilon \in p \cap \rho] \to \rho \in p] \text{ and } [Y \subseteq X \times X \\ &\quad \text{and } \forall p, q \in X[(q,p) \in Y \text{ iff } [p = \emptyset \text{ or } q \cap (\max(p)+1) = p]]]] \end{aligned}$$

This holds in $H(\theta)$, and (i) follows.

For (ii), let $p \in \mathbb{P} \cap M$. Let $\langle D_n : n \in \omega \rangle$ list all of the dense open subsets of $\mathbb{P}$ which are in M. For each $n \in \omega$ define $f : \omega \times \mathbb{P} \times S \to \mathbb{P}$ by setting, for each $n \in \omega$, $q \in \mathbb{P}$, and $\alpha \in S$, $f(n, q, \alpha) =$ some $p \in D_n$ such that $p \leq q \cup \{\alpha\}$. Thus $f \in H(\theta)$. So

$$\begin{aligned} H(\theta) \models \exists f[&f \text{ is a function and } \mathrm{dmn}(f) = \omega \times \mathbb{P} \times S \text{ and } \forall n \in \omega \forall q \in \mathbb{P} \forall \alpha \in S \\ &[f(n,q,\alpha) \in D_n \text{ and } f(n,q,\alpha) \leq q \cup \{\alpha\}] \end{aligned}$$

It follows that this statement holds in M, and hence the function f is in M. Now let $\langle \beta_n : n \in \omega \rangle$ be a strictly increasing sequence of ordinals with supremum γ. We now define $\langle q_n : n \in \omega \rangle$. Let $q_0 = p$. If $q_0 \geq \cdots \geq q_n$ have been defined, let $q_{n+1} = f(n, q_n, \beta_n)$. Since $f \in M$, it follows that each $q_n \in M$, and hence $\max(q_n) < \gamma$. Finally, let $r = \bigcup_{n \in \omega} q_n \cup \{\gamma\}$. Then clearly $r \in \mathbb{P}$. Now we claim that r is $(M, \mathbb{P})$-generic. We need to show that $r \leq (D_n \cap M)$ for each $n \in \omega$. Suppose that $s \leq r$. Then $s \leq q_{n+1} = f(n, q_n, \beta_n) \in D_n$, so s is compatible with the member q_{n+1} of D_n. $\square$

Proposition 31.180. (V.7.16) *Let $S \subseteq \omega_1$ be stationary, and let $\mathbb{P}$ be the poset described in Proposition 31.33. Suppose that $M \preceq H(\theta)$, M countable, and $S \in M$. Let $\gamma = \sup(M \cap \omega_1)$, and assume that $\gamma \notin S$. Then*

(i) $\mathbb{P} \in M$.

(ii) There is no $q \in \mathbb{P}$ which is $(M, \mathbb{P})$-generic.

Proof. (i) holds as in the proof of Proposition 31.179. Now note:

(1) $r, s \in \mathbb{P}$ are compatible iff $r \leq s$, where $\max(s) \leq \max(r)$.

In fact, $\Leftarrow$ is obvious. For $\Rightarrow$, suppose that $t \leq r, s$. Then $r \cap (\max(s)+1) = t \cap (\max(r)+1) \cap (\max(s)+1) = t \cap (\max(s)+1) = s$.

Now for (ii), suppose that $q \in \mathbb{P}$ is $(M, \mathbb{P})$-generic. Let $r = q \cup \{\delta\}$, where $\max(q), \gamma < \delta \in S$. So $r \in \mathbb{P}$ and $r \leq q$. $\langle \beta_n : n \in \omega \rangle$ be a strictly increasing sequence of ordinals with supremum γ. Take any $n \in \omega$. Then $D_n \stackrel{\text{def}}{=} \{s \in \mathbb{P} : \beta_n \leq \max(s)\}$ is dense, and hence $q \leq D_n \cap M$. It follows that there is an $s_n \in D \cap M$ such that r and s_n are compatible. Hence by (1), $r \leq s_n$. Now $\beta_n \leq \max(s) < \gamma$. Since n is arbitrary, $r \cap \gamma$ is unbounded in γ. Since $\gamma \notin S$, this contradicts r being closed. $\square$

Lemma 31.181. (V.7.18) *In M suppose that θ is uncountable and regular and $\mathbb{P} \in H(\theta)$.*

Then in any generic extension $M[G]$ there is a club $C \subseteq [(H(\theta))^M]^{\leq\omega}$ such that for all $N \in C$ and $p \in \mathbb{P}$ the following conditions hold:

(i) $N \preceq (H(\theta))^M$ and $\mathbb{P} \in N$.

(ii) For all $D \in N$, if D is a dense subset of $\mathbb{P}$, then $D \cap N \cap G \neq \emptyset$.

(iii) If $N \in M$ and $p \in G$, then there is a $q \le p$ that is $(N, \mathbb{P})$-generic.

Proof. Let C be the set of all $N \in [(H(\theta))^M]^{\le\omega}$ satisfying (i) and (ii). Clearly C is closed. To show that it is unbounded, let $K \in [(H(\theta))^M]^{\le\omega}$. Let c be a choice funtion for nonempty subsets of $(H(\theta))^M$. Let $\langle \exists x_n \varphi_n(x_n, \overline{y}_n) : n \in \omega\rangle$ list all formulas in the language of set theory that begin with an existential quantifier. We define $Y_0 = K$ and

$$\begin{aligned} Y_{m+1} = Y_m &\cup \{a \in (H(\theta))^M : n \in \omega, \exists \overline{b} \subseteq Y_m[(H(\theta))^M \models \varphi_n(a, \overline{b})]\} \\ &\cup \{c(D \cap G) : D \subseteq \mathbb{P},\ D \text{ dense},\ D \in Y_m\}. \end{aligned}$$

Finally, let $N = \bigcup_{m\in\omega} Y_m$. Clearly N satisfies (i) and (ii).

For (iii), suppose that $N \in M$ (and $N \in C$), and $p \in G$. Then there is a $q \in G$ with $q \le p$ such that $q \Vdash \forall D[D \text{ dense in } \check{\mathbb{P}} \to \exists x[x \in \check{N} \cap D \cap \Gamma]]$. Thus for all $D \in N$ such that $D \subseteq \mathbb{P}$ is dense in $\mathbb{P}$ we have $q \Vdash [(D \cap N)^v \cap \Gamma \ne \emptyset]$. Hence by Lemma 31.172, $q \le \bigvee(D \cap N)$ for all $D \in M$ such that $D \subseteq \mathbb{P}$ is dense in $\mathbb{P}$. □

Lemma 31.182. (V.7.19) *Assume that $M \in [H(\theta)]^{\le\omega}$ and $\mathbb{P}, \kappa, p \in M$ and $M \preceq H(\theta)$. Let $q \le p$ be $(M, \mathbb{P})$-generic, and let τ be a $\mathbb{P}$-name such that $p \Vdash [\tau \in [\kappa]^{\le\omega}]$. Then $q \Vdash [\tau \subseteq M \cap \kappa]$.*

Proof. Fix $\dot{f} \in M$ such that $p \Vdash [\dot{f} \in {}^\omega\kappa \text{ and } \tau \subseteq \text{rng}(\dot{f})]$. For each $n \in \omega$ let $D_n = \{r \in \mathbb{P} : r \perp p \text{ or } \exists\alpha[r \Vdash [\dot{f}(n) = \alpha]]\}$. Clearly D_n is dense and $D_n \in M$; so $q \le \bigvee(D_n \cap M)$. Hence $q \le \bigvee\{r \le p : \exists\alpha \in M[r \Vdash [\dot{f}(n) = \alpha]]\}$. Then $q \Vdash [\dot{f}(n) \in M]$. In fact, let $q \in G$, generic. Suppose that $\dot{f}_G(n) \notin M$. Then there is an $s \le q$ with $s \in G$ such that $s \Vdash [\dot{f}(n) \notin M]$. But $q \le \bigvee\{r \le p : \exists\alpha \in M[r \Vdash [\dot{f}(n) = \alpha]]\}$ implies that there is an $r \le p$ compatible with s and an $\alpha \in M$ such that $r \Vdash [\dot{f}(n) = \alpha]$, contradiction. □

Let A be an uncountable set and $f : [A]^{<\omega} \to [A]^{\le\omega}$. A set $x \in [A]^{\le\omega}$ is a *closure point* of f iff $f(e) \subseteq x$ for every $e \in [x]^{<\omega}$. $\text{Cl}(f)$ is the collection of all closure points of f.

If A is uncountable, $f : [A]^{<\omega} \to [A]^{\le\omega}$, and $x \in [A]^{\le\omega}$, we define $y_0 = x$ and $y_{i+1} = y_i \cup \bigcup\{f(e) : e \in [y_i]^{<\omega}\}$, and $\text{Cl}_f(x) = \bigcup_{i\in\omega} y_i$.

Lemma 31.183. *If A is an uncountable set, $C \subseteq [A]^{\le\omega}$ is club, and $D \subseteq C$ is countable and directed, then $\bigcup D \in C$.*

Proof. Let $D = \{d_i : i < \omega\}$. For each $i < \omega$ let $e_i \in D$ be such that $d_j \subseteq e_i$ for each $j < i$ and also $e_j \subseteq e_i$ for each $j < i$. Then $\bigcup D = \bigcup_{i<\omega} e_i \in C$. □

Lemma 31.184. *If A is uncountable, $f : [A]^{<\omega} \to [A]^{\le\omega}$ and $\emptyset \ne x \in [A]^{\le\omega}$, then:*

(i) $\text{Cl}_f(x) \in \text{Cl}(f)$*;*
(ii) If $z \subseteq x$, then $\text{Cl}_f(z) \subseteq \text{Cl}_f(x)$*;*
(iii) $\text{Cl}_f(x) = \bigcup\{\text{Cl}_f(z) : z \in [x]^{<\omega}\}$*;*
(iv) If $y \subseteq x \in \text{Cl}(f)$, then $\text{Cl}_f(y) \subseteq x$*;*
(v) If $x \in \text{Cl}(f)$, then $\text{Cl}_f(x) = x$*.*

Proof. (i): suppose that $e \in [\text{Cl}_f(x)]^{<\omega}$. With $\langle y_i : i \in \omega\rangle$ as in the definition of $\text{Cl}_f(x)$, there is an $i \in \omega$ such that $e \in [y_i]^{<\omega}$. Hence $f(e) \subseteq y_{i+1} \subseteq \text{Cl}_f(x)$. This proves (i).

(ii): if $\langle y_i^z : i \in \omega\rangle$ is as in the definition of $\mathrm{Cl}_f(z)$ and $\langle y_i^x : i \in \omega\rangle$ is as in the definition of $\mathrm{Cl}_f(x)$, then by induction $y_i^z \subseteq y_i^x$ for all $i \in \omega$, and (ii) follows.

(iii): $\supseteq$ holds by (ii). For $\subseteq$, with $\langle y_i : i \in \omega\rangle$ as in the definition of $\mathrm{Cl}_f(x)$, we prove that $y_i \subseteq$ rhs for all $i \in \omega$ by induction on i, where rhs is the right-hand side of (iii). Since $z \subseteq \mathrm{Cl}_f(z)$ for each $z \in [x]^{<\omega}$ we have $y_0 = x \subseteq$ rhs. Now suppose that $y_i \subseteq$ rhs. If $e \in [y_i]^{<\omega}$, then there is a $z \in [x]^{<\omega}$ such that $e \in \mathrm{Cl}_f(z)$, by (ii). Then $f(e) \subseteq \mathrm{Cl}_f(z) \subseteq$ rhs. Hence $y_{i+1} \subseteq$ rhs. This proves (iii).

(iv): clear.

(v): $x \subseteq \mathrm{Cl}_f(x) \subseteq x$ by (iv). □

Lemma 31.185. *If A is an uncountable set and $f : [A]^{<\omega} \to [A]^{\leq\omega}$, then $\mathrm{Cl}(f)$ is a club of $[A]^{\leq\omega}$.*

Proof. Clearly $\mathrm{Cl}(f)$ is closed. If $x \in [A]^{\leq\omega}$, then $x \subseteq \mathrm{Cl}_f(x) \in \mathrm{Cl}(f)$ by Lemma 31.184, so $\mathrm{Cl}(f)$ is unbounded. □

Lemma 31.186. *If A is an uncountable set and C is club in $[A]^{\leq\omega}$, then there is an $f : [A]^{<\omega} \to [A]^{\leq\omega}$ such that $\mathrm{Cl}(f) \subseteq C$.*

Proof. We define $f(e) \in C$ for all $e \in [A]^{<\omega}$ by recursion on $|e|$. For each $a \in A$ choose $f(\{a\}) \in C$ with $\{a\} \subseteq f(\{a\})$. Now suppose that $f(e)$ has been defined for all $e \in [A]^n$, and $e \in [A]^{n+1}$. Choose $f(e) \in C$ such that $e \cup \bigcup_{a\in e} f(e\backslash\{a\}) \subseteq f(e)$. Note that $e_1 \subseteq e_2$ implies that $f(e_1) \subseteq f(e_2)$.

We claim that $\mathrm{Cl}(f) \subseteq C$. For, suppose that $x \in \mathrm{Cl}(f)$. Then $\{f(e) : e \in [x]^{<\omega}\}$ is directed and $x = \bigcup\{f(e) : e \in [x]^{<\omega}\}$, so $x \in C$. □

Lemma 31.187. *Suppose that θ is uncountable and regular, and λ is uncountable and $\lambda \in H(\theta)$. Suppose that $C \subseteq [H(\theta)]^{\leq\omega}$ is club. Then there is a club C' in $[\lambda]^{\leq\omega}$ such that $C' \subseteq \{x \cap \lambda : x \in C\}$.*

Proof. Let $f : [H(\theta)]^{<\omega} \to [H(\theta)]^{\leq\omega}$ be such that $\mathrm{Cl}(f) \subseteq C$, by Lemma 31.186. Define $g : [\lambda]^{<\omega} \to [\lambda]^{\leq\omega}$ by setting $g(e) = \mathrm{Cl}_f(e) \cap \lambda$ for all $e \in [\lambda]^{<\omega}$. We claim that $\mathrm{Cl}(g) \subseteq \{x \cap \lambda : x \in \mathrm{Cl}(f)\}$. For, suppose that $y \in \mathrm{Cl}(g)$. Then

$$\begin{aligned}\mathrm{Cl}_f(y) \cap \lambda = \bigcup\{\mathrm{Cl}_f(z) \cap \lambda : z \in [y]^{<\omega}\} &= \bigcup\{g(z) : z \in [y]^{<\omega}\}\\ &\subseteq \bigcup\{\mathrm{Cl}_g(z) : z \in [y]^{<\omega}\} = y,\end{aligned}$$

so $y = \mathrm{Cl}_f(y) \cap \lambda$. □

Theorem 31.188. (V.7.17) *Suppose that $\mathbb{P}$ is a forcing order. Then the following are equivalent:*

(i) $\mathbb{P}$ is proper,

(ii) For every uncountable regular cardinal θ with $\mathbb{P} \in H(\theta)$ there is a club $C \subseteq [H(\theta)]^{\leq\omega}$ such that for all $N \in C$ the following conditions hold:

(a) $\mathbb{P} \in N$.

(b) $N \preceq H(\theta)$

(c) for all $p \in \mathbb{P} \cap N$ *there is a* $q \leq p$ *which is* (N, P)*-generic.*

(iii) There is an infinite cardinal ρ *such that for every uncountable regular cardinal* $\theta \geq \rho$ *with* $\mathbb{P} \in H(\theta)$ *there is a club* $C \subseteq [H(\theta)]^{\leq\omega}$ *such that for all* $N \in C$ *the following conditions hold:*

(a) $\mathbb{P} \in N$.

(b) $N \preceq H(\theta)$

(c) for all $p \in \mathbb{P} \cap N$ *there is a* $q \leq p$ *which is* (N, P)*-generic.*

Proof. (i)⇒(ii): Assume that $\mathbb{P}$ is proper and θ is a regular uncountable cardinal such that $\mathbb{P} \in H(\theta)$. Let

$$S = \{N \subseteq [H(\theta)]^{\leq\omega} : \mathbb{P} \in N,\ N \preceq H(\theta),\ \exists p \in \mathbb{P} \cap N \forall q \leq p[q \text{ is not } (N, \mathbb{P})\text{-generic}]\}.$$

We claim that S is not stationary. For, suppose that it is. For each $N \in S$ choose $p \in \mathbb{P} \cap N$ as indicated. Then by Fodor's theorem, Theorem 23.23, there exist a $p \in \mathbb{P}$ and a stationary subset T of S such that for all $N \in T$, $p \in \mathbb{P} \cap N$ and $\forall q \leq p[q$ is not $(N, \mathbb{P})$-generic]. Suppose that G is $\mathbb{P}$-generic over M with $p \in G$. In $M[G]$ let C be a club given by Lemma 31.181. Since $\mathbb{P}$ is proper, T is stationary in $M[G]$, and so we can choose $N \in T \cap C$. Thus $N \in M$. By Lemma 31.181(iii), there is a $q \leq p$ which is $(N, \mathbb{P})$-generic. This contradicts the definition of T. So S is not stationary.

It follows that there is a club C in $[H(\theta)]^{\leq\omega}$ such that $C \cap S = \emptyset$. Let $C' = \{N \in [H(\theta)]^{\leq\omega} : \mathbb{P} \in N \text{ and } N \preceq H(\theta)\}$. Then C' is club. Now $C \cap C'$ is as desired in the theorem.

(ii)⇒(iii): obvious.

(iii)⇒(i): Assume (iii). Let λ be an uncountable cardinal, and suppose that $S \subseteq [\lambda]^{\leq\omega}$ is stationary, G is generic, and S is not stationary in $M[G]$. Say $C \subseteq [\lambda]^{\leq\omega}$ is club with $S \cap C = \emptyset$. Let $\langle f^n_i : i \in \omega, n \in \omega\rangle$ be a system of members of $M[G]$ with $f^n_i : {}^n\lambda \to \lambda$ and, with $M = (\lambda, \langle f^n_i : i \in \omega, n \in \omega\rangle)$ we have $\mathrm{Sm}(M) \subseteq C$. (See Theorem 30.84.) Then there is a name σ and a $p \in G$ such that $\forall i \in \omega \forall n \in \omega[p \Vdash [\sigma^n_i : {}^n\lambda \to \lambda$ and $\mathrm{Sm}(\mathrm{op}(\lambda, \sigma)) \cap S = \emptyset]$. Let $\theta \geq \rho$ be an uncountable cardinal such that $\lambda \in H(\theta)$. By (iii) let $C' \subseteq [H(\theta)]^{\leq\omega}$ satisfy the indicated conditions. By Lemma 31.187, let C'' be a club in $[\lambda]^{\leq\omega}$ such that $C'' \subseteq \{N \cap \lambda : N \in C'\}$. Since S is stationary, choose $N \in C' \cap C''$ such that $N \cap \lambda \in C'' \cap C' \cap S$. Now $H(\theta) \models \exists p[p \Vdash [\sigma^n_i : {}^n\lambda \to \lambda$ and $\mathrm{Sm}(\mathrm{op}(\lambda, \sigma)) \cap S = \emptyset]]$, so there is a $p \in N$ and $\sigma^n_i \in N$ such that $p \Vdash [\sigma^n_i : {}^n\lambda \to \lambda$ and $\mathrm{Sm}(\mathrm{op}(\lambda, \sigma)) \cap S = \emptyset]$. Choose $q \leq p$ such that q is $(N, \mathbb{P})$-generic. We claim that $q \Vdash N \cap \lambda \in \mathrm{Sm}(\mathrm{op}(\lambda, \sigma))$. Let $e \in {}^n(N \cap \lambda)$; we show that $q \Vdash \sigma^n_i(e) \in N \cap \lambda$. Let

$$A = \{r : \exists \alpha < \lambda[r \Vdash \sigma^n_i(e) = \alpha]\} \cup \{r : r \perp q\}.$$

Then A is dense and $A \in N$. From q being $(N, \mathbb{P})$-generic it follows that for all $r \leq q$, r is compatible with some $s \in A \cap N$. Hence for any $r \leq q$ there exist s, t such that $t \leq r, s$ and $s \in A \cap N$. Therefore there is an $\alpha < \lambda$ such that $s \Vdash \sigma^n_i(e) = \alpha$. Since α is definable from s, σ^n_i, e, it follows that $\alpha \in N$.

Thus we have shown that $q \Vdash N \cap \lambda \in \mathrm{Sm}(\mathrm{op}(\lambda, \sigma))$, $q \Vdash \mathrm{Sm}(\mathrm{op}(\lambda, \sigma)) \cap S = \emptyset]$, and $N \cap \lambda \in S$, contradiction. $\square$

The *proper forcing axiom*, PFA, is the statement that $MA_{\mathbb{P}}(\omega_1)$ holds for every proper poset $\mathbb{P}$.

Lim is the set of all countable limit ordinals. A *ladder system* is a sequence $\langle C_\gamma : \gamma \in \text{Lim}\rangle$ such that each C_γ is a cofinal subset of γ of order type ω.

A ladder system $\langle C_\gamma : \gamma \in \text{Lim}\rangle$ is *club guessing* iff for every club $D \subseteq \omega_1$ there is a $\gamma \in \text{Lim}$ such that $C_\gamma \subseteq D$.

Lemma 31.189. (V.7.21) *$\diamondsuit$ implies that there is a club guessing system.*

Proof. Let $\langle A_\alpha : \alpha < \omega_1\rangle$ be a $\diamondsuit$ sequence. For each $\gamma < \omega_1$ let $C_\gamma \subseteq A_\gamma$ be cofinal in γ if $\sup(A_\gamma) = \gamma$, and $C_\gamma = \emptyset$ otherwise. Now let $D \subseteq \omega_1$ be club. Let $D' = \{\gamma \in D : \gamma$ is limit and $D \cap \gamma$ is cofinal in $\gamma\}$. Then $\{\gamma < \omega_1 : D \cap \gamma = A_\gamma\}$ is stationary. Choose $\gamma \in D'$ with $D \cap \gamma = A_\gamma$. Then $D \cap \gamma$ is cofinal in γ, so also A_γ is cofinal in γ, so $C_\gamma \subseteq A_\gamma \subseteq D$. $\square$

Lemma 31.190. (V.7.22) *If C is a club guessing system and $\mathbb{P}$ is ccc, then C remains a club guessing system in any generic extension $M[G]$ using $\mathbb{P}$.*

Proof. Suppose that C is a club guessing system and $\mathbb{P}$ is ccc. Let D be a club in $M[G]$. By Proposition 30.81, there is a club D' in M such that $D' \subseteq D$. Then there is a limit γ such that $C_\gamma \subseteq D'$. $\square$

Theorem 31.191. (V.7.24) *Assume PFA, and suppose that C is a ladder system. Then there is a club $D \subseteq \omega_1$ such that $C_\gamma \backslash D$ is infinite for every limit $\gamma < \omega_1$.*

Proof. Let $\mathbb{P}$ be the set of all countable closed $p \subseteq \omega_1$ such that $\sup(p) \in \text{Lim}\cup\{0\}$ and $\forall$limit $\gamma < \omega_1[C_\gamma\backslash p$ is infinite]. We order $\mathbb{P}$ by saying $q \le p$ iff $p \subseteq q$ and $(q\backslash p)\cap\sup(p) = \emptyset$. Clearly $\le$ is reflexive. Now suppose that $r \le q \le p$. Then obviously $p \subseteq r$. Also,

$$(r\backslash p) \cap \sup(p) = ((r\backslash q) \cup (q\backslash p)) \cap \sup(p) \subseteq ((r\backslash q) \cap \sup(q)) \cup ((q\backslash p) \cap \sup(p)) = \emptyset.$$

Hence $r \le p$.

For each $\alpha < \omega_1$ let $D_\alpha = \{p \in \mathbb{P} : \sup(p) > \alpha\}$.

(1) D_α is dense.

For, let $q \in \mathbb{P}$. Let γ be a limit ordinal greater than $\max(\alpha, \sup(q))$. Let $\langle \delta_i : i < \omega\rangle$ enumerate in order all the members of C_γ which are greater than $\max(\alpha, \sup(q))$. Set $p = q \cup \{\delta_{2i} : i < \omega\} \cup \{\gamma\}$. Then $p \le q$ and $p \in D_\alpha$.

Assuming that $\mathbb{P}$ is proper, let G be a filter meeting all the sets D_α, and let $D = \bigcup G$. Now D is club in ω_1. In fact, it is unbounded since $G \cap D_\alpha \ne \emptyset$ for all $\alpha < \omega_1$. Now suppose that $\gamma < \omega_1$ is limit and $D\cap\gamma$ is unbounded in γ. Choose $p \in G\cap D_\gamma$. Then $p\cap\gamma$ is unbounded in γ. For, if $\alpha < \gamma$ choose $\beta \in D \cap \gamma$ with $\alpha < \beta < \gamma$. Say $\beta \in q \in G$. Say $r \in G$ and $r \le p, q$. Then $\beta \in r$ since $r \le q$, and $\beta \in p$ since r is an end extension of p and $\beta < \gamma < \sup(p)$. Since $p \cap \gamma$ is unbounded in γ and p is closed, it follows that $\gamma \in p$ and hence $\gamma \in D$. So D is club. If $\gamma \in \text{Lim}$, choose $p \in G$ with $\sup(p) > \gamma$. Then $C_\gamma\backslash p$ is infinite, and so clearly $C_\gamma\backslash D$ is infinite.

Now to prove that $\mathbb{P}$ is proper, we apply Theorem 31.188. So, suppose that θ is an uncountable regular cardinal and $\mathbb{P} \in H(\theta)$. Take any countable N such that $\mathbb{P} \in N$ and $N \preceq H(\theta)$. Suppose that $p \in \mathbb{P} \cap N$. Let $\langle D_n : n \in \omega \rangle$ list all of the dense subsets of $\mathbb{P}$ which are in N.

(1) There is a limit ordinal $\gamma < \omega_1$ which is a limit of limit ordinals such that $\sup(p) < \gamma$ and for all $n \in \omega$ and all $q \in N$ if $\sup(q) < \gamma$ then there is an $r \in D_n$ such that $r \leq q$ and $\sup(r) < \gamma$.

In fact, for each $q \in \mathbb{P}$ and $n \in \omega$ let $f(q, n) \in D_n$ be such that $f(q, n) \leq q$. Now we define $\alpha_0 = \sup(p)$ and $\alpha_{m+1} = \sup\{\sup(f(q, n)) : \sup(q) \leq \alpha_m, n \in \omega\} + \omega$, and $\gamma = \sup_{m\in\omega} \alpha_m$. Clearly (1) holds.

Let $\langle \alpha_n : n \in \omega \rangle$ be the strictly increasing enumeration of C_γ. We now define two sequences $\langle q_n : n \in \omega \rangle$ and $\langle \beta_n : n \in \omega \rangle$. Let $q_0 = p$ and let β_0 be a limit ordinal less than γ, with $\sup(p) < \beta_0$, with some member of C_γ in the interval $(\sup(p), \beta_0)$. Having defined q_m and β_m, with β_m a limit ordinal less than γ with $\sup(q_m) < \beta_m$, let $q'_{m+1} = q_m \cup \{\beta_m\}$, and let $q_{m+1} \leq q'_{m+1}$ be a member of $D_m \cap N$ with $\sup(q_{m+1}) < \gamma$. Let β_{m+1} be a limit ordinal less than γ with $\sup(q_{m+1}) < \beta_{m+1}$, and with some member of C_γ in the interval $(\sup(q_{m+1}), \beta_{m+1})$.

Now let $r = \bigcup_{m\in\omega} q_m \cup \{\gamma\}$. We claim that r is $(N, \mathbb{P})$-generic. For, suppose that $m \in \omega$; we claim that $r \leq (D_m \cap N)$. Suppose that $s \leq r$. Then $s \leq q'_{m+1} \in D_m \cap N$, so s and q'_{m+1} are compatible. □

Theorem 31.192. (V.7.2) *Every countably closed poset is proper.*

Proof. We apply Theorem 31.188. Let θ be an uncountable regular cardinal such that $\mathbb{P} \in H(\theta)$. Let $C = \{N : N \text{ countable}, \mathbb{P} \in N \text{ and } N \preceq H(\theta)\}$. Suppose that $p \in \mathbb{P} \cap N$. Let $\langle D_n : n \in \omega \rangle$ list all of the dense subsets of $\mathbb{P}$ which are in N. Define $\langle q_n : n \in \omega \rangle$ by recursion: $q_0 = p$. If q_n has been defined, let $q_{n+1} \leq q_n$ with $q_{n+1} \in D_n$. Let $r \leq q_n$ for all n. Clearly r is $(N, \mathbb{P})$-generic. □

We now give an equivalent definition of properness involving a game. Let $\mathbb{P}$ be a forcing order; we describe a game $\Gamma(\mathbb{P})$ played between players I and II. First I chooses $p_0 \in P$ and a maximal antichain A_0 of $\mathbb{P}$. Then II chooses a countable subset B^0_0 of A_0. At the n-th pair of moves, I chooses a maximal antichain A_n and then II chooses countable sets $B^n_i \subseteq A_i$ for each $i \leq n$. Then we say that II wins iff there is a $q \leq p_0$ such that for every $i \in \omega$, the set

$$\bigcup_{i\leq n\in\omega} B^n_i$$

is predense below q.

We give a rigorous formulation of these ideas, not relying on informal notions of games. A *play* of the game $\Gamma(\mathbb{P})$ is an infinite sequence

$$\langle p_0, A_0, C_0, A_1, C_1, \ldots, A_n, C_n \ldots \rangle$$

satisfying the following conditions for each $n \in \omega$:

(1) $p_0 \in \mathbb{P}$.

(2) A_n is a maximal antichain of $\mathbb{P}$.

(3) $C_n = \langle B_i^n : i \leq n \rangle$, where each B_i^n is a countable subset of A_i.

Given such a play, we say that II *wins* iff there is a $q \leq p_0$ such that for every $i \in \omega$, the set

$$\bigcup_{i \leq n \in \omega} B_i^n$$

is predense below q.

A *partial play of length m of* $\Gamma(\mathbb{P})$ is a sequence

$$\langle p_0, A_0, C_0, A_1, C_1, \ldots, A_{m-1}, C_{m-1}, A_m \rangle$$

satisfying the above conditions. Note that the partial play ends with one of the maximal antichains A_m. A *strategy* for II is a function S whose domain is the set of all partial plays of $\Gamma(\mathbb{P})$, such that if $\mathbb{P}$ is a partial play as above, then $S(\mathbb{P})$ is a set C_m satisfying the condition (3). A play is said to be *according to S* iff for every m, $C_m = S(\langle p_0, A_0, C_0, A_1, C_1, \ldots, A_{m-1}, C_{m-1}, A_m \rangle)$. The strategy S is *winning* iff II wins every play which is played according to S.

Proposition 31.193. *$\mathbb{P}$ is proper iff II has a winning strategy in $\Gamma(\mathbb{P})$.*

Proof. First suppose that $\mathbb{P}$ is proper. Let θ be an uncountable regular cardinal such that $\mathbb{P} \in H(\theta)$, and by Theorem 31.188 let $C \subseteq [H(\theta)]^{\leq \omega}$ be a club with the indicated properties.

Now a strategy for II is as follows. After I chooses A_0, II chooses a countable $N_0 \in C$ with $N_0 \preceq H(\lambda)$ and with $A_0 \in N_0$; and II sets $B_0^0 = A_0 \cap N_0$. Suppose that I chooses A_n, and II has chosen $N_0 \preceq \cdots \preceq N_{n-1} \preceq H(\lambda)$ with each $N_i \in C$. Then II chooses $N_n \in C$ so that $N_{n-1} \preceq N_n \preceq H(\lambda)$, and sets $B_i^n = A_i \cap N_n$ for all $i \leq n$. When the game is finished, let $N_\omega = \bigcup_{n \in \omega} N_n$. So $N_\omega \preceq H(\lambda)$ and $N_\omega \in C$. By Theorem 31.188 choose $q \leq p_0$ so that q is (N_ω, P)-generic. Since $N_\omega \preceq H(\lambda)$, we may assume that $q \in N_\omega$. Take any $i \in \omega$; we claim that $q \leq \bigvee \bigcup_{i \leq n} B_i^n$. Say $q \in N_n$ with $i \leq n$. Again since $N_n \preceq H(\lambda)$, $A_n \in N_n$, and $B_i^n = A_i \cap N_n$, it follows that B_i^n is a maximal antichain in $P \cap N_n$. Let $D = \{r \in P \cap N_n : r \leq s \text{ for some } s \in B_i^n\}$. Then D is dense in $P \cap N_n$. Take any $r \leq q$. Then r is compatible with some $s \in D \cap N_n$. Hence r is compatible with some $t \in B_i^n$. This shows that II wins.

Conversely, suppose that II has a winning strategy σ. Let λ be sufficiently large, and let $N \preceq H(\lambda)$ be such that $P, p_0, \sigma \in N$, N countable. Then we take the game in which I lists all of the maximal antichains of P which are in N, and II plays using his strategy. All of the sets B_i^n which II plays are in N, since $\sigma \in N$. Since II wins, choose $q \leq p_0$ such that for each $i \in \omega$ the set $\bigcup_{i \leq n} B_i^n$ is predense below q. We claim that q is (P, N)-generic. For, let $D \subseteq P$, $D \in N$, be dense. Let C be maximal such that C is an antichain and $\forall p \in C \exists d \in D[p \leq d]$. Then C is a maximal antichain. For, suppose that $p \perp C$. Choose $d \in D$ such that $d \leq p$. Then $d \perp C$, so that $C \cup \{d\}$ still satisfies the conditions on C, contradiction. Say $C = A_i$. Say $q \in N_n$ with $i \le n$. Now $\bigcup_{i\le m} B_i^m$ is predense below q. Hence there is an $s \in \bigcup_{i\le m} B_i^m$ such that s and q are compatible. Say $t \le s, q$. Say $s \in B_i^m$ with $n \le m$. Now $B_i = A_i \cap N$, so $s \in A_i = C$. Choose $d \in D$ such that $s \le d$. Now t is compatible with $d \in D \cap N$ and $t \le q$. This shows that q is (P, N)-generic. So P is proper. $\square$

For the following results, recall the notion of *full name* from just before Lemma 31.129; *proper* from just before Proposition 31.163; $(N, \mathbb{P})$-*generic* from just before Proposition 31.173. i_* from just before Lemma 30.3.

If M is a c.t.m. and $N \in M$, for G a generic filter over M we let $N[G] = \{\sigma_G : \sigma$ is a $\mathbb{P}$-name and $\sigma \in N$.

Theorem 31.194. *For any forcing poset $\mathbb{P}$ the following conditions are equivalent:*

(i) $\mathbb{P}$ is proper.

(ii) For every regular $\theta > 2^{|\mathrm{trcl}(\mathbb{P})|}$, every countable $N \preceq H(\theta)$ with $\mathbb{P} \in N$, and every $p \in \mathbb{P} \cap N$, there is a $q \le p$ such that q is $(N, \mathbb{P})$-generic.

Proof. (i)$\Rightarrow$(ii): Assume that $\mathbb{P}$ is proper. Let $\theta > 2^{|\mathrm{trcl}(\mathbb{P})|}$, θ regular, let $N \preceq H(\theta)$ with $\mathbb{P} \in N$, N countable, and let $r \in \mathbb{P} \cap N$.

By Theorem 31.188 let $C \subseteq [H(|\mathrm{trcl}(\mathbb{P})|^+)]^{\le\omega}$ be club such that for all $Q \in C$ the following conditions hold:

(1) $\mathbb{P} \in Q$.

(2) $Q \preceq H(|\mathrm{trcl}(\mathbb{P})|^+)$

(3) for all $p \in \mathbb{P} \cap Q$ there is a $q \le p$ which is $(Q, \mathbb{P})$-generic.

Now by Lemma 31.186 there is an $f : [H(|\mathrm{trcl}(\mathbb{P})|^+]^{<\omega} \to [H(|\mathrm{trcl}(\mathbb{P})|^+)]^{\le\omega}$ such that $\mathrm{Cl}(f) \subseteq C$. Now note that if $(M, N) \in f$ then $M \in [H(|\mathrm{trcl}(\mathbb{P})|^+)]^{<\omega}$ and $N \in [H(|\mathrm{trcl}(\mathbb{P})|^+)]^{\le\omega}$. So $|M| < |\mathrm{trcl}(\mathbb{P})|^+$ and each member of M is in $H(|\mathrm{trcl}(\mathbb{P})|^+)$ and hence has size $< |\mathrm{trcl}(\mathbb{P})|^+$. So $M \in H(|\mathrm{trcl}(\mathbb{P})|^+)$. Similarly, $N \in H(|\mathrm{trcl}(\mathbb{P})|^+)$. Now $|f| = |H(|\mathrm{trcl}(\mathbb{P})|^+)]^{<\omega}| = |H(|\mathbb{P}|^+)| = 2^{<|\mathrm{trcl}(\mathbb{P})|^+} = 2^{|\mathrm{trcl}(\mathbb{P})|} < \theta$, using Lemma 12.53. So $f \in H(\theta)$.

Now for all $Q \in \mathrm{Cl}(f)$ the conditions (1)–(3) hold. We may assume that f is the least function in $H(\theta)$ with this property. Then clearly N is closed under f, and so $N \cap H(\theta)$ is closed under f. So $N \cap H(\theta) \in \mathrm{Cl}(f)$. Thus by (3) for $N \cap H(\theta)$, since $p \in H(\theta)$ because $p \in \mathbb{P} \in H(\theta)$, it follows that there is a $q \le p$ which is $(N \cap H(\theta), \mathbb{P})$-generic. We claim that q is $(N, \mathbb{P})$-generic. For, suppose that $D \subseteq \mathbb{P}$ is dense and $D \in N$. Since $\mathbb{P} \in H(\theta)$, we have $D \in N \cap H(\theta)$. Hence $q \le \bigvee(D \cap N \cap H(\theta))$. Hence for all $r \le q$ there is an $s \in D \cap N \cap H(\theta)$ such that r and s are compatible. So $q \le \bigvee(D \cap N)$, as desired.

(ii)$\Rightarrow$(i): Assume (ii), and let C be the set of all countable $N \preceq H(\theta)$ such that $\mathbb{P} \in N$. Then Theorem 31.188(iii) holds, and so $\mathbb{P}$ is proper. $\square$

Lemma 31.195. *If $N \preceq H(\lambda)$, then $N[G] \preceq H(\lambda)^{M[G]}$.*

Proof. We apply Tarski's criterion. Suppose that $H(\lambda)^{M[G]} \models \exists x\varphi(x, y_1, \ldots, y_n)$ with each y_i in $N[G]$. Say $y_i = \tau_G^i$ with $\tau_i \in N$. Thus $H(\lambda)^{M[G]} \models \exists x\varphi(x, \tau_G^1, \ldots, \tau_G^n)$.

Choose $p \in G$ such that $p \Vdash (\exists x\varphi(\sigma, \tau^1, \ldots, \tau^n))^{H(\lambda)}$. Then since $N \preceq H(\lambda)$, it follows that $p \Vdash (\exists x\varphi(\sigma, \tau^1, \ldots, \tau^n))^N$. Hence $N[G] \models \exists x\varphi(x, y_1, \ldots, y_n)$. □

Proposition 31.196. *Let $\mathbb{P}$ be a forcing order, σ a $\mathbb{P}$-name, $p \in \mathbb{P}$, and $p \Vdash [\sigma$ is an ordinal$]$. Then the set $\{q \in \mathbb{P} : \exists\alpha[q \Vdash [\sigma = \check{\alpha}]]\}$ is dense below p.*

Proof. Assume the hypotheses. Suppose that $r \leq p$. Let G be generic with $r \in G$. Then σ_G is an ordinal α. Thus there is an $s \in G$ such that $s \Vdash [\sigma = \check{\alpha}]$. Choose $q \in G$ with $q \leq r, s$. Then q is in the set of the proposition. □

Proposition 31.197. *Suppose that $N \preceq H(\lambda)$, $\mathbb{P}$ is a forcing order in N, and $p \in \mathbb{P}$. Then the following are equivalent:*

(i) p is $(N, \mathbb{P})$-generic.

(ii) If $D \subseteq \mathbb{P}$, D is dense, and $D \in N$, then for every generic G, $p \in G$ implies that $D \cap N \cap G \neq \emptyset$.

Proof. (i)⇒(ii): Assume (i), $D \subseteq \mathbb{P}$, D is dense, $D \in N$, and $p \in G$ generic. By (i), $D \cap N$ is predense below p, so $D \cap N \cap G \neq \emptyset$.

(ii)⇒(i): Assume (ii) and suppose that $D \subseteq \mathbb{P}$ is dense, $D \in N$. Take any $q \leq p$. Let G be generic with $q \in G$. By (ii), choose $r \in D \cap N \cap G$. Thus q is compatible with a member of $D \cap N$, as desired. □

Proposition 31.198. *Suppose that $N \preceq H(\lambda)$, $\mathbb{P}$ is a forcing order in N, and $p \in \mathbb{P}$. Then the following are equivalent:*

(i) p is $(N, \mathbb{P})$-generic.

(ii) For every $\mathbb{P}$-name $\sigma \in N$ and every $q \leq p$, if $q \Vdash \sigma$ is an ordinal, then for every generic G with $q \in G$, $\sigma_G \in N$.

Proof. (i)⇒(ii): Assume (i), $\sigma \in N$ is a $\mathbb{P}$-name, $q \leq p$, $q \Vdash \sigma$ is an ordinal, and $q \in G$ generic. Then by Proposition 31.196, $D \stackrel{\text{def}}{=} \{r \in \mathbb{P} : \exists\alpha[r \Vdash \sigma = \check{\alpha}]\}$ is dense below q. Let f be a one-one function mapping D into **On** such that for each $r \in D$, $r \Vdash \sigma = f(r)^v$. Since $N \preceq H(\lambda)$, $f \in N$. Let $D' = D \cup \{r : r \perp q\}$. Then $D' \in N$ and D' is dense. So by (i) $D' \cap N$ is pre-dense below p, hence also pre-dense below q, so we can choose $r \in D \cap N \cap G$ with $r \leq q$. Now $r \Vdash \sigma = f(r)^v$. Hence $\sigma_G = f(r) \in N$.

(ii)⇒(i): Assume (ii); we verify Proposition 31.197(ii). Suppose that $D \subseteq \mathbb{P}$, D is dense, and $D \in N$. Let $p \in G$ generic. Let f be a bijection from a cardinal κ onto D. Then $f, \kappa \in H(\lambda)$, so we get that f and κ are in N. Choose $a \in D \cap G$, and let $\alpha < \kappa$ be such that $f(\alpha) = a$. Let σ be a $\mathbb{P}$-name such that $\sigma_G = \alpha$. Choose $q \leq p$ with $q \in G$ such that $q \Vdash \sigma$ is an ordinal. Then by (ii), $\sigma_G \in N$. Hence also $a \in N$, so $a \in D \cap N \cap G$. □

Proposition 31.199. *Suppose that $\mathbb{P}$ is a forcing order, $N \preceq H(\lambda)$, and $p \in \mathbb{P}$. Then the following are equivalent:*

(i) p is $(N, \mathbb{P})$-generic.

(ii) If $p \in G$ generic and $\alpha \in N[G] \cap \mathrm{On}$, then $\alpha \in N$.

Proof. (i)$\Rightarrow$(ii): Assume (i), and suppose that $p \in G$ generic and $\alpha \in N[G] \cap \mathrm{On}$. Say $\alpha = \sigma_G$ with $\sigma \in N$. Choose $q \in G$ such that $q \Vdash \sigma$ is an ordinal, and choose $r \in G$ with $r \leq p, q$. Then by Proposition 31.198(ii), $\sigma_G \in N$.

(ii)$\Rightarrow$(i): Assume (ii). We will check Proposition 31.198(ii). Suppose that σ is a $\mathbb{P}$-name, $\sigma \in N$, $q \leq p$, $q \Vdash \sigma$ is an ordinal, and $q \in G$ generic. Then $\sigma_G \in N[G] \cap \mathrm{On}$, so by (ii), $\sigma_G \in N$. $\square$

Now suppose that $\mathbb{P}$ is a forcing order and π is a $\mathbb{P}$-name for a forcing order. We now associate with each $(\mathbb{P} * \pi)$-name τ a $\mathbb{P}$-name τ^*, by recursion:

$$\tau^* = \{(\eta, p) : \exists \mu \exists \theta[\theta \in \mathrm{dmn}(\pi) \wedge \eta = \mathrm{op}(\mu^*, \theta) \wedge (\mu, (p, \theta)) \in \tau\}.$$

Proposition 31.200. *Suppose that $\mathbb{P}$ is a forcing order, π is a $\mathbb{P}$-name for a forcing order, τ is a $(\mathbb{P} * \pi)$-name, and G is $\mathbb{P}$-generic over M. Then τ^*_G is a π_G-name.*

Proof. By induction:

$$\begin{aligned}
\tau^*_G &= \{\eta_G : \exists p \in G[(\eta, p) \in \tau^*]\} \\
&= \{\eta_G : \exists p \in G \exists \mu \exists \theta[\theta \in \mathrm{dmn}(\pi) \wedge \eta = \mathrm{op}(\mu^*, \theta) \wedge (\mu, (p, \theta)) \in \tau]\} \\
&= \{(\mu^*_G, \theta_G) : \theta \in \mathrm{dmn}(\pi) \wedge \exists p \in G[(\mu, (p, \theta)) \in \tau]\}.
\end{aligned}$$

$\square$

Proposition 31.201. *Suppose that $\mathbb{P}$ is a forcing order and π is a $\mathbb{P}$-name for a forcing order. Let $G * H$ be $(\mathbb{P} * \pi)$-generic. Then for any $(\mathbb{P} * \pi)$-name τ we have $\tau_{G*H} = (\tau^*_G)_H$.*

Proof. By induction:

$$\begin{aligned}
\tau_{G*H} &= \{\sigma_{G*H} : \exists (p, \xi) \in G * H[(\sigma, (p, \xi)) \in \tau]\} \\
&= \{(\sigma^*_G)_H : \exists p \in G \exists \xi \in \mathrm{dmn}(\pi)[\xi_G \in H \wedge (\sigma, (p, \xi)) \in \tau]\}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(\tau^*_G)_H &= \{\rho_H : \exists q \in H[(\rho, q) \in \tau^*_G]\} \\
&= \{\mu^*_G : \exists p \in G \exists \theta \in \mathrm{dmn}(\pi)[\theta_G \in H \wedge (\mu, (p, \theta)) \in \tau]\}
\end{aligned}$$

which is the same as above. $\square$

Proposition 31.202. *Let P be a forcing order, π a P-name for a forcing order, $N \preceq H(\lambda)$, p is (N, P)-generic, $(p, \sigma) \in P * \pi$, and for all P-generic G, if $p \in G$ then σ_G is $(N[G], \pi_G)$-generic.*

Then (p, σ) is $(N, P * \pi)$-generic.

Proof. We will apply Proposition 31.199. Suppose that $G*H$ is generic, $(p, \sigma) \in G*H$, and $\alpha \in N[G * H] \cap \mathrm{On}$. Then there is a $\mathbb{P} * \pi$ name $\tau \in N$ such that $\alpha = \tau_{G*H}$. By Proposition 31.201 we have $\alpha = (\tau^*_G)_H$. Clearly $\tau^* \in N$, so $\tau^*_G \in N[G]$. Now $\sigma_G \in H$, $\alpha \in N[G][H]$ and σ_G is $(N[G], \pi_G)$-generic, it follows that $\alpha \in N[G] \cap \mathrm{On}$. Let ξ be a

P-name, $\xi \in N$, such that $\xi_G = \alpha$. Since $p \in G$ and p is (N, P)-generic, it follows that $\alpha \in N$. $\square$

Lemma 31.203. *Suppose that $\mathbb{P}$ is a forcing order, π is a $\mathbb{P}$-name for a forcing order, $i(p) = (p, 1)$ for all $p \in P$; so i is a complete embedding of $\mathbb{P}$ into $\mathbb{P} * \pi$. Suppose that σ is a P-name, and $G * H$ is generic. Then $(i_*(\sigma))_{G*H} = \sigma_G$.*

Proof. By induction on σ:

$$\begin{aligned} x \in (i_*(\sigma))_{G*H} \quad &\text{iff} \quad \exists q \in G * H \exists \nu[(\nu, q) \in i_*(\sigma) \wedge x = \nu_{G*H}] \\ &\text{iff} \quad \exists q \in G * H \exists \nu \exists (\rho, r) \in \sigma[(\nu, q) = (i_*(\rho), i(r)) \wedge x = \nu_{G*H}] \\ &\text{iff} \quad \exists q \in G * H \exists (\rho, r) \in \sigma[x = (i_*(\rho))_{G*H} \wedge i(r) = q] \\ &\text{iff} \quad \exists (\rho, r) \in \sigma[x = \rho_G \wedge r \in G] \\ &\text{iff} \quad x \in \sigma_G. \end{aligned}$$ $\square$

Theorem 31.204. *Suppose that $\mathbb{P}$ is a forcing poset and π is a full P-name such that $\Vdash [\pi$ is proper$]$. Let θ be a regular uncountable cardinal such that $\mathrm{trcl}(\mathbb{P}), \pi \in H(\theta)$, and suppose that $N \preceq H((2^\theta)^+)$, N countable, and $\mathbb{P} * \pi \in N$. Let i be the complete embedding of $\mathbb{P}$ into $\mathbb{P} * \pi$. Suppose that $p \in \mathbb{P}$ is $(N, \mathbb{P})$-generic, σ and η are $\mathbb{P}$-names, $\eta \in N$, and*

$$p \Vdash \sigma \in N^v \wedge \eta \in N^v \wedge \mathrm{op}(\sigma, \eta) \in (\mathbb{P} * \pi)^v \wedge \sigma \in \Gamma.$$

Then there is a $\xi \in \mathrm{dmn}(\pi)$ such that (p, ξ) is $(N, \mathbb{P} * \pi)$-generic and $(p, \xi) \Vdash i_*(\mathrm{op}(\sigma, \xi)) \in \Gamma$.

Proof. Let $p \in G$ generic. Then $\sigma_G \in N$, $\eta_G \in N$, $(\sigma_G, \eta_G) \in P * \pi$, and $\sigma_G \in G$. By Lemma 31.195, $N[G] \preceq (H((2^\theta)^+))^{M[G]}$. Now $\eta \in N$, so $\eta_G \in N[G]$. Also, $\pi \in H(\theta)$, so $\pi_G \in (H(\theta))^{M[G]}$. Note that π_G is a proper forcing order in $M[G]$. Now by Theorem 31.194 there is a $q \leq \eta_G$ such that q is $(N[G], \pi_G)$-generic. Thus

$$p \Vdash \exists \chi[\chi \in \pi \wedge \chi \leq \eta \wedge \chi \text{ is } (N[\Gamma], \pi) - \text{generic}.$$

By the maximal principle let τ be a $\mathbb{P}$-name such that

$$p \Vdash \tau \in \pi \wedge \tau \leq \eta \wedge \tau \text{ is } (N[\Gamma], \pi) - \text{generic}.$$

By the definition of full names, let $\xi \in \mathrm{dmn}(\pi)$ be such that $p \Vdash \xi = \tau$ and $(\xi, p) \in \pi$. Then $(p, \xi) \in \mathbb{P} * \pi$ and $p \Vdash \xi$ is $(N[\Gamma], \pi) -$ generic. Hence by Proposition 31.202, (p, ξ) is $(N, \mathbb{P} * \pi)$-generic. It remains to show that $(p, \xi) \Vdash i_*(\mathrm{op}(\sigma, \xi)) \in \Gamma$. Let $(p, \xi) \in G * H$, generic. Thus $p \in G$ and $\xi_G \in H$. Also, $\sigma_G \in G$ by assumption. Hence $(i_*(\mathrm{op}(\sigma, \xi)))_{G*H} = (\mathrm{op}(\sigma, \xi))_G = (\sigma_G, \xi_G) \in G * H$. $\square$

Lemma 31.205. *Let $\alpha > 0$, and let $(\langle(\mathbb{P}_\xi : \xi \leq \alpha\rangle, \langle \pi_\xi : \xi < \alpha\rangle)$ be a countable support iteration, with each $\mathbb{P}_\xi$ for $\xi < \alpha$ proper, and each name π_ξ full. For $\xi \leq \eta \leq \alpha$, let $i_{\xi\eta}$ be the complete embedding of $\mathbb{P}_\xi$ into $\mathbb{P}_\eta$. For each $\xi \leq \alpha$ let Γ_ξ be the standard name for* *a generic filter over* $\mathbb{P}_\xi$. *Let* λ *be sufficiently large, and let* $N \preceq H(\lambda)$ *be countable with* $\alpha, \mathbb{P}_\alpha, \langle \mathbb{P}_\xi : \xi \le \alpha\rangle, \langle \pi_\xi : \xi < \alpha\rangle \in N$. *Let* $\gamma_0 \in \alpha \cap N$, *and assume that* $p_{\gamma_0} \in \mathbb{P}_{\gamma_0}$ *is* $(N, \mathbb{P}_{\gamma_0})$*-generic and* σ *and* τ *are* $\mathbb{P}_{\gamma_0}$*-names such that*

$$p_{\gamma_0} \Vdash_{\mathbb{P}_{\gamma_0}} \sigma \in N^{\mathrm{v}} \wedge \sigma \in \mathbb{P}_\alpha^{\mathrm{v}} \wedge \tau \in \mathbb{P}_{\gamma_0}^v \wedge \tau \subseteq \sigma \wedge \tau \in \Gamma_{\gamma_0}.$$

Then there is a $(N, \mathbb{P}_\alpha)$*-generic condition* q *such that* $q \restriction \gamma_0 = p_{\gamma_0}$ *and* $q \Vdash_{\mathbb{P}_\alpha} (i_{\gamma_0\alpha})_*(\sigma) \in \Gamma_\alpha$.

Proof. Induction on α; so assume that the lemma holds for any positive ordinal less than α. First suppose that α is a successor ordinal $\beta + 1$, and suppose that $\gamma_0 = \beta$. Let G be $\mathbb{P}_{\gamma_0}$-generic with $p_{\gamma_0} \in G$. Then $\sigma_G \in N$, $\sigma_G \in \mathbb{P}_\alpha$, $\tau_G \in \mathbb{P}_{\gamma_0}$, $\tau_G \subseteq \sigma_G$, and $\tau_G \in G$. Hence there is a ξ such that $\sigma_G = \tau_G{}^\frown\langle \xi_G\rangle$. Then

$$p_{\gamma_0} \Vdash \tau \in N^v \wedge \xi \in N^v \wedge \mathrm{op}(\tau, \xi) \in (\mathbb{P} * \pi_{\gamma_0})^v \wedge \tau \in \Gamma.$$

Then by Theorem 31.204 there is a $\xi \in \mathrm{dmn}(\pi_{\gamma_0})$ such that (p_{γ_0}, ξ) is $(N, \mathbb{P}_{\gamma_0} * \pi_{\gamma_0})$-generic and $(p_{\gamma_0}, \xi) \Vdash i_*(\mathrm{op}(\tau, \xi)) \in \Gamma$. Now for each $q \in \mathbb{P}_\alpha$ let $f(q) = (q \restriction \gamma_0, q(\gamma_0))$. Then f is an isomorphism from $\mathbb{P}_\alpha$ onto $\mathbb{P}_{\gamma_0} \times \pi_{\gamma_0}$. It follows that $p_{\gamma_0}{}^\frown\langle\xi\rangle$ is $(N, \mathbb{P}_\alpha)$-generic. We claim that $p_{\gamma_0}{}^\frown\langle\xi\rangle \Vdash ((i_{\gamma_0\alpha})_*)(\sigma) \in \Gamma$. For, suppose that $p_{\gamma_0}{}^\frown\langle\xi\rangle \in G$ generic. Then $(p_{\gamma_0}, \xi) \in f[G]$, so $(i_*(\mathrm{op}(\tau, \xi)))_{f[G]} \in f[G]$. Now $f[G] = (G \restriction \gamma_0) * H$ for some H, so by Lemma 31.203, $(i_*(\mathrm{op}(\tau, \xi)))_{f[G]} = (\tau_{G\restriction\gamma_0}, \xi_{G\restriction\gamma_0}) = f(\sigma_{G\restriction\gamma_0})$. Since $(i_*(\mathrm{op}(\tau, \xi)))_{f[G]} \in f[G]$, it follows that $\sigma_{G\restriction\gamma_0} \in G$. Hence it remains only to show that $(((i_{\gamma_0\alpha})_*)(\sigma))_G = \sigma_{G\restriction\gamma_0}$:

$$\begin{aligned}
x \in (((i_{\gamma_0\alpha})_*)(\sigma))_G \quad &\text{iff} \quad \exists q \in G \exists \nu[(\nu, q) \in (i_{\gamma_0\alpha})_*(\sigma) \wedge x = \nu_G]\\
&\text{iff} \quad \exists q \in G \exists \nu \exists (\rho, r) \in \sigma[(\nu, q) = ((i_{\gamma_0\alpha})_*(\rho), i_{\gamma_0\alpha}(r)) \wedge x = \nu_G]\\
&\text{iff} \quad \exists q \in G \exists (\rho, r) \in \sigma[q = i_{\gamma_0\alpha}(r) \wedge x = ((i_{\gamma_0\alpha})_*(\rho))_G]\\
&\text{iff} \quad \exists q \in G \exists (\rho, r) \in \sigma[i_{\gamma_0,\alpha}(r) \in G \wedge x = \rho_{G\restriction\gamma_0}]\\
&\text{iff} \quad \exists (\rho, r) \in \sigma[r \in G \restriction \gamma_0 \wedge x = \rho_{G\restriction\gamma_0}]\\
&\text{iff} \quad x \in \sigma_{G\restriction\gamma_0}.
\end{aligned}$$

Now suppose that $\alpha = \beta + 1$ and $\gamma_0 < \beta$. Now For any $\mathbb{P}_\alpha$-generic G let $\sigma_G = \rho_G{}^\frown\langle\xi_G\rangle$. Now

$$p_{\gamma_0} \Vdash \rho \in N^v \wedge \rho \in \mathbb{P}_\beta \wedge \tau \in \mathbb{P}_{\gamma_0}^v \wedge \tau \subseteq \rho \wedge \tau \in \Gamma_{\gamma_0},$$

so by the inductive hypothesis we get a $(N, \mathbb{P}_\beta)$-generic q such that $q \restriction \gamma_0 = p_{\gamma_0}$ and $q \Vdash (i_{\gamma_0\beta})_*(\rho) \in \Gamma_\beta$. Thus

$$\begin{aligned}
q \Vdash &(i_{\gamma_0\beta})_*(\sigma) \in N^v \wedge (i_{\gamma_0\beta})_*(\sigma) \in \mathbb{P}_\alpha^v \wedge (i_{\gamma_0\beta})_*(\rho) \in \mathbb{P}_\beta^v \wedge\\
&(i_{\gamma_0\beta})_*(\rho) \subseteq (i_{\gamma_0\beta})_*(\sigma) \wedge (i_{\gamma_0\beta})_*(\sigma) \in \Gamma_\beta.
\end{aligned}$$

Then by the first special case of this proof we get a $(N, \mathbb{P}_\alpha)$-generic r such that $r \restriction \beta = q$ and $r \Vdash (i_{\beta\alpha})_*((i_{\gamma_0\beta})_*(\sigma)) \in \Gamma_\alpha$. Now Proposition 31.202 finishes this part of the induction.

Now suppose that α is a limit ordinal. Now since $N \preceq H(\lambda)$, there is no largest ordinal in $\alpha \cap N$. Moreover, N is countable. Hence there is an increasing sequence $\langle \gamma_i : i \in \omega \rangle$ of ordinals in $\alpha \cap N$, cofinal in $\alpha \cap N$, starting with our given γ_0. Thus $\sup_{i\in\omega} \gamma_i = \alpha$. Let $\langle D_i : i \in \omega \rangle$ list all of the dense subsets of $\mathbb{P}_\alpha$ which are in N. Now we are going to define sequences $\langle q_i : i \in \omega \rangle$, $\langle \tau_i : i \in \omega \rangle$, $\langle \mu_i : i \in \omega \rangle$ so that the following conditions hold:

(1) $q_n \in \mathbb{P}_{\gamma_n}$ for each $n \in \omega$.

(2) $q_0 = p_{\gamma_0}$, q_n is $(N, \mathbb{P}_{\gamma_n})$-generic, and $q_{n+1} \restriction \gamma_n = q_n$.

(3) $\tau_0 = \sigma$, and for $n > 0$, τ_n is a $\mathbb{P}_{\gamma_n}$-name such that q_n forces (in $\mathbb{P}_{\gamma_n}$) each of the following:

 (a) $\tau_n \in N^{\mathrm{v}} \wedge \tau_n \in \mathbb{P}^{\mathrm{v}}_\alpha$.
 (b) μ_n is a $\mathbb{P}_{\gamma_n}$-name, $\mu_n \subseteq \tau_n$, and $\mu_n \in \Gamma_{\gamma_n}$.
 (c) $\tau_n \leq_\alpha (i_{\gamma_{n-1}\gamma_n})_*(\tau_{n-1})$.
 (d) $\tau_n \in D^{\mathrm{v}}_{n-1}$.

We define $q_0 = p_{\gamma_0}$ and $\tau_0 = \sigma$. Now suppose that q_n and τ_n have been defined so that (1)–(3) hold. We claim

$$(4) \qquad q_n \Vdash_{\mathbb{P}_{\gamma_n}} \exists\chi[\chi \in \mathbb{P}^{\mathrm{v}}_\alpha \wedge \chi \in N^{\mathrm{v}} \wedge \exists\mu[\mu \in \Gamma_{\gamma_n} \wedge \mu \subseteq \chi] \wedge \chi \in D^{\mathrm{v}}_n \wedge \chi \leq \tau_n]$$

To prove (4), let G be generic over $\mathbb{P}_{\gamma_n}$ with $q_n \in G$. Since (a)–(d) hold for n, we have $\tau_{nG} \in N$, $\tau_{nG} \in \mathbb{P}_\alpha$, $\mu_{nG} \subseteq \tau_{nG}$, $\mu_{nG} \in G$, $\tau_{nG} \leq_\alpha ((i_{\gamma_{n-1}\gamma_n})_*(\tau_{n-1}))_G$, and $\tau_{nG} \in D_{n-1}$. Now let

$$D'_n = \{p \restriction \gamma_n : p \in D_n \wedge [p \leq_\alpha \tau_{nG} \vee (p \restriction \gamma_n) \perp \mu_{nG})]\}.$$

Then $D'_n \in N$ since $N \preceq H(\lambda)$. We claim that D'_n is dense in $\mathbb{P}_{\gamma_n}$. To see this, let $r \in \mathbb{P}_{\gamma_n}$.

Case 1. $r \perp \mu_{nG}$. By the density of D_n, choose $s \in D_n$ such that $s \leq_\alpha (i_{\gamma_n\alpha})_*(r)$. Then $s \restriction \gamma_n \leq r$, and so $(s \restriction \gamma_n) \perp \mu_{nG}$. Hence $s \restriction \gamma_n \in D'_n$, as desired.

Case 2. r and μ_{nG} are compatible. Hence $(i_{\gamma_n\alpha})_*(r)$ and τ_{nG} are compatible; say $s \leq (i_{\gamma_n\alpha})_*(r), \tau_{nG}$. By the density of D_n, let t be such that $t \in D_n$ and $t \leq s$. Thus $t \leq \tau_{nG}$, so $t \restriction \gamma_n \in D'_n$, and $t \restriction \gamma_n \leq r$, as desired.

So D'_n is dense and $D'_n \in N$. Since q_n is $(N, \mathbb{P}_{\gamma_n})$-generic, it follows by definition that $D'_n \cap N$ is pre-dense below q_n. So we can choose $x \in G \cap D'_n \cap N$. Say $x = p \restriction \gamma_n$ with $p \in D_n$. Thus $H(\lambda) \models \exists p \in D_n[x = p \restriction \gamma_n]$, so since $N \preceq H(\lambda)$, we may assume that $p \in N$. Now $x, \mu_{nG} \in G$, so they are compatible. Hence $p \leq_\alpha \tau_{nG}$. Thus with $\chi_G = p$ we have verified the conclusion of (4). So (4) holds.

By the maximal principle we get a $\mathbb{P}_{\gamma_n}$-name ρ such that

(5) $q_n \Vdash_{\mathbb{P}_{\gamma_n}} \rho \in \mathbb{P}^{\mathrm{v}}_\alpha \wedge \rho \in N^{\mathrm{v}} \wedge \exists\mu[\mu \in \Gamma_{\gamma_n} \wedge \mu \subseteq \rho] \wedge \rho \in D^{\mathrm{v}}_n \wedge \rho \leq \tau_n$.

Now $q_n \Vdash_{\mathbb{P}_{\gamma_n}} \exists\xi[\xi \in \mathbb{P}^{\mathrm{v}}_{\gamma_{n+1}} \wedge \xi \in N^{\mathrm{v}} \wedge \xi \subseteq \rho]$. Hence by the maximal principle again, there is a $\mathbb{P}_{\gamma_n}$-name ξ such that $q_n \Vdash \xi \in \mathbb{P}^{\mathrm{v}}_{\gamma_{n+1}} \wedge \xi \in N^{\mathrm{v}} \wedge \xi \subseteq \rho$. Thus

$$q_n \Vdash_{\mathbb{P}_{\gamma_n}} \xi \in N^{\mathrm{v}} \wedge \xi \in \mathbb{P}^{\mathrm{v}}_{\gamma_{n+1}} \wedge \exists\mu[\mu \in \Gamma_{\gamma_n} \wedge \mu \subseteq \xi].$$

We now apply the inductive hypothesis to $\gamma_n, \gamma_{n+1}, q_n, \xi$ in place of $\gamma_0, \alpha, p_{\gamma_0}, \sigma$ to obtain a $(N, \mathbb{P}_{\gamma_{n+1}})$-generic condition q_{n+1} such that $q_{n+1} \restriction \gamma_n = q_n$ and $q_{n+1} \Vdash_{\mathbb{P}_{\gamma_{n+1}}} (i_{\gamma_n\gamma_{n+1}})_*(\xi) \in \Gamma_{\gamma_{n+1}}$. Let $\tau_{n+1} = (i_{\gamma_n\gamma_{n+1}})_*(\rho)$. Then we claim

$$\text{(6)} \qquad \begin{aligned} q_{n+1} \Vdash & \tau_{n+1} \in N^{\text{v}} \wedge \tau_{n+1} \in \mathbb{P}_\alpha^{\text{v}} \wedge \exists\mu[\mu \subseteq \tau_{n+1} \wedge \mu \in \Gamma_{\gamma_{n+1}}] \\ & \wedge \tau_{n+1} \leq i_{\gamma_n\gamma_{n+1}}(\tau_n) \wedge \tau_{n+1} \in D_n^{\text{v}} \end{aligned}$$

To prove (6) we apply Theorem 26.4. Let G be generic on $\mathbb{P}_{\gamma_{n+1}}$ with $q_{n+1} \in G$. Let $H = (i_{\gamma_n\gamma_{n+1}})^{-1}[G]$. Then H is $\mathbb{P}_{\gamma_n}$-generic by Theorem 26.3. Since $q_{n+1} \restriction \gamma_n = q_n$, we have $q_{n+1} \leq i_{\gamma_n\gamma_{n+1}}(q_n)$, hence $i_{\gamma_n\gamma_{n+1}}(q_n) \in G$ and $q_n \in H$. Hence by (5), $\rho_H \in N$, $\rho_H \in \mathbb{P}_\alpha$, $\rho_H \restriction \gamma_n \in H$, $\rho_H \leq_\alpha \tau_{nH}$, and $\rho_H \in D_n$. Now by Theorem 26.4 we have $\tau_{n+1,G} = ((i_{\gamma_n\gamma_{n+1}})_*(\rho))_G = \rho_H$. It follows that $\tau_{n+1,G} \in N$, $\tau_{n+1,G} \in \mathbb{P}_\alpha$, $\tau_{n+1,G} \in D_n$, and $\tau_{n+1,G} \leq_\alpha (i_{\gamma_n\gamma_{n+1}})_*(\tau_{nG})$. Finally,

$$\begin{aligned} \tau_{n+1,G} \restriction \gamma_{n+1} &= ((i_{\gamma_n\gamma_{n+1}})_*(\rho))_G \restriction \gamma_{n+1} \\ &= \rho_H \restriction \gamma_{n+1} \\ &= \xi_H \\ &= ((i_{\gamma_n\gamma_{n+1}})_*(\xi))_G \\ &\in G \quad \text{since } q_{n+1} \Vdash_{\mathbb{P}_{\gamma_{n+1}}} (i_{\gamma_n\gamma_{n+1}})_*(\xi) \in \Gamma_{\gamma_{n+1}}. \end{aligned}$$

This finishes the construction.

Let $r = \bigcup_{n\in\omega} q_n$. We claim that r is as desired in the Lemma. Clearly r has countable support. By (3), using Theorem 26.4, we have

(7) $i_{\gamma_n\alpha}(q_n) \Vdash (i_{\gamma_n\alpha})_*(\tau_n) \in N^{\text{v}} \wedge (i_{\gamma_n\alpha})_*(\tau_n) \in \mathbb{P}_\alpha^{\text{v}}$.

(8) $i_{\gamma_n\alpha}(q_n) \Vdash (i_{\gamma_n\alpha})_*(\tau_n) \restriction \gamma_n \in (i_{\gamma_n\alpha})_*(\Gamma_{\gamma_n})$.

(9) $i_{\gamma_{n+1}\alpha}(q_{n+1}) \Vdash (i_{\gamma_{n+1}\alpha})_*(\tau_{n+1}) \leq_\alpha (i_{\gamma_n\alpha})_*(\tau_n))$.

(10) $i_{\gamma_{n+1}\alpha}(q_{n+1}) \Vdash (i_{\gamma_{n+1}\alpha})_*(\tau_{n+1}) \in D_n^{\text{v}}$.

Since $r \leq i_{\gamma_n\alpha}(q_n)$ for all n, this gives

(11) $r \Vdash (i_{\gamma_n\alpha})_*(\tau_n) \in N^{\text{v}} \wedge (i_{\gamma_n\alpha})_*(\tau_n) \in \mathbb{P}_\alpha^{\text{v}}$.

(12) $r \Vdash (i_{\gamma_n\alpha})_*(\tau_n) \restriction \gamma_n \in (i_{\gamma_n\alpha})_*(\Gamma_{\gamma_n})$.

(13) $r \Vdash (i_{\gamma_{n+1}\alpha})_*(\tau_{n+1}) \leq_\alpha (i_{\gamma_n\alpha})_*(\tau_n))$.

(14) $r \Vdash (i_{\gamma_{n+1}\alpha})_*(\tau_{n+1}) \in D_n^{\text{v}}$.

Now to show that r is $(N, \mathbb{P}_\alpha)$-generic, we will apply Lemma 31.197. So suppose that G is generic on $\mathbb{P}_\alpha$, $r \in G$, D is dense in $\mathbb{P}_\alpha$, and $D \in N$; we want to show that $D \cap N \cap G \neq \emptyset$. Choose n so that $D = D_n$. Let $s = ((i_{\gamma_{n+1}\alpha})_*(\tau_{n+1}))_G$. Thus by (11) and (14), $s \in D \cap N$. By Theorem 26.4 and (12), $s = \tau_{n+1,G_{\eta_{n+1}}} \in G_{\eta_{n+1}} \subseteq G$. This finishes the proof that r is $(N, \mathbb{P}_\alpha)$-generic.

Clearly $r \restriction \gamma_0 = p_{\gamma_0}$, and $r \Vdash (i_{\gamma_0\alpha})_*(\sigma) \in \Gamma_\alpha$ since $\sigma = \tau_0$. $\square$

Theorem 31.206. *Let $\alpha > 0$, and let $(\langle(\mathbb{P}_\xi : \xi \leq \alpha\rangle, \langle\pi_\xi : \xi < \alpha\rangle)$ be a countable support iteration, with each $\mathbb{P}_\xi$ for $\xi < \alpha$ proper, and each name π_ξ full. Then $\mathbb{P}_\alpha$ is proper.*

Proof. Let N and λ be as in the statement of Lemma 31.205. We are going to apply Lemma 31.194 So, let $p \in \mathbb{P}_\alpha$ with $p \in N$. Let $\gamma_0 = 0$. Recall that $\mathbb{P}_0 = \{0\}$. Trivially, 0 is $(N, \mathbb{P}_0)$-generic. The hypothesis of Lemma 31.205 holds, with p^{v} in place of σ. Hence by Lemma 31.205 we get a $q \in \mathbb{P}_\alpha$ such that q is $(N, \mathbb{P}_\alpha)$-generic and $q \Vdash_\alpha (i_{0\alpha})_*(p^{\mathrm{v}}) \in \Gamma_\alpha$. Let G be $\mathbb{P}_\alpha$-generic with $q \in G$. Then $((i_{0\alpha})_*(p^{\mathrm{v}}))_G \in G$, i.e., $p \in G$. Choose $r \in G$ such that $r \leq p, q$. Then r is clearly $(N, \mathbb{P}_\alpha)$-generic. $\square$

For the proof we need many-sorted forms of Łoś's theorem on ultraproducts. We give a proof of this theorme in the case of certain two-sorted structures; the general case of finitely many sorts is treated similarly. The language is as follows. There are two sorts of variables: $v_0, v_1, \ldots$ and $w_0, w_1, \ldots$. There is a four-place relation symbol Q. Atomic formulas have the form $v_i = v_j$, $w_i = w_j$, or $Qv_iw_jw_kw_l$. We have connectives $\neg$, $\rightarrow$, $\forall v_i$, and $\forall w_i$. A structure for this language is a triple (A, B, C) such that A and B are nonempty sets and $C \subseteq A \times B \times B \times B$. Given $a \in {}^\omega A$ and $b \in {}^\omega B$ and any formula φ, we define $(A, B, C) \models \varphi[a, b]$ as follows:

$$\begin{aligned}
&(A,B,C) \models (v_i = v_j)[a,b] \quad \text{iff} \quad a_i = a_j;\\
&(A,B,C) \models (w_i = w_j)[a,b] \quad \text{iff} \quad b_i = b_j;\\
&(A,B,C) \models (\neg\varphi)[a,b] \quad \text{iff} \quad \text{not}((A,B,C) \models \varphi[a,b])\\
&(A,B,C) \models (\varphi \rightarrow \psi)[a,b] \quad \text{iff} \quad \text{not}((A,B,C) \models \varphi[a,b]) \text{ or } ((A,B,C) \models \varphi)[a,b])\\
&(A,B,C) \models \forall v_i\varphi[a,b] \quad \text{iff} \quad \text{for all } u \in A((A,B,C) \models \varphi[a^i_u, b])\\
&(A,B,C) \models \forall w_i\varphi[a,b] \quad \text{iff} \quad \text{for all } u \in B((A,B,C) \models \varphi[a, b^i_u])
\end{aligned}$$

Suppose that $\langle (A_i, B_i, C_i) : i \in I\rangle$ is a system of structures, and F is an ultrafilter on I. For each $i \in I$ let $M_i = (A_i, B_i, C_i)$. Further, let $A' = \prod_{i\in I} A_i$, $B' = \prod_{i\in I} B_i$, and $C' = \{(a, b, c, d) : a \in A',\ b, c, d \in B'\}$. We define

$$\begin{aligned}
a \equiv_0 b \quad &\text{iff} \quad a, b \in A' \text{ and } \{i \in I : a_i = b_i\} \in F;\\
c \equiv_1 d \quad &\text{iff} \quad c, d \in B' \text{ and } \{i \in I : c_i = d_i\} \in F.
\end{aligned}$$

It is easy to check that $\equiv_0$ is an equivalence relation on A' and $\equiv_1$ is an equivalence relation on B'. We let A'' be the set of all $\equiv_0$-classes, and B'' the set of all $\equiv_1$-classes. We also define

$$C'' = \{([a], [b], [c], [d]) : \{i \in I : (a_i, b_i, c_i, d_i) \in C_i\} \in F\}.$$

Proposition 32.1. *For any $(a, b, c, d) \in A' \times B' \times B' \times B'$ the following are equivalent:*
(i) $([a], [b], [c], [d]) \in C''$.
(ii) $\{i \in I : (a_i, b_i, c_i, d_i) \in C_i\} \in F$.

Proof. (ii)⇒(i) holds by definition. Now assume (i). Then there are $a' \in A'$ and $b', c', d' \in B'$ such that $[a] = [a']$, $[b] = [b']$, $[c] = [c']$, $[d] = [d']$ and $\{i \in I : (a'_i, b'_i, c'_i, d'_i) \in C_i\} \in F$. Then $\{i \in I : a_i = a'_i\} \in F$, $\{i \in I : b_i = b'_i\} \in F$, $\{i \in I : c_i = c'_i\} \in F$, and $\{i \in I : d_i = d'_i\} \in F$. Now

$$\begin{aligned}
&\{i \in I : a_i = a'_i\} \cap \{i \in I : b_i = b'_i\} \cap \{i \in I : c_i = c'_i\} \cap \{i \in I : d_i = d'_i\} \cap\\
&\{i \in I : (a'_i, b'_i, c'_i, d'_i) \in C_i\} \subseteq \{i \in I : (a_i, b_i, c_i, d_i) \in C_i\};
\end{aligned}$$

it follows that $\{i \in I : (a_i, b_i, c_i, d_i) \in C_i\} \in F$. □

The *ultraproduct* of $\langle M_i : i \in I\rangle$ is the structure (A'', B'', C''); it is denoted by $\prod_{i\in I} M_i/F$.

Theorem 32.2. (Łoś) *Let $\langle M_i : i \in I\rangle$ be a system of structures as above, and let F be an ultrafilter on I. Suppose that $a \in {}^\omega A'$ and $b \in {}^\omega B'$. Let $\pi : A' \to A''$ be the natural map; we use π also for the natural map from B' to B''. Then the following conditions are equivalent:*

(i) $\prod_{i\in I} M_i/F \models \varphi[\pi\circ a, \pi\circ b]$.

(ii) $\{i \in I : M_i \models \varphi[\mathrm{pr}_i \circ a, \mathrm{pr}_i \circ b]\} \in F$.

Proof. For brevity let $N = \prod_{i\in I} M_i/F$. We prove the theorem by induction on φ:

$$\begin{aligned}
N \models (v_k = v_j)[\pi\circ a, \pi\circ b] \quad &\text{iff} \quad (\pi\circ a)(k) = (\pi\circ a)(j)\\
&\text{iff} \quad [a_k] = [a_j]\\
&\text{iff} \quad \{i \in I : a_k(i) = a_j(i)\} \in F\\
&\text{iff} \quad \{i \in I : (\mathrm{pr}_i\circ a)(k) = (\mathrm{pr}_i\circ a)(j)\} \in F\\
&\text{iff} \quad \{i \in I : M_i \models (v_k = v_j)[\mathrm{pr}_i\circ a, \mathrm{pr}_i\circ b]\} \in F;
\end{aligned}$$

similarly for $w_k = w_j$

$$\begin{aligned}
N \models \neg\varphi[\pi\circ a, \pi\circ b] \quad &\text{iff} \quad \text{not}(N \models \varphi[\pi\circ a, \pi\circ b])\\
&\text{iff} \quad \text{not}(\{i \in I : M_i \models \varphi[\mathrm{pr}_i\circ a, \mathrm{pr}_i\circ b]\} \in F)\\
&\text{iff} \quad (I\backslash\{i \in I : M_i \models \varphi[\mathrm{pr}_i\circ a, \mathrm{pr}_i\circ b]\}) \in F\\
&\text{iff} \quad \{i \in I : M_i \models \neg\varphi[\mathrm{pr}_i\circ a, \mathrm{pr}_i\circ b]\} \in F\\
N \models (\varphi\to\psi)[\pi\circ a, \pi\circ b] \quad &\text{iff}\\
&\text{not}(N \models \varphi[\pi\circ a, \pi\circ b]) \text{ or } N \models \psi[\pi\circ a, \pi\circ b]\\
&\text{iff}\\
&\{i \in I : M_i \models \neg\varphi[\mathrm{pr}_i\circ a, \mathrm{pr}_i\circ b]\} \in F\\
&\text{or } \{i \in I : M_i \models \psi[\mathrm{pr}_i\circ a, \mathrm{pr}_i\circ b]\} \in F\\
&\text{iff}\\
&\{i \in I : M_i \models \neg\varphi[\mathrm{pr}_i\circ a, \mathrm{pr}_i\circ b]\} \cup \{i \in I : M_i \models \psi[\mathrm{pr}_i\circ a, \mathrm{pr}_i\circ b]\} \in F\\
&\text{iff} \quad \{i \in I : M_i \models (\neg\varphi\vee\psi)[\mathrm{pr}_i\circ a, \mathrm{pr}_i\circ b]\} \in F\\
&\text{iff} \quad \{i \in I : M_i \models (\varphi\to\psi)[\mathrm{pr}_i\circ a, \mathrm{pr}_i\circ b]\} \in F.
\end{aligned}$$

Now suppose that not$(N \models (\forall v_k\varphi)[\pi\circ a, \pi\circ b])$. Then there is a $u \in M'$ such that not$(N \models \varphi[\pi\circ a^k_u, \pi\circ b])$, so by the inductive hypothesis we get $\{i \in I : M_i \models \varphi[\mathrm{pr}_i\circ a^k_u, \mathrm{pr}_i\circ b]\} \notin F$. Since

$$\{i \in I : M_i \models \forall v_k\varphi[\mathrm{pr}_i\circ a, \mathrm{pr}_i\circ b]\} \subseteq \{i \in I : M_i \models \varphi[\mathrm{pr}_i\circ a^k_u, \mathrm{pr}_i\circ b]\},$$

it follows that $\{i \in I : M_i \models \forall v_k\varphi[\mathrm{pr}_i\circ a, \mathrm{pr}_i\circ b]\} \notin F$.

On the other hand, suppose that $\{i \in I : M_i \models \forall v_k\varphi[\mathrm{pr}_i\circ a, \mathrm{pr}_i\circ b]\} \notin F$. Then $P \stackrel{\text{def}}{=} \{i \in I : M_i \models \exists v_k\neg\varphi[\mathrm{pr}_i\circ a, \mathrm{pr}_i\circ b]\} \in F$. For each $i \in P$ choose $u_i \in A_i$ such that $M_i \models \neg\varphi[(\mathrm{pr}_i\circ a)^k_{u_i}, \mathrm{pr}_i\circ b]$. For $i \in I\backslash P$ let $u_i \in A_i$ be arbitrary. Then for each $i \in P$ we have $M_i \models \neg\varphi[\mathrm{pr}_i\circ a^k_u, \mathrm{pr}_i\circ b]$, so $\{i \in I : M_i \models \neg\varphi[\mathrm{pr}_i\circ a^k_u, \mathrm{pr}_i\circ b]\} \in F$, hence $\{i \in I :$

$M_i \models \varphi[\mathrm{pr}_i \circ a^k_u, \mathrm{pr}_i \circ b]\} \notin F$, hence by the inductive hypothesis not$(N \models \varphi[\pi \circ a^k_u, \pi \circ b])$, so not$(N \models (\forall v_k \varphi)[\pi \circ a, \pi \circ b])$.

The case $\forall w_k \varphi$ is similar. □

Let μ be an infinite cardinal. We define

$$L(\mu, F) = \left\{ \prod_{\alpha<\mu} L_\alpha/F : \text{each } L_\alpha \text{ is a finite linear order, and} \right.$$

$$\left. F \text{ is an ultrafilter on } \mu \text{ and } \forall n \in \omega[\{\alpha < \mu : |L_\alpha| > n\} \in F] \right\}$$

$$P(\mu, F) = \left\{ \prod_{\alpha<\mu} P_\alpha/F : \text{each } P_\alpha \text{ is a finite tree with a unique root, and} \right.$$

$$\left. F \text{ is an ultrafilter on } \mu \text{ and } \forall n \in \omega[\{\alpha < \mu : |P_\alpha| > n\} \in F] \right\}$$

A *pseudo-tree* is a partial order P such that for each $x \in P$, $\{y \in P : y \leq x\}$ is linearly ordered.

Proposition 32.3. *(i)* $L(\mu, F) \subseteq P(\mu, F)$.

(ii) If $A \in P(\mu, F)$, then A has a maximal element, a unique minimum element, every non maximal element has at least one immediate successor, and every non-minimum element has a unique immediate predecessor. A is a pseudo-tree, and any two elements of A have a glb.

(iii) If $A \in L(\mu, F)$, then A has a maximum element, a minimum element, and every non maximum element has an immediate successor, and every non minimum element has an immediate predecessor. A is a linear order.

Proof. (i) is clear. For (ii), suppose that $A \in P(\mu, F)$ as in the definition.
maximal element: $P_\alpha \models \exists x \forall y[x \leq y \to x = y]$
unique minimal element: $P_\alpha \models \exists x \forall y[x \leq y]$.
every non maximal element has at least one immediate successor:

$$P_\alpha \models \forall x[\exists y[x < y] \to \exists y[x < y \wedge \forall z[z < y \to z \leq x]]].$$

every non-minimum element has a unique immediate predecessor:

$$P_\alpha \models \forall x[\exists y[y < x] \to \exists! y[y < x \wedge \forall z[z < x \to z \leq y]]].$$

A is a pseudo-tree: Suppose that $[x], [y], [z] \in A$ and $[x], [y] \leq [z]$. Thus $M \stackrel{\text{def}}{=} \{\alpha \in \mu : x_\alpha \leq z_\alpha\} \in F$ and $N \stackrel{\text{def}}{=} \{\alpha \in \mu : y_\alpha \leq z_\alpha\} \in F$. If $\alpha \in M \cap N$ then $x_\alpha, y_\alpha \leq z_\alpha$, so $x_\alpha \leq y_\alpha$ or $y_\alpha \leq x_\alpha$. Thus

$$M \cap N \subseteq \{\alpha \in \mu : x_\alpha \leq y_\alpha\} \cup \{\alpha \in \mu : y_\alpha \leq x_\alpha\}.$$

It follows easily that $[x] \leq [y]$ or $[y] \leq [x]$.

any two elements $[x], [y]$ have a glb:

$$P_\alpha \models \forall x, y \exists z[z \leq x, y \wedge \forall w[w \leq x, y \to w \leq z]].$$

(iii) is clear. $\square$

If $A \in L(\mu, F)$, then 1 is the greatest element of A, and 0 the least element. If $x \in A$ and it has at least n successors, we denote its n-th successor by $x + n$. x is *near* 1 iff there is an $n \in \omega$ such that $x + n = 1$.

Proposition 32.4. *In any $A \in L(\mu, F)$, 0 is not near 1.*

Proof. Suppose that 0 is near 1; say $0+n = 1$. Then A has only $n+1$ elements. $\square$

For infinite regular cardinals κ, λ, and a linear order $(X, <)$, a (κ, λ)-*gap in* $(X, <)$ is a pair (a, b) with $a \in {}^\kappa X$ and $b \in {}^\lambda X$ such that:

(1) $\forall \alpha, \beta < \kappa \forall \gamma, \delta < \lambda[\alpha < \beta$ and $\gamma < \delta$ imply that $a_\alpha < \alpha_\beta < b_\delta < b_\gamma]$.

(2) There is no $x \in X$ such that $\forall \alpha < \kappa \forall \beta < \lambda[a_\alpha < x < b_\beta]$.

We define

$$\begin{aligned}
C(\mu, F) &= \{(\kappa, \lambda) : \text{there is a } (\kappa, \lambda)\text{-gap in some } (X, <) \in L(\mu, F)\};\\
\mathfrak{p}(\mu, F) &= \min\{\kappa : \exists(\kappa_1, \kappa_2) \in C(\mu, F)[\max(\kappa_1, \kappa_2) = \kappa]\};\\
\mathfrak{t}(\mu, F) &= \min\{\kappa \geq \omega : \kappa \text{ is regular and there is a strictly increasing}\\
&\qquad \text{unbounded } x \in {}^\kappa A \text{ for some } (A, \leq) \in P(\mu, F)\};\\
D(\mu, F) &= \{(\kappa, \lambda) \in C(\mu, F) : \max(\kappa, \lambda) < \mathfrak{t}(\mu, F)\}
\end{aligned}$$

Proposition 32.5. *If $X \in L(\mu, F)$ and $\langle a_\xi : \xi < \mu\rangle$ is a strictly increasing sequence of elements of X, with μ a limit ordinal less than $\mathfrak{p}(\mu, F)$, then there is a $b \in X$ not near to 1 such that $\forall \xi < \mu[a_\xi < b]$.*

Proof. Clearly each a_ξ is not near to 1. Let c_0 be the maximum element of X, and let $c_{n+1} = c_n - 1$ for all $n \in \omega$. Then (a, c) is not a gap, and this gives the desired element b. $\square$

Proposition 32.6. *Suppose that $\langle A_\alpha : \alpha < \mu\rangle$ is a system of finite linear orders, and $\forall \alpha < \mu[\emptyset \neq B_\alpha \subseteq A_\alpha]$.*

(i) There is an isomorphism f of $\prod_{\alpha<\mu} B_\alpha/F$ into $\prod_{\alpha<\mu} A_\alpha/F$ such that $f([x]_B) = [x]_A$ for all $x \in \prod_{\alpha<\mu} B_\alpha$.

(ii) For any $X \subseteq \prod_{\alpha<\mu} A_\alpha/F$ the following are equivalent:

(a) There exist $B_\alpha \subseteq A_\alpha$ for all $\alpha < \mu$ such that, with f as in (i), $X = f[\prod_{\alpha<\mu} B_\alpha]$.

(b) There exist $B_\alpha \subseteq A_\alpha$ for all $\alpha < \mu$ such that $X = \{[x]_A : \{\alpha < \mu : x_\alpha \in B_\alpha\} \in F\}$.

Proof. (i): Suppose that $[x]_B = [y]_B$. Thus $\{\alpha < \mu : x_\alpha = y_\alpha\} \in F$. Hence $[x]_A = [y]_A$. Preservation of $<$ is clear.

(ii)(a)⇒(ii)(b): Clear.

(ii)(b)⇒(ii)(a): Assume (ii)(b). Suppose that $u \in X$. Say $u = [x]_A$ such that $T \stackrel{\text{def}}{=} \{\alpha < \mu : x_\alpha \in B_\alpha\} \in F$. Let $y \in \prod_{\alpha<\mu} B_\alpha$ such that $\forall \alpha \in T[x_\alpha = y_\alpha]$. Then $f([y]_B) = [y]_A = [x]_A = u$. Conversely, suppose that $u \in f[\prod_{\alpha<\mu} B_\alpha]$. Say $x \in \prod_{\alpha<\mu} B_\alpha$ and $u = f([x]_B)$. So $u = [x]_A$ and $\{\alpha < \mu : x_\alpha \in B_\alpha\} = \mu \in F$, so $u \in X$. $\square$

A subset M of $\prod_{\alpha<\mu} A_\alpha/F$ is *internal* iff there is a system $\langle B_\alpha : \alpha < \mu\rangle$ such that $\forall \alpha < \mu[\emptyset \neq B_\alpha \subseteq A_\alpha]$ and $M = f[\prod_{\alpha<\mu} B_\alpha/F]$, with f as in Proposition 32.6.

Proposition 32.7. *Let $\langle A_\alpha : \alpha < \mu\rangle$ be a system of finite linear orders. Then every nonempty internal subset of $\prod_{\alpha<\mu} A_\alpha/F$ has a greatest and least element.*

Proof. Say M is internal, as above. For each $\alpha < \mu$ let x_α be the least element of B_α. Then $\{\alpha < \mu : x_\alpha$ is the least element of $B_\alpha\} = \mu \in F$. Thus $\prod_{\alpha<\mu} B_\alpha/F \models [x]$ is the least element. Similarly for the greatest element. $\square$

Proposition 32.8. *The collection of internal subsets of $\prod_{\alpha<\mu} A_\alpha/F$ is a field of subsets of $\prod_{\alpha<\mu} A_\alpha/F$ containing all singletons.*

Proof. Let X and Y be internal subsets of $\prod_{\alpha<\mu} A_\alpha/F$, say given by $\langle B_\alpha : \alpha < \mu\rangle$ and $\langle C_\alpha : \alpha < \mu\rangle$. For each $\alpha < \mu$ let $D_\alpha = B_\alpha \cup C_\alpha$. Let $f([x]_B) = [x]_A$ for all $x \in \prod_{\alpha<\mu} B_\alpha$, $g([x]_C) = [x]_A$ for all $x \in \prod_{\alpha<\mu} C_\alpha$, and $h([x]_D) = [x]_A$ for all $x \in \prod_{\alpha<\mu} D_\alpha$. Say $Z = h[\prod_{\alpha<\mu} D_\alpha]$. We claim that $X \cup Y = Z$. Suppose that $u \in X$. Say $u = f([x])$ with $x \in \prod_{\alpha<\mu} B_\alpha$. Then $u = [x]_A = h([x])$, so $u \in Z$. Thus $X \subseteq Z$. Similarly $Y \subseteq Z$. Now suppose that $u \in Z$. Say $u = h([x]_D) = [x]_A$ with $x \in \prod_{\alpha<\mu} D_\alpha$. Thus $\mu = \{\alpha < \mu : x_\alpha \in B_\alpha\} \cup \{\alpha < \mu : x_\alpha \in C_\alpha\}$. Wlog $T \stackrel{\text{def}}{=} \{\alpha < \mu : x_\alpha \in B_\alpha\} \in F$. Let $y \in \prod_{\alpha<\mu} B_\alpha$ be such that $x_\alpha = y_\alpha$ for all $\alpha \in T$. Then $f([y]_B) = [y]_A = [x]_A = u \in X$. So $X \cup Y = Z$.

Now let X be an internal subset of $\prod_{\alpha<\mu} A_\alpha/F$, say given by $\langle B_\alpha : \alpha < \mu\rangle$ and the function f. Let $C_\alpha = A_\alpha\backslash B_\alpha$ for all $\alpha < \mu$, and let $g([x]_C) = [x]_A$ for all $x \in \prod_{\alpha<\mu} C_\alpha$. Let $Y = g[\prod_{\alpha<\mu} C_\alpha]$. We claim that $(\prod_{\alpha<\mu} A_\alpha/F)\backslash X = Y$. First suppose that $u \in \prod_{\alpha<\mu} A_\alpha/F)\backslash X$. Say $u = [x]_A$ with $x \in \prod_{\alpha<\mu} A_\alpha$.

(1) $T \stackrel{\text{def}}{=} \{\alpha < \mu : x_\alpha \in B_\alpha\} \notin F$.

In fact, otherwise let $y \in \prod_{\alpha<\mu} B_\alpha$ be such that $x_\alpha = y_\alpha$ for all $\alpha \in T$. Then $u = [x]_A = [y]_A = f([y]_B) \in X$, contradiction.

So (1) holds. It follows that $S \stackrel{\text{def}}{=} \{\alpha < \mu : x_\alpha \in C_\alpha\} \in F$. Let $z \in \prod_{\alpha<\mu} C_\alpha$ be such that $z_\alpha = x_\alpha$ for all $\alpha \in S$. Then $u = [x]_A = [z]_A = g([z]_C) \in Y$. Thus $(\prod_{\alpha<\mu} A_\alpha/F)\backslash X \subseteq Y$.

Now suppose that $u \in Y$; say $u = g([x]_C)$ with $x \in \prod_{\alpha<\mu} C_\alpha$. Suppose that $u \in X$. Say $u = f([y]_B)$ with $y \in \prod_{\alpha<\mu} B_\alpha$. Thus $[x]_A = [y]_A$. Now $\forall \alpha < \mu[x_\alpha \notin B_\alpha \wedge t_\alpha \in B_\alpha$, so $\{\alpha < \mu : x_\alpha = y_\xi\} = \emptyset$, contradiction. Thus $(\prod_{\alpha<\mu} A_\alpha/F)\backslash X = Y$, and we have a field of sets.

Given $[f] \in \prod_{\alpha<\mu} A_\alpha/F$, $\{[f]\}$ is internal, given by $\langle \{f_\alpha\} : \alpha < \mu\rangle$. □

If $\psi(v_0)$ is a formula with one free variable v_0 and X is a structure, then $\psi(X) = \{x \in X : X \models \psi[x]\}$.

Proposition 32.9. *Let $X = \prod_{\alpha<\mu} A_\alpha/F$, and let $\psi(v_0)$ be a formula with one free variable v_0. Then $\psi(X)$ is an internal subset of X.*

Proof. We claim that $\langle \psi(A_\alpha) : \alpha < \mu\rangle$ shows that $\psi(X)$ is internal. In fact,

$$\begin{aligned} [x] \in \psi(X) \quad &\text{iff} \quad X \models \psi[[x]] \\ &\text{iff} \quad \{\alpha < \mu : A_\alpha \models [x_\alpha]\} \in F \\ &\text{iff} \quad \{\alpha < \mu : x_\alpha \in \psi(A_\alpha)\} \in F. \end{aligned}$$ □

Proposition 32.10. *If κ is uncountable regular, $\kappa < \mathfrak{t}(\mu, F)$, and $\kappa \leq \mathfrak{p}(\mu, F)$, then $(\kappa, \kappa) \notin C(\mu, F)$.*

Proof. Suppose not. Let $\langle L_\alpha : \alpha < \mu\rangle$ be a system of finite linear orders, and $X = \prod_{\alpha<\mu} L_\alpha/F$. Suppose that X has a (κ, κ)-gap $a, b \in {}^\kappa X$. For each $\alpha < \mu$ let P_α be the set of all functions p such that:

(1) $\mathrm{dmn}(p)$ is an initial segment of L_α, and $\mathrm{rng}(p) \subseteq L_\alpha \times L_\alpha$.

(2) $\forall d, d' \in \mathrm{dmn}(p)[d <_\alpha d' \rightarrow \pi_0(p(d)) <_\alpha \pi_0(p(d')) <_\alpha \pi_1(p(d')) <_\alpha \pi_1(p(d))]$.

The partial order on P_α is inclusion. Clearly this gives a tree with the unique root $\emptyset$. Let $(Q, \leq_Q) = \prod_{\alpha<\mu} P_\alpha/F$.

For each $\alpha < \mu$ let $G_\alpha = \{(p, c, d, e) : p \in P_\alpha,\ c, d, e \in L_\alpha,\ c \in \mathrm{dmn}(p),\ p(c) = (d, e)\}$. Let $H = \{([p], [c].[d], [e]) : \{\alpha < \mu : (p_\alpha, c_\alpha, d_\alpha, e_\alpha) \in G_\alpha\} \in F\}$. For details below it is convenient to apply Łoś's Theorem to the two-sorted structure $\overline{A} = (Q, X, \leq_Q, \leq_X, H)$. Note that $H \subseteq Q \times X \times X \times X$. Now suppose that $q \in Q$, $x, y, z \in X$, and $(q, x, y, z) \in H$. Say $q = [q^*]$, $x = [x^*]$, $y = [y^*]$, and $z = [z^*]$. Then

$$\begin{aligned} &\{\alpha < \mu : (q^*_\alpha, x^*_\alpha, y^*_\alpha, z^*_\alpha) \in G_\alpha\} \in F, \text{i.e.,} \\ &\{\alpha < \mu : q^*_\alpha \in P_\alpha, x^*_\alpha, y^*_\alpha, z^*_\alpha \in L_\alpha, x^*_\alpha \in \mathrm{dmn}(q^*_\alpha), q^*_\alpha(x^*_\alpha) = (y^*_\alpha, z^*_\alpha)\} \in F \\ &\{\alpha < \mu : q^*_\alpha \in P_\alpha, x^*_\alpha, y^*_\alpha, z^*_\alpha \in L_\alpha, x^*_\alpha \in \mathrm{dmn}(q^*_\alpha), q^*_\alpha(x^*_\alpha) = (y^*_\alpha, z^*_\alpha)\} \\ &\quad \subseteq \{\alpha < \mu : q^*_\alpha \in P_\alpha, x^*_\alpha \in L_\alpha, x^*_\alpha \in \mathrm{dmn}(q^*_\alpha), \exists \text{ unique } u, v \in P_\alpha[q^*_\alpha(x^*_\alpha) = (u, v)\}. \end{aligned}$$

Hence using $\overline{A}$, for all $q \in Q$ and $x \in X$, if there are $y, z \in X$ such that $(q, x, y, z) \in H$, then there are unique $u, v \in X$ such that $(q, x, u, v) \in H$. Hence we can make the following definition. For any $q \in Q$ let $\mathrm{dmn}(f_q) = \{x \in X : \exists y, z \in X[(q, x, y, z) \in H]\}$, and set $f_q(x) = (y, z)$ with $(q, x, y, z) \in H$.

(3) If $d, d' \in \mathrm{dmn}(f_q)$ and $d < d'$, then $\pi_0(f_q(d)) < \pi_0(f_q(d')) < \pi_1(f_q(d')) < \pi_1(f_q(d))$.

In fact, let $f_q(d) = (r,s)$ and $f_q(d') = (t,u)$. Thus $(q,d,r,s),(q,d',t,u) \in H$. Write $x = [x^*]$ for any x in X or Q. Hence the following sets are in F:

$$\{\alpha < \mu : q^*_\alpha \in P_\alpha, d^*_\alpha, r^*_\alpha, s^*_\alpha \in L_\alpha, d^*_\alpha \in \mathrm{dmn}(q^*_\alpha), q^*_\alpha(d^*_\alpha) = (r^*_\alpha, s^*_\alpha)\};$$
$$\{\alpha < \mu : q^*_\alpha \in P_\alpha, d'^*_\alpha, t^*_\alpha, u^*_\alpha \in L_\alpha, d'^*_\alpha \in \mathrm{dmn}(q^*_\alpha), q^*_\alpha(d'^*_\alpha) = (t^*_\alpha, u^*_\alpha)\};$$
$$\{\alpha < \mu : d^*_\alpha < d'^*_\alpha\}.$$

Take any α in the intersection of these three sets. Then by (2) we have $r^*_\alpha < t^*_\alpha < u^*_\alpha < s^*_\alpha$. Hence (3) follows.

(4) $\forall q \in Q[\mathrm{dmn}(f_q)$ has a maximal element$]$.

For, let $q \in Q$. Say $q = [q']$. Then for all $\alpha < \mu$, $q'_\alpha \in P_\alpha$ and so $\mathrm{dmn}(q'_\alpha)$ has a maximal element d_α. Thus

$$\exists b, c[(q'_\alpha, d_\alpha, b, c) \in G_\alpha \wedge \forall e, b', c'[(q'_\alpha, e, b', c') \in G_\alpha \to e \le d_\alpha].$$

It follows that

$$\exists b, c[(q, [d_\alpha], b, c) \in H] \wedge \forall e, b', c'[(q, e, b', c') \in H \to d_\alpha \le e].$$

Now (4) follows.

(5) $\forall q, r \in Q[q < r \to f_q \subseteq f_r]$.

In fact, suppose that $q, r \in Q$ and $q < r$. Write $q = [q']$ and $r = [r']$. Then $M \stackrel{\mathrm{def}}{=} \{\alpha < \mu : q'_\alpha \subseteq r'_\alpha\} \in F$. For any $\alpha \in M$ we have

$$\forall a, b, c[(q'_\alpha, a, b, c) \in G_\alpha \to (r'_\alpha, a, b, c) \in G_\alpha];$$

hence

$$\forall a, b, c[(q, a, b, c) \in H \to (r, a, b, c) \in H];$$

(5) follows.

Now we define $c_\alpha \in Q$ for $\alpha < \kappa$. Let $a_\alpha = [a'_\alpha]$ and $b_\alpha = [b'_\alpha]$ for all $\alpha < \kappa$. For each $\alpha < \mu$ let $p_\alpha = \{(\min L_\alpha, (a'_{0\alpha}, b'_{0\alpha}))\}$, and set $c_0 = [\langle p_\alpha : \alpha < \mu\rangle]$. Note that for each $\alpha < \mu$ we have $(p_\alpha, \min L_\alpha, a'_{0\alpha}, b'_{0\alpha}) \in G_\alpha$, so $(c_0, 0, a_0, b_0) \in H$. Hence $f_{c_0} = \{(0, (a_0, b_0))\}$. By Proposition 32.4, 0 is not near to 1. Now suppose that $c_\alpha = [c'_\alpha]$ has been defined so that $\mathrm{dmn}(f_{c_\alpha})$ has a maximum element $d_\alpha = [d'_\alpha]$ which is not near to 1. Let

$$M \stackrel{\mathrm{def}}{=} \{\beta < \mu : d'_{\alpha\beta} \text{ is the maximum element of } \mathrm{dmn}(c'_{\alpha\beta}) \text{ and } d'_{\alpha\beta} \text{ is not the maximum element of } L_\alpha\} \in F.$$

For $\beta \in M$ let $c'_{\alpha+1,\beta} = c'_{\alpha\beta} \cup \{(d'_{\alpha\beta}+1, (a'_{\alpha+1,\beta}, b'_{\alpha+1,\beta}))\}$, with $c'_{\alpha+1,\beta}$ arbitrary otherwise. Let $c_{\alpha+1} = [\langle c'_{\alpha+1,\beta} : \beta < \mu\rangle]$. Then $f_{c_{\alpha+1}} = f_{c_\alpha} \cup \{(d_\alpha + 1, (a_{\alpha+1}, b_{\alpha+1}))\}$. Note that the maximum element $d_\alpha + 1$ of $\mathrm{dmn}(f_{c_{\alpha+1}})$ is not near to 1.

Now suppose that α is a limit ordinal less than κ and c_β has been defined for all $\beta < \alpha$, with $c_\beta < c_\gamma$ for $\beta < \gamma < \alpha$; also, for all $\beta < \alpha$ the domain of the function f_{c_β} has a maximum element d_β which is not near to 1. Since $\alpha < \kappa < \mathfrak{t}(\mu, F)$, there is an $e \in Q$ such that $c_\beta < e$ for all $\beta < \alpha$. Say $e = [e']$. For each $\gamma < \mu$ let g'_γ be the maximal member of $\mathrm{dmn}(e'_\gamma)$, and set $g = [g']$. Let

$$N = \{\gamma < \mu : g'_\gamma \text{ is the maximum member of } \mathrm{dmn}(e'_\gamma) \text{ and } a'_{0\gamma} < a'_{\alpha\gamma} < b'_{\alpha\gamma} < b'_{0\gamma}\}.$$

Thus $N \in F$. Now

$$N \subseteq \{\gamma < \mu : \exists s \leq g'_\gamma \exists u, v[(e'_\gamma, s, u, v) \in G_\gamma \wedge u < a'_{\alpha\gamma} < b'_{\alpha\gamma} < v]\}.$$

The set on the right is thus in F. For each $\gamma \in N$ let w'_γ be the maximum s as indicated. and let w_γ be arbitrary for γ not in this set. Then $w \stackrel{\text{def}}{=} [\langle w'_\gamma : \gamma < \mu\rangle]$ is maximum such that $w \leq g$ and $(e, w, u, v) \in H$ for some u, v with $u < a_\alpha < b_\alpha < v$. Now $\{\gamma < \mu : w'_\gamma \in \mathrm{dmn}(e'_\gamma)\} \in F$. For any γ in this set, let $q'_\gamma = e'_\gamma \restriction w'_\gamma$. Thus $q'_\gamma \in P_\gamma$. Then

$$\forall s, t, u[G_\gamma(q'_\gamma, s, t, u) \rightarrow G_\gamma(e'_\gamma, s, t, u)] \wedge \forall s[\exists t, u[G_\gamma(q'_\gamma, s, t, u)] \leftrightarrow s < w'_\gamma].$$

Then by Łoś's theorem we have

$$\forall s, t, h[H([q'], s, t, u) \rightarrow H(e, s, t, u)] \wedge \forall s[\exists t, u[H([q'], s, t, u) \leftrightarrow s < w].$$

Thus $f_{[q']} = f_e \restriction w$.

Case 1. w is not near to 1. For any $\gamma < \mu$ let $c'_{\alpha\gamma} = q'_\gamma \cup \{(w'_\gamma, (a'_{\alpha\gamma}, b'_{\alpha\gamma}))\}$. Let $c_\alpha = [c'_\alpha]$. Now if $\beta < \alpha$, then $c_\beta < e$ and $d_\beta < w$, so $c_\beta < c_\alpha$. This completes the recursive definition. Since $\kappa < t(\mu, F)$, there is a $u \in Q$ such that $c_\alpha < u$ for all $\alpha < \kappa$. Let v be the largest element of the domain of f_u. Then for any $\alpha < \kappa$ we have

$$a_\alpha = \pi_0(f_u(\alpha)) < \pi_0(f_u(v)) < p_1(f_u(v)) < \pi_1(f_u(\alpha)) = b_\alpha,$$

contradicting a, b being a gap.

Case 2. w is near to 1. Let $\langle \beta_\xi : \xi < \mathrm{cf}(\alpha)\rangle$ be strictly increasing with supremum α. Now $\langle d_{\beta_\xi} : \xi < \mathrm{cf}(\alpha)\rangle$, $\langle w - n : n \in \omega\rangle$ is not a $(\mathrm{cf}(\alpha), \omega)$-gap since κ is uncountable and $\leq p(\mu, F)$, so there is a $\tilde{d}_\alpha \in X$ such that $\forall \xi < \mathrm{cf}(\alpha) \forall n \in \omega[d_{\beta_\xi} < \tilde{d}_\alpha < w - n\rangle$. Now we get a contradiction as in Case 1. Namely, for each $\gamma < \mu$ let $q''_\gamma = e'_\gamma \restriction \tilde{d}_{\alpha\gamma}$ and $c'_{\alpha\gamma} = q''_\gamma \cup \{(\tilde{d}_{\alpha\gamma}, (a'_{\alpha\gamma}, b'_{\alpha\gamma}))\}$ and $c_\alpha = [c']$, then proceed as before to get a contradiction. $\square$

If L is a linear order and X is any nonempty set, then $X^{<L}$ is the set of all functions f such that the domain of f is an initial segment of L and the range of f is contained in X. Under $\subseteq$, $X^{<L}$ is a pseudo-tree with a unique root, the empty set.

Theorem 32.11. *Every finite tree with a unique root can be isomorphically embedded in $X^{<L}$ for some finite linear order L and some finite set X.*

Proof. Let T be a finite tree with a unique root r. Let κ be the height of T, and let $L = \kappa$ under its natural order. Let $X = Y\backslash\{r\}$. For each $t \in T$ let $f(t) = \langle a_0, \ldots, a_m\rangle$ where $a_0, \ldots, a_m$ is a list in strictly increasing order of all elements different from r which are $\leq t$. In particular, $f(r) = \emptyset$. Clearly f is the desired isomorphic embedding. $\square$

Proposition 32.12. *If* $\kappa = \mathfrak{t}(\mu, F)$*, then* $(\kappa, \kappa) \in C(\mu, F)$.

Proof. Let $P = \prod_{\alpha<\mu} P_\alpha/F$, each P_α a finite tree with a unique root, with $c \in {}^\kappa P$ strictly increasing and unbounded. By Theorem 32.11 we may assume that $P_\alpha \subseteq X_\alpha^{<M_\alpha}$ where M_α is a finite linear order and X_α is a finite set. For each $\alpha < \mu$ let $<_\alpha$ be a well-order on X_α and $\ll_\alpha$ a well-order on M_α. If $p, q \in P_\alpha$ are incomparable, then $\chi(p, q)$ is the $\ll_\alpha$-least $i \in M_\alpha$ such that $p_i \neq q_i$. Now we define a relation $\prec_\alpha$ on $Q_\alpha \stackrel{\text{def}}{=} P_\alpha \times \{0, 1\}$:

$(p, \varepsilon) \prec_\alpha (q, \delta)$ iff one of the following holds:

(1) p and q are comparable, $\varepsilon = 0$, and $\delta = 1$.
(2) $p \subset q$ and $\varepsilon = \delta = 0$.
(3) $p \supset q$ and $\varepsilon = \delta = 1$.
(4) p and q are incomparable and $p_{\chi(p,q)} <_\alpha q_{\chi(p,q)}$.
(5) $(p, 0) \preceq_\alpha (q, \varepsilon) \preceq_\alpha (p, 0)$ is impossible.

For, assume that $(p, 0) \preceq_\alpha (q, \varepsilon) \preceq_\alpha (p, 0$.

Case 1. p and q are comparable and $\varepsilon = 1$. This is impossible.
Case 2. $p \subset q$ and $\varepsilon = 0$. This is impossible.
Case 3. p and q are incomparable and $p_{\chi(p,q)} <_\alpha q_{\chi(p,q)}$. this is impossible.

Thus (5) holds.

(6) $(p, 1) \preceq_\alpha (q, \varepsilon) \preceq_\alpha (p, 1)$ is impossible.

This is proved similarly.

Now let $Q = \prod_{\alpha<\mu} Q_\alpha/F$. For each $\alpha < \kappa$ let $c_\alpha = [c'_\alpha]$. Note that $c'_\alpha \in \prod_{\beta<\mu} P_\beta$. and so $c'_{\alpha\beta} \in P_\beta$ for all $\beta < \mu$. Now define

$$c'^0_\alpha = \langle (c'_{\alpha\beta}, 0) : \beta < \mu\rangle \in \prod_{\beta<\mu} Q_\beta,$$
$$c'^1_\alpha = \langle (c'_{\alpha\beta}, 1) : \beta < \mu\rangle \in \prod_{\beta<\mu} Q_\beta.$$

Now we claim

(7) $(\langle [c'^0_\alpha] : \alpha < \kappa\rangle, \langle [c'^1_\alpha] : \alpha < \kappa\rangle)$ is a (κ, κ)-gap in Q.

In fact, take $\beta < \alpha < \kappa$. Then $c_\beta < c_\alpha$, so $[c'_\beta] < [c'_\alpha]$, hence $\{\gamma < \mu : c'_{\beta\gamma} \subset c'_{\alpha\gamma}\} \in F$. Now

$$\{\gamma < \mu : c'_{\beta\gamma} \subset c'_{\alpha\gamma}\} \subseteq \{\gamma < \mu : c'^0_{\beta\gamma} \prec_\gamma c'^0_{\alpha\gamma} \prec_\gamma c'^1_{\alpha\gamma} \prec_\gamma c'^1_{\beta\gamma}\}.$$

It follows that for $\beta < \alpha < \kappa$ we have $[c'^0_\beta] < [c'^0_\alpha] < [c'^1_\alpha] < [c'^1_\beta]$.

Now suppose that $q \in Q$ and $[c'^0_\alpha] < q < [c'^1_\alpha]$ for all $\alpha < \kappa$. Write $q = [q']$ with $q' \in \prod_{\beta<\mu} Q_\beta$. Say $q'_\beta = (p_\beta, \varepsilon_\beta)$ for all $\beta < \mu$. Now $[\langle p_\beta : \beta < \mu\rangle]$ is not a bound for

$\langle c_\alpha : \alpha < \kappa\rangle$, so there is an $\alpha < \kappa$ such that $c_\alpha \not< [\langle p_\beta : \beta < \mu\rangle]$. Hence the following sets are in F:

$$R \stackrel{\text{def}}{=} \{\beta < \mu : c'_{\alpha\beta} \not\subseteq p_\beta\};$$
$$N \stackrel{\text{def}}{=} \{\beta < \mu : (c'_{\alpha\beta}, 0) \prec_\beta (p_\beta, \varepsilon_\beta)\};$$
$$S \stackrel{\text{def}}{=} \{\beta < \mu : (p_\beta, \varepsilon_\beta) \prec_\beta (c'_{\alpha\beta}, 1)\}.$$

Now $\{\beta < \mu : \varepsilon_\beta = 0\} \cup \{\beta < \mu : \varepsilon_\beta = 1\} = \mu \in F$, so we have two cases.

Case 1. $M \stackrel{\text{def}}{=} \{\beta < \mu : \varepsilon_\beta = 0\} \in F$. Now suppose that $\beta \in M \cap N \cap R \cap S$. Then $(c'_{\alpha\beta}, 0) \prec_\beta (p_\beta, 0)$ and $c'_{\alpha\beta} \not\subseteq p_\beta$.

Subcase 1.1. $p_\beta \subset c'_{\alpha\beta}$. Then $(p_\beta, 0) \prec_\beta (c'_{\alpha\beta}, 0) \prec_\beta (p_\beta, 0)$. This contradicts (5)

Subcase 1.2. p_β and $c'_{\alpha\beta}$ are incomparable. Since $(c'_{\alpha\beta}, 0) \prec_\beta (p_\beta, 0)$, it follows that

$$(c'_{\alpha\beta})_{\chi(c'_{\alpha\beta}, p_\beta)} <_\beta (p_\beta)_{\chi(c'_{\alpha\beta}, p_\beta)}.$$

Then $(c'_{\alpha\beta}, 1) \prec_\beta (p_\beta, 0) \prec_\beta (c'_{\alpha\beta}, 1)$, which contradicts (6).

Case 2. $M \stackrel{\text{def}}{=} \{\beta < \mu : \varepsilon_\beta = 1\} \in F$. Now suppose that $\beta \in M \cap N \cap R \cap S$. Then $(p_\beta, 1) \prec_\beta (c'_{\alpha\beta}, 1)$. Since $c'_{\alpha\beta} \not\subseteq p_\beta$, it follows that $c'_{\alpha\beta}$ and p_β are incomparable, and

$$(p_\beta)_{\chi(p_\beta, c'_{\alpha\beta})} <_\beta (c'_{\alpha\beta})_{\chi(p_\beta, c'_{\alpha\beta})}$$

Hence $(c'_{\alpha\beta}, 0) \prec_\beta (p_\beta, 1) \prec_\beta (c'_{\alpha\beta}, 0)$, which contradicts (5). □

Corollary 32.13. $p(\mu, F) \le t(\mu, F)$. □

Proposition 32.14. *If L is a linear order, a, b is a κ, λ-gap in L, and a, c is a κ, μ-gap in L, then $\lambda = \mu$.*

Proof. Say $\lambda < \mu$. For each $\xi < \lambda$ there is an $\eta < \mu$ such that $c_\eta < b_\xi$, as otherwise b_ξ would be below each c_η and hence would fill the gap a, c. So for each $\xi < \lambda$ choose $\eta_\xi < \mu$ such that $c_{\eta_\xi} < b_\xi$. Let $\theta = \sup_{\xi<\lambda} \eta_\xi + 1$. Then c_θ is below each b_ξ, again a contradiction. (Recall that κ, λ, μ are regular.) □

Proposition 32.15. *If L_α is a finite linear order for each $\alpha < \mu$, then we have*

$$\left(\prod_{\alpha<\mu} L_\alpha/F, \le\right) \cong \left(\prod_{\alpha<\mu} L_\alpha/F, \ge\right).$$

Proof. For each $\alpha < \mu$ let h_α be an isomorphism of $(L_\alpha, \le)$ onto $(L_\alpha, \ge)$. Define $k : \prod_{\alpha<\mu} L_\alpha/F \to \prod_{\alpha<\mu} L_\alpha/F$ by setting $h([x]) = [y]$, where $y_\alpha = h_\alpha(x_\alpha)$ for all $\alpha < \mu$. Clearly k is a well-defined bijection. Now take any $x, y \in \prod_{\alpha<\mu} L_\alpha$.

$$\begin{aligned}
[x] \le [y] \quad &\text{iff} \quad \{\alpha < \mu : x_\alpha \le y_\alpha\} \in F \\
&\text{iff} \quad \{\alpha < \mu : h_\alpha(x_\alpha) \ge h_\alpha(y_\alpha)\} \in F \\
&\text{iff} \quad k([x]) \ge k([y]).
\end{aligned}$$
□

Theorem 32.16. *Suppose that κ is regular uncountable and $\kappa \le p(\mu, F)$. Let $\langle L_\alpha : \alpha < \mu\rangle$ be a system of finite linear orders, let F be an ultrafilter on μ, and suppose that $\forall n \in \omega[\{\alpha < \mu : |L_\alpha| > n\} \in F]$. Let $(L, \le) = \prod_{\alpha<\mu} L_\alpha/F$. Then there is a regular θ such that $(L, \le)$ has a (κ, θ)-gap; and there is a regular θ' such that $(L, \le)$ has a (θ', κ)-gap.*

Proof. By Proposition 32.3, L has an infinite increasing sequence $\langle c_n : n \in \omega\rangle$. Thus each c_n is not near to 1. Now suppose that $\alpha < \kappa$ and c_α has been defined and is not near to 1. Let $c_{\alpha+1} = c_\alpha + 1$. Suppose that $\alpha < \kappa$ is limit and c_β has been defined for all $\beta < \alpha$, each c_β not near to 1. Now $(\langle c_\beta : \beta < \alpha\rangle, \langle 1 - n : n \in \omega\rangle)$ is not a gap, since $|\alpha|, \omega < \kappa \le p(\mu, F)$. Hence there is a c_α not near to 1 such that $c_\beta < c_\alpha$ for all $\beta < \alpha$. So we have constructed $c \in {}^\kappa L$ strictly increasing with no element near to 1. Let $A = \{d \in L : \forall \alpha < \kappa[c_\alpha < d]\}$. Note that A is nonempty, since e.g. $1 \in A$. If A has a first element d, then there is an $\alpha < \kappa$ such that $d - 1 \le c_\alpha < c_{\alpha+1} < d$, contradiction. So A does not have a first element. Let $\langle e_\xi : \xi < \theta\rangle$ be a strictly decreasing coinitial sequence of elements of A. This gives θ as required in the theorem; for the second conclusion, apply Proposition 32.15. □

Theorem 32.17. *Suppose that κ is uncountable and regular, $\kappa < t(\mu, F)$, and $\kappa \le p(\mu, F)$. Then there is a unique regular θ such that $(\kappa, \theta) \in C(\mu, F)$.*

Proof. Existence was proved in Theorem 32.16. Now suppose that $\langle M_\alpha : \alpha < \mu\rangle$ and $\langle N_\alpha : \alpha < \mu\rangle$ are systems of finite linear orders, $\forall n \in \omega[\{\alpha < \mu : |M_\alpha| > n\} \in F]$, $\forall n \in \omega[\{\alpha < \mu : |N_\alpha| > n\} \in F]$, $M = \prod_{\alpha<\mu} M_\alpha/F$, $N = \prod_{\alpha<\mu} N_\alpha/F$, $a^0 \in {}^\kappa M$, $b^0 \in {}^{\theta_0}M$, (a^0, b^0) is a (κ, θ_0)-gap, $a^1 \in {}^\kappa N$, $b^1 \in {}^{\theta_1}N$, and (a^1, b^1) is a (κ, θ_1)-gap. We may assume that $M_\alpha \cap N_\alpha = \emptyset$ for all $\alpha < \mu$, and we define an order on $M_\alpha \cup N_\alpha$ by putting each member of M_α before each member of N_α. For each $\alpha < \mu$ let P_α be the set of all functions p such that

(1) $\mathrm{dmn}(p)$ is a nonempty initial segment of $M_\alpha \cup N_\alpha$, and $\mathrm{rng}(p) \subseteq M_\alpha \times N_\alpha$.

(2) $\forall d, d' \in \mathrm{dmn}(p)[d < d' \to \pi_0(p(d)) < \pi_0(p(d'))$ and $\pi_1(p(d)) < \pi_1(p(d'))]$.

We now consider the two-sorted structure $(P_\alpha, M_\alpha \cup N_\alpha, G_\alpha)$ where $G_\alpha = \{(p, a, b, c, d) : p \in P_\alpha, a \in \mathrm{dmn}(p), p(a) = (b, c), d = \max(\mathrm{dmn}(p))\}$. Let $P = \prod_{\alpha<\mu} P_\alpha/F$ and $X = \prod_{\alpha<\mu}(M_\alpha \cup N_\alpha)/F$. For each $\alpha < \mu$ let

$$H = \{([p], [x], [y], [z], [w]) : \{\alpha < \mu : (p_\alpha, x_\alpha, y_\alpha, z_\alpha, w_\alpha) \in G_\alpha\} \in F\}.$$

Thus $H \subseteq P \times X \times X \times X \times X$. We claim

(3) $\forall p \in P \forall x \in X \forall y, z, u, v, s, t \in X[(p, x, y, z, s) \in H$ and $(p, x, u, v, t) \in H \to y = u$, $z = v$ and $s = t]$.

In fact, suppose that $p \in P$, $x, y, z, u, v, s, t \in X$, $(p, x, y, z, s) \in H$, and $(p, x, u, v, t) \in H$. Say $p = [p']$, $x = [x']$, $y = [y']$, $u = [u']$, $v = [v']$, $s = [s']$, and $t = [t']$. Then

$$\{\alpha < \mu : (p'_\alpha, x'_\alpha, y'_\alpha, z'_\alpha, s'_\alpha) \in G_\alpha\} \in F \text{ and } \{\alpha < \mu : (p'_\alpha, x'_\alpha, u'_\alpha, v'_\alpha, t'_\alpha) \in G_\alpha\} \in F.$$

If α is in both of the sets here, then $p'_\alpha \in P_\alpha$, $x'_\alpha \in \mathrm{dmn}(p'_\alpha)$, $p'_\alpha(x'_\alpha) = (y'_\alpha, z'_\alpha) = (u'_\alpha, v'_\alpha)$, and $s'_\alpha = t'_\alpha = \max(\mathrm{dmn}(p'_\alpha)$. Hence $y'_\alpha = u'_\alpha$ and $z'_\alpha = v'_\alpha$. So $\{\alpha < \mu : y'_\alpha = u'_\alpha\} \in F$; so $y = u$. Similarly $z = v$ and $s = t$. So (3) holds.

Now for any $p \in P$ let $\mathrm{dmn}(f_p) = \{x : \exists y, z, s[(p, x, y, z, s) \in H\}$, and set $f_p(x) = (y, z)$. This is justified by (3). Note that $\mathrm{dmn}(f_p)$ has a maximum element; we denote it by s^p.

(4) Let $d, e \in \mathrm{dmn}(f_p)$ and $d < e$. Say $f_p(d) = (x, y)$ and $f_p(e) = (u, v)$. Then $x < u$ and $y < v$.

In fact, write $p = [p']$, $d = [d']$, $x = [x']$, $y = [y']$, $e = [e']$, $u = [u']$, $v = [v']$ and $s^p = [s^{p\prime}]$. Now $(p, d, x, y, s^p), (p, e, u, v, s^p) \in H$, so

$$\{\alpha < \mu : (p'_\alpha, d'_\alpha, x'_\alpha, y'_\alpha, s^{p\prime}_\alpha) \in G_\alpha\} \in F \text{ and } \{\alpha < \mu : (p'_\alpha, e'_\alpha, u'_\alpha, v'_\alpha, s^{p\prime}_\alpha) \in G_\alpha\} \in F.$$

For α in the intersection of these two sets we have $p'_\alpha(d'_\alpha) = (x'_\alpha, y'_\alpha)$ and $p'_\alpha(e'_\alpha) = (u'_\alpha, v'_\alpha)$. Hence by (2), $x'_\alpha < u'_\alpha$ and $y'_\alpha < v'_\alpha$. Then (4) follows.

We now construct $\langle c_\alpha : \alpha < \kappa\rangle \in {}^\kappa P$ stictly increasing such that the following condition holds:

(5) s^{c_α} is not near to 1.

Now if $\xi < \kappa$, then $a^0_\xi \in M$ and $a^1_\xi \in N$. Say $a^0_\xi = [a^{0\prime}_\xi]$ and $a^1_\xi = [a^{1\prime}_\xi]$. Thus $\forall \xi < \kappa[a^{0\prime}_\xi \in \prod_{\alpha<\mu} M_\alpha]$ and $\forall \xi < \kappa[a^{1\prime}_\xi \in \prod_{\alpha<\mu} N_\alpha]$. For each $\alpha < \mu$ let 0_α be the smallest element of $M_\alpha \cup N_\alpha$. For each $\alpha < \mu$ let $p_\alpha = \{(0_\alpha, ((a^{0\prime}_0)_\alpha, (a^{1\prime}_0)_\alpha))\}$. Thus $p_\alpha \in P_\alpha$. Let $c_0 = [\langle p_\alpha : \alpha < \mu\rangle]$. Now for any $\alpha < \mu$ we have $(p_\alpha, 0, (a^{0\prime}_0)_\alpha, (a^{1\prime}_0)_\alpha, 0) \in G_\alpha$. Hence $(c_0, 0, a^0_0, a^1_0, 0) \in H$. Hence $f_{c_0} = \{(0, (a^0_0, a^1_0))\}$. Thus $s^{c_0} = 0$. Note that 0 is not near to 1, since F is nonprincipal; so (5) holds for $\alpha = 0$.

If c_α has been defined so that the maximum member s^{c_α} of $\mathrm{dmn}(f_{c_\alpha})$ is not near to 1, let $c_\alpha = [c'_\alpha]$, $s^{c_\alpha} = [d'_\alpha]$. Then

$$\begin{aligned} Q \stackrel{\text{def}}{=} \{\beta < \mu : d'_{\alpha\beta} &\text{ is the maximum element of } \mathrm{dmn}(c'_\alpha) \\ &\text{but is not the greatest element of } M_\alpha \cup N_\alpha\} \in F. \end{aligned}$$

Then for $\beta \in Q$ let $c'_{\alpha+1,\beta} = c'_{\alpha\beta} \cup \{(d'_{\alpha\beta} + 1, (a^{0\prime}_{\alpha+1}, a^{1\prime}_{\alpha+1}))\}$. $c'_{\alpha+1,\beta}$ is arbitrary otherwise. Then $(c'_{\alpha+1,\beta}, d'_{\alpha\beta}, a^{0\prime}_{\alpha+1,\beta}, a^{1\prime}_{\alpha+1,\beta}, d'_{\alpha\beta}) \in G_\beta$, and so, with $c_{\alpha+1} = [\langle c'_{\alpha+1,\beta} : \beta < \mu\rangle]$,

$$f_{c_{\alpha+1}} = f_{c_\alpha} \cup \{(s^{c_\alpha} + 1, (a^0_{\alpha+1}, a^1_{\alpha+1}))\}.$$

Thus $s^{c_{\alpha+1}} = s^{c_\alpha} + 1$, and (5) clearly holds for $\alpha + 1$.

Now suppose that α is limit. Since $\alpha < \kappa < t(\mu, F)$, there is an $e \in P$ such that $\forall \beta < \alpha[c_\beta < e]$. Let $s^e = [d']$ and $e = [e']$. Let

$$Q = \{\gamma < \mu : d'_\gamma \text{ is the maximum member of } \mathrm{dmn}(e'_\gamma) \text{ and } a^{0\prime}_0 < a^{0\prime}_\gamma \text{ and } a^{1\prime}_0 < a^{1\prime}_\gamma\}.$$

Thus $Q \in F$. Now

$$Q \subseteq \{\gamma < \mu : \exists s \leq d'_\gamma \exists u, v, w[(e'_\gamma, s, u, v, w) \in G_\gamma \wedge u < a^{0\prime}_{\alpha\gamma} \text{ and } v < a^{1\prime}_{\alpha\gamma}\}.$$

The set on the right is thus in F. For each $\gamma \in Q$ let w'_γ be the maximum s as indicated. and let w_γ be arbitrary for γ not in this set. Then $w \stackrel{\text{def}}{=} [\langle w'_\gamma : \gamma < \mu\rangle]$ is maximum such that $w \le d$ and $(e, w, u, v, x) \in H$ for some u, v, x with $u < a^0_\alpha$ and $v < a^1_\alpha$. Thus w is the maximum member of P which is $\le \mathrm{dmn}(f_e)$ such that $\pi_0(f_e(w)) < a^0_\alpha$ and $\pi_1(f_e(w)) < a^1_\alpha$. Now $\{\gamma < \mu : w'_\gamma \in \mathrm{dmn}(e'_\gamma)\} \in F$. For any γ in this set, let $q_\gamma = e'_\gamma \restriction w'_\gamma$. Then

$$\forall s, t, u, v[G_\gamma(q_\gamma, s, t, u, v) \to G_\gamma(e'_\gamma, s, t, u, v)] \wedge \forall s[\exists t, u, v[G_\gamma(q_\gamma, s, t, u, v)] \leftrightarrow s < w'_\gamma].$$

Then by Łoś's theorem we have

$$\forall s, t, u, v[H([q], s, t, u, v) \to H(e, s, t, u, v)] \wedge \forall s[\exists t, u, v[H([q], s, t, u, v) \leftrightarrow s < w]].$$

Thus $f_{[q]} = f_e \restriction w$.

Case 1. w is not near to 1. For any $\gamma < \mu$ let $c'_{\alpha\gamma} = q_\gamma \cup \{(w'_\gamma, (a'_{\alpha\gamma}, b'_{\alpha\gamma}))\}$. Then $[c'_\alpha] = [q] \cup \{(w, (a_\alpha, b_\alpha))\}$. Let $s^{c_\alpha} = w$. Let $c_\alpha = [c'_\alpha]$.

Case 2. w is near to 1. Let $\langle \beta_\xi : \xi < \mathrm{cf}(\alpha)\rangle$ be strictly increasing with supremum α. Now $\langle s^{c_{\beta_\xi}} : \xi < \mathrm{cf}(\alpha)\rangle$, $\langle w - n : n \in \omega\rangle$ is not a $(\mathrm{cf}(\alpha), \omega)$-gap since κ is uncountable and $\le p(\mu, F)$, so there is an $s^{c_\alpha} \in X$ such that $\forall \xi < \mathrm{cf}(\alpha) \forall n \in \omega[s^{c_{\beta_\xi}} < s^{c_\alpha} < w - n]\rangle$. As in Case 1 we get c' such that $[c'_\alpha] = [q] \cup \{(s^{c_\alpha}, (a_\alpha, b_\alpha))\}$. Let $c_\alpha = [c'_\alpha]$.

Note that for each $\alpha < \kappa$, s^{c_α} is the maximum member of $\mathrm{dmn}(c_\alpha)$. For brevity we let $u = \langle s^{c_\gamma} : \gamma < \kappa\rangle$.

Since $\kappa < t(\mu, F)$, there is an $e \in P$ such that $\forall \alpha < \kappa[c_\alpha < e]$. Let $t = \max(\mathrm{dmn}(f_e))$. We construct $d^0 \in {}^{\theta_0}(\prod_{\alpha<\mu}(M_\alpha \cup N_\alpha)/F)$ so that (c, d^0) is a (κ, θ_0)-gap. A similar construction will give a (κ, θ_1)-gap (c, d^1), so $\theta_0 = \theta_1$ by Proposition 32.14. Let $d^0_0 = t$. Then $\forall \gamma < \kappa[s^{c_\gamma} < s]$. Suppose that $\xi < \theta_0$ and d^0_ξ has been constructed so that $\forall \gamma < \kappa[s^{c_\gamma} < d^0_\xi]$.

(6) There is an $x \in \mathrm{dmn}(f_e)$ such that $\pi_0(f_e(x)) \le b^0_{\xi+1}$ and $x < d^0_\xi$.

In fact, take any $\eta < \kappa$. Then $\pi_0(f_e(d_\eta)) = a^0_\eta \le b^0_{\xi+1}$ and $s^{c_\eta} < d^0_\xi$. So (6) holds. Let

$$d^0_{\xi+1} = \max\{x \in \mathrm{dmn}(f_e) : \pi_0(f_e(x)) \le b^0_{\xi+1} \text{ and } x < d^0_\xi\}.$$

Note that $s^{c_\eta} < d^0_{\xi+1}$ for all $\eta < \kappa$.

Now suppose that ξ is limit and d^0_η has been defined for every $\eta < \xi$ so that $s^{c_\theta} < d^0_\eta$ for all $\theta < \kappa$ and $\eta < \xi$. We claim that there is a x such that the following conditions hold:

(7) $x \in \mathrm{dmn}(f_e)$.

(8) $\pi_0(f_e(x)) \le b^0_\xi$.

(9) $s^{c_\gamma} < x$ for all $\gamma < \kappa$.

(10) $x < d^0_\eta$ for all $\eta < \xi$.

Suppose there is no such x. Now we claim

(11) $\forall \eta < \xi \exists \gamma < \theta_0[b^0_\gamma < \pi_0(f_e(d^0_\eta))]$.

For, suppose that $\eta < \xi$. Then for any $\gamma < \kappa$, $s^{c_\gamma} < d^0_\eta$, and hence $\pi_0(f_e(d^0_\eta)) > \pi_0(f_e(d_\gamma)) = a^0_\gamma$. Now since a^0, b^0 is a gap, it follows that there is a $\delta < \theta_0$ such that $b^0_\delta < \pi_0(f_e(d^0_\eta)$, as desired in (11).

Now for each $\eta < \xi$ let

$$g(\eta) = \min\{\gamma < \theta_0 : \pi_0(f_e(d^0_\eta)) > b^0_\gamma\}.$$

We claim that $\mathrm{rng}(g)$ is cofinal in θ_0 (contradicting θ_0 regular). For, suppose that $\xi < \gamma < \theta_0$. Let

$$y = \max\{u \in \mathrm{dmn}(f_e) : \pi_0(f_e(u)) \le b^0_\gamma\}.$$

Clearly y satisfies (7) and (9). Now $\pi_0(f_e(y)) \le b^0_\gamma < b^0_\xi$, so (8) holds. Hence by assumption, (10) does not hold. Hence there is an $\eta < \xi$ such that $d^0_\eta \le y$. Now $\pi_0(f_e(d^0_\eta)) \le \pi_0(f_e(y)) \le b^0_\gamma$. It follows that $g(\eta) > \gamma$, proving the claim.

So there is a x satisfying (7)–(10); we let d^0_ξ be such a x.

This finishes the construction of d^0. We claim that (u, d^0) is a (κ, θ_0)-gap. Suppose that x fills the gap. Take any $\xi < \kappa$ and $\eta < \theta_0$. Then

$$a^0_\xi = \pi_0(f_e(d_\xi)) < \pi_0(f_e(x)) < \pi_0(f_e(d^0_\eta)) \le b^0_\eta.$$

Thus $\pi_0(f_\chi(x))$ fills the gap (a^0, b^0), contradiction.

So (u, d^0) is a (κ, θ_0)-gap. Similarly we get a (k, θ_1)-gap. By Proposition 32.14, $\theta_0 = \theta_1$. $\square$

Corollary 32.18. *If κ is a regular cardinal, $\kappa < t(\mu, F)$, $\kappa \le p(\mu, F)$, and there is an $L \in L(\mu, F)$ such that there is no (κ, θ)-gap in L, then $(\kappa, \theta) \notin C(\mu, F)$.*

Proof. Assume the hypothesis, but suppose that $(\kappa, \theta) \in C(\mu, F)$. Let $(X, <) \in L(\mu, F)$ be such that it has a (κ, θ)-gap. By Theorem 32.16, there is a (κ, θ')-gap in L for some regular θ'. Thus $(\kappa, \theta), (\kappa, \theta') \in C(\mu, F)$, so by Theorem 32.17, $\theta = \theta'$. Hence L has a (κ, θ)-gap, contradiction. $\square$

Theorem 32.19. *Suppose that $\langle (X_\alpha, \le_\alpha) : \alpha < \mu\rangle$ is a system of finite linear orders, $X = \prod_{\alpha<\mu} X_\alpha/F$, U is an infinite subset of X, Z is a nonempty family of nonempty internal subsets of X, $|U|, |Z| < t(\mu, F), p(\mu, F)$, and $U \subseteq z$ for all $z \in Z$.*

Then there is an internal Y such that $U \subseteq Y \subseteq \bigcap Z$.

Proof. Let $z \in {}^\kappa Z$ enumerate Z. For each $\alpha < \kappa$ let $z_\alpha = [z'_\alpha]$. For each $\alpha < \mu$ let Q_α be the set of all functions f satisfying the following conditions:

(1) $\mathrm{dmn}(f)$ is an initial segment of X_α.

(2) $\mathrm{rng}(f) \subseteq \mathscr{P}(X_\alpha)$.

(3) $\forall x, y \in X_\alpha[x \le y \in \mathrm{dmn}(f) \to f(y) \subseteq f(x)]$.

Let $Q = \prod_{\alpha<\mu}(Q_\alpha, \subseteq)/F$. For $\alpha < \mu$ let $G_\alpha = \{(f, a, b) : f \in Q_\alpha,\ a \in \mathrm{dmn}(f),\ b \in f(a)\}$. Let

$$H = \Bigg\{([f], [a], [b]) : f \in \prod_{\alpha<\mu} Q_\alpha,\ a \in X,\ b \in \prod_{\alpha<\mu} \mathscr{P}(X_\alpha), \\ \{\alpha :\ (f_\alpha, a_\alpha, b_\alpha) \in G_\alpha\} \in F\Bigg\}.$$

For $c \in Q$ let $\mathrm{dmn}(f_c) = \{a : \exists b[(c, a, b) \in H]\}$ and for any $a \in \mathrm{dmn}(f_c)$ let $f_c(a) = \{b : (c, a, b) \in H\}$. Now we construct $q \in {}^{\kappa+1}Q$ by recursion so that the following conditions hold for all $\alpha \leq \kappa$:

(4) $\mathrm{dmn}(f_{q_\alpha})$ has a maximal element d_α.

(5) d_α is not near to 1.

(6) $U \subseteq f_{q_\alpha}(d)$ for all $d \leq d_\alpha$.

(7) $f_{q_\alpha}(d_\alpha) \subseteq z_\alpha$ for $\alpha < \kappa$.

(8) $f_{q_\alpha}(d_\alpha)$ is an internal subset of X.

(9) If $\beta < \alpha$, then $q_\beta \leq q_\alpha$. (Hence $f_{q_\beta} \subseteq f_{q_\alpha}$.)

Now z_0 is an internal subset of X, so let $\langle B_\alpha : \alpha < \mu\rangle$ be such that $\forall\alpha < \mu[B_\alpha \subseteq X_\alpha]$ and $z_0 = \{[x] : \{\alpha < \mu : x_\alpha \in B_\alpha\} \in F\}$. Define $q'_{0\alpha} = \{(0_\alpha, B_\alpha)\}$ for all $\alpha < \mu$, where 0_α is the zero of X_α. Thus $q'_{0\alpha} \in Q_\alpha$ for all $\alpha < \mu$. Let $q_0 = [\langle {q'_0}_{\alpha\alpha} : \alpha < \mu\rangle]$ Then $q_0 \in Q$. Now

$$\begin{aligned}\mathrm{dmn}(f_{q_0}) &= \Bigg\{[a] : a \in X,\ \exists b \in \prod_{\alpha<\mu} \mathscr{P}(X_\alpha)[(q_0, [a], [b]) \in H]\Bigg\} \\ &= \Bigg\{[a] : a \in X,\ \exists b \in \prod_{\alpha<\mu} \mathscr{P}(X_\alpha) \\ &\qquad [\{\alpha < \mu : (q'_{0\alpha}, a_\alpha, b_\alpha) \in G_\alpha\} \in F]\Bigg\}.\end{aligned}$$

Now for all $\alpha < \mu$, $a \in X$, and $b \in \prod_{\alpha<\mu} \mathscr{P}(X_\alpha)$, $(q'_{0\alpha}, a_\alpha, b_\alpha) \in G_\alpha$ iff $a_\alpha = 0$ and $b_\alpha \in B_\alpha$. Thus for $a \in X$,

$$\begin{aligned}[a] \in \mathrm{dmn}(f) \quad &\text{iff} \quad \exists b \in \prod_{\alpha<\mu} \mathscr{P}(X_\alpha)[\{\alpha < \mu : (q'_{0\alpha}, a_\alpha, b_\alpha) \in G_\alpha\} \in F] \\ &\text{iff} \quad \{\alpha < \mu : a_\alpha = 0\} \in F.\end{aligned}$$

In fact, if $b \in \prod_{\alpha<\mu} \mathscr{P}(X_\alpha)$ and $\{\alpha < \mu : (q'_{0\alpha}, a_\alpha, b_\alpha) \in G_\alpha\} \in F$, then $\{\alpha < \mu : (q'_{0\alpha}, a_\alpha, b_\alpha) \in G_\alpha\} \subseteq \{\alpha < \mu : a_\alpha = 0\}$, so $\{\alpha < \mu : a_\alpha = 0\} \in F$. Conversely, if $\{\alpha < \mu : a_\alpha = 0\} \in F$, choose $b \in \prod_{\alpha<\mu} \mathscr{P}(X_\alpha)$ such that $\{\alpha < \mu : b_\alpha \neq \emptyset\} \in F$. Then $\{\alpha < \mu : b_\alpha \neq \emptyset\} \cap \{\alpha < \mu : a_\alpha = 0\} \subseteq \{\alpha < \mu : (q'_{0\alpha}, a_\alpha, b_\alpha) \in G_\alpha\}$, and so $(q'_{0\alpha}, a_\alpha, b_\alpha) \in G_\alpha\} \in F$.

It follows that

$$\begin{aligned} f_{q_0}(0) &= \{b : (q_0, 0, b) \in H\} = \{[b] : \{\alpha : (q'_{0\alpha}, 0, b_\alpha) \in G_\alpha\} \in F\} \\ &= \{[b] : \{\alpha : b_\alpha \in q'_{0\alpha}(0)\} \in F\} = \{[b] : \{\alpha : b_\alpha \in B_\alpha\} \in F\} = z_0 \end{aligned}$$

Hence (4)–(9) hold for $\alpha = 0$.

Now assume that q_α has been defined satisfying (4)–(9). By (8) let $\langle C_\beta : \beta < m\rangle$ be such that $\forall\beta < \mu[C_\beta \subseteq X_\beta]$ and $f_{q_\alpha}(d_\alpha) = \{[x] : \{\beta < \mu : x_\beta \in C_\beta\} \in F\}$. Let $q_\alpha = [q'_\alpha]$ and $d_\alpha = [d'_\alpha]$. For each $\beta < \mu$ let

$$q'_{\alpha+1,\beta} = q'_{\alpha\beta} \cup \{(d'_{\alpha\beta} + 1, z'_{\alpha+1,\beta} \cap q'_{\alpha\beta}(d'_{\alpha\beta}))\}.$$

Then $q'_{\alpha+1,\beta} \in Q_\beta$. Let $q_{\alpha+1} = [q'_{\alpha+1}]$. Then $f_{q_{\alpha+1}} = f_{q_\alpha} \cup \{(d_\alpha + 1, z_{\alpha+1} \cap f_{q_\alpha}(d_\alpha))\}$. Then (4)–(7) and (9) clearly hold for $\alpha+1$. For (8), we have $f_{q_{\alpha+1}}(d_{\alpha+1}) = z_{\alpha+1} \cap f_{q_\alpha}(d_\alpha)$; as an intersection of two internal sets, this is internal by Proposition 32.7.

Now suppose that α is a limit ordinal $\leq \kappa$. Then $\langle f_{q_\beta} : \beta < \alpha\rangle$ is strictly increasing by (9), and $\kappa < t(\mu, F)$, so there is an $r \in Q$ such that $q_\beta \leq r$ for all $\beta < \alpha$. Now note that if $\beta < \alpha$ and $u \in U$, then by (6), $u \in f_{q_\beta}(d_\beta) \subseteq f_r(d_\beta)$. Now let $u \in U$. Say $r = [r']$ and $u = [u']$. Since $u \in f_r(d_0)$, the set $\{\beta < \mu : \exists x \in \mathrm{dmn}(r'_\beta)[u'_\beta \in r'_\beta(x)]\}$ is nonempty and is in F, and for a given β in this set there are only finitely many such x. (Since r'_β is finite.) Hence there is a maximum $d \in \mathrm{dmn}(f_r)$ such that $u \in f_r(d)$; denote this d by e_u. Thus $d_\beta < e_u$ for all $\beta < \alpha$. Now $\mathrm{dmn}(f_r) \subseteq X$. Since $|U|, |\alpha| < p(\mu, F)$, there is a d_α such that $d_\beta \leq d_\alpha \leq e_u$ for all $\beta < \alpha$ and $u \in U$. Let

$$q'_{\alpha\beta} = \begin{cases} r'_\beta \restriction d'_{\alpha\beta} \cup \{(d'_{\alpha\beta}, z'_{\alpha\beta} \cap r'_\beta(d'_{\alpha\beta}))\} & \text{if } \alpha < \kappa, \\ r'_\beta \restriction d'_{\alpha\beta} \cup \{(d'_{\alpha\beta}, r'_\beta(d'_{\alpha\beta}))\} & \text{if } \alpha = \kappa \end{cases}$$

Then let $q_\alpha = [q'_\alpha]$. Then

$$f_{c_\alpha} = \begin{cases} f_r \cup \{(d_\alpha, z_\alpha \cap r(d_\alpha))\} & \text{if } \alpha < \kappa, \\ f_r \cup \{(d_\alpha, r(d_\alpha))\} & \text{if } \alpha = \kappa \end{cases}$$

Then (4)–(7) and (9) clearly hold. For (8), for each $\beta < \mu$ let

$$B_\beta = \{u \in X_\beta : d'_{\alpha\beta} \in \mathrm{dmn}(q'_{\alpha\beta}) \text{ and } u \in q'_{\alpha\beta}(d'_{\alpha\beta})\}.$$

Then

$$\begin{aligned} [x] \in f_{c_\alpha}(d_\alpha) \quad &\text{iff} \quad [x] \in r(d_\alpha) \\ &\text{iff} \quad (r, d_\alpha, [x]) \in H \\ &\text{iff} \quad \{\beta < \mu : (r'_\beta, d'_{\alpha\beta}, x_\beta) \in G_\beta\} \in F \\ &\text{iff} \quad [x] \in B_\beta. \end{aligned}$$

This finishes the construction. Clearly $f_{q_\kappa}(d_\kappa)$ is as desired. □

A function $f : {}^m(\prod_{\alpha<\mu} A_\alpha/F) \to (\prod_{\alpha<\mu} A_\alpha/F$ is *internal* iff there is a system $\langle f_\alpha : \alpha < \mu\rangle$ such that each $f_\alpha : {}^m A_\alpha \to A_\alpha$ and for all $x_0, \ldots, x_{m-1}, y \in \prod_{\alpha<\mu} A_\alpha$ we have

$$f([x_0], \ldots, [x_{m-1}]) = [y] \quad \text{iff} \quad \{\alpha < \mu : f_\alpha(x_{0\alpha}, \ldots, x_{m-1,\alpha}) = y_\alpha\} \in F.$$

Theorem 32.20. *Suppose that $\langle X_\alpha : \alpha < \mu\rangle$ is a system of finite linear orders. Let $X = \prod_{\alpha<\mu} X_\alpha/F$. Suppose that $d \in {}^\kappa X$ is strictly decreasing, with $\kappa < t(\mu, F)$, and let $D = \operatorname{rng}(d)$. Suppose that $G : D \to X$. Then there is an internal $H : X \to X$ such that $G \subseteq H$.*

Proof. Write $d_\beta = [d'_\beta]$ for all $\beta < \kappa$, and let $G' : D \to \prod_{\alpha<\mu} X_\alpha$ be such that $G(d_\beta) = [G'(d_\beta)]$ for all $\beta < \kappa$.

For each $\alpha < \mu$ let P_α be the set of all functions f such that for some $d \in X_\alpha$, $\operatorname{dmn}(f) = \{x \in X_\alpha : d < x\}$, and $\operatorname{rng}(f) \subseteq X_\alpha$. Let $K_\alpha = \{(f, x, y) : f \in P_\alpha,\ x, y \in X_\alpha,\ x \in \operatorname{dmn}(f),\ f(x) = y\}$. We take the two-sorted structure $\overline{A}_\alpha = (X_\alpha, \leq, P_\alpha, \subseteq, K_\alpha)$. So $K_\alpha \subseteq P_\alpha \times X_\alpha \times X_\alpha$. Let $X = \prod_{\alpha<\mu} X_\alpha/F$, $P = \prod_{\alpha<\mu} P_\alpha/F$, and $H = \{([f], [x], [y]) : \{\alpha < \mu : (f_\alpha, x_\alpha, y_\alpha) \in K_\alpha\} \in F\}$. $\overline{B} = \prod_{\alpha<\mu} \overline{A}_\alpha/F = (X, P, H)$. Then for any $\alpha < \mu$,

$$\begin{aligned}\overline{A}_\alpha \models &\forall f \in P_\alpha[\forall x, y, z \in X_\alpha[(f, x, y) \in K_\alpha \wedge (f, x, z) \in K_\alpha \to y = z]\\ &\wedge \exists! d \in X_\alpha \forall x \in X_\alpha[\exists y \in X_\alpha[(f, x, y) \in K_\alpha] \leftrightarrow d < x].\end{aligned}$$

It follows by Łoś's theorem that for every $c \in P$ there exist $d \in X$ and $f_c \in P$ such that f_c is a function with domain $\{x \in X : d < x\}$ and range contained in X, with $f_{[c']}([x']) = [y']$ iff $\{\alpha < \mu : c'_\alpha(x'_\alpha) = y'_\alpha\} \in F$.

We now define by recursion $c \in {}^\kappa P$ so that the following conditions hold:

(1) $\operatorname{dmn}(f_{c_\beta}) = \{x \in X : d_\beta < x\}$.

(2) $\forall \gamma < \beta[f_{c_\gamma}(d_\gamma) = G(d_\gamma)$.

(3) $\forall \gamma < \beta[f_{c_\gamma} \subseteq f_{c_\beta}]$.

Now define $c'_{0\alpha}(x) = x$ for all $\alpha < \mu$ and $x > d'_{0\alpha}$, and $c_0 = [c'_0]$. To check (1), first suppose that $[x'] \in \operatorname{dmn}(f_{c_0})$. Let $[y'] = f_{c_0}([x'])$. Then $\{\alpha < \mu : c'_{0\alpha}(x'_\alpha) = y'_\alpha\} \in F$. Thus $\{\alpha < \mu : d'_{0\alpha} < x'_\alpha\} \in F$, so $[d'_0] < [x']$. Conversely, suppose that $[d'_0] < [x']$. Then $M \overset{\text{def}}{=} \{\alpha < \mu : d'_{0\alpha} < x'_\alpha\} \in F$. Then $c'_\alpha(x'_\alpha) = x'_\alpha$. It follows that $[x'] \in \operatorname{dmn}(f_{c_0})$. This proves (1). (2) and (3) hold vacuously.

Suppose that $c_\beta = [c'_\beta]$ has been defined satisfying (1)–(3). For any $\alpha < \mu$ and $d'_{\beta+1,\alpha} < x$ define

$$c'_{\beta+1,\alpha}(x) = \begin{cases} c'_{\beta\alpha}(x) & \text{if } d'_{\beta\alpha} < x,\\ (G'(d_\beta))_\alpha & \text{if } x \leq d'_{\beta\alpha}.\end{cases}$$

Let $c_{\beta+1} = [c'_{\beta+1}]$. To check (1), first suppose that $[x'] \in \operatorname{dmn}(f_{c_{\beta+1}})$. Let $[y'] = f_{c_{\beta+1}}([x'])$. Then $\{\alpha < \mu : c'_{\beta+1,\alpha}(x'_\alpha) = y'_\alpha\} \in F$. Then $\{\alpha < \mu : d'_{\beta+1,\alpha} < x'_\alpha\} \in F$, so $d_{\beta+1} < [x']$. Conversely, suppose that $d_{\beta+1} < [x']$. Then $M = \{\alpha < \mu : d'_{\beta+1,\alpha} < x'_\alpha\} \in F$. If $\alpha \in M$, then $d'_{\beta+1,\alpha} < x'_\alpha$, and so $x'_\alpha \in \mathrm{dmn}(c'_{\beta+1,\alpha})$. Say $c'_{\beta+1,\alpha}(x'_\alpha) = y'_\alpha$. Then $f_{c_{\beta+1}}([x'] = [y']$, so $[x'] \in \mathrm{dmn}(f_{c_{\beta+1}}$. Thus (1) holds for $\beta+1$.

For (3), if $\gamma < \beta$, then $f_{c_\gamma} \subseteq f_{c_\beta}$ by the inductive hypothesis. So it suffices to show that $f_{c_\beta} \subseteq f_{c_{\beta+1}}$. If $[x'] > d_\beta$, then $M \stackrel{\mathrm{def}}{=} \{\alpha < \mu : x'_\alpha > d'_{\beta\alpha}\} \in F$, and for $\alpha \in M$, $c'_{\beta+1,\alpha}(x'_\alpha) = c'_\beta(x'_\alpha)$, so $f_{c_\beta}([x']) = f_{c_{\beta+1}}([x']$. So (3) holds.

Now suppose that β is limit and $c_\gamma = [c'_\gamma]$ has been defined for all $\gamma < \beta$ so that (1)–(3) hold. Since $\kappa < t(\mu, F)$, there is an $r = [r'] \in P$ such that $c_\gamma < r$ for all $\gamma < \beta$. Now for each $\alpha < \mu$ let $\mathrm{dmn}(c'_{\beta\alpha}) = \{x \in X_\alpha : d'_{\beta\alpha} < x\}$. For any $x \in X_\alpha$ with $d'_{\beta\alpha} < x$ let

$$c'_{\beta\alpha}(x) = \begin{cases} r'_\alpha(x) & \text{if } x \in \mathrm{dmn}(r'_\alpha), \\ x & \text{otherwise.} \end{cases}$$

To check (1), suppose that $[x'] \in \mathrm{dmn}(f_{c_\beta})$. Let $[y'] = f_{c_\beta}([x'])$. Then $M \stackrel{\mathrm{def}}{=} \{\alpha < \mu : c'_{\beta\alpha}(x'_\alpha) = y'_\alpha\} \in F$. For $\alpha \in M$ we have $x'_\alpha \in \mathrm{dmn}(c'_{\beta\alpha})$, so $d'_{\beta,\alpha} < x'_\alpha$. Hence $[d'_\beta] < [x']$.

Conversely, suppose that $[d'_\beta] < [x']$. Then $M \stackrel{\mathrm{def}}{=} \{\alpha < \mu : d'_{\beta\alpha} < x'_\alpha\} \in F$. For any $\alpha \in M$ we have $x'_\alpha \in \mathrm{dmn}(c'_{\beta\alpha})$; say $c'_{\beta\alpha}(x'_\alpha) = y'_\alpha$. Then $f_{c_\beta}([x']) = [y']$. This proves (1).

For (2), suppose that $\gamma < \beta$. Then $f_{c_\gamma}(d_\gamma) = G(d_\gamma)$ by the inductive hypothesis, since $\gamma + 1 < \beta$.

For (3), suppose that $\gamma < \beta$ and $[x'] > d_\gamma$. Then $[x'] \in \mathrm{dmn}(r)$, so $M \stackrel{\mathrm{def}}{=} \{\alpha < \mu : x'_\alpha \in \mathrm{dmn}(r'_\alpha)\} \in F$. Also, $c_{\gamma+1} < r$, so $N = \{\alpha < \mu : c_{\gamma+1,\alpha}(x'_\alpha) = r'_\alpha(x'_\alpha)\} \in F$. For any $\alpha \in M \cap N$, $c'_{\beta\alpha}(x'_\alpha) = r'_\alpha(x'_\alpha) = c_{\gamma+1,\alpha}(x'_\alpha)$. Hence $c_\beta([x']) = c_{\gamma+1}([x']) = c_\gamma([x'])$. So (3) holds.

This completes the construction. Since $\kappa < t(\mu, F)$, choose $e = [e'] \in P$ such that $c_\beta < e$ for all $\beta < \kappa$. Now define for any $x \in \prod_{\alpha<\mu} X_\alpha$ and $\alpha < \mu$

$$(H'(x))_\alpha = \begin{cases} e'_\alpha(x_\alpha) & \text{if } x_\alpha \in \mathrm{dmn}(e'_\alpha), \\ x_\alpha & \text{otherwise.} \end{cases}$$

Next define for any $x \in \prod_{\alpha<\mu} X_\alpha$, $H([x]) = [H'(x)]$. This is well-defined. In fact, suppose that $x, y \in \prod_{\alpha<\mu} X_\alpha$ and $[x] = [y]$. Then $M \stackrel{\mathrm{def}}{=} \{\alpha < \mu : x_\alpha = y_\alpha\} \in F$. If $\alpha \in M$, clearly $(H'(x))_\alpha = (H'(y))_\alpha$.

Now $G \subseteq H$. For, suppose that $\beta < \kappa$. Then $H(d_\beta) = H([d'_\beta]) = [H'(d'_\beta)]$. Now $f_{c_\beta}(d_\beta) = G(d_\beta)$, so $M \stackrel{\mathrm{def}}{=} \{\alpha < \mu : c'_\beta(d_{\beta\alpha}) = (G'(d_\beta))_\alpha\} \in F$. For any $\alpha \in M$,

$$(H'(d'_\beta))_\alpha = e'_\alpha(d'_{\beta\alpha}) = c'_\beta(d'_{\beta\alpha}) = (G'(d_\beta))_\alpha).$$

Hence $H(d_\beta) = G(d_\beta)$.

If remains only to show that H is internal. For each $\alpha < \mu$ and $x \in X_\alpha$ let

$$f_\alpha(x) = \begin{cases} e'_\alpha(x) & \text{if } x \in \mathrm{dmn}(e'_\alpha), \\ x & \text{otherwise.} \end{cases}$$

Then for any $x, y \in \prod_{\alpha<\mu} X_\alpha$,

$$\begin{aligned} H([x]) = [y] \quad &\text{iff} \quad [H'(x)] = [y] \quad \text{iff} \quad \{\alpha < \mu : (H'(x))_\alpha = y_\alpha\} \in F \\ &\text{iff} \quad \{\alpha < \mu : f_\alpha(x_\alpha) = y_\alpha\} \in F. \end{aligned}$$

□

Theorem 32.21. *Suppose that $\langle X_\alpha : \alpha < \mu\rangle$ is a system of finite linear orders. Let $X = \prod_{\alpha<\mu} X_\alpha/F$. Suppose that $d \in {}^\kappa X$ is strictly decreasing, with $\kappa < t(\mu, F)$, and let $D = \mathrm{rng}(d)$. Suppose that $G : {}^2D \to X$. Then there is an internal $H : {}^2X \to X$ such that $G \subseteq H$.*

Proof. Write $d_\beta = [d'_\beta]$ for all $\beta < \kappa$, and let $G' : D \times D \to \prod_{\alpha<\mu} X_\alpha$ be such that $G(d_\beta, d_\gamma) = [G'(d_\beta, d_\gamma)]$ for all $\beta, \gamma < \kappa$.

For each $\alpha < \mu$ let P_α be the set of all functions f such that for some $e \in X_\alpha$, $\mathrm{dmn}(f) = \{(x, y) \in X_\alpha \times X_\alpha : x, y > e\}$ and $\mathrm{rng}(f) \subseteq X_\alpha$. Let $K_\alpha = \{(f, x, y, z) : f \in P_\alpha,\ x, y, z \in X_\alpha,\ f(x, y) = z\}$. We take the two-sorted structure $\overline{A}_\alpha = (X_\alpha, \leq, P_\alpha, \subseteq, K_\alpha)$ and let $\overline{B} = \prod_{\alpha<\mu} \overline{A}_\alpha/F$. Write $\overline{B} = (X, P, H)$. Then for any $\alpha < \mu$,

$$\begin{aligned}\overline{A}_\alpha \models & \forall f \in P_\alpha[\forall x, y, z, w \in X_\alpha[K_\alpha(f, x, y, z) \wedge K_\alpha(f, x, y, w) \to z = w] \\ & \wedge \exists! d \in X_\alpha \forall x, y \in X_\alpha[\exists z \in X_\alpha[K_\alpha(f, x, y, z)] \leftrightarrow d < x, y].\end{aligned}$$

It follows by Łoś's theorem that for every $c \in P$ there exist $d \in X$ and $f_c \in P$ such that f_c is a function with domain $\{(x, y) \in X \times X : d < x, y\}$ and range contained in X, with $f_{[c']}([x'], [y']) = [z']$ iff $\{\alpha < \mu : c'_\alpha(x'_\alpha, y'_\alpha) = z'_\alpha\} \in F$.

Now we define by recursion $c \in {}^\kappa P$ so that the following conditions hold:

(1) $\mathrm{dmn}(f_{c_\beta}) = \{(x, y) \in P \times P : d_\beta < x, y\}$.

(2) If $\gamma, \delta < \beta$, then $f_{c_\beta}(d_\gamma, d_\delta) = G(d_\gamma, d_\delta)$.

(3) If $\gamma < \beta$, then $c_\gamma < c_\beta$.

Let $c'_{0\alpha}(x, y) = x$ for all $x, y > d'_{0\alpha}$ and all $\alpha < \mu$; and let $c_0 = [c'_0]$. To check (1), first suppose that $([x'], [y']) \in \mathrm{dmn}(f_{c_0})$. Let $[z'] = f_{c_0}([x'], [y'])$. Then $\{\alpha < \mu : c'_{0\alpha}(x'_\alpha, y'_\alpha) = z'_\alpha\} \in F$. Thus $\{\alpha < \mu : d'_{0\alpha} \leq x'_\alpha, y'_\alpha\} \in F$, so $[d'_0] < [x'], [y']$. Conversely, suppose that $[d'_0] < [x'], [y']$. Then $M \stackrel{\text{def}}{=} \{\alpha < \mu : d'_{0\alpha} < x'_\alpha, y'_\alpha\} \in F$. Then $c'_{0\alpha}(x'_\alpha, y'_\alpha) = x'_\alpha$. It follows that $([x'], [y']) \in \mathrm{dmn}(f_{c_0})$ and $f_{c_0}([x'], [y']) = G(d_0, d_0)$. This proves (1). (2) and (3) hold vacuously.

Now suppose that c_β has been defined. By Theorem 32.19 let $g_\beta : X \to X$ be internal such that $g_\beta(d_\eta) = G(d_\beta, d_\eta)$ for all $\eta < \kappa$, and let $h_\beta : X \to X$ be internal such that $h_\beta(x) = G(d_\eta, d_\beta)$ for all $\eta < \kappa$. Then there is a system $\langle g'_\beta : \beta < \mu\rangle$ such that each $g'_\beta : X_\beta \to X_\beta$ and

$$\forall x, y \in \prod_{\alpha<\mu} X_\alpha[g_\beta([x]) = [y] \quad \text{iff} \quad \{\alpha < \mu : g'_\beta(x_\beta) = y_\beta\} \in F].$$

Similarly, there is a system $\langle h'_\beta : \beta < \mu\rangle$ such that each $h'_\beta : X_\beta \to X_\beta$ and

$$\forall x, y \in \prod_{\alpha<\mu} X_\alpha[h_\beta([x]) = [y] \quad \text{iff} \quad \{\alpha < \mu : h'_\beta(x_\beta) = y_\beta\} \in F].$$

Now we define, for $x, y \in \prod_{\alpha<\mu} X_\alpha$,

$$c'_{\beta+1,\alpha}(x_\alpha, y_\alpha) = \begin{cases} c'_{\beta\alpha}(x_\alpha, y_\alpha) & \text{if } x_\alpha, y_\alpha > d'_{\beta\alpha}, \\ g'_{\beta\alpha}(y) & \text{if } d'_{\beta+1,\alpha} < x_\alpha \leq d'_{\beta\alpha} \text{ and } y_\alpha > d'_{\beta+1,\alpha}, \\ h'_{\beta\alpha}(x) & \text{if } x_\alpha > d'_{\beta+1,\alpha} \text{ and } d_{\beta+1,\alpha} < y_\alpha \leq d'_{\beta\alpha} \end{cases}$$

Then let $c_{\beta+1} = [c'_{\beta+1}]$. To check (1), first suppose that $([x'],[y']) \in \mathrm{dmn}(f_{\beta+1})$. Say $f_{\beta+1}([x'],[y']) = [z']$. Then $\{\alpha < \mu : c'_{\beta+1,\alpha}(x',y') = z'\} \in F$. Hence $\{\alpha < \mu : d'_{\beta+1,\alpha} < x'_\alpha, y'_\alpha\} \in F$. So $d_{\beta+1} < [x'],[y']$. Conversely, suppose that $d_{\beta+1} < [x'],[y']$. Then $M \stackrel{\mathrm{def}}{=} \{\alpha < \mu : d_{\beta+1,\alpha} < x'_\alpha, y'_\alpha\} \in F$. For $\alpha \in M$ there is a z'_α such that $c'_{\beta+1,\alpha}(x'_\alpha, y'_\alpha) = z'_\alpha$. Hence $([x'],[y']) \in \mathrm{dmn}(c_{\beta+1})$. So (1) holds.

For (2), suppose that $\gamma, \delta < \beta + 1$.

Case 1. $\gamma, \delta < \beta$. Then $d_\beta < d_\gamma, d_\delta$, so $M \stackrel{\mathrm{def}}{=} \{\alpha < \mu : d'_{\beta\alpha} < d'_{\gamma\alpha}, d'_{\delta\alpha}\} \in F$. Now $f_{c_\beta}(d_\gamma, d_\delta) = G(d_\gamma, d_\delta)$, so $N \stackrel{\mathrm{def}}{=} \{\alpha < \mu : c'_{\beta\alpha}(d'_{\gamma\alpha}, d'_{\delta\alpha}) = (G'(d_\gamma, d_\delta))_\alpha\} \in F$. For $\alpha \in M \cap N$ we have $c'_{\beta+1,\alpha}(d'_{\gamma\alpha}, d'_{\delta\alpha}) = c'_{\beta\alpha}(d'_{\gamma\alpha}, d'_{\delta\alpha}) = (G'(d_\gamma, d_\delta))_\alpha$. Hence $f_{c_{\beta+1}}(d_\gamma, d_\delta) = G(d_\gamma, d_\delta)$.

Case 2. $\gamma = \beta$, $\beta > \delta$. Then

$$M \stackrel{\mathrm{def}}{=} \{\alpha < \mu : c'_{\beta+1,\alpha}(d'_{\beta\alpha}, d'_{\delta\alpha}) = g'_{\beta\alpha}(d'_{\delta\alpha}) = (G(d_\beta, d_\delta))_\alpha\} \in F.$$

Hence $f_{c_{\beta+1}}(d_\beta, d_\delta) = G(d_\beta, d_\delta)$.

Case 3. $\gamma \le \beta$, $\beta = \delta$. Then

$$M \stackrel{\mathrm{def}}{=} \{\alpha < \mu : c'_{\beta+1,\alpha}(d'_{\gamma\alpha}, d'_{\beta\alpha}) = h'_{\gamma\alpha}(d'_{\beta\alpha}) = (G(d_\gamma, d_\beta))_\alpha\} \in F.$$

Hence $f_{c_{\beta+1}}(d_\gamma, d_\beta) = G(d_\gamma, d_\beta)$.

Now for (3), suppose that $\gamma < \beta + 1$.

Case 1. $\gamma < \beta$. Then by the inductive hypothesis, $c_\gamma < c_\beta$. Hence it suffices to show (3) for $\gamma = \beta$.

Case 2. $\gamma = \beta$. Now for any $\alpha < \mu$, if $x, y \in \prod_{\gamma<\mu} X_\gamma$ and $x_\alpha, y_\alpha > d'_{\beta\alpha}$, then $c'_{\beta+1,\alpha}(x_\alpha, y_\alpha) = c'_{\beta\alpha}(x_\alpha, y_\alpha)$. Hence $f_{c_{\beta+1}})([x],[y]) = f_{c_\beta}([x],[y])$.

Now suppose that β is limit and $c_\gamma = [c'_\gamma]$ has been defined for all $\gamma < \beta$. Since $\kappa < t(\mu, F)$, there is an $r = [r'] \in P$ such that $c_\gamma < r$ for all $\gamma < \beta$. Say $r = [r']$. For $[x'],[y'] > d_\beta$ let

$$c'_{\beta\alpha}(x'_\alpha, y'_\alpha) = \begin{cases} r'(x'_\alpha, y'_\alpha) & \text{if } (x'_\alpha, y'_\alpha) \in \mathrm{dmn}(r'), \\ x'_\alpha & \text{otherwise.} \end{cases}$$

For (1), first suppose that $([x'],[y']) \in \mathrm{dmn}(f_{c_\beta})$. Say $f_{c_\beta}([x'],[y']) = [z']$. Then $\{\alpha < \mu : c'_{\beta\alpha}(x',y') = z'\} \in F$. Hence $\{\alpha < \mu : d'_{\beta\alpha} < x'_\alpha, y'_\alpha\} \in F$. So $d_\beta < [x'],[y']$. Conversely, suppose that $d_\beta < [x'],[y']$. Then $M \stackrel{\mathrm{def}}{=} \{\alpha < \mu : d_{\beta\alpha} < x'_\alpha, y'_\alpha\} \in F$. For $\alpha \in M$ there is a z'_α such that $c'_{\beta\alpha}(x'_\alpha, y'_\alpha) = z'_\alpha$. Hence $([x'],[y']) \in \mathrm{dmn}(c_\beta)$. So (1) holds.

For (2), suppose that $\gamma, \delta < \beta$. Choose ε with $\gamma, \delta < \varepsilon < \beta$. Then

$$c'_{\beta\alpha}(d'_{\gamma\alpha}, d'_{\delta\alpha}) = r'(d'_{\gamma\alpha}, d'_{\delta\alpha}) = c'_\varepsilon(d'_{\gamma\alpha}, d'_{\delta\alpha}) = ((G(d_\gamma, d_\delta))_\alpha.$$

Hence $f_{c_\beta}(d_\gamma, d_\delta) = G(d_\gamma, d_\delta)$.

For (3), if $\gamma < \beta$ and $x'_\alpha, y'_\alpha > d'_\gamma$ then $c'_{\beta\alpha}(x'_\alpha, y'_\alpha) = r'(x'_\alpha, y'_\alpha) = c'_{\gamma\alpha}(x'_\alpha, y'_\alpha)$. So $f_{c_\alpha}([x'],[y']) = f_{c_\gamma}([x'],[y'])$.

This completes the construction. Since $\kappa < t(\mu, F)$, choose $e \in P$ such that $c_\alpha < e$ for all $\alpha < \kappa$. Now define

$$H'(x,y)_\beta = \begin{cases} c'_{\gamma\beta}(x,y) & \text{if } d_{\gamma\beta} \le x, y \text{ for some } \gamma < \kappa, \\ x & \text{otherwise.} \end{cases}$$

Clearly $H = [H']$ is as desired. □

Theorem 32.22. *If κ is an infinite cardinal, then there is a function $F : [\kappa^+]^2 \to \kappa$ such that for every cofinal $A \subseteq \kappa^+$ we have $|F[[A]^2]| = \kappa$.*

Proof. For each $\alpha < \kappa^+$ let $g_\alpha : \alpha \to \kappa$ be an injection. Define $F(\{\alpha, \beta\}) = g_\alpha(\beta)$ where $\beta < \alpha$. Suppose that $A \subseteq \kappa^+$ is cofinal. Let $\langle \alpha_\xi : \xi < \kappa^+ \rangle$ be the strictly increasing enumeration of A. Then for any $\xi < \kappa$ we have $F(\{\alpha_\xi, \alpha_\kappa\}) = g_{\alpha_\kappa}(\alpha_\xi)$, and so

$$\kappa = |\text{rng}(g_\kappa)| = |\{F[\{\alpha_\xi, \alpha_\kappa\}] : \xi < \kappa\}| \le |F[[A]^2]|$$ □

Lemma 32.23. *Assume that $\langle X_\alpha : \alpha < \mu \rangle$ is a system of nonempty sets. For each $\alpha < \mu$ let*

$$P_\alpha = \{f : f \text{ is a function and } \exists D \subseteq X_\alpha[\text{dmn}(f) = {}^2D \text{ and } \text{rng}(f) \subseteq X_\alpha]\}.$$

Let $X = \prod_{\alpha<\mu} X_\alpha$, $P = \prod_{\alpha<\mu} P_\alpha$, and

$$R_\alpha = \{(f, D, x, y, z) : f \in P_\alpha,\ D \subseteq X_\alpha,\ \text{dmn}(f) = {}^2D,\ x, y \in \text{dmn}(f),\ f(x,y) = z\}.$$

We consider the three-sorted structure $(X_\alpha, P_\alpha, \mathscr{P}(X_\alpha), R_\alpha)$. For each $[p] \in P$ let $f_{[p]}$ be the function such that

$$\text{dmn}(f_{[p]}) = \{([x],[y]) \in X \times X : \{\alpha < \mu : \exists D \subseteq X_\alpha \exists z \in X_\alpha[(p_\alpha, D, x_\alpha, y_\alpha, z) \in R_\alpha]\} \in F\}$$

and

$$\forall([x],[y]) \in \text{dmn}(f_{[p]}) \forall [z] \in X[f_{[p]}([x],[y]) = [z] \quad \textit{iff} \quad \{\alpha < \mu : p_\alpha(x_\alpha, y_\alpha) = z_\alpha\} \in F.$$

Assume that $\xi < \kappa$, $w \in X$, and

(a) $u \in {}^\xi X$ is strictly decreasing.

(b) $H : {}^2\xi \to \{x \in X : x > w\}$.

(c) $\overline{p} \in P$ is such that $f_{\overline{p}}(u_\alpha, u_\beta) = H(\alpha, \beta)$ for all $\alpha, \beta \in \xi$ such that $(u_\alpha, u_\beta) \in \text{dmn}(f_{\overline{p}})$.

Then there is a $p \in P$ such that:

(d) $\forall \alpha, \beta \in \xi[(u_\alpha, u_\beta) \in \text{dmn}(f_p)$ and $f_p(u_\alpha, u_\beta) = H(\alpha, \beta)]$.

(e) $f_p(x,y) > w$ for all x, y such that $(x,y) \in \text{dmn}(f_p)$.

(f) If $(x,y) \in \text{dmn}(f_p) \cap \text{dmn}(f_{\overline{p}})$, then $f_p(x,y) = f_{\overline{p}}(x,y)$.

Proof. Define $G : {}^2(\text{rng}(u)) \to X$ by setting, for any $\varphi, \psi < \xi$, $G(u_\varphi, u_\psi) = H(\varphi, \psi)$. Then we apply Theorem 32.20 to get an internal $\rho_1 : {}^2X \to X$ such that for all $\varphi, \psi < \xi$, $\rho_1(u_\varphi, u_\psi) = G(u_\varphi, u_\psi) = H(\varphi, \psi)$. Now for any $x, y \in X$ let

$$\rho_2(x,y) = \begin{cases} f_{\overline{p}}(x,y) & \text{if } (x,y) \in \text{dmn}(f_{\overline{\rho}}), \\ \rho_1(x,y) & \text{otherwise.} \end{cases}$$

Note that ρ_2 is internal. In fact, since ρ_1 is internal there is a system $\langle l_\alpha : \alpha < \mu\rangle$ such that $\forall\alpha < \mu[l_\alpha : {}^2X_\alpha \to X_\alpha]$ and for any $[x], [y], [z] \in X$,

$$\rho_1([x], [y]) = [z] \quad \text{iff} \quad \{\alpha < \mu : l_\alpha(x_\alpha, y_\alpha) = z_\alpha\} \in F.$$

Now for any $\alpha < \mu$ and $x, y \in X_\alpha$, with $\overline{p} = [p']$ define

$$a_\alpha(x, y) = \begin{cases} p'_\alpha(x, y) & \text{if } \exists d[(p'_\alpha, d, x, y, p'_\alpha(x.y)) \in R_\alpha \\ l_\alpha(x, y) & \text{otherwise.} \end{cases}$$

Then clearly for all $[x], [y], [z] \in P$,]

$$\rho_2([x], [y]) = [z] \quad \text{iff} \quad \{\alpha < \mu : a_\alpha(x_\alpha, y_\alpha) = z_\zeta\} \in F.$$

This shows that ρ_2 is internal.

For each $\beta < \xi$ let $Z_\beta = \{x \in X : \rho_2(x, u_\beta) > w\}$. Then Z_β is internal since ρ_2 is internal. In fact, say $w = [w']$. For each $\alpha < \mu$ let $B_\alpha = \bigcup_{t > w'_\alpha}\{x \in X_\alpha : a_\alpha(x, u'_{\beta\alpha}) = t\}$. Then by Łoś's theorem,

$$[x] \in \prod_{\alpha<\mu} B_\alpha/F \quad \text{iff} \quad \overline{A} \models \exists t > w[\rho_2([x], u_\beta) = t] \quad \text{iff} \quad [x] \in Z_\beta.$$

Now let $U = \{u_\beta : \beta < \xi\}$. Then $U \subseteq Z_\beta$ for each $\beta < \xi$ by (c) and the definition of ρ_2. Now by Theorem 32.18 there is an internal Y such that $U \subseteq Y \subseteq \bigcap_{\beta<\xi} Z_\beta$. Define

$$Y^* = Y\backslash\{y \in Y : \exists y' \in Y[\rho_2(y', y) \le w]\}.$$

Then Y^* is internal. In fact, since Y is internal, there is a system $\langle C_\alpha : \alpha < \mu\rangle$ such that each $C_\alpha \subseteq X_\alpha$ and $[x] \in Y$ iff $\{\alpha < \mu : x_\alpha \in C_\alpha\} \in F$. For each $\alpha < \mu$ let $D_\alpha = \{x \in C_\alpha : \forall y, z \in X_\alpha[a_\alpha(x, y) = z \to z > w'_\alpha\}$. Then for any $[x] \in X$,

$$\{\alpha < \mu : x_\alpha \in D_\alpha\} \in F \quad \text{iff} \quad \overline{A} \models \forall y, z \in X[\rho_2(x, y) = z \to z > w] \quad \text{iff} \quad [x] \in Y^*.$$

$(*)$ $U \subseteq Y^*$.

For, let $\beta < \xi$. Then $u_\beta \in Y$. Suppose that $y' \in Y$. Then $y' \in Z_\beta$, so $\rho_2(y', u_\beta) > w$. It follows that $u_\beta \in Y^*$. So $(*)$ holds.

Clearly $\rho_2(x, y) > w$ for all $x, y \in Y^*$. Let $p = \rho_2 \restriction {}^2(Y^*)$. Then p is internal, i.e., $p \in P$. For, if $\alpha < \mu$ let $\text{dmn}(a') = {}^2D_\alpha$ and $a'(x, y) = a(x, y)$ for any $x, y \in D_\alpha$. Then

$$\begin{aligned}\{\alpha < \mu : a'(x'_\alpha, y'_\alpha) = z'_\alpha\} = \{\alpha < \mu : x'_\alpha \in D_\alpha\} &\cap \{\alpha < \mu : y'_\alpha \in D_\alpha\} \\ &\cap \{\alpha < \mu : a(x'_\alpha, y'_\alpha) = z'_\alpha\}\end{aligned}$$

and so

$$\begin{aligned}\{\alpha < \mu : a'(x'_\alpha, y'_\alpha) = z'_\alpha\} \in F \quad \text{iff} \quad & \{\alpha < \mu : x'_\alpha \in D_\alpha\} \in F \text{ and } \{\alpha < \mu : y'_\alpha \in D_\alpha\} \in F \\ & \text{and } \{\alpha < \mu : a(x'_\alpha, y'_\alpha) = z'_\alpha\} \in F \\ \text{iff} \quad & [x], [y] \in Y^* \text{ and } \rho_2([x], [y]) = [z] \\ \text{iff} \quad & p([x], [y]) = [z].\end{aligned}$$

Clearly (d)–(f) hold. $\square$

Theorem 32.24. $\mathfrak{p}(\mu, F) = \mathfrak{t}(\mu, F)$.

Proof. By Corollary 32.12 we have $\mathfrak{p}(\mu, F) \leq \mathfrak{t}(\mu, F)$, so we just need to show that $\mathfrak{t}(\mu, F) \leq \mathfrak{p}(\mu, F)$. Let $\langle X_\alpha : \alpha < \mu\rangle$ be a system of finite linear orders such that $X \stackrel{\text{def}}{=} \prod_{\alpha<\mu} X_\alpha/F$ has a (κ, θ)-gap with $\theta \leq \kappa = \mathfrak{p}(\mu, F)$. If $\theta = \kappa$, then $\mathfrak{t}(\mu, F) = \mathfrak{p}(\mu, F)$ by Proposition 32.9. Now suppose that $\theta < \kappa = \mathfrak{p}(\mu, F) < \mathfrak{t}(\mu, F)$; we want to get a contradiction. Let (x^1, x^0) be a (κ, θ)-gap in X. If $x \in X_\alpha$ let $X_\alpha \restriction x = \{x' \in X_\alpha : x' \leq x\}$. Define $P_\alpha = \{f : f$ is a function and $\exists D \subseteq X_\alpha[\text{dmn}(f) = {}^2D]$ and $\text{rng}(f) \subseteq X_\alpha\}$. Set

$$R_\alpha = \{(f, D, x, y, z) : f \in P_\alpha,\ D \subseteq X_\alpha,\ \text{dmn}(f) = {}^2D,\ x, y \in \text{dmn}(f),\ f(x, y) = z\}.$$

Let Q_α be the set of all functions ψ such that

(1) $\text{dmn}(\psi) = X_\alpha \restriction x$ for some $x \in X_\alpha$.

(2) $\text{rng}(\psi) \subseteq X_\alpha \times P_\alpha$.

(3) $\forall z \in \text{dmn}(\psi)\forall(a, b) \in \text{dmn}(2^{nd}(\psi(z))[2^{nd}(\psi(z)))(a, b) \geq 1^{st}(\psi(z))]$.

(4) $\forall z, z'[z \leq z' \in \text{dmn}(\psi) \to 1^{st}(\psi(z)) \leq 1^{st}(\psi(z'))$.

(5) $\forall z, z', a, b[z \leq z' \in \text{dmn}(\psi) \wedge \forall w[z \leq w \leq z' \to (a, b) \in \text{dmn}(2^{nd}(\psi(w)))] \to$
$\forall w[z \leq w \leq z' \to (2^{nd}(\psi(z)))(a, b) = (2^{nd}(\psi(w)))(a, b) = (2^{nd}(\chi(z')))(a, b)]]$.

$(*)$ Q_α is a finite tree with a unique minimum element.

For, we can take the unique root to be $\emptyset$. Now suppose that $\chi, \varphi, \psi \in Q_\alpha$ and $\chi, \varphi \leq \psi$. Say $\text{dmn}(\chi) = X_\alpha \restriction x'$, $\text{dmn}(\varphi) = X_\alpha \restriction x''$, and $\text{dmn}(\psi) = X_\alpha \restriction x$. Then $x', x'' \leq x$; say $x' \leq x''$. For any $y \leq x'$, $\chi(y) = \psi(y)$ and $\varphi(y) = \psi(y)$; so $\chi(y) = \varphi(y)$. Thus $\chi \leq \varphi$. So $(*)$ holds.

For each $\alpha < \mu$ let

$$\begin{aligned} T_\alpha = \{&(\psi, x, y, a, p, u, v, w) : \psi \in Q_\alpha,\ \text{dmn}(\psi) = X_\alpha \restriction x,\\ &y \in X_\alpha,\ y \leq x,\ \psi(y) = (a, p),\ (u, v) \in \text{dmn}(p), p(u, v) = w\}.\end{aligned}$$

We consider the 4-sorted structure $(X_\alpha, P_\alpha, \mathscr{P}(P_\alpha), Q_\alpha, R_\alpha, T_\alpha)$. Thus

$$\begin{aligned} &R_\alpha \subseteq P_\alpha \times \mathscr{P}(X_\alpha) \times X_\alpha \times X_\alpha \times X_\alpha,\\ &T_\alpha \subseteq Q_\alpha \times X_\alpha \times X_\alpha \times P_\alpha \times X_\alpha \times X_\alpha \times X_\alpha.\end{aligned}$$

Let $P = \prod_{\alpha<\mu} P_\alpha/F$, $V = \prod_{\alpha<\mu} \mathscr{P}(X_\alpha)/F$, $Q = \prod_{\alpha<\mu} Q_\alpha/F$,

$$\begin{aligned} R =&\{([f], [D], [x], [y], [z]) : \{\alpha < \mu : (f_\alpha, D_\alpha, x_\alpha, y_\alpha, z_\alpha) \in R_\alpha\} \in F\},\\ T =&\{([\psi], [x], [y], [a], [p], [u], [v], [w]) :\\ &\quad \{\alpha < \mu : (\psi_\alpha, x_\alpha, y_\alpha, a_\alpha, p_\alpha, u_\alpha.v_\alpha, w_\alpha) \in T_\alpha\} \in F\},\\ \overline{A} =&(X, P, V, Q, R, T).\end{aligned}$$

Applying Łoś's theorem to the structure $\overline{A}$, looking in particular at P, we see that for each $p \in P$ there is a function f_p whose domain is ${}^2D$ for some $D \subseteq X$ and whose range is a subset of X. We denote D by D_p. Looking at T, we see that for each $\psi \in Q$ there is a function g_ψ with the following properties:

(6) $\mathrm{dmn}(g_\psi) = X \upharpoonright x$ for some $x \in X$.

(7) $\mathrm{rng}(g_\psi) \subseteq X \times P$.

(8) $\forall z \leq x$ let $g_\psi(z) = (y, p)$. Then $\forall (a, b) \in \mathrm{dmn}(f_p)[f_p(a, b) \geq y]$.

(9) $\forall z, z' \in X[z \leq z' \in \mathrm{dmn}(g_\psi) \to 1^{st}(g_\psi(z)) \leq 1^{st}(g_\psi(z'))]$.

(10) $\forall z, z', a, b \in X[\forall w[z \leq w \leq z' \in \mathrm{dmn}(g_\psi) \to (a, b) \in \mathrm{dmn}(2^{nd}(g_\psi(w))] \to \forall w[z \leq w \leq z' \in \mathrm{dmn}(g_\psi) \to f_{2^{nd}(g_\psi(z))}(a, b) = f_{2^{nd}(g_\psi(w))}(a, b) \ = f_{2^{nd}(g_\psi(z'))}(a, b)]$.

For any $\psi \in Q$ let r_ψ be the maximum element of $\mathrm{dmn}(g_\psi)$. For any $\psi \in Q$ and $z \in \mathrm{dmn}(g_\psi)$ let $D_\psi(z) = D_{2^{nd}(g_\psi(z))}$. Let $D_\psi = D_\psi(r_\psi)$. Further, let $g_\psi(z) = (\psi^1(z), \psi^2(z))$ and $g_\psi(r_\psi) = (\psi^1, \psi^2)$.

By Theorem 32.21, let $G_0 : [\theta^+]^2 \to \theta$ be such that for every cofinal $A \subseteq \theta^+$ we have $|G_0[[A]^2]| = \theta$. For $\alpha, \beta \in \kappa$ with $\alpha \neq \beta$ define

$$G(\{\alpha, \beta\}) = \begin{cases} G_0(\{\alpha, \beta\}) & \text{if } \alpha, \beta < \theta^+, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly

(11) If $\psi, \chi \in Q$ and $\psi \leq \chi$, then
  (a) $r_\psi \leq r_\chi$.
  (b) $\forall z \in \mathrm{dmn}(g_\psi)[\psi(z) = \chi(z)]$.

Now we construct $c \in {}^\kappa Q$, $y \in {}^\kappa X$, and $\mu \in {}^\kappa\kappa$ by recursion so that the following conditions hold for all $\beta < \kappa$:

(12) y_β is not near to 0.

(13) If $\gamma < \beta$, then $c_\gamma < c_\beta$ and $y_\beta < y_\gamma$.

(14) $\forall \gamma \leq \beta[y_\gamma \in D_{c_\beta}]$.

(15) For all $\gamma, \delta \leq \beta[f_{c^2_\beta}(y_\gamma, y_\delta) = x^0_{G(\gamma,\delta)}]$.

(16) $\forall \gamma < \beta[\mu(\gamma) < \mu(\beta)$ and $c^1_\beta = x^1_{\mu(\beta)} + 1$.

(17) $\forall z < r_{c_\beta}[1^{st}(g_{c_\beta}(z)) \leq 1^{st}(g_{c_\beta}(r_{c_\beta}))]$.

Let $y_0 \in X$ be such that y_0 is not near to 0, and let $c_0 = \{(0, (x^1_0 + 1, \{((y_0, y_0), x^0_{G(0,0)})\})\}$. Clearly $c_0 \in Q$ and (12)–(17) hold.

Now suppose that $c_\gamma \in Q$ has been defined for all $\gamma \leq \beta$ satisfying (12)–(17). Let $y_{\beta+1} = y_\beta - 1$. We apply Lemma 32.22 with ξ replaced by $\beta+2$, u replaced by $\langle y_\gamma : \gamma \leq \beta + 1\rangle$, H given by $H(\gamma, \delta) = x^0_{G(\gamma,\delta)}$ for all $\gamma, \delta \leq \beta+1$, w replaced by $x^1_{\beta+1}$, with $\overline{\rho}$ replaced by

c^2_β. Note that if $\gamma,\delta\in\beta+2$ and $(y_\gamma,y_\delta)\in\mathrm{dmn}(f_{c^2_\beta})$, then $f_{c^2_\beta}(y_\gamma,y_\delta)=x^0_{G(\gamma,\delta)}=H(\gamma,\delta)$. So Lemma 32.22 gives a function ρ satisfying

(18) $\forall\gamma,\delta\in\beta+2[(y_\gamma,y_\delta)\in\mathrm{dmn}(f_\rho)$ and $f_\rho(y_\gamma,y_\delta)=H(\gamma,\delta)]$.

(19) $f_\rho(x,y)>x^1_{\beta+1}$ for all x,y such that $(x,y)\in\mathrm{dmn}(f_\rho)$.

(20) If $(x,y)\in\mathrm{dmn}(f_\rho)\cap\mathrm{dmn}(f_{c^2_\beta})$, then $f_\rho(x,y)=f_{c^2_\beta}(x,y)$.

Let $\rho'=\rho\restriction{}^2\{w:y_{\beta+1}\le w\}$, $\mu(\beta+1)=\mu(\beta)+1$ and $c_{\beta+1}=c_\beta\cup\{(r_{c_\beta}+1,(x^1_{\mu(\beta+1)}+1,\rho'))\}$. Now we check (6)–(10) and (12)–(17) for $\beta+1$. (6) and (7) are clear. For (8), we have $g_{c_{\beta+1}}(r_{c_\beta+1})=(x^1_{\beta+1}+1,\rho')$. If $(a,b)\in\mathrm{dmn}(f_{\rho'})$, then $y_{\beta+1}\le a,b$ and $f_{\rho'}(a,b)=f_\rho(a,b)>x^1_{\beta+1}$ by (19); so (8) holds. If $z\le r_{c_\beta}$, then by (17) and (16),

$$1^{st}(g_{c_{\beta+1}}(z))=1^{st}(g_{c_\beta}(z))\le 1^{st}(g_{c_\beta}(r_{c_\beta}))=c^1_\beta=x^1_\beta+1<x^1_{\beta+1}+1=1^{st}(g_{c_{\beta+1}}(r_{\beta+1}).$$

This proves (9). For (10), suppose that $z\le r_{c_{\beta+1}}$, $a,b\in X$, and $\forall w[z\le w\le r_{c_{\beta+1}}\to(a,b)\in\mathrm{dmn}(2^{nd}(g_{c_{\beta+1}}(w))]$. Then $\forall w[z\le w\le r_{c_\beta}\to(a,b)\in\mathrm{dmn}(2^{nd}(g_{c_{\beta+1}}(w))=\mathrm{dmn}(2^{nd}(g_{c_\beta}(w))]$, and so (10) for β gives

$$\forall w[z\le w\le r_{c_\beta}\to f_{2^{nd}(g_{c_\beta}(z))}(a,b)=f_{2^{nd}(g_{c_\beta}(w))}(a,b)=f_{2^{nd}(g_{c_\beta}(r_{c_\beta}))}(a,b)]$$

and hence

$(\star)$ $\forall w[z\le w\le r_{c_\beta}\to f_{2^{nd}(g_{c_{\beta+1}}(z))}(a,b)=f_{2^{nd}(g_{c_{\beta+1}}(w))}(a,b)=f_{2^{nd}(g_{c_{\beta+1}}(r_{c_{\beta+1}}))}(a,b)]$

Now if $z\le r_{c_\beta}$, then our assumption for proving (10) implies that

$$(a,b)\in\mathrm{dmn}(2^{nd}(g_{c_{\beta+1}}(r_{c_\beta}))\cap\mathrm{dmn}(2^{nd}(g_{c_{\beta+1}}(r_{c_{\beta+1}}))],$$

and hence by (20), $f_{2^{nd}(g_{c_\beta}(r_{c_\beta}))}(a,b)=f_{2^{nd}(g_{c_{\beta+1}}(r_{c_{\beta+1}}))}(a,b)$. Together with $(\star)$ this gives (10) for $\beta+1$.

(12) and (13) are clear. (14) follows from (18). For (15), suppose that $\gamma,\delta\le\beta+1$. Then by (18), $f_{c^2_{\beta+1}}(y_\gamma,y_\delta)=H(\gamma,\delta)=x^0_{G(\gamma,\delta)}$. (16) is clear. (17) follows from (16) for β and $\beta+1$. Thus we have checked (6)–(10) and (12)–(17) for $\beta+1$.

Now suppose that β is limit $<\kappa$. Then $\beta<\mathfrak{p}(\mu,F)$, so there is an $e\in Q$ such that $\forall\gamma<\beta[c_\gamma<e]$. For each $\gamma<\beta$ we have $r_{c_\gamma}\in\mathrm{dmn}(g_{c_\gamma})\cap\mathrm{dmn}(g_e)$, and by (11)(b), $c_\gamma(r_{c_\gamma})=e(r_{c_\gamma})$. Hence $D_{c_\gamma}=D_{c_\gamma}(r_{c_\gamma})=D_e(r_{c_\gamma})$. By (14), $y_\gamma\in D_{c_\gamma}=D_e(r_{c_\gamma})$. For each $\gamma<\beta$ let

$$d_\gamma=\max\{z\in\mathrm{dmn}(e):y_\gamma\in D_e(z)]\}.$$

(21) $\forall\gamma,\delta<\beta[r_{c_\gamma}<d_\delta]$.

In fact, suppose that $\delta<\beta$. Suppose that $\delta<\varepsilon<\beta$. By (14), $y_\delta\in D_{c_\varepsilon}$. Thus $y_\delta\in D_{c_\varepsilon}(r_{c_\varepsilon})=D_e(r_{c_\varepsilon})$. Hence $r_{c_\varepsilon}\le d_\delta$. This is true for all $\varepsilon\in(\delta,\beta)$. So (21) holds.

Now $\{r_{c_\gamma} : \gamma < \beta\}$ has cofinality less than κ and also $\{d_\gamma : \gamma < \beta\}$ has coinitiality less than κ. Hence the assumption that $\kappa = \mathfrak{p}(\mu, F)$ gives an element r_β of X such that $\forall\gamma < \kappa[r_{c_\gamma} < r_\beta < d_\gamma]$ Now let $\mathrm{dmn}(e') = \mathrm{dmn}(e) \cap (X \restriction r_\beta)$; $e' = e \restriction \mathrm{dmn}(e')$.

Now since $\beta < \mathfrak{p}(\mu, F)$, there is a y_β less than each y_γ for $\gamma < \beta$, with y_β not near 0. We apply Lemma 32.22 with ξ replaced by $\beta + 1$, u replaced by $\langle y_\gamma : \gamma \le \beta\rangle$, H given by $H(\gamma, \delta) = x^0_{G(\gamma,\delta)}$ for all $\gamma, \delta \le \beta$, w replaced by x^1_β, $\overline{\rho}$ replaced by $2^{nd}(e(r_\beta))$. Note that by (15) $\forall\gamma, \delta < \beta[f_{2^{nd}(e(r_\beta))}(y_\gamma, y_\delta) = x^0_{G(\gamma,\delta)}]$. So we get $p \in P$ such that

(22) $\forall\gamma, \delta \le \beta[(y_\gamma, y_\delta) \in \mathrm{dmn}(f_p)$ and $f_p(y_\gamma, y_\delta) = H(\gamma, \delta)]$.

(23) $f_p(x, y) > x^1_\beta$ for all x, y such that $(x, y) \in \mathrm{dmn}(f_p)$.

(24) If $(x, y) \in \mathrm{dmn}(f_p) \cap \mathrm{dmn}(f_{2^{nd}(e(r_\beta))})$, then $f_p(x, y) = f_{2^{nd}(e(r_\beta))}(x, y)$.

Let $\rho' = p \restriction {}^2\{s : r_\beta \le s\}$. Since $\beta < \kappa$ and κ is regular, let $\mu(\beta) = \sup\{\mu(\gamma) : \gamma < \beta\}$ Let $c_\beta = e' \cup \{(r_\beta, (x^1_{\mu(\beta)} + 1, \rho'))\}$. Now we check (6)–(10) and (12)–(17). (6) and (7) are clear. For (8), suppose that $z \le r_\beta$. If $z \in \mathrm{dmn}(c')$ the conclusion is clear. Suppose that $z = r_\beta$. Then $g_{c_\beta}(r_\beta) = (x^1_\beta + 1, \rho')$, and $\forall(a, b) \in \mathrm{dmn}(f_{\rho'})[f_{\rho'}(a, b) \ge x^1_\beta + 1$ by (23). So (8) holds. (9) is clear. (10) follows from (24). Clearly (12) and (13) hold. (14) and (15) follow from (22). (16) holds by definition. (17) is clear.

This finishes the construction of $\langle c_\beta : \beta < \kappa\rangle$.

Suppose that $\langle c_\beta : \beta < \kappa\rangle$ is bounded; say $c_\beta < s$ for all $\beta < \kappa$. For each $\eta < \theta^+$ we have by (14) $y_\eta \in D_{c_\eta} = D_{c_\eta}(r_{c_\eta}) = D_s(r_{s_\eta})$, so we can let z_η be the maximum element of

$$H_\eta \stackrel{\text{def}}{=} \{z \in \mathrm{dmn}(g_c) : \forall z'[r_{c_\eta} \le z' \le z \to y_\eta \in D_s(z')]\}.$$

(25) $r_{c_\beta} \le z_\eta$ for all $\beta < \kappa$.

For, suppose that $\beta < \kappa$. Wlog $\eta < \beta$. Suppose that $r_{c_\eta} \le z' \le r_{c_\beta}$. By (14), $y_\eta \in D_{c_\beta}(z') = D_c(z')$, so by the definition of H_η we have $r_{c_\beta} \le z_\eta$, and (25) holds.

By (16) and (25), for each $\eta < \theta^+$, $c^1(z_\eta) \ge c^1(r_{c_\beta}) = c^1_\beta > x^1_\beta$ for all $\beta < \kappa$. So there is a $K(\eta) < \theta$ such that $x^0_{K(\eta)} < c^1_{z_\eta}$. Let $A \in [\theta^+]^{\theta^+}$ and $\gamma \in \theta$ be such that $\forall\eta \in A[K(\eta) = \gamma]$. Choose $\zeta, \eta \in A$ such that $G(\eta, \zeta) > \gamma$. Let $z^* = \min(z_\eta, z_\zeta)$. So $r_{c_\eta}, r_{c_\zeta} \le z^* \le z_\eta, z_\zeta$, so $[y_\eta, y_\zeta\} \subseteq D_c(z^*)$. Hence with $\mu = \max(\eta, \zeta + 1)$,

$$\begin{aligned}
&(g_c(z^*))(y_\eta, y_\zeta) = (g_c(d_{c_\mu}))(y_\eta, y_\zeta) = c^2_\mu(y_\eta, y_\zeta) = x^0_{G(\eta,\zeta)} < x^0_\gamma;\\
&(g_c(z^*))(y_\eta, y_\zeta) \ge c^1(z^*) > x^0_{K(\eta)} = x^0_\gamma,
\end{aligned}$$

contradiction.

Since $\langle c_\beta : \beta < \kappa\rangle$ is unbounded, it follows that $\mathfrak{t}(\mu, F) \le \kappa < \mathfrak{t}(\mu, F)$, contradiction. $\square$

Theorem 32.25.

$$\begin{aligned}
\mathfrak{p} = \min\{\kappa : \exists A \in {}^\kappa([\omega]^\omega)[\forall\xi, \eta < \kappa\exists\rho < \kappa[A_\rho \subseteq A_\xi \cap A_\eta]\\
\wedge \neg\exists C \in [\omega]^\omega\forall\xi < \kappa[|C\backslash A_\xi| < \omega]]\}.
\end{aligned}$$

Proof. Clearly $\leq$ holds. Now suppose that $\mathscr{A} \subseteq [\omega]^\omega$, $|\mathscr{A}| = \mathfrak{p}$, $\forall F \in [\mathscr{A}]^{<\omega}[\bigcap F$ is infinite], and there is no $C \in [\omega]^\omega$ such that $C \backslash A$ is finite for all $A \in \mathscr{A}$. Let $\mathscr{B} = \{\bigcap F : F \in \mathscr{A}^{<\omega}\}$, and let $A \in {}^{\mathfrak{p}}\mathscr{B}$ be a bijection. Clearly A satisfies the conditions of the theorem. ☐

Theorem 32.26. *Assume that $\mathfrak{p} < \mathfrak{t}$. Suppose that $A \in {}^{\mathfrak{p}}([\omega]^\omega)$ is as in Theorem 32.24 with κ replaced by $\mathfrak{p}$. Then there exist an uncountable regular $\kappa < \mathfrak{p}$ and a $B \in {}^{\kappa}([\omega]^\omega)$ such that:*

(i) $\forall \xi < \mathfrak{p} \forall \alpha < \kappa[B_\alpha \cap A_\xi$ is infinite].

(ii) $\forall \alpha, \beta < \kappa[\beta \leq \alpha \rightarrow B_\alpha \backslash B_\beta$ is finite].

(iii) $\neg \exists C \in [\omega]^\omega[\forall \alpha < \kappa[|C \backslash B_\alpha| < \omega]$ and $\forall \xi < \mathfrak{p}[C \cap A_\xi$ is infinite]].

Proof. We define ζ and $B' \in {}^{\zeta}([\omega]^\omega)$ by recursion so that

$$\forall \xi < \zeta \forall \alpha < \mathfrak{p}[B'_\xi \cap A_\alpha \text{ is infinite}]. \tag{1}$$

$B'_0 = \omega$. Obviously (1) holds. Now assume that B'_ξ has been defined so that (1) holds. Let $B'_{\xi+1} = B'_\xi \cap A_\xi$. Thus $B'_{\xi+1}$ is infinite, by (1) for ξ. Suppose that $\eta < \mathfrak{p}$. By the condition in Theorem 32.24 there is a $\rho < \mathfrak{p}$ such that $A_\rho \subseteq A_\xi \cap A_\eta$. By (1) for ξ, $B' \cap A_\rho$ is infinite; so $B' \cap A_\xi \cap A_\eta$ is infinite. So (1) holds for $\xi + 1$. For ξ limit we consider two cases.

Case 1. There is an infinite $C \subseteq \omega$ such that $\forall \eta < \xi[C \backslash B'_\eta$ is finite] and $\forall \gamma < \mathfrak{p}[C \cap A_\gamma$ is infinite]. Then we let B'_ξ be such a C. Clearly (1) holds for ξ.

Case 2. Otherwise let $\zeta = \xi$ and stop.

This finishes the construction. Clearly ζ is not a successor ordinal.

(2) $\zeta \leq \mathfrak{p}$.

In fact, suppose not. Then $\forall \eta < \mathfrak{p}[B'_\mathfrak{p} \backslash B'_\eta$ is finite]. Now for any $\xi < \mathfrak{p}$ we have $B'_{\xi+1} \subseteq A_\xi$, so $B'_\mathfrak{p} \backslash A_\xi \subseteq B'_\mathfrak{p} \backslash B'_{\xi+1}$; so $B'_\mathfrak{p} \backslash A_\xi$ is finite. This contradicts the hypothesis on A.

(3) If $\xi < \eta < \zeta$, then $B'_\eta \backslash B'_\xi$ is finite.

In fact, this is clear if η is limit, or if $\eta = \xi + m$ for some $m \in \omega$. So suppose that $\eta = \omega \cdot \alpha + m$ with $m \in \omega \backslash \{0\}$ and $\xi < \omega \cdot \alpha$. Then $B'_\eta \backslash B'_\xi \subseteq B'_{\omega \cdot \alpha} \backslash B'_\xi$, and the latter is finite. So (3) holds.

(4) $\zeta < \mathfrak{p}$.

For, suppose that $\zeta = \mathfrak{p}$. Since $\mathfrak{p} < \mathfrak{t}$, there is a $C \in [\omega]^\omega$ such that $\forall \xi < \zeta[C \subseteq^* B'_\xi]$. If $\xi < \mathfrak{p}$, then $C \subseteq^* B'_{\xi+1} \subseteq A_\xi$. This contradicts Theorem 32.24. So (4) holds.

Let $\kappa = \text{cf}(\zeta)$.

(5) $\omega < \kappa$.

For, assume that $\omega = \kappa$. Let $\langle \rho_i : i < \omega \rangle$ be strictly increasing with supremum ζ. For each $n \in \omega$ let $\hat{B}_n = (\bigcap_{i \leq n} B'_{\rho_i}) \backslash n$. Then $\forall m, n \in \omega[m \leq n \rightarrow \hat{B}_n \subseteq \hat{B}_m]$. Clearly

$\bigcap_{n\in\omega} \hat{B}_n = \emptyset$. For any $n \in \omega$,

$$B'_{\rho_n} \triangle \hat{B}_n = \left(B'_{\rho_n} \backslash \left(\left(\left(\bigcap_{i\leq n} B'_{\rho_i}\right) \backslash n\right)\right)\right) \cup \left(\left(\left(\left(\bigcap_{i\leq n} B'_{\rho_i}\right) \backslash n\right) \backslash B'_{\rho_n}\right)\right)$$
$$= \bigcup_{i\leq n} (B'_{\rho_n} \backslash B'_{\rho_i}) \cup (B'_{\rho_n} \cap n)$$

and this last set is finite. Thus

(6) $\forall n \in \omega[B'_n \triangle \hat{B}_n$ is finite].

From (1) and (6) it follows that $\forall n \in \omega \forall \xi < \mathfrak{p}[\hat{B}_n \cap A_\xi$ is infinite]. Now for each $\xi < \mathfrak{p}$ define $f_\xi \in {}^\omega\omega$ by setting $f_\xi(n) = \min(\hat{B}_n \cap A_\xi)$. Now $\mathfrak{p} < \mathfrak{t} \leq \mathfrak{b}$, so there is an $f \in {}^\omega\omega$ such that $\forall \xi < \mathfrak{p}[f_\xi \leq^* f]$. Let $C = \bigcup_{n\in\omega}((f(n) + 1) \cap \hat{B}_n)$.

(7) $\forall \xi < \mathfrak{p}[C \cap A_\xi$ is infinite].

In fact, let $\xi < \mathfrak{p}$. Choose $m \in \omega$ so that $\forall n \geq m[f_\xi(n) \leq f(n)]$. Then for all $n \geq m$, $\min(\hat{B}_n \cap A_\xi) = f_\xi(n) \leq f(n)$ and so $\min(\hat{B}_n \cap A_\xi) \in C$. Since $\min(\hat{B}_n \cap A_\xi) > geqn$, this proves (7).

Now for all $n \in \omega$, $C \backslash \hat{B}_n \subseteq f(n) + 1$, so $C \cap \hat{B}_n$ is finite. Hence also $C \cap B'_n$ is finite, by (6). But this means that B'_ζ is defined, contradiction.

So (5) holds, and the theorem is proved. □

Theorem 32.27. *Suppose that $\zeta < \mathfrak{t}$ is an ordinal, $A \in {}^\zeta([\omega]^\omega)$, $B \in {}^\omega([\omega]^\omega)$, and:*
(i) $B_n \backslash B_m$ is finite if $m < n$.
(ii) $\forall \xi < \zeta \forall n \in \omega[A_\xi \cap B_n$ is infinite].
(iii) $\forall \xi, \eta < \zeta[\eta \leq \xi \to \exists n \in \omega[B_n \cap (A_\xi \backslash A_\eta)$ is finite]].

Then there is a $C \in [\omega]^\omega$ such that $\forall \xi < \zeta \forall n \in \omega[C \backslash A_\xi$ and $C \backslash B_n$ are finite].

Proof. For each $n \in \omega$ let $B'_n = (\bigcap_{m\leq n} B_m) \backslash n$. Then clearly

(1) $B'_n \subseteq B'_m$ if $m < n$.

(2) $\forall \xi < \zeta \forall n \in \omega[A_\xi \cap B'_n$ is infinite].

We prove (2) for fixed $\xi < \zeta$ by induction on n. It is clear for $n = 0$. Assuming it for n,

$$A_\xi \cap B'_{n+1} = A_\xi \cap \left(\left(\bigcap_{m\leq n+1} B_m\right) \backslash (n+1)\right)$$
$$= \left(A_\xi \cap \left(\left(\bigcap_{m\leq n} B_m\right) \backslash n\right)\right) \cup (A_\xi \cap (B_{n+1} \backslash (n+1))),$$

and the latter is infinite by the inductive hypothesis.

(3) $\forall \xi, \eta < \zeta[\eta \le \xi \to \exists n \in \omega[B'_n \cap (A_\xi \backslash A_\eta)$ is finite]].

This is clear.

For all $\xi < \zeta$ let $f_\xi \in {}^\omega\omega$ be strictly increasing such that $\forall n \in \omega[f_\xi(n) \in B'_n \cap A_\xi]$. Since $\zeta < \mathfrak{t} \le \mathfrak{b}$, there is an $f \in {}^\omega\omega$ such that $f_\xi <^* f$ for all $\xi < \zeta$. Define $B^* = \bigcup_{n\in\omega}(B'_n \cap f(n))$.

(4) $\forall \xi < \zeta[A_\xi \cap B^*$ is infinite].

In fact, let $\xi < \zeta$. Choose N so that $\forall n \ge N[f_\xi(n) < f(n)]$. Then for any $n \ge N$ we have $f_\xi(n) \in A_\xi \cap f(n)$. So (4) holds.

(5) $\forall \xi, \eta < \zeta[\eta \le \xi \to B^* \cap (A_\xi \backslash A_\eta)$ is finite].

In fact, for any $n \in \omega$ we have $B^* \backslash B'_n \subseteq \bigcup_{i<n}(B'_i \cap f(i))$, so $B^* \backslash B'_n$ is finite. Now suppose that $\xi, \eta < \zeta$ and $\eta \le \xi$. By (3), choose $n \in \omega$ such that $B'_n \cap (A_\xi \backslash A_\eta)$ is finite. Then, in $\mathscr{P}(\omega)/\mathrm{fin}$ we have $[B^*] \le [B'_n] \le [\omega \backslash (A_\xi \backslash A_\eta)]$, and (5) follows.

Now by (5), $\langle B^* \cap A_\xi : \xi < \zeta\rangle$ is decreasing mod finite. Since $\zeta < t$, choose $C \in [\omega]^\omega$ such that $[C] \le [B^* \cap A_\xi]$ for all $\xi < \zeta$. Thus $\forall \xi < \zeta[C \backslash A_\xi$ is finite]. Also, for any $n \in \omega$, $B^* \backslash B'_n$ is finite. (See above, after (5).) Now $B^* \backslash B_n \subseteq B^* \backslash B'_n$, so $B^* \backslash B_n$ is finite. Now $C \backslash B_n \subseteq (C \backslash B^*) \cup (B^* \backslash B_n)$, so $C \backslash B_n$ is finite. $\square$

If σ and τ are finite subsets of ω, we write $\sigma \triangleleft \tau$ if σ is a proper initial segment of τ. Now we define:

$$\Sigma = \{S \in [[\omega]^{<\omega} \backslash \{\emptyset\}]^\omega : \forall \sigma, \tau \in S[\sigma \ne \tau \to \min(\sigma) \ne \min(\tau)]\}$$
$$S \prec_\Sigma S' \quad \text{iff} \quad S, S' \in \Sigma \text{ and } \exists F \in [S']^{<\omega} \forall \tau \in S' \backslash F \exists \sigma \in S[\sigma \triangleleft \tau].$$

Note that the $\sigma \in S$ asserted to exist here is unique, since $\sigma \triangleleft \tau$ implies that the first element of σ is the same as the first element of τ, and distinct members of S have different first members.

Proposition 32.28. *If $S \in \Sigma$ and $m \in \omega$, then $\{\sigma \in S : \min(\sigma) < m\}$ and $\{\sigma \in S : \max(\sigma) < m\}$ are finite.*

Proof. For, obviously $\{\sigma \in S : \min(\sigma) < m\}$ is finite. Since $\{\sigma \in S : \max(\sigma) < m\} \subseteq \{\sigma \in S : \min(\sigma) < m\}$, also $\{\sigma \in S : \max(\sigma) < m\}$ is finite. $\square$

Proposition 32.29. $\forall S, S' \in \Sigma[S \prec_\Sigma S'$ *iff* $\exists n \in \omega \forall \tau \in S'[\min(\tau) > n \to \exists \sigma \in S[\sigma \triangleleft \tau]]$.

Proof. $\Rightarrow$: Assume that $S \prec_\Sigma S'$, and choose $F \in [S']^{<\omega}$ correspondingly. Let n be greater than each $\min \sigma)$ for $\sigma \in F$. Suppose that $\tau \in S'$ and $\min(\tau) > n$. Then $\tau \notin F$, so $\exists \sigma \in S[\sigma \triangleleft \tau]$.

$\Leftarrow$: Assume the indicated condition, and choose n correspondingly. Let $F = \{\sigma \in S' : \min(\sigma) \le n\}$. So F is a finite subset of S'. If $\tau \in S' \backslash F$, then $\min(\sigma) > n$, and hence $\exists \sigma \in S[\sigma \triangleleft \tau]$. $\square$

Theorem 32.30. *$\prec_\Sigma$ is transitive.*

Proof. Suppose that $S \prec S' \prec S''$. Choose n_0, n_1 such that

$$\forall \tau \in S'[\min(\tau) > n_0 \rightarrow \exists \sigma \in S[\sigma \triangleleft \tau]];$$
$$\forall \nu \in S''[\min(\nu) > n_1 \rightarrow \exists \tau \in S'[\tau \triangleleft \nu]].$$

Let

$$m = \max\{n_1, \sup\{\max(\tau) : \tau \in S',\ \min(\tau) \leq n_0\}.$$

Now suppose that $\nu \in S''$ and $\min(\nu) > m$. Since $\min(\nu) > n_1$, it follows that there is a $\tau \in S'$ such that $\tau \triangleleft \nu$. If $\min(\tau) \leq n_0$, then $\min(\nu) > m \geq \max(\tau)$, contradiction. So $\min(\tau) > n_0$ and so there is a $\sigma \in S$ such that $\sigma \triangleleft \tau$, hence $\sigma \triangleleft \nu$. □

Theorem 32.31. *Suppose that $\zeta < \mathfrak{t}$ is an ordinal, $S \in {}^{\zeta}\Sigma$, and $\forall \eta, \xi < \zeta[\eta < \xi \rightarrow S_\eta \prec S_\xi]$. Then there is a $T \in \Sigma$ such that $S_\xi \prec T$ for all $\xi < \zeta$.*

Proof. For each $n \in \omega$ let $B_n = \{\sigma \in [\omega]^{<\omega} \backslash \{\emptyset\} : \min(\sigma) \geq n\}$. For each $\xi < \zeta$ let

$$A_\xi = \{\tau \in [\omega]^{<\omega} \backslash \{\emptyset\} : \exists \sigma \in S_\xi[\sigma \triangleleft \tau]\}$$

Suppose that $\eta < \xi < \zeta$. Since $S_\eta \prec S_\xi$, choose n_0 so that $\forall \tau \in S_\xi[\min(\tau) \geq n_0 \rightarrow \exists \sigma \in S_\eta[\sigma \triangleleft \tau]]$. Thus $S_\xi \cap B_{n_0} \subseteq A_\eta$. Choose n such that $\max(\sigma) < n$ whenever $\sigma \in S_\xi \backslash B_{n_0}$.

(1) $A_\xi \cap B_n \subseteq A_\eta$.

In fact, suppose that $\tau \in A_\xi \cap B_n$. So $\min(\tau) \geq n$. Choose $\sigma \in S_\xi$ such that $\sigma \triangleleft \tau$. Then $\min(\sigma) \geq n$, so by the choice of n, $\sigma \in B_{n_0}$. So $\min(\sigma) \geq n_0$. Hence $\sigma \in S_\xi \cap B_{n_0} \subseteq A_\eta$, so there is a $\rho \in S_\eta$ such that $\rho \triangleleft \sigma$. Thus $\rho \triangleleft \tau$. This shows that $\tau \in A_\eta$, proving (1).

Now obviously each A_ξ is infinite, so also $A_\xi \cap B_n$ is infinite. Clearly $B_m \subseteq B_n$ if $n < m$. Let $f : \omega \rightarrow [\omega]^{<\omega}$ be a bijection, $A'_\xi = \{m \in \omega : f(m) \in A_\xi\}$ for all $\xi < \zeta$, and $B'_n = \{m \in \omega : f(m) \in B_n\}$. Then the hypotheses of Theorem 32.26 hold for A'_ξ and B'_n. So let $C \in ([\omega]^{<\omega})^\omega$ be such that $\forall \xi < \zeta \forall n \in \omega[C \backslash B_n$ and $C \backslash A_\xi$ are finite].

Now suppose that $\xi < \zeta$; we show that $S_\xi \prec C$. Choose $n \in \omega$ such that $\forall \sigma \in C[\min(\sigma) \geq n \rightarrow \sigma \in A_\xi]$. Now take any $\tau \in C$ such that $\min(\tau) \geq n$. Then $\tau \in A_\xi$, so there is a $\sigma \in S_\xi$ such that $\sigma \triangleleft \tau$. □

Theorem 32.32. *Suppose that $\mathfrak{p} < \mathfrak{t}$, $A \in {}^{\mathfrak{p}}([\omega]^\omega)$ as in Theorem 32.24, and κ and $B \in {}^{\kappa}([\omega]^\omega)$ as in Theorem 32.25. Then there is a function S with domain $\mathfrak{p}+1$ such that:*

(i) $\forall \xi \leq \mathfrak{p}[S_\xi \in \Sigma]$.

(ii) $\forall \xi < \mathfrak{p} \forall \alpha < \kappa[S_\xi \backslash [B_\alpha]^{<\omega}$ *is finite*].

(iii) $\forall \eta, \xi \leq \mathfrak{p}[\eta < \xi \rightarrow S_\eta \prec S_\xi]$.

(iv) $\forall \xi < \mathfrak{p} \forall \sigma \in S_{\xi+1}[\max(\sigma) \in A_\xi]$.

Proof. We define S_ξ for $\xi \leq \mathfrak{p}$ by recursion.

$\xi = 0$: Since $\kappa < \mathfrak{p} < \mathfrak{t}$ and $\forall \alpha, \beta < \kappa[\alpha < \beta \rightarrow B_\beta \backslash B_\alpha$ is finite], there is an infinite $C \subseteq \omega$ such that $C \backslash B_\alpha$ is finite for all $\alpha < \kappa$. Let $S_0 = \{\{n\} : n \in C\}$. Clearly (i) holds for $\xi = 0$. Suppose that $\alpha < \kappa$. Then $C \backslash B_\alpha$ is finite, so there is an N such that $\forall n \geq N[n \in C \rightarrow n \in B_\alpha]$. Hence $\forall x \in S_0 \backslash \{\{n\} : n < N\}[x \in [B_\alpha]^{<\omega}]$. So $S_0 \backslash [B_\alpha]^{<\omega}$ is finite. So (ii) holds. (iii) and (iv) hold vacuously for $\xi = 0$.

ξ to $\xi+1$: If $\alpha < \beta < \kappa$ then $B_\beta \cap A_\xi \backslash (B_\alpha \cap A_\xi)$ is finite; and by (i) of Theorem 32.25, $B_\alpha \cap A_\xi$ is infinite for all $\alpha < \kappa$. Hence since $\kappa < \mathfrak{p} < t$, there is an infinite $C \subseteq A_\xi$ such that $C \backslash B_\alpha$ is finite for all $\alpha < \kappa$. For each $\sigma \in S_\xi$ there is a $\sigma' \in [\omega]^{<\omega} \backslash \{\emptyset\}$ such that $\sigma \triangleleft \sigma' \subseteq \sigma \cup C$; namely one can take any $n \in C$ with $n > \sigma$ (possible since σ is finite and C is infinite), and let $\sigma' = \sigma \cup \{n\}$. Let $S_{\xi+1} = \{\sigma' : \sigma \in S_\xi\}$. Then (i) is clear for $\xi+1$. Concerning (ii), suppose that $\alpha < \kappa$. Choose N so that $\forall n \geq N[n \in C \rightarrow n \in B_\alpha]$, and choose M so that $\forall \sigma \in S_\xi[\min(\sigma) \geq M \rightarrow \sigma \in [B_\alpha]^{<\omega}]$. Suppose that $\tau \in S_{\xi+1}$ and $\min(\tau) \geq M, N$. Say $\tau = \sigma'$ with $\sigma \in S_\xi$. Since $\min(\sigma) \geq M$ it follows that $\sigma \in [B_\alpha]^{<\omega}$. Now $\sigma' \backslash \sigma \subseteq C$ and $\min(\sigma' \backslash \sigma) \geq N$, so $\sigma' \backslash \sigma \subseteq B_\alpha$. Hence $\tau \in [B_\alpha]^{<\omega}$. This proves (ii). Clearly $S_\xi \prec S_{\xi+1}$. Finally, (iv) holds since for $\sigma \in S_{\xi+1}$ we have $\max(\sigma) \in C \subseteq A_\xi$.

ξ limit, $\xi < \mathfrak{p}$: Let $\zeta \max(\xi, \kappa)$ and

$$\begin{aligned}
&\mathbb{I} = \{I \in [[\omega]^{<\omega} \backslash \{\emptyset\}]^{<\omega} : \forall \sigma, \tau \in I[\sigma \neq \tau \rightarrow \min(\sigma) \neq \min(\tau)]\};\\
&\mathbb{P} = \mathbb{I} \times [\zeta]^{<\omega};\\
(I, J) \leq (I', J') \quad \text{iff} \quad &(I, J), (I', J') \in \mathbb{P},\ I \subseteq I',\ J \subseteq J',\ \text{and } \forall \sigma \in I' \backslash I:\\
&\quad (1)\ \forall \eta \in J \cap \xi \exists \tau \in S_\eta[\tau \triangleleft \sigma];\\
&\quad (2)\ \forall \alpha \in J \cap \kappa[\sigma \subseteq B_\alpha].
\end{aligned}$$

Clearly $\leq$ is reflexive on $\mathbb{P}$. Suppose that $(I, J) \leq (I', J') \leq (I'', J'')$ and $\sigma \in I'' \backslash I$.

Case 1. $\sigma \notin I'$. Then (1) and (2) hold for J', hence also for J.

Case 2. $\sigma \in I'$. Clearly (1) and (2) hold for J.

It follows that $\leq$ is transitive. Clearly $\leq$ is antisymmetric. So $\leq$ is a partial order on $\mathbb{P}$. Also, $\mathbb{P}$ is σ-centered upwards. For, $\mathbb{P} = \bigcup_{I_0 \in \mathbb{I}} \{(I, J) \in \mathbb{P} : I = I_0\}$, and for any $I_0 \in \mathbb{I}$ the set $\{(I, J) \in \mathbb{P} : I = I_0\}$ is centered: if $(I_0, J_0), \ldots, (I_0, J_m) \in \mathbb{P}$, then for each $k \leq m$, $(I_0, J_k) \leq (I_0, J_0 \cup \ldots \cup J_m)$.

(3) $\forall \eta < \zeta[Q'_\eta \stackrel{\text{def}}{=} \{(I, J) \in \mathbb{P} : \eta \in J\}$ is cofinal in $\mathbb{P}]$.

For, given $\eta < \zeta$ and $(I, J) \in \mathbb{P}$ we have $(I, J) \leq (I, J \cup \{\eta\})$.

(4) $\forall k \in \omega[Q_k \stackrel{\text{def}}{=} \{(I, J) \in \mathbb{P} : \exists \sigma \in I[\sigma \not\subseteq k]\}$ is cofinal in $\mathbb{P}]$.

To prove (4), let $k \in \omega$ and $(I, J) \in \mathbb{P}$. By (3) we may assume that $0 \in J$. Let $\eta^* = \max(J \cap \xi)$ and $B^* = \bigcap_{\alpha \in J \cap \kappa} B_\alpha$.

(5) $\bigcup S_{\eta^*} \backslash B^*$ is finite.

For, let α be the maximum element of $J \cap \kappa$. Then by Theorem 32.25(ii), $B_\alpha \backslash B^*$ is finite. Choose $m \in \omega$ such that $\forall p \geq m[p \in B_\alpha \rightarrow p \in B^*]$. Also by (ii), there is an $n \in \omega$ such that $\forall \sigma \in S_{\eta^*}[\max(\sigma) \geq n \rightarrow \sigma \in [B_\alpha]^{<\omega}]$. Take any $p \geq m, n$ and suppose that $l \in \bigcup S_{\eta^*}$ with $l \geq p$. Say $l \in \sigma \in S_{\eta^*}$. Then $\max(\sigma) \geq l \geq p \geq n$, so $\sigma \in [B_\alpha]^{<\omega}$. Hence $l \in B_\alpha$ and $l \geq m$, so $l \in B^*$. This proves (5).

By (5) there is a $k' \geq k$ such that $\bigcup S_{\eta^*} \backslash B^* \subseteq k'$.

(6) $\exists \sigma \in S_{\eta^*}[k' \leq \min(\sigma)$ and $\forall \eta \in J \cap \eta^* \exists \tau \in S_\eta[\tau \triangleleft \sigma]]$.

In fact, $S_\eta \prec S_{\eta^*}$ for all $\eta \in J \cap \eta^*$, so for every $\eta \in J \cap \eta^*$ there is an $s_\eta \in \omega$ such that $\forall \tau \in S_{\eta^*}[\min(\tau) \geq s_\eta \rightarrow \exists \sigma \in S_\eta[\sigma \triangleleft \tau]]$. Let $t = \max\{s_\eta : \eta \in J \cap \eta^*\}$. Let $\sigma \in S_{\eta^*}$ be such that $\min(\sigma) \geq k', t$. Suppose that $\eta \in J \cap \eta^*$. Then there is a $\tau \in S_\eta$ such that $\tau \triangleleft \sigma$. So (6) holds.

Take σ as in (6). Since $\sigma \in S_{\eta^*}$ and $k' \leq \min(\sigma)$, it follows that $\sigma \subseteq B^*$. Since B^* is infinite, choose $m \in B^*$ such that $m > \max(\sigma)$. Let $\sigma' = \sigma \cup \{m\}$. Now $m > \max(\sigma) \geq \min(\sigma) \geq k' \geq k$, so $(I, J) \leq (I \cup \{\sigma'\}, J) \in Q_k$. This proves (4).

The following two statements are clear:

(7) If $\eta < \xi$, $(I, J) \in Q'_\eta$, and $(I, J) \leq (I', J')$, then $(I', J') \in Q'_\eta$.

(8) If $k \in \omega$, $(I, J) \in Q_k$, and $(I, J) \leq (I', J')$, then $(I', J') \in Q_k$.

Now we apply $\mathfrak{m}_\sigma = \mathfrak{p}$ to get $R \subseteq \mathbb{P}$ such that R is upwards directed and R intersects every Q'_η, $\eta < \zeta$ and every Q_k, $k \in \omega$. Let $T = \bigcup\{I : (I, J) \in R\}$.

(9) T is infinite.

For, suppose that $\sigma_i \in T$ for each $i < m$. Say $\sigma_i \in I_i$ with $(I_i, J_i) \in R$, for each $i < m$. Let k be larger than each member of $\bigcup_{i<m} \sigma_i$, and choose $(I', J') \in R \cap Q_k$. Choose $\tau \in I'$ such that $\tau \not\subseteq k$. Then $\tau \neq \sigma_i$ for all $i < m$. So (9) holds.

(10) $\forall \eta < \xi[S_\eta \prec T]$.

For, suppose that $\eta < \xi$. Choose $(I, J) \in R \cap Q'_\eta$. Thus $\eta \in J$. Let $k \in \omega$ be greater than $\max(\sigma)$ for all $\sigma \in I$. Suppose that $\tau \in T$ and $\min(\tau) \geq k$. Then $\tau \notin I$. Say $\tau \in I'$ with $(I', J') \in R$. Choose $(I'', J'') \in R$ such that $(I, J), (I', J') \leq (I'', J'')$. Then $\tau \in I''$ since $I' \subseteq I''$. Since $(I, J) \leq (I'', J'')$ it follows that there is a $\sigma \in S_\eta$ such that $\sigma \triangleleft \tau$. This proves (10).

(11) $\forall \alpha < \kappa[T \backslash [B_\alpha]^{<\omega}$ is finite].

For, let $\alpha < \kappa$. Choose $(I, J) \in R \cap Q'_\alpha$. Let k be greater than $\max(\sigma)$ for all $\sigma \in I$. Suppose that $\sigma \in T$ and $\min(\sigma) \geq k$. Say $\sigma \in I'$ with $(I', J') \in R$. So $\sigma \notin I$. Choose $(I'', J'') \in R$ such that $(I, J), (I', J') \leq (I'', J'')$. Hence $\alpha \in J''$. It follows that $\sigma \subseteq B_\alpha$, i.e., $\sigma \in [B_\alpha]^{<\omega}$. So (11) holds.

Let $S_\xi = T$. Clearly (i)–(iii) hold. (iv) holds vacuously.

$\xi = \mathfrak{p}$. Since $\mathfrak{p} < \mathfrak{t}$, we can apply Theorem 32.30 to get $S_\mathfrak{p}$ such that $S_\xi \prec S_\mathfrak{p}$ for all $\xi < \mathfrak{p}$. This completes the construction. □

A *strict partial order* is a pair $(P, <)$ such that $<$ is irreflexive and transitive. Generalizing the notion for linear orderings, for infinite regular cardinals κ, λ, and a strict partial order $(X, <)$, a (κ, λ)-*gap in* $(X, <)$ is a pair (a, b) with $a \in {}^\kappa X$ and $b \in {}^\lambda X$ such that:

(1) $\forall \alpha, \beta < \kappa \forall \gamma, \delta < \lambda[\alpha < \beta$ and $\gamma < \delta$ imply that $a_\alpha < \alpha_\beta < b_\delta < b_\gamma]$.

(2) There is no $x \in X$ such that $\forall \alpha < \kappa \forall \beta < \lambda[a_\alpha < x < b_\beta]$.

A gap $(a, b) \in ({}^\kappa X) \times ({}^\lambda X)$ is *linear* provided the following conditions hold:

$$\forall x \in X[\forall \xi < \kappa[a_\xi < x] \rightarrow \exists \eta < \lambda[b_\eta < x]];$$
$$\forall x \in X[\forall \xi < \lambda[x < b_\xi] \rightarrow \exists \eta < \kappa[x < a_\eta]].$$

The gap is then called a *linear* (κ, λ)-*gap*.

Note that $\mathscr{P}(\omega)/\text{fin}$ under $<$ is a strict partial order. So is ${}^{\omega}\omega$ under the relation $<^*$, where $f <^* g$ iff $\exists k \forall n \geq k[f(n) \leq g(n)]$ and $\{n : f(n) < g(n)\}$ is infinite.

Theorem 32.33. *If $\mathfrak{p} < \mathfrak{t}$, then there is a regular uncountable $\kappa < \mathfrak{p}$ such that there is a linear $(\mathfrak{p}, \kappa)$-gap in $(\mathscr{P}(\omega)/\text{fin}, <)$ and also one in $({}^{\omega}\omega, <^*)$.*

Proof. Let $A \in {}^{\mathfrak{p}}([\omega]^{\omega})$ be as in Theorem 32.24, κ and $B \in {}^{\kappa}([\omega]^{\omega})$ as in Theorem 32.25, $S \in {}^{\mathfrak{p}+1}\Sigma$ as in Theorem 32.31. Recall that $\mathfrak{p}$ is regular and uncountable. Let $A' = \{\min(\sigma) : \sigma \in S_{\mathfrak{p}}\}$, and for each $n \in A'$ let $\sigma_n \in S_{\mathfrak{p}}$ be such that $n = \min(\sigma_n)$. Clearly A' is infinite. For any $\alpha < \kappa$ and $n \in A'$ let

$$f_\alpha(n) = \begin{cases} \min(\sigma_n \backslash B_\alpha) & \text{if } \sigma_n \backslash B_\alpha \neq \emptyset, \\ 1 + \max(\sigma_n) & \text{otherwise.} \end{cases}$$

For each $\xi < \mathfrak{p}$ and each $n \in A'$ let

$$g_\xi(n) = \begin{cases} \max(\sigma) \text{ such that } \sigma \in S_{\xi+1} \text{ and } \sigma \lhd \sigma_n\} & \text{if there is such a } \sigma \\ 1 + \max(\sigma_n) & \text{otherwise.} \end{cases}$$

If $\alpha < \kappa$ and $n \in A'$, then $(\sigma_n \backslash B_\alpha) \cap f_\alpha(n) = \emptyset$, so

(1) For all $\alpha < \kappa$ and $n \in A'$ we have $\sigma_n \cap f_\alpha(n) \subseteq B_\alpha$.

(2) $\forall \xi < \mathfrak{p} \exists k \in \omega \forall n \in A' \backslash k[g_\xi(n) \in A_\xi]$.

In fact, $S_{\xi+1} \prec S_{\mathfrak{p}}$ by Theorem 32.31(iii). Hence there is a $k \in \omega$ such that for all $\tau \in S_{\mathfrak{p}}$, if $\min(\tau) \geq k$ then $\exists \mu \in S_{\xi+1}[\mu \lhd \tau]$. Suppose that $n \geq k$ and $n \in A'$. Then $\min(\sigma_n) = n \geq k$, so there is a $\tau \in S_{\xi+1}$ such that $\tau \lhd \sigma_n$. Hence the first clause in the definition of $g_\xi(n)$ applies, and we then get $g_\xi(n) \in A_\xi$ by Theorem 32.31(iv).

(3) $\forall n \in A'[n \leq g_\xi(n)]$.

In fact, let $n \in A'$. If the first clause in the definition of $g_\xi(n)$ holds, then $n = \min(\sigma_n) \leq \max(\sigma) = g_\xi(n)$, where $\sigma \lhd \sigma_n$ as in the definition. If the second clause holds, then $n = \min(\sigma_n) \leq 1 + \max(\sigma_n) = g_\xi(n)$. This proves (3).

(4) If $\alpha \leq \beta < \kappa$, then there is an $n_0 \in \omega$ such that $\forall n \in A' \backslash n_0[\sigma_n \cap (B_\beta \backslash B_\alpha) = \emptyset]$.

For, assume that $\alpha \leq \beta < \kappa$. Then $B_\beta \backslash B_\alpha$ is finite by Theorem 32.25(ii). Let $n_0 > (B_\beta \backslash B_\alpha)$. Suppose that $n \in A' \backslash n_0$. Then $\min(\sigma_n) = n \geq n_0$, so $\sigma_n \cap (B_\beta \backslash B_\alpha) = \emptyset$.

(5) If $\alpha \leq \beta < \kappa$, and with n_0 as in (4), we have $\forall n \in A' \backslash n_0[f_\beta(n) \leq f_\alpha(n)]$.

In fact, by (4) we have $\sigma_n \backslash B_\alpha \subseteq \sigma_n \backslash B_\beta$.

Case 1. $\sigma_n \backslash B_\alpha \neq \emptyset$. Then $f_\beta(n) \leq f_\alpha(n)$ by definition.

Case 2. $\sigma_n \backslash B_\alpha = \emptyset \neq \sigma_n \backslash B_\beta$. Then $f_\beta(n) = \min(\sigma_n \backslash B_\beta) \leq 1 + \max(\sigma_n) = f_\alpha(n)$.

Case 3. $\sigma_n \backslash B_\beta = \emptyset$. Clearly $f_\beta(n) = f_\alpha(n)$.

So (5) holds.

(6) $\forall \alpha < \kappa \exists \beta \in (\alpha, \kappa)[\{n \in A' : f_\beta(n) \geq f_\alpha(n)\}$ is finite].

Suppose not; so there is an $\alpha < \kappa$ such that for all $\beta \in (\alpha, \kappa)[\{n \in A' : f_\beta(n) \geq f_\alpha(n)\}$ is infinite]. For each $\beta < \kappa$ let $C_\beta = \{n \in A' : f_\beta(n) \geq f_\alpha(n)\}$.

(7) $\forall \beta < \kappa[C_\beta$ is infinite].

For, suppose that $\beta < \kappa$ and C_β is finite. Then by definition, $\beta \leq \alpha$. Choose $m \in \omega$ such that $\forall n \geq m[n \in A' \to f_\beta(n) < f_\alpha(n)]$. This contradicts (5).

(8) $\forall \beta, \gamma < \kappa[\beta < \gamma \to C_\gamma \subseteq^* C_\beta]$.

In fact, suppose that $\beta < \gamma < \kappa$. By (5) choose $k \in \omega$ so that $\forall n \in A' \backslash k[f_\gamma(n) \leq f_\beta(n)]$. Hence if $n \geq k$ and $n \in C_\gamma$ then $f_\alpha(n) \leq f_\gamma(n) \leq f_\beta(n)$. So (8) holds.

Since $\kappa < \mathfrak{t}$ (applied to A' rather than ω), let $D \in [A']^\omega$ be such that $\forall \beta \in \kappa[D \subseteq^* C_\beta]$. In particular, $D \subseteq^* C_0$, so $D \backslash C_0$ is finite. Now let

$$E = \bigcup_{n \in D} (\sigma_n \cap [n, f_\alpha(n))).$$

(9) $\forall \beta < \mathfrak{p} \exists k \in \omega \forall n \in A' \backslash k \exists \tau_n \in S_\beta[\tau_n \triangleleft \sigma_n$ and $\tau_n \subseteq B_\alpha]]$.

In fact, let $\beta < \mathfrak{p}$. Since $S_\beta \prec S_\mathfrak{p}$ by Theorem 32.31(iii), by Proposition 32.28 choose $k \in \omega$ such that $\forall n \in A'[k < n \to \exists \tau \in S_\beta[\tau \prec \sigma_n]]$. Also, by Theorem 32.31(ii) choose $l \in \omega$ so that $\forall \tau \in S_\beta[l < \min(\tau) \to \tau \subseteq B_\alpha]$. Then $\forall n > k, l[n \in A' \to \exists \tau_n \in S_\beta[\tau_n \triangleleft \sigma_n$ and $\tau_n \subseteq B_\alpha]]$.

(10) E is infinite.

For, take any $\beta < \mathfrak{p}$ and let k be as in (9). Take any $n \in D \backslash k$. Now if $\sigma_n \backslash B_\alpha \neq \emptyset$, then $\max(\tau_n) < \min(\sigma_n \backslash B_\alpha)$, hence $\tau_n \subseteq f_\alpha(n)$. If $\sigma_n \backslash B_\alpha = \emptyset$, then clearly $\tau_n \subseteq f_\alpha(n)$. Hence $\sigma_n \cap [n, f_\alpha(n)) \neq \emptyset$. Thus (10) holds.

(11) $\forall \beta \in (\alpha, \kappa)[D \subseteq^* \{n \in A' : f_\beta(n) = f_\alpha(n)\}]$.

In fact, suppose that $\alpha < \beta < \kappa$. By the choice of D we have $D \subseteq^* C_\beta$, and by (5) $\forall n \geq n_0[f_\beta(n) \leq f_\alpha(n)]$. Hence (11) holds.

(12) $\forall \beta \in \kappa[E \subseteq^* B_\beta]$.

For, suppose that $\beta \in \kappa$. Now by (1), $\forall n \in A'[\sigma_n \cap f_\alpha(n) \subseteq B_\alpha]$. Thus $E \subseteq B_\alpha$. If $\beta \leq \alpha$, then by Theorem 32.25(ii), $B_\alpha \subseteq^* B_\beta$; hence $E \subseteq^* B_\beta$. So we may assume that $\alpha < \beta$. By (11) choose $k \in \omega$ such that $\forall n \geq k[n \in D \to f_\beta(n) = f_\alpha(n)]$. Suppose that m is greater than $\max(\sigma_p)$ for all $p < k$, and $m \in E$. Say $p \in D$ and $m \in \sigma_p \cap [p, f_\alpha(p))$. Then $p \geq k$ and so $f_\alpha(p) = f_\beta(p)$. Now $\sigma_p \cap [p, f_\alpha(p)) = \sigma_p \cap [p, f_\beta(p)) \subseteq B_\beta$ by (1). So $m \in B_\beta$. This proves (12).

(13) $\forall \xi < \mathfrak{p}[E \cap A_\xi$ is infinite].

For, suppose that $\xi < \mathfrak{p}$. By (9) choose $k \in \omega$ such that $\forall n \in A' \backslash k \exists \tau_n \in S_{\xi+1}[\tau_n \triangleleft \sigma_n$ and $\tau_n \subseteq B_\alpha]$. Take any $n \in D \backslash k$. Then $\tau_n \in S_{\xi+1}$, so $\max(\tau_n) \in A_\xi$ by Theorem 32.31(iv). Also $\max(\tau_n) \in \sigma_n$. Since $\tau_n \subseteq f_\alpha(n)$, because $\tau_n \subseteq B_\alpha$, we have $\max(\tau_n) \in E$. So (13) holds.

Now (10), (12), (13) contradict Theorem 32.25(iii). Hence (6) holds.

(14) If $\xi < \eta < \mathfrak{p}$, then $\exists k \in \omega \forall n \in A'\backslash k[g_\xi(n) < g_\eta(n)]$.

For, assume that $\xi < \eta < \mathfrak{p}$. Then $S_{\xi+1} \prec S_{\eta+1} \prec S_\mathfrak{p}$, so by Proposition 32.28,

$$\exists k \in \omega \forall \sigma[\min(\sigma) \geq k \to \exists \tau \in S_{\eta+1}[\tau \triangleleft \sigma]];$$
$$\exists l \in \omega \forall \tau[\min(\tau) \geq l \to \exists \rho \in S_{\xi+1}[\rho \triangleleft \tau]].$$

Let $s = \max(k, l)$. Suppose that $n \geq s$ and $n \in A'$. Then $\min(\sigma_n) = n \geq k$, so there is a $\tau \in S_{\eta+1}$ such that $\tau \triangleleft \sigma_n$. Also, $\min(\tau) = \min(\sigma_n) = n \geq l$, so there is a $\rho \in S_{\xi+1}$ such that $\rho \triangleleft \tau$. Now $g_\xi(n) = \max(\rho) < \max(\tau) = g_\eta(n)$. This proves (14).

(15) $\forall \xi < \mathfrak{p} \forall \alpha < \kappa[g_\xi \leq^* f_\alpha]$.

For, suppose that $\xi < \mathfrak{p}$ and $\alpha < \kappa$. By (9), $\exists N \forall n \in A'\backslash N \exists \tau \in S_{\xi+1}[\tau \triangleleft \sigma_n$ and $\tau \subseteq B_\alpha]$. Thus for any $n \in A'\backslash N$,

$$\begin{aligned} g_\xi(n) = \max(\tau) \leq \min(\sigma_n\backslash B_\alpha) = f_\alpha(n) \quad &\text{if} \quad \sigma_n\backslash B_\alpha \neq \emptyset \\ \text{or } g_\xi(n) = \max(\tau) \leq 1 + \max(\sigma_n) = f_\alpha(n) \quad &\text{if} \quad \sigma_n\backslash B_\alpha = \emptyset, \end{aligned}$$

proving (15).

(16) Suppose that $f \in {}^{A'}\omega$ and $\forall \alpha < \kappa[f \leq^* f_\alpha]$. Then $\exists \xi < \mathfrak{p}[f \leq^* g_\xi]$.

For, suppose that $f \in {}^{A'}\omega$ and $\forall \alpha < \kappa[f \leq^* f_\alpha]$. For any $n \in A'$ let

$$f'(n) = \begin{cases} f(n) & \text{if } f(n) \leq 1 + \max(\sigma_n), \\ 1 + \max(\sigma_n) & \text{otherwise.} \end{cases}$$

(17) $\forall \alpha < \kappa[f' \leq^* f_\alpha]$.

For, choose $k \in \omega$ such that $\forall n \in A'\backslash k[f(n) \leq f_\alpha(n)]$. Take any $n \in A'\backslash k$.

Case 1. $f(n) \leq 1 + \max(\sigma_n)$. Then $f'(n) = f(n) \leq f_\alpha(n)$.

Case 2. $f(n) > 1 + \max(\sigma_n)$. Then $f'(n) = 1 + \max(\sigma_n) < f(n) \leq f_\alpha(n)$.

So (17) holds.

(18) If $\xi < \mathfrak{p}$ and $f' \leq^* g_\xi$, then $f \leq^* g_\xi$.

In fact, suppose that $\xi < \mathfrak{p}$ and $f' \leq^* g_\xi$. Choose k so that $\forall n \in A'\backslash k[f'(n) \leq g_\xi(n)]$. Since always $g_\xi(n) \leq 1 + \max(\sigma_n)$, it follows that $\forall n \in A'\backslash k[f(n) \leq g_\xi(n)]$. Thus $f \leq^* g_\xi$.

By (17) and (18) we may assume that $f(n) \leq 1 + \max(\sigma_n)$ for all $n \in A'$. Now let $D' = \bigcup_{n \in A'}(\sigma_n \cap f(n))$.

(19) $\forall \alpha < \kappa[D'\backslash B_\alpha$ is finite$]$.

In fact, fix $\alpha < \kappa$. Then $D'\backslash B_\alpha = \bigcup_{n \in A'}((\sigma_n \cap f(n))\backslash B_\alpha)$. Choose k so that $\forall n \in A'\backslash k[f(n) \leq f_\alpha(n)]$. Then if $n \in A'\backslash k$ and $\sigma_n\backslash B_\alpha \neq \emptyset$, then $f(n) \leq \min(\sigma_n\backslash B_\alpha)$, and so $(\sigma_n\backslash B_\alpha) \cap f(n) = \emptyset$. It follows that

$$D'\backslash B_\alpha = \bigcup\{\sigma_n \cap f(n) : n \in A' \text{ and } n < k\},$$

and so $D'\backslash B_\alpha$ is finite.

It now follows by Theorem 32.25(iii) that there is a $\xi < \mathfrak{p}$ such that $D' \cap A_\xi$ is finite. By (2) let k be such that $\forall n \in A'\backslash k[g_\xi(n) \in A_\xi]$. Let $l \in \omega$ be such that $\forall n \in A'\backslash l \exists \tau \in S_{\xi+1}[\tau \triangleleft \sigma_n]]$. Hence if $n \in A'$, $n \geq k, l$, and $g_\xi(n) < f(n)$, then $g_\xi(n) = \max(\tau) \in \sigma_n \cap f(n) \cap A_\xi$, i.e., $g_\xi(n) \in D' \cap A_\xi$. Since $D' \cap A_\xi$ is finite and $g_\xi(n) \geq n$ for all n by (3), it follows that $f \leq^* g_\xi$, proving (16).

(20) Suppose that $f \in {}^{A'}\omega$ and $\forall \xi < \mathfrak{p}[g_\xi \leq^* f]$. Then $\exists \alpha < \kappa[f_\alpha \leq^* f]$.

In fact, suppose that $f \in {}^{A'}\omega$ and $\forall \xi < \mathfrak{p}[g_\xi \leq^* f]$, but $\forall \alpha < \kappa[f_\alpha \not\leq^* f]$.

(21) There is an infinite $C \subseteq A'$ such that $\forall \alpha < \kappa[C\backslash\{n \in A' : f_\alpha(n) > f(n)\}$ is finite].

For, let $a_\alpha = \{n \in A' : f_\alpha(n) > f(n)\}$ for each $\alpha < \kappa$. Suppose that F is a finite nonempty subset of κ, and let β be the largest member of F. By (5), $\{n \in A' : f_\beta(n) > f_\alpha(n)\}$ is finite for all $\alpha \in F$, hence also $\bigcup_{\alpha \in F}\{n \in A' : f_\beta(n) > f_\alpha(n)\}$ is finite. Hence the first set in the following sequence is infinite:

$$\begin{aligned}&\{n \in A' : f_\beta(n) > f(n)\} \cap \bigcap_{\alpha \in F}\{n \in A' : f_\beta(n) \leq f_\alpha(n)\}\\ &\subseteq \{n \in A' : \forall \alpha \in F[f_\alpha(n) > f(n)]\}\\ &= \bigcap_{\alpha \in F} a_\alpha.\end{aligned}$$

Now since $\kappa < \mathfrak{p}$, the existence of C as in (21) follows.

Let $D = \bigcup_{n \in C}(\sigma_n \cap f(n))$.

(22) $\forall \alpha < \kappa[D\backslash B_\alpha$ is finite].

In fact, fix $\alpha < \kappa$. Then $D\backslash B_\alpha = \bigcup_{n \in C}((\sigma_n \cap f(n))\backslash B_\alpha)$. By the definition of C, choose k so that $\forall n \geq k[n \in C \to f(n) < f_\alpha(n)]$. Then if $n \geq k$, $n \in C$, and $\sigma_n\backslash B_\alpha \neq \emptyset$, then $f(n) \leq \min(\sigma_n\backslash B_\alpha)$, and so $(\sigma_n\backslash B_\alpha) \cap f(n) = \emptyset$. It follows that

$$D\backslash B_\alpha = \bigcup\{\sigma_n \cap f(n) : n \in C \text{ and } n < k\},$$

and so $D\backslash B_\alpha$ is finite.

Now by Theorem 32.25(iii) there is a $\xi < \mathfrak{p}$ such that $D \cap A_\xi$ is finite. By (2), let $k \in \omega$ be such that $\forall n \in A'\backslash k[g_\xi(n) \in A_\xi]$. Let $l \in \omega$ be such that $\forall n \in A'\backslash l \exists \tau \in S_{\xi+1}[\tau \triangleleft \sigma_n]$. Since $g_\xi <^* g_{\xi+1} \leq^* f$, choose s so that $\forall n \geq s[g_\xi(n) < f(n)]$. Hence if $n \in C$, $n \geq s, k, l$, $g_\xi(n) = \max(\tau) \in \sigma_n \cap f(n) \cap A_\xi$, i.e., $g_\xi(n) \in D \cap A_\xi$. Since C is infinite and $g_\xi(n) \geq n$ for all $n \in C$, this contradicts $D \cap A_\xi$ being finite. So (20) holds.

Now by (6) and using $\mathfrak{p} < \mathfrak{t}$, there is a strictly increasing $\alpha \in {}^{\mathfrak{p}}\mathfrak{p}$ such that $\forall \gamma, \delta < \mathfrak{p}[\gamma < \delta \to f_{\alpha_\delta} <^* f_{\alpha_\gamma}]$. Let k be the strictly increasing enumeration of A'; thus k is a bijection from ω onto A'. Now by (14), (15), (16), and (20), $(\langle g_\xi \circ k : \xi < \mathfrak{p}\rangle, \langle f_{\alpha_\gamma} \circ k : \gamma < \kappa\rangle)$ is a linear $(\mathfrak{p}, \kappa)$-gap in $({}^\omega\omega, <^*)$.

Now let $M = A' \times \omega$ and define

$$U_\xi = \{(n,i) : n \in A',\ i \le g_\xi(n)\} \quad \text{for all } \xi < \mathfrak{p}$$
$$V_\gamma = \{(n,i) : n \in A',\ i \le f_{\alpha_\gamma}(n)\} \quad \text{for all } \gamma < \kappa.$$

(23) $(\langle [U_\xi] : \xi < \mathfrak{p}\rangle, \langle [V_\gamma] : \gamma < \kappa\rangle$ is a linear $(\mathfrak{p},\kappa)$-gap in $(\mathscr{P}(M)/\mathrm{fin}, <)$.

To prove (23), first note:

(24) $\forall \beta, \gamma < \kappa[\beta < \gamma \to [V_\gamma] < [V_\beta]]$.

In fact, suppose that $\beta < \gamma < \kappa$. Then $f_{\alpha_\gamma} <^* f_{\alpha_\beta}$. Hence there is a $k \in \omega$ such that $\forall n \in A'\backslash k[f_{\alpha_\gamma}(n) \le f_{\alpha_\beta}(n)]$. Hence $V_\gamma \backslash V_\beta = \{(n,i) : n \in A' \cap k, f_{\alpha_\beta}(n) < i \le f_{\alpha_\gamma}(n)\}$ is finite. So $[V_\gamma] \le [V_\beta]$.

Now because $f_{\alpha_\gamma} <^* f_{\alpha_\beta}$, the set $\{n \in A' : f_{\alpha_\gamma}(n) < f_{\alpha_\beta}(n)\}$ is infinite. Hence $V_\beta \backslash V_\gamma = \{(n,i) : n \in A' \cap k, f_{\alpha_\gamma}(n) < i \le f_{\alpha_\beta}(n)\}$ is infinite. So $[V_\gamma] < [V_\beta]$. So (24) holds.

(25) $\forall \xi, \eta < \mathfrak{p}[\xi < \eta \to [U_\xi] < [U_\eta]]$.

For, assume $\xi < \eta < \mathfrak{p}$. By (14) $\exists k \in \omega \forall n \in A'\backslash k[g_\xi(n) < g_\eta(n)]$. Hence $U_\xi \backslash U_\eta = \{(n,i) : n < k, g_\eta(n) < i \le g_\xi(n)\}$ is finite, so $[U_\xi] \le [U_\eta]$.

Now because $g_\xi <^* g_\eta$, the set $\{n \in A' : g_\xi(n) < g_\eta(n)\}$ is infinite. Hence $U_\eta \backslash U_\xi = \{(n,i) : n \in A' \cap k, g_\xi(n) < i \le g_\eta(n)\}$ is infinite. So $[U_\xi] < [U_\eta]$. This proves (25)

(26) $\forall \xi < \mathfrak{p} \forall \gamma < \kappa[[U_\xi] \le [V_\gamma]]$.

In fact, let $\xi < \mathfrak{p}$ and $\gamma < \kappa$. By (15) choose k such that $\forall n \in A'\backslash k[g_\xi(n) \le f_{\alpha_\gamma}(n)]$. Hence $U_\xi \backslash V_\gamma = \{(n,i) : n < k, f_{\alpha_\gamma}(n) < i \le g_\xi(n)\}$ is finite, and so $[U_\xi] \le [V_\gamma]$. So (26) holds.

Now suppose that $W \subseteq M$ and $[W] \le [V_\gamma]$ for all $\gamma < \kappa$.

(27) For all $n \in A'$, the set $\{i : (n,i) \in W\}$ is finite.

For, let $F \in [M]^{<\omega}$ be such that $\forall (m,i) \in M[(m,i) \notin F$ and $(m,i) \in W \to (m,i) \in V_0]$. Then

$$\{i : (n,i) \in W\} \subseteq \{i : \exists m[(m,i) \in F]\} \cup \{i : i \le f_{\alpha_0}(n)\}.$$

So (27) holds.

Now for each $n \in A'$ let $f(n) = \sup\{i : (n,i) \in W\}$, with $\sup \emptyset = 0$.

(28) $f \le^* f_{\alpha_\gamma}$ for all $\gamma < \kappa$.

In fact, with $\gamma < \kappa$ let $F \in [M]^{<\omega}$ be such that $\forall (n,i) \in M[(n,i) \notin F$ and $(n,i) \in W \to (n,i) \in V_\gamma]$. Let k be greater than all m such that $(m,i) \in F$ for some i. Suppose that $n \in A'$ and $n \ge k$. If $(n,i) \in W$, then $(n,i) \notin F$, hence $(n,i) \in V_\gamma$; it follows that $i \le f_{\alpha_\gamma}(n)$. Hence $f(n) \le f_{\alpha_\gamma}(n)$. This proves (28).

Now α is strictly increasing, so $\gamma \le \alpha_\gamma$ for all $\gamma < \kappa$. Hence by (28) we have $f \le^* f_\gamma$ for all $\gamma < \kappa$. Hence by (16) there is a $\xi < \mathfrak{p}$ such that $f \le^* g_\xi$. Say $k \in \omega$ and $\forall n \ge k[f(n) \le g_\xi(n)]$. Let $F = \{(m,i) : m < k$ and $(m,i) \in W\}$. Suppose that $(n,i) \in M\backslash F$ and $(n,i) \in W$. Then $n \ge k$, so $f(n) \le g_\xi(n)$. Also $i \le f(n)$, so $(n,i) \in U_\xi$. Thus we have shown:

(29) $[W] \leq [U_\xi]$.

Now suppose that $X \subseteq M$ and $[U_\xi] \leq [X]$ for all $\xi < \mathfrak{p}$. For each $n \in \omega$ let

$$f(n) = \begin{cases} \min\{i \in \omega : (n,i) \notin X\} & \text{if this set is nonempty,} \\ 2 + \max(\sigma_n) & \text{otherwise.} \end{cases}$$

Suppose that $\xi < \mathfrak{p}$. Then $U_\xi \backslash X$ is finite. So for $F = U_\xi \backslash X$ we have $\forall (n,i) \in M \backslash F[(n,i) \in U_\xi \rightarrow (n,i) \in X]$. Let k be greater than each $n \in A'$ such that $(n,i) \in F$ for some i. Suppose that $n \geq k$. Then $(n,i) \in U_\xi \backslash F$ for all $i \leq g_\xi(n)$, so $(n,i) \in X$ for all $i \leq g_\xi(n)$. Hence $f(n) > g_\xi(n)$. Thus $g_\xi \leq^* f$.

This is true for all $\xi < \mathfrak{p}$. By (20) there is a $\gamma < \kappa$ such that $f_\gamma <^* f$. Hence $f_{\alpha_\gamma} <^* f$. So there is a $k \in \omega$ such that $\forall n \geq k[f_{\alpha_\gamma}(n) < f(n)]$. Let $F = \{(m,i) : m < k, i \leq f_{\alpha_\gamma}(m)\}$. Suppose that $(n,i) \in M \backslash F$ and $(m,i) \in V_\gamma$. Then $i \leq f_{\alpha_\gamma}(n)$. It follows that $n \geq k$. Hence $f_{\alpha_\gamma}(n) < f(n)$. Hence $(n,i) \in X$. Thus $[V_\gamma] \leq [X]$.

This finishes the proof of (23). $\square$

Proposition 32.34. *The forcing order $([\omega]^\omega, \leq^*, \omega)$ is $\mathfrak{t}$-closed.* $\square$

For brevity let $\mathbb{P} = ([\omega]^\omega, \leq^*, \omega)$.

Proposition 32.35. *Suppose that G is M-generic over $\mathbb{P}$. Then G is an ultrafilter on ω.*

Proof. By the definition of generic filter, $G \subseteq [\omega]^\omega$. If $A, B \in G$ then there is a $C \in G$ such that $C \subseteq A, B$; hence $A \cap B \in G$. G is closed upwards since it is a filter on $\mathbb{P}$. Obviously $\emptyset \notin G$. Now suppose that $A \subseteq \omega$ in $M[G]$; we want to show that $A \in G$ or $(\omega \backslash A) \in G$. By Theorem 16.10 of setth or Theorem 29.9 of full, $A \in M$. Let $D = \{a \in P : a \leq^* A \text{ or } a \leq^* (\omega \backslash A)\}$. Then D is dense in P, since if $B \in [\omega]^\omega$ then $B \cap A$ or $B \backslash A$ is infinite. Take $a \in G \cap D$. Then $A \in G$ or $(\omega \backslash A) \in G$. $\square$

Proposition 32.36. *If $A, B \in [\omega]^\omega$ in M and G is M generic over $\mathbb{P}$, then $A \leq^* B$ iff $M[G] \models \check{A} \leq^* \check{B}$.*

Proof. We take $A \leq^* B$ to mean that there exist an $m \in \omega$ and a bijection f from m onto $A \backslash B$. If such f, g exist in M, then $m, f \in M[G]$ and so $A \leq^* B$ in $M[G]$. If they exist in $M[G]$, then $f, g \in M$ by Theorem 16.10 of setth or Theorem 29.9 of full.

Proposition 32.37. *If $A, B \in [\omega]^\omega$, and G is M generic over $\mathbb{P}$, then $A <^* B$ in M iff $M[G] \models \check{A} <^* \check{B}$.*

Proof. We take $A <^* B$ to mean that $A \leq^* B$ and there is an injection g from ω into $B \backslash A$. Hence the result follows by the above argument. $\square$

Proposition 32.38. *If G is M generic over $\mathbb{P}$, then $M[G] \models \mathfrak{t} \leq \mathfrak{t}^M$.*

Proof. Suppose in M that $\langle A_\alpha : \alpha < \mathfrak{t} \rangle$ is strictly decreasing under $\leq^*$, each $A_\alpha \in [\omega]^\omega$, such that there is no $B \in [\omega]^\omega$ such that $\forall \alpha < \mathfrak{t}[B \leq^* A_\alpha]$. Then in $M[G]$ the sequence $\langle A_\alpha : \alpha < \mathfrak{t} \rangle$ is strictly decreasing under $\leq^*$ by Proposition 32.36. Suppose in

$M[G]$ that $B \in [\omega]^\omega$ such that $\forall \alpha < \mathfrak{t}[B \leq^* A_\alpha]$. Then by Theorem 16.10 of setth or Theorem 29.9 of full, $B \in M$, contradiction. $\square$

Proposition 32.39. *If G is M generic over $\mathbb{P}$, then $M[G] \models \mathfrak{t} = \mathfrak{t}^M$.*

Proof. Suppose that $\kappa < \mathfrak{t}$ and $A \in {}^\kappa([\omega]^\omega) \in M[G]$ is strictly decreasing under $\leq^*$. Let $B : \kappa \times \omega \to 2$ be defined by

$$B(\xi, n) = \begin{cases} 1 & \text{if } n \in A_\xi, \\ 0 & \text{if } n \notin A_\xi. \end{cases}$$

Then $B \in M$ by Theorem 16.10. Hence $A \in M$. Let $C \in [\omega]^\omega$ be such that $C \leq^* A_\xi$ for all $\xi < \kappa$. Then this is true in $M[G]$ also by Proposition 32.36. Since κ is arbitrary, $M[G] \models \mathfrak{t}^M \leq \mathfrak{t}$. The other inequailty holds by Proposition 32.37. $\square$

Proposition 32.40. *If G is M generic over $\mathbb{P}$, then $M[G] \models \mathfrak{p} = \mathfrak{p}^M$.*

Proof. Let A be as in Theorem 32.24 (in M). Suppose that $B \in [\omega]^\omega$ in $M[G]$ and $B \leq^* A_\xi$ for all $\xi < \mathfrak{p}$. Then $B \in M$ by Theorem 16.10 of setth or Theorem 29.9 of full, contradiction. Hence $M[G] \models \mathfrak{p} \leq \mathfrak{p}^M$.

Now suppose that A is as in Theorem 32.24 (in $M[G]$) for any $\kappa < \mathfrak{p}$. Then $A \in M$ by the argument in the proof of Proposition 32.37. Hence there is a $C \in [\omega]^\omega$ in M such that $\forall \xi < \kappa[C \leq^* A_\xi]$. Since $C \in M[G]$, it follows that $\kappa < \mathfrak{p}$ in the sense of $M[G]$. Since κ is arbitrary, $\mathfrak{p}^M \leq \mathfrak{p}$. $\square$

Theorem 32.41. *$M[G] \models \mathfrak{t} \leq \mathfrak{t}(\omega, G)$.*

Proof. Working in $M[G]$, suppose that $\kappa < \mathfrak{t}$, $\langle P_n : n \in \omega \rangle$ is a sequence of finite trees each with a single root, and $A \in {}^\kappa(\prod_{n\in\omega} P_n)$ is such that $\langle [A_\xi] : \xi < \kappa \rangle$ is strictly increasing and unbounded. Wlog $\forall n \in \omega[P_n \subseteq \omega]$ and $\forall m, n \in \omega[m \neq n \to P_n \cap P_m = \emptyset]$. Let $B : \kappa \times \omega \to \omega$ be defined by $B(\xi, n) = A_\xi(n)$. Then $B \in M$ by Theorem 16.10 of setth or Theorem 29.9 of full. Hence $A \in M$. This is a contradiction. $\square$

Theorem 32.42. *Suppose that $\mathfrak{p} < \mathfrak{t}$ and G is M-generic over $\mathbb{P}$. Then $M[G] \models \mathfrak{p}(\omega, G) \leq \mathfrak{p}$.*

Proof. By Theorem 32.32 let κ be an uncountable regular cardinal less than $\mathfrak{p}$ and (f, g) be a linear $(\mathfrak{p}, \kappa)$-gap in $({}^\omega\omega, <^*)$, in $M[G]$. Note that f_ξ and g_α are in M, by Theorem 16.10. Let $k \in \omega$ be such that $\forall n \geq k[g_1(n) < g_0(n)]$. Define

$$g'_0(n) = \begin{cases} 1 & \text{if } n < k, \\ g_0(n) & \text{if } n \geq k. \end{cases}$$

Thus $\forall n \in \omega[g'_0(n) > 0]$. Now for each $\alpha \in \kappa\backslash\{0\}$ let $l_\alpha \in \omega$ be such that $\forall n \geq l_\alpha[g_\alpha(n) < g_0(n)]$. Define

$$g'_\alpha(n) = \begin{cases} g_\alpha(n) & \text{if } n \geq l_\alpha, \\ 0 & \text{otherwise}. \end{cases}$$

Thus $\forall\alpha\in\kappa\backslash\{0\}\forall n\in\omega[g'_\alpha(n)<g_0(n)]$.

For each $\xi<\mathfrak{p}$ let $m_\xi\in\omega$ be such that $\forall n\ge m_\xi[f_\xi(n)<g_0(n)]$. Define

$$f'_\xi(n)=\begin{cases}f_\xi(n) & \text{if } n\ge m_\xi,\\ 0 & \text{otherwise.}\end{cases}$$

Thus $\forall\xi<\kappa\forall n\in\omega[f'_\xi(n)<g_0(n)]$.

Thus we may assume that $\forall n\in\omega[g_0(n)>0]$, $\forall\alpha\in\kappa\backslash\{0\}\forall n\in\omega[g_\alpha(n)<g_0(n)]$, and $\forall\xi<\mathfrak{p}\forall n\in\omega[f_\xi(n)<g_0(n)]$.

Now for any $n\in\omega$ let $X_n=(g_0(n),\le)$ and $X=\prod_{n\in\omega}X_n/G$. Let $h_\alpha(n)=g_{1+\alpha}(n)$ for all $\alpha<\kappa$. It suffices now to show that $(\langle[f_\xi]:\xi<\mathfrak{p}\rangle,\langle[h_\eta]:\eta<\kappa\rangle)$ is a gap in X.

If $\xi<\eta<\mathfrak{p}$, then $\{n\in\omega:f_\xi(n)\ge f_\eta(n)\}$ is finite, and hence its complement is in G; so $[f_\xi]<[f_\eta]$. Similarly, $\alpha<\beta<\kappa$ implies that $[h_\beta]<[h_\alpha]$. Also, by the same argument $[f_\xi]<[h_\alpha]$ for all $\xi<\mathfrak{p}$ and $\alpha<\kappa$.

Suppose that $k\in{}^\omega\omega$ and $\forall\eta<\kappa[[k]\le[h_\eta]]$. Thus $\forall\eta<\kappa[\{n\in\omega:k(n)\le h_\eta(n)\}\in G]$. Let $p\in G$ such that

$$p\Vdash\forall\eta<\kappa[\{n\in\omega:\check{k}(n)\le\check{h}_\eta(n)\}\in\Gamma].$$

For each $\eta<\kappa$ let $A_\eta=\{n\in\omega:k(n)\le h_\eta(n)\}$.

(1) $\forall\eta<\kappa[p\subseteq^* A_\eta]$.

For, fix $\eta<\kappa$ and suppose that $p\not\subseteq^* A_\eta$. Then $p\backslash A_\eta$ is infinite. Let H be M-generic over $\mathbb{P}$ with $p\backslash A_\eta\in H$. Now $p\in H$, so $[k]_H\le[h_\eta]_H$. Also $\omega\backslash A_\eta=\{n\in\omega:h_\eta(n)<k(n)\}$ and $(\omega\backslash A_\eta)\in H$, so $[h_\eta]_H<[k]_H$, contradiction. This proves (1).

Now for any $n\in\omega$ let

$$\tilde{k}(n)=\begin{cases}k(n) & \text{if } n\in p,\\ 0 & \text{otherwise.}\end{cases}$$

(2) $\forall\eta<\kappa[\tilde{k}\le^* h_\eta]$.

For, take any $\eta<\kappa$. By (1) let $k\in\omega$ be such that $\forall n\ge k[n\in p\to n\in A_\eta]$. Thus $\forall n\ge k[n\in p\to k(n)\le h_\eta(n)]$. So $\forall n\ge k[\tilde{k}(n)\le h_\eta(n)]$. So (2) holds.

It follows that there is a $\gamma<\mathfrak{p}$ such that $\tilde{k}<^* f_\gamma$. Since $p\in G$, we get $[k]=[\tilde{k}]<[f_\gamma]$. This proves that $(\langle[f_\xi]:\xi<\mathfrak{p}\rangle,\langle[h_\eta]:\eta<\kappa\rangle)$ is a gap in X. □

Theorem 32.43. $\mathfrak{p}=\mathfrak{t}$.

Proof. Suppose that $\mathfrak{p}<\mathfrak{t}$. Let G be M-generic over $\mathbb{P}$. Then

$$M[G]\models\mathfrak{p}(\omega,G)\le\mathfrak{p}<\mathfrak{t}\le\mathfrak{t}(\omega,G)=\mathfrak{p}(\omega,G),$$

contradiction. □

33. Cofinality of posets

We begin the study of possible cofinalities of partially ordered sets—the PCF theory. In this chapter we develop some combinatorial principles needed for the main results.

Ordinal-valued functions and their orderings

A *filter* on a set A is a collection F of subsets of A with the following properties:

(1) $A \in F$.
(2) If $X \in F$ and $X \subseteq Y \subseteq A$, then $Y \in F$.
(3) If $X, Y \in F$ then $X \cap Y \in F$.

A filter F is *proper* iff $F \neq \mathscr{P}(A)$.

Suppose that F is a filter on a set A and $R \subseteq \mathbf{On} \times \mathbf{On}$. Then for functions $f, g \in {}^A\mathbf{On}$ we define

$$f \, R_F \, g \quad \text{iff} \quad \{i \in A : f(i) \, R \, g(i)\} \in F.$$

The most important cases of this notion that we will deal with are $f <_F g$, $f \leq_F g$, and and $f =_F g$. Thus

$$\begin{aligned} f <_F g \quad &\text{iff} \quad \{i \in A : f(i) < g(i)\} \in F; \\ f \leq_F g \quad &\text{iff} \quad \{i \in A : f(i) \leq g(i)\} \in F; \\ f =_F g \quad &\text{iff} \quad \{i \in A : f(i) = g(i)\} \in F. \end{aligned}$$

Sometimes we use this notation for ideals rather than filters, using the duality between ideals and filters, which we now describe. An *ideal* on a set A is a collection I of subsets of A such that the following conditions hold:

(4) $\emptyset \in I$
(5) If $X \subseteq Y \in I$ then $X \in I$.
(6) If $X, Y \in I$ then $X \cup Y \in I$.

An ideal I is *proper* iff $I \neq \mathscr{P}(A)$.

If F is a filter on A, let $F' = \{X \subseteq A : A \backslash X \in F\}$. Then F' is an ideal on A. If I is an ideal on A, let $I^* = \{X \subseteq A : A \backslash X \in I\}$. Then I^* is a filter on A. If F is a filter on A, then $F'^* = F$. If I is an ideal on A, then $I^{*\prime} = I$.

Now if I is an ideal on A, then

$$\begin{aligned} f \, R_I \, g \quad &\text{iff} \quad \{i \in A : \neg(f(i) \, R_I \, g(i))\} \in I; \\ f <_I g \quad &\text{iff} \quad \{i \in A : f(i) \geq g(i)\} \in I; \\ f \leq_I g \quad &\text{iff} \quad \{i \in A : f(i) > g(i)\} \in I; \\ f =_I g \quad &\text{iff} \quad \{i \in A : f(i) \neq g(i)\} \in I. \end{aligned}$$

Some more notation: $R_I(f, g) = \{i \in I : f(i) \, R \, g(i)\}$. In particular, $<_I (f, g) = \{i \in I : f(i) < g(i)\}$ and $\leq_I (f, g) = \{i \in I : f(i) \leq g(i)\}$.

The following trivial proposition is nevertheless important in what follows.

Proposition 33.1. *Let F be a proper filter on A. Then*

(i) $<_F$ is irreflexive and transitive.

(ii) $\leq_F$ is reflexive on ${}^A\mathbf{On}$, and it is transitive.

(iii) $f \leq_F g <_F h$ implies that $f <_F h$.

(iv) $f <_F g \leq_F h$ implies that $f <_F h$.

(v) $f <_F g$ or $f =_F g$ implies $f \leq_F g$.

(vi) If $f =_F g$, then $g \leq_F f$.

(vii) If $f \leq_F g \leq_F f$, then $f =_F g$. $\square$

Some care must be taken in working with these notions. The following examples illustrate this.

(1) An example with $f \leq_F g$ but neither $f <_F g$ nor $f =_F g$ nor $f = g$: Let $A = \omega$, $F = \{A\}$, and define $f, g \in {}^\omega\omega$ by setting $f(n) = n$ for all n and

$$g(n) = \begin{cases} n & \text{if } n \text{ is even,} \\ n+1 & \text{if } n \text{ is odd.} \end{cases}$$

(2) An example where $f =_F g$ but neither $f <_F g$ nor $f = g$: Let $A = \omega$ and let F consist of all subsets of ω that contain all even natural numbers. Define f and g by

$$f(n) = \begin{cases} n & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd;} \end{cases} \qquad g(n) = \begin{cases} n & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Products and reduced products

In the preceding section we were considering ordering-type relations on the proper classes ${}^A\mathbf{On}$. Now we restrict ourselves to sets. Suppose that $h \in {}^A\mathbf{On}$. We specialize the general notion by considering $\prod_{a\in A} h(a) \subseteq {}^A\mathbf{On}$. To eliminate trivialities, we usually assume that $h(a)$ is a limit ordinal for every $a \in A$; then we call h *non-trivial.*

Proposition 33.2. *If F is a proper filter on A, $g, h \in {}^A\mathbf{On}$, h is non-trivial, and $g <_F h$, then there is a $k \in \prod_{a\in A} h(a)$ such that $g =_F k$.*

Proof. For any $a \in A$ let

$$k(a) = \begin{cases} g(a) & \text{if } g(a) < h(a), \\ 0 & \text{otherwise.} \end{cases}$$

Thus $k \in \prod_{a\in A} h(a)$. Moreover,

$$\{a \in A : g(a) = k(a) \supseteq \{a \in A : g(a) < h(a)\} \in F,$$

so $g =_F k$. $\square$

We will frequently consider the structure $(\prod_{a\in A} h(a), <_F, \leq_F)$ in what follows. For most considerations it is equivalent to consider the associated *reduced product*, which we define as follows. Note that $=_F$ is an equivalence relation on the set $\prod_{a\in A} h(a)$. We define the underlying set of the reduced product to be the collection of all equivalence classes under $=_F$; it is denoted by $\prod_{a\in A} h(a)/F$. Further, we define, for $x, y \in \prod_{a\in A} h(a)/F$,

$$\begin{aligned} x <_F y \quad &\text{iff} \quad \exists f, g \in \prod A[x = [f],\ y = [g],\ \text{and}\ f <_F g]; \\ x \leq_F y \quad &\text{iff} \quad \exists f, g \in \prod A[x = [f],\ y = [g],\ \text{and}\ f \leq_F g]. \end{aligned}$$

Here $[h]$ denotes the equivalence class of $h \in \prod A$ under $=_F$.

Proposition 33.3. *Suppose that $h \in {}^A\mathbf{On}$ is nontrivial, and $f, g \in \prod_{a\in A} h(a)$. Then*
(i) $[f] <_F [g]$ iff $f <_F g$.
(ii) $[f] \leq_F [g]$ iff $f \leq_F g$.

Proof. (i): The direction $\Leftarrow$ is obvious. Now suppose that $[f] <_F [g]$. Then there are $f', g' \in \prod A$ such that $[f] = [f']$, $[g] = [g']$, and $f' <_F g'$. Hence

$$\begin{aligned} &\{\kappa \in A : f(\kappa) = f'(\kappa)\} \cap \{\kappa \in A : g(\kappa) = g'(\kappa)\} \cap \{\kappa \in A : f'(\kappa) < g'(\kappa)\} \\ &\quad \subseteq \{\kappa \in A : f(\kappa) < g(\kappa)\}, \end{aligned}$$

and it follows that $\{\kappa \in A : f(\kappa) < g(\kappa)\} \in F$, and so $f <_F g$.

(ii): similarly. □

A filter F on A is an *ultrafilter* iff F is proper, and is maximal under all the proper filters on A. Equivalently, F is proper, and for any $X \subseteq A$, either $X \in F$ or $A\backslash X \in F$. The dual notion to an ultrafilter is a maximal ideal.

If F is an ultrafilter on A, then $\prod_{a\in A} h(a)/F$ is an *ultraproduct* of h.

Proposition 33.4. *If $h \in^A$ **On** is nontrivial and F is an ultrafilter on A, then $<_F$ is a linear order on $\prod_{a\in A} h(a)/F$, and $[f] \leq_F [g]$ iff $[f] <_F [g]$ or $[f] = [g]$.*

Proof. By Proposition 33.1(iii) and Proposition 33.3, $<_F$ is transitive. Also, from Proposition 33.3 it is clear that $<_F$ is irreflexive. Now suppose that $f, g \in \prod A$; we want to show that $[f]$ and $[g]$ are comparable. Assume that $[f] \neq [g]$. Thus $\{\kappa \in A : f(\kappa) = g(\kappa)\} \notin F$, so $\{\kappa \in A : f(\kappa) \neq g(\kappa)\} \in F$. Since

$$\{\kappa \in A : f(\kappa) \neq g(\kappa)\} = \{\kappa \in A : f(\kappa) < g(\kappa)\} \cup \{\kappa \in A : g(\kappa) < f(\kappa)\},$$

it follows that $[f] < [g]$ or $[g] < [f]$.

Thus $<_F$ is a linear order on $\prod A/F$.

Next,

$$\{\kappa \in A : f(\kappa) \leq g(\kappa)\} = \{\kappa \in A : f(\kappa) = g(\kappa)\} \cup \{\kappa \in A : f(\kappa) < g(\kappa)\},$$

so it follows by Proposition 33.3 that $[f] \leq_F [g]$ iff $[f] = [g]$ or $[f] <_F [g]$. □

In this section we allow partial orders P to be proper classes. We may speak of a partial ordering P if the relation $<_P$ is clear from the context. Recall the essential equivalence of the notion of a partial ordering with the "$\le$" version; see the easy exercise E13.15.

A *double ordering* is a system $(P, \le_P, <_P, =_P)$ such that the following conditions hold (cf. Proposition 33.1):

(i) $<_P$ is irreflexive and transitive.

(ii) $\le_P$ is reflexive on P, and it is transitive.

(iii) $f \le_P g <_P h$ implies that $f <_P h$.

(iv) $f <_P g \le_P h$ implies that $f <_P h$.

(v) $f <_P g$ or $f =_P g$ implies $f \le_P g$.

(vi) If $f =_P g$, then $g \le_P f$.

(vii) If $f \le_P g \le_P f$, then $f =_P g$.

Proposition 33.5. *For any set A any proper filter F on A, and any $P \subseteq {}^A\mathbf{On}$ the system $(P, \le_F, <_F, =_F)$ is a double ordering.* $\square$

Proposition 33.6. *Let $h \in {}^A\mathbf{On}$, with h taking only limit ordinal values, and let F be a proper filter on A. Then $(\prod_{a\in A} h(a)/F, \le_F, <_F, =)$ is a double ordering.* $\square$

We now give some general definitions, applying to any double ordering $(P, \le_P, <_P)$ unless otherwise indicated.

• A subclass $X \subseteq P$ is *cofinal* in P iff $\forall p \in P \exists q \in X(p \le_P q)$. By the condition (3) above, this is equivalent to saying that X is cofinal in P iff $\forall p \in P \exists q \in X(p <_P q)$.

• Since clearly P itself is cofinal in P, we can make the basic definition of the cofinality $\mathrm{cf}(P)$ of P, for a *set* P:

$$\mathrm{cf}(P) = \min\{|X| : X \text{ is cofinal in } P\}.$$

Note that $\mathrm{cf}(P)$ can be singular. For, let $A = \omega$, $h(a) = \omega_a$ for all $a \in \omega$, $I = \{\emptyset\}$, and $Y = \prod_{a\in A} h(a)$.. Suppose that X is cofinal in $\prod_{a\in A} h(a)$. Take any $a \in \omega$; we show that $\omega_a \le |X|$. We define a one-one sequence $\langle f_\alpha : \alpha < \omega_i\rangle$ of elements of X by recursion. Suppose that f_β has been defined for all $\beta < \alpha$. Let k be the member of $\prod_{a\in A} h(a)$ such that $k(b) = 0$ for all $b \ne a$, while $k(a) \in \omega_a \backslash \{f_\beta(a) : \beta < \alpha\}$. Choose $f_\alpha \in X$ such that $k <_I f_\alpha$.

• A sequence $\langle p_\xi : \xi < \lambda\rangle$ of elements of P is *$<_P$-increasing* iff $\forall \xi, \eta < \lambda(\xi < \eta \to p_\xi <_P p_\eta)$. Similarly for *$\le_P$-increasing*.

• Suppose that P is a double order and is a set. We say that P *has true cofinality* iff P has a linearly ordered subset which is cofinal.

Proposition 33.7. *Suppose that a set P is a double order, and $\langle p_\alpha : \alpha < \lambda\rangle$ is strictly increasing in the sense of P, is cofinal in P, and λ is regular. Then P has true cofinality, and its cofinality is λ.*

Proof. Obviously P has true cofinality. If X is a subset of P of size less than λ, for each $q \in X$ choose $\alpha_q < \lambda$ such that $q < p_{\alpha_q}$. Let $\beta = \sup_{q\in X} \alpha_q$. Then $\beta < \lambda$ since λ is regular. For any $q \in X$ we have $q < p_\beta$. This argument shows that $\mathrm{cf}(P) = \lambda$. □

Proposition 33.8. *Suppose that P is a double ordering, P a set, and P has true cofnality. Then:*

(i) $\mathrm{cf}(P)$ *is regular.*

(ii) $\mathrm{cf}(P)$ *is the least size of a linearly ordered subset which is cofinal in P.*

(iii) There is a $<_P$-increasing, cofinal sequence in P of length $\mathrm{cf}(P)$.

Proof. Let X be a linearly ordered subset of P which is cofinal in P, and let $\{y_\alpha : \alpha < \mathrm{cf}(P)\}$ be a subset of P which is cofinal in P; we do not assume that $\langle y_\alpha : \alpha < \mathrm{cf}(P)\rangle$ is $<_P$- or $\leq_P$-increasing.

(iii): We define a sequence $\langle x_\alpha : \alpha < \mathrm{cf}(P)\rangle$ by recursion. Let x_0 be any element of X. If x_α has been defined, let $x_{\alpha+1} \in X$ be such that $x_\alpha, y_\alpha < x_{\alpha+1}$; it exists since X is cofinal, using condition (3). Now suppose that $\alpha < \mathrm{cf}(P)$ is limit and x_β has been defined for all $\beta < \alpha$. Then $\{x_\beta : \beta < \alpha\}$ is not cofinal in P, so there is a $z \in P$ such that $z \not\leq x_\beta$ for all $\beta < \alpha$. Choose $x_\alpha \in X$ so that $z < x_\alpha$. Since X is linearly ordered, we must then have $x_\beta < x_\alpha$ for all $\beta < \alpha$. This finishes the construction. Since $y_\alpha < x_{\alpha+1}$ for all $\alpha < \mathrm{cf}(P)$, it follows that $\{x_\alpha : \xi < \mathrm{cf}(P)\}$ is cofinal in P. So (iii) holds.

(i): Suppose that $\mathrm{cf}(P)$ is singular, and let $\langle \beta_\xi : \xi < \mathrm{cf}(\mathrm{cf}(P))\rangle$ be a strictly increasing sequence cofinal in $\mathrm{cf}(P)$. With $\langle x_\alpha : \alpha < \mathrm{cf}(P)\rangle$ as in (iii), it is then clear that $\{x_{\beta_\xi} : \xi < \mathrm{cf}(\mathrm{cf}(P))\}$ is cofinal in P, contradiction (since $\mathrm{cf}(\mathrm{cf}(P)) < \mathrm{cf}(P)$ because $\mathrm{cf}(P)$ is singular).

(ii): By (iii), there is a linearly ordered subset of P of size $\mathrm{cf}(P)$ which is cofinal in P; by the definition of cofinality, there cannot be one of smaller size. □

For P with true cofinality, the cardinal $\mathrm{cf}(P)$ is called the *true cofinality* of P, and is denoted by $\mathrm{tcf}(P)$. We write $\mathrm{tcf}(P) = \lambda$ to mean that P has true cofinality, and it is equal to λ.

• P is λ-*directed* iff for any subset Q of P such that $|Q| < \lambda$ there is a $p \in P$ such that $q \leq_P p$ for all $q \in Q$; equivalently, there is a $p \in P$ such that $q <_P p$ for all $q \in Q$.

Proposition 33.9. (Pouzet) *Assume that P is a double ordering which is a set. For any infinite cardinal λ, we have* $\mathrm{tcf}(P) = \lambda$ *iff the following two conditions hold:*

(i) P has a cofinal subset of size λ.

(ii) P is λ-directed.

Proof. $\Rightarrow$ is clear, remembering that λ is regular. Now assume that (i) and (ii) hold, and let X be a cofinal subset of P of size λ.

First we show that λ is regular. Suppose that it is singular. Write $X = \bigcup_{\alpha<\mathrm{cf}(\lambda)} Y_\alpha$ with $|Y_\alpha| < \lambda$ for each $\alpha < \mathrm{cf}(\lambda)$. Let p_α be an upper bound for Y_α for each $\alpha < \mathrm{cf}(\lambda)$, and let q be an upper bound for $\{p_\alpha : \alpha < \mathrm{cf}(\lambda)\}$. Choose $r > q$. Then choose $s \in X$ with $r \le s$. Say $s \in Y_\alpha$. Then $s \le p_\alpha \le q < r \le s$, contradiction.

So, λ is regular. Let $X = \{r_\alpha : \alpha < \lambda\}$. Now we define a sequence $\langle p_\alpha : \alpha < \lambda\rangle$ by recursion. Having defined p_β for all $\beta < \alpha$, by (ii) let p_α be such that $p_\beta < p_\alpha$ for all $\beta < \alpha$, and $r_\beta < p_\alpha$ for all $\beta < \alpha$. Clearly this sequence shows that $\mathrm{tcf}(P, <_P) = \lambda$. $\square$

Proposition 33.10. *Let P be a set. If G is a cofinal subset of P, then* $\mathrm{cf}(P) = \mathrm{cf}(G)$. *Moreover,* $\mathrm{tcf}(P) = \mathrm{tcf}(G)$, *in the sense that if one of them exists then so does the other, and they are equal.* (That is what we mean in the future too when we assert the equality of true cofinalities.)

Proof. Let H be a cofinal subset of P of size $\mathrm{cf}(P)$. For each $p \in H$ choose $q_p \in G$ such that $p \le_P q_p$. Then $\{q_p : p \in H\}$ is cofinal in G. In fact, if $r \in G$, choose $p \in H$ such that $r \le_P p$. Then $r \le_P q_p$, as desired. This shows that $\mathrm{cf}(G) \le \mathrm{cf}(P)$.

Now suppose that K is a cofinal subset of G. Then it is also cofinal in P. For, if $p \in P$ choose $q \in G$ such that $p \le_P q$, and then choose $r \in K$ such that $q \le_P r$. So $p \le_P r$, as desired. This shows the other inequality.

For the true cofinality, we apply Theorem 33.9. So suppose that P has true cofinality λ. By Theorem 33.9 and the first part of this proof, G has a cofinal subset of size λ, since cofinality is the same as true cofinality when the latter exists. Now suppose that $X \subseteq G$ is of size $< \lambda$. Choose an upper bound p for it in P. Then choose $q \in G$ such that $p \le_P q$. So q is an upper bound for X, as desired. Thus since Theorem 33.9(i) and 33.9(ii) hold for G, it follows from that theorem that $\mathrm{tcf}(G) = \lambda$.

The other implication, that the existence of $\mathrm{tcf}(G, <)$ implies that of $\mathrm{tcf}(P, <)$ and their equality, is even easier, since a sequence cofinal in G is also cofinal in P. $\square$

• A sequence $\langle p_\xi : \xi < \lambda\rangle$ of elements of P is *persistently cofinal* iff

$$\forall h \in P \exists \xi_0 < \lambda \forall \xi(\xi_0 \le \xi < \lambda \Rightarrow h <_P p_\xi).$$

Proposition 33.11. *(i) If $\langle p_\xi : \xi < \lambda\rangle$ is $<_P$-increasing and cofinal in P, then it is persistently cofinal.*

(ii) If $\langle p_\xi : \xi < \lambda\rangle$ and $\langle p'_\xi : \xi < \lambda\rangle$ are two sequences of members of P, $\langle p_\xi : \xi < \lambda\rangle$ is persistently cofinal in P, and $p_\xi \le_P p'_\xi$ for all $\xi < \lambda$, then also $\langle p'_\xi : \xi < \lambda\rangle$ is persistently cofinal in P. $\square$

• If $X \subseteq P$, then an *upper bound* for X is an element $p \in P$ such that $q \le_P p$ for all $q \in X$.

• If $X \subseteq P$, then a *least upper bound* for X is an upper bound a for X such that $a \le_P a'$ for every upper bound a' for X. So if a and b are least upper bounds for X, then $a \le_P b \le_P a$.

It is possible here to have $a \ne b$. For example, let $A = \omega$, $h(a) = \omega + \omega$ for all $a \in \omega$, $f_n(m) = m + n$ for all $m, n \in \omega$, $I = \{Y \subseteq \omega : \text{each member of } Y \text{ is odd}\}$. $X = \{f_n : n \in \omega\}$. We consider the double order $(\prod_{a\in\omega} h(a), \le_I, <_I)$. Let

$$g(m) = \begin{cases} \omega & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd} \end{cases} \qquad h(m) = \begin{cases} \omega & \text{if } m \text{ is even,} \\ 1 & \text{if } m \text{ is odd} \end{cases}$$

Then g and h are least upper bounds for X, while $g \neq h$.

• If $X \subseteq P$, then a *minimal upper bound* for X is an upper bound a for X such that if b is an upper bound for X and $b \leq_P a$, then $a \leq_P b$.

Proposition 33.12. *If $X \subseteq P$ and a is a least upper bound for X, then a is a minimal upper bound for X.* □

Now we come to an ordering notion which is basic for pcf theory.

• If $X \subseteq P$ and for every $x \in X$ there is an $x' \in X$ such that $x <_P x'$, then an element $a \in P$ is an *exact upper bound* of X provided

(1) a is a least upper bound for X, and

(2) X is cofinal in $\{p \in P : p <_P a\}$.

Note that under the hypothesis here, $a \notin X$, and hence $x <_F a$ for all $x \in X$ by (1).

Here is an example of a set X with a least upper bound but no exact upper bound. Let $A = \omega$, $h(a) = \omega + \omega$ for all $a \in \omega$, and for $m, n \in \omega$,

$$f_n(m) = \begin{cases} n & \text{if } m \neq n, \\ 0 & \text{if } m = n, \end{cases}$$

$X = \{f_n : n \in \omega\}$, $I = \{\emptyset\}$. We consider the double order $(\prod_{a\in\omega} h(a), \leq_I, <_I)$. Then a least upper bound for X is the function a such that $a(m) = \omega$ for all $m \in \omega$, but X does not have an exact upper bound.

Ordinal-valued functions and exact upper bounds

In this section we give some simple facts about exact upper bounds in the case of most interest to us—the partial ordering of ordinal-valued functions.

First we note that the rough equivalence between products and reduced products continues to hold for the cofinality notions introduced above. We state this for the most important properties above:

Proposition 33.13. *Suppose that $h \in {}^A\mathbf{On}$, and h takes only limit ordinal values. Then*

(i) If $X \subseteq \prod_{a\in A} h(a)$, then X is cofinal in $(\prod_{a\in A} h(a), <_I, \leq_I)$ iff $\{[f] : f \in X\}$ is cofinal in $(\prod_{a\in A} h(a)/I, <_I, \leq_I)$.

(ii) $\mathrm{cf}(\prod_{a\in A} h(a), <_I, \leq_I) = \mathrm{cf}(\prod_{a\in A} h(a)/I, <_I, \leq_I)$.

(iii) $\mathrm{tcf}(\prod_{a\in A} h(a), <_I, \leq_I) = \mathrm{tcf}(\prod_{a\in A} h(a)/I, <_I, \leq_I)$.

(iv) If $X \subseteq \prod_{a\in A} h(a)$ and $f \in \prod_{a\in A} h(a)$, then f is an exact upper bound for X iff $[f]$ is an exact upper bound for $\{[g] : g \in X\}$.

Proof. (i) is immediate from Proposition 33.3. For (ii), if X is cofinal in the system $(\prod_{a\in A} h(a), <_I, \leq_I)$, then clearly $\{[f] : f \in X\}$ is cofinal in $(\prod_{a\in A} h(a)/I, <_I, \leq_I)$, by Proposition 33.3 again; so $\geq$ holds. Now suppose that $\{[f] : f \in Y\}$ is cofinal in $(\prod_{a\in A} h(a)/I, <_I, \leq_I)$. Given $g \in \prod_{a\in A} h(a)$, choose $f \in Y$ such that $[g] <_I [f]$. Then $g <_I f$. So Y is cofinal in $(\prod_{a\in A} h(a), <_I, \leq_I)$, and $\leq$ holds.

(iii) and (iv) are proved similarly. □

The following obvious proposition will be useful.

Proposition 33.14. *Suppose that $F \cup \{f, g\} \subseteq {}^A\mathbf{On}$, I is an ideal on A, and $f =_I g$. Suppose that f is an upper bound, least upper bound, minimal upper bound, or exact upper bound for F under $\leq_I$. Then also g is an upper bound, least upper bound, minimal upper bound, or exact upper bound for F under $\leq_I$, respectively.*

Here is our simplest existence theorem for exact upper bounds.

• If X is a collection of members of ${}^A\mathbf{On}$, then $\sup X \in {}^A\mathbf{On}$ is defined by

$$(\sup X)(a) = \sup\{f(a) : f \in X\}.$$

Proposition 33.15. *Suppose that $\lambda > |A|$ is a regular cardinal, and $f = \langle f_\xi : \xi < \lambda\rangle$ is an increasing sequence of members of ${}^A\mathbf{On}$ in the partial ordering $<$ of everywhere dominance. (That is, $f < g$ iff $f(a) < g(a)$ for all $a \in A$.) Then* $\sup f$ *is an exact upper bound for f, and* $\mathrm{cf}((\sup f)(a)) = \lambda$ *for every $a \in A$.*

Proof. For brevity let $h = \sup f$. Then clearly h is an upper bound for f. Now suppose that $f_\xi \leq g \in {}^A\mathbf{On}$ for all $\xi < \lambda$. Then for any $a \in A$ we have $h(a) = \sup_{\xi<\lambda} f_\xi(a) \leq g(a)$, so $h \leq g$. Thus h is a least upper bound for f. Now suppose that $k \in {}^A\mathbf{On}$ and $k < h$. Then for every $a \in A$ we have $k(a) < h(a)$, and hence there is a $\xi_a < \lambda$ such that $k(a) < f_{\xi_a}(a)$. Let $\eta = \sup_{a\in A} \xi_a$. So $\eta < \lambda$ since λ is regular and greater than $|A|$. Clearly $k < f_\eta$, as desired. □

The next proposition gives equivalent definitions of least upper bounds for our special partial order.

Proposition 33.16. *Suppose that I is a proper ideal on A, $F \subseteq {}^A\mathbf{On}$, and $f \in {}^A\mathbf{On}$. Then the following conditions are equivalent.*

(i) f is a least upper bound of F under $\leq_I$.

(ii) f is an upper bound of F under $\leq_I$, and for any $f' \in {}^A\mathbf{On}$, if f' is an upper bound of F under $\leq_I$ and $f' \leq_I f$, then $f =_I f'$.

(iii) f is a minimal upper bound of F under $\leq_I$.

Proof. (i)⇒(ii): Assume (i) and the hypotheses of (ii). Hence $f \leq_I f'$, so $f =_I f'$ by Proposition 33.1(vii).

(ii)⇒(iii): Assume (ii), and suppose that $g \in {}^A\mathbf{On}$ is an upper bound for F and $g \leq_I f$. Then $g =_I f$ by (ii), so $f \leq_I g$.

(iii)⇒(i): Assume (iii). Let $g \in {}^A\mathbf{On}$ be any upper bound for F. Define $h(a) = \min(f(a), g(a))$ for all $a \in A$. Then h is an upper bound for F, since if $k \in F$, then $\{a \in A : k(a) > f(a)\} \in I$ and also $\{a \in A : k(a) > g(a)\} \in I$, and

$$\{a \in A : k(a) > \min(f(a), g(a))\} \subseteq \{a \in A : k(a) > f(a)\} \cup \{a \in A : k(a) > g(a)\} \in I,$$

so $k \leq_I h$. Also, clearly $h \leq_I f$. So by (iii), $f \leq_I h$, and hence $f \leq_I g$, as desired. $\square$

In the next proposition we see that in the definition of exact upper bound we can weaken the condition (1), under a mild restriction on the set in question.

Proposition 33.17. *Suppose that F is a nonempty set of functions in ${}^A\mathbf{On}$ and $\forall f \in F \exists f' \in F[f <_I f']$. Suppose that h is an upper bound of F, and $\forall g \in {}^A\mathbf{On}$, if $g <_I h$ then there is an $f \in F$ such that $g <_I f$. Then h is an exact upper bound for F.*

Proof. First note that $\{a \in A : h(a) = 0\} \in I$. In fact, choose $f \in F$. Then $f <_I h$, and so $\{a \in A : h(a) = 0\} \subseteq \{a \in A : f(a) \geq h(a)\} \in I$, as desired.

Now we show that h is a least upper bound for F. Let k be any upper bound. Let

$$l(a) = \begin{cases} k(a) & \text{if } k(a) < h(a), \\ 0 & \text{otherwise.} \end{cases}$$

Since $\{a \in A : l(a) \geq h(a)\} \subseteq \{a \in A : h(a) = 0\}$, it follows by the above that $\{a \in A : l(a) \geq h(a)\} \in I$, and so $l <_I h$. So by assumption, choose $f \in F$ such that $l <_I f$. Now $f \leq_I k$, so $l <_I k$ and hence

$$\{a \in A : k(a) < h(a)\} \subseteq \{a \in A : l(a) \geq k(a)\} \in I,$$

so $h \leq_I k$, as desired.

For the other property in the definition of exact upper bound, suppose that $g <_I h$. Then by assumption there is an $f \in F$ such that $g <_I f$, as desired. $\square$

Corollary 33.18. *If $h \in {}^A\mathbf{On}$ is non trivial and $F \subseteq \prod_{a \in A} h(a)$, then h is an exact upper bound of F with respect to an ideal I on A iff F is cofinal in $\prod_{a \in A} h(a)$.* $\square$

In the next proposition we use the standard notation I^+ for $A \backslash I$. The proposition shows that exact upper bounds restrict to smaller sets A.

Proposition 33.19. *Suppose that F is a nonempty subset of ${}^A\mathbf{On}$, I is a proper ideal on A, h is an exact upper bound for F with respect to I, and $\forall f \in F \exists f' \in F(f <_I f')$. Also, suppose that $A_0 \in I^+$. Then:*

(i) $J \stackrel{\text{def}}{=} I \cap \mathscr{P}(A_0)$ is a proper ideal on A_0.

(ii) For any $f, f' \in {}^A\mathbf{On}$, if $f <_I f'$ then $f \restriction A_0 <_J f' \restriction A_0$.

(iii) $h \restriction A_0$ is an exact upper bound for $\{f \restriction A_0 : f \in F\}$.

(i) is clear. Assume the hypotheses of (ii). Then

$$\{a \in A_0 : f'(a) \leq f(a)\} \subseteq \{a \in A : f'(a) \leq f(a)\} \in I,$$

and so $f \restriction A_0 <_J f' \restriction A_0$.

For (iii), by (ii) we see that $h \restriction A_0$ is an upper bound for $\{f \restriction A_0 : f \in F\}$. To see that it is an exact upper bound, we will apply Proposition 33.18. So, suppose that $k <_J h \restriction A_0$. Fix $f \in F$. Now define $g \in {}^A\mathbf{On}$ by setting

$$g(a) = \begin{cases} f(a) & \text{if } a \in A \backslash A_0, \\ k(a) & \text{if } a \in A_0. \end{cases}$$

Then

$$\{a \in A : g(a) \geq h(a)\} \subseteq \{a \in A : f(a) \geq h(a)\} \cup \{a \in A_0 : k(a) \geq h(a)\} \in I,$$

so $g <_I h$. Hence there is an $l \in F$ such that $g <_I l$. Hence

$$\{a \in A_0 : k(a) \geq l(a)\} \subseteq \{a \in A : g(a) \geq l(a)\} \in I,$$

so $k <_J l$, as desired. $\square$

Next, increasing the ideal maintains exact upper bounds:

Proposition 33.20. *Suppose that F is a nonempty subset of ${}^A\mathbf{On}$, I is a proper ideal on A, h is an exact upper bound for F with respect to I, and $\forall f \in F \exists f' \in F(f <_I f')$.*

Let J be a proper ideal on A such that $I \subseteq J$. Then h is an exact upper bound for F with respect to J.

Proof. We will apply Proposition 33.17. Note that h is clearly an upper bound for F with respect to J. Now suppose that $g <_J h$. Let $f \in F$. Define g' by

$$g'(a) = \begin{cases} g(a) & \text{if } g(a) < h(a), \\ f(a) & \text{otherwise.} \end{cases}$$

Then $\{a \in A : g'(a) \geq h(a)\} \subseteq \{a \in A : f(a) \geq h(a)\} \in I$, since $f <_I h$. So $g' <_I h$. Hence by the exactness of h there is a $k \in F$ such that $g' <_I k$. So

$$\begin{aligned}\{a : g(a) \geq k(a)\} &\subseteq \{a \in A : h(a) > g(a) \geq k(a)\} \cup \{a \in A : h(a) \leq g(a)\} \\ &\subseteq \{a \in A : g'(a) \geq k(a)\} \cup \{a \in A : h(a) \leq g(a)\},\end{aligned}$$

and this union is in J since the first set is in I and the second one is in J. Hence $g <_J k$, as desired. $\square$

Again we turn from the general case of proper classes ${}^A\mathbf{On}$ to the sets $\prod_{a \in A} h(a)$, where $h \in^A \mathbf{On}$ has only limit ordinal values. We prove some results which show that under a weak hypothesis we can restrict attention to $\prod A$ for A a nonempty set of infinite regular cardinals instead of $\prod_{a \in A} h(a)$, as far as cofinality notions are concerned. Here $\prod A$ consists of all choice functions f with domain A; $f(a) \in a$ for all $a \in A$.

Proposition 33.21. *Suppose that $h \in {}^A\mathbf{On}$ and $h(a)$ is a limit ordinal for every $a \in A$. For each $a \in A$, let $S(a) \subseteq h(a)$ be cofinal in $h(a)$ with order type $\mathrm{cf}(h(a))$. Suppose that I is a proper ideal on A. Then*

(i) $\mathrm{cf}(\prod_{a \in A} h(a), <_I) = \mathrm{cf}(\prod_{a \in A} S(a), <_I)$ *and*

(ii) $\mathrm{tcf}(\prod_{a \in A} h(a), <_I) = \mathrm{tcf}(\prod_{a \in A} S(a), <_I)$.

Proof. For each $f \in \prod h$ define $g_f \in \prod_{a \in A} S(a)$ by setting

$$g_f(a) = \text{ least } \alpha \in S(a) \text{ such that } f(a) \leq \alpha.$$

We prove (i): suppose that $X \subseteq \prod h$ and X is cofinal in $(\prod h, <_I)$; we show that $\{g_f : f \in X\}$ is cofinal in $\mathrm{cf}(\prod_{a\in A} S(a), <_I)$, and this will prove $\geq$. So, let $k \in \prod_{a\in A} S(a)$. Thus $k \in \prod h$, so there is an $f \in X$ such that $k <_I f$. Since $f \leq g_f$, it follows that $k <_I g_f$, as desired. Conversely, suppose that $Y \subseteq \prod_{a\in A} S(a)$ and Y is cofinal in $(\prod_{a\in A} S(a), <_I)$; we show that also Y is cofinal in $\prod h$, and this will prove $\leq$ of the claim. Let $f \in \prod h$. Then $f \leq g_f$, and there is a $k \in Y$ such that $g_f <_I k$; so $f <_I k$, as desired.

This finishes the proof of (i).

For (ii), first suppose that $\mathrm{tcf}(\prod h, <_I)$ exists; call it λ. Thus λ is an infinite regular cardinal. Let $\langle f_i : i < \lambda\rangle$ be a $<_I$-increasing cofinal sequence in $\prod h$. We claim that $g_{f_i} \leq g_{f_j}$ if $i < j < \lambda$. In fact, if $a \in A$ and $f_i(a) < f_j(a)$, then $f_i(a) < f_j(a) \leq g_{f_j}(a) \in S(a)$, and so by the definition of g_{f_i} we get $g_{f_i}(a) \leq g_{f_j}(a)$. This implies that $g_{f_i} \leq_I g_{f_j}$. Now $\mathrm{cf}(\prod h, <_I) = \lambda$, so for any $B \in [\lambda]^{<\lambda}$ there is a $j < \lambda$ such that $g_{f_i} <_I f_j \leq g_{f_j}$. It follows that we can take a subsequence of $\langle g_{f_i} : i < \lambda\rangle$ which is strictly increasing modulo I; it is also clearly cofinal, and hence $\lambda = \mathrm{tcf}(\prod_{a\in A} S(a), <_I)$.

Conversely, suppose that $\mathrm{tcf}(\prod_{a\in A} S(a), <_I)$ exists; call it λ. Let $\langle f_i : i < \lambda\rangle$ be a $<_I$-increasing cofinal sequence in $\prod_{a\in A} S(a)$. Then it is also a sequence showing that $\mathrm{tcf}(\prod h, <_I)$ exists and equals $\mathrm{tcf}(\prod_{a\in A} S(a), <_I)$. □

Proposition 33.22. *Suppose that $\langle L_a : a \in A\rangle$ and $\langle M_a : a \in A\rangle$ are systems of linearly ordered sets such that each L_a and M_a has no last element. Suppose that L_a is isomorphic to M_a for all $a \in A$. Let I be any ideal on A. Then*

$$\left(\prod_{a\in A} L_a, <_I, \leq_I\right) \cong \left(\prod_{a\in A} M_a, <_I, \leq_I\right). \qquad \square$$

Putting the last two propositions together, we see that to determine cofinality and true cofinality of $(\prod h, <_I, \leq_I)$, where $h \in {}^A\mathbf{On}$ and $h(a)$ is a limit ordinal for all $a \in A$, it suffices to take the case in which each $h(a)$ is an infinite regular cardinal. (One passes from $h(a)$ to $S(a)$ and then to $\mathrm{cf}(h(a))$.) We can still make a further reduction, given in the following useful lemma.

Lemma 33.23. (Rudin-Keisler) *Suppose that c maps the set A into the class of regular cardinals, and $B = \{c(a) : a \in A\}$ is its range. For any ideal I over A, define its Rudin-Keisler projection J on B by*

$$X \in J \quad \textit{iff} \quad X \subseteq B \textit{ and } c^{-1}[X] \in I.$$

Then J is an ideal on B, and there is an isomorphism h of $\prod B/J$ into $\prod_{a\in A} c(a)/I$ such that for any $e \in \prod B$ we have $h(e/J) = \langle e(c(a)) : a \in A\rangle/I$.

If $|A| < \min(B)$, then the range of h is cofinal in $\prod_{a\in A} c(a)/I$, and we have
(i) $\mathrm{cf}(\prod B/J) = \mathrm{cf}(\prod_{a\in A} c(a)/I$ *and*
(ii) $\mathrm{tcf}(\prod B/J) = \mathrm{tcf}(\prod_{a\in A} c(a)/I)$.

Proof. Clearly J is an ideal. Next, for any $e \in \prod B$ let $\overline{e} = \langle e(c(a)) : a \in A \rangle$. Then for any $e_1, e_2 \in \prod B$ we have

$$\begin{aligned}
e_1 =_J e_2 \quad &\text{iff} \quad \{b \in B : e_1(b) \neq e_2(b)\} \in J \\
&\text{iff} \quad c^{-1}[\{b \in B : e_1(b) \neq e_2(b)\}] \in I \\
&\text{iff} \quad \{a \in A : e_1(c(a)) \neq e_2(c(a))\} \in I \\
&\text{iff} \quad \overline{e_1} =_I \overline{e_2}.
\end{aligned}$$

This shows that h exists as indicated and is one-one. Similarly, h preserves $<_I$ in each direction. So the first part of the lemma holds.

Now suppose that $|A| < \min(B)$. Let G be the range of h. By Proposition 33.11, (i) and (ii) follow from G being cofinal in $\prod_{a \in A} c(a)/I$. Let $g \in \prod_{a \in A} c(a)$. Define $e \in \prod B$ by setting, for any $b \in B$,

$$e(b) = \sup\{g(a) : a \in A \text{ and } c(a) = b\}.$$

The additional supposition implies that $e \in \prod B$. Now note that $\{a \in A : g(a) > e(c(a))\} = \emptyset \in I$, so that $g/I \leq h(e/J)$, as desired. $\square$

According to these last propositions, the calculation of true cofinalities for partial orders of the form $(\prod_{a \in A} h(a), <_I)$, with $h \in {}^A\mathbf{On}$ and $h(a)$ a limit ordinal for every $a \in A$, and with $|A| < \min(\mathrm{cf}(h(a))$, reduces to the calculation of true cofinalities of partial orders of the form $(\prod B, <_J)$ with B a set of regular cardinals with $|B| < \min(B)$.

Lemma 33.24. *If $(P_i, <_i)$ is a partial order with true cofinality λ_i for each $i \in I$ and D is an ultrafilter on I, then* $\mathrm{tcf}(\prod_{i \in I} \lambda_i/D) = \mathrm{tcf}(\prod_{i \in I} P_i/D)$.

Proof. Note that $\prod_{i \in I} \lambda_i/D$ is a linear order, and so its true cofinality μ exists and equals its cofinality. So the lemma is asserting that the ultraproduct $\prod_{i \in I} P_i/D$ has μ as true cofinality.

Let $\langle g_\xi : \xi < \mu \rangle$ be a sequence of members of $\prod_{i \in I} \lambda_i$ such that $\langle g_\xi/D : \xi < \mu \rangle$ is strictly increasing and cofinal in $\prod_{i \in I} \lambda_i/D$. For each $i \in I$ let $\langle f_{\xi,i} : \xi < \lambda_i \rangle$ be strictly increasing and cofinal in $(P_i, <_i)$. For each $\xi < \mu$ define $h_\xi \in \prod_{i \in I} P_i$ by setting $h_\xi(i) = f_{g_\xi(i),i}$. We claim that $\langle h_\xi/D : \xi < \mu \rangle$ is strictly increasing and cofinal in $\prod_{i \in I} P_i/D$ (as desired).

To prove this, first suppose that $\xi < \eta < \mu$. Then

$$\{i \in I : h_\xi(i) < h_\eta(i)\} = \{i \in I : f_{g_\xi(i),i} <_i f_{g_\eta(i),i}\} = \{i \in I : g_\xi(i) < g_\eta(i)\} \in D;$$

so $h_\xi/D < h_\eta/D$.

Now suppose that $k \in \prod_{i \in I} P_i$; we want to find $\xi < \mu$ such that $k/D < h_\xi/D$. Define $l \in \prod_{i \in I} \lambda_i$ by letting $l(i)$ be the least $\xi < \mu$ such that $k(i) < f_{\xi,i}$. Choose $\xi < \mu$ such that $l/D < g_\xi/D$. Now if $l(i) < g_\xi(i)$, then $k(i) < f_{l(i),i} <_i f_{g_\xi(i),i} = h_\xi(i)$. So $k/D < h_\xi/D$. $\square$

We introduce several notions leading up to an existence theorem for exact upper bounds: projections, strongly increasing sequences, a partition property, and the bounding projection property.

We start with the important notion of **projections**. By a *projection framework* we mean a triple (A, I, S) consisting of a nonempty set A, an ideal I on A, and a sequence $\langle S_a : a \in A\rangle$ of nonempty sets of ordinals. Suppose that we are given such a framework. We define $\sup S$ in the natural way: it is a function with domain A, and $(\sup S)(a) = \sup(S_a)$ for every $a \in A$. Thus $\sup S \in {}^A\mathbf{On}$. Now suppose also that we have a function $f \in {}^A\mathbf{On}$. Then we define the *projection* of f onto $\prod_{a\in A} S_a$, denoted by $f^+ = \mathrm{proj}(f, S)$, by setting, for any $a \in A$,

$$f^+(a) = \begin{cases} \min(S_a\backslash f(a)) & \text{if } f(a) < \sup(S_a), \\ \min(S_a) & \text{otherwise.} \end{cases}$$

Thus

$$f^+(a) = \begin{cases} f(a) & \text{if } f(a) \in S_a \text{ and } f(a) \text{ is not} \\ & \text{the largest element of } S_a, \\ \text{least } x \in S_a \text{ such that } f(a) < x & \text{if } f(a) \notin S_a \text{ and } f(a) < \sup(S_a), \\ \min(S_a) & \text{if } \sup(S_a) \leq f(a). \end{cases}$$

Proposition 33.25. *Let a projection framework be given, with the notation above.*

(i) If $f \in {}^A\mathbf{On}$, then $f^+ \in \prod_{a\in A} S_a$.

(ii) If $f_1, f_2 \in {}^A\mathbf{On}$ and $f_1 =_I f_2$, then $f_1^+ =_I f_2^+$.

(iii) If $f \in {}^A\mathbf{On}$ and $f <_I \sup S$, then $f \leq_I f^+$, and for every $g \in \prod_{a\in A} S_a$, if $f \leq_I g$ then $f^+ \leq_I g$.

Proof. (i) and (ii) are clear. For (iii), suppose that $f \in {}^A\mathbf{On}$ and $f <_I \sup S$. Then if $f(a) > f^+(a)$ we must have $f(a) \geq \sup(S_a)$. Hence $f \leq_I f^+$. Now suppose that $g \in \prod_{a\in A} S_a$ and $f \leq_I g$. If $f(a) \leq g(a)$ and $f(a) < \sup(S_a)$, then $f^+(a) \leq g(a)$. Hence

$$\{a \in A : g(a) < f^+(a)\} \subseteq \{a \in A : f(a) > g(a)\} \cup \{a \in A : f(a) \geq \sup(S_a)\} \in I,$$

so $f^+ \leq_I g$. □

Another important notion in discussing exact upper bounds is as follows. Let I be an ideal over A, L a set of ordinals, and $f = \langle f_\xi : \xi \in L\rangle$ a sequence of members of ${}^A\mathbf{On}$. Then we say that f is *strongly increasing under I* iff there is a system $\langle Z_\xi : \xi \in L\rangle$ of members of I such that

$$\forall \xi, \eta \in L[\xi < \eta \Rightarrow \forall a \in A\backslash(Z_\xi \cup Z_\eta)[f_\xi(a) < f_\eta(a)]].$$

Under the same assumptions we say that f is *very strongly increasing under I* iff there is a system $\langle Z_\xi : \xi \in L\rangle$ of members of I such that

$$\forall \xi, \eta \in L[\xi < \eta \Rightarrow \forall a \in A\backslash Z_\eta[f_\xi(a) < f_\eta(a)].$$

Proposition 33.26. *Under the above assumptions, f is very strongly increasing under I iff for every $\xi \in L$ we have*

$$(*) \qquad \sup\{f_\alpha + 1 : \alpha \in L \cap \xi\} \leq_I f_\xi.$$

Proof. $\Rightarrow$: suppose that f is very strongly increasing under I, with sets Z_ξ as indicated. Let $\xi \in L$. Suppose that $a \in A \backslash Z_\xi$. Then for any $\alpha \in L \cap \xi$ we have $f_\alpha(a) < f_\xi(a)$, and so $\sup\{f_\alpha(a) + 1 : \alpha \in L \cap \xi\} \leq f_\xi(a)$; it follows that $(*)$ holds.

$\Leftarrow$: suppose that $(*)$ holds for each $\xi \in L$. For each $\xi \in L$ let

$$Z_\xi = \{a \in A : \sup\{f_\alpha(a) + 1 : \alpha \in L \cap \xi\} > f_\xi(a)\};$$

it follows that $Z_\xi \in I$. Now suppose that $\alpha \in L$ and $\alpha < \xi$. Suppose that $a \in A \backslash Z_\xi$. Then $f_\alpha(a) < f_\alpha(a) + 1 \leq \sup\{f_\beta(a) + 1 : \beta \in L \cap \xi\} \leq f_\xi(a)$, as desired. □

Lemma 33.27. (The sandwich argument) *Suppose that $h = \langle h_\xi : \xi \in L \rangle$ is strongly increasing under I, L has no largest element, and ξ' is the successor in L of ξ for every $\xi \in L$. Also suppose that $f_\xi \in {}^A\mathbf{On}$ is such that*

$$h_\xi <_I f_\xi \leq_I h_{\xi'} \text{ for every } \xi \in L.$$

Then $\langle f_\xi : \xi \in L \rangle$ is also strongly increasing under I.

Proof. Let $\langle Z_\xi : \xi \in L \rangle$ testify that h is strongly increasing under I. For every $\xi \in L$ let

$$W_\xi = \{a \in A : h_\xi(a) \geq f_\xi(a) \text{ or } f_\xi(a) > h_{\xi'}(a)\}.$$

Thus by hypothesis we have $W_\xi \in I$. Let $Z^\xi = W_\xi \cup Z_\xi \cup Z_{\xi'}$ for every $\xi \in L$; so $Z_\xi \in I$. Then if $\xi_1 < \xi_2$, both in L, and if $a \in A \backslash (Z^{\xi_1} \cup Z^{\xi_2})$, then

$$f_{\xi_1}(a) \leq h_{\xi_1'}(a) \leq h_{\xi_2}(a) < f_{\xi_2}(a);$$

these three inequalities hold because $a \in A \backslash W_{\xi_1}$, $a \in A \backslash (Z_{\xi_1'} \cup Z_{\xi_2})$, and $a \in A \backslash W_{\xi_2}$ respectively. □

Now we give a proposition connecting the notion of strongly increasing sequence with the existence of exact upper bounds.

Proposition 33.28. *Let I be a proper ideal over A, let $\lambda > |A|$ be a regular cardinal, and let $f = \langle f_\xi : \xi < \lambda \rangle$ be a $<_I$ increasing sequence of functions in ${}^A\mathbf{On}$. Then the following conditions are equivalent:*

(i) f has a strongly increasing subsequence of length λ under I.

(ii) f has an exact upper bound h such that $\{a \in A : \mathrm{cf}(h(a)) \neq \lambda\} \in I$.

(iii) f has an exact upper bound h such that $\mathrm{cf}(h(a)) = \lambda$ for all $a \in A$.

(iv) There is a sequence $g = \langle g_\xi : \xi < \lambda \rangle$ such that $g_\xi < g_\eta$ (everywhere) for $\xi < \eta$, and f is cofinally equivalent to g, in the sense that $\forall \xi < \lambda \exists \eta < \lambda (f_\xi <_I g_\eta)$ and $\forall \xi < \lambda \exists \eta < \lambda (g_\xi <_I f_\eta)$.

Proof. (i)⇒(ii): Let $\langle \eta(\xi) : \xi < \lambda \rangle$ be a strictly increasing sequence of ordinals less than λ, thus with supremum λ since λ is regular, and assume that $\langle f_{\eta(\xi)} : \xi < \lambda \rangle$ is strongly increasing under I. Hence for each $\xi < \lambda$ let $Z_\xi \in I$ be chosen correspondingly. We define for each $a \in A$

$$h(a) = \sup\{f_{\eta(\xi)}(a) : \xi < \lambda, a \notin Z_\xi\}.$$

To see that h is an exact upper bound for f, we are going to apply Proposition 33.17. If $f_{\eta(\xi)}(a) > h(a)$, then $a \in Z_\xi \in I$. Hence $f_{\eta(\xi)} \leq_I h$ for each $\xi < \lambda$. Then for any $\xi < \lambda$ we have $f_\xi \leq_I f_{\eta(\xi)} \leq_I h$, so h bounds every f_ξ. Now suppose that $d <_I h$. Let $M = \{a \in A : d(a) \geq h(a)\}$; so $M \in I$. For each $a \in A \backslash M$ we have $d(a) < h(a)$, and so there is a $\xi_a < \lambda$ such that $d(a) < f_{\eta(\xi_a)}(a)$ and $a \notin Z_{\xi_a}$. Since $|A| < \lambda$ and λ is regular, the ordinal $\rho \stackrel{\text{def}}{=} \sup_{a \in A \backslash M} \xi_a$ is less than λ. We claim that $d <_I f_{\eta(\rho)}$. In fact, suppose that $a \in A \backslash (M \cup Z_\rho)$. Then $a \in A \backslash (Z_{\xi_a} \cup Z_\rho)$, and so $d(a) < f_{\eta(\xi_a)}(a) \leq f_{\eta(\rho)}(a)$. Thus $d <_I f_{\eta(\rho)}$, as claimed. Now it follows easily from Proposition 33.17 that h is an exact upper bound for f.

For the final portion of (ii), it suffices to show

(1) There is a $W \in I$ such that $\mathrm{cf}(h(a)) = \lambda$ for all $a \in A \backslash W$.

In fact, let

$$W = \{a \in A : \exists \xi_a < \lambda \forall \xi' \in [\xi_a, \lambda)[a \in Z_{\xi'}]\}.$$

Since $|A| < \lambda$, the ordinal $\rho \stackrel{\text{def}}{=} \sup_{a \in W} \xi_a$ is less than λ. Clearly $W \subseteq Z_\rho$, so $W \in I$. For $a \in A \backslash W$ we have $\forall \xi < \lambda \exists \xi' \in [\xi, \lambda)[a \notin Z_{\xi'}]$. This gives an increasing sequence $\langle \sigma_\nu : \nu < \lambda \rangle$ of ordinals less than λ such that $a \notin Z_{\sigma_\nu}$ for all $\nu < \lambda$. By the strong increasing property it follows that $f_{\eta(\sigma_0)}(a) < f_{\eta(\sigma_1)}(a) < \cdots$, and so $h(a)$ has cofinality λ. This proves (1), and with it, (ii).

(ii)⇒(iii): Let $W = \{a \in A : \mathrm{cf}(h(a)) \neq \lambda\}$; so $W \in I$ by (ii). Since I is a proper ideal, choose $a_0 \in A \backslash W$, and define

$$h'(a) = \begin{cases} h(a) & \text{if } a \in A \backslash W, \\ h(a_0) & \text{if } a \in W. \end{cases}$$

Then $h =_I h'$, and it follows that h' satisfies the properties needed.

(iii)⇒(iv): For each $a \in A$, let $\langle \mu^a_\xi : \xi < \lambda \rangle$ be a strictly increasing sequence of ordinals with supremum $h(a)$. Define $g_\xi(a) = \mu^a_\xi$ for all $a \in A$ and $\xi < \lambda$. Clearly $g_\xi < g_\eta$ if $\xi < \eta$. Now suppose that $\xi < \lambda$. Then $f_\xi <_I h$. For each $a \in A$ such that $f_\xi(a) < h(a)$ choose $\rho_a < \lambda$ such that $f_\xi(a) < \mu^a_{\rho_a}$. Since $|A| < \lambda$, choose $\eta < \lambda$ such that $\rho_a < \eta$ for all $a \in A$. Then for any $a \in A$ such that $f_\xi(a) < h(a)$ we have $f_\xi(a) < \mu^a_\eta = g_\eta(a)$. Hence $f_\xi <_I g_\eta$, which is half of what is desired in (iv).

Now suppose that $\xi < \lambda$. Then $g_\xi < h$, so by the exactness of h, there is an $\eta < \lambda$ such that $g_\xi <_I f_\eta$, as desired.

(iv)⇒(i): Assume (iv). Define strictly increasing continuous sequences $\langle \eta(\xi) : \xi < \lambda \rangle$ and $\langle \rho(\xi) : \xi < \lambda \rangle$ of ordinals less than λ as follows. Let $\eta(0) = 0$, and choose $\rho(0)$ so that $g_0 <_I f_{\rho(0)}$. If $\eta(\xi)$ and $\rho(\xi)$ have been defined, choose $\eta(\xi + 1) > \eta(\xi)$ such that

$f_{\rho(\xi)} \leq_I g_{\eta(\xi+1)}$, and choose $\rho(\xi+1) > \rho(\xi)$ such that $g_{\eta(\xi+1)} <_I f_{\rho(\xi+1)}$. Thus for every $\xi < \lambda$ we have

$$g_{\eta(\xi)} <_I f_{\rho(\xi)} \leq_I g_{\eta(\xi+1)}.$$

since obviously $\langle g_{\eta(\xi)} : \xi < \lambda \rangle$ is strongly increasing under I, Lemma 33.27 gives (i). $\square$

The notion of a strongly increasing sequence is clarified by giving an example of a sequence such that no subsequence is strongly increasing. This example depends on the following well-known lemma.

Lemma 33.29. *If κ is a regular cardinal and I is the ideal $[\kappa]^{<\kappa}$ on κ, then there is a sequence $f \stackrel{\text{def}}{=} \langle f_\xi : \xi < \kappa^+ \rangle$ of members of ${}^\kappa\kappa$ such that $f_\xi <_I f_\eta$ whenever $\xi < \eta < \kappa$.*

Proof. We construct the sequence by recursion. Let $f_0(\alpha) = 0$ for all $\alpha < \kappa$. If f_ξ has been defined, let $f_{\xi+1}(\alpha) = f_\xi(\alpha) + 1$ for all $\alpha < \kappa$. Now suppose that $\xi < \kappa$ is a limit ordinal, and f_η has been defined for every $\eta < \xi$. Let $\langle \eta(\beta) : \beta < \gamma \rangle$ be a strictly increasing sequence of ordinals with supremum ξ, where $\gamma = \text{cf}(\xi)$. Thus $\gamma \leq \kappa$. Define

$$f_\xi(\alpha) = (\sup_{\beta \leq \alpha} f_{\eta(\beta)}(\alpha)) + 1.$$

The sequence constructed this way is as desired. For example, if ξ is a limit ordinal as above, then for each $\rho < \kappa$ we have $\{\alpha < \kappa : f_{\eta(\rho)}(\alpha) \geq f_\xi(\alpha)\} \subseteq \rho$, and so $f_{\eta(\rho)} <_I f_\xi$. $\square$

Now let $A = \kappa$ and let I and f be as in the lemma. Suppose that f has a strongly increasing subsequence of length κ^+ under I. Then by proposition 33.28, f has an exact upper bound h such that $\text{cf}(h(\alpha)) = \kappa^+$ for all $\alpha < \kappa$. Now the function k with domain κ taking the constant value κ is clearly an upper bound for f. Hence $h \leq_I k$. Hence there is an $\alpha < \kappa$ such that $h(\alpha) \leq k(\alpha) = \kappa$, contradiction.

A further fact along these lines is as follows.

Lemma 33.30. *Suppose that $I = [\omega]^{<\omega}$ and $f \stackrel{\text{def}}{=} \langle f_\xi : \xi < \lambda \rangle$ is a $<_I$-increasing sequence of members of ${}^\omega\omega$ which has an exact upper bound h, where λ is an infinite cardinal. Then $\langle f_\xi : \xi < \lambda \rangle$ is a scale, i.e., for any $g \in {}^\omega\omega$ there is a $\xi < \lambda$ such that $g <_I f_\xi$.*

Proof. Let $k(m) = \omega$ for all $m < \omega$. Then k is an upper bound for f under $<_I$, and so $h \leq_I k$. Letting $h'(m) = \min(h(m), k(m))$ for all $m \in \omega$, we thus get $h =_I h'$. So by Proposition 33.14, h' is also an exact upper bound for f. Hence we may assume that $h(m) \leq \omega$ for every $m < \omega$. Now we claim

(1) $\exists n < \omega \forall p \geq n(0 < h(p))$.

In fact, the set $\{p \in \omega : f_0(p) \geq h(p)\}$ is in I, so there is an n such that $f_0(p) < h(p)$ for all $p \geq n$, as desired in (1).

Let n_0 be as in (1).

(2) $M \stackrel{\text{def}}{=} \{p \in \omega : h(p) \neq \omega\}$ is finite.

For, suppose that M is infinite. Define

$$l(p) = \begin{cases} h(p) - 1 & \text{if } 0 < h(p) < \omega, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $l <_I h$. For, $\{p : l(p) \geq h(p)\} \subseteq \{p : h(p) = 0\} \in I$. So our claim holds. Now by exactness, choose $\xi < \kappa$ such that $l <_I f_\xi$. Then we can choose $p \in M$ such that $l(p) < f_\xi(p) < h(p)$, contradiction.

Thus M is finite. Hence we may assume that $h(p) = \omega$ for all p, and the desired conclusion of the lemma follows. □

Now there is a model M of ZFC in which there are no scales (see for example Blass [∞]), and yet it is easy to see that there is a sequence $f \stackrel{\text{def}}{=} \langle f_\xi : \xi < \omega_1 \rangle$ which is $<_I$-increasing. Hence by Lemma 33.30, this sequence does not have an exact upper bound.

Another fact which helps the intuition on exact upper bounds is as follows.

Lemma 33.31. *Let κ be a regular cardinal, and let $I = [\kappa]^{<\kappa}$. For each $\xi < \kappa$ let $f_\xi \in {}^\kappa\kappa$ be defined by $f_\xi(\alpha) = \xi$ for all $\alpha < \kappa$. Thus $f \stackrel{\text{def}}{=} \langle f_\xi : \xi < \kappa \rangle$ is increasing everywhere. Claim: f does not have a least upper bound under $<_I$. (Hence it does not have an exact upper bound.)*

Proof. Suppose that h is an upper bound for f under $<_I$. We find another upper bound k for f under $<_I$ such that h is not $\leq_I k$. First we claim

(1) $\forall \alpha < \kappa \exists \beta < \kappa \forall \gamma \geq \beta (\alpha \leq h(\gamma))$.

In fact, otherwise we get a $\xi < \kappa$ such that for all $\beta < \kappa$ there is a $\gamma > \beta$ such that $\xi > h(\gamma)$. But then $|\{\alpha < \kappa : f_\xi(\alpha) > h(\alpha)\}| = \kappa$, contradiction.

By (1) there is a strictly increasing sequence $\langle \beta_\alpha : \alpha < \kappa \rangle$ of ordinals less than κ such that for all $\alpha < \kappa$ and all $\gamma \geq \beta_\alpha$ we have $\alpha < h(\gamma)$. Now we define $k \in {}^\kappa\kappa$ by setting, for each $\gamma < \kappa$,

$$k(\gamma) = \begin{cases} \alpha & \text{if } \beta_{\alpha+1} \leq \gamma < \beta_{\alpha+2}, \\ h(\gamma) & \text{otherwise.} \end{cases}$$

To see that k is an upper bound for f under $<_I$, take any $\xi < \kappa$. If $\beta_{\xi+1} \leq \gamma$, then $h(\gamma) \geq \xi + 1$, and hence $k(\gamma) \geq \xi = f_\xi(\gamma)$, as desired. For each $\xi < \kappa$ we have $k(\beta_{\xi+1}) = \xi < h(\beta_{\xi+1})$, so h is not $\leq_I k$. □

Now we define a partition property. Suppose that I is an ideal over a set A, λ is an uncountable regular cardinal $> |A|$, $f = \langle f_\xi : \xi < \lambda \rangle$ is a $<_I$-increasing sequence of members of ${}^A\mathbf{On}$, and κ is a regular cardinal such that $|A| < \kappa \leq \lambda$. The following property of these things is denoted by $(*)_\kappa$:

$(*)_\kappa$ For all unbounded $X \subseteq \lambda$ there is an $X_0 \subseteq X$ of order type κ such that $\langle f_\xi : \xi \in X_0 \rangle$ is strongly increasing under I.

Proposition 33.32. *Assume the above notation, with $\kappa < \lambda$. Then $(*)_\kappa$ holds iff the set*

$$\{\delta < \lambda : \text{cf}(\delta) = \kappa \text{ and } \langle f_\xi : \xi \in X_0 \rangle \textit{ is strongly increasing under } I$$
$$\textit{for some unbounded } X_0 \subseteq \delta\}$$

is stationary in λ.

Proof. Let S be the indicated set of ordinals δ.

$\Rightarrow$: Assume $(*)_\kappa$ and suppose that $C \subseteq \lambda$ is a club. Choose $C_0 \subseteq C$ of order type κ such that $\langle f_\xi : \xi \in C_0 \rangle$ is strongly increasing under I. Let $\delta = \sup(C_0)$. Clearly $\delta \in C \cap S$.

$\Leftarrow$: Assume that S is stationary in λ, and suppose that $X \subseteq \lambda$ is unbounded. Define

$$C = \{\alpha \in \lambda : \alpha \text{ is a limit ordinal and } X \cap \alpha \text{ is unbounded in } \alpha\}.$$

We check that C is club in λ. For closure, suppose that $\alpha < \lambda$ is a limit ordinal and $C \cap \alpha$ is unbounded in α; we want to show that $\alpha \in C$. So, we need to show that $X \cap \alpha$ is unbounded in α. To this end, take any $\beta < \alpha$; we want to find $\gamma \in X \cap \alpha$ such that $\beta < \gamma$. Since $C \cap \alpha$ is unbounded in α, choose $\delta \in C \cap \alpha$ such that $\beta < \delta$. By the definition of C we have that $X \cap \delta$ is unbounded in δ. So we can choose $\gamma \in X \cap \delta$ such that $\beta < \gamma$. Since $\gamma < \delta < \alpha$, γ is as desired. So, indeed, C is closed.

To show that C is unbounded in λ, take any $\beta < \lambda$; we want to find an $\alpha \in C$ such that $\beta < \alpha$. Since X is unbounded in λ, we can choose a sequence $\gamma_0 < \gamma_1 < \cdots$ of elements of X with $\beta < \gamma_0$. Now λ is uncountable and regular, so $\sup_{n\in\omega} \gamma_n < \lambda$, and it is the member of C we need.

Now choose $\delta \in C \cap S$. This gives us an unbounded set X_0 in δ such that $\langle f_\xi : \xi \in X_0 \rangle$ is strongly increasing under I. Now also $X \cap \delta$ is unbounded, since $\delta \in C$. Hence we can define by induction two increasing sequences $\langle \eta(\xi) : \xi < \kappa \rangle$ and $\langle \nu(\xi) : \xi < \kappa \rangle$ such that each $\eta(\xi)$ is in X_0, each $\nu(\xi)$ is in X, and $\eta(\xi) < \nu(\xi) \leq \eta(\xi + 1)$ for all $\xi < \kappa$. It follows by the sandwich argument, Lemma 33.28, that $X_1 \stackrel{\text{def}}{=} \{\nu(\xi) : \xi < \kappa\}$ is a subset of X as desired in $(*)_\kappa$. $\square$

Finally, we introduce the bounding projection property.

Suppose that $f = \langle f_\xi : \xi < \lambda \rangle$ is a $<_I$-increasing sequence of functions in ${}^A\mathbf{On}$, with λ a regular cardinal $> |A|$. Also suppose that κ is a regular cardinal and $|A| < \kappa \leq \lambda$.

We say that f has the *bounding projection property for κ* iff whenever $\langle S(a) : a \in A \rangle$ is a system of nonempty sets of ordinals such that each $|S(a)| < \kappa$ and for each $\xi < \lambda$ we have $f_\xi <_I \sup(S(a))$, then for some $\xi < \lambda$, the function $\text{proj}(f_\xi, \langle S(a) : a \in A \rangle)$ $<_I$-bounds f.

We need the following simple result.

Proposition 33.33. *Suppose that $f = \langle f_\xi : \xi < \lambda \rangle$ is a $<_I$-increasing sequence of functions in $\mathbf{On}^A$, with λ a regular cardinal $> |A|$. Also suppose that κ is a regular cardinal and $|A| < \kappa \leq \lambda$. Assume that f has the bounding projection property for κ.*

Also suppose that $f' = \langle f'_\xi : \xi < \lambda \rangle$ is a sequence of functions in $\mathbf{On}^A$, and $f_\xi =_I f'_\xi$ for every $\xi < \lambda$.

Then f' has the bounding projection property for κ.

Proof. Clearly f' is $<_I$-increasing, so that the setup for the bounding projection property holds. Now suppose that $\langle S(a) : a \in A \rangle$ is a system of nonempty sets of ordinals such that each $|S(a)| < \kappa$ and for each $\xi < \lambda$ we have $f'_\xi <_I \sup(S)$. Then the same is true for f, so by the bounding projection property for f we can choose $\xi < \lambda$ such that the function $\text{proj}(f_\xi, \langle S(a) : a \in A \rangle)$ $<_I$-bounds f. Now suppose that $\eta < \lambda$. Then

$f_\eta \le_I \operatorname{proj}(f_\xi, \langle S(a) : a \in A\rangle)$. Hence $f'_\eta \le_I \operatorname{proj}(f_\xi, \langle S(a) : a \in A\rangle)$, and $\operatorname{proj}(f_\xi, \langle S(a) : a \in A\rangle) = \operatorname{proj}(f'_\xi, \langle S(a) : a \in A\rangle)$, as desired. $\square$

The following proposition shows that we can weaken the bounded projection property somewhat, by replacing "$<_I$" by "$<$ (everywhere)".

Proposition 33.34. *Suppose that $f = \langle f_\xi : \xi < \lambda\rangle$ is a $<_I$-increasing sequence of functions in $\mathbf{On}^A$, with λ a regular cardinal $> |A|$. Also suppose that κ is a regular cardinal and $|A| < \kappa \le \lambda$. Then the following conditions are equivalent:*

(i) f has the bounding projection property for κ.

(ii) If $\langle S(a) : a \in A\rangle$ is a system of nonempty sets of ordinals such that each $|S(a)| < \kappa$ and for each $\xi < \lambda$ we have $f_\xi < \sup(S)$ (everywhere), then for some $\xi < \lambda$, the function proj$(f_\xi, \langle S(a) : a \in A\rangle)$ $<_I$-bounds f.

Proof. Obviously (i)⇒(ii). Now assume that (ii) holds, and suppose that $\langle S(a) : a \in A\rangle$ is a system of sets of ordinals such that each $|S(a)| < \kappa$ and for each $\xi < \lambda$ we have $f_\xi <_I \sup(S)$. Now for each $a \in A$ let

$$\gamma(a) = \begin{cases} \sup\{f_\xi(a) + 1 : \xi < \lambda \text{ and } f_\xi(a) \ge \sup(S(a))\} & \text{if this set is nonempty,}\\ \sup(S(a)) + 1 & \text{otherwise;}\end{cases}$$

$$S'(a) = S(a) \cup \{\gamma(a)\}.$$

Note that $f_\xi < \sup(S')$ everywhere. Hence by (ii), there is a $\xi < \lambda$ such that the function $\operatorname{proj}(f_\xi, \langle S'(a) : a \in A\rangle)$ $<_I$-bounds f. Now let $\eta < \lambda$. If $f_\xi(a) < \sup(S(a))$ and $f_\eta(a) < (\operatorname{proj}(f_\xi, \langle S'(a) : a \in A\rangle))(a)$, then

$$\begin{aligned}(\operatorname{proj}(f_\xi, \langle S'(a) : a \in A\rangle))(a) &= \min(S'(a)\backslash f_\xi(a))\\ &= \min(S(a)\backslash f_\xi(a))\\ &= (\operatorname{proj}(f_\xi, \langle S(a) : a \in A\rangle))(a).\end{aligned}$$

Hence $f_\eta <_I \operatorname{proj}(f_\xi, \langle S(a) : a \in A\rangle)$, as desired. $\square$

Lemma 33.35. (Bounding projection lemma) *Suppose that I is an ideal over A, $\lambda > |A|$ is a regular cardinal, $f = \langle f_\xi : \xi < \lambda\rangle$ is a $<_I$-increasing sequence satisfying $(*)_\kappa$ for a regular cardinal κ such that $|A| < \kappa \le \lambda$. Then f has the bounding projection property for κ.*

Proof. Assume the hypothesis of the lemma and of the bounding projection property for κ. For every $\xi < \lambda$ let

$$f^+_\xi = \operatorname{proj}(f_\xi, S).$$

Suppose that the conclusion of the bounding projection property fails. Then for every $\xi < \lambda$, the function f^+_ξ is not a bound for f, and so there is a $\xi' < \lambda$ such that $f_{\xi'} \not\le_I f^+_\xi$. Since $f_\xi \le_I f^+_\xi$, we must have $\xi < \xi'$. Clearly for any $\xi'' \ge \xi'$ we have $f_{\xi''} \not\le_I f^+_\xi$. Thus for every $\xi'' \ge \xi'$ we have $<(f^+_\xi, f_{\xi''}) \in I^+$. Now we define a sequence $\langle \xi(\mu) : \mu < \lambda\rangle$ of elements of λ by recursion. Let $\xi(0) = 0$. Suppose that $\xi(\mu)$ has been defined. Choose $\xi(\mu+1) > \xi(\mu)$ so that $<(f^+_{\xi(\mu)}, f_{\xi''}) \in I^+$ for every $\xi'' \geq \xi(\mu+1)$. If ν is limit and $\xi(\mu)$ has been defined for all $\mu < \nu$, let $\xi(\nu) = \sup_{\mu<\nu} \xi(\mu)$. Then let X be the range of this sequence. Thus

$$\text{if } \xi, \xi' \in X \text{ and } \xi < \xi', \text{ then } <(f^+_\xi, f_{\xi'}) \in I^+.$$

Since $(*)_\kappa$ holds, there is a subset $X_0 \subseteq X$ of order type κ such that $\langle f_\xi : \xi \in X_0\rangle$ is strongly increasing under I. Let $\langle Z_\xi : \xi \in X_0\rangle$ be as in the definition of strongly increasing under I.

For every $\xi \in X_0$, let ξ' be the successor of ξ in X_0. Note that

$$<(f^+_\xi, f_{\xi'})\backslash(Z_\xi \cup Z_{\xi'} \cup \{a \in A : f_\xi(a) \geq \sup(S(a))\}) \in I^+,$$

and hence it is nonempty. So, choose

$$a_\xi \in <(f^+_\xi, f_{\xi'})\backslash(Z_\xi \cup Z_{\xi'} \cup \{a \in A : f_\xi(a) \geq \sup(S(a))\}).$$

Note that this implies that $f^+_\xi(a_\xi) \in S(a_\xi)$. Since $\kappa > |A|$, we can find a single $a \in A$ such that $a = a_\xi$ for all ξ in a subset X_1 of X_0 of size κ. Now for $\xi_1 < \xi_2$ with both in X_1, we have

$$f^+_{\xi_1}(a) < f_{\xi'_1}(a) \leq f_{\xi_2}(a) \leq f^+_{\xi_2}(a).$$

[The first inequality is a consequence of $a = a_{\xi_1} \in <(f^+_{\xi_1}, f_{\xi'_1})$, the second follows from $\xi'_1 \leq \xi_2$ and the fact that

$$a = a_{\xi_1} = a_{\xi_2} \in A\backslash(Z_{\xi'_1} \cup Z_{\xi_2}),$$

and the third is true by the definition of $f^+_{\xi_2}$.]

Thus $\langle f^+_\xi(a) : \xi \in X_1\rangle$ is a strictly increasing sequence of members of $S(a)$. This contradicts our assumption that $|S(a)| < \kappa$. □

The next lemma reduces the problem of finding an exact upper bound to that of finding a least upper bound.

Lemma 33.36. *Suppose that I is a proper ideal over A, $\lambda \geq |A|^+$ is a regular cardinal, and $f = \langle f_\xi : \xi \in \lambda\rangle$ is a $<_I$-increasing sequence of functions in ${}^A\mathbf{On}$ satisfying the bounding projection property for $|A|^+$. Suppose that h is a least upper bound for f. Then h is an exact upper bound.*

Proof. Assume the hypotheses, and suppose that $g <_I h$; we want to find $\xi < \lambda$ such that $g <_I f_\xi$. By increasing h on a subset of A in the ideal, we may assume that $g < h$ everywhere. Define $S_a = \{g(a), h(a)\}$ for every $a \in A$. By the bounding projection property we get a $\xi < \lambda$ such that $f^+_\xi \stackrel{\text{def}}{=} \text{proj}(f_\xi, \langle S_a : a \in A\rangle)$ is an upper bound for f. We shall prove that $g <_I f_\xi$, as required.

Since h is a least upper bound, it follows that $h \leq_I f^+_\xi$. Thus $M \stackrel{\text{def}}{=} \{a \in A : h(a) > f^+_\xi(a)\} \in I$. Also, the set $N \stackrel{\text{def}}{=} \{a \in A : f_\xi(a) \geq \sup S_a\}$ is in I. Suppose that

$a \in A \backslash (M \cup N)$. Then $g(a) < h(a) \leq f^+_\xi(a) = \min(S_a \backslash f_\xi(a))$, and this implies that $g(a) < f_\xi(a)$. So $g <_I f_\xi$, as desired. $\square$

Here is our first existence theorem for exact upper bounds.

Theorem 33.37. (Existence of exact upper bounds) *Suppose that I is a proper ideal over A, $\lambda > |A|^+$ is a regular cardinal, and $f = \langle f_\xi : \xi \in \lambda \rangle$ is a $<_I$-increasing sequence of functions in ${}^A\mathbf{On}$ that satisfies the bounding projection property for $|A|^+$. Then f has an exact upper bound.*

Proof. Assume the hypotheses. By Lemma 33.36 it suffices to show that f has a least upper bound, and to do this we will apply Proposition 33.16(ii). Suppose that f does not have a least upper bound. Since it obviously has an upper bound, this means, by Proposition 33.16(ii):

(1) For every upper bound $h \in {}^A\mathbf{On}$ for f there is another upper bound h' for f such that $h' \leq_I h$ and $\{a \in A : h'(a) < h(a)\} \in I^+$.

In fact, Proposition 33.16(ii) says that there is another upper bound h' for f such that $h' \leq_I h$ and it is not true that $h =_I h'$. Hence $\{a \in A : h(a) < h'(a)\} \in I$ and $\{a \in A : h(a) \neq h'(a)\} \in I^+$. So

$$\{a \in A : h(a) \neq h'(a)\} \backslash \{a \in A : h(a) < h'(a)\} \in I^+ \quad \text{and}$$
$$\{a \in A : h(a) \neq h'(a)\} \backslash \{a \in A : h(a) < h'(a)\} = \{a \in A : h'(a) < h(a)\},$$

so (1) follows.

Now we shall define by induction on $\alpha < |A|^+$ a sequence $S^\alpha = \langle S^\alpha(a) : a \in A \rangle$ of sets of ordinals satisfying the following conditions:

(2) $0 < |S^\alpha(a)| \leq |A|$ for each $a \in A$;

(3) $f_\xi(a) < \sup S^\alpha(a)$ for all $\xi \in \lambda$ and $a \in A$;

(4) If $\alpha < \beta$, then $S^\alpha(a) \subseteq S^\beta(a)$, and if δ is a limit ordinal, then $S^\delta(a) = \bigcup_{\alpha<\delta} S^\alpha(a)$.

We also define sequences $\langle h_\alpha : \alpha < |A|^+ \rangle$ and $\langle h'_\alpha : \alpha < |A|^+ \rangle$ of functions and $\langle \xi(\alpha) : \alpha < |A|^+ \rangle$ of ordinals.

The definition of S^α for α limit is fixed by (4), and the conditions (2)–(4) continue to hold. To define S^0, pick any function k that bounds f (everywhere) and define $S^0(a) = \{k(a)\}$ for all $a \in A$; so (2)–(4) hold.

Suppose that $S^\alpha = \langle S^\alpha(a) : a \in A \rangle$ has been defined, satisfying (2)–(4); we define $S^{\alpha+1}$. By the bounding projection property for $|A|^+$, there is a $\xi(\alpha) < \lambda$ such that $h_\alpha \stackrel{\text{def}}{=} \text{proj}(f_{\xi(\alpha)}, S^\alpha)$ is an upper bound for f under $<_I$. Then

(5) if $\xi(\alpha) \leq \xi' < \lambda$, then $h_\alpha =_I \text{proj}(f_{\xi'}, S^\alpha)$.

In fact, recall that $h_\alpha(a) = \min(S^\alpha(a) \backslash f_{\xi(\alpha)}(a))$ for every $a \in A$, using (3). Now suppose that $\xi(\alpha) < \xi' < \lambda$. Let $M = \{a \in A : f_{\xi(\alpha)}(a) \geq f_{\xi'}(a)\}$. So $M \in I$. For any $a \in A \backslash M$ we have $f_{\xi(\alpha)}(a) < f_{\xi'}(a)$, and hence

$$\min(S^\alpha(a) \backslash f_{\xi(\alpha)}(a)) \leq \min(S^\alpha(a) \backslash f_{\xi'}(a));$$

it follows that $h_\alpha \leq_I \text{proj}(f_{\xi'}, S^\alpha)$. For the other direction, recall that h_α is an upper bound for f under $<_I$. So $f_{\xi'} \leq_I h_\alpha$. If a is any element of A such that $f_{\xi'}(a) \leq h_\alpha(a)$ then, since $h_\alpha(a) \in S^\alpha(a)$, we get $\min(S^\alpha(a)\backslash f_{\xi'}(a)) \leq h_\alpha(a)$. Thus $\text{proj}(f_{\xi'}, S^\alpha) \leq_I h_\alpha$.

This checks (5).

Now we apply (1) to get an upper bound h'_α for f such that $h'_\alpha \leq_I h_\alpha$ and $< (h'_\alpha, h_\alpha) \in I^+$. We now define $S^{\alpha+1}(a) = S^\alpha(a) \cup \{h'_\alpha(a)\}$ for any $a \in A$.

(6) If $\xi(\alpha) \leq \xi < \lambda$, then $\text{proj}(f_\xi, S^{\alpha+1}) =_I h'_\alpha$.

For, we have $f_\xi \leq_I h'_\alpha$ and, by (5), $h_\alpha =_I \text{proj}(f_\xi, S^\alpha)$. If $a \in A$ is such that $f_\xi(a) \leq h'_\alpha(a)$, $h'_\alpha(a) \leq h_\alpha(a)$, and $h_\alpha(a) = \text{proj}(f_\xi, S^\alpha)(a)$, then $\min(S^\alpha(a)\backslash f_\xi(a)) = h_\alpha(a) \geq h'_\alpha(a) \geq f_\xi(a)$, and hence

$$\text{proj}(f_\xi, S^{\alpha+1})(a) = \min(S^{\alpha+1}(a)\backslash f_\xi(a)) = h'_\alpha(a).$$

It follows that $\text{proj}(f_\xi, S^{\alpha+1}) =_I h'_\alpha$, as desired in (6).

Now since $|A|^+ < \lambda$, let $\xi < \lambda$ be greater than each $\xi(\alpha)$ for $\alpha < |A|^+$. Define $H_\alpha = \text{proj}(f_\xi, S^\alpha)$ for each $\alpha < |A|^+$. Since $\xi > \xi(\alpha)$, we have $H_\alpha =_I h_\alpha$ by (5). Note that $H_{\alpha+1} = \text{proj}(f_\xi, S^{\alpha+1}) =_I h'_\alpha$; so $< (H_{\alpha+1}, H_\alpha) \in I^+$. Now clearly by the construction we have $S^{\alpha_1}(a) \subseteq S^{\alpha_2}(a)$ for all $a \in A$ when $\alpha_1 < \alpha_2 < |A|^+$. Hence we get

(7) if $\alpha_1 < \alpha_2 < |A|^+$, then $H_{\alpha_2} \leq H_{\alpha_1}$, and $< (H_{\alpha_2}, H_{\alpha_1}) \in I^+$.

Now for every $\alpha < |A|^+$ pick $a_\alpha \in A$ such that $H_{\alpha+1}(a_\alpha) < H_\alpha(a_\alpha)$. We have $a_\alpha = a_\beta$ for all α, β in some subset of $|A|^+$ of size $|A|^+$, and this gives an infinite decreasing sequence of ordinals, contradiction. □

Lemma 33.38. *Suppose that I is a proper ideal over A, $\lambda \geq |A|^+$ is a regular cardinal, $f = \langle f_\xi : \xi < \lambda \rangle$ is a $<_I$-increasing sequence of functions in ${}^A\mathbf{On}$, $|A|^+ \leq \kappa \leq \lambda$, f satisfies the bounding projection property for κ, and g is an exact upper bound for f. Then*

$$\{a \in A : g(a) \textit{ is non-limit, or } \text{cf}(g(a)) < \kappa\} \in I.$$

Proof. Let $P = \{a \in A : g(a) \text{ is non-limit, or } \text{cf}(g(a)) < \kappa\}$. If $a \in P$ and $g(a)$ is a limit ordinal, choose $S(a) \subseteq g(a)$ cofinal in $g(a)$ and of order type $< \kappa$. If $g(a) = 0$ let $S(a) = \{0\}$, and if $g(a) = \beta + 1$ for some β let $S(a) = \{\beta\}$. Finally, if $g(a)$ is limit but is not in P, let $S(a) = \{g(a)\}$.

Now for any $\xi < \lambda$ let

$$\begin{aligned} N_\xi &= \{a \in A : f_\xi(a) \geq f_{\xi+1}(a)\} \quad \text{and} \\ Q_\xi &= \{a \in A : f_{\xi+1}(a) \geq g(a)\}. \end{aligned}$$

Then clearly

(∗) If $a \in A\backslash(N_\xi \cup Q_\xi)$, then $f_\xi(a) < \sup(S(a))$.

It follows that $\{a \in A : f_\xi(a) \geq \sup S(a)\} \subseteq N_\xi \cup Q_\xi \in I$. Hence the hypothesis of the bounding projection property holds. Applying it, we get $\xi < \lambda$ such that $f^+_\xi \stackrel{\text{def}}{=}$

$\text{proj}(f_\xi, \langle S(a) : a \in A\rangle)$ $<_I$-bounds f. Since g is a least upper bound for f, we get $g \leq_I f^+_\xi$, and hence $M \stackrel{\text{def}}{=} \{a \in A : f^+_\xi(a) < g(a)\} \in I$. By $(*)$, for any $a \in P\backslash(N_\xi \cup Q_\xi)$ we have $f^+_\xi(a) = \min(S(a)\backslash f_\xi(a)) < g(a)$. This shows that $P\backslash(N_\xi \cup Q_\xi) \subseteq M$, hence $P \subseteq N_\xi \cup Q_\xi \cup M \in I$, so $P \in I$, as desired. □

Now we give our main theorem on the existence of exact upper bounds.

Theorem 33.39. *Suppose that I is a proper ideal over A, $\lambda > |A|^+$ is a regular cardinal, $f = \langle f_\xi : \xi < \lambda\rangle$ is a $<_I$-increasing sequence of functions in ${}^A\mathbf{On}$, and $|A|^+ \leq \kappa$. Then the following are equivalent:*

(i) $(*)_\kappa$ holds for f.

(ii) f satisfies the bounding projection property for κ.

(iii) f has an exact upper bound g such that

$$\{a \in A : g(a) \textit{ is non-limit, or } \operatorname{cf}(g(a)) < \kappa\} \in I.$$

Proof. (i)⇒(ii): By the bounding projection lemma, Lemma 33.35.

(ii)⇒(iii): Since the bounding projection property for κ clearly implies the bounding projection property for $|A|^+$, this implication is true by Theorem 33.37 and Lemma 33.38.

(iii)⇒(i): Assume (iii). By modifying g on a set in the ideal we may assume that $g(a)$ is a limit ordinal and $\operatorname{cf}(g(a)) \geq \kappa$ for all $a \in A$. Choose a club $S(a) \subseteq g(a)$ of order type $\operatorname{cf}(g(a))$. Thus the order type of $S(a)$ is $\geq \kappa$. We prove that $(*)_\kappa$ holds. So, assume that $X \subseteq \lambda$ is unbounded; we want to find $X_0 \subseteq X$ of order type κ over which f is strongly increasing under I. To do this, we intend to define by induction on $\alpha < \kappa$ a function $h_\alpha \in \prod S$ and an index $\xi(\alpha) \in X$ such that

(1) $h_\alpha <_I f_{\xi(\alpha)} \leq_I h_{\alpha+1}$.

(2) The sequence $\langle h_\alpha : \alpha < \kappa\rangle$ is $<$-increasing (increasing everywhere; and hence it certainly is strongly increasing under I).

(3) $\langle \xi(\alpha) : \alpha < \kappa\rangle$ is strictly increasing.

After we have done this, the sandwich argument (Lemma 33.27) shows that $\langle f_{\xi(\alpha)} : \alpha < \kappa\rangle$ is strongly increasing under I and of order type κ, giving the desired result.

The functions h_α are defined as follows.

$h_0 \in \prod S$ is arbitrary.

For a limit ordinal $\delta < \kappa$ let $h_\delta = \sup_{\alpha<\delta} h_\alpha$.

Having defined h_α, we define $h_{\alpha+1}$ as follows. Since g is an exact upper bound and $h_\alpha < g$, choose $\xi(\alpha)$ greater than all $\xi(\beta)$ for $\beta < \alpha$ such that $h_\alpha <_I f_{\xi(\alpha)}$. Also, since $f_\xi <_I g$ for all $\xi < \lambda$, the projections $f^+_\xi = \text{proj}(f, S)$ are defined. We define

$$h_{\alpha+1}(a) = \begin{cases} \max(h_\alpha(a), f^+_{\xi(\alpha)}(a)) + 1 & \text{if } f_{\xi(\alpha)}(a) < g(a), \\ h_\alpha(a) + 1 & \text{if } f_{\xi(\alpha)}(a) \geq g(a). \end{cases}$$

Thus we have

$$h_\alpha <_I f_{\xi(\alpha)} \leq_I h_{\alpha+1}, \text{ for every } \alpha.$$

So conditions (1)–(3) hold. $\square$

Now we apply some infinite combinatorics to get information about $(*)_\kappa$.

Theorem 33.40. (Club guessing) *Suppose that κ is a regular cardinal, λ is a cardinal such that $\mathrm{cf}(\lambda) \geq \kappa^{++}$, and $S^\lambda_\kappa = \{\delta \in \lambda : \mathrm{cf}(\delta) = \kappa\}$. Then there is a sequence $\langle C_\delta : \delta \in S^\lambda_\kappa\rangle$ such that:*

(i) For every $\delta \in S^\lambda_\kappa$ the set $C_\delta \subseteq \delta$ is club, of order type κ.

(ii) For every club $D \subseteq \lambda$ there is a $\delta \in D \cap S^\lambda_\kappa$ such that $C_\delta \subseteq D$.

The sequence $\langle C_\delta : \delta \in S^\lambda_\kappa\rangle$ is called a *club guessing sequence* for S^λ_κ.

Proof. First we take the case of uncountable κ. Fix a sequence $C' = \langle C'_\delta : \delta \in S^\lambda_\kappa\rangle$ such that $C'_\delta \subseteq \delta$ is club in δ of order type κ, for every $\delta \in S^\lambda_\kappa$. For any club E of λ, let

$$C' \restriction E = \langle C'_\delta \cap E : \delta \in S^\lambda_\kappa \cap E'\rangle,$$

where $E' = \{\delta \in E : E \cap \delta \text{ is unbounded in } \delta\}$. Clearly E' is also club in λ. Also note that $C'_\delta \cap E$ is club in δ for each $\delta \in S^\lambda_\kappa \cap E'$. We claim:

(1) There is a club E of λ such that for every club D of λ there is a $\delta \in D \cap E' \cap S^\lambda_\kappa$ such that $C'_\delta \cap E \subseteq D$.

Note that if we prove (1), then the theorem follows by defining $C_\delta = C'_\delta \cap E$ for all $\delta \in E' \cap S^\lambda_\kappa$, and $C_\delta = C'_\delta$ for $\delta \in S^\kappa_\lambda \backslash E'$.

Assume that (1) is false. Hence for every club $E \subseteq \lambda$ there is a club $D_E \subseteq \lambda$ such that for every $\delta \in D_E \cap E' \cap S^\lambda_\kappa$ we have

$$C'_\delta \cap E \not\subseteq D_E.$$

We now define a sequence $\langle E^\alpha : \alpha < \kappa^+\rangle$ of clubs of λ decreasing under inclusion, by induction on α:

(2) $E^0 = \lambda$.

(3) If $\gamma < \kappa^+$ is a limit ordinal and E^α has been defined for all $\alpha < \gamma$, we set $E^\gamma = \bigcap_{\alpha<\gamma} E^\alpha$. Since $\gamma < \kappa^+ < \mathrm{cf}(\lambda)$, E^γ is club in λ.

(4) If E^α has been defined, let $E^{\alpha+1}$ be the set of all limit points of $E^\alpha \cap D_{E^\alpha}$, i.e., the set of all $\varepsilon < \lambda$ such that $E^\alpha \cap D_{E^\alpha} \cap \varepsilon$ is unbounded in ε.

This defines the sequence. Let $E = \bigcap_{\alpha<\kappa^+} E^\alpha$. Then E is club in λ. Take any $\delta \in S^\lambda_\kappa \cap E$. Since $|C'_\delta| = \kappa$ and the sequence $\langle E^\alpha : \alpha < \kappa^+\rangle$ is decreasing, there is an $\alpha < \kappa^+$ such that $C'_\delta \cap E = C'_\delta \cap E^\alpha$. So $C'_\delta \cap E^\alpha = C'_\delta \cap E^{\alpha+1}$. Hence $C'_\delta \cap E^\alpha \subseteq D_{E^\alpha}$, contradiction.

Thus the case κ uncountable has been finished.

Now we take the case $\kappa = \omega$. For $S = S^\lambda_{\aleph_0}$ fix $C = \langle C_\delta : \delta \in S\rangle$ so that C_δ is club in δ with order type ω. We denote the n-th element of C_δ by $C_\delta(n)$. For any club $E \subseteq \lambda$ and any $\delta \in S \cap E'$ we define

$$C^E_\delta = \{\max(E \cap (C_\delta(n) + 1)) : n \in \omega\},$$

where again E' is the set of limit points of members of E. This set is cofinal in δ. In fact, given $\alpha < \delta$, there is a $\beta \in E \cap \delta$ such that $\alpha < \beta$ since $\delta \in E'$, and there is an $n \in \omega$ such that $\beta < C_\delta(n)$. Then $\alpha < \max(E \cap (C_\delta(n) + 1))$, as desired. There may be repetitions in the description of C^E_δ, but $\max(E \cap (C_\delta(n) + 1)) \le \max(E \cap (C_\delta(m) + 1))$ if $n < m$, so C^E_δ has order type ω. We claim

(5) There is a closed unbounded $E \subseteq \lambda$ such that for every club $D \subseteq \lambda$ there is a $\delta \in D \cap S \cap E'$ such that $C^E_\delta \subseteq D$. [This proves the club guessing property.]

Suppose that (5) fails. Thus for every closed unbounded $E \subseteq \lambda$ there exist a club $D_E \subseteq \lambda$ such that for every $\delta \in D_E \cap S \cap E'$ we have $C^E_\delta \not\subseteq D$. Then we construct a descending sequence E^α of clubs in λ as in the case $\kappa > \omega$, for $\alpha < \omega_1$. Thus for each $\alpha < \omega_1$ and each $\delta \in D_{E^\alpha} \cap S \cap (E^\alpha)'$ we have $C^{E^\alpha}_\delta \not\subseteq D_{E^\alpha}$. Let $E = \bigcap_{\alpha<\omega_1} E^\alpha$. Take any $\delta \in S \cap E$. For $n \in \omega$ and $\alpha < \beta$ we have

$$E^\alpha \cap (C_\delta(n) + 1) \supseteq E^\beta \cap (C_\delta(n) + 1),$$

and so $\max(E^\alpha \cap (C_\delta(n)+1)) \ge \max(E^\beta \cap (C_\delta(n)+1))$; it follows that there is an $\alpha_n < \omega_1$ such that $\max(E^\beta \cap (C_\delta(n) + 1)) = \max(E^{\alpha_n} \cap (C_\delta(n) + 1))$ for all $\beta > \alpha_n$. Choose γ greater than all α_n. Thus

(6) For all $\varepsilon > \gamma$ and all $n \in \omega$ we have $\max(E^\varepsilon \cap (C_\delta(n) + 1)) = \max(E^\gamma \cap (C_\delta(n) + 1))$.

But there is a $\rho \in C^{E^\gamma}_\delta \setminus D_{E^\gamma}$; say that $\rho = \max(E^\gamma \cap (C_\delta(n) + 1))$. Then $\rho = \max(E^{\gamma+1} \cap (C_\delta(n) + 1)) \in E^{\gamma+1} = (E^\gamma \cap D_{E^\gamma})' \in D_{E^\gamma}$, contradiction. □

Lemma 33.41. *Suppose that:*

(i) I is an ideal over A.

(ii) κ and λ are regular cardinals such that $|A| < \kappa$ and $\kappa^{++} < \lambda$.

(iii) $f = \langle f_\xi : \xi < \lambda\rangle$ is a sequence of length λ of functions in ${}^A\mathbf{On}$ that is $<_I$-increasing and satisfies the following condition:

For every $\delta < \lambda$ with $\mathrm{cf}(\delta) = \kappa^{++}$ there is a club $E_\delta \subseteq \delta$ such that for some $\delta' \ge \delta$ with $\delta' < \lambda$,

$$(\star) \qquad \sup\{f_\alpha : \alpha \in E_\delta\} \le_I f_{\delta'}.$$

Under these assumptions, $(*)_\kappa$ holds for f.

Proof. Assume the hypotheses. Let $S = S^{\kappa^{++}}_\kappa$; so S is stationary in κ^{++}. By Theorem 33.40, let $\langle C_\delta : \delta \in S\rangle$ be a club guessing sequence for S; thus

(1) For every $\delta \in S$, the set $C_\delta \subseteq \delta$ is a club of order type κ.

(2) For every club $D \subseteq \kappa^{++}$ there is a $\delta \in D \cap S$ such that $C_\delta \subseteq D$.

Now let $U \subseteq \lambda$ be unbounded; we want to find $X_0 \subseteq U$ of order type κ such that $\langle f_\xi : \xi \in X_0 \rangle$ is strongly increasing under I. To do this we first define an increasing continuous sequence $\langle \xi(i) : i < \kappa^{++} \rangle \in {}^{\kappa^{++}}\lambda$ recursively.

Let $\xi(0) = 0$. For i limit, let $\xi(i) = \sup_{k<i} \xi(k)$.

Now suppose for some $i < \kappa^{++}$ that $\xi(k)$ has been defined for every $k \leq i$; we define $\xi(i+1)$. For each $\alpha \in S$ we define

$$h_\alpha = \sup\{f_\eta : \eta \in \xi[C_\alpha \cap (i+1)]\} \quad \text{and}$$
$$\sigma_\alpha = \begin{cases} \text{least } \sigma \in (\xi(i), \lambda) \text{ such that } h_\alpha \leq_I f_\sigma & \text{if there is such a } \sigma, \\ \xi(i) + 1 & \text{otherwise.} \end{cases}$$

Now we let $\xi(i+1)$ be the least member of U which is greater than $\sup\{\sigma_\alpha : \alpha \in S\}$. It follows that

(3) If $\alpha \in S$ and the first case in the definition of σ_α holds, then $h_\alpha <_I f_{\xi(i+1)}$.

Now the set $F \stackrel{\text{def}}{=} \{\xi(k) : k \in \kappa^{++}\}$ is closed, and has order type κ^{++}. Let $\delta = \sup F$. Then F is a club of δ, and $\mathrm{cf}(\delta) = \kappa^{++}$. Hence by the hypothesis (iii) of the lemma, there is a club $E_\delta \subseteq \delta$ and a $\delta' \in [\delta, \lambda)$ such that $(\star)$ in the lemma holds. Note that $F \cap E_\delta$ is club in δ.

Let $D = \xi^{-1}[F \cap E_\delta]$. Since ξ is strictly increasing and continuous, it follows that D is club in κ^{++}. Hence by (2) there is an $\alpha \in D \cap S$ such that $C_\alpha \subseteq D$. Hence

$$\overline{C}_\alpha \stackrel{\text{def}}{=} \xi[C_\alpha] \subseteq F \cap E_\delta$$

is club in $\xi(\alpha)$ of order type κ. Then by $(\star)$ we have

$$\sup\{f_\rho : \rho \in \overline{C}_\alpha\} \leq_I f_{\delta'}.$$

Now

(4) For every $\rho < \rho'$ both in $\overline{C}_\alpha$, we have $\sup\{f_\zeta : \zeta \in \overline{C}_\alpha \cap (\rho+1)\} <_I f_{\rho'}$.

To prove this, note that there is an $i < \kappa^{++}$ such that $\rho = \xi(i)$. Now follow the definition of $\xi(i+1)$. There C_α was considered (among all other closed unbounded sets in the guessing sequence), and h_α was formed at that stage. Now

$$h_\alpha = \sup\{f_\eta : \eta \in \xi[C_\alpha \cap (i+1)]\} \leq \sup\{f_\eta : \eta \in \xi[C_\alpha]\} = \sup\{f_\eta : \eta \in \overline{C}_\alpha\} \leq_I f_{\delta'},$$

so the first case in the definition of σ_α holds. Thus by (3), $h_\alpha <_I f_{\xi(i+1)}$. Clearly $\xi(i+1) \leq \rho'$, so (4) follows.

Now let $\langle \eta(\nu) : \nu < \kappa \rangle$ be the strictly increasing enumeration of $\overline{C}_\alpha$, and set

$$X_0 = \{\eta(\omega \cdot \rho + 2m + 1) : \rho < \kappa, 0 < m < \omega\},$$
$$X_1 = \{\eta(\omega \cdot \rho + 2m) : \rho < \kappa, 0 < m < \omega\},$$

and for each $\beta \in X_1$ let $f'_\beta = \sup\{f_\sigma + 1 : \sigma \in X_0 \cap \beta\}$. Then for $\beta < \beta'$, both in X_1, we have $f'_\beta < f'_{\beta'}$. Now suppose that $\zeta \in X_0$; say $\zeta = \eta(\omega \cdot \rho + 2m + 1)$ with $\rho < \kappa$ and $0 < m < \omega$. Then

$$\begin{aligned} f'_{\eta(\omega\cdot\rho+2m)} &= \sup\{f_\sigma + 1 : \sigma \in X_0 \cap \eta(\omega \cdot \rho + 2m)\} <_I f_\zeta \quad \text{by (4)} \\ &\le \sup\{f_\sigma + 1 : \sigma \in X_0 \cap \eta(\omega \cdot \rho + 2m + 2)\} \\ &= f'_{\eta(\omega\cdot\rho+2m+2)}. \end{aligned}$$

Hence by Proposition 33.27, $\langle f_\zeta : \zeta \in X_0\rangle$ is very strongly increasing under I. □

Now we need a purely combinatorial proposition.

Proposition 33.42. *Suppose that κ and λ are regular cardinals, and $\kappa^{++} < \lambda$. Suppose that F is a function with domain contained in $[\lambda]^{<\kappa}$ and range contained in λ. Suppose that for every $\delta \in S^\lambda_{\kappa^{++}}$ there is a closed unbounded set $E_\delta \subseteq \delta$ such that $[E_\delta]^{<\kappa} \subseteq \mathrm{dmn}(F)$. Then the following set is stationary:*

$$\begin{aligned} \{\alpha \in S^\lambda_\kappa :\ &\text{there is a closed unbounded } D \subseteq \alpha \text{ such that for any } a, b \in D \\ &\text{with } a < b,\ \{d \in D : d \le a\} \in \mathrm{dmn}(F) \text{ and } F(\{d \in D : d \le a\}) < b\} \end{aligned}$$

Proof. We follow the proof of Theorem 33.41 closely. Call the indicated set T. Let U be a closed unbounded subset of λ. We want to find a member of $T \cap U$.

Let $S = S^{\kappa^{++}}_\kappa$; so S is stationary in κ^{++}. By Theorem 33.40, let $\langle C_\delta : \delta \in S\rangle$ be a club guessing sequence for S; thus

(1) For every $\delta \in S$, the set $C_\delta \subseteq \delta$ is a club of order type κ.

(2) For every club $D \subseteq \kappa^{++}$ there is a $\delta \in D \cap S$ such that $C_\delta \subseteq D$.

We define an increasing continuous sequence $\langle \xi(i) : i < \kappa^{++}\rangle \in {}^{\kappa^{++}}\lambda$ recursively.

Let $\xi(0)$ be the least member of U. For i limit, let $\xi(i) = \sup_{k<i} \xi(k)$.

Now suppose for some $i < \kappa^{++}$ that $\xi(k)$ has been defined for every $k \le i$; we define $\xi(i+1)$. For each $\alpha \in S$ we consider two possibilities. If $\xi[C_\alpha \cap (i+1)] \in \mathrm{dmn}(F)$, we let σ_α be any ordinal greater than both $\xi(i)$ and $F(\xi[C_\alpha \cap (i+1)])$. Otherwise, we let $\sigma_\alpha = \xi(i) + 1$. Since $|S| < \lambda$, we can let $\xi(i+1)$ be the least member of U greater than all σ_α for $\alpha \in S$. Hence

(3) If $\alpha \in S$ and the first case in the definition of σ_α holds, then $\xi[C_\alpha \cap (i+1)] \in \mathrm{dmn}(F)$ and $F(\xi[C_\alpha \cap (i+1)]) < \xi(i+1)$.

Now the set $G = \mathrm{rng}(\xi)$ is closed and has order type κ^{++}. Let $\delta = \sup(G)$. Hence by the hypothesis of the proposition, there is a closed unbounded set $E_\delta \subseteq \delta$ such that $[E_\delta]^{<\kappa} \subseteq \mathrm{dmn}(F)$. Note that $G \cap E_\delta$ is also closed unbounded in δ.

Let $H = \xi^{-1}[G \cap E_\delta]$. Thus H is club in κ^{++}. Hence by (2) there is an $\alpha \in H \cap S$ such that $C_\alpha \subseteq H$. Hence $\overline{C}_\alpha \stackrel{\text{def}}{=} \xi[C_\alpha] \subseteq G \cap E_\delta$ is club in $\xi(\alpha)$ of order type κ. We claim that $\overline{C}_\alpha$ is as desired in the proposition. For, suppose that $a, b \in \overline{C}_\alpha$ and $a < b$. Write

$a = \xi(i)$. Then $\{d \in \overline{C}_\alpha : d \le a\} = \xi[C_\alpha \cap (i+1)] \subseteq E_\delta$, and so (3) gives the desired conclusion. $\square$

Next we give a condition under which $(*)_\kappa$ holds.

Lemma 33.43. *Suppose that I is a proper ideal over a set A of regular cardinals such that $|A| < \min(A)$. Assume that $\lambda > |A|$ is a regular cardinal such that $(\prod A, <_I)$ is λ-directed, and $\langle g_\xi : \xi < \lambda\rangle$ is a sequence of members of $\prod A$.*

Then there is a $<_I$-increasing sequence $f = \langle f_\xi : \xi < \lambda\rangle$ of length λ in $\prod A$ such that:

(i) $g_\xi < f_{\xi+1}$ for every $\xi < \lambda$.

(ii) $(*)_\kappa$ holds for f, for every regular cardinal κ such that $\kappa^{++} < \lambda$ and $\{a \in A : a \le \kappa^{++}\} \in I$.

Proof. Let f_0 be any member of $\prod A$. At successor stages, if f_ξ is defined, let $f_{\xi+1}$ be any function in $\prod A$ that $<$-extends f_ξ and g_ξ.

At limit stages δ, there are three cases. In the first case, $\mathrm{cf}(\delta) \le |A|$. Fix some $E_\delta \subseteq \delta$ club of order type $\mathrm{cf}(\delta)$, and define

$$f_\delta = \sup\{f_i : i \in E_\delta\}.$$

For any $a \in A$ we have $\mathrm{cf}(\delta) \le |A| < \min(A) \le a$, and so $f_\delta(a) < a$. Thus $f_\delta \in \prod A$.

In the second case, $\mathrm{cf}(\delta) = \kappa^{++}$, where κ is regular, $|A| < \kappa$, and $\{a \in A : a \le \kappa^{++}\} \in I$. Then we define f'_δ as in the first case. Then for any $a \in A$ with $a > \kappa^{++}$ we have $f'_\delta(a) < a$, and so $\{a \in A : a \le f'_\delta(a)\} \in I$, and we can modify f'_δ on this set which is in I to obtain our desired f_δ.

In the third case, neither of the first two cases holds. Then we let f_δ be any $\le_I$-upper bound of $\{f_\xi : \xi < \delta\}$; it exists by the λ-directedness assumption.

This completes the construction. Obviously (i) holds. For (ii), suppose that κ is a regular cardinal such that $\kappa^{++} < \lambda$ and $\{a \in A : a \le \kappa^{++}\} \in I$. If $|A| < \kappa$, the desired conclusion follows by Lemma 33.41. In case $\kappa \le |A|$, note that $\langle f_\xi : \xi < \kappa\rangle$ is $<$-increasing, and so is certainly strongly increasing under I. $\square$

Now we apply these results to the determination of true cofinality for some important concrete partial orders.

Notation. *For any set X of cardinals, let*

$$X^{(+)} = \{\alpha^+ : \alpha \in X\}.$$

Theorem 33.44. (Representation of μ^+ as a true cofinality, I) *Suppose that μ is a singular cardinal with uncountable cofinality. Then there is a club C in μ such that C has order type $\mathrm{cf}(\mu)$, every element of C is greater than $\mathrm{cf}(\mu)$, and*

$$\mu^+ = \mathrm{tcf}\left(\prod C^{(+)}, <_{J^{\mathrm{bd}}}\right),$$

where J^{bd} is the ideal of all bounded subsets of $C^{(+)}$.

Proof. Let C_0 be any closed unbounded set of limit cardinals less than μ such that $|C_0| = \mathrm{cf}(\mu)$ and all cardinals in C_0 are above $\mathrm{cf}(\mu)$. Then

(1) all members of C_0 which are limit points of C_0 are singular.

In fact, suppose on the contrary that $\kappa \in C_0$, κ is a limit point of C_0, and κ is regular. Thus $C_0 \cap \kappa$ is unbounded in κ, so $|C_0 \cap \kappa| = \kappa$. But $\mathrm{cf}(\mu) < \kappa$ and $|C_0| = \mathrm{cf}\mu$, contradiction. So (1) holds. Hence wlog every member of C_0 is singular.

Now we claim

(2) $(\prod C_0^{(+)}, <_{J^{\mathrm{bd}}})$ is μ-directed.

In fact, suppose that $F \subseteq \prod C_0^{(+)}$ and $|F| < \mu$. For $a \in C_0^{(+)}$ with $|F| < a$ let $h(a) = \sup_{f \in F} f(a)$; so $h(a) \in a$. For $a \in C_0^{(+)}$ with $a \leq |F|$ let $h(a) = 0$. Clearly $f \leq_{J^{\mathrm{bd}}} h$ for all $f \in F$. So (2) holds.

(3) $(\prod C_0^{(+)}, <_{J^{\mathrm{bd}}})$ is μ^+-directed.

In fact, by (2) it suffices to find a bound for a subset F of $\prod C_0^{(+)}$ such that $|F| = \mu$. Write $F = \bigcup_{\alpha < \mathrm{cf}(\mu)} G_\alpha$, with $|G_\alpha| < \mu$ for each $\alpha < \mathrm{cf}(\mu)$. By (2), each G_α has an upper bound k_α under $<_{J^{\mathrm{bd}}}$. Then $\{k_\alpha : \alpha < \mathrm{cf}(\mu)\}$ has an upper bound h under $<_{J^{\mathrm{bd}}}$. Clearly h is an upper bound for F.

Now we are going to apply Lemma 33.43 to J^{bd}, $C_0^{(+)}$, and μ^+ in place of I, A, and λ; and with anything for g. Clearly the hypotheses hold, so we get a $<_{J^{\mathrm{bd}}}$-increasing sequence $f = \langle f_\xi : \xi < \mu^+ \rangle$ in $\prod C_0^{(+)}$ such that $(*)_\kappa$ holds for f and the bounding projection property holds for κ, for every regular cardinal $\kappa < \mu$. It also follows that the bounding projection property holds for $|A|^+$, and hence by 33.37, f has an exact upper bound h. Then by Lemma 33.38, for every regular $\kappa < \mu$ we have

$$(\star) \qquad \{a \in C_0^{(+)} : h(a) \text{ is non-limit, or } \mathrm{cf}(h(a)) < \kappa\} \in J^{\mathrm{bd}}.$$

Now the identity function k on $C_0^{(+)}$ is obviously is an upper bound for f, so $h \leq_{J^{\mathrm{bd}}} k$. By modifying h on a set in J^{bd} we may assume that $h(a) \leq a$ for all $a \in C_0^{(+)}$. Now we claim

($\star\star$) The set $C_1 \stackrel{\mathrm{def}}{=} \{\alpha \in C_0 : h(\alpha^+) = \alpha^+\}$ contains a club of μ.

Assume otherwise. Then for every club K, $K \cap (\mu \backslash C_1) \neq 0$. This means that $\mu \backslash C_1$ is stationary, and hence $S \stackrel{\mathrm{def}}{=} C_0 \backslash C_1$ is stationary. For each $\alpha \in S$ we have $h(\alpha^+) < \alpha^+$. Hence $\mathrm{cf}(h(\alpha^+)) < \alpha$ since α is singular. Hence by Fodor's theorem $\langle \mathrm{cf}(h(\alpha^+)) : \alpha \in C_0 \rangle$ is bounded by some $\kappa < \mu$ on a stationary subset of S. This contradicts ($\star$).

Thus ($\star\star$) holds, and so there is a club $C \subseteq C_0$ such that $h(\alpha^+) = \alpha^+$ for all $\alpha \in C$. Now $\langle f_\xi \restriction C^{(+)} : \xi < \mu^+ \rangle$ is $<_{J^{\mathrm{bd}}}$-increasing. We claim that it is cofinal in $(\prod C^{(+)}, <_{J^{\mathrm{bd}}})$. For, suppose that $g \in \prod C^{(+)}$. Let g' be the extension of g to $\prod C_0^{(+)}$ such that $g'(a) = 0$ for any $a \in C_0 \backslash C$. Then $g' <_{J^{\mathrm{bd}}} h$, and so there is a $\xi < \mu^+$ such that $g' <_{J^{\mathrm{bd}}} f_\xi$. So $g <_{J^{\mathrm{bd}}} f_\xi \restriction C^{(+)}$, as desired. This shows that $\mu^+ = \mathrm{tcf}(\prod C^{(+)}, <_{J^{\mathrm{bd}}})$. $\square$

Theorem 33.45. (Representation of μ^+ as a true cofinality, II) *If μ is a singular cardinal of countable cofinality, then there is an unbounded set $D \subseteq \mu$ of regular cardinals such that*

$$\mu^+ = \operatorname{tcf}\left(\prod D, <_{J^{\mathrm{bd}}}\right).$$

Proof. Let C_0 be a set of uncountable regular cardinals with supremum μ, of order type ω.

(1) $\prod C_0/J^{\mathrm{bd}}$ is μ-directed.

For, let $X \subseteq \prod C_0$ with $|X| < \mu$. For each $a \in C_0$ such that $|X| < a$, let $h(a) = \sup\{f(a) : f \in X\}$, and extend h to all of C_0 in any way. Clearly $h \in \prod C_0$ and it is an upper bound in the $<_{J^{\mathrm{bd}}}$ sense for X.

From (1) it is clear that $\prod C_0/J^{\mathrm{bd}}$ is also μ^+-directed. By Lemma 33.43 we then get a $<_{J^{\mathrm{bd}}}$-increasing sequence $\langle f_\xi : \xi < \mu^+\rangle$ which satisfies $(*)_\kappa$ for every regular $\kappa < \mu^+$. By Theorems 33.37 and 33.38 f has an exact upper bound h such that $\{a \in C_0 : h(a)$ is non-limit or $\operatorname{cf}(h(a)) < \kappa\} \in J^{\mathrm{bd}}$ for every regular $\kappa < \mu^+$. We may assume that $h(a) \le a$ for all $a \in C_0$, since the identity function is clearly an upper bound for f; and we may assume that each $h(a)$ is a limit ordinal of uncountable cofinality since $\{a \in C_0 : \operatorname{cf}(h(a)) < \omega_1\} \in J^{\mathrm{bd}}$.

(2) $\operatorname{tcf}\left(\prod_{a \in C_0} \operatorname{cf}(h(a)), <_{J^{\mathrm{bd}}}\right) = \mu^+$.

To prove this, for each $a \in C_0$ let D_a be club in $h(a)$ of order type $\operatorname{cf}(h(a))$, and let $\langle \eta_{a\xi} : \xi < \operatorname{cf}(h(a))\rangle$ be the strictly increasing enumeration of D_a. For each $\xi < \mu^+$ we define $f'_\xi \in \prod_{a \in C_0} \operatorname{cf}(h(a))$ as follows. Since $f_\xi <_{J^{\mathrm{bd}}} h$, the set $\{a \in C_0 : f_\xi(a) \ge h(a)\}$ is bounded, so choose $a_0 \in C_0$ such that for all $b \in C_0$ with $a_0 \le b$ we have $f_\xi(b) < h(b)$. For such a b we define $f'_\xi(b)$ to be the least ν such that $f_\xi(b) < \eta_{b\nu}$. Then we extend f'_α in any way to a member of $\prod_{a \in C_0} \operatorname{cf}(h(a)))$.

(3) $\xi < \sigma < \mu^+$ implies that $f'_\xi \le_{J^{\mathrm{bd}}} f'_\sigma$.

This is clear by the definitions.

Now for each $l \in \prod_{a \in C_0} \operatorname{cf}(h(a)))$ define $k_l \in \prod C_0$ by setting $k_l(a) = \eta_{al(a)}$ for all a. So $k_l < h$. Since h is an exact upper bound for f, choose $\xi < \mu^+$ such that $k_l <_{J^{\mathrm{bd}}} f_\xi$. Choose a such that $k_l(b) < f_\xi(b) < h(b)$ for all $b \ge a$. Then for all $b \ge a$, $\eta_{bl(b)} < \eta_{bf'_\xi(b)}$, and hence $l(b) < f'_\xi(b)$. This proves that $l <_{J^{\mathrm{bd}}} f'_\xi$. This proves the following statement.

(4) $\{f'_\xi : \xi < \mu^+\}$ is cofinal in $\left(\prod_{a \in C_0} \operatorname{cf}(h(a)), <_{J^{\mathrm{bd}}}\right)$.

Now (3) and (4) yield (2).

Now let $B = \{\operatorname{cf}(h(a)) : a \in C_0\}$. Define

$$X \in J \text{ iff } X \subseteq B \text{ and } h^{-1}[\operatorname{cf}^{-1}[X]] \in J^{\mathrm{bd}}.$$

By Lemma 33.24 we get $\operatorname{tcf}(\prod B/J) = \mu^+$. It suffices now to show that J is the ideal of bounded subsets of B. Suppose that $X \in J$, and choose $a \in C_0$ such that $h^{-1}[\operatorname{cf}^{-1}[X]] \subseteq \{b \in C_0 : b < a\}$. Thus $X \subseteq \{b \in A : \operatorname{cf}(h(b)) < a\} \in J^{\mathrm{bd}}$, so X is bounded. Conversely, if X is bounded, choose $a \in B$ such that $X \subseteq \{b \in B : b \le a\}$. Now

$$\begin{aligned} h^{-1}[\operatorname{cf}^{-1}[X]] &= \{b \in C_0 : \operatorname{cf}(h(b)) \in X\} \\ &\subseteq \{b \in C_0 : \operatorname{cf}(h(b)) \le a\}, \end{aligned}$$

and this is bounded by the choice of h. □

Proposition 33.46. (The trichotomy theorem) *Suppose that $\lambda > |A|^+$ is a regular cardinal and $f = \langle f_\xi : \xi < \lambda\rangle$ is a $<_I$-increasing sequence. Consider the following properties of f and a regular cardinal κ such that $|A| < \kappa \leq \lambda$:*

Bad$_\kappa$: *There exist:*

(a) sets S_a of ordinals for $a \in A$ such that $f_\alpha <_I \sup S$ for all $\alpha < \lambda$ and $|S_a| < \kappa$; and

(b) an ultrafilter D over A extending the dual of I

such that for every $\alpha < \lambda$ there is a $\beta < \lambda$ such that proj$(f_\alpha, S) <_D f_\beta$.

Ugly *There exists a function $g \in^A$ Ord such that, defining $t_\alpha = \{a \in A : g(a) < f_\alpha(a)\}$, the sequence $\langle t_\alpha : \alpha < \lambda\rangle$ does not stabilize modulo I. That is, for every α there is a $\beta > \alpha$ in λ such that $t_\beta \backslash t_\alpha \in I^+$. Note here that $\langle t_\alpha : \alpha < \lambda\rangle$ is $\subseteq_I$-increasing.*

Good$_\kappa$ *There exists an exact upper bound g for f such that $\mathrm{cf}(g(a)) \geq \kappa$ for every $a \in A$.*

*Then the assertion of this theorem is that the bounding projection property for κ is equivalent to $\neg$**Bad$_\kappa$** $\wedge\, \neg$**Ugly**. Hence if neither **Bad$_\kappa$** nor **Ugly**, then **Good$_\kappa$**.*

Proof. We use the abbreviation f^+_α for proj(f_α, S).

First assume the bounding projection property for κ. Suppose that **Bad$_\kappa$** holds, and assume the notation of it. Choose $\alpha < \lambda$ such that f^+_α is a $<_I$-upper bound for f. Choose $\beta > \alpha$ as in the definition of **Bad$_\kappa$**. Then $f_\beta <_D f^+_\alpha <_D f_\beta$, contradiction.

To prove $\neg$**Ugly**, suppose that g is as in the definition of **Ugly**. By 2.13, let h be an exact upper bound for f, and for each $a \in A$ let $S(a) = \{g(a), h(a)\}$. Thus $f_\alpha <_I \sup S$ for each $\alpha < \lambda$. By the bounding projection property, choose $\alpha < \lambda$ such that f^+_α $<_I$-bounds f. Take any $\beta > \alpha$. Then $f_\beta <_I f^+_\alpha$, so $\{a : f_\beta(a) \geq f^+_\alpha(a)\} \in I$. Now

$$t_\beta \backslash t_\alpha = \{a : f_\alpha(a) \leq g(a) < f_\beta(a)\} \subseteq \{a : f_\beta(a) \geq f^+_\alpha(a)\} \in I,$$

contradiction.

Conversely, assume $\neg$**Bad$_\kappa$** and $\neg$**Ugly**, but also suppose that the bounding projection property for κ fails to hold. By the last supposition we get the hypothesis of the bounding projection property, but there is no $\xi < \lambda$ such that f^+_ξ bounds f. For all $\xi, \alpha < \lambda$ let $t^\xi_\alpha = \{a \in A : f^+_\xi(a) < f_\alpha(a)\}$.

(1) For every $\xi < \lambda$ there is a $\beta_\xi > \xi$ such that $t^\xi_{\beta_\xi} \in I^+$ and for all $\gamma > \beta_\xi$ we have $t^\xi_\gamma \backslash t^\xi_{\beta_\xi} \in I$.

In fact, since f^+_ξ does not bound f, we can choose $\beta_\xi > \xi$ such that $f_{\beta_\xi} \not\leq_I f^+_\xi$; and since $\neg$**Ugly**, we can choose $\delta_\xi > \xi$ such that for all $\gamma > \delta_\xi$ we have $t^\xi_\gamma \backslash t^\xi_{\delta_\xi} \in I$. We may assume that $\beta_\xi = \delta_\xi$, and this gives the desired conclusion of (1).

By (1) we can define strictly increasing sequences $\langle \xi(\nu) : \nu < \lambda\rangle$ and $\langle \beta(\nu) : \nu < \lambda\rangle$ such that for all $\nu < \lambda$, $t^{\xi(\nu)}_{\beta(\nu)} \in I^+$, $\xi(\nu) < \beta(\nu)$, $\beta(\nu) < \xi(\rho)$ if $\nu < \rho < \lambda$, and $t^{\xi(\nu)}_\gamma \backslash t^{\xi(\nu)}_{\beta(\nu)} \in I$ for all $\gamma > \beta(\nu)$.

(2) If $\nu < \rho < \lambda$, then

$$t^{\xi(\rho)}_{\beta(\rho)} \subseteq \left(t^{\xi(\nu)}_{\beta(\nu)} \cap t^{\xi(\rho)}_{\beta(\rho)}\right) \cup \left(t^{\xi(\nu)}_{\beta(\rho)} \backslash t^{\xi(\nu)}_{\beta(\nu)}\right) \cup \left\{a \in A : f^+_{\xi(\rho)}(a) < f^+_{\xi(\nu)}(a)\right\}.$$

To prove this, suppose that a is not a member of the right side. Then the following conditions hold:

(3) $f_{\beta(\nu)}(a) \le f^+_{\xi(\nu)}(a)$ or $f_{\beta(\rho)}(a) \le f^+_{\xi(\rho)}(a)$.

(4) $f_{\beta(\rho)}(a) \le f^+_{\xi(\nu)}(a)$ or $f^+_{\xi(\nu)}(a) < f_{\beta(\nu)}(a)$.

(5) $f^+_{\xi(\nu)}(a) \le f^+_{\xi(\rho)}(a)$.

Clearly then, $f_{\beta(\rho)}(a) \le f^+_{\xi(\rho)}(a)$, which shows that a is not in the left side. So (2) holds.

(6) If $\nu_1 < \cdots < \nu_m < \lambda$, then $t^{\xi(\nu_1)}_{\beta(\nu_1)} \cap \ldots \cap t^{\xi(\nu_m)}_{\beta(\nu_m)} \in I^+$.

We prove this by induction on m. It is clear for $m = 1$. Assume it for m, and suppose that $\nu_1 < \cdots < \nu_{m+1}$. Then by (2),

$$\begin{aligned} t^{\xi(\nu_2)}_{\beta(\nu_2)} \cap \ldots \cap t^{\xi(\nu_m)}_{\beta(\nu_m)} \subseteq &\left(t^{\xi(\nu_1)}_{\beta(\nu_1)} \cap \ldots \cap t^{\xi(\nu_{m+1})}_{\beta(\nu_{m+1})}\right) \\ &\cup \left(t^{\xi(\nu_1)}_{\beta(\nu_2)} \backslash t^{\xi(\nu_1)}_{\beta(\nu_1)}\right) \cup \left\{a \in A : f^+_{\xi(\nu_2)}(a) < f^+_{\xi(\nu_1)}(a)\right\}, \end{aligned}$$

and the last two sets are in I, so our conclusion follows by the inductive hypothesis.

By (6), the set $I^* \cup \{t^{\xi(\nu)}_{\beta(\nu)} : \nu < \lambda\}$ has fip, and hence is contained in an ultrafilter D.

By $\neg\mathbf{Bad}_\kappa$, choose $\alpha < \lambda$ such that f^+_α is a $<_D$-bound for f. Take ν with $\alpha < \xi(\nu)$. Now $t^{\xi(\nu)}_{\beta(\nu)} \in D$, so $f^+_{\xi(\nu)} <_D f_{\beta(\nu)}$. Thus $f^+_\alpha \le_D f^+_{\xi(\nu)} <_D f_{\beta(\nu)} <_D f^+_\alpha$, contradiction.

The final assertion of the theorem follows by 2.15. □

Proposition 33.47. *Suppose that λ is a regular cardinal, A is an infinite set such that $\forall\mu < \lambda(\mu^{|A|} < \lambda$, and $\langle f_\alpha : \alpha < \lambda\rangle$ is a system of members of ^{A}Ord.*

Then there is a stationary subset E of λ such that for all $\alpha, \beta \in E$, if $\alpha < \beta$ then $f_\alpha \le f_\beta$. Moreover, for all $a \in A$, either $\langle f_\alpha(a) : \alpha \in E\rangle$ is a constant sequence, or it is strictly increasing.

If in addition I is a proper ideal on A and $\langle f_\alpha : \alpha < \lambda\rangle$ is $<_I$-increasing, then $(*)_\lambda$ holds.

Proof. For each $a \in A$ fix $\gamma_a > \sup\{f_\alpha(a) : \alpha < \lambda\}$. For $\alpha < \lambda$ and $a \in A$ let $S^\alpha(a) = \{f_\beta(a) : \beta < \alpha\} \cup \{\gamma_a\}$. For any $\alpha < \lambda$ and $a \in A$, let $g_\alpha(a) = \min(S^\alpha \backslash f_\alpha(a))$. Thus either $g_\alpha(a) = \gamma_a$ or $g_\alpha(a) = f_\beta(a)$ for some $\beta < \alpha$. In the second case, choose such a β; call it β_a.

Let $T = \{\delta < \lambda : \mathrm{cf}(\delta) = |A|^+\}$. Suppose that $\alpha \in T$. Since $\mathrm{cf}(\alpha) = |A|^+$, we can choose $\mu_\alpha < \alpha$ such that $\beta_a < \mu_\alpha$ for all $a \in A$ for which β_a is defined. Hence $g_\alpha = \mathrm{proj}(f_\alpha, S^{\mu_\alpha})$. By Fodor's theorem we may assume that $\mu = \mu_\alpha$ is fixed on a stationary subset T' of T. Since S^μ has size $\mu < \lambda$, $\mu^{|A|} < \lambda$, and g_α maps A into a set of size at most μ, we may assume that $g = g_\alpha$ is fixed for all α in a stationary subset T'' of T'. Now

$$T'' = \bigcup_{h \in {}^A 2} \{\alpha \in T'' : \forall a \in A(f_\alpha(a) < g(a) \leftrightarrow h(a) = 1\}.$$

Since $2^{|A|} < \lambda$, it follows that there is an $h \in {}^A 2$ such that

$$E \stackrel{\text{def}}{=} \{\alpha \in T'' : \forall a \in A(f_\alpha(a) < g(a) \leftrightarrow h(a) = 1\}$$

is stationary. Suppose that $\alpha, \beta \in E$, $\alpha < \beta$, $a \in A$, and $f_\beta(a) < f_\alpha(a)$. Then

$$f_\alpha(a) \leq \min(S^\alpha(a) \backslash f_\alpha(a)) = g(a) = \min(S^\beta(a) \backslash f_\beta(a)) \leq f_\alpha(a),$$

and so $f_\alpha(a) = g(a)$. It follows that $h(a) = 0$. But $f_\beta(a) < f_\alpha(a) = g(a)$, contradiction.

So we have proved that if $\alpha, \beta \in E$ and $\alpha < \beta$, then $f_\alpha \leq f_\beta$.

We claim that also for each $a \in A$, either $f_\alpha(a) = g(a)$ for all $\alpha \in E$, or $f_\alpha(a) < f_\beta(a)$ for all $\alpha, \beta \in E$ such that $\alpha < \beta$. Otherwise, there is an $\alpha \in E$ with $f_\alpha(a) < g(a)$ and there are $\beta, \delta \in E$ with $\beta < \delta$ and $f_\beta(a) = f_\delta(a)$. Then $g(a) = \min(S^\delta(a) \backslash f_\delta(a)) = f_\delta(a)$, since $f_\beta(a) \in S^\delta(a)$. But then $h(a) = 0$, contradicting $f_\alpha(a) < g(a)$.

For the last statement of the theorem, assume that $f_\alpha <_I f_\beta$ for all $\alpha < \beta < \lambda$. Now if $\alpha, \beta \in E$ and $\alpha < \beta$, then $\{a \in A : f_\alpha(a) \geq f_\beta(a)\} \in I$. Since $f_\alpha \leq f_\beta$, this means that $\{a \in A : f_\alpha(a) = f_\beta(a)\} \in I$. But this set is $B \stackrel{\text{def}}{=} \{a \in A : f_\delta(a) = f_\varepsilon(a)$ for all $\delta, \varepsilon \in E\}$. For $a \notin B$ and $\alpha < \beta$, both in E, we have $f_\alpha(a) < f_\beta(a)$. So $\langle f_\alpha : \alpha \in E \rangle$ is strongly increasing mod I. By 2.6 it follows then that $\langle f_\alpha : \alpha < \lambda \rangle$ has an exact upper bound h such that $\mathrm{cf}(h(a)) = \lambda$ for all $a \in A$. Hence by 2.15, $(*)_\lambda$ holds for $\langle f_\alpha : \alpha < \lambda \rangle$. $\square$

34. Basic properties of PCF

For any set A of regular cardinals define

$$\text{pcf}(A) = \left\{\text{cf}\left(\prod A/D\right) : D \text{ is an ultrafilter on } A\right\}.$$

By definition, $\text{pcf}(\emptyset) = \emptyset$. We begin with a very easy proposition which will be used a lot in what follows.

Proposition 34.1. *Let A and B be sets of regular cardinals.*

(i) $A \subseteq \text{pcf}(A)$.

(ii) If $A \subseteq B$, *then* $\text{pcf}(A) \subseteq \text{pcf}(B)$.

(iii) $\text{pcf}(A \cup B) = \text{pcf}(A) \cup \text{pcf}(B)$.

(iv) If $B \subseteq A$, *then* $\text{pcf}(A) \backslash \text{pcf}(B) \subseteq \text{pcf}(A \backslash B)$.

(v) If A is finite, then $\text{pcf}(A) = A$.

(vi) If $B \subseteq A$, B is finite, and A is infinite, then $\text{pcf}(A) = \text{pcf}(A \backslash B) \cup B$.

(vii) $\min(A) = \min(\text{pcf}(A))$.

(viii) If A is infinite, then the first ω members of A are the same as the first ω members of $\text{pcf}(A)$.

Proof. (i): For each $a \in A$, the principal ultrafilter with $\{a\}$ as a member shows that $a \in \text{pcf}(A)$.

(ii): Any ultrafilter F on A can be extended to an ultrafilter G on B. The mapping $[f] \mapsto [f]$ is easily seen to be an isomorphism of $\prod A/F$ onto $\prod B/G$. Note here that $[f]$ is used in two senses, one for an element of $\prod A/F$, where each member of $[f]$ is in $\prod A$, and the other for an element of $\prod B/G$, with members in the larger set $\prod B$.

(iii): $\supseteq$ holds by (ii). Now suppose that D is an ultrafilter on $A \cup B$. Then $A \in D$ or $B \in D$, and this proves $\subseteq$.

(iv): Suppose that $B \subseteq A$ and $\lambda \in \text{pcf}(A) \backslash \text{pcf}(B)$. Let D be an ultrafilter on A such that $\lambda = \text{cf}(\prod A/D)$. Then $B \notin D$, as otherwise $\lambda \in \text{pcf}(B)$. So $A \backslash B \in D$, and so $\lambda \in \text{pcf}(A \backslash B)$.

(v): If A is finite, then every ultrafilter on A is principal.

(vi): We have

$$\begin{aligned} \text{pcf}(A) &= \text{pcf}(A \backslash B) \cup \text{pcf}(B) \quad \text{by (iii)} \\ &= \text{pcf}(A \backslash B) \cup B \quad \text{by (v)} \end{aligned}$$

(vii): Let $a = \min(A)$. Thus $a \in \text{pcf}(A)$ by (i). Suppose that $\lambda \in \text{pcf}(A)$ with $\lambda < a$; we want to get a contradiction. Say $\langle [g_\xi] : \xi < \lambda \rangle$ is strictly increasing and cofinal in $\prod A/D$. Now define $h \in \prod A$ as follows: for any $b \in A$, $h(b) = \sup\{g_\xi(b) + 1 : \xi < \lambda\}$. Thus $[g_\xi] < [h]$ for all $\xi < \lambda$, contradiction.

(viii): Suppose that $\lambda \in \text{pcf}(A) \backslash A$. Suppose that $\lambda \cap A$ is finite, and let $a = \min(A \backslash \lambda)$. So $\lambda \leq a$, and if $b \in A \cap a$ then $b < \lambda$. Thus $A \cap \lambda = A \cap a$. Hence $\lambda \in \text{pcf}(A) = \text{pcf}(A \backslash a) \cup (A \cap \lambda)$ by (vi), and so $a \leq \lambda$ by (vii). So $\lambda = a$, contradiction. Thus $\lambda \cap A$ is infinite, and this proves (viii). $\square$

The following result gives a connection with earlier material; of course there will be more connections shortly.

Proposition 34.2. *If A is a collection of regular cardinals, F is a proper filter on A, and $\lambda = \mathrm{tcf}(\prod A/F)$, then $\lambda \in \mathrm{pcf}(A)$.*

Proof. Let $\langle f_\xi : \xi < \lambda\rangle$ be a $<_F$-increasing cofinal sequence in $\prod A/F$. Let D be any ultrafilter containing F. Then clearly $\langle f_\xi : \xi < \lambda\rangle$ is a $<_D$-increasing cofinal sequence in $\prod A/D$. $\square$

Definitions. A set A is *progressive* iff A is an infinite set of regular cardinals and $|A| < \min(A)$.

If $\alpha < \beta$ are ordinals, then $(\alpha, \beta)_{\mathrm{reg}}$ is the set of all regular cardinals κ such that $\alpha < \kappa < \beta$. Similarly for $[\alpha, \beta)_{\mathrm{reg}}$, etc. All such sets are called *intervals of regular cardinals.*

Proposition 34.3. *Assume that A is a progressive set, then*

(i) Every infinite subset of A is progressive.

(ii) If α is an ordinal and $A \cap \alpha$ is unbounded in α, then α is a singular cardinal.

(iii) If A is an infinite interval of regular cardinals, then A does not have any weak inaccessible as a member, except possibly its first element. Moreover, there is a singular cardinal λ such that $A \cap \lambda$ is unbounded in λ and $A \backslash \lambda$ is finite.

Proof. (i): Obvious.

(ii): Obviously α is a cardinal. Now $A \cap \alpha$ is cofinal in α and $|A \cap \alpha| \le |A| < \min(A) < \alpha$. Hence α is singular.

(iii): If $\kappa \in A$, then by (ii), $A \cap \kappa$ cannot be unbounded in κ; hence κ is a successor cardinal, or is the first element of A. For the second assertion of (iii), let $\sup(A) = \aleph_{\alpha+n}$ with α a limit ordinal. Since A is an infinite interval of regular cardinals, it follows that $A \cap \aleph_\alpha$ is unbounded in $\aleph_\alpha$, and hence by (ii), $\aleph_\alpha$ is singular. Hence the desired conclusion follows. $\square$

Theorem 34.4. (Directed set theorem) *Suppose that A is a progressive set, and λ is a regular cardinal such that $\sup(A) < \lambda$. Suppose that I is a proper ideal over A containing all proper initial segments of A and such that $(\prod A, <_I)$ is λ-directed. Then there exist a set A' of regular cardinals and a proper ideal J over A' such that the following conditions hold:*

(i) $A' \subseteq [\min(A), \sup(A))$ and A' is cofinal in $\sup(A)$.

(ii) $|A'| \le |A|$.

(iii) J contains all bounded subsets of A'.

(iv) $\lambda = \mathrm{tcf}(\prod A', <_J)$.

Proof. First we note:

$(*)$ A does not have a largest element.

For, suppose that a is the largest element of A. Note that then $I = \mathscr{P}(A\backslash\{a\})$. For each $\xi < a$ define $f_\xi \in \prod A$ by setting

$$f_\xi(b) = \begin{cases} 0 & \text{if } b \ne a, \\ \xi & \text{if } b = a. \end{cases}$$

Since $a < \lambda$, choose $g \in \prod A$ such that $f_\xi <_I g$ for all $\xi \in a$. Thus $\{b \in A : f_\xi(b) \geq g(b)\} \in I$, so $f_\xi(a) < g(a)$ for all $\xi < a$. This is clearly impossible. So $(*)$ holds.

Now by Lemma 34.43 there is a $<_I$-increasing sequence $f = \langle f_\xi : \xi < \lambda \rangle$ in $\prod A$ which satisfies $(*)_\kappa$ for every $\kappa \in A$. Hence by 34.37–34.39, f has an exact upper bound $h \in {}^A\mathbf{On}$ such that

$$\{a \in A : h(a) \text{ is non-limit or } \mathrm{cf}(h(a)) < \kappa\} \in I \tag{1}$$

for every $\kappa \in A$. Now the identity function k on A is clearly an upper bound for f, so $h \leq_I k$; and by (1), $\{a \in A : h(a) \text{ is non-limit or } \mathrm{cf}(h(a)) < \min(A)\} \in I$. Hence by changing h on a set in the ideal we may assume that

$$\min(A) \leq \mathrm{cf}(h(a)) \leq a \quad \text{for all } a \in A. \tag{2}$$

Now f shows that $(\prod h, <_I)$ has true cofinality λ. Let $A' = \{\mathrm{cf}(h(a)) : a \in A\}$. By Lemma 34.23 there is a proper ideal J on A' such that $(\prod A', <_J)$ has true cofinality λ; namely,

$$X \in J \quad \text{iff} \quad X \subseteq A' \text{ and } h^{-1}[\mathrm{cf}^{-1}[X]] \in I.$$

Clearly (ii) and (iv) hold. By (2) we have $A' \subseteq [\min(A), \sup(A))$. Now to show that A' is cofinal in $\sup(A)$, suppose that $\kappa \in A$; we find $\mu \in A'$ such that $\kappa \leq \mu$. In fact, $\{a \in A : \mathrm{cf}(h(a)) < \kappa\} \in I$ by (1). Let $X = \{b \in A' : b < \kappa\}$. Then

$$h^{-1}[\mathrm{cf}^{-1}[X]] = \{a \in A : \mathrm{cf}(h(a)) < \kappa\} \in I,$$

and so $X \in J$. Taking any $\mu \in A' \backslash X$ we get $\kappa \leq \mu$. Thus (i) holds. Finally, for (iii), suppose that $\mu \in J$; we want to show that $Y \overset{\text{def}}{=} \{b \in A' : b < \mu\} \in J$. By (i), choose $\kappa \in A$ such that $\mu \leq \kappa$. Then $Y \subseteq \{b \in A' : b < \kappa\}$, and by the argument just given, the latter set is in J. So (iii) holds. $\square$

Corollary 34.5. *Suppose that A is progressive, is an interval of regular cardinals, and λ is a regular cardinal $> \sup(A)$. Assume that I is a proper ideal over A such that $(\prod A, <_I)$ is λ-directed. Then $\lambda \in \mathrm{pcf}(A)$.*

Proof. We may assume that I contains all proper initial segments of A. For, suppose that this is not true. Then there is a proper initial segment B of A such that $B \notin I$. With $a \in A \backslash B$ we then have $B \subseteq A \cap a$, and so $A \cap a \notin I$. Let a be the smallest element of A such that $A \cap a \notin I$. Then $J \overset{\text{def}}{=} I \cap \mathscr{P}(A \cap a)$ is a proper ideal that contains all proper initial segments of $A \cap a$. we claim that $(\prod(A \cap a), J)$ is λ-directed. For, suppose that $X \subseteq \prod(A \cap a)$ with $|X| < \lambda$. For each $g \in X$ let $g^+ \in \prod A$ be such that $g^+ \supseteq g$ and $g^+(b) = 0$ for all $b \in A \backslash a$. Choose $f \in \prod A$ such that $g^+ \leq_I f$ for all $g \in X$. So if $g \in X$ we have

$$\{b \in A \cap a : g(b) > f(b)\} = \{b \in A : g^+(b) > f(b)\} \in I \cap \mathscr{P}(A \cap a),$$

and so $g \leq_J (f \restriction (A \cap a)$ for all $g \in X$, as desired.

Now the corollary follows from the theorem. ☐

The ideal $J_{<\lambda}$

Let A be a set of regular cardinals. We define

$$J_{<\lambda}[A] = \{X \subseteq A : \mathrm{pcf}(X) \subseteq \lambda\}.$$

In words, $X \in J_{<\lambda}[A]$ iff X is a subset of A such that for any ultrafilter D over A, if $X \in D$, then $\mathrm{cf}(\prod A, <_D) < \lambda$. Thus X "forces" the cofinalities of ultraproducts to be below λ.

Clearly $J_{<\lambda}[A]$ is an ideal of A. If $\lambda < \min(A)$, then $J_{<\lambda}[A] = \{\emptyset\}$ by 34.1(vii). If $\lambda < \mu$, then $J_{<\lambda}[A] \subseteq J_{<\mu}[A]$. If $\lambda \notin \mathrm{pcf}(A)$, then $J_{<\lambda}[A] = J_{<\lambda^+}[A]$. If λ is greater than each member of $\mathrm{pcf}(A)$, then $J_{<\lambda}[A]$ is the improper ideal $\mathscr{P}(A)$. If $\lambda \in \mathrm{pcf}(A)$, then $A \notin J_{<\lambda}[A]$.

If A is clear from the context, we simply write $J_{<\lambda}$.

If I and J are ideals on a set A, then $I + J$ is the smallest ideal on A which contains $I \cup J$; it consists of all X such that $X \subseteq Y \cup Z$ for some $Y \in I$ and $Z \in J$.

Lemma 34.6. *If A is an infinite set of regular cardinals and B is a finite subset of A, then for any cardinal λ we have*

$$J_{<\lambda}[A] = J_{<\lambda}[A\backslash B] + \mathscr{P}(B \cap \lambda).$$

Proof. Let $X \in J_{<\lambda}[A]$. Thus $\mathrm{pcf}(X) \subseteq \lambda$. Using 34.1(vi) we have $\mathrm{pcf}(X) = \mathrm{pcf}(X\backslash B) \cup (X \cap B)$, so $X\backslash B \in J_{<\lambda}[A\backslash B]$ and $X \cap B \subseteq B \cap \lambda$, and it follows that $X \in J_{<\lambda}[A\backslash B] + \mathscr{P}(B \cap \lambda)$.

Now suppose that $X \in J_{<\lambda}[A\backslash B] + \mathscr{P}(B \cap \lambda)$. Then there is a $Y \in J_{<\lambda}[A\backslash B]$ such that $X \subseteq Y \cup (B \cap \lambda)$. Hence by 34.1(vi) again, $\mathrm{pcf}(X) \subseteq \mathrm{pcf}(Y) \cup (B \cap \lambda) \subseteq \lambda$, so $X \in J_{<\lambda}[A]$. ☐

Recall that for any ideal on a set Y, $I^* = \{a \subseteq Y : Y\backslash a \in I\}$ is the filter corresponding to I.

Proposition 34.7. *If A is a collection of regular cardinals and λ is a cardinal, then*

$$J^*_{<\lambda}[A] = \bigcap \left\{ D : D \textit{ is an ultrafilter and } \mathrm{cf}\left(\prod A/D\right) \geq \lambda \right\}.$$

The intersection is to be understood as being equal to $\mathscr{P}(A)$ if there is no ultrafilter D such that $\mathrm{cf}(\prod A/D) \geq \lambda$.

Proof. Note that for any $X \subseteq A$, $X \in J^*_{<\lambda}[A]$ iff $A\backslash X \in J_{<\lambda}[A]$ iff $\mathrm{pcf}(A\backslash X) \subseteq \lambda$. Now suppose that $X \in J^*_{<\lambda}[A]$ and D is an ultrafilter such that $\mathrm{cf}(\prod A/D) \geq \lambda$. If $X \notin D$, then $A\backslash X \in D$ and hence $\mathrm{pcf}(A\backslash X) \not\subseteq \lambda$, contradiction. Thus X is in the indicated intersection.

If X is in the indicated intersection, we want to show that $A \backslash X \subseteq \lambda$. To this end, suppose that D is an ultrafilter such that $A \backslash X \in D$, and to get a contradiction suppose that $\mathrm{cf}(\prod A/D) \geq \lambda$. Then $X \in D$ by assumption, contradiction.

Note that the argument gives the desired result in case there are no ultrafilters D as indicated in the intersection; in this case, $\mathrm{pcf}(A \backslash X) \subseteq \lambda$ for every $X \subseteq A$, and so $J^*_{<\lambda}[A] = \mathscr{P}(A)$. $\square$

Theorem 34.8. (λ-directedness) *Assume that A is progressive. Then for every cardinal λ, the partial order $(\prod A, <_{J_{<\lambda}[A]})$ is λ-directed.*

Proof. We may assume that there are infinitely many members of A less than λ. For, suppose not. Let $F \subseteq \prod A$ with $|F| < \lambda$. We define $g \in \prod A$ by setting, for any $a \in A$,

$$g(a) = \begin{cases} \sup\{f(a) : f \in F\} & \text{if } |F| < a, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $f \leq g \bmod J_{<\lambda}[A]$ for all $f \in F$. For, if $f(a) > g(a)$, then $\lambda > |F| \geq a$; thus $\{a : f(a) > g(a)\} \subseteq \lambda \cap A$. Now $\mathrm{pcf}(\lambda \cap A) = \lambda \cap A \subseteq \lambda$, so $\{a : f(a) > g(a)\} \in J_{<\lambda}[A]$.

So, we make the indicated assumption. By this assumption, the set $B \stackrel{\text{def}}{=} A \cap \{|A|^+, |A|^{++}, |A|^{+++}, |A|^{++++}\} \subseteq \lambda$. Suppose that we have shown that $(\prod(A\backslash B), J_{<\lambda}(A\backslash B))$ is λ-directed. Now let $Y \subseteq \prod A$ with $|Y| < \lambda$. Choose $g \in \prod(A\backslash B)$ such that $f \restriction (A\backslash B) <_{J_{<\lambda}[A\backslash B]} g$ for all $f \in Y$. Let $g^+ \in \prod A$ be an extension of g. Then

$$\begin{aligned} \{a : f(a) > g^+(a)\} &= \{a \in A\backslash B : f(a) > g(a)\} \cup \{a \in B : f(a) > g^+(a)\} \\ &\in J_{<\lambda}[A\backslash B] + \mathscr{P}(B \cap \lambda) \\ &= J_{<\lambda}[A] \quad \text{by Lemma 34.6.} \end{aligned}$$

Thus g^+ is an upper bound for $Y \bmod J_{<\lambda}[A]$.

Hence we may assume that $|A|^{+3} < \min(A)$.

Now we prove by induction on the cardinal λ_0 that if $\lambda_0 < \lambda$ and $F = \{f_i : i < \lambda_0\} \subseteq \prod A$ is a family of functions of size λ_0, then F has an upper bound in $(\prod A, <_{J_{<\lambda}})$. So, we assume that this is true for all cardinals less than λ_0. If $\lambda_0 < \min(A)$, then $\sup(F)$ is as desired. So, assume that $\min(A) \leq \lambda_0$.

First suppose that λ_0 is singular. Let $\langle \alpha_i : i < \mathrm{cf}(\lambda_0)\rangle$ be increasing and cofinal in λ_0, each α_i a cardinal. By the inductive hypothesis, let g_i be a bound for $\{f_\xi : \xi < \alpha_i\}$ for each $i < \mathrm{cf}\lambda_0$, and then let h be a bound for $\{g_i : i < \mathrm{cf}\lambda_0\}$. Clearly h is a bound for F.

So assume that λ_0 is regular. We are now going to define a $<_{J_{<\lambda}}$-increasing sequence $\langle f'_\xi : \xi < \lambda_0\rangle$ which satisfies $(*)_\kappa$, with $\kappa = |A|^+$, and such that $f_i \leq f'_i$ for all $i < \lambda_0$. To do this choose, for every $\delta \in S^{\lambda_0}_{\kappa^{++}}$ a club $E_\delta \subseteq \delta$ of order type κ^{++}. Now for such a δ we define

$$f'_\delta = \sup(\{f'_j : j \in E_\delta\} \cup \{f_\delta\}).$$

For ordinals $\delta < \lambda_0$ of cofinality $\neq \kappa^{++}$ we apply the inductive hypothesis to get f'_δ such that $f'_\xi <_{J_{<\lambda}} f'_\delta$ for every $\xi < \delta$ and also $f_\delta <_{J_{<\lambda}} f'_\delta$.

This finishes the construction. By Lemma 34.41, $(*)_{|A|^+}$ holds for f, and hence by Theorem 34.39, f has an exact upper bound $g \in {}^A\mathbf{On}$ with respect to $<_{J_{<\lambda}}$. The identity function on A is an upper bound for f, so we may assume that $g(a) \leq a$ for all $a \in A$. Now we shall prove that $B \stackrel{\text{def}}{=} \{a \in A : g(a) = a\} \in J_{<\lambda}[A]$, so a further modification of g yields the desired upper bound for f.

To get a contradiction, suppose that $B \notin J_{<\lambda}[A]$. Hence $\mathrm{pcf}(B) \not\subseteq \lambda$, and so there is an ultrafilter D over A such that $B \in D$ and $\mathrm{cf}(\prod A/D) \geq \lambda$. Clearly $D \cap J_{<\lambda}[A] = \emptyset$, as otherwise $\mathrm{cf}(\prod A/D) < \lambda$. Now f has length $\lambda_0 < \lambda$, and so it is bounded in $\prod A/D$; say that $f_i <_D h \in \prod A$ for all $i < \lambda_0$. Thus $h(a) < a = g(a)$ for all $a \in B$. Now we define $h' \in \prod A$ by

$$h'(a) = \begin{cases} h(a) & \text{if } a \in B, \\ 0 & \text{otherwise.} \end{cases}$$

Then $h' <_{J_{<\lambda}} g$, since

$$\{a \in A : h'(a) \geq g(a)\} = \{a \in A : g(a) = 0\} \subseteq \{a \in A : f_0(a) \geq g(a)\} \in J_{<\lambda}.$$

Hence by the exactness of g it follows that $h' <_{J_{<\lambda}} f_i$ for some $i < \lambda_0$. But $B \in D$ and hence $h =_D h'$. So $h <_D f_i$, contradiction. □

Corollary 34.9. *Suppose that A is progressive, D is an ultrafilter over A, and λ is a cardinal. Then:*

(i) $\mathrm{cf}(\prod A/D) < \lambda$ *iff* $J_{<\lambda}[A] \cap D \neq \emptyset$.

(ii) $\mathrm{cf}(\prod A/D) = \lambda$ *iff* $J_{<\lambda^+} \cap D \neq \emptyset = J_{<\lambda} \cap D$.

(iii) $\mathrm{cf}(\prod A/D) = \lambda$ *iff λ^+ is the first cardinal μ such that* $J_{<\mu} \cap D \neq \emptyset$.

Proof. (i): $\Rightarrow$: Assuming that $J_{<\lambda}[A] \cap D = \emptyset$, the fact from Theorem 34.8 that $<_{J_{<\lambda}}$ is λ-directed implies that also $\prod A/D$ is λ-directed, and hence $\mathrm{cf}(\prod A/D) \geq \lambda$.

$\Leftarrow$: Assume that $J_{<\lambda}[A] \cap D \neq \emptyset$. Choose $X \in J_{<\lambda} \cap D$. Then by definition, $\mathrm{pcf}(A) \subseteq \lambda$, and hence $\mathrm{cf}(\prod A/D) < \lambda$.

(ii): Immediate from (i).

(iii): Immediate from (ii). □

We now give two important theorems about pcf.

Theorem 34.10. *If A is progressive, then* $|\mathrm{pcf}(A)| \leq 2^{|A|}$.

Proof. By Corollary 34.9, for each $\lambda \in \mathrm{pcf}(A)$ we can select an element $f(\lambda) \in J_{<\lambda^+} \backslash J_{<\lambda}$. Clearly f is a one-one function from $\mathrm{pcf}(A)$ into $\mathscr{P}(A)$. □

Notation. We write $J_{\leq\lambda}$ in place of $J_{<\lambda^+}$.

Theorem 34.11. (The max pcf theorem) *If A is progressive, then* $\mathrm{pcf}(A)$ *has a largest element.*

Proof. Let

$$I = \bigcup_{\lambda \in \mathrm{pcf}(A)} J_{<\lambda}[A].$$

Now clearly each ideal $J_{<\lambda}$ is proper (since for example $\{\lambda\} \notin J_{<\lambda}$), so I is also proper. Extend the dual of I to an ultrafilter D, and let $\mu = \mathrm{cf}(\prod A/D)$. Then for each $\lambda \in \mathrm{pcf}(A)$ we have $J_{<\lambda} \cap D = \emptyset$ since $I \cap D = \emptyset$, and by Corollary 34.9 this means that $\mu \geq \lambda$. □

Corollary 34.12. *Suppose that A is progressive. If λ is a limit cardinal, then*

$$J_{<\lambda}[A] = \bigcup_{\theta<\lambda} J_{\leq\theta}[A].$$

Proof. The inclusion $\supseteq$ is clear. Now suppose that $X \in J_{<\lambda}[A]$. Thus $\mathrm{pcf}(X) \subseteq \lambda$. Let μ be the largest element of $\mathrm{pcf}(X)$. Then $\mu \in \lambda$, and $\mathrm{pcf}(X) \subseteq \mu^+$, so $X \in J_{<\mu^+}$, and the latter is a subset of the right side. □

Theorem 34.13. (The interval theorem) *If A is a progressive interval of regular cardinals, then* $\mathrm{pcf}(A)$ *is an interval of regular cardinals.*

Proof. Let $\mu = \sup(A)$. By 34.3(iii) and 34.1(vi) we may assume that μ is singular. By Theorem 34.11 let $\lambda_0 = \max(\mathrm{pcf}(A))$. Thus we want to show that every regular cardinal λ in (μ, λ_0) is in $\mathrm{pcf}(A)$. By Theorem 34.8, the partial order $(\prod A, <_{J_{<\lambda}})$ is λ-directed. Clearly $J_{<\lambda}$ is a proper ideal, so $\lambda \in \mathrm{pcf}(A)$ by Corollary 34.5. □

Definition. If κ is a cardinal $\leq |A|$, then we define

$$\mathrm{pcf}_\kappa(A) = \bigcup\{\mathrm{pcf}(X) : X \subseteq A \text{ and } |X| = \kappa\}.$$

Theorem 34.14. *If A is an interval of regular cardinals and $\kappa < \min(A)$, then* $\mathrm{pcf}_\kappa(A)$ *is an interval of regular cardinals.*

Note here that we do not assume that A is progressive.

Proof. Let $\lambda_0 = \sup \mathrm{pcf}_\kappa(A)$. Note that each subset X of A of cardinality κ is progressive, and so $\max(\mathrm{pcf}(X))$ exists by Theorem 34.11. Thus

$$\lambda_0 = \sup\{\max(\mathrm{pcf}(X)) : X \subseteq A \text{ and } |X| = \kappa\}.$$

To prove the theorem it suffices to take any regular cardinal λ such that $\min(A) < \lambda < \lambda_0$ and show that $\lambda \in \mathrm{pcf}_\kappa(A)$. In fact, this will show that $\mathrm{pcf}_\kappa(A)$ is an interval of regular cardinals, whether or not λ_0 is regular. Since $\lambda < \lambda_0$, there is an $X \subseteq A$ of size κ such that $\lambda \leq \max(\mathrm{pcf}(X))$. Hence $X \notin J_{<\lambda}[X]$. If there is a proper initial segment Y of X which is not in $J_{<\lambda}[X]$, we can choose the smallest $a \in X$ such that $X \cap a \notin J_{<\lambda}[X]$ and work with $X \cap a$ rather than X. So we may assume that every proper initial segment of X is in $J_{<\lambda}[X]$. If $\lambda \in A$, clearly $\lambda \in \mathrm{pcf}_\kappa(A)$. So we may assume that $\lambda \notin A$. If $\lambda < \sup(X)$, then $\lambda \in A$, contradiction. If $\lambda = \sup(X)$, then $\lambda = \sup(A)$ since $\lambda \notin A$, and this contradicts Proposition 34.3(ii). So $\sup(X) < \lambda$. Since $J_{<\lambda}[X]$ is λ-directed by Theorem 34.8, we can apply 34.4 to obtain $\lambda \in \mathrm{pcf}(X)$, and hence $\lambda \in \mathrm{pcf}_\kappa(A)$, as desired. □

Another of the central results of pcf theory is as follows.

Theorem 34.15. (Closure theorem.) *Suppose that A is progressive, $B \subseteq \mathrm{pcf}(A)$, and B is progressive. Then* $\mathrm{pcf}(B) \subseteq \mathrm{pcf}(A)$. *In particular, if* $\mathrm{pcf}(A)$ *itself is progressive, then* $\mathrm{pcf}(\mathrm{pcf}(A)) = \mathrm{pcf}(A)$.

Proof. Suppose that $\mu \in \text{pcf}(B)$, and let E be an ultrafilter on B such that $\mu = \text{cf}(\prod B/E)$. For every $b \in B$ fix an ultrafilter D_b on A such that $b = \text{cf}(\prod A/D_b)$. Define F by

$$X \in F \quad \text{iff} \quad X \subseteq A \text{ and } \{b \in B : X \in D_b\} \in E.$$

It is straightforward to check that F is an ultrafilter on A. The rest of the proof consists in showing that $\mu = \text{cf}(\prod A/F)$.

By Proposition 34.22 we have

$$\mu = \text{cf}\left(\prod_{b\in B}\left(\prod A/D_b\right)/E\right).$$

Hence it suffices by Proposition 34.10 to show that $\prod A/F$ is isomorphic to a cofinal subset of this iterated ultraproduct. To do this, we consider the Cartesian product $B \times A$ and define

$$H \in P \quad \text{iff} \quad H \subseteq B \times A \text{ and } \{b \in B : \{a \in A : (b, a) \in H\} \in D_b\} \in E.$$

Again it is straightforward to check that P is an ultrafilter over $B \times A$. Let $r(b, a) = a$ for any $(b, a) \in B \times A$. Then

$$(*) \qquad \left(\prod_{(b,a)\in B\times A} a\right)/P \cong \prod_{b\in B}\left(\prod A/D_b\right)/E.$$

To prove $(*)$, for any $f \in \prod_{\langle b,a\rangle\in B\times A} a$ we define $f' \in \prod_{b\in B}(\prod A/D_b)$ by setting

$$f'(b) = \langle f(b, a) : a \in A\rangle/D_b.$$

Then for any $f, g \in \prod_{\langle b,a\rangle\in B\times A} a$ we have

$$\begin{aligned} f =_P g \quad &\text{iff} \quad \{(b, a) : f(b, a) = g(b, a)\} \in P \\ &\text{iff} \quad \{b : \{a : f(b, a) = g(b, a)\} \in D_b\} \in E \\ &\text{iff} \quad \{b : f'(b) = g'(b)\} \in E \\ &\text{iff} \quad f' =_E g'. \end{aligned}$$

Hence we can define $k(f/P) = f'/E$, and we get a one-one function. To show that it is a surjection, suppose that $h \in \prod_{b\in B}(\prod A/D_b)$. For each $b \in B$ write $h(b) = h'_b/D_b$ with $h'_b \in \prod A$. Then define $f(b, a) = h'_b(a)$. Then

$$f'(b) = \langle f(b, a) : a \in A\rangle/D_b = \langle h'_b(a) : a \in A\rangle/D_b = h'_b/D_b = h(b),$$

as desired. Finally, k preserves order, since

$$\begin{aligned} f/P < g/P \quad &\text{iff} \quad \{(b, a) : f(b, a) < g(b, a)\} \in P \\ &\text{iff} \quad \{b : \{a : f(b, a) < g(b, a)\} \in D_b\} \in E \\ &\text{iff} \quad \{b : f'(b) < g'(b)\} \in E \\ &\text{iff} \quad k(f/P) < k(g/P). \end{aligned}$$

So $(*)$ holds.

Now we apply Lemma 34.23, with $r, B \times A, A, P$ in place of c, A, B, I respectively. Then F is the Rudin-Keisler projection on A, since for any $X \subseteq A$,

$$\begin{aligned} X \in F \quad &\text{iff} \quad \{b \in B : X \in D_b\} \in E \\ &\text{iff} \quad \{b \in B : \{a \in A : r(b,a) \in X\} \in D_b\} \in E \\ &\text{iff} \quad \{b \in B : \{a \in A : (b,a) \in r^{-1}[X]\} \in D_b\} \in E \\ &\text{iff} \quad r^{-1}[X] \in P. \end{aligned}$$

Thus by Lemma 34.23 we get an isomorphism h of $\prod A/F$ into $\prod_{(b,a)\in B\times A} a/P$ such that $h(e/F) = \langle e(r(b,a)) : (b,a) \in B \times A\rangle/P$ for any $e \in \prod A$. So now it suffices now to show that the range of h is cofinal in $\prod_{(b,a)\in B\times A} a/P$. Let $g \in \prod_{(b,a)\in B\times A} a$. For every $b \in B$ define $g_b \in \prod A$ by $g_b(a) = g(b,a)$. Let $\lambda = \min(B)$. Since B is progressive, we have $|B| < \lambda$. Hence by the λ-directness of $\prod A/J_{<\lambda}[A]$ (Theorem 34.8), there is a function $k \in \prod A$ such that $g_b <_{J_{<\lambda}} k$ for each $b \in B$. Now $\lambda \leq b$ for all $b \in B$, so $J_{<\lambda} \cap D_b = \emptyset$, and so $g_b <_{D_b} k$. It follows that $g/P <_P h(k/D)$. In fact, let $H = \{(b,a) : g(b,a) < k(r(b,a))\}$. Then

$$\{b \in B : \{a \in A : (b,a) \in H\} \in D_b\} = \{b \in B : \{a \in A : g_b(a) < k(a)\} \in D_b\} = B \in E,$$

as desired. $\square$

Generators for $J_{<\lambda}$

If I is an ideal on a set A and $B \subseteq A$, then $I + B$ is the ideal generated by $I \cup \{B\}$; that is, it is the intersection of all ideals J on A such that $I \cup \{B\} \subseteq J$.

Proposition 34.16. *Suppose that I is an ideal on A and $B, X \subseteq A$. Then the following conditions are equivalent:*

(i) $X \in I + B$.

(ii) There is a $Y \in I$ such that $X \subseteq Y \cup B$.

(iii) $X \backslash B \in I$.

Proof. Clearly (ii)⇒(i). The set

$$\{Z \subseteq A : \exists Y \in I[Z \subseteq Y \cup B]\}$$

is clearly an ideal containing $I \cup \{B\}$, so (i)⇒(ii). If Y is as in (ii), then $X\backslash B \subseteq Y$, and hence $X\backslash B \in I$; so (ii)⇒(iii). If $X\backslash B \in I$, then $X \subseteq (X\backslash B) \cup B$, so X satisfies the condition of (ii). So (iii)⇒(ii). $\square$

The following easy lemma will be useful later.

Lemma 34.17. *Suppose that A is progressive and $B \subseteq A$.*

(i) $\mathscr{P}(B) \cap J_{<\lambda}[A] = J_{<\lambda}[B]$.

(ii) If $f, g \in \prod A$ and $f <_{J_{<\lambda}[A]} g$, then $(f \restriction B) <_{J_{<\lambda}[B]} (g \restriction B)$.

Proof. (i): Suppose that $X \in \mathscr{P}(B)\cap J_{<\lambda}[A]$ and $X \in D$, an ultrafilter on B. Extend D to an ultrafilter E on A. Then $\prod B/D \cong \prod A/E$, and $\mathrm{cf}(\prod A/E) < \lambda$. So $X \in J_{<\lambda}[B]$. The converse is proved similarly.

(ii): Assume that $f, g \in \prod A$ and $f <_{J_{<\lambda}[A]} g$. Then

$$\{a \in B : g(b) \le f(b)\} \in \mathscr{P}(B) \cap J_{<\lambda}[A] = J_{<\lambda}[B]$$

by (i), as desired. □

Definitions. If there is a set X such that $J_{\le\lambda}[A] = J_{<\lambda} + X$, then we say that λ is *normal.*

Let A be a set of regular cardinals, and λ a cardinal. A subset $B \subseteq A$ is a λ-*generator over* A iff $J_{\le\lambda}[A] = J_{<\lambda}[A] + B$. We omit the qualifier "over A" if A is understood from the context.

Suppose that $\lambda \in \mathrm{pcf}(A)$. A *universal sequence for* λ is a sequence $f = \langle f_\xi : \xi < \lambda\rangle$ which is $<_{J_{<\lambda}[A]}$-increasing, and for every ultrafilter D over A such that $\mathrm{cf}(\prod A/D) = \lambda$, the sequence f is cofinal in $\prod A/D$.

Theorem 34.18. (Universal sequences) *Suppose that A is progressive. Then every $\lambda \in \mathrm{pcf}(A)$ has a universal sequence.*

Proof. First,

(1) We may assume that $|A|^+ < \min(A) < \lambda$.

In fact, suppose that we have proved the theorem under the assumption (1), and now take the general situation. Recall from Proposition 3.19(vii) that $\min(A) \le \lambda$. If $\lambda = \min(A)$, define $f_\xi \in \prod A$, for $\xi < \lambda$, by $f_\xi(a) = \xi$ for all $a \in A$. Thus f is $<$-increasing, hence $<_{J_{<\lambda}[A]}$-increasing. Suppose that D is an ultrafilter on A such that $\mathrm{cf}(\prod A/D) = \lambda$. Then $\{\min(A)\} \in D$, as otherwise $A\backslash\{\min(A)\} \in D$ and hence $\mathrm{cf}(\prod A/D) > \lambda$ by Proposition 34.1(vii). Thus for any $g \in \prod A$, let $\xi = g(\min(A)) + 1$. Then $\{a \in A : g(a) < f_\xi(a)\} \supseteq \{\min(A)\} \in D$, so $[g] < [f_\xi]$. Hence $\langle [f_\xi] : \xi < \lambda\rangle$ is cofinal in $\prod A/D$.

Now suppose that $\min(A) < \lambda$. Let $a_0 = \min A$. Let $A' = A\backslash\{a_0\}$. If D is an ultrafilter such that $\lambda = \mathrm{cf}(\prod A/D)$, then $A' \in D$ since $a_0 < \lambda$, hence $\{a_0\} \notin D$. It follows that $\lambda \in \mathrm{pcf}(A')$. Clearly $|A'|^+ < \min A' \le \lambda$. Hence by assumption we get a system $\langle f_\xi : \xi < \lambda\rangle$ of members of $\prod A'$ which is increasing in $<_{J_{<\lambda}[A']}$ such that for every ultrafilter D over A' such that $\lambda = \mathrm{cf}(\prod A'/D)$, f is cofinal in $\prod A'/D$. Extend each f_ξ to $g_\xi \in \prod A$ by setting $g_\xi(a_0) = 0$. If $\xi < \eta < \lambda$, then

$$\{a \in A : g_\xi(a) \ge g_\eta(a)\} \subseteq \{a \in A' : f_\xi(a) \ge f_\eta(a)\} \cup \{a_0\},$$

and $\{a \in A' : f_\xi(a) \ge f_\eta(a)\} \in J_{<\lambda}[A'] \subseteq J_{<\lambda}[A]$ and also $\{a_0\} \in J_{<\lambda}[A]$ since $a_0 < \lambda$, so $g_\xi <_{J_{<\lambda}} g_\eta$. Now let D be an ultrafilter over A such that $\lambda = \mathrm{cf}(\prod A/D)$. As above, $A' \in D$; let $D' = D \cap \mathscr{P}(A')$. Then $\lambda = \mathrm{cf}(\prod A'/D')$. To show that g is cofinal in $\prod A/D$, take any $h \in \prod A$. Choose $\xi < \lambda$ such that $(h \restriction A')/D' < f_\xi/D'$. Then

$$\{a \in A : h(a) \ge g_\xi(a)\} \supseteq \{a \in A' : h(a) \ge f_\xi(a)\},$$

so $h/D < g_\xi/D$, as desired.

Thus we can make the assumption as in (1). Suppose that there is no universal sequence for λ. Thus

(2) For every $<_{J_{<\lambda}}$-increasing sequence $f = \langle f_\xi : \xi < \lambda\rangle$ there is an ultrafilter D over A such that $\mathrm{cf}(\prod A/D) = \lambda$ but f is not cofinal in $\prod A/D$.

We are now going to construct a $<_{J_{<\lambda}}$-increasing sequence $f^\alpha = \langle f^\alpha_\xi : \xi < \lambda\rangle$ for each $\alpha < |A|^+$. We use the fact that $\prod A/J_{<\lambda}$ is λ-directed (Theorem 34.8).

Using this directedness, we start with any $<_{J_{<\lambda}}$-increasing sequence $f^0 = \langle f^0_\xi : \xi < \lambda\rangle$.

For δ limit $< |A|^+$ we are going to define f^δ_ξ by induction on ξ so that the following conditions hold:

(3) $f^\delta_i <_{J_{<\lambda}} f^\delta_\xi$ for $i < \xi$,

(4) $\sup\{f^\alpha_\xi : \alpha < \delta\} \le f^\delta_\xi$.

Suppose that f^δ_i has been defined for all $i < \xi$. By λ-directedness, choose g such that $f^\delta_i <_{J_{<\lambda}} g$ for all $i < \xi$. Now for any $a \in A$ we have $\sup\{f^\alpha_\xi(a) : \alpha < \delta\} < a$, since $\delta < |A|^+ < \min A \le a$. Hence we can define

$$f^\delta_\xi(a) = \max\{g(a), \sup\{f^\alpha_\xi(a) : \alpha < \delta\}\}.$$

Clearly the conditions (3), (4) hold.

Now suppose that f^α has been defined and is $<_{J_{<\lambda}}$-increasing; we define $f^{\alpha+1}$. By (2), choose an ultrafilter D_α over A such that

(5) $\mathrm{cf}(\prod A/D_\alpha) = \lambda$;

(6) The sequence f^α is bounded in $<_{D_\alpha}$.

By (6), choose $f^{\alpha+1}_0$ which bounds f^α in $<_{D_\alpha}$; in addition, $f^{\alpha+1}_0 \ge f^\alpha_0$. Let $\langle h_\xi/D_\alpha : \xi < \lambda\rangle$ be strictly increasing and cofinal in $\prod A/D_\alpha$. Now we define $f^{\alpha+1}_\xi$ by induction on ξ when $\xi > 0$. First, by λ-directness, choose k such that $f^{\alpha+1}_i <_{J_{<\lambda}} k$ for all $i < \xi$. Then for any $a \in A$ let

$$f^{\alpha+1}_\xi(a) = \max(k(a), h_\xi(a), f^\alpha_\xi(a)).$$

Then the following conditions hold:

(7) $f^{\alpha+1}$ is strictly increasing and cofinal in $\prod A/D_\alpha$;

(8) $f^{\alpha+1}_i \ge f^\alpha_i$ for every $i < \lambda$.

This finishes the construction. Clearly we then have

(9) If $i < \lambda$ and $\alpha_1 < \alpha_2 < |A|^+$, then $f^{\alpha_1}_i \le f^{\alpha_2}_i$.

(10) f^α is bounded in $\prod A/D_\alpha$ by $f^{\alpha+1}_0$.

(11) $f^{\alpha+1}$ is cofinal in $\prod A/D_\alpha$.

Now let $h = \sup_{\alpha<|A|^+} f_0^\alpha$. Then $h \in \prod A$, since $|A|^+ < \min(A)$. By (11), for each $\alpha < |A|^+$ choose $i_\alpha < \lambda$ such that $h <_{D_\alpha} f_{i_\alpha}^{\alpha+1}$. Since $\lambda > |A|^+$ is regular, we can choose $i < \lambda$ such that $i_\alpha < i$ for all $\alpha < |A|^+$. Now define

$$A^\alpha = \{a \in A : h(a) \le f^\alpha(a)\}.$$

By (9) we have $A^\alpha \subseteq A^\beta$ for $\alpha < \beta < |A|^+$. We are going to get a contradiction by showing that $A^\alpha \subset A^{\alpha+1}$ for every $\alpha < |A|^+$.

In fact, this follows from the following two statements.

(12) $A^\alpha \notin D_\alpha$.

This holds because $f_i^\alpha <_{D_\alpha} f_i^{\alpha+1} \le h$.

(13) $A^{\alpha+1} \in D_\alpha$.

This holds because $h <_{D_\alpha} f_i^{\alpha+1}$ by the choice of i and (7). $\square$

Proposition 34.19. *If A is a set of regular cardinals, λ is the largest member of* $\mathrm{pcf}(A)$*, and $\langle f_\xi : \xi < \lambda\rangle$ is universal for λ, then it is cofinal in $(\prod A, J_{<\lambda})$.*

Proof. Assume the hypotheses. Fix $g \in \prod A$; we want to find $\xi < \lambda$ such that $g <_{J_{<\lambda}} f_\xi$. Suppose that no such ξ exists. Then, we claim, the set

$$J^*_{<\lambda} \cup \{\{a \in A : g(a) \ge f_\xi(a)\} : \xi < \lambda\} \tag{1}$$

has fip. For, suppose that it does not have fip. Then there is a finite nonempty subset F of λ such that

$$\bigcup_{\xi\in F} \{a \in A : g(a) < f_\xi(a)\} : \xi < \lambda\} \in J^*_{<\lambda}. \tag{2}$$

Let η be the largest member of F. Note that the set

$$\{a \in A : f_\xi(a) < f_\rho(a) \text{ for all } \xi, \rho \in F \text{ such that } \xi < \rho\}$$

is also a member of $J^*_{<\lambda}$; intersecting this set with the set of (2), we get a member of $J^*_{<\lambda}$ which is a subset of $\{a \in A : g(a) < f_\eta(a)\}$, so that $g <_{J_{<\lambda}} f_\eta$, contradiction.

Thus the set (1) has fip. Let D be an ultrafilter containing it. Then $\mathrm{cf}(\prod A/D) = \lambda$, so by hypothesis there is a $\xi < \lambda$ such that $g <_D f_\xi$. Thus $\{a \in A : g(a) < f_\xi(a)\} \in D$. But also $\{a \in A : g(a) \ge f_\xi(a)\} \in D$, contradiction. $\square$

Theorem 34.20. *If A is progressive, then* $\mathrm{cf}(\prod A, <) = \max(\mathrm{pcf}(A))$*. In particular,* $\mathrm{cf}(\prod A, <)$ *is regular.*

Proof. First we prove $\ge$. Let $\lambda = \max(\mathrm{pcf}(A))$, and let D be an ultrafilter on A such that $\lambda = \mathrm{cf}(\prod A/D)$. Now for any $f, g \in \prod A$, if $f < g$ then $f <_D g$. Hence any cofinal set in $(\prod A, <)$ is also cofinal in $(\prod A, <_D)$, and so $\lambda = \mathrm{cf}(\prod A, <_D) \le \mathrm{cf}(\prod A, <)$.

To prove $\leq$, we exhibit a cofinal subset of $(\prod A, <)$ of size λ. For every $\mu \in \mathrm{pcf}(A)$ fix a universal sequence $f^\mu = \langle f_i^\mu : i < \mu\rangle$ for μ, by Theorem 34.18. Let F be the set of all functions of the form

$$\sup\{f_{i_1}^{\mu_1}, f_{i_2}^{\mu_2}, \ldots, f_{i_n}^{\mu_n}\},$$

where $\mu_1, \mu_2, \ldots, \mu_n$ is a finite sequence of members of $\mathrm{pcf}(A)$, possibly with repetitions, and $i_k < \mu_k$ for each $k = 1, \ldots, n$. We claim that F is cofinal in $(\prod A, <)$; this will complete the proof.

To prove this claim, let $g \in \prod A$. Let

$$I = \{>(f, g) : f \in F\}.$$

(Recall that $>(f, g) = \{a \in A : f(a) > g(a)\}$.) Now I is closed under unions, since

$$>(f_1, g) \cup >(f_2, g) = >(\sup(f_1, f_2), g).$$

If $A \in I$, then $A = >(f, g)$ for some $f \in F$, as desired. So, suppose that $A \notin I$. Now $J \stackrel{\text{def}}{=} \{A \backslash X : X \in I\}$ has fip since I is closed under unions, and so this set can be extended to an ultrafilter D over A. Let $\mu = \mathrm{cf}(\prod A/D)$. Then f^μ is cofinal in $(\prod A, <_D)$ since it is universal for μ. But $f_i^\mu \leq_I g$ for all $i < \mu$, since $f_i^\mu \in F$ and so $>(f_i^\mu, g) \in I$. This is a contradiction. $\square$

Note that Theorem 34.20 is not talking about true cofinality. In fact, clearly any increasing sequence of elements of $\prod A$ under $<$ must have order type at most $\min(A)$, and so true cofinality does not exist if A has more than one element.

Lemma 34.21. *Suppose that A is progressive, $\lambda \in \mathrm{pcf}(A)$, and $f' = \langle f'_\xi : \xi < \lambda\rangle$ is a universal sequence for λ. Suppose that $f = \langle f_\xi : \xi < \lambda\rangle$ is $<_{J_{<\lambda}}$-increasing, and for every $\xi' < \lambda$ there is a $\xi < \lambda$ such that $f'_{\xi'} \leq_{J_{<\lambda}} f_\xi$. Then f is universal for λ.*

Proof. This is clear, since for any ultrafilter D over A such that $\mathrm{cf}(\prod A/D) = \lambda$ we have $D \cap J_{<\lambda} = \emptyset$, and hence $f'_{\xi'} \leq_{J_{<\lambda}} f_\xi$ implies that $f'_{\xi'} \leq_D f_\xi$. $\square$

For the next result, note that if A is progressive, then $|A| < \min(A)$, and hence $|A|^+ \leq \min(A)$. So $A \cap |A|^+ = \emptyset \in J_{<\lambda}$ for any λ. So if μ is an ordinal and $A \cap \mu \notin J_{<\lambda}$, then $|A|^+ < \mu$.

Lemma 34.22. *Suppose that A is a progressive set of regular cardinals and $\lambda \in \mathrm{pcf}(A)$.*

(i) Let μ be the least ordinal such that $A \cap \mu \notin J_{<\lambda}[A]$. Then there is a universal sequence for λ that satisfies $(*)_\kappa$ for every regular cardinal κ such that $\kappa < \mu$.

(ii) There is a universal sequence for λ that satisfies $(*)_{|A|^+}$.

Proof. First note that (ii) follows from (i) by the remark preceding this lemma. Now we prove (i). Note by the minimality of μ that either $\mu = \rho + 1$ for some $\rho \in A$, or μ is a limit cardinal and $A \cap \mu$ is unbounded in μ.

(1) $\mu \leq \lambda + 1$.

For, let D be an ultrafilter such that $\lambda = \mathrm{cf}(\prod A/D)$. Then $A \cap (\lambda+1) \in D$, as otherwise $\{a \in A : \lambda < a\} \in D$, and so $\mathrm{cf}(\prod A/D) > \lambda$ by 34.1(vii), contradiction. Thus $\lambda \in \mathrm{pcf}(A \cap (\lambda+1))$, and hence $\mathrm{pcf}((A \cap (\lambda+1)) \not\subseteq \lambda$, proving (1).

(2) $\mu \neq \lambda$.

For, $|A| < \min(A) \leq \lambda$, so $A \cap \lambda$ is bounded in λ because λ is regular. Hence $\mu \neq \lambda$ by an initial remark of this proof.

Now we can complete the proof for the case in which μ is $\rho+1$ for some $\rho \in A$. In this case, actually $\rho = \lambda$. For, we have $A \cap \rho \in J_{<\lambda}[A]$ while $A \cap (\rho+1) \notin J_{<\lambda}[A]$. Let D be an ultrafilter on A such that $A \cap (\rho+1) \in D$ and $\mathrm{cf}(\prod A/D) \geq \lambda$. Then $A \cap \rho \notin D$, since $A \cap \rho \in J_{<\lambda}[A]$, so $\{\rho\} \in D$, and so $\rho \geq \lambda$. By (1) we then have $\rho = \lambda$.

Now define, for $\xi < \lambda$ and $a \in A$,

$$f_\xi(a) = \begin{cases} 0 & \text{if } a < \lambda, \\ \xi & \text{if } \lambda \leq a. \end{cases}$$

Thus $f_\xi \in \prod A$. The sequence $\langle f_\xi : \xi < \lambda \rangle$ is $<_{J_{<\lambda}[A]}$-increasing, since if $\xi < \eta < \lambda$ then $\{a \in A : f_\xi(a) \geq f_\eta(a)\} \subseteq A \cap \lambda \in J_{<\lambda}[A]$. It is also universal for λ. For, suppose that D is an ultrafilter on A such that $\mathrm{cf}(\prod A/D) = \lambda$. Suppose that $g \in \prod A$. Now $|A| < \min(A) \leq \lambda$, so $\xi \stackrel{\text{def}}{=} (\sup_{a \in A} g(a)) + 1$ is less than λ. Now $\{a \in A : g(a) < f_\xi(a)\} = A \in D$, so $g <_D f_\xi$, as desired. Finally, $\langle f_\xi : \xi < \lambda \rangle$ satisfies $(*)_\lambda$, since it is itself strongly increasing under $J_{<\lambda}[A]$. In fact, if $\xi < \eta < \lambda$ and $a \in A \backslash \lambda$, then $f_\xi(a) = \xi < \eta = f_\eta(a)$, and $A \cap \lambda \in J_{<\lambda}[A]$.

Hence the case remains in which $\mu < \lambda$ and $A \cap \mu$ is unbounded in μ. Let $\langle f'_\xi : \xi < \lambda \rangle$ be any universal sequence for λ. We now apply Lemma 34.43 with I replaced by $J_{<\lambda}[A]$. (Recall that $(\prod A, I_{<\lambda}[A]$ is λ-directed by Theorem 34.8.) This gives us a $<_{J_{<\lambda}[A]}$-increasing sequence $f = \langle f_\xi : \xi < \lambda \rangle$ such that $f'_\xi < f_{\xi+1}$ for every $\xi < \lambda$, and $(*)_\kappa$ holds for f, for every regular cardinal κ such that $\kappa^{++} < \lambda$ and $\{a \in A : a \leq \kappa^{++}\} \in J_{<\lambda}[A]$. Clearly then f is universal for λ. If κ is a regular cardinal less than μ, then $\kappa^{++} < \mu < \lambda$, and $\{a \in A : a \leq \kappa^{++}\} \subseteq J_{<\lambda}[A]$ by the minimality of μ, so the conclusion of the lemma holds. □

Lemma 34.23. *Suppose that A is a progressive set of regular cardinals, $B \subseteq A$, and λ is a regular cardinal. Then the following conditions are equivalent:*

(i) $J_{\leq\lambda}[A] = J_{<\lambda}[A] + B$.

(ii) $B \in J_{\leq\lambda}[A]$, and for every ultrafilter D on A, if $\mathrm{cf}(\prod A/D) = \lambda$, then $B \in D$.

Proof. (i)⇒(ii): Assume (i). Obviously, then, $B \in J_{\leq\lambda}[A]$. Now suppose that D is an ultrafilter on A and $\mathrm{cf}(\prod A/D) = \lambda$. By Corollary 34.9(ii) we have $J_{\leq\lambda}[A] \cap D \neq \emptyset = J_{<\lambda}[A] \cap D$. Choose $X \in J_{\leq\lambda}[A] \cap D$. Then by Proposition 34.16, $X \backslash B \in J_{<\lambda}[A]$, so since $J_{<\lambda}[A] \cap D = \emptyset$, we get $B \in D$.

(ii)⇒(i): $\supseteq$ is clear. Now suppose that $X \in J_{\leq\lambda}[A]$. If $X \subseteq B$, then obviously $X \in J_{<\lambda}[A] + B$. Suppose that $X \not\subseteq B$, and let D be any ultrafilter such that $X \backslash B \in D$. Then $\mathrm{cf}(\prod A/D) \leq \lambda$ since $\mathrm{pcf}(X) \subseteq \lambda^+$, and so $\mathrm{cf}(\prod A/D) < \lambda$ by the second assumption in (ii). This shows that $\mathrm{pcf}(X \backslash B) \subseteq \lambda$, so $X \backslash B \in J_{<\lambda}[A]$, and hence $X \in J_{<\lambda}[A] + B$ by Proposition 34.16. □

Theorem 34.24. *If A is progressive, then every member of* $\mathrm{pcf}(A)$ *has a generator.*

Proof. First suppose that we have shown the theorem if $|A|^+ < \min(A)$. We show how it follows when $|A|^+ = \min(A)$. The least member of $\mathrm{pcf}(A)$ is $|A|^+$ by 34.1(vii). We have $J_{<|A|^+}[A] = \{\emptyset\}$ and $J_{\leq|A|^+}[A] = \{\emptyset, \{|A|^+\}\} = J_{<|A|^+}[A] + |A|^+$, so $|A|^+$ is a $|A|^+$-generator. Now suppose that $\lambda \in \mathrm{pcf}(A)$ with $\lambda > |A|^+$. Let $A' = A\backslash\{|A|^+\}$. By 34.1(vi) we also have $\lambda \in \mathrm{pcf}(A')$. By the supposed result there is a $b \subseteq A'$ such that $J_{\leq\lambda}[A'] = J_{<\lambda}[A'] + b$. Hence, applying Lemma 34.6 to λ^+ and $\{|A|^+\}$,

$$\begin{aligned} J_{\leq\lambda}[A] &= J_{\leq\lambda}[A'] + \{|A|^+\} \\ &= J_{<\lambda}[A'] + b + \{|A|^+\} \\ &= J_{<\lambda}[A] + b, \end{aligned}$$

as desired.

Thus we assume henceforth that $|A|^+ < \min(A)$. Suppose that $\lambda \in \mathrm{pcf}(A)$. First we take the case $\lambda = |A|^{++}$. Hence by Lemma 34.1(vii) we have $\lambda \in A$. Clearly

$$J_{\leq\lambda}[A] = \{\emptyset, \{\lambda\}\} = \{\emptyset\} + \{\lambda\} = J_{<\lambda}[A] + \{\lambda\},$$

so λ has a generator in this case. So henceforth we assume that $|A|^{++} < \lambda$.

By Lemma 34.22, there is a universal sequence $f = \langle f_\xi : \xi < \lambda\rangle$ for λ such that $(*)_{|A|^+}$ holds. Hence by Lemma 8.40, f has an exact upper bound h with respect to $<_{J_{<\lambda}}$. Since h is a least upper bound for f and the identity function on A is an upper bound for f, we may assume that $h(a) \leq a$ for all $a \in A$. We now define

$$B = \{a \in A : h(a) = a\}.$$

Thus we can finish the proof by showing that

$$(\star) \qquad J_{\leq\lambda}[A] = J_{<\lambda}[A] + B$$

First we show that $B \in J_{\leq\lambda}[A]$, i.e., that $\mathrm{pcf}(B) \subseteq \lambda^+$. Let D be any ultrafilter over A having B as an element; we want to show that $\mathrm{cf}(\prod A/D) \leq \lambda$. If $D \cap J_{<\lambda} \neq \emptyset$, then $\mathrm{cf}(\prod A/D) < \lambda$ by the definition of $J_{<\lambda}$. Suppose that $D \cap J_{<\lambda} = \emptyset$. Now since f is $<_{J_{<\lambda}}$-increasing and $D \cap J_{<\lambda} = \emptyset$, the sequence f is also $<_D$-increasing. It is also cofinal; for let $g \in \prod A$. Define

$$g'(a) = \begin{cases} g(a) & \text{if } a \in B, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{a \in A : g'(a) \geq h(a)\} \subseteq \{a \in A : h(a) = 0\} \subseteq \{a \in A : f_0(a) \geq h(a)\} \in J_{<\lambda}$. So $g' <_{J_{<\lambda}} h$. Since h is an exact upper bound for f, there is hence a $\xi < \lambda$ such that $g' <_{J_{<\lambda}} f_\xi$. Hence $g' <_D f_\xi$, and clearly $g =_D g'$, so $g <_D f_\xi$. This proves that $\mathrm{cf}(\prod A/D) = \lambda$. So we have proved $\supseteq$ in $(\star)$.

For $\subseteq$, we argue by contradiction and suppose that there is an $X \in J_{\leq\lambda}$ such that $X \notin J_{<\lambda}[A] + B$. Hence (by Proposition 34.16), $X\backslash B \notin J_{<\lambda}$. Hence $J^*_{<\lambda} \cup \{X\backslash B\}$ has fip, so we extend it to an ultrafilter D. Since $D \cap J_{<\lambda} = \emptyset$, we have $\mathrm{cf}(\prod A/D) \geq \lambda$. But also $X \in D$ since $X \backslash B \in D$, and $X \in J_{\leq\lambda}$, so $\mathrm{cf}(\prod A/D) = \lambda$. By the universality of f it follows that f is cofinal in $\mathrm{cf}(\prod A/D)$. But $A \backslash B \in D$, so $\{a \in A : h(a) < a\} \in D$, and so there is a $\xi < \lambda$ such that $h <_D f_\xi$. This contradicts the fact that h is an upper bound of f under $<_{J_{<\lambda}}$. $\square$

Now we state some important properties of generators.

Lemma 34.25. *Suppose that A is progressive, $\lambda \in \mathrm{pcf}(A)$, and $B \subseteq A$.*

(i) If B is a λ-generator, D is an ultrafilter on A, and $\mathrm{cf}(\prod A/D) = \lambda$, then $B \in D$.

(ii) If B is a λ-generator, then $\lambda \notin \mathrm{pcf}(A \backslash B)$.

(iii) If $B \in J_{\leq\lambda}$ and $\lambda \notin \mathrm{pcf}(A \backslash B)$, then B is a λ-generator.

(iv) If $\lambda = \max(\mathrm{pcf}(A))$, then A is a λ-generator on A.

(v) If B is a λ-generator, then the restrictions to B of any universal sequence for λ are cofinal in $(\prod B, <_{J_{<\lambda}[B]})$.

(vi) If B is a λ-generator, then $\mathrm{tcf}(\prod B, <_{J_{<\lambda}[B]}) = \lambda$.

(vii) If B is a λ-generator on A, then $\lambda = \max(\mathrm{pcf}(B))$.

(viii) If B is a λ-generator on A and D is an ultrafilter on A, then $\mathrm{cf}(\prod A/D) = \lambda$ iff $B \in D$ and $D \cap J_{<\lambda} = \emptyset$.

(ix) If B is a λ-generator on A and $B =_{J_{<\lambda}} C$, then C is a λ-generator on A. [Here $X =_I Y$ means that the symmetric difference of X and Y is in I, for any ideal I.]

(x) If B is a λ-generator, then so is $B \cap (\lambda + 1)$.

(xi) If B and C are λ-generators, then $B =_{J_{<\lambda}} C$.

(xii) If $\lambda = \max(\mathrm{pcf}(A))$ and B is a λ-generator, then $A \backslash B \in J_{<\lambda}$.

Proof. (i): By Corollary 34.9(ii), choose $C \in J_{\leq\lambda} \cap D$. Hence $C \subseteq X \cup B$ for some $X \in J_{<\lambda}$. By Corollary 34.9(ii) again, $J_{<\lambda} \cap D = \emptyset$, so $X \notin D$. Thus $C \backslash X \subseteq B$ and $C \backslash X \in D$, so $B \in D$.

(ii): Clear by (i).

(iii): Assume the hypothesis. We need to show that every member C of $J_{\leq\lambda}$ is a member of $J_{<\lambda} + B$. Now $\mathrm{pcf}(C) \subseteq \lambda^+$. Hence $\mathrm{pcf}(C \backslash B) \subseteq \lambda$, so $C \backslash B \in J_{<\lambda}$, and the desired conclusion follows from Proposition 34.16.

(iv): By (iii).

(v): Suppose not. Let $f = \langle f_\xi : \xi < \lambda \rangle$ be a universal sequence for λ such that there is an $h \in \prod B$ such that h is not bounded by any $f_\xi \restriction B$. Thus $\leq (f_\xi \restriction B, h) \notin J_{<\lambda}[B]$ for all $\xi < \lambda$. Now suppose that $\xi < \eta < \lambda$. Then

$$\begin{aligned} \leq (f_\eta \restriction B, h) \backslash (\leq (f_\xi \restriction B, h)) &= \{a \in B : f_\eta(a) \leq h(a) < f_\xi(a)\} \\ &\subseteq \{a \in A : f_\eta(a) < f_\xi(a)\} \in J_{<\lambda}[A]. \end{aligned}$$

Hence by Lemma 34.17(i) we have $\leq (f_\eta \restriction B, h) \backslash (\leq (f_\xi \restriction B, h)) \in J_{<\lambda}[B]$. It follows that if N is a finite subset of λ with largest element less than η, then

$$(*) \qquad (\leq (f_\eta \restriction B, h)) \backslash \bigcap_{\xi \in N} (\leq (f_\xi \restriction B, h)) \in J_{<\lambda}[B].$$

We claim now that

$$M \overset{\text{def}}{=} \{\leq (f_\xi \restriction B, h) : \xi < \lambda\} \cup (J_{<\lambda}[B])^*$$

has fip. Otherwise, there is a finite subset N of λ and a set $C \in J_{<\lambda}[B]$ such that

$$\left(\bigcap_{\xi \in N} \leq (f_\xi \restriction B, h)\right) \cap (B \backslash C) = \emptyset;$$

hence if ξ is the largest member of N we get $\leq (f_\xi \restriction B, h) \in J_{<\lambda}[B]$ by $(*)$, contradiction. So we extend the set M to an ultrafilter D on B, then to an ultrafilter E on A. Note that $B \in E$. Also, $E \cap J_{<\lambda}[A] = \emptyset$. In fact, if $X \in E \cap J_{<\lambda}[A]$, then $X \cap B \in J_{<\lambda}[A]$, so $X \cap B \in D \cap J_{<\lambda}[B]$ by Lemma 34.17(i). But $D \cap J_{<\lambda}[B] = \emptyset$ by construction. Now $B \in E \cap J_{\leq\lambda}[A]$, so $\operatorname{cf}(\prod A/E) = \lambda$, and h bounds all f_ξ in this ultraproduct, contradicting the universality of f.

(vi): By Lemma 34.17 and (v).

(vii): By (i) we have $\lambda \in \operatorname{pcf}(B)$. Now $B \in J_{\leq\lambda}[A]$, so $\operatorname{pcf}(B) \subseteq \lambda^+$. The desired conclusion follows.

(viii): For $\Rightarrow$, suppose that $\operatorname{cf}(\prod A/D) = \lambda$. Then $B \in D$ by (i), and obviously $D \cap J_{<\lambda} = \emptyset$. For $\Leftarrow$, assume that $B \in D$ and $D \cap J_{<\lambda} = \emptyset$. Now $B \in J_{\leq\lambda}$, so $\operatorname{cf}(\prod A/D) = \lambda$ by Corollary 34.9(ii).

(ix): We have $B \in J_{\leq\lambda}$ and $C = (C\backslash B) \cup (C \cap B)$, so $C \in J_{\leq\lambda}$. Suppose that $\lambda \in \operatorname{pcf}(A\backslash C)$. Let D be an ultrafilter on A such that $\operatorname{cf}(\prod A/D) = \lambda$ and $A\backslash C \in D$. Now $B \in D$ by (i), so $B\backslash C \in D$. This contradicts $B\backslash C \in J_{<\lambda}$. So $\lambda \notin \operatorname{pcf}(A\backslash C)$. Hence C is a λ-generator, by (iii).

(x): Let $B' = B \cap (\lambda + 1)$. Clearly $B' \in J_{\leq\lambda}$. Suppose that $\lambda \in \operatorname{pcf}(A\backslash B')$. Say $\lambda = \operatorname{cf}(\prod A/D)$ with $A\backslash B' \in D$. Also $A \cap (\lambda + 1) \in D$, since $A\backslash(\lambda + 1) \in D$ would imply that $\operatorname{cf}(\prod A/D) > \lambda$ by Proposition 34.1(vii). Since clearly

$$(A\backslash B') \cap (A \cap (\lambda + 1)) \subseteq A\backslash B,$$

this yields $A\backslash B \in D$, contradicting (ii). Therefore, $\lambda \notin \operatorname{pcf}(A\backslash B')$. So B' is a λ-generator, by (iii).

(xi): This is clear from Proposition 34.16.

(xii): Clear by (iv) and (xi). □

Lemma 34.26. *Suppose that A is a progressive set, F is a proper filter over A, and λ is a cardinal. Then the following are equivalent.*

(i) $\operatorname{tcf}(\prod A/F) = \lambda$.

(ii) $\lambda \in \operatorname{pcf}(A)$, *$F$ has a λ-generator on A as an element, and $J^*_{<\lambda} \subseteq F$.*

(iii) $\operatorname{cf}(\prod A/D) = \lambda$ *for every ultrafilter D extending F.*

Proof. (i)⇒(iii): obvious.

(iii)⇒(ii): Obviously $\lambda \in \operatorname{pcf}(A)$. Let B be a λ-generator on A. Suppose that $B \notin F$. Then there is an ultrafilter D on A such that $A\backslash B \in D$ and D extends F. Then $\operatorname{cf}(\prod A/D) = \lambda$ by (iii), and this contradicts Lemma 34.25(i).

(ii)⇒(i): Let $B \in F$ be a λ-generator. By Lemma 34.25(vi) we have $\operatorname{tcf}(\prod B/J_{<\lambda}) = \lambda$, and hence $\operatorname{tcf}(\prod A/F) = \lambda$ since $B \in F$ and $J^*_{<\lambda} \subseteq F$. □

Proposition 34.27. *Suppose that A is a progressive set of regular cardinals, and λ is any cardinal. Then the following conditions are equivalent:*

(i) $\lambda = \max(\text{pcf}(A))$.

(ii) $\lambda = \text{tcf}(\prod A/J_{<\lambda}[A])$.

(iii) $\lambda = \text{cf}(\prod A/J_{<\lambda}[A])$.

Proof. (i)⇒(ii): By Lemma 34.25(iv),(vi).

(ii)⇒(iii): Obvious.

(iii)⇒(ii): Assume (iii). Let $\mu = \max(\text{pcf}(A))$. By (i)⇒(iii) we have $\lambda = \mu$. □

Lemma 34.28. *Suppose that A is progressive, $A_0 \subseteq A$, and $\lambda \in \text{pcf}(A_0)$. Suppose that B is a λ-generator on A. Then $B \cap A_0$ is a λ-generator on A_0.*

Proof. Since $B \in J_{\leq\lambda}[A]$, we have $\text{pcf}(B) \subseteq \lambda^+$ and hence $\text{pcf}(B \cap A_0) \subseteq \lambda^+$ and so $B \cap A_0 \in J_{\leq\lambda}[A_0]$. If $\lambda \in \text{pcf}(A_0 \backslash B)$, then also $\lambda \in \text{pcf}(A \backslash B)$, and this contradicts Lemma 34.25(ii). Hence $\lambda \notin \text{pcf}(A_0 \backslash B)$, and hence $B \cap A_0$ is a λ-generator for A_0 by Lemma 34.25(iii). □

Definition. If A is progressive, a *generating sequence for A* is a sequence $\langle B_\lambda : \lambda \in \text{pcf}(A)\rangle$ such that B_λ is a λ-generator on A for each $\lambda \in \text{pcf}(A)$.

Theorem 34.29. *Suppose that A is progressive, $\langle B_\lambda : \lambda \in \text{pcf}(A)\rangle$ is a generating sequence for A, and $X \subseteq A$. Then there is a finite subset N of $\text{pcf}(X)$ such that $X \subseteq \bigcup_{\mu\in N} B_\mu$.*

Proof. We show that for all $X \subseteq A$, if $\lambda = \max(\text{pcf}(X))$, then there is a finite subset N as indicated, using induction on λ. So, suppose that this is true for every cardinal $\mu < \lambda$, and now suppose that $X \subseteq A$ and $\max(\text{pcf}(X)) = \lambda$. Then $\lambda \notin \text{pcf}(X \backslash B_\lambda)$ by Lemma 34.25(ii), and so $\text{pcf}(X \backslash B_\lambda) \subseteq \lambda$. Hence $\max(\text{pcf}(X \backslash B_\lambda)) < \lambda$. Hence by the inductive hypothesis there is a finite subset N of $\text{pcf}(X \backslash B_\lambda)$ such that $X \backslash B_\lambda \subseteq \bigcup_{\mu\in N} B_\mu$. Hence

$$X \subseteq B_\lambda \cup \bigcup_{\mu\in N} B_\mu,$$

and $\{\lambda\} \cup N \subseteq \text{pcf}(X)$. □

Corollary 34.30. *Suppose that A is progressive, $\langle B_\lambda : \lambda \in \text{pcf}(A)\rangle$ is a generating sequence for A, and $X \subseteq A$. Suppose that λ is any infinite cardinal. Then $X \in J_{<\lambda}[A]$ iff $X \subseteq \bigcup_{\mu\in N} B_\mu$ for some finite subset N of $\lambda \cap \text{pcf}(A)$.*

Proof. ⇒: Assume that $X \in J_{<\lambda}[A]$. Thus $\text{pcf}(X) \subseteq \lambda$, and Theorem 34.29 gives the desired conclusion.

⇐: Assume that a set N is given as indicated. Suppose that $\rho \in \text{pcf}(X)$. Say $\rho = \text{cf}(\prod A/D)$ with $X \in D$. Then $B_\mu \in D$ for some $\mu \in N$. By the definition of generator, $B_\mu \in J_{\leq\mu}[A]$, and hence $\rho \leq \mu < \lambda$. Thus we have shown that $\text{pcf}(X) \subseteq \lambda$, so $X \in J_{<\lambda}[A]$. □

Lemma 34.31. *Suppose that A is progressive and $\langle B_\lambda : \lambda \in \text{pcf}(A)\rangle$ is a generating sequence for A. Suppose that D is an ultrafilter on A. Then there is a $\lambda \in \text{pcf}(A)$ such that $B_\lambda \in D$, and if λ is minimum with this property, then $\lambda = \text{cf}(\prod A/D)$.*

Proof. Let $\mu = \text{cf}(\prod A/D)$. Then $\mu \in \text{pcf}(A)$ and $B_\mu \in D$ by Lemma 34.25(i). Suppose that $B_\lambda \in D$ with $\lambda < \mu$. Now $B_\lambda \in J_{\leq\lambda} \subseteq J_{<\mu}$, contradicting Lemma 34.25(viii), applied to μ. $\square$

Lemma 34.32. *If A is progressive and also* $\text{pcf}(A)$ *is progressive, and if* $\lambda \in \text{pcf}(A)$ *and B is a λ-generator for A, then* $\text{pcf}(B)$ *is a λ-generator for* $\text{pcf}(A)$.

Proof. Note by Theorem 34.15 that $\text{pcf}(\text{pcf}(B)) = \text{pcf}(B)$ and $\text{pcf}(\text{pcf}(A\backslash B)) = \text{pcf}(A\backslash B)$. Since $B \in J_{\leq\lambda}[A]$, we have $\text{pcf}(B) \subseteq \lambda^+$, and hence $\text{pcf}(\text{pcf}(B)) \subseteq \lambda^+$ and so $\text{pcf}(B) \in J_{\leq\lambda}[\text{pcf}(A)]$. Now suppose that $\lambda \in \text{pcf}(\text{pcf}(A)\backslash\text{pcf}(B))$. Then by Lemma 34.1(iv) we have $\lambda \in \text{pcf}(\text{pcf}(A\backslash B)) = \text{pcf}(A\backslash B)$, contradicting Lemma 34.25(ii). So $\lambda \notin \text{pcf}(\text{pcf}(A)\backslash\text{pcf}(B))$. It now follows by Lemma 34.25(iii) that $\text{pcf}(B)$ is a λ-generator for $\text{pcf}(A)$. $\square$

The following result is relevant to Theorem 34.44. Let μ be a singular cardinal, C a club of μ, and suppose that $X \in J_{<\mu}[C^{(+)}]$. Now $\text{pcf}(X)$ has a maximal element, and so there is an $\alpha < \mu$ such that $X \subseteq \text{pcf}(X) \subseteq \alpha$. Thus $J_{<\mu}[C^{(+)}] \subseteq J^{\text{bd}}$.

Lemma 34.33. *If μ is a singular cardinal of uncountable cofinality, then there is a club $C \subseteq \mu$ such that* $\text{tcf}(\prod C^{(+)}/J_{<\mu}[C^{(+)}]) = \mu^+$.

Proof. Let C_0 be a club in μ such that such that $\mu^+ = \text{tcf}(\prod C_0^{(+)}/J^{bd})$, by Theorem 34.44. Let $C_1 \subseteq C_0$ be such that the order type of C_1 is $\text{cf}(\mu)$, C_1 is cofinal in μ, and $\forall\kappa \in C_1[\text{cf}(\mu) < \kappa]$. Hence $C_1^{(+)}$ is progressive. Now $\mu^+ \in \text{pcf}(C_1^{(+)})$ by Lemma 34.26. Let B be a μ^+-generator for $C_1^{(+)}$. Define $C = \{\delta \in C_1 : \delta^+ \in B\}$. Now $C_1\backslash C$ is bounded. Otherwise, let $X = C_1^{(+)}\backslash B = (C_1\backslash C)^{(+)}$. So X is unbounded, and hence clearly $\mu^+ = \text{tcf}(\prod X/J^{\text{bd}})$. Hence $\mu^+ \in \text{pcf}(X)$. This contradicts Lemma 34.25(ii).

So, choose $\varepsilon < \mu$ such that $C_1\backslash C \subseteq \varepsilon$. Hence $C_1\backslash\varepsilon \subseteq C\backslash\varepsilon \subseteq C_1\backslash\varepsilon$, so $C_1\backslash\varepsilon = C\backslash\varepsilon$. Clearly $\mu^+ = \text{tcf}(\prod(C_1\backslash\varepsilon)^{(+)}/J^{\text{bd}})$, so $\mu^+ \in \text{pcf}((C_1\backslash\varepsilon)^{(+)})$. We claim that $\text{tcf}(\prod(C_1\backslash\varepsilon)^{(+)}/J_{<\mu^+}[(C_1\backslash\varepsilon)^{(+)}]) = \mu^+$. To show this, we apply Lemma 34.26. Suppose that D is any ultrafilter on $(C_1\backslash\varepsilon)^{(+)}$ such that $J_{<\mu^+}[(C_1\backslash\varepsilon)^{(+)}] \cap D = \emptyset$. Now by Lemma 34.28, $B \cap (C_1\backslash\varepsilon)^{(+)}$ is a μ^+-generator for $(C_1\backslash\varepsilon)^{(+)}$. Note that $C^+ \subseteq B$. Now $B \cap (C_1\backslash\varepsilon)^{(+)} = B \cap (C\backslash\varepsilon)^{(+)} = (C\backslash\varepsilon)^{(+)}$. It follows by Lemma 34.25(viii) that $\text{cf}(\prod(C_1\backslash\varepsilon)^{(+)}/D) = \mu^+$. This proves that $\text{tcf}(\prod(C_0\backslash\varepsilon)^{(+)}/J_{<\mu^+}[(C_1\backslash\varepsilon)^{(+)}]) = \mu^+$. Now we claim that $J_{<\mu^+}[(C_1\backslash\varepsilon)^{(+)}] = J_{<\mu}[(C_1\backslash\varepsilon)^{(+)}]$. For, suppose that $X \in J_{<\mu^+}[(C_1\backslash\varepsilon)^{(+)}]$. So $\text{pcf}(X) \subseteq \mu^+$. Since X is progressive (because $C_1\backslash\varepsilon)^{(+)}$ is), we have $\max(\text{pcf}(X)) < \mu$, hence $\text{pcf}(X) \subseteq \mu$. $\square$

By essentially the same proof as for Lemma 34.33 we get

Lemma 34.34. *If μ is a singular cardinal of countable cofinality, then there is an unbounded subset C of μ consisting of regular cardinals such that* $\text{tcf}(\prod C/J_{<\mu}[C]) = \mu^+$.

Proof. Let C_0 be an unbounded collection of regular cardinals in μ such that $\mu^+ = \text{tcf}(\prod C_0/J^{bd})$, by Theorem 34.45. Let $C_1 \subseteq C_0$ be such that the order type of C_1 is $\text{cf}(\mu)$, C_1 is cofinal in μ, and $\forall\kappa \in C_1[\omega < \kappa]$. Hence C_1 is progressive. Now $\mu^+ \in \text{pcf}(C_1)$

by Lemma 34.26. Let B be a μ^+-generator for C_1. Define $C = B \cap C_1$. Now $C_1 \backslash C$ is bounded. Otherwise, let $X = C_1 \backslash B = C_1 \backslash C$. So X is unbounded, and hence clearly $\mu^+ = \mathrm{tcf}(\prod X/J^{\mathrm{bd}})$. Hence $\mu^+ \in \mathrm{pcf}(X)$. This contradicts Lemma 34.25(ii).

So, choose $\varepsilon < \mu$ such that $C_1 \backslash C \subseteq \varepsilon$. Hence $C_1 \backslash \varepsilon \subseteq C \backslash \varepsilon \subseteq C_1 \backslash \varepsilon$, so $C_1 \backslash \varepsilon = C \backslash \varepsilon$. Clearly $\mu^+ = \mathrm{tcf}(\prod (C_1 \backslash \varepsilon)/J^{\mathrm{bd}})$, so $\mu^+ \in \mathrm{pcf}(C_1 \backslash \varepsilon)$. We claim that $\mathrm{tcf}(\prod (C_1 \backslash \varepsilon)/J_{<\mu^+}[C_1 \backslash \varepsilon] = \mu^+$. To show this, we apply Lemma 34.26. Suppose that D is any ultrafilter on $C_1 \backslash \varepsilon$ such that $J_{<\mu^+}[C_1 \backslash \varepsilon] \cap D = \emptyset$. Now by Lemma 34.28, $B \cap (C_1 \backslash \varepsilon)$ is a μ^+-generator for $C_1 \backslash \varepsilon$. Note that $C \subseteq B$. Now $B \cap (C_1 \backslash \varepsilon) = B \cap (C \backslash \varepsilon) = (C \backslash \varepsilon)$. It follows by Lemma 34.25(viii) that $\mathrm{cf}(\prod (C_1 \backslash \varepsilon)/D) = \mu^+$. This proves that $\mathrm{tcf}(\prod (C_0 \backslash \varepsilon)/J_{<\mu^+}[C_1 \backslash \varepsilon]) = \mu^+$. Now we claim that $J_{<\mu^+}[C_1 \backslash \varepsilon] = J_{<\mu}[C_1 \backslash \varepsilon]$. For, suppose that $X \in J_{<\mu^+}[C_1 \backslash \varepsilon]$. So $\mathrm{pcf}(X) \subseteq \mu^+$. Since X is progressive (because $C_1 \backslash \varepsilon$ is), we have $\max(\mathrm{pcf}(X)) < \mu$, hence $\mathrm{pcf}(X) \subseteq \mu$. $\square$

Proposition 34.35. *Suppose that F is a proper filter over a progressive set A of regular cardinals. Define*

$$\mathrm{pcf}_F(A) = \left\{ \mathrm{cf}\left(\prod A/D\right) : D \text{ is an ultrafilter extending } F \right\}.$$

Then:

(i) $\max(\mathrm{pcf}_F(A))$ *exists.*

(ii) $\mathrm{cf}(\prod A/F) = \max(\mathrm{pcf}_F(A))$*.*

(iii) If $B \subseteq \mathrm{pcf}_F(A)$ *is progressive, then* $\mathrm{pcf}(B) \subseteq \mathrm{pcf}_F(A)$*.*

(iv) If A is a progressive interval of regular cardinals with no largest element, and

$$F = \{X \subseteq A : A \backslash X \text{ is bounded}\}$$

is the filter of co-bounded subsets of A, then $\mathrm{pcf}_F(A)$ *is an interval of regular cardinals.*

Proof. (i): Clearly $\mathrm{pcf}_F(A) \subseteq \mathrm{pcf}(A)$, and so if $\lambda = \max(\mathrm{pcf}(A))$, then $A \in F \cap J_{<\lambda^+}[A]$. Hence we can choose μ minimum such that $F \cap J_{<\mu}[A] \neq \emptyset$. By Corollary 34.12, μ is not a limit cardinal; write $\mu = \lambda^+$. Then $F \cap J_{<\lambda} = \emptyset$, and so $F \cup J^*_{<\lambda}$ has fip; let D be an ultrafilter containing this set. Then $D \cap J_{\leq\lambda} \supseteq F \cap J_{\leq\lambda} \neq \emptyset$, while $D \cap J_{<\lambda} = \emptyset$. Hence $\mathrm{cf}(\prod A/D) = \lambda$ by Corollary 34.9. On the other hand, since $F \cap J_{\leq\lambda}[A] \neq \emptyset$, any ultrafilter E containing F must be such that $\mathrm{cf}(\prod A/E) \leq \lambda$.

(ii): Cf. the proof of Theorem 34.20. Let $\lambda = \max(\mathrm{pcf}_F(A))$, and let D be an ultrafilter extending F such that $\lambda = \mathrm{cf}(\prod A/D)$. Let $\langle f_\alpha : \alpha < \lambda \rangle$ be strictly increasing and cofinal mod D. Now if $g < h$ mod F, then also $g < h$ mod D. So a cofinal subset of $\prod A$ mod F is also a cofinal subset mod D, so $\lambda \leq \mathrm{cf}(\prod A/F)$. Hence it suffices to exhibit a cofinal subset of $\prod A$ mod F of size λ. For every $\mu \in \mathrm{pcf}_F(A)$ fix a universal sequence $f^\mu = \langle f^\mu_i : i < \mu \rangle$ for μ, by Theorem 34.18. Let G be the set of all functions of the form

$$\sup\{f^{\mu_1}_{i_1}, f^{\mu_2}_{i_2}, \ldots, f^{\mu_n}_{i_n}\},$$

where $\mu_1, \mu_2, \ldots, \mu_n$ is a finite sequence of members of $\mathrm{pcf}_F(A)$, possibly with repetitions, and $i_k < \mu_k$ for each $k = 1, \ldots, n$. We claim that G is cofinal in $(\prod A, <_F)$; this will complete the proof of (ii).

To prove this claim, let $g \in \prod A$. Suppose that $g \not< f \bmod F$ for all $f \in G$. Then, we claim, the set

$$(*) \qquad F \cup \{\{a \in A : f(a) \leq g(a)\} : f \in G\}$$

has fip. For, suppose not. Then there is a finite subset G' of G such that $\bigcup_{g \in G'}\{a \in A : g(a) < f(a)\} \in F$. Let $h = \sup_{f \in G'} f$. Then $g < h \bmod F$ and $h \in G$, contradiction. Thus $(*)$ has fip, and we let D be an ultrafilter containing it. Let $\mu = \mathrm{cf}(\prod A/D)$. Then $\mu \in \mathrm{pcf}_F(A)$, and $f \leq g \bmod D$ for all $f \in G$. Since the members of a universal sequence for μ are in G, this is a contradiction. This completes the proof of (ii).

For (iii), we look at the proof of Theorem 34.15. Let F' be the ultrafilter named F at the beginning of that proof. Since $B \subseteq \mathrm{pcf}_F(A)$, each $b \in B$ is in $\mathrm{pcf}_F(A)$, and hence the ultrafilters D_b can be taken to extend F. Hence $F \subseteq F'$, and so $\mu \in \mathrm{pcf}_F(A)$, as desired in (iii).

Finally, we prove (iv). Let $\lambda_0 = \min(\mathrm{pcf}_F(A))$ and $\lambda_1 = \max(\mathrm{pcf}_F(A))$, and suppose that μ is a regular cardinal such that $\lambda_0 < \mu < \lambda_1$. Let D be an ultrafilter such that $F \subseteq D$ and $\mathrm{cf}(\prod A/D) = \lambda_1$. Then by Corollary 34.9(ii), $D \cap J_{<\lambda_1} = \emptyset$, so $J^*_{\lambda_1} \subseteq D$. Thus $F \cup J^*_{<\mu} \subseteq F \cup J^*_{<\lambda_1} \subseteq D$, so $F \cup J^+_{<\mu}$ generates a proper filter G. Since $(\prod A, <_{J_{<\mu}})$ is μ-directed by Theorem 34.8, so is $(\prod A, <_G)$. Note that if $a \in A$, then $\{b \in A : a < b\} \in F$. It follows that $\sup(A) \leq \lambda_0 < \mu$. Hence we can apply Theorem 34.4 and get a subset A' of A (since A is an interval of regular cardinals) and a proper ideal K over A' such that A' is cofinal in A, K contains all proper initial segments of A', and $\mathrm{tcf}(\prod A, <_K) = \mu$. Let $\langle f_\alpha : \alpha < \mu\rangle$ be strictly increasing and cofinal mod K. Extend K^* to a filter L on A, and extend each function f_α to a function f^+_α on A. Then clearly $\langle f^+_\alpha : \alpha < \mu\rangle$ is strictly increasing and cofinal mod L, and L contains F. This shows that $\mu \in \mathrm{pcf}_F(A)$. $\square$

35. Main cofinality theorems

The sets H_Ψ

We will shortly give several proofs involving the important general idea of making elementary chains inside the sets H_Ψ. Recall that H_Ψ, for an infinite cardinal Ψ, is the collection of all sets hereditarily of size less than Ψ, i.e., with transitive closure of size less than Ψ. We consider H_Ψ as a structure with $\in$ together with a well-ordering $<^*$ of it, possibly with other relations or functions, and consider elementary substructures of such structures.

Recall that A is an *elementary substructure* of B iff A is a subset of B, and for every formula $\varphi(x_0,\ldots,x_{m-1})$ and all $a_0,\ldots,a_{m-1}\in A$, $A\models\varphi(a_0,\ldots,a_{m-1})$ iff $B\models\varphi(a_0,\ldots,a_{m-1})$.

The basic downward Löwenheim-Skolem theorem will be used a lot. This theorem depends on the following lemma.

Lemma 35.1. (Tarski) *Suppose that A and B are first-order structures in the same language, with A a substructure of B. Then the following conditions are equivalent:*

(i) A is an elementary substructure of B.

(ii) For every formula of the form $\exists y\varphi(x_0,\ldots,x_{m-1},y)$ and all $a_0,\ldots,a_{m-1}\in A$, if $B\models\exists y\varphi(a_0,\ldots,a_{m-1},y)$ then there is a $b\in A$ such that $B\models\varphi(a_0,\ldots,a_{m-1},b)$.

Proof. (i)⇒(ii): Assume (i) and the hypotheses of (ii). Then by (i) we see that $A\models\exists y\varphi(a_0,\ldots,a_{m-1},y)$, so we can choose $b\in A$ such that $A\models\varphi(a_0,\ldots,a_{m-1},b)$. Hence $B\models\varphi(a_0,\ldots,a_{m-1},b)$, as desired.

(ii)⇒(i): Assume (ii). We show that for any formula $\varphi(x_0,\ldots,x_{m-1})$ and any elements $a_0,\ldots,a_{m-1}\in A$, $A\models\varphi(a_0,\ldots,a_{m-1})$ iff $B\models\varphi(a_0,\ldots,a_{m-1})$, by induction on φ. It is true for φ atomic by our assumption that A is a substructure of B. The induction steps involving $\neg$ and $\vee$ are clear. Now suppose that $A\models\exists y\varphi(a_0,\ldots,a_{m-1},y)$, with $a_0,\ldots,a_{m-1}\in A$. Choose $b\in A$ such that $A\models\varphi(a_0,\ldots,a_{m-1},b)$. By the inductive assumption, $B\models\varphi(a_0,\ldots,a_{m-1},b)$. Hence $B\models\exists y\varphi(a_0,\ldots,a_{m-1},y)$, as desired.

Conversely, suppose that $B\models\exists y\varphi(a_0,\ldots,a_{m-1},y)$. By (ii), choose $b\in A$ such that $B\models\varphi(a_0,\ldots,a_{m-1},b)$. By the inductive assumption, $A\models\varphi(a_0,\ldots,a_{m-1},b)$. Hence $A\models\exists y\varphi(a_0,\ldots,a_{m-1},y)$, as desired. □

Theorem 35.2. *Suppose that A is an L-structure, X is a subset of A, κ is an infinite cardinal, and κ is $\geq$ both $|X|$ and the number of formulas of $\mathscr{L}$, while $\kappa\leq|A|$. Then A has an elementary substructure B such that $X\subseteq B$ and $|B|=\kappa$.*

Proof. Let a well-order $\prec$ of A be given. We define $\langle C_n:n\in\omega\rangle$ by recursion. Let C_0 be a subset of A of size κ with $X\subseteq C_0$. Now suppose that C_n has been defined. Let M_n be the collection of all pairs of the form $(\exists y\varphi(x_0,\ldots,x_{m-1},y),a)$ such that a is a sequence of elements of C_n of length m. For each such pair we define $f(\exists y\varphi(x_0,\ldots,x_{m-1},y),a)$ to be the $\prec$-least element b of A such that $A\models\varphi(a_0,\ldots,a_{m-1},b)$, if there is such an element, and otherwise let it be the least element of C_n. Then we define

$$C_{n+1}=C_n\cup\{f(\exists y\varphi(x_0,\ldots,x_{m-1},y),a):(\exists y\varphi(x_0,\ldots,x_{m-1},y),a)\in M_n\}.$$

Finally, let $B = \bigcup_{n\in\omega} C_n$.

By induction it is clear that $|C_n| = \kappa$ for all $n \in \omega$, and so also $|B| = \kappa$.

Now to show that B is an elementary substructure of A we apply Lemma 35.1. First we show that B is a substructure of A; this amounts to showing that B is closed under each fundamental operation F^A. Say F is m-ary, and $b_0, \ldots, b_{m-1} \in B$. Then there is an n such that $b_0, \ldots, b_{m-1} \in C_n$. Now $(\exists y[Fx_0 \ldots x_{m-1} = y], \langle b_0, \ldots, b_{m-1}\rangle) \in M_n$. Let $c = F^A(b_0, \ldots, b_{m-1})$; so $f((\exists y[Fx_0 \ldots x_{m-1} = y], \langle b_0, \ldots, b_{m-1}\rangle) = c \in C_{n+1} \subseteq B$.

Now suppose that we are given a formula of the form $\exists y\varphi(x_0, \ldots, x_{m-1}, y)$ and elements $a_0, \ldots, a_{m-1}$ of B, and $A \models \exists y\varphi(a_0, \ldots, a_{m-1}, y)$. Clearly there is an $n \in \omega$ such that $a_0, \ldots, a_{m-1} \in C_n$. Then $(\exists y\varphi(x_0, \ldots, x_{m-1}, y), a) \in M_n$, and $f(\exists y\varphi(x_0, \ldots, x_{m-1}, y), a)$ is an element b of $C_{n+1} \subseteq B$ such that $A \models \varphi(a_0, \ldots, a_{m-1}, b)$. This is as desired in Lemma 35.1. $\square$

Given an elementary substructure A of a set H_Ψ, we will frequently use an argument of the following kind. A set theoretic formula holds in the real world, and involves only sets in A. By absoluteness, it holds in H_Ψ, and hence it holds in A. Thus we can transfer a statement to A even though A may not be transitive; and the procedure can be reversed.

To carry this out, we need some facts about transitive closures first of all.

Lemma 35.3. *(i) If $X \subseteq A$, then $\operatorname{tr\,cl}(X) \subseteq \operatorname{tr\,cl}(A)$.*

(ii) $\operatorname{tr\,cl}(\mathscr{P}(A)) = \mathscr{P}(A) \cup \operatorname{tr\,cl}(A)$.

(iii) If $\operatorname{tr\,cl}(A)$ is infinite, then $|\operatorname{tr\,cl}(\mathscr{P}(A))| \le 2^{|\operatorname{tr\,cl}(A)|}$.

(iv) $\operatorname{tr\,cl}(A \cup B) = \operatorname{tr\,cl}(A) \cup \operatorname{tr\,cl}(B)$.

(v) $\operatorname{tr\,cl}(A \times B) = (A \times B) \cup \{\{a\} : a \in A\} \cup \{\{a, b\} : a \in A, b \in B\} \cup \operatorname{tr\,cl}(A) \cup \operatorname{tr\,cl}(B)$.

(vi) If $\operatorname{tr\,cl}(A)$ or $\operatorname{tr\,cl}(B)$ is infinite, then $|\operatorname{tr\,cl}(A \times B)| \le \max(\operatorname{tr\,cl}(A), \operatorname{tr\,cl}(B)$.

(vii) $\operatorname{tr\,cl}({}^AB) \subseteq ({}^AB) \cup \operatorname{tr\,cl}(A \times B)$.

(viii) If $\operatorname{tr\,cl}(A)$ or $\operatorname{tr\,cl}(B)$ is infinite, then $|\operatorname{tr\,cl}({}^AB)| \le 2^{\max(|\operatorname{tr\,cl}(A)|, |\operatorname{tr\,cl}(A)|)}$.

(ix) If $\operatorname{tr\,cl}(A)$ is infinite, then $|\operatorname{tr\,cl}(\prod A)| \le 2^{|\operatorname{tr\,cl}(A)|}$.

(x) If $\operatorname{tr\,cl}(A)$ or $\operatorname{tr\,cl}(B)$ is infinite, then $|\operatorname{tr\,cl}({}^A(\prod B))| \le 2^{2^{\max(|\operatorname{tr\,cl}(A)|, |\operatorname{tr\,cl}(B)|)}}$.

(xi) If A is an infinite set of regular cardinals, then $|\operatorname{tr\,cl}(\operatorname{pcf}(A))| \le 2^{|\operatorname{tr\,cl}(A)|}$.

Proof. (i)–(viii) are clear. For (ix), note that $\prod A \subseteq {}^A\bigcup A$, so (ix) follow from (viii). For (x),

$$\begin{aligned}
|\operatorname{tr\,cl}\left({}^A\left(\prod B\right)\right)| &\le 2^{\max\left(|\operatorname{tr\,cl}(A), |\operatorname{tr\,cl}\left(\prod B\right)|\right)} \quad \text{by (viii)} \\
&\le 2^{\max(|\operatorname{tr\,cl}(A), 2^{|\operatorname{tr\,cl}(B)|})} \\
&\le 2^{2^{\max(|\operatorname{tr\,cl}(A)|, |\operatorname{tr\,cl}(B)|)}}.
\end{aligned}$$

Finally, for (xi), note that $\operatorname{tr\,cl}(\operatorname{pcf}(A)) = \operatorname{pcf}(A) \cup \bigcup \operatorname{pcf}(A)$. Now $|\operatorname{pcf}(A)| \le 2^{|A|} \le 2^{|\operatorname{tr\,cl}(A)|}$ by Theorem 34.10. $\square$

We also need the fact that some rather complicated formulas and functions are absolute for sets H_Ψ. Note that H_Ψ is transitive. Many of the indicated formulas are not absolute for H_Ψ in general, but only under the assumptions given that Ψ is much larger than the sets in question.

Lemma 35.4. *Suppose that Ψ is an uncountable regular cardinal. Then the following formulas (as detailed in the proof) are absolute for H_Ψ.*

(i) $B = \mathscr{P}(A)$.

(ii) "D is an ultrafilter on A".

(iii) κ is a cardinal.

(iv) κ is a regular cardinal.

(v) "κ and λ are cardinals, and $\lambda = \kappa^+$".

(vi) $\kappa = |A|$.

(vii) $B = \prod A$.

(viii) $A = {}^B C$.

(ix) "A is infinite", if Ψ is uncountable.

(x) "A is an infinite set of regular cardinals and D is an ultrafilter on A and λ is a regular cardinal and $f \in {}^\lambda \prod A$ and f is strictly increasing and cofinal modulo D", provided that $2^{|\mathrm{tr\,cl}(A)|} < \Psi$.

(xi) "A is an infinite set of regular cardinals, and $B = \mathrm{pcf}(A)$", if $2^{|\mathrm{tr\,cl}(A)|} < \Psi$.

(xii) "A is an infinite set of regular cardinals and $f = \langle J_{<\lambda}[A] : \lambda \in \mathrm{pcf}(A)\rangle$", provided that $2^{|\mathrm{tr\,cl}(A)|} < \Psi$.

(xiii) "A is an infinite set of regular cardinals and $B = \langle B_\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ and $\forall \lambda \in \mathrm{pcf}(A)(B_\lambda$ is a λ-generator)", if $2^{2^{|\mathrm{tr\,cl}(A)|}} < \Psi$.

Proof. Absoluteness follows by easy arguments upon producing suitable formulas, as follows.

(i): Suppose that $A, B \in H_\Psi$. We may take the formula $B = \mathscr{P}(A)$ to be

$$\forall x \in B[\forall y \in x(y \in A)] \wedge \forall x[\forall y \in x(y \in A) \to x \in B].$$

The first part is obviously absolute for H_Ψ. If the second part holds in V it clearly holds in H_Ψ. Now suppose that the second part holds in H_Ψ. Suppose that $x \subseteq A$. Hence $x \in H_\Psi$ and it follows that $x \in B$.

(ii): Assume that $A, D \in H_\Psi$. We can take the statement "D is an ultrafilter on A" to be the following statement:

$$\begin{aligned}&\forall X \in D(X \subseteq A) \wedge A \in D \wedge \forall X, Y \in D(X \cap Y \in D) \wedge \emptyset \notin D\\ &\wedge \forall Y \forall X \in D[X \subseteq Y \wedge Y \subseteq A \to Y \in D] \wedge \forall Y[Y \subseteq A \to Y \in D \vee (A \backslash Y) \in D].\end{aligned}$$

Again this is absolute because $Y \subseteq A$ implies that $Y \in H_\Psi$.

(iii): Suppose that $\kappa \in H_\Psi$. Then

$$\begin{aligned}\kappa \text{ is a cardinal} \quad \text{iff} \quad & \kappa \text{ is an ordinal and } \forall f[f \text{ is a function and}\\ & \mathrm{dmn}(f) = \kappa \text{ and } \mathrm{rng}(f) \in \kappa \to f \text{ is not one-to-one}].\end{aligned}$$

Note here that if f is a function with $\mathrm{dmn}(f) = \kappa$ and $\mathrm{rng}(f) \subseteq \kappa$, then $f \subseteq \kappa \times \kappa$, and hence $f \in H_\Psi$.

(iv): Assume that $\kappa \in H_\Psi$. Then

$$\kappa \text{ is a regular cardinal} \quad \text{iff} \quad \begin{array}{l} \kappa \text{ is a cardinal, } 1 < \kappa, \text{ and } \forall f[f \text{ is a function} \\ \text{and } \mathrm{dmn}(f) \in \kappa \text{ and } \mathrm{rng}(f) \subseteq \kappa \text{ and} \\ \forall \alpha, \beta \in \mathrm{dmn}(f)(\alpha < \beta \rightarrow f(\alpha) < f(\beta)) \\ \rightarrow \exists \gamma < \kappa \forall \alpha \in \mathrm{dmn}(f)(f(\alpha) \in \gamma)]. \end{array}$$

(v): Assume that $\kappa, \lambda \in H_\Psi$. Then ($\kappa$ and λ are cardinals and $\lambda = \kappa^+$) iff

$$\begin{array}{l} \kappa \text{ is a cardinal and } \lambda \text{ is a cardinal and } \kappa < \lambda \\ \text{and } \forall \alpha < \lambda[\kappa < \alpha \rightarrow \exists f[f \text{ is a function and } \mathrm{dmn}(f) = \kappa \\ \text{and } \mathrm{rng}(f) = \alpha \text{ and } f \text{ is one-one and } \mathrm{rng}(f) = \alpha]]. \end{array}$$

(vi): Suppose that $\kappa, A \in H_\Psi$. Then

$$\kappa = |A| \quad \text{iff} \quad \begin{array}{l} \kappa \text{ is a cardinal and } \exists f[f \text{ is a function} \\ \text{and } \mathrm{dmn}(f) = \kappa \text{ and } \mathrm{rng}(f) = A \text{ and } f \text{ is one-to-one}] \end{array}$$

(vii): Assume that $A, B \in H_\Psi$. Then

$$B = \prod A \quad \text{iff} \quad \begin{array}{l} \forall f \in B[f \text{ is a function and } \mathrm{dmn}(f) = A \text{ and} \\ \forall x \in A[f(x) \in x]] \text{ and } \forall f[f \text{ is a function and} \\ \mathrm{dmn}(f) = A \text{ and } \forall x \in A[f(x) \in x] \rightarrow f \in B]. \end{array}$$

Note that if f is a function with domain A and $f(x) \in x$ for all $x \in A$, then $f \subseteq A \times \bigcup A$, and hence $f \in H_\Psi$.

(viii): Suppose that $A, B, C \in H_\Psi$. Then

$$A = {}^B C \quad \text{iff} \quad \begin{array}{l} \forall f \in A[f \text{ is a function and } \mathrm{dmn}(f) = B \\ \text{and } \mathrm{rng}(f) \subseteq C] \text{ and } \forall f[f \text{ is a function} \\ \text{and } \mathrm{dmn}(f) = B \text{ and } \mathrm{rng}(f) \subseteq C \rightarrow f \in A]. \end{array}$$

(ix): "A is infinite" iff $\exists f(f$ is a one-one function, $\mathrm{dmn}(f) = \omega$, and $\mathrm{rng}(f) \subseteq A)$.

(x): Suppose that $A, D, \lambda, f \in H_\Psi$, and $2^{|\mathrm{tr\,cl}(A)|} < \Psi$. Then $\prod A \in H_\Psi$ by Lemma 35.3(ix). Now

$$\begin{array}{l} A \text{ is an infinite set of regular cardinals and } D \text{ is an ultrafilter on } A \\ \text{and } \lambda \text{ is a regular cardinal and } f \in {}^\lambda \prod A \text{ and } f \text{ is strictly} \\ \text{increasing and cofinal modulo } D \end{array}$$

iff

A is infinite and $\forall x \in A[x$ is a regular cardinal] and D is an ultrafilter on A and

λ is a regular cardinal and $\exists B\Big[B = \prod A$ and f is a function

and $\mathrm{dmn}(f) = \lambda$ and $\mathrm{rng}(f) \subseteq B$ and

$\forall \xi, \eta < \lambda \forall X \subseteq A[\forall a \in A[a \in X \Leftrightarrow f_\xi(a) < f_\eta(a)] \to X \in D]$

and $\forall g \in B \exists \xi < \lambda \forall X \subseteq A[\forall a \in A[a \in X \Leftrightarrow g(a) < f_\xi(a)] \to X \in D]\Big]$.

(xi): Assume that $2^{|\mathrm{tr\,cl}(A)|)} < \Psi$, and $A, B \in H_\Psi$. Let $\varphi(A, D, \lambda, f)$ be the statement of (x). Note:

(1) If $\varphi(A, D, \lambda, f)$, then $D, \lambda, f \in H_\Psi$, and $\max(\lambda, |\mathrm{tr\,cl}(A)|) \le 2^{|\mathrm{tr\,cl}(A)|}$.

In fact, $D \subseteq \mathscr{P}(A)$, so $\mathrm{tr\,cl}(D) \subseteq \mathrm{tr\,cl}(\mathscr{P}(A)) = \mathscr{P}(A) \cup \mathrm{tr\,cl}(A)$, and so $|\mathrm{tr\,cl}(D)| < \Psi$ by Lemma 35.3(iii); so $D \in H_\Psi$. Now f is a one-one function from λ into $\prod A$, so $\lambda \le |\prod A| < \Psi$, and hence $\lambda \in H_\Psi$ and $\max(\lambda, |\mathrm{tr\,cl}(A)|) \le 2^{|\mathrm{tr\,cl}(A)|}$. Finally, $f \subseteq \lambda \times \prod A$, so it follows that $f \in H_\Psi$.

Thus (1) holds. Hence the following equivalence shows the absoluteness of the statement in (xi):

A is an infinite set of regular cardinals and $B = \mathrm{pcf}(A)$

iff

A is infinite, and $\forall \mu \in A(\mu$ is a regular cardinal$) \wedge \forall \lambda \in B \exists D \exists f \varphi(A, D, \lambda, f)$
$\wedge \forall D \forall \lambda \forall f[\varphi(A, D, \lambda, f) \to \lambda \in B]$.

(xii): Assume that $2^{|\mathrm{tr\,cl}(A)|)} < \Psi$. By Lemma 35.3(xi) we have $\mathrm{pcf}(A) \in H_\Psi$. Hence

A is an infinite set of regular cardinals $\wedge f = \langle J_{<\lambda}[A] : \lambda \in \mathrm{pcf}(A)\rangle$

iff

A is infinite and $\forall \kappa \in A(\kappa$ is a regular cardinal and
f is a function and $\exists B[B = \mathrm{pcf}(A) \wedge B = \mathrm{dmn}(f)]$
$\forall \lambda \in \mathrm{dmn}(f) \forall X \subseteq A[A \in f(\lambda)$ iff $\exists C[C = \mathrm{pcf}(X) \wedge C \subseteq \lambda]]$

(xiii): Assume that $2^{2^{|\mathrm{tr\,cl}(A)|}} < \Psi$, and $A, B \in H_\Psi$. Note as above that $\mathrm{pcf}(A) \in H_\Psi$. Note that for any cardinal λ we have $J_{<\lambda}[A] \subseteq \mathscr{P}(A)$ and, with f as in (xi),

$f \subseteq \text{pcf}(A) \times \mathscr{P}(\mathscr{P}(A))$; so $f \in H_\Psi$. Let $\varphi(f, A)$ be the formula of (xii). Thus

$$\begin{aligned}
&A \text{ is a set of regular cardinals and } B = \langle B_\lambda : \lambda \in \text{pcf}(A)\rangle\\
&\text{and } \forall\lambda \in \text{pcf}(A)(B_\lambda \text{ is a } \lambda\text{-generator})\\
&\\
&\text{iff}\\
&\\
&B \text{ is a function and } \exists C[C = \text{pcf}(A) \wedge C = \text{dmn}(B)] \wedge \exists f[\varphi(f, A)\wedge\\
&\forall\lambda \in \text{dmn}(B)\forall\mu \in \text{dmn}(B)[\lambda \text{ is a cardinal and } \mu \text{ is a cardinal and}\\
&\mu = \lambda^+ \rightarrow B_\lambda \in f(\mu) \wedge \forall X \subseteq A[X \in f(\mu) \text{ iff } X\backslash B_\lambda \in f(\lambda)]]]
\end{aligned}$$

□

Now we turn to the consideration of elementary substructures of H_Ψ. The following lemma gives basic facts used below.

Lemma 35.5. *Suppose that Ψ is an uncountable cardinal, and N is an elementary substructure of H_Ψ (under $\in$ and a well-order of H_Ψ).*

(i) For every ordinal α, $\alpha \in N$ iff $\alpha + 1 \in N$.

(ii) $\omega \subseteq N$.

(iii) If $a \in N$, then $\{a\} \in N$.

(iv) If $a, b \in N$, then $\{a, b\}, (a, b) \in N$.

(v) If $A, B \in N$, then $A \times B \in N$.

(vi) If $A \in N$ then $\bigcup A \in N$.

(vii) If $f \in N$ is a function, then $\text{dmn}(f), \text{rng}(f) \in N$.

(viii) If $f \in N$ is a function and $a \in N \cap \text{dmn}(f)$, then $f(a) \in N$.

(ix) If $X, Y \in N$, $X \subseteq N$, and $|Y| \leq |X|$, then $Y \subseteq N$.

(x) If $X \in N$ and $X \neq \emptyset$, then $X \cap N \neq \emptyset$.

(xi) $\mathscr{P}(A) \in N$ if $A \in N$ and $2^{|\text{tr cl}(A)|} < \Psi$.

(xii) If ρ is an infinite ordinal, $|\rho|^+ < \Psi$, and $\rho \in N$, then $|\rho| \in N$ and $|\rho|^+ \in N$.

(xiii) If $A \in N$, then $\prod A \in N$ if $2^{|\text{tr cl}(A)|} < \Psi$.

(xiv) If $A \in N$, A is a set of regular cardinals, and $A \subseteq H_\Psi$, then $\text{pcf}(A) \in N$ if $2^{|\text{tr cl}(A)|} < \Psi$.

(xv) If $A \in N$, A is a set of regular cardinals, then $\langle J_{<\lambda}[A] : \lambda \in \text{pcf}(A)\rangle \in N$ if $2^{2^{|\text{tr cl}(A)|}} < \Psi$.

(xvi) If $A \in N$ and A is a set of regular cardinals, then there is a function $\langle B_\lambda : \lambda \in \text{pcf}(A)\rangle \in N$, where for each $\lambda \in \text{pcf}(A)$, the set B_λ is a λ-generator, if $2^{2^{|\text{tr cl}(A)|}} < \Psi$.

Proof. (i): Let α be an ordinal, and suppose that $\alpha \in N$. Then $\alpha \in H_\Psi$, and hence $\alpha \cup \{\alpha\} \in H_\Psi$. By absoluteness, $H_\Psi \models \exists x(x = \alpha \cup \{\alpha\})$, so $N \models \exists x(x = \alpha \cup \{\alpha\})$. Choose $b \in N$ such that $N \models b = \alpha \cup \{\alpha\}$. Then $H_\Psi \models b = \alpha \cup \{\alpha\}$, so by absoluteness, $b = \alpha \cup \{\alpha\}$. This proves that $\alpha \cup \{\alpha\} \in N$.

The method used in proving (i) can be used in the other parts; so it suffices in most other cases just to indicate a formula which can be used.

(ii): An easy induction, using the formulas $\exists x \forall y \in x(y \neq y)$ and $\exists x[a \subseteq x \wedge a \in x \wedge \forall y \in x[y \in a \vee y = a]]$.

(iii): Use the formula $\exists x[\forall y \in x(y = a) \wedge a \in x]$.

(iv): Similar to (ii).

(v): Use the formula

$$\exists C[\forall a \in A \forall b \in B[(a, b) \in C] \wedge \forall x \in C \exists a \in A \exists b \in B[x = (a, b)]].$$

(vi): Use the formula $\exists B[\forall x \in A[x \subseteq B] \wedge \forall y \in B \exists x \in A(y \in x)]$.

(vii): Use the formula $\exists A[\forall x \forall y[(x, y) \in f \to x \in A] \wedge \forall x \in A \exists y[(x, y) \in f]]$. Note that this formula is absolute for H_Ψ for example $(x, y) \in f \in H_\Psi$ implies that $x, y \in H_\Psi$.

(viii): Use the formula $\exists x[(a, x) \in f]$.

(ix): Let f be a function mapping X onto Y (assuming, as we may, that $Y \neq \emptyset$). Then $f \in H_\Psi$, so by the above method, we get another function $g \in N$ which maps X onto Y. Now (viii) gives the conclusion of (ix).

(x): Use the formula $\exists x \in X[x = x]$.

(xi): $\mathscr{P}(A) \in H_\Psi$ by Lemma 35.3(iii). Hence we can use the formula

$$\exists B[\forall x \in B(x \subseteq A) \wedge \forall x[x \subseteq A \to x \in B]].$$

(xii): Assume that ρ is an infinite ordinal and $\rho \in N$. Then

$$H_\Psi \models \exists \alpha \leq \rho[(\exists f : \rho \to \alpha,\ \text{a bijection}) \wedge \forall \beta \leq \rho[(\exists g : \rho \to \beta,\ \text{a bijection}) \to \alpha \leq \beta]].$$

Hence by the standard argument, there are $\alpha, f \in N$ such that

$$H_\Psi \models f : \rho \to \alpha \text{ is a bijection } \wedge \forall \beta \leq \rho[(\exists g : \rho \to \beta,\ \text{a bijection}) \to \alpha \leq \beta].$$

Clearly then $\alpha = |\rho|$.

For $|\rho|^+$, use the formula

$$\begin{aligned}\exists \beta \exists \Gamma \Big[&\forall \gamma \in \Gamma \exists f[f \text{ is a bijection from } \rho \text{ onto } \gamma] \\ &\wedge \forall \gamma \forall f[f \text{ is a bijection from } \rho \text{ onto } \gamma \to \gamma \in \Gamma] \\ &\wedge \beta = \bigcup \Gamma \Big].\end{aligned}$$

(xiii): Note that $\prod A \in H_\Psi$ by Lemma 35.3(ix). Then use the formula

$$\begin{aligned}\exists B \Big[&\forall f \in B(f \text{ is a function } \wedge \operatorname{dmn}(f) = A \wedge \forall a \in A(f(a) \in a)) \\ &\wedge \forall f[f \text{ is a function } \wedge \operatorname{dmn}(f) = A \wedge \forall a \in A(f(a) \in a) \to f \in B] \Big].\end{aligned}$$

(xiv): $\operatorname{pcf}(A) \in H_\Psi$ by Lemma 35.3(xi), so by Lemma 35.4(xi) we can use the formula $\exists B[B = \operatorname{pcf}(A)]$.

(xv): We have $\text{pcf}(A) \in H_\Psi$ and $\mathscr{P}(\mathscr{P}(H_\Psi))$ by Lemma 35.3(iii),(xi). It follows that $\langle J_{<\lambda}[A] : \lambda \in \text{pcf}(A)\rangle \in H_\Psi$. Hence by Lemma 35.4(xii) we can use the formula $\exists f[f = \langle J_{<\lambda}[A] : \lambda \in \text{pcf}(A)\rangle]$.

(xvi): By Lemma 35.3(iii),(xi) and Lemma 35.4(xiii) we can use the formula

$$\exists B[B : \text{pcf}(A) \to \mathscr{P}(A) \wedge \forall\lambda \in \text{pcf}(A)[B_\lambda \text{ is a } \lambda \text{ generator for } A]]. \qquad \square$$

Definition. Let κ be a regular cardinal. An elementary substructure N of H_Ψ is *κ-presentable* iff there is an increasing and continuous chain $\langle N_\alpha : \alpha < \kappa\rangle$ of elementary substructures of H_Ψ such that:

(1) $|N| = \kappa$ and $\kappa + 1 \subseteq N$.

(2) $N = \bigcup_{\alpha<\kappa} N_\alpha$.

(3) For every $\alpha < \kappa$, the function $\langle N_\beta : \beta \leq \alpha\rangle$ is a member of $N_{\alpha+1}$.

It is obvious how to construct a κ-presentable substructure of H_Ψ.

Lemma 35.6. *If N is a κ-presentable substructure of H_Ψ, with notation as above, and if $\alpha < \kappa$, then $\alpha + \omega \subseteq N_\alpha \in N_{\alpha+1}$.*

Proof. First we show that $\alpha \subseteq N_\alpha$ for all $\alpha < \kappa$, by induction. It is trivial for $\alpha = 0$, and the successor step is immediate from the induction hypothesis and Lemma 35.5(vii). The limit step is clear.

Now it follows that $\alpha + \omega \subseteq N_\alpha$ by an inductive argument using Lemma 35.5(i). Finally, $N_\alpha \in N_{\alpha+1}$ by (3) and Lemma 35.5(viii). $\square$

For any set M, we let $\overline{M}$ be the set of all ordinals α such that $\alpha \in M$ or $M \cap \alpha$ is unbounded in α.

Lemma 35.7. *If N is a κ-presentable substructure of H_Ψ, with notation as above, then*

(i) If $\alpha < \kappa$, then $\overline{N_\alpha} \subseteq N$.

(ii) If $\kappa < \alpha \in \overline{N}\backslash N$, then α is a limit ordinal and $\text{cf}(\alpha) = \kappa$, and in fact there is a closed unbounded subset E of α such that $E \subseteq N$ and E has order type κ.

Proof. First we consider (i). Suppose that $\gamma \in \overline{N_\alpha}$. We may assume that $\gamma \notin N_\alpha$.

Case 1. $\gamma = \sup(N_\alpha \cap \mathbf{On})$. Then

$$H_\Psi \models \exists\gamma'[\forall\delta(\delta \in N_\alpha \to \delta \leq \gamma') \wedge \forall\varepsilon[\forall\delta(\delta \in N_\alpha \to \delta \leq \varepsilon) \to \gamma' \leq \varepsilon]];$$

in fact, our given γ is the unique γ' for which this holds. Hence this statement holds in N, as desired.

Case 2. $\exists\theta \in N_\alpha(\gamma < \theta)$. We may assume that θ is minimum with this property. Now for any $\beta \in N_\alpha$ we can let $\rho(\beta)$ be the supremum of all ordinals in N_α which are less than β. So $\rho(\theta) = \gamma$. By absoluteness we get

$$\begin{aligned} H_\Psi \models &\forall\beta \in N_\alpha \exists\rho[\forall\varepsilon \in N_\alpha(\varepsilon < \beta \to \varepsilon < \rho) \\ &\wedge \forall\chi[\forall\varepsilon \in N_\alpha(\varepsilon < \beta \to \varepsilon < \chi) \to \rho \leq \chi]]; \end{aligned}$$

Hence N models this formula too; applying it to θ in place of β, we get $\rho \in N$ such that

$$\begin{aligned}N \models & \forall\varepsilon \in N_\alpha(\varepsilon < \theta \to \varepsilon < \rho) \\ & \wedge \forall\chi[\forall\varepsilon \in N_\alpha(\varepsilon < \theta \to \varepsilon < \chi) \to \rho \leq \chi].\end{aligned}$$

Thus $\gamma = \rho \in N$, as desired. This proves (i).

For (ii), suppose that $\kappa < \alpha \in \overline{N}\backslash N$. Let $E = \{\sup(\alpha \cap N_\xi) : \xi < \kappa\}$. Note that if $\xi < \kappa$, then by (i), $\sup(\alpha \cap N_\xi) \in N$. So $E \subseteq N$. It is clearly closed in α. It is unbounded, since for any $\beta \in \alpha \cap N$ there is a $\xi < \kappa$ such that $\beta \in N_\xi$, and so $\beta \leq \sup(\alpha \cap N_\xi) \in N$. $\square$

For any set N we define the *characteristic function* of N; it is defined for each regular cardinal μ as follows:

$$\mathrm{Ch}_N(\mu) = \sup(N \cap \mu).$$

Proposition 35.8. *Let κ be a regular cardinal, let N be a κ-presentable substructure of H_Ψ, and let μ be a regular cardinal.*

(i) If $\mu \leq \kappa$, then $\mathrm{Ch}_N(\mu) = \mu \in N$.

(ii) If $\kappa < \mu$, then $\mathrm{Ch}_N(\mu) \notin N$, $\mathrm{Ch}_N(\mu) < \mu$, and $\mathrm{Ch}_N(\mu)$ has cofinality κ.

(iii) For every $\alpha \in \overline{N} \cap \mu$ we have $\alpha \leq \mathrm{Ch}_N(\mu)$.

Proof. (i): True since $\kappa + 1 \subseteq N$.

(ii): Since $|N| = \kappa < \mu$ and μ is regular, we must have $\mathrm{Ch}_N(\mu) \notin N$ and $\mathrm{Ch}_N(\mu) < \mu$. Then $\mathrm{Ch}_N(\mu)$ has cofinality κ by Lemma 35.7.

(iii): clear. $\square$

Theorem 35.9. *Suppose that M and N are elementary substructures of H_Ψ and $\kappa < \mu$ are cardinals, with $\mu < \Psi$.*

(i) If $M \cap \kappa \subseteq N \cap \kappa$ and $\sup(M \cap \nu^+) = \sup(M \cap N \cap \nu^+)$ for every successor cardinal $\nu^+ \leq \mu$ such that $\nu^+ \in M$, then $M \cap \mu \subseteq N \cap \mu$.

(ii) If M and N are both κ-presentable and if $\sup(M \cap \nu^+) = \sup(N \cap \nu^+)$ for every successor cardinal $\nu^+ \leq \mu$ such that $\nu^+ \in M$, then $M \cap \mu = N \cap \mu$.

Proof. (i): Assume the hypothesis. We prove by induction on cardinals δ in the interval $[\kappa, \mu]$ that $M \cap \delta \subseteq N \cap \delta$. This is given for $\delta = \kappa$. If, inductively, δ is a limit cardinal, then the desired conclusion is clear. So assume now that δ is a cardinal, $\kappa \leq \delta < \mu$, and $M \cap \delta \subseteq N \cap \delta$. If $\delta^+ \notin M$, then by Lemma 35.5(xii), $[\delta, \delta^+] \cap M = \emptyset$, so the desired conclusion is immediate from the inductive hypothesis. So, assume that $\delta^+ \in M$. Then the hypothesis of (i) implies that there are ordinals in $[\delta, \delta^+]$ which are in $M \cap N$, and hence by Lemma 35.5(xii) again, $\delta^+ \in N$. Now to show that $M \cap [\delta, \delta^+] \subseteq N \cap [\delta, \delta^+]$, take any ordinal $\gamma \in M \cap [\delta, \delta^+]$. We may assume that $\gamma < \delta^+$. Since $\sup(M \cap \delta^+) = \sup(M \cap N \cap \delta^+)$ by assumption, we can choose $\beta \in M \cap N \cap \delta^+$ such that $\gamma < \beta$. Let f be the $<^*$-smallest bijection from β to δ. So $f \in M \cap N$. Since $\gamma \in M$, we also have $f(\gamma) \in M$ by Lemma 35.5(viii). Now $f(\gamma) < \delta$, so by the inductive assumption that $M \cap \delta \subseteq N \cap \delta$, we have $f(\gamma) \in N$. Since $f \in N$, so is f^{-1}, and $f^{-1}(f(\gamma)) = \gamma \in N$, as desired. This finishes the proof of (i).

(ii): Assume the hypothesis. Now we want to check the hypothesis of (i). By the definition of κ-presentable we have $\kappa = M \cap \kappa = N \cap \kappa$. Now suppose that ν is a cardinal and $\nu^+ \leq \mu$ with $\nu^+ \in M$. We may assume that $\kappa < \nu^+$. Let $\gamma = \mathrm{Ch}_M(\nu^+)$; this is the same as $\mathrm{Ch}_N(\nu^+)$ by the hypothesis of (ii). By Lemma 35.8 we have $\gamma \notin M \cup N$; hence by Lemma 35.7 there are clubs P, Q in γ such that $P \subseteq M$ and $Q \subseteq N$. Hence $\sup(M \cap \nu^+) = \sup(M \cap \nu^+) = \sup(M \cap N \cap \nu^+)$. This verifies the hypothesis of (i) for the pair M, N and also for the pair N, M. So our conclusion follows. $\square$

Minimally obedient sequences

Suppose that A is progressive, $\lambda \in \mathrm{pcf}(A)$, and B is a λ-generator for A. A sequence $\langle f_\xi : \xi < \lambda \rangle$ of members of $\prod A$ is called *persistently cofinal* for λ, B provided that $\langle (f_\xi \restriction B) : \xi < \lambda \rangle$ is persistently cofinal in $(\prod B, <_{J_{<\lambda}[B]})$. Recall that this means that for all $h \in \prod B$ there is a $\xi_0 < \lambda$ such that for all ξ, if $\xi_0 \leq \xi < \lambda$, then $h <_{J_{<\lambda}[B]} (f_\xi \restriction B)$.

Lemma 35.10. *Suppose that A is progressive, $\lambda \in \mathrm{pcf}(A)$, and B and C are λ-generators for A. A sequence $\langle f_\xi : \xi < \lambda \rangle$ of members of $\prod A$ is persistently cofinal for λ, B iff it is persistently cofinal for λ, C.*

Proof. Suppose that $\langle f_\xi : \xi < \lambda \rangle$ is persistently cofinal for λ, B, and suppose that $h \in \prod C$. Let $k \in \prod B$ be any function such that $h \restriction (B \cap C) = k \restriction (B \cap C)$. Choose $\xi_0 < \lambda$ such that for all $\xi \in [\xi_0, \lambda)$ we have $k <_{J_{<\lambda}[B]} (f_\xi \restriction B)$. Then for any $\xi \in [\xi_0, \lambda)$ we have

$$\begin{aligned} \{a \in C : h(a) \geq f_\xi(a)\} &= \{a \in B \cap C : h(a) \geq f_\xi(a)\} \cup \{a \in C \backslash B : h(a) \geq f_\xi(a)\} \\ &\subseteq \{a \in B : k(a) \geq f_\xi(a)\} \cup (C \backslash B); \end{aligned}$$

Now $(C \backslash B) \in J_{<\lambda}[A]$ by Lemma 34.25(xi), so $h <_{J_{<\lambda}[C]} (f_\xi \restriction C)$. By symmetry the lemma follows. $\square$

Because of this lemma we say that f is persistently cofinal for λ iff it is persistently cofinal for λ, B for some λ-generator B.

Lemma 35.11. *Suppose that A is progressive, $\lambda \in \mathrm{pcf}(A)$, and $f \stackrel{\mathrm{def}}{=} \langle f_\xi : \xi < \lambda \rangle$ is universal for λ. Then f is persistently cofinal for λ.*

Proof. Let B be a λ-generator. Then by Lemma 34.25(vii), λ is the largest member of $\mathrm{pcf}(B)$. By Lemma 34.17, $\langle (f_\xi \restriction B) : \xi < \lambda \rangle$ is strictly increasing under $<_{J_{<\lambda}[B]}$, and by Lemma 34.25(v) it is cofinal in $(\prod B, <_{J_{<\lambda}[B]})$. By Proposition 34.11, it is thus persistently cofinal in $(\prod B, <_{J_{<\lambda}[B]})$. $\square$

Lemma 35.12. *Suppose that A is progressive, $\lambda \in \mathrm{pcf}(A)$, and $A \in N$, where N is a κ-presentable elementary substructure of H_Ψ, with $|A| < \kappa < \min(A)$ and $2^{|\mathrm{tr\,cl}(A)|} < \Psi$. Suppose that $f = \langle f_\xi : \xi < \lambda \rangle$ is a sequence of functions in $\prod A$.*

Then for every $\xi < \lambda$ there is an $\alpha < \kappa$ such that for any $a \in A$,

$$f_\xi(a) < \mathrm{Ch}_N(a) \quad \text{iff} \quad f_\xi(a) < \mathrm{Ch}_{N_\alpha}(a).$$

Proof.

$$\begin{aligned}
\mathrm{Ch}_N(a) &= \sup(N \cap a) \\
&= \bigcup (N \cap a) \\
&= \bigcup \left(a \cap \bigcup_{\alpha<\kappa} N_\alpha \right) \\
&= \bigcup_{\alpha<\kappa} \bigcup (N_\alpha \cap a) \\
&= \bigcup_{\alpha<\kappa} \mathrm{Ch}_{N_\alpha}(a).
\end{aligned}$$

Hence for every $a \in A$ for which $f_\xi(a) < \mathrm{Ch}_N(a)$, there is an $\alpha_a < \kappa$ such that $f_\xi(a) < Ch_{N_{\alpha_a}}(a)$. Hence the existence of α as indicated follows. □

Lemma 35.13. *Suppose that A is progressive, κ is regular, $\lambda \in \mathrm{pcf}(A)$, and $A, \lambda \in N$, where N is a κ-presentable elementary substructure of H_Ψ, with $|A| < \kappa < \min(A)$ and Ψ is big. Suppose that $f = \langle f_\xi : \xi < \lambda \rangle \in N$ is a sequence of functions in $\prod A$ which is persistently cofinal in λ. Then for every $\xi \geq \mathrm{Ch}_N(\lambda)$ the set*

$$\{a \in A : \mathrm{Ch}_N(a) \leq f_\xi(a)\}$$

is a λ-generator for A.

Proof. Assume the hypothesis, including $\xi \geq \mathrm{Ch}_N(\lambda)$. Let α be as in Lemma 35.12. We are going to apply Lemma 34.25(ix). Since $A, f, \lambda \in N$, we may assume that $A, f, \lambda \in N_0$, by renumbering the elementary chain if necessary. Now $\kappa \subseteq N$, and $|A| < \kappa$, so we easily see that there is a bijection $f \in N$ mapping an ordinal $\alpha < \kappa$ onto A; hence $A \subseteq N$ by Lemma 35.5(viii), and so $A \subseteq N_\beta$ for some $\beta < \kappa$. We may assume that $A \subseteq N_0$. By Lemma 35.5(xvi),(viii), there is a λ-generator B which is in N_0.

Now the sequence f is persistently cofinal in $\prod B/J_{<\lambda}$, and hence

$$H_\Psi \models \forall h \in \prod B \exists \eta < \lambda \forall \rho \geq \eta [h \restriction B <_{J_{<\lambda}} f_\rho \restriction B]; \quad \text{hence}$$
$$N \models \forall h \in \prod B \exists \eta < \lambda \forall \rho \geq \eta [h \restriction B <_{J_{<\lambda}} f_\rho \restriction B];$$

Hence for every $h \in N$, if $h \in \prod B$ then there is an $\eta < \lambda$ with $\eta \in N$ such that $N \models \forall \rho \geq \eta [h \restriction B <_{J_{<\lambda}} f_\varphi \restriction B]$; going up, we see that really for every $h \in N \cap \prod A$ there is an $\eta_h \in N \cap \lambda$ such that for all ρ with $\rho \geq \eta_h$ we have $h \restriction B <_{J_{<\lambda}} f_\rho \restriction B$. Since ξ, as given in the statement of the Lemma, is $\geq$ each member of $N \cap \lambda$, hence $\geq \eta_h$ for each $h \in N \cap \prod A$, we see that

(1) $$h \restriction B <_{J_{<\lambda}} f_\xi \restriction B \text{ for every } h \in N \cap \prod A.$$

Now we can apply (1) to $h = \mathrm{Ch}_{N_\alpha}$, since this function is clearly in N. So $\mathrm{Ch}_{N\alpha} \restriction B <_{J_{<\lambda}[B]} f_\xi \restriction B$. Hence by the choice of α (see Lemma 35.12)

(2) $$\mathrm{Ch}_N \restriction B \leq_{J_{<\lambda}[B]} f_\xi \restriction B.$$

Note that (2) says that $B\backslash\{a \in A : \mathrm{Ch}_N(a) \le f_\xi(a)\} \in J_{<\lambda}[A]$.

Now $\lambda \notin \mathrm{pcf}(A\backslash B)$ by Lemma 34.25(ii), and hence $J_{<\lambda}[A\backslash B] = J_{\le\lambda}[A\backslash B]$. So by Theorem 34.8 we see that $\prod(A\backslash B)/J_{<\lambda}[A\backslash B]$ is λ^+-directed, so $\langle f_\xi \restriction (A\backslash B) : \xi < \lambda\rangle$ has an upper bound $h \in \prod(A\backslash B)$. We may assume that $h \in N$, by the usual argument. Hence

$$f_\xi \restriction (A\backslash B) <_{J_{<\lambda}[A\backslash B]} h < \mathrm{Ch}_N \restriction (A\backslash B);$$

hence $\{a \in A\backslash B : \mathrm{Ch}_N(a) \le f_\xi(a)\} \in J_{<\lambda}[A]$, and together with (2) and using Lemma 34.25(ix) this finishes the proof. □

Now suppose that A is progressive, δ is a limit ordinal, $f = \langle f_\xi : \xi < \delta\rangle$ is a sequence of members of $\prod A$, $|A|^+ \le \mathrm{cf}(\delta) < \min(A)$, and E is a club of δ of order type $\mathrm{cf}(\delta)$. Then we define

$$h_E = \sup\{f_\xi : \xi \in E\}.$$

We call h_E the *supremum along E of f*. Thus $h_E \in \prod A$, since $\mathrm{cf}(\delta) < \min(A)$. Note that if $E_1 \subseteq E_2$ then $h_{E_1} \le h_{E_2}$.

Lemma 35.14. *Let A, δ, f be as above. Then there is a unique function g in $\prod A$ such that the following two conditions hold.*

(i) There is a club C of δ of order type $\mathrm{cf}(\delta)$ *such that $g = h_C$.*

(ii) If E is any club of C of order type $\mathrm{cf}(\delta)$*, then $g \le h_E$.*

Proof. Clearly such a function g is unique if it exists.

Now suppose that there is no such function g. Then for every club C of δ of order type $\mathrm{cf}(\delta)$ there is a club D of order type $\mathrm{cf}(\delta)$ such that $h_C \not\le h_D$, hence $h_C \not\le h_{C\cap D}$. Hence there is a decreasing sequence $\langle E_\alpha : \alpha < |A|^+\rangle$ of clubs of δ such that for every $\alpha < |A|^+$ we have $h_{E_\alpha} \not\le h_{E_{\alpha+1}}$. Now note that

$$|A|^+ = \bigcup_{a\in A}\{\alpha < |A|^+ : h_{E_\alpha}(a) > h_{E_{\alpha+1}}(a)\}.$$

Hence there is an $a \in A$ such that $M \stackrel{\text{def}}{=} \{\alpha < |A|^+ : h_{E_\alpha}(a) > h_{E_{\alpha+1}}(a)\}$ has size $|A|^+$. Now $h_{E_\alpha}(a) \ge h_{E_\beta}(a)$ whenever $\alpha < \beta < |A|^+$, so this gives an infinite decreasing sequence of ordinals, contradiction. □

The function g of this lemma is called the *minimal club-obedient bound* of f.

Corollary 35.15. *Suppose that A is progressive, δ is a limit ordinal, $f = \langle f_\xi : \xi < \delta\rangle$ is a sequence of members of $\prod A$, $|A|^+ \le \mathrm{cf}(\delta) < \min(A)$, J is an ideal on A, and f is $<_J$-increasing. Let g be the minimal club-obedient bound of f. Then g is a $\le_J$-bound for f.* □

Now suppose that A is progressive, $\lambda \in \mathrm{pcf}(A)$, and κ is a regular cardinal such that $|A| < \kappa < \min(A)$. We say that $f = \langle f_\alpha : \alpha < \lambda\rangle$ is *κ-minimally obedient for λ* iff f is a universal sequence for λ and for every $\delta < \lambda$ of cofinality κ, f_δ is the minimal club-obedient bound of f.

A sequence f is *minimally obedient for* λ iff $|A|^+ < \min(A)$ and f is minimally obedient for every regular κ such that $|A| < \kappa < \min(A)$.

Lemma 35.16. *Suppose that* $|A|^+ < \min(A)$ *and* $\lambda \in \mathrm{pcf}(A)$. *Then there is a minimally obedient sequence for* λ.

Proof. By Theorem 34.18, let $\langle f^0_\xi : \xi < \lambda\rangle$ be a universal sequence for λ. Now by induction we define functions f_ξ for $\xi < \lambda$. Let $f_0 = f^0_0$, and choose $f_{\xi+1}$ so that $\max(f_\xi, f^0_\xi) < f_{\xi+1}$.

For limit $\delta < \lambda$ such that $|A| < \mathrm{cf}(\delta) < \min(A)$, let f_δ be the minimally club-obedient bound of $\langle f_\xi : \xi < \delta\rangle$.

For other limit $\delta < \lambda$, use the λ-directedness (Theorem 34.8) to get f_δ as a $<_{J_{<\lambda}}$-bound of $\langle f_\xi : \xi < \delta\rangle$.

Thus we have assured the minimally obedient property, and it is clear that $\langle f_\xi : \xi < \lambda\rangle$ is universal. $\square$

Lemma 35.17. *Suppose that* A *is progressive, and* κ *is a regular cardinal such that* $|A| < \kappa < \min(A)$. *Also assume the following:*

(i) $\lambda \in \mathrm{pcf}(A)$.

(ii) $f = \langle f_\xi : \xi < \lambda\rangle$ *is a* κ*-minimally obedient sequence for* λ.

(iii) N *is a* κ*-presentable elementary substructure of* H_Ψ, *with* Ψ *large, such that* $\lambda, f, A \in N$.

Then the following conditions hold:

(iv) For every $\gamma \in \overline{N} \cap \lambda \backslash N$ *we have:*

(a) $\mathrm{cf}(\gamma) = \kappa$.

(b) There is a club C *of* γ *of order type* κ *such that* $f_\gamma = \sup\{f_\xi : \xi \in C\}$ *and* $C \subseteq N$.

(c) $f_\gamma(a) \in \overline{N} \cap a$ *for every* $a \in A$.

(v) If $\gamma = \mathrm{Ch}_N(\lambda)$, *then:*

(a) $\gamma \in \overline{N} \cap \lambda \backslash N$; *hence we let* C *be as in (iv)(b), with* $f_\gamma = \sup\{f_\xi : \xi \in C\}$.

(b) $f_\xi \in N$ *for each* $\xi \in C$.

(c) $f_\gamma \le (\mathrm{Ch}_N \restriction A)$.

(vi) $\gamma = \mathrm{Ch}_N(\lambda)$ *and* C *is as in (iv)(b), with* $f_\gamma = \sup\{f_\xi : \xi \in C\}$, *and* B *is a* λ *generator, then for every* $h \in N \cap \prod A$ *there is a* $\xi \in C$ *such that* $(h \restriction B) <_{J_{<\lambda}} (f_\xi \restriction B)$.

Proof. Assume (i)–(iii). Note that $A \subseteq N$, by Lemma 35.5(ix).

For (iv), suppose also that $\gamma \in \overline{N} \cap \lambda \backslash N$. Then by Lemma 35.7 we have $\mathrm{cf}(\gamma) = \kappa$, and there is a club E in γ of order type κ such that $E \subseteq N$. By (ii), we have $f_\gamma = f_C$ for some club C of γ of order type κ. By the minimally obedient property we have $f_C = f_{C\cap E}$, and thus we may assume that $C \subseteq E$. For any $\xi \in C$ and $a \in A$ we have $f_\xi(a) \in N$ by Lemma 35.5(viii). So (iv) holds.

For (v), suppose that $\gamma = \mathrm{Ch}_N(\lambda)$. Then $\gamma \in \overline{N} \cap \lambda \backslash N$ because $|N| = \kappa < \min(A) \le \lambda$. For each $\xi \in C$ we have $f_\xi \in N$ by Lemma 35.5(viii). For (c), if $a \in A$, then $f_\gamma(a) = \sup_{\xi\in C} f_\xi(a) \le \mathrm{Ch}_N(a)$, since $f_\xi(a) \in N \cap a$ for all $\xi \in C$.

Next, assume the hypotheses of (vi). By Lemma 35.11, f is persistently cofinal in λ, so by Lemma 35.13, B' is a λ-generator. By Lemma 34.25(v) there is a $\xi \in C$ such that $h \restriction B' <_{J_{<\lambda}} f_\xi \restriction B'$. Now $B =_{J_{<\lambda}[A]} B'$ by Lemma 34.25(xi), so

$$\{a \in B : h(a) \geq f_\xi(b)\} \subseteq (B \backslash B') \cup \{a \in B' : h(a) \geq f_\xi(b)\} \in J_{<\lambda}[A]. \qquad \square$$

We now define some abbreviations.

$H_1(A, \kappa, N, \Psi)$ abbreviates

A is a progressive set of regular cardinals, κ is a regular cardinal such that $|A| < \kappa <$ $\min(A)$, and N is a κ-presentable elementary substructure of H_Ψ, with Ψ big and $A \in N$.

$H_2(A, \kappa, N, \Psi, \lambda, f, \gamma)$ abbreviates

$H_1(A, \kappa, N, \Psi)$, $\lambda \in \mathrm{pcf}(A)$, $f = \langle f_\xi : \xi < \lambda \rangle$ is a sequence of members of $\prod A$, $f \in N$, and $\gamma = \mathrm{Ch}_N(\lambda)$.

$P_1(A, \kappa, N, \Psi, \lambda, f, \gamma)$ abbreviates

$H_2(A, \kappa, N, \Psi, \lambda, f, \gamma)$ and $\{a \in A : \mathrm{Ch}_N(a) \leq f_\gamma(a)\}$ is a λ-generator.

$P_2(A, \kappa, N, \Psi, \lambda, f, \gamma)$ abbreviates

$H_2(A, \kappa, N, \Psi, \lambda, f, \gamma)$ and the following hold:
(i) $f_\gamma \leq (\mathrm{Ch}_N \restriction A)$.
(ii) For every $h \in N \cap \prod A$ there is a $d \in N \cap \prod A$ such that for any λ-generator B,

$$(h \restriction B) <_{J_{<\lambda}} (d \restriction B) \quad \text{and} \quad d \leq f_\gamma.$$

Thus $H_1(A, \kappa, N, \Psi)$ is part of the hypothesis of Lemma 35.17, and $H_2(A, \kappa, N, \Psi, \lambda, f, \gamma)$ is a part of the hypotheses of Lemma 35.17(v).

Lemma 35.18. *If $H_2(A, \kappa, N, \Psi, \lambda, f, \gamma)$ holds and f is persistently cofinal for λ, then $P_1(A, \kappa, N, \Psi, \lambda, f, \gamma)$ holds.*

Proof. This follows immediately from Lemma 35.13. $\square$

Lemma 35.19. *If $H_2(A, \kappa, N, \Psi, \lambda, f, \gamma)$ holds and f is κ-minimally obedient for λ, then both $P_1(A, \kappa, N, \Psi, \lambda, f, \gamma)$ and $P_2(A, \kappa, N, \Psi, \lambda, f, \gamma)$ hold.*

Proof. Since f is κ-minimally obedient for λ, it is a universal sequence for λ, by definition. Hence by Lemma 35.11 f is persistently cofinal for λ, and so property P_1 follows from Lemma 35.18.

For P_2, note that $\lambda, A \in N$ since $f \in N$, by Lemma 35.5(vii),(ix). Hence the hypotheses of Lemma 35.17(v) hold. So (i) in P_2 holds by Lemma 35.17(v)(c). For condition (ii), suppose that $h \in N \cap \prod A$. Take B and C as in Lemma 35.17(vi), and choose $\xi \in C$ such that $h \restriction B <_{J_{<\lambda}} f_\xi \restriction B$. Let $d = f_\xi$. Clearly this proves condition (ii). $\square$

The following obvious extension of Lemma 35.19 will be useful below.

Lemma 35.20. *Assume $H_1(A, \kappa, N, \Psi)$, and also assume that $\gamma = \mathrm{Ch}_N(\lambda)$ and*

(i) $f \stackrel{\text{def}}{=} \langle f^\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ is a sequence of sequences $\langle f^\lambda_\xi : \xi < \lambda\rangle$ each of which is a κ-minimally obedient for λ.

Then for each $\lambda \in N \cap \mathrm{pcf}(A)$, $P_1(A, \kappa, N, \Psi, \lambda, f^\lambda, \gamma)$ and $P_2(A, \kappa, N, \Psi, \lambda, f^\lambda, \gamma)$ hold. □

Lemma 35.21. *Suppose that $P_1(A, \kappa, N, \Psi, \lambda, f, \gamma)$ and $P_2(A, \kappa, N, \Psi, \lambda, f, \gamma)$ hold. Then*

(i) $\{a \in A : \mathrm{Ch}_N(a) = f_\gamma(a)\}$ is a λ-generator.

(ii) If $\lambda = \max(\mathrm{pcf}(A))$, then

$$< (f_\gamma, \mathrm{Ch}_N \restriction A) = \{a \in A : f_\gamma(a) < \mathrm{Ch}_N(a)\} \in J_{<\lambda}[A].$$

Proof. By (i) of $P_2(A, \kappa, N, \Psi, \lambda, f, \gamma)$ we have $f_\gamma \leq (\mathrm{Ch}_N \restriction A)$, so (i) holds by $P_1(A, \kappa, N, \Psi, \lambda, f, \gamma)$. (ii) follows from $P_1(A, \kappa, N, \Psi, \lambda, f, \gamma)$ and Lemma 34.25(xii). □

Lemma 35.22. *Assume that $P_1(A, \kappa, N, \Psi, \lambda, f, \gamma)$ and $P_2(A, \kappa, N, \Psi, \lambda, f, \gamma)$ hold. Let*

$$b = \{a \in A : \mathrm{Ch}_N(a) = f_\gamma(a)\}.$$

Then

(i) b is a λ-generator.

(ii) There is a set $b' \subseteq b$ such that:

(a) $b' \in N$;

(b) $b \backslash b' \in J_{<\lambda}[A]$;

(c) b' is a λ-generator.

Proof. (i) holds by Lemma 35.21(i). For (ii), by Lemma 35.12 choose $\alpha < \kappa$ such that, for every $a \in A$,

$$f_\gamma(a) < \mathrm{Ch}_N(a) \quad \text{iff} \quad f_\gamma(a) < \mathrm{Ch}_{N_\alpha}(a). \tag{1}$$

Now by (i) of $P_2(A, \kappa, N, \Psi, \lambda, f, \gamma)$ we have $f_\gamma \leq (\mathrm{Ch}_N \restriction A)$. Hence by (1) we see that for every $a \in A$,

$$a \in b \quad \text{iff} \quad \mathrm{Ch}_{N_\alpha}(a) \leq f_\gamma(a). \tag{2}$$

Now by (ii) of $P_2(A, \kappa, N, \Psi, \lambda, f, \gamma)$ applied to $h = \mathrm{Ch}_{N_\alpha} \restriction A$, there is a $d \in N \cap \prod A$ such that the following conditions hold:

(3) $(\mathrm{Ch}_{N_\alpha} \restriction b) <_{J_{<\lambda}} (d \restriction b)$.

(4) $d \leq f_\gamma$.

Now we define

$$b' = \{a \in A : \mathrm{Ch}_{N_\alpha}(a) \leq d(a)\}.$$

Clearly $b' \in N$. Also, by (3),

$$b \backslash b' = \{a \in b : d(a) < \mathrm{Ch}_{N_\alpha}(a)\} \in J_{<\lambda},$$

and so (ii)(b) holds. Thus $b \subseteq_{J_{<\lambda}} b'$. If $a \in b'$, then $\mathrm{Ch}_{N_\alpha}(a) \leq d(a) \leq f_\gamma(a)$ by (4), so $a \in b$ by (2). Thus $b' \subseteq b$. Now (ii)(c) holds by Lemma 34.25(ix). $\square$

Lemma 35.23. *Assume $H_1(A, \kappa, N, \Psi)$ and $A \in N$. Suppose that $\langle f^\lambda : \lambda \in \mathrm{pcf}(A)\rangle \in N$ is an array of sequences $\langle f^\lambda_\xi : \xi < \lambda\rangle$ with each $f^\lambda_\xi \in \prod A$. Also assume that for every $\lambda \in N \cap \mathrm{pcf}(A)$, both $P_1(A, \kappa, N, \Psi, \lambda, f^\lambda, \gamma(\lambda))$ and $P_2(A, \kappa, N, \Psi, \lambda, f^\lambda, \gamma(\lambda))$ hold.*

Then there exist cardinals $\lambda_0 > \lambda_1 > \cdots > \lambda_n$ in $\mathrm{pcf}(A) \cap N$ such that

$$(\mathrm{Ch}_N \restriction A) = \sup\{f^{\lambda_0}_{\gamma(\lambda_0)}, \ldots, f^{\lambda_n}_{\gamma(\lambda_n)}\}.$$

Proof. We will define by induction a descending sequence of cardinals $\lambda_i \in \mathrm{pcf}(A) \cap N$ and sets $A_i \in \mathscr{P}(A) \cap N$ (strictly decreasing under inclusion as i grows) such that if $A_i \neq \emptyset$ then $\lambda_i = \max(\mathrm{pcf}(A_i))$ and

$$(1) \qquad (\mathrm{Ch}_N \restriction (A \backslash A_{i+1})) = \sup\{(f^{\lambda_0}_{\gamma(\lambda_0)} \restriction (A \backslash A_{i+1})), \ldots, (f^{\lambda_i}_{\gamma(\lambda_i)} \restriction (A \backslash A_{i+1}))\}.$$

Since the cardinals are decreasing, there is a first i such that $A_{i+1} = \emptyset$, and then the lemma is proved. To start, $A_0 = A$ and $\lambda_0 = \max(\mathrm{pcf}(A))$. Clearly $\lambda_0 \in N$. Now suppose that λ_i and A_i are defined, with $A_i \neq 0$. By Lemma 35.22(i) and Lemma 34.25(x), the set

$$\{a \in A \cap (\lambda_i + 1) : \mathrm{Ch}_N(a) = f^{\lambda_i}_{\gamma(\lambda_i)}(a)\}$$

is a λ_i-generator. Hence by Lemma 35.22(ii) we get another λ_i-generator b'_{λ_i} such that

(2) $b'_{\lambda_i} \in N$.

(3) $b'_{\lambda_i} \subseteq \{a \in A \cap (\lambda_i + 1) : \mathrm{Ch}_N(a) = f^{\lambda_i}_{\gamma(\lambda_i)}(a)\}$.

Note that $b'_{\lambda_i} \neq \emptyset$. Let $A_{i+1} = A_i \backslash b'_{\lambda_i}$. Thus $A_{i+1} \in N$. Furthermore,

(4) $A \backslash A_{i+1} = (A \backslash A_i) \cup b'_{\lambda_1}$.

Now by Lemma 9.25(ii) and $\lambda_i = \max(\mathrm{pcf}(A_i))$ we have $\lambda_i \notin \mathrm{pcf}(A_{i+1})$. If $A_{i+1} \neq \emptyset$, we let $\lambda_{i+1} = \max(\mathrm{pcf}(A_{i+1}))$. Now by (i) of $P_2(A, \kappa, N, \Psi, \lambda, f^{\lambda_j}, \gamma(\lambda_j))$ we have

(5) $f^{\lambda_j}_{\gamma(\lambda_j)} \leq (\mathrm{Ch}_N \restriction A)$ for all $j \leq i$.

Now suppose that $a \in A \backslash A_{i+1}$. If $a \in A_i$, then by (4), $a \in b'_{\lambda_1}$, and so by (3), $\mathrm{Ch}_N(a) = f^{\lambda_i}_{\gamma(\lambda_1)}(a)$, and (1) holds for a. If $a \notin A_i$, then $A \neq A_i$, so $i \neq 0$. Hence by the inductive hypothesis for (1),

$$\mathrm{Ch}_N(a) = \sup\{f^{\lambda_0}_{\gamma(\lambda_0)}(a), \ldots, f^{\lambda_{i-1}}_{\gamma(\lambda_{i-1})}(a)\},$$

and (1) for a follows by (5). $\square$

The cofinality of $([\mu]^\kappa, \subseteq)$

First we give some simple properties of the sets $[\mu]^\kappa$, not involving pcf theory.

Proposition 35.24. *If $\kappa \le \mu$ are infinite cardinals, then*

$$(*) \qquad |[\mu]^\kappa| = \mathrm{cf}([\mu]^\kappa, \subseteq) \cdot 2^\kappa.$$

Proof. Let $\lambda = \mathrm{cf}([\mu]^\kappa, \subseteq)$, and let $\langle Y_i : i < \lambda \rangle$ be an enumeration of a cofinal subset of $\mathrm{cf}([\mu]^\kappa, \subseteq)$. For each $i < \lambda$ let f_i be a bijection from Y_i to κ. Now the inequality $\ge$ in $(*)$ is clear. For the other direction, we define an injection g of $[\mu]^\kappa$ into $\lambda \times \mathscr{P}(\kappa)$, as follows. Given $E \in [\mu]^\kappa$, let $i < \lambda$ be minimum such that $E \subseteq Y_i$, and define $g(E) = (i, f_i[E])$. Clearly g is one-one. $\square$

Proposition 35.25. *(i) If $\kappa_1 < \kappa_2 \le \mu$, then*

$$\mathrm{cf}([\mu]^{\kappa_1}, \subseteq) \le \mathrm{cf}([\mu]^{\kappa_2}, \subseteq) \cdot \mathrm{cf}([\kappa_2]^{\kappa_1}, \subseteq).$$

(ii) $\mathrm{cf}([\kappa^+]^\kappa, \subseteq) = \kappa^+$.
(iii) If $\kappa^+ \le \mu$, then $\mathrm{cf}([\mu]^\kappa, \subseteq) \le \mathrm{cf}([\mu]^{\kappa^+}, \subseteq) \cdot \kappa^+$.
(iv) If $\kappa \le \mu_1 < \mu_2$, then $\mathrm{cf}([\mu_1]^\kappa, \subseteq) \le \mathrm{cf}([\mu_2]^\kappa, \subseteq)$.
(v) If $\kappa \le \mu$, then $\mathrm{cf}([\mu^+]^\kappa, \subseteq) \le \mathrm{cf}([\mu]^\kappa, \subseteq) \cdot \mu^+$.
(vi) $\mathrm{cf}([\aleph_0]^{\aleph_0}, \subseteq) = 1$, *while for $m \in \omega \backslash 1$,* $\mathrm{cf}([\aleph_m]^{\aleph_0}) = \aleph_m$.
(vii) $\mathrm{cf}([\mu]^{\le\kappa}, \subseteq) = \mathrm{cf}([\mu]^\kappa, \subseteq)$.

Proof. (i): Let $M \subseteq [\mu]^{\kappa_2}$ be cofinal in $([\mu]^{\kappa_2}, \subseteq)$ of size $\mathrm{cf}([\mu]^{\kappa_2}, \subseteq)$, and let $N \subseteq ([\kappa_2]^{\kappa_1}, \subseteq)$ be cofinal in $([\kappa_2]^{\kappa_1}, \subseteq)$ of size $\mathrm{cf}([\kappa_2]^{\kappa_1}, \subseteq)$. For each $X \in M$ let $f_X : \kappa_2 \to X$ be a bijection. It suffices now to show that $\{f_X[Y] : X \in M, Y \in N\}$ is cofinal in $([\mu]^{\kappa_1}, \subseteq)$. Suppose that $W \in [\mu]^{\kappa_1}$. Choose $X \in M$ such that $W \subseteq X$. Then $f_X^{-1}[W] \in [\kappa_2]^{\kappa_1}$, so there is a $Y \in N$ such that $f_X^{-1}[W] \subseteq Y$. Then $W \subseteq f_X[Y]$, as desired.

(ii): The set $\{\gamma < \kappa^+ : |\gamma \backslash \kappa| = \kappa\}$ is clearly cofinal in $([\kappa^+]^\kappa$. If M is a nonempty subset of $[\kappa^+]^\kappa$ of size less than κ^+, then $|\bigcup M| = \kappa$, and $(\bigcup M) + 1$ is a member of $[\kappa^+]^\kappa$ not covered by any member of M. So (ii) holds.

(iii): Immediate from (i) and (ii).

(iv): Let $M \subseteq [\mu_2]^\kappa$ be cofinal of size $\mathrm{cf}([\mu_2]^\kappa, \subseteq)$. Let $N = \{X \cap \mu_1 : X \in M\} \backslash [\mu_1]^{<\kappa}$. It suffices to show that N is cofinal in $\mathrm{cf}([\mu_1]^\kappa, \subseteq)$. Suppose that $X \in [\mu_1]^\kappa$. Then also $X \in [\mu_2]^\kappa$, so we can choose $Y \in M$ such that $X \subseteq Y$. Clearly $X \subseteq Y \cap \mu_1 \in N$, as desired.

(v): For each $\gamma \in [\mu, \mu^+)$ let f_γ be a bijection from γ to μ. Let $E \subseteq [\mu]^\kappa$ be cofinal in $([\mu]^\kappa, \subseteq)$ and of size $\mathrm{cf}([\mu]^\kappa, \subseteq)$. It suffices to show that $\{f_\gamma^{-1}[X] : \gamma \in [\mu, \mu^+), X \in E\}$ is cofinal in $([\mu^+]^\kappa, \subseteq)$. So, take any $Y \in [\mu^+]^\kappa$. Choose $\gamma \in [\mu, \mu^+)$ such that $Y \subseteq \gamma$. Then $f_\gamma[Y] \in [\mu]^\kappa$, so we can choose $X \in E$ such that $f[Y] \subseteq X$. Then $Y \subseteq f_\gamma^{-1}[X]$, as desired.

(vi): Clearly $\mathrm{cf}([\aleph_0]^{\aleph_0}, \subseteq) = 1$. By induction it is clear from (v) that $\mathrm{cf}([\aleph_m]^{\aleph_0}) \le \aleph_m$. For $m > 0$ equality must hold, since if $X \subseteq [\aleph_m]^{\aleph_0}$ and $|X| < \aleph_m$, then $\bigcup X < \aleph_m$, and no denumerable subset of $\aleph_m \backslash \bigcup X$ is contained in a member of X.

(vii): Clear. $\square$

The following elementary lemmas will also be needed.

Lemma 35.26. *If $\alpha < \beta$ are limit ordinals, then*

$$|[\alpha, \beta]| = |\{\gamma : \alpha < \gamma < \beta,\ \gamma \text{ a successor ordinal}\}|.$$

Proof. For every $\delta \in [\alpha, \beta)$ let $f(\delta) = \delta + 1$. Then f is a one-one function from $[\alpha, \beta)$ onto $\{\gamma : \alpha < \gamma < \beta,\ \gamma \text{ a successor ordinal}\}$. $\square$

Lemma 35.27. *If $\alpha < \theta \leq \beta$ with θ limit, then*

$$|[\alpha, \beta]| = |\{\gamma : \alpha \leq \gamma \leq \beta,\ \gamma \text{ a successor ordinal}\}|.$$

Proof. Write $\beta = \delta + m$ with δ limit and $m \in \omega$. Then

$$[\alpha, \beta] = [\alpha, \alpha + \omega) \cup [\alpha + \omega, \delta] \cup (\delta, \beta],$$

and the desired conclusion follows easily from Lemma 35.26. $\square$

Theorem 35.28. *Suppose that μ is singular and $\kappa < \mu$ is an uncountable regular cardinal such that $A \stackrel{\text{def}}{=} (\kappa, \mu)_{\text{reg}}$ has size $< \kappa$. Then*

$$\text{cf}([\mu]^\kappa, \subseteq) = \max(\text{pcf}(A)).$$

Proof. Note by the progressiveness of A that every limit cardinal in the interval (κ, μ) is singular, and hence every member of A is a successor cardinal.

First we prove $\geq$. Suppose to the contrary that $\text{cf}([\mu]^\kappa, \subseteq) < \max(\text{pcf}(A))$. For brevity write $\max(\text{pcf}(A)) = \lambda$. let $\{X_i : i \in I\} \subseteq [\mu]^\kappa$ be cofinal and of cardinality less than λ. Pick a universal sequence $\langle f_\xi : \xi < \lambda \rangle$ for λ by Theorem 34.18. For every $\xi < \lambda$, $\text{rng}(f_\xi)$ is a subset of μ of size $\leq |A| \leq \kappa$, and hence $\text{rng}(f_\xi)$ is covered by some X_i. Thus $\lambda = \bigcup_{i \in I} \{\xi < \lambda : \text{rng}(f_\xi) \subseteq X_i\}$, so by $|I| < \lambda$ and the regularity of λ we get an $i \in I$ such that $|\{\xi < \lambda : \text{rng}(f_\xi) \subseteq X_i\}| = \lambda$. Now define for any $a \in A$,

$$h(a) = \sup(a \cap X_i).$$

Since $\kappa < a$ for each $a \in A$, we have $h \in \prod A$. Now the sequence $\langle f_\xi : \xi < \lambda \rangle$ is cofinal in $\prod A$ under $<_{J_{<\lambda}}$ by Lemma 34.25(v),(iv). So there is a $\xi < \lambda$ such that $h <_{J_{<\lambda}} f_\xi$. Thus there is an $a \in A$ such that $h(a) < f_\xi(a) \in X_i$, contradicting the definition of h.

Second we prove $\leq$, by exhibiting a cofinal subset of $[\mu]^\kappa$ of size at most $\max(\text{pcf}(A))$. Take N and Ψ so that $H_1(A, \kappa, N, \Psi)$. Let $\mathscr{M}$ be the set of all κ-presented elementary substructures M of H_Ψ such that $A \subseteq M$, and let

$$F = \{M \cap \mu : M \in \mathscr{M}\} \backslash [\mu]^{<\kappa}.$$

Since $|M| = \kappa$, we have $|M \cap \mu| \leq \kappa$, and so $\forall M \in F(|M \cap \mu| = \kappa)$.

(1) F is cofinal in $[\mu]^\kappa$.

In fact, for any $X \in [\mu]^\kappa$ we can find $M \in \mathscr{M}$ such that $X \subseteq M$, and (1) follows.

By (1) it suffices to prove that $|F| \leq \max(\mathrm{pcf}(A))$.

Claim. If $M, N \in \mathscr{M}$ are such that $\mathrm{Ch}_M \restriction A = \mathrm{Ch}_N \restriction A$, then $M \cap \mu = N \cap \mu$.

For, if ν^+ is a successor cardinal $\leq \mu$, then $\sup(M \cap \nu^+) = \mathrm{Ch}_M(\nu^+) = \mathrm{Ch}_N(\nu^+) = \sup(N \cap \nu^+)$. So the claim holds by Theorem 35.9.

Now for each $M \in \mathscr{M}$, let $g(M)$ be the sequence $\langle(\lambda_0, \gamma_0), \ldots, (\lambda_n, \gamma_n)\rangle$ given by Lemma 35.23. Clearly the range of g has size $\leq \max(\mathrm{pcf}(A))$. Now for each $X \in F$, choose $M_X \in \mathscr{M}$ such that $X = M_X \cap \mu$. Then for $X, Y \in F$ and $X \neq Y$ we have $M_X \cap \mu \neq M_Y \cap \mu$, hence by the claim $\mathrm{Ch}_{M_X} \restriction A \neq \mathrm{Ch}_{M_Y} \restriction A$, and hence by Lemma 35.23, $g(M_X) \neq g(M_Y)$. This proves that $|F| \leq \max(\mathrm{pcf}(A))$. $\square$

Corollary 35.29. *Let $A = \{\aleph_m : 0 < m < \omega\}$. Then for any $m \in \omega$ we have $\mathrm{cf}([\aleph_\omega]^{\aleph_m}) = \max(\mathrm{pcf}(A))$.* $\square$

Elevations and transitive generators

We start with some simple general notions about cardinals. If B is a set of cardinals, then a *walk* in B is a sequence $\lambda_0 > \lambda_1 > \cdots > \lambda_n$ of members of B. Such a walk is necessarily finite. Given cardinals $\lambda_0 > \lambda$ in B, a *walk from λ_0 to λ* is a walk as above with $\lambda_n = \lambda$. We denote by $F_{\lambda_0,\lambda}(B)$ the set of all walks from λ_0 to λ.

Now suppose that A is progressive and $\lambda_0 \in \mathrm{pcf}(A)$. A *special walk from λ_0 to λ_n in* $\mathrm{pcf}(A)$ is a walk $\lambda_0 > \cdots > \lambda_n$ in $\mathrm{pcf}(A)$ such that $\lambda_i \in A$ for all $i > 0$. We denote by $F'_{\lambda_0,\lambda}(A)$ the collection of all special walks from λ_0 to λ in $\mathrm{pcf}(A)$.

Next, suppose in addition that $f \stackrel{\text{def}}{=} \langle f^\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ is a sequence of sequences, where each f^λ is a sequence $\langle f^\lambda_\xi : \xi < \lambda\rangle$ of members of $\prod A$. If $\lambda_0 > \cdots > \lambda_n$ is a special walk in $\mathrm{pcf}(A)$, and $\gamma_0 \in \lambda_0$, then we define an associated sequence of ordinals by setting

$$\gamma_{i+1} = f^{\lambda_i}_{\gamma_i}(\lambda_{i+1})$$

for all $i < n$. Note that $\gamma_i < \lambda_i$ for all $i = 0, \ldots, n$. Then we define

$$\mathrm{El}_{\lambda_0,\ldots,\lambda_n}(\gamma_0) = \gamma_n.$$

Now we define the *elevation* of the sequence f, denoted by $f^e \stackrel{\text{def}}{=} \langle f^{\lambda,e} : \lambda \in \mathrm{pcf}(A)\rangle$, by setting, for any $\lambda_0 \in \mathrm{pcf}(A)$, any $\gamma_0 \in \lambda_0$, and any $\lambda \in A$,

$$f^{\lambda_0,e}_{\gamma_0}(\lambda) = \begin{cases} f^{\lambda_0}_{\gamma_0}(\lambda) & \text{if } \lambda_0 \leq \lambda, \\ \max(\{\mathrm{El}_{\lambda_0,\ldots,\lambda_n}(\gamma_0) : (\lambda_0, \ldots, \lambda_n) \in F'_{\lambda_0,\lambda}\}) & \text{if } \lambda < \lambda_0, \\ & \quad \text{and this maximum exists,} \\ f^{\lambda_0}_{\gamma_0}(\lambda) & \text{if } \lambda < \lambda_0, \text{ otherwise.} \end{cases}$$

Note here that the superscript ^{e} is only notational, standing for "elevated".

Lemma 35.30. *Assume the above notation. Then $f^{\lambda_0}_{\gamma_0} \le f^{\lambda_0,e}_{\gamma_0}$ for all $\lambda_0 \in \mathrm{pcf}(A)$ and all $\gamma_0 \in \lambda_0$.*

Proof. Take any $\gamma_0 \in \lambda_0$ and any $\lambda \in A$. If $\lambda_0 \le \lambda$, then $f^{\lambda_0,e}_{\gamma_0}(\lambda) = f^{\lambda_0}_{\gamma_0}(\lambda)$. Suppose that $\lambda < \lambda_0$. If the above maximum does not exist, then again $f^{\lambda_0,e}_{\gamma_0}(\lambda) = f^{\lambda_0}_{\gamma_0}(\lambda)$. Suppose the maximum exists. Now $(\lambda_0, \lambda) \in F'_{\lambda_0,\lambda}(A)$, so

$$f^{\lambda_0}_{\gamma_0}(\lambda) = \mathrm{El}_{\lambda_0,\lambda}(\gamma_0) \le \max(\{\mathrm{El}_{\lambda_0,\ldots,\lambda_n}(\gamma_0) : (\lambda_0, \ldots, \lambda_n) \in F'_{\lambda_0,\lambda}\}) = f^{\lambda_0,e}_{\gamma_0}(\lambda). \qquad \square$$

Lemma 35.31. *Suppose that A is progressive, κ is a regular cardinal such that $|A| < \kappa < \min(A)$, and $f \overset{\text{def}}{=} \langle f^\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ is a sequence of sequences f^λ such that f^λ is κ-minimally obedient for λ. Assume also $H_1(A, \kappa, N, \Psi)$ and $f \in N$.*

Then also $f^e \in N$.

Proof. The proof is a more complicated instance of our standard procedure for going from V to H_Ψ to N and then back. We sketch the details.

Assume the hypotheses. In particular, $A \in N$. Hence also $\mathrm{pcf}(A) \in N$. Also, $|A| < \kappa$, so $A \subseteq N$. Now clearly $F' \in N$. Also, $\mathrm{El} \in N$. (Note that El depends upon A.) Then by absoluteness,

$$H_\Psi \models \exists g \;\; g \text{ is a function, } \mathrm{dmn}(g) = \mathrm{pcf}(A) \wedge \forall \lambda_0 \in \mathrm{pcf}(A) \forall \gamma_0 \in \lambda_0 \forall \lambda \in A$$

$$g(\lambda) = \begin{cases} f^{\lambda_0}_{\gamma_0}(\lambda) & \text{if } \lambda_0 \le \lambda, \\ \max(\{\mathrm{El}_{\lambda_0,\ldots,\lambda_n}(\gamma_0) : (\lambda_0, \ldots, \lambda_n) \in F'_{\lambda_0,\lambda}\}) & \text{if } \lambda < \lambda_0, \\ & \quad\text{and this maximum exists,} \\ f^{\lambda_0}_{\gamma_0}(\lambda) & \text{if } \lambda < \lambda_0, \text{ otherwise.} \end{cases}$$

Now the usual procedure can be applied. $\square$

Lemma 35.32. *Suppose that A is progressive, κ is a regular cardinal such that $|A| < \kappa < \min(A)$, and $f \overset{\text{def}}{=} \langle f^\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ is a sequence of sequences f^λ such that f^λ is κ-minimally obedient for λ. Assume $H_1(A, \kappa, N, \Psi)$ and $f \in N$.*

Suppose that $\lambda_0 \in \mathrm{pcf}(A) \cap N$, and let $\gamma_0 = \mathrm{Ch}_N(\lambda_0)$.

(i) If $\lambda_0 > \cdots > \lambda_n$ is a special walk in $\mathrm{pcf}(A)$, and $\gamma_1, \ldots, \gamma_n$ are formed as above, then $\gamma_i \in \overline{N}$ for all $i = 0, \ldots, n$.

(ii) For every $\lambda \in A \cap \lambda_0$ we have $f^{\lambda_0,e}_{\gamma_0}(\lambda) \in \overline{N}$.

Proof. (i): By Lemma 35.17(iv)(c), $f^{\lambda_0}_{\gamma_0}(\lambda) \in \overline{N}$, and (i) follows by induction using Lemma 35.17(iv)(c).

(ii): immediate from (i). $\square$

Lemma 35.33. *Assume the hypotheses of Lemma 35.35. Then* *(i) For any special walk $\lambda_0 > \cdots > \lambda_n = \lambda$ in $F'_{\lambda_0,\lambda}$, we have*

$$El_{\lambda_0,\ldots,\lambda_n}(\gamma_0) \leq \mathrm{Ch}_N(\lambda).$$

(ii) $f^{\lambda_0,e}_{\gamma_0} \leq \mathrm{Ch}_N \restriction A$ for every $\gamma_0 < \lambda_0$.
(iii) If there is a special walk $\lambda_0 > \cdots > \lambda_n = \lambda$ in $F'_{\lambda_0,\lambda}$ such that

$$El_{\lambda_0,\ldots,\lambda_n}(\gamma_0) = \mathrm{Ch}_N(\lambda),$$

then

$$\mathrm{Ch}_N(\lambda) = f^{\lambda_0,e}_{\gamma_0}(\lambda).$$

(iv) Suppose that $\mathrm{Ch}_N(\lambda) = f^{\lambda_0,e}_{\gamma_0}(\lambda) = \gamma$. If there is an $a \in A \cap \lambda$ such that $f^{\lambda,e}_{\gamma}(a) = \mathrm{Ch}_N(a)$, then also $f^{\lambda_0,e}_{\gamma_0}(a) = \mathrm{Ch}_N(a)$.

Proof. (i) is immediate from Lemma 35.32(i) and Lemma 35.8(iii). (ii) and (iii) follow from (i). For (iv), by Lemma 35.32(i) and (i) there are special walks $\lambda_0 > \cdots > \lambda_n = \lambda$ and $\lambda = \lambda'_0 > \cdots > \lambda'_m = a$ such that

$$f^{\lambda_0,e}_{\gamma_0}(\lambda) = \mathrm{Ch}_N(\lambda) = \mathrm{El}_{\lambda_0,\ldots,\lambda_n}(\gamma_0) \quad \text{and}$$
$$f^{\lambda,e}_{\gamma}(a) = \mathrm{Ch}_N(a) = \mathrm{El}_{\lambda'_0,\ldots,\lambda'_m}(a).$$

It follows that

$$\mathrm{El}_{\lambda_0,\ldots,\lambda_n,\lambda'_1,\ldots,a}(\gamma_0) = \mathrm{Ch}_N(a),$$

and (iii) then gives $f^{\lambda_0,e}_{\gamma_0}(a) = \mathrm{Ch}_N(a)$. □

Definition. Suppose that A is progressive and $A \subseteq P \subseteq \mathrm{pcf}(A)$. A system $\langle b_\lambda : \lambda \in P\rangle$ of subsets of A is *transitive* iff for all $\lambda \in P$ and all $\mu \in b_\lambda$ we have $b_\mu \subseteq b_\lambda$.

Theorem 35.34. *Suppose that $H_1(A, \kappa, N, \Psi)$, and $f = \langle f^\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ is a system of functions, and each f^λ is κ-minimally obedient for λ. Let f^e be the derived elevated array. For every $\lambda_0 \in \mathrm{pcf}(A) \cap N$ put $\gamma_0 = \mathrm{Ch}_N(\lambda_0)$ and define*

$$b_{\lambda_0} = \{a \in A : \mathrm{Ch}_N(a) = f^{\lambda_0,e}_{\gamma_0}(a)\}.$$

Then the following hold for each $\lambda_0 \in \mathrm{pcf}(A) \cap N$:
(i) b_{λ_0} is a λ_0-generator.
(ii) There is a $b'_{\lambda_0} \subseteq b_{\lambda_0}$ such that
(a) $b_{\lambda_0} \backslash b'_{\lambda_0} \in J_{<\lambda_0}[A]$.
(b) $b'_{\lambda_0} \in N$ (each one individually, not the sequence).
(c) b'_{λ_0} is a λ_0-generator.
(iii) The system $\langle b_\lambda : \lambda \in \mathrm{pcf}(A) \cap N\rangle$ is transitive.

Proof. Note that $H_2(A, \kappa, N, \Psi, \lambda_0, f^{\lambda_0,e}, \gamma_0)$ holds by Lemma 35.34. By definition, minimally obedient implies universal, so f^{λ_0} is persistently cofinal by Lemma 35.11. Hence by Lemma 35.24, $f^{\lambda_0,e}$ is persistently cofinal, and so $P_1(A, \kappa, N, \Psi, \lambda_0, f^{\lambda_0,e}, \gamma_0)$ holds by

Lemma 35.18. Also, by Lemma 35.19 $P_2(A,\kappa,N,\Psi,\lambda_0,f^{\lambda_0},\gamma_0)$ holds, so the condition $P_2(A,\kappa,N,\Psi,\lambda_0,f^{\lambda_0,e},\gamma_0)$ holds by Lemmas 35.30 and 35.33(ii). Now (i) and (ii) hold by Lemma 35.22.

Now suppose that $\lambda_0\in\mathrm{pcf}(A)\cap N$ and $\lambda\in b_{\lambda_0}$. Thus

$$\mathrm{Ch}_N(\lambda)=f^{\lambda_0,e}_{\gamma_0}(\lambda),$$

where $\gamma_0=\mathrm{Ch}_N(\lambda_0)$. Write $\gamma=\mathrm{Ch}_N(\lambda)$. We want to show that $b_\lambda\subseteq b_{\lambda_0}$. Take any $a\in b_\lambda$. So $\mathrm{Ch}_N(a)=f^{\lambda,e}_\gamma(a)$. By Lemma 35.33(iv) we get $f^{\lambda_0,e}_{\gamma_0}(a)=\mathrm{Ch}_N(a)$, so $a\in b_{\lambda_0}$, as desired. □

Localization

Theorem 35.35. *Suppose that A is a progressive set. Then there is no subset $B\subseteq\mathrm{pcf}(A)$ such that $|B|=|A|^+$ and, for every $b\in B$, $b>\max(\mathrm{pcf}(B\cap b))$.*

Proof. Assume the contrary. We may assume that $|A|^+<\min(A)$. In fact, if we know the result under this assumption, and now $|A|^+=\min(A)$, suppose that $B\subseteq\mathrm{pcf}(A)$ with $|B|=|A|^+$ and $\forall b\in B[b>\max(\mathrm{pcf}(B\cap b))]$. Let $A'=A\backslash\{|A|^+\}$. Then let $B'=B\backslash\{|A|^+\}$. Hence we have $B'\subseteq\mathrm{pcf}(A')$. Clearly $|B'|=|A'|^+$ and $\forall b\in B'[b>\max(\mathrm{pcf}(B'\cap b))]$, contradiction.

Also, clearly we may assume that B has order type $|A|^+$.

Let $E=A\cup B$. Then $|E|<\min(E)$. Let $\kappa=|E|$. By Lemma 35.16, we get an array $\langle f^\lambda:\lambda\in\mathrm{pcf}(E)\rangle$, with each f^λ κ-minimally obedient for λ. Choose N and Ψ so that $H_1(A,\kappa,N,\Psi)$, with N containing $A,B,E,\langle f^\lambda:\lambda\in\mathrm{pcf}(E)\rangle$. Now let $\langle b_\lambda:\lambda\in\mathrm{pcf}(E)\cap N\rangle$ be the set of transitive generators as guaranteed by Theorem 35.34. Let $b'_\lambda\in N$ be such that $b'_\lambda\subseteq b_\lambda$ and $b_\lambda\backslash b'_\lambda\in J_{<\lambda}$.

Now let F be the function with domain $\{a\in A:\exists\beta\in B(a\in b_\beta)\}$ such that for each such a, $F(a)$ is the least $\beta\in B$ such that $a\in b_\beta$. Define $B_0=\{\gamma\in B:\exists a\in\mathrm{dmn}(F)(\gamma\leq F(a)\}$. Thus B_0 is an initial segment of B of size at most $|A|$. Clearly $B_0\in N$. We let $\beta_0=\min(B\backslash B_0)$; so $B_0=B\cap\beta_0$.

Now we claim

(1) There exists a finite descending sequence $\lambda_0>\cdots>\lambda_n$ of cardinals in $N\cap\mathrm{pcf}(B_0)$ such that $B_0\subseteq b_{\lambda_0}\cup\ldots\cup b_{\lambda_n}$.

We prove more: we find a finite descending sequence $\lambda_0>\cdots>\lambda_n$ of cardinals in $N\cap\mathrm{pcf}(B_0)$ such that $B_0\subseteq b'_{\lambda_0}\cup\ldots\cup b'_{\lambda_n}$. Let $\lambda_0=\max(\mathrm{pcf}(B_0))$. Since $B_0\in N$, we clearly have $\lambda_0\in N$ and hence $b'_{\lambda_0}\in N$. So $B_1\stackrel{\mathrm{def}}{=}B_0\backslash b'_{\lambda_0}\in N$. Now suppose that $B_k\subseteq B_0$ has been defined so that $B_k\in N$. If $B_k=\emptyset$, the construction stops. Suppose that $B_k\neq\emptyset$. Let $\lambda_k=\max(\mathrm{pcf}(B_k))$. Clearly $\lambda_k\in N$, so $b'_{\lambda_k}\in N$ and $B_{\kappa+1}\stackrel{\mathrm{def}}{=}B_k\backslash b'_{\lambda_k}\in N$. Since $B_{\kappa+1}=B_k\backslash b'_{\lambda_k}$ and b'_{λ_k} is a λ_k-generator, from Lemma 9.25(xii) it follows that

$\lambda_0 > \lambda_1 > \cdots$. So the construction eventually stops; say that $B_{n+1} = \emptyset$. So $B_n \subseteq b'_{\lambda_n}$. So

$$\begin{aligned} B_0 &\subseteq b'_{\lambda_0} \cup (B_0 \backslash b'_{\lambda_0}) \\ &= b'_{\lambda_0} \cup B_1 \\ &\subseteq b'_{\lambda_0} \cup b'_{\lambda_1} \cup B_2 \\ &\ldots\ldots\ldots \\ &\subseteq b'_{\lambda_0} \cup b'_{\lambda_1} \cup \ldots \cup B_n \\ &\subseteq b'_{\lambda_0} \cup b'_{\lambda_1} \cup \ldots \cup b'_{\lambda_n}. \end{aligned}$$

This proves (1).

Note that $\beta_0 > \max(\mathrm{pcf}(B \cap \beta_0) = \max(\mathrm{pcf}(B_0)) \geq \lambda_0, \ldots, \lambda_n$ by the initial assumption of the proof. Next, we claim

(2) $b_{\beta_0} \subseteq b_{\lambda_0} \cup \ldots \cup b_{\lambda_n}$.

To prove this, first note that $b_{\beta_0} \subseteq A \cup B_0$. For, $b_{\beta_0} \subseteq E$ by definition, and $E = A \cup B$; $b_{\beta_0} \cap B = B_0$, so indeed $b_{\beta_0} \subseteq A \cup B_0$. Also, $B_0 \subseteq b_{\lambda_0} \cup \ldots \cup b_{\lambda_n}$. So it suffices to prove that $b_{\beta_0} \cap A \subseteq b_{\lambda_0} \cup \ldots \cup b_{\lambda_n}$.

Consider any cardinal $a \in b_{\beta_0} \cap A$. Since $\beta_0 \in B$, we have $a \in \mathrm{dmn}(F)$, and since $\beta_0 \notin B_0$ we have $F(a) < \beta_0$. Let $\beta = F(a)$. So $a \in b_\beta$, and $\beta < \beta_0$, so by the minimality of β_0, $\beta \in B_0$. Since $B_0 \subseteq b_{\lambda_0} \cup \ldots \cup b_{\lambda_n}$, it follows that $\beta \in b_{\lambda_i}$ for some $i = 0, \ldots, n$. But transitivity implies that $b_\beta \subseteq b_{\lambda_i}$, and hence $a \in b_{\lambda_i}$, as desired. So (2) holds.

By (2) we have

$$\mathrm{pcf}(b_{\beta_0}) \subseteq \mathrm{pcf}(b_{\lambda_0}) \cup \ldots \cup \mathrm{pcf}(b_{\lambda_n}),$$

and hence by Lemma 34.25(vii) we get $\beta_0 = \max(\mathrm{pcf}(b_{\beta_0})) \leq \max\{\lambda_i : i = 0, \ldots, n\} < \beta_0$, contradiction. $\square$

Theorem 35.36. (Localization) *Suppose that A is a progressive set of regular cardinals. Suppose that $B \subseteq \mathrm{pcf}(A)$ is also progressive. Then for every $\lambda \in \mathrm{pcf}(B)$ there is a $B_0 \subseteq B$ such that $|B_0| \leq |A|$ and $\lambda \in \mathrm{pcf}(B_0)$.*

Proof. We prove by induction on λ that if A and B satisfy the hypotheses of the theorem, then the conclusion holds. Let C be a λ-generator over B. Thus $C \subseteq B$ and $\lambda = \max(\mathrm{pcf}(C))$ by Lemma 34.25(vii). Now $C \subseteq \mathrm{pcf}(A)$ and C is progressive. It suffices to find $B_0 \subseteq C$ with $|B_0| \leq |A|$ and $\lambda \in \mathrm{pcf}(B_0)$.

Let $C_0 = C$ and $\lambda_0 = \lambda$. Suppose that $C_0 \supseteq \cdots \supseteq C_i$ and $\lambda_0 > \cdots > \lambda_i$ have been constructed so that $\lambda = \max(\mathrm{pcf}(C_i))$ and C_i is a λ-generator over B. If there is no maximal element of $\lambda \cap \mathrm{pcf}(C_i)$ we stop the construction. Otherwise, let λ_{i+1} be that maximum element, let D_{i+1} be a λ_{i+1}-generator over B, and let $C_{i+1} = C_i \backslash D_{i+1}$. Now $D_{i+1} \in J_{\leq \lambda_{i+1}}[B] \subseteq J_{<\lambda}[B]$, so C_{i+1} is still a λ-generator of B by Lemma 9.25(ix), and $\lambda = \max(\mathrm{pcf}(C_{i+1}))$ by Lemma 34.25(vii). Note that $\lambda_{i+1} \notin \mathrm{pcf}(C_{i+1})$, by Lemma 34.25(ii).

This construction must eventually stop, when $\lambda \cap C_i$ does not have a maximal element; we fix the index i.

(1) There is an $E \subseteq \lambda \cap \text{pcf}(C_i)$ such that $|E| \leq |A|$ and $\lambda \in \text{pcf}(E)$.

In fact, suppose that no such E exists. We now construct a strictly increasing sequence $\langle \gamma_j : j < |A|^+ \rangle$ of elements of $\text{pcf}(C_i)$ such that $\gamma_k > \max(\text{pcf}(\{\gamma_j : j < k\})$ for all $k < |A|^+$. (This contradicts Theorem 35.35.) Suppose that $\{\gamma_j : j < k\} = E$ has been defined. Now $\lambda \notin \text{pcf}(E)$ by the supposition after (1), and $\lambda < \max(\text{pcf}(E))$ is impossible since $\text{pcf}(E) \subseteq \text{pcf}(C_i)$ and $\lambda = \max(\text{pcf}(C_i))$. So $\lambda > \max(\text{pcf}(E))$. Hence, because $\lambda \cap C_i$ does not have a maximal element, we can choose $\gamma_k \in \lambda \cap C_i$ such that $\gamma_k > \max(\text{pcf}(E))$, as desired. Hence (1) holds.

We take E as in (1). Apply the inductive hypothesis to each $\gamma \in E$ and to A, E in place of A, B; we get a set $G_\gamma \subseteq E$ such that $|G_\gamma| \leq |A|$ and $\gamma \in \text{pcf}(G_\gamma)$. Let $H = \bigcup_{\gamma \in E} G_\gamma$. Note that $|H| \leq |A|$. Thus $E \subseteq \text{pcf}(H)$. Since $\text{pcf}(E) \subseteq \text{pcf}(H)$ by Theorem 9.15, we have $\lambda \in \text{pcf}(H)$, completing the inductive proof.

The size of $\text{pcf}(A)$

Theorem 35.37. *If A is a progressive interval of regular cardinals, then* $|\text{pcf}(A)| < |A|^{+4}$.

Proof. Assume that A is a progressive interval of regular cardinals but $|\text{pcf}(A)| \geq |A|^{+4}$. Let $\rho = |A|$. We will define a set B of size ρ^+ consisting of cardinals in $\text{pcf}(A)$ such that each cardinal in B is greater than $\max(\text{pcf}(B \cap b))$. This will contradict Theorem 35.35.

Let $S = S^{\rho^{+3}}_{\rho^+}$; so S is a stationary subset of ρ^{+3}. By Theorem 34.40 let $\langle C_k : k \in S \rangle$ be a club guessing sequence. Thus

(1) C_k is a club in k of order type ρ^+, for each $k \in S$.

(2) If D is a club in ρ^{+3}, then there is a $k \in D \cap S$ such that $C_k \subseteq D$.

Let σ be the ordinal such that $\aleph_\sigma = \sup(A)$. Now $\text{pcf}(A)$ is an interval of regular cardinals by Theorem 34.13. So $\text{pcf}(A)$ contains all regular cardinals in the set $\{\aleph_{\sigma+\alpha} : \alpha < \rho^{+4}\}$.

Now we are going to define a strictly increasing continuous sequence $\langle \alpha_i : i < \rho^{+3} \rangle$ of ordinals less than ρ^{+4}.

1. Let $\alpha_0 = \rho^{+3}$.
2. For i limit let $\alpha_i = \bigcup_{j<i} \alpha_j$.
3. Now suppose that α_j has been defined for all $j \leq i$; we define α_{i+1}. For each $k \in S$ let $e_k = \{\aleph_{\sigma+\alpha_j} : j \in C_k \cap (i+1)\}$. Thus $e_k^{(+)}$ is a subset of $\text{pcf}(A)$. If $\max(\text{pcf}(e_k^{(+)})) < \aleph_{\sigma+\rho^{+4}}$, let β_k be an ordinal such that $\max(\text{pcf}(e_k^{(+)})) < \aleph_{\sigma+\beta_k}$ and $\beta_k < \rho^{+4}$; otherwise let $\beta_k = 0$. Let α_{i+1} be greater than α_i and all β_k for $k \in S$, with $\alpha_{i+1} < \rho^{+4}$. This is possible because $|S| = \rho^{+3}$. Thus

(3) For every $k \in S$, if $\max(\text{pcf}(e_k^{(+)})) < \aleph_{\sigma+\rho^{+4}}$, then $\max(\text{pcf}(e_k^{(+)})) < \aleph_{\sigma+\alpha_{i+1}}$.

This finishes the definition of the sequence $\langle \alpha_i : i < \rho^{+3} \rangle$. Let $D = \{\alpha_i : i < \rho^{+3}\}$, and let $\delta = \sup(D)$. Then D is club in δ. Let $\mu = \aleph_{\sigma+\delta}$. Thus μ has cofinality ρ^{+3}, and it is singular since $\delta > \alpha_0 = \rho^{+3}$. Now we apply Corollary 9.35: there is a club C_0 in μ such that $\mu^+ = \max(\text{pcf}(C_0^{(+)}))$. We may assume that $C_0 \subseteq [\aleph_\sigma, \mu)$. so we can write $C_0 = \{\aleph_{\sigma+i} : i \in D_0\}$ for some club D_0 in δ. Let $D_1 = D_0 \cap D$. So D_1 is a club of δ. Let

$E = \{i \in \rho^{+3} : \alpha_i \in D_1\}$. It is clear that E is a club in ρ^{+3}. So by (2) choose $k \in E \cap S$ such that $C_k \subseteq E$. Let $C'_k = \{\beta \in C_k :$ there is a largest $\gamma \in C_k$ such that $\gamma < \beta\}$. Set $B = \{\aleph^+_{\sigma+\alpha_i} : i \in C'_k\}$. We claim that B is as desired. Clearly $|B| = \rho^+$.

Take any $j \in C'_k$. We want to show that

$$(*) \qquad \aleph^+_{\sigma+\alpha_j} > \max(\mathrm{pcf}(B \cap \aleph^+_{\sigma+\alpha_j})).$$

Let $i \in C_k$ be largest such that $i < j$. So $i+1 \leq j$. We consider the definition given above of α_{i+1}. We defined $e_k = \{\aleph_{\sigma+\alpha_l} : l \in C_k \cap (i+1)\}$. Now

(4) $B \cap \aleph^+_{\sigma+\alpha_j} \subseteq e_k^{(+)}$.

For, if $b \in B \cap \aleph^+_{\sigma+\alpha_j}$, we can write $b = \aleph^+_{\sigma+\alpha_l}$ with $l \in C'_k$ and $l < j$. Hence $l \leq i$ and so $b = \aleph^+_{\sigma+\alpha_l} \in e_k^{(+)}$. So (4) holds.

Now if $l \in C_k \cap (i+1)$, then $l \in E$, and so $\alpha_l \in D_1 \subseteq D_0$. Hence $\aleph_{\sigma+\alpha_l} \in C_0$. This shows that $e_k^{(+)} \subseteq C_0^{(+)}$. So $\max(\mathrm{pcf}(e_k^{(+)})) \leq \max(\mathrm{pcf}(C_0^{(+)})) = \mu^+ < \aleph_{\sigma+\rho^{+4}}$. Hence by (3) we get $\max(\mathrm{pcf}(e_k^{(+)})) < \aleph_{\sigma+\alpha_{i+1}}$. So

$$\begin{aligned}\max(\mathrm{pcf}(B \cap \aleph^+_{\sigma+\alpha_j})) &\leq \max(\mathrm{pcf}(e_k^{(+)})) \quad \text{by (4)}\\ &< \aleph^+_{\sigma+\alpha_{i+1}}\\ &\leq \aleph^+_{\sigma+\alpha_j},\end{aligned}$$

which proves $(*)$. $\square$

Theorem 35.38. *If $\aleph_\delta$ is a singular cardinal such that $\delta < \aleph_\delta$, then*

$$\mathrm{cf}([\aleph_\delta]^{|\delta|}, \subseteq) < \aleph_{|\delta|^{+4}}.$$

Proof. Let $\kappa = |\delta|^+$ and $A = (\kappa, \aleph_\delta)_{\mathrm{reg}}$. By Lemma 35.25(iii) and Lemma 35.28,

$$\begin{aligned}\mathrm{cf}([\aleph_\delta]^{|\delta|}, \subseteq) &\leq \max(|\delta|^+, \mathrm{cf}([\aleph_\delta]^{|\delta|^+}, \subseteq))\\ &\leq \max(|\delta|^+, \max(\mathrm{pcf}(A))).\end{aligned}$$

Hence it suffices to show that $\max(\mathrm{pcf}(A)) < \aleph_{|\delta|^{+4}}$.

By Theorem 35.37, $|\mathrm{pcf}(A)| < |A|^{+4}$. Write $\max(\mathrm{pcf}(A)) = \aleph_\alpha$ and $\kappa = \aleph_\beta$. We want to show that $\alpha < |\delta|^{+4}$. Now $\mathrm{pcf}(A) = (\kappa, \max(\mathrm{pcf}(A))]_{\mathrm{reg}} = (\aleph_\beta, \aleph_\alpha]_{\mathrm{reg}}$. By Lemma 35.27, $|(\beta, \alpha)| = |\mathrm{pcf}(A)| < |A|^{+4} \leq |\delta|^{+4}$. Also, $\beta \leq \aleph_\beta = \kappa = |\delta|^+ < |\delta|^{+4}$. So $|\alpha| < |\delta|^{+4}$, and hence $\alpha < |\delta|^{+4}$. $\square$

Theorem 35.39. *If δ is a limit ordinal, then*

$$\aleph_\delta^{\mathrm{cf}(\delta)} < \max\left(\left(|\delta|^{\mathrm{cf}(\delta)}\right)^+, \aleph_{|\delta|^{+4}}\right).$$

Proof. If $\delta = \aleph_\delta$, then $|\delta| = \aleph_\delta$ and the conclusion is obvious. So assume that $\delta < \aleph_\delta$. Now

(1) $\aleph_\delta^{\mathrm{cf}(\delta)} \le |\delta|^{\mathrm{cf}(\delta)} \cdot \mathrm{cf}([\aleph_\delta]^{|\delta|}, \subseteq)$.

In fact, let $B \subseteq [\aleph_\delta]^{|\delta|}$ be cofinal and of size $\mathrm{cf}([\aleph_\delta]^{|\delta|}, \subseteq)$. Now $\mathrm{cf}(\delta) \le |\delta|$, so

$$[\aleph_\delta]^{\mathrm{cf}(\delta)} = \bigcup_{Y \in B} [Y]^{\mathrm{cf}(\delta)},$$

and (1) follows. Hence the theorem follows by Theorem 35.38. $\square$

Corollary 35.40. $\aleph_\omega^{\aleph_0} < \max\left((2^{\aleph_0})^+, \aleph_{\omega_4}\right)$. $\square$

36. Various forcing orders

In this section we briefly survey various forcing orders which have been used. Many of them give rise to new real numbers, i.e., new subsets of ω. (It is customary to identify real numbers with subsets of ω, since these are simpler objects than Dedekind cuts; and a bijection in the ground model between $\mathbb{R}$ and $\mathscr{P}(\omega)$ transfers the newness to "real" real numbers.) For each kind of forcing we give a reference for further results concerning it. Of course our list of forcing orders is not complete, but we hope the treatment here can be a guide to further study.

Cohen forcing

The forcing used in Chapter 16 is, as indicated there, called *Cohen forcing.* If M is a c.t.m. of ZFC, $\mathbb{P}$ is $\mathrm{Fin}(\omega, 2)$, and G is $\mathbb{P}$-generic over M, then $\bigcup G$ is a *Cohen real.* More generally, if N is a c.t.m. of ZFC and $M \subseteq N$, then a Cohen real in N is a function $f : \omega \to 2$ in N such that there is a $\mathbb{P}$-generic filter G over M such that $M[G] \subseteq N$ and $f = \bigcup G$.

Theorem 36.1. *Suppose that M is a c.t.m. of ZFC, $I \in M$, $I = J_0 \cup J_1$ with $J_0 \cap J_1 = \emptyset$, and G is $\mathrm{Fin}(I, 2)$-generic over M.*

(i) Let $H_0 = G \cap \mathrm{Fin}(J_0, 2)$. Then H_0 is $\mathrm{Fin}(J_0, 2)$-generic over M.

(ii) Let $H_1 = G \cap \mathrm{Fin}(J_1, 2)$. Then H_1 is $\mathrm{Fin}(J_1, 2)$-generic over $M[H_0]$.

(iii) $M[G] = M[H_0][H_1]$.

Proof. We are going to use Theorem 25.13. Let $\mathbb{P}$ be the partial order $\mathrm{Fin}(J_0, 2)$ and $\mathbb{Q}$ the partial order $\mathrm{Fin}(J_1, 2)$. We claim that $\mathrm{Fin}(I, 2)$ is isomorphic to $\mathbb{P} \times \mathbb{Q}$. Define $f(p) = (p \restriction J_0, p \restriction J_1)$. Clearly this is an isomorphism. We claim that $f[G] = H_0 \times H_1$. For, suppose that $p \in G$. then $p \restriction J_0 \subseteq p$, so $p \restriction J_0 \in G$, and hence $p \restriction J_0 \in H_0$. Similarly, $p \restriction J_1 \in H_1$. So $f(p) \in H_0 \times H_1$. Conversely, if $(p, q) \in H_0 \times H_1$, then $p \in G$ and $q \in G$, so there is an $r \in G$ such that $p, q \subseteq r$. Now $p \cup q \subseteq r$, so $p \cup q \in G$. Clearly $f(p \cup q) = (p, q)$. So this proves that $f[G] = H_0 \times H_1$.

Now it follows from Lemma 25.9 that $H_0 \times H_1$ is $\mathbb{P} \times \mathbb{Q}$-generic over M, and $M[G] = M[H_0 \times H_1]$. Now we can apply Theorem 25.13 to get:

(1) H_0 is $\mathbb{P}$-generic over M.

(2) H_1 is $\mathbb{Q}$-generic over $M[H_0]$.

(3) $M[G] = M[H_0][H_1]$.

This proves our theorem. □

It follows that all of the subsets of ω given in the proof of Theorem 16.1 are Cohen reals:

Corollary 36.2. *Let M be a c.t.m. of ZFC and let κ be a cardinal of M such that $\kappa^\omega = \kappa$. Let $\mathbb{P} = \mathrm{Fin}(\kappa, 2)$ in M, and let G be $\mathbb{P}$-generic over M, and let $g = \bigcup G$. Let $h : \kappa \times \omega \to \kappa$*

be a bijection in M. Then for each $\alpha < \kappa$, the set $\{m \in \omega : g(h(\alpha, m)) = 1\}$ is a Cohen real.

Proof. Remember that subsets of ω and their characteristic functions are both considered as reals. Implicitly, one is a Cohen real iff the other is, by definition. So we will show that the function $l \stackrel{\text{def}}{=} \langle g(h(\alpha, m)) : m \in \omega \rangle$ is a Cohen real.

Fix $\alpha < \kappa$, and let $J = \{\beta < \kappa : h^{-1}(\beta)$ has the form (α, m) for some $m \in \omega\}$. Let $k(m) = h(\alpha, m)$ for all $m \in \omega$. Then k is a bijection from ω onto J. By 36.1, $G \cap \text{Fin}(J, 2)$ is $\text{Fin}(J, 2)$-generic over M. Define $k' : \text{Fin}(J, 2) \to \text{Fin}(\omega, 2)$ by setting $k'(p) = p \circ k$ for any $p \in \text{Fin}(J, 2)$. So k' is an isomorphism from $\text{Fin}(J, 2)$ onto $\text{Fin}(\omega, 2)$. Clearly then $k'[G \cap \text{Fin}(J, 2)]$ is $\text{Fin}(\omega, 2)$-generic over M. So the proof is completed by checking that $\bigcup k'[G \cap \text{Fin}(J, 2)] = l$. Take any $m \in \omega$. Then

$$\begin{aligned}
(m, \varepsilon) \in \bigcup k'[G \cap \text{Fin}(J, 2)] \quad & \text{iff} \quad \text{there is a } p \in k'[G \cap \text{Fin}(J, 2)] \\
& \qquad \text{such that } (m, \varepsilon) \in p \\
& \text{iff} \quad \text{there is a } q \in G \cap \text{Fin}(J, 2) \\
& \qquad \text{such that } (m, \varepsilon) \in k'(q) \\
& \text{iff} \quad \text{there is a } q \in G \cap \text{Fin}(J, 2) \\
& \qquad \text{such that } (m, \varepsilon) \in q \circ k \\
& \text{iff} \quad g(k(m)) = \varepsilon \\
& \text{iff} \quad g(h(\alpha, m)) = \varepsilon \\
& \text{iff} \quad (m, \varepsilon) \in l
\end{aligned}$$

□

Theorem 36.3. *Suppose that M is a c.t.m. of ZFC and G is $\text{Fin}(\omega, 2)$-generic over M. Let $g = \bigcup G$ (so that g is a Cohen real). Then for any $f \in {}^\omega 2$ which is in M, the set $\{m \in \omega : f(m) < g(m)\}$ is infinite.*

Proof. For each $n \in \omega$ let in M

$$D_n = \{h \in \text{Fin}(\omega, 2) : \text{there is an } m > n \text{ such that } m \in \text{dmn}(h) \text{ and } f(m) < h(m)\}.$$

Clearly D_n is dense. Hence the desired result follows. □

Thus if g is a Cohen real, then there is no f in the ground model such that $\{m \in \omega : g(m) \leq f(m)\}$ is finite. Put another way, if $A \subseteq \omega$ is a Cohen real, then there is no $B \subseteq \omega$ in the ground model such that $A \backslash B$ is finite.

Let $\langle P_i : i \in I \rangle$ be a system of forcing orders. We define the *product* of these orders to be the set

$$\prod_{i \in I}^{\text{w}} P_i = \left\{ f \in \prod_{i \in I} P_i : \{j \in I : f(j) \neq 1\} \text{ is finite} \right\}$$

with the order

$$f \leq g \quad \text{iff} \quad \forall i \in I[f_i \leq g_i].$$

Theorem 36.4. *For any infinite cardinal κ, $\mathrm{Fin}(\kappa, 2)$ is isomorphic to $\prod_{\alpha<\kappa} \mathrm{Fin}(\omega, 2)$.*

Proof. Let $k : \kappa \to \kappa \times \omega$ be a bijection. For each $f \in \prod_{\alpha<\kappa} \mathrm{Fin}(\omega, 2)$ let

$$\begin{aligned}\mathrm{dmn}(F(f)) &= \{\alpha < \kappa : 2^{\mathrm{nd}}(k(\alpha)) \in \mathrm{dmn}(f(1^{\mathrm{st}}(k(\alpha))) \text{ and}\\ (F(f))(\alpha) &= (f(1^{\mathrm{st}}(k(\alpha))))(2^{\mathrm{nd}}(k(\alpha))).\end{aligned}$$

Clearly F maps $\prod_{\alpha<\kappa} \mathrm{Fin}(\omega, 2)$ into $\mathrm{Fin}(\kappa, 2)$. To show that F is one-one, suppose that $f, g \in \prod_{\alpha<\kappa} \mathrm{Fin}(\omega, 2)$ and $f \neq g$; say $f(\alpha) \neq g(\alpha)$. Say $(n, \varepsilon) \in f(\alpha)\backslash g(\alpha)$. Let $\beta = k^{-1}(\alpha, n)$. Thus $\beta \in \mathrm{dmn}(F(f))$. We may assume that $\beta \in \mathrm{dmn}(F(g))$. It follows that $(F(f))(\beta) \neq (F(g))(\beta)$. So $F(f) \neq F(g)$.

To show that F maps onto, let $h \in \mathrm{Fin}(\kappa, 2)$. Define $f \in \prod_{\alpha<\kappa} \mathrm{Fin}(\omega, 2)$ by setting

$$\begin{aligned}\mathrm{dmn}(f(\alpha)) &= \{n \in \omega : k^{-1}(\alpha, n) \in \mathrm{dmn}(h)\}\\ (f(\alpha))(n) &= h(k^{-1}(\alpha, n)) \quad \text{if } k^{-1}(\alpha, n) \in \mathrm{dmn}(h).\end{aligned}$$

Clearly $F(f) = h$.

Clearly $f \leq g$ iff $F(f) \subseteq F(g)$. □

Cohen reals are widely used in set theory.

Roitman, J. *Adding a random or a Cohen real.* Fund. Math. 103 (1979), 47–60.

Random forcing

The general idea of random forcing is to take a σ-algebra of measurable sets with respect to some measure, divide by the ideal of sets of measure zero, obtaining a complete Boolean algebra, and use it as the forcing algebra; the partially ordered set of nonzero elements is the forcing partial order.

We give fairly complete details for the case of the product measure on ${}^{\kappa}2$, for any infinite cardinal κ. To make our treatment self-contained we give a standard development of this measure, following

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Let κ be an infinite cardinal. For each $f \in \mathrm{Fn}(\kappa, 2, \omega)$ let $U_f = \{g \in {}^{\kappa}2 : f \subseteq g\}$. Hence $U_\emptyset = {}^{\kappa}2$. Note that the function taking f to U_f is one-one. For each $f \in \mathrm{Fn}(\kappa, 2, \omega)$ let $\theta_0(U_f) = 1/2^{|\mathrm{dmn}(f)|}$. Thus $\theta_0(U_\emptyset) = 1$. Let $\mathcal{C} = \{U_f : f \in \mathrm{Fn}(\kappa, 2, \omega)\}$. Note that ${}^{\kappa}2 \in \mathcal{C}$. For any $A \subseteq {}^{\kappa}2$ let

$$\theta(A) = \inf\left\{\sum_{n\in\omega} \theta_0(C_n) : C \in {}^{\omega}\mathcal{C} \text{ and } A \subseteq \bigcup_{n\in\omega} C_n\right\}.$$

An *outer measure* on a set X is a function $\mu : \mathscr{P}(X) \to [0, \infty]$ satisfying the following conditions:

(1) $\mu(\emptyset) = 0$.

(2) If $A \subseteq B \subseteq X$, then $\mu(A) \leq \mu(B)$.

(3) For every $A \in {}^{\omega}\mathscr{P}(X)$, $\mu(\bigcup_{n\in\omega} A_n) \leq \sum_{n\in\omega} \mu(A_n)$.

Proposition 36.4. *θ is an outer measure on ${}^{\kappa}2$.*

Proof. For (1), for any $m \in \omega$ let $f \in \mathrm{Fn}(\kappa, 2, \omega)$ have domain of size m. Then $\emptyset \subseteq U_f$ and $\theta_0(U_f) = \frac{1}{m}$. Hence $\theta(\emptyset) = 0$.

For (2), if $A \subseteq B \subseteq {}^{\kappa}2$, then

$$\left\{C \in {}^{\omega}\mathcal{C} : B \subseteq \bigcup_{n\in\omega} C_n\right\} \subseteq \left\{C \in {}^{\omega}\mathcal{C} : A \subseteq \bigcup_{n\in\omega} C_n\right\},$$

and hence $\mu(A) \leq \mu(B)$.

For (3), assume that $A \in {}^{\omega}\mathscr{P}({}^{\kappa}2)$. We may assume that $\sum_{n\in\omega} \theta(A_n) < \infty$. Let $\varepsilon > 0$; we show that $\theta(\bigcup_{n\in\omega} A_n) \leq \sum_{n\in\omega} \theta(A_n) + \varepsilon$, and the arbitrariness of ε then gives the desired result. For each $n \in \omega$ choose $C^n \in {}^{\omega}\mathcal{C}$ such that $A_n \subseteq \bigcup_{m\in\omega} C^n_m$ and $\sum_{m\in\omega} \theta_0(C^n_m) \leq \theta(A_n) + \frac{\varepsilon}{2^n}$. Then $\bigcup_{n\in\omega} A_n \subseteq \bigcup_{n\in\omega}\bigcup_{m\in\omega} C^n_m$ and

$$\theta\left(\bigcup_{n\in\omega} A_n\right) \leq \sum_{n\in\omega}\sum_{n\in\omega} \theta_0(C^n_m) \leq \sum_{n\in\omega} \theta(A_n) + \varepsilon,$$

as desired. $\square$

If A is a σ-algebra of subsets of X, then a *measure* on A is a function $\mu : A \to [0, \infty]$ such that $\mu(\emptyset) = 0$ and $\mu(\bigcup_{i\in\omega} a_i) = \sum_{i\in\omega} \mu(a_i)$ if $a \in {}^{\omega}A$ and $a_i \cap a_j = \emptyset$ for all $i \neq j$. Note that $a_i = \emptyset$ is possible for some $i \in \omega$.

We give some important properties of measures:

Proposition 36.5. *Suppose that μ is a measure on a σ-algebra A of subsets of X. Then:*

(i) If $Y, Z \in A$ and $Y \subseteq Z$, then $\mu(Y) \leq \mu(Z)$.

(ii) If $Y \in {}^{\omega}A$, then $\mu(\bigcup_{n\in\omega} Y_n) \leq \sum_{n\in\omega} \mu(Y_n)$.

(iii) If $Y \in {}^{\omega}A$ and $Y_n \subseteq Y_{n+1}$ for all $n \in \omega$, then $\mu(\bigcup_{n\in\omega} Y_n) = \sup_{n\in\omega} \mu(Y_n)$.

Proof. (i): We have $\mu(Z) = \mu(Y) + \mu(Z \backslash Y) \geq \mu(Y)$.

(ii): Let $Z_n = Y_n \backslash \bigcup_{m<n} Y_m$. By induction, $\bigcup_{m\leq n} Z_m = \bigcup_{m\leq n} Y_m$, and hence $\bigcup_{m\in\omega} Z_m = \bigcup_{m\in\omega} Y_m$. Now

$$\mu\left(\bigcup_{m\in\omega} Y_m\right) = \mu\left(\bigcup_{m\in\omega} Z_m\right) = \sum_{m\in\omega} \mu(Z_m) \leq \sum_{m\in\omega} \mu(Y_m).$$

(iii): Again let $Z_n = Y_n \backslash \bigcup_{m<n} Y_m$. By induction, $Y_n = \bigcup_{m\leq n} Z_m$. Hence

$$\mu\left(\bigcup_{n\in\omega} Y_n\right) = \mu\left(\bigcup_{n\in\omega} Z_n\right)$$

$$
\begin{aligned}
&= \sum_{n\in\omega} \mu(Z_n) \\
&= \lim_{n\to\infty} \sum_{m\le n} \mu(Z_m) \\
&= \lim_{n\to\infty} \mu\left(\bigcup_{m\le n} Z_m\right) \\
&= \lim_{n\to\infty} \mu(Y_n) \\
&= \sup_{n\in\omega} \mu(Y_n).
\end{aligned}
$$

□

Proposition 36.6. *Let*

$$A = \{E \subseteq {}^{\kappa}2 : \forall X \subseteq {}^{\kappa}2[\theta(X) = \theta(X\cap E) + \theta(X\backslash E)]\}.$$

Then A is a σ-algebra of subsets of ${}^{\kappa}2$, and $\theta \restriction A$ is a measure on A.

Proof. $\emptyset \in A$ since for any $X \subseteq {}^{\kappa}2$ we have

$$\theta(X\cap\emptyset) + \theta(X\backslash\emptyset) = \theta(\emptyset) + \theta(X) = 0 + \theta(X) = \theta(X).$$

If $E \in A$, obviously also ${}^{\kappa}2\backslash E \in A$.

Next we show that if $E_1, E_2 \in A$ then $E_1 \cup E_2 \in A$. For any $X \subseteq {}^{\kappa}2$,

$$
\begin{aligned}
&\theta(X\cap(E_1\cup E_2)) + \theta(X\backslash(E_1\cup E_2)) \\
&\qquad = \theta(X\cap(E_1\cup E_2)\cap E_1) + \theta(X\cap(E_1\cup E_2)\backslash E_1) + \theta(X\backslash(E_1\cup E_2)) \\
&\qquad = \theta(X\cap E_1) + \theta((X\backslash E_1)\cap E_2) + \theta((X\backslash E_1)\backslash E_2) \\
&\qquad = \theta(X\cap E_1) + \theta(X\backslash E_1) \\
&\qquad = \theta(X).
\end{aligned}
$$

Now suppose that $E \in {}^{\omega}A$. Let $F = \bigcup_{i\in\omega} E_i$; we want to show that $F \in A$. For each $n \in \omega$ let $G_n = \bigcup_{i\le n} E_i$. So $G_n \in A$ by the binary case already considered. Let $H_0 = G_0$ and $H_{n+1} = G_{n+1}\backslash G_n$ for all $n \in \omega$. Hence $H_0 = E_0$ and $H_{n+1} = E_{n+1}\backslash G_n$ for all $n \in \omega$. Moreover, by induction $\bigcup_{i\le n} H_i = \bigcup_{i\le n} G_i$ for all $n \in \omega$, and hence $\bigcup_{i\in\omega} H_i = \bigcup_{i\in\omega} G_i = \bigcup_{i\in\omega} E_i = F$.

Now suppose that $n \ge 1$ and $X \subseteq {}^{\kappa}2$. Then

$$
\begin{aligned}
\theta(X\cap G_n) &= \theta(X\cap G_n\cap G_{n-1}) + \theta(X\cap G_n\backslash G_{n-1}) \\
&= \theta(X\cap G_{n-1}) + \theta(X\cap H_n).
\end{aligned}
$$

Hence by induction we get

(1) $$\theta(X\cap G_n) = \sum_{m\le n} \theta(X\cap H_m) \quad \text{for all } n\in\omega.$$

Now $X = (X \cap F) \cup (X \backslash F)$, so by the outer measure property we have

$$\theta(X) \leq \theta(X \cap F) + \theta(X \backslash F). \tag{2}$$

Now $X \cap F = \bigcup_{n \in \omega}(X \cap H_n)$, so by the outer measure property we have

$$\begin{aligned}\theta(X \cap F) &\leq \sum_{n \in \omega} \theta(X \cap H_n) \\ &= \lim_{m \to \infty} \sum_{n \leq m} \theta(X \cap H_n) \\ &= \lim_{m \to \infty} \theta(G_m) \quad \text{by (1)}\end{aligned}$$

Thus

$$\theta(X \cap F) \leq \lim_{m \to \infty} \theta(X \cap G_m) \tag{3}$$

Next, note that $G_n \subseteq \bigcup_{m \in \omega} G_m$ and hence $X \backslash \bigcup_{m \in \omega} G_m \subseteq X \backslash G_n$. Also $G_m \subseteq G_n$ for $m \leq n$, and hence $X \backslash G_n \subseteq X \backslash G_m$. Thus by (2) in the definition of outer measure we have

$$\theta(X \backslash F) = \theta\left(X \backslash \bigcup_{m \in \omega} G_m\right) \leq \inf_{m \in \omega} \theta(X \backslash G_m) = \lim_{m \to \omega} \theta(X \backslash G_m).$$

Together with (3) it then follows that

$$\theta(X \cap F) + \theta(X \backslash F) \leq \lim_{m \to \omega} (\theta(X \cap G_m) + \theta(X \backslash G_m)) = \theta(X).$$

Together with (2) this implies that $F \in A$.

Thus A is a σ-algebra of subsets of ${}^{\kappa}2$.

Now suppose that $E_n \cap E_m = \emptyset$ for $n \neq m$. Then

$$\theta(G_{n+1}) = \theta(G_{n+1} \cap E_{n+1}) + \theta(G_{n+1} \backslash E_{n+1}) = \theta(E_{n+1}) + \theta(G_n).$$

It follows by induction that $\theta(G_n) = \sum_{m \leq n} \theta(E_m)$ for every $n \in \omega$. Hence

$$\theta(F) \leq \sum_{m \in \omega} \theta(E_m) \quad \text{since } \theta \text{ is an outer measure} \tag{4}$$

and

$$\theta(F) \geq \theta(\bigcup_{m \leq n} E_m) = \theta(G_n) = \sum_{m \leq n} \theta(E_m)$$

for each n, and hence $\theta(F) \geq \sum_{m \in \omega} E_m$. Now (4) gives $\theta(F) = \sum_{m \in \omega} \theta(E_m)$. □

Proposition 36.7. *If $\varepsilon \in 2$ and $\alpha < \kappa$, then $\{f \in {}^{\kappa}2 : f(\alpha) = \varepsilon\} \in A$.*

Proof. Let $E = \{f \in {}^\kappa 2 : f(\alpha) = \varepsilon\}$, and let $X \subseteq {}^\kappa 2$; we want to show that $\theta(X) = \theta(X \cap E) + \theta(X \backslash E)$. $\leq$ holds by the definition of outer measure. Now suppose that $\delta > 0$. Choose $C \in {}^\omega\mathcal{C}$ such that $X \subseteq \bigcup_{n\in\omega} C_n$ and $\sum_{n\in\omega} \theta_0(C_n) < \theta(X) + \delta$. For each $n \in \omega$ let $C_n = U_{f_n}$ with $f_n \in \mathrm{Fn}(\kappa, 2, \omega)$. For each $n \in \omega$, if $\alpha \notin \mathrm{dmn}(f_n)$, replace C_n by U_g and U_h, where $g = f \cup \{(\alpha, 0)\}$ and $h = f \cup \{(\alpha, 1)\}$; let the new sequence be $C' \in {}^\omega\mathcal{C}$. Then $\sum_{n\in\omega} \theta(C_n) = \sum_{n\in\omega} \theta(C'_n)$ and $X \subseteq \bigcup_{n\in\omega} C'_n$. Then there is a partition M, N of ω such that $X \cap E \subseteq \bigcup_{n\in M} C'_n$ and $X \backslash E \subseteq \bigcup_{n\in N} C'_n$. Hence

$$\theta(X \cap E) + \theta(X \backslash E) \leq \sum_{n\in M} \theta(C'_n) + \sum_{n\in N} \theta(C'_n) = \sum_{n\in\omega} \theta(C'_n) < \theta(X) + \delta.$$

Since δ is arbitrary, it follows that $\theta(X) = \theta(X \cap E) + \theta(X \backslash E)$. $\square$

For $f : 2 \to \mathbb{R}$ we define $\int f = \frac{1}{2} f(0) + \frac{1}{2} f(1)$.

Proposition 36.8. *If $f_n : 2 \to [0, \infty)$ for each $n \in \omega$ and $\forall t < 2[\sum_{n\in\omega} f_n(t) < \infty]$, then $\sum_{n\in\omega} \int f_n < \infty$, and $\sum_{n\in\omega} \int f_n = \int \sum_{n\in\omega} f_n$.*

Proof.

$$\int \sum_{n\in\omega} f_n = \frac{1}{2} \sum_{n\in\omega} f_n(0) + \frac{1}{2} \sum_{n\in\omega} f_n(1) = \sum_{n\in\omega} \left(\frac{1}{2} f_n(0) + \frac{1}{2} f_n(1) \right) = \sum_{n\in\omega} \int f_n.$$

Proposition 36.9. $\theta({}^\kappa 2) = 1$.

Proof. It is obvious that ${}^\kappa 2 \in A$, and that $\theta({}^\kappa 2) \leq \theta_0({}^\kappa 2) = 1$. Suppose that $\theta({}^\kappa 2) < 1$. Choose $C \in {}^\omega\mathcal{C}$ such that $2^\kappa = \bigcup_{n\in\omega} C_n$ and $\sum_{n\in\omega} \theta_0(C_n) < 1$. For each $n \in \omega$ let $C_n = U_{f_n}$, where $f_n \in \mathrm{Fn}(\kappa, 2, \omega)$.

(1) $\forall g \in \mathrm{Fn}(\kappa, 2, \omega) \exists n \in \omega [f_n \subseteq g \text{ or } g \subseteq f_n]$.

In fact, let $g \in \mathrm{Fn}(\kappa, 2, \omega)$. Let $h \in {}^\kappa 2$ with $g \subseteq h$. Choose n such that $h \in C_n$. Then $f_n \subseteq h$. So $f_n \subseteq g$ or $g \subseteq f_n$.

(2) Let $M = \{n \in \omega : \forall m \neq n [f_m \not\subseteq f_n]\}$. Then ${}^\kappa 2 \subseteq \bigcup_{n\in M} U_{f_n}$.

For, given $g \in {}^\kappa 2$ choose $n \in \omega$ such that $g \in C_n$. Thus $f_n \subseteq g$. Let $m \in \omega$ with $f_m \subseteq f_n$ and $|\mathrm{dmn}(f_m)|$ minimum. Then $f_m \subseteq g$ and $m \in M$, as desired.

(3) $|M| \geq 2$.

In fact, obviously $M \neq \emptyset$. Suppose that $M = \{n\}$. Since $\sum_{n\in M} \theta_0(C_n) < 1$, we have $f_n \neq \emptyset$. Then ${}^\kappa 2 \subseteq U_{f_n}$, contradiction.

(4) M is infinite.

In fact, suppose that M is finite, and let $m = \sup\{|\mathrm{dmn}(f_n)| : n \in M\}$. Let $g \in \mathrm{Fn}(\kappa, 2, \omega)$ be such that $|\mathrm{dmn}(g)| = m + 1$. Then by (1), $f_n \subseteq g$ for all $n \in M$. Because of (3), this contradicts (2).

Let $J = \bigcup_{n\in M} \mathrm{dmn}(f_n)$.

(5) J is infinite.

For, suppose that J is finite. Now $M = \bigcup_{G \subseteq J} \{n \in M : \mathrm{dmn}(f_n) = G\}$, so there is a $G \subseteq J$ such that $\{n \in M : \mathrm{dmn}(f_n) = G\}$ is infinite. But clearly $|\{n \in M : \mathrm{dmn}(f_n) = G\}| \leq 2^{|G|}$, contradiction.

Let $i : \omega \to J$ be a bijection. For $n, k \in \omega$ let f'_{nk} be the restriction of f_n to the domain $\{\alpha \in \mathrm{dmn}(f_n) : \forall j < k[\alpha \neq i_j]\}$, and let

$$\alpha_{nk} = \frac{1}{2^{|\mathrm{dmn}(f'_{nk})|}}.$$

Now for $n, k \in \omega$ and $t < 2$ we define

$$f_{nk}(t) = \begin{cases} \alpha_{n,k+1} & \text{if } i_k \notin \mathrm{dmn}(f_n), \\ \alpha_{n,k+1} & \text{if } i_k \in \mathrm{dmn}(f_n) \text{ and } f_n(i_k) = t, \\ 0 & \text{otherwise.} \end{cases}$$

(6) $\int f_{nk} = \alpha_{nk}$ for all $n, k \in \omega$.

In fact,

$$\begin{aligned} \int f_{nk} &= \frac{1}{2} f_{nk}(0) + \frac{1}{2} f_{nk}(1) \\ &= \begin{cases} \alpha_{n,k+1} & \text{if } i_k \notin \mathrm{dmn}(f_n), \\ \frac{1}{2}\alpha_{n,k+1} & \text{if } i_k \in \mathrm{dmn}(f_n) \end{cases} \\ &= \alpha_{nk}. \end{aligned}$$

Now we define by induction elements $t_k \in 2$ and subsets M_k of M. Let $M_0 = M$. Now suppose that M_k and t_i have been defined for all $i < k$, so that $\sum_{n \in M_k} \alpha_{nk} < 1$. Note that this holds for $k = 0$. Now

$$\begin{aligned} 1 > \sum_{n \in M_k} \alpha_{nk} &= \sum_{n \in M_k} \int f_{nk} \quad \text{by (6)} \\ &= \int \sum_{n \in M_k} f_{nk} \quad \text{by Proposition 1.} \end{aligned}$$

It follows that there is a $t_k < 2$ such that $\left(\sum_{n \in M_k} f_{nk}\right)(t_k) < 1$. Let

$$M_{k+1} = \{n \in M : \forall j < k+1[i_j \notin \mathrm{dmn}(f_n), \text{ or } i_j \in \mathrm{dmn}(f_n) \text{ and } f_n(i_j) = t_j]\}.$$

If $n \in M_{k+1}$, then $f_{nk}(t_k) = \alpha_{n,k+1}$. Hence

$$\sum_{n \in M_{k+1}} \alpha_{n,k+1} = \sum_{n \in M_{k+1}} f_{mk}(t_k) \leq \left(\sum_{n \in M_k} f_{nk}\right)(t_k) < 1.$$

Also, $M_{k+1} \neq \emptyset$. For, let $g \in {}^{\kappa}2$ such that $g(i_j) = t_j$ for all $j \leq k$. Say $g \in C_n$ with $n \in M$. Then $f_n \subseteq g$. Hence $i_j \notin \mathrm{dmn}(f_n)$, or $i_j \in \mathrm{dmn}(f_n)$ and $f_n(i_j) = t_j$. Thus $n \in M_{k+1}$.

This finishes the construction. Now let $g \in {}^{\kappa}2$ be such that $g(i_j) = t_j$ for all $j \in \omega$. Say $g \in C_n$ with $n \in M$. Then $f_n \subseteq g$. The domain of f_n is a finite subset of J. Choose $k \in \omega$ so that $\mathrm{dmn}(f_n) \subseteq \{i_j : j < k\}$. Then $n \in M_k$. Hence $f'_{nk} = \emptyset$ and so $\alpha_{nk} = 1$. This contradicts $\sum_{m \in M_k} \alpha_{mk} < 1$. □

Let ν be the tiny function with domain 2 which interchanges 0 and 1. For any $f \in {}^{\kappa}2$ let $F(f) = \nu \circ f$.

Proposition 36.10.

(i) F is a permutation of ${}^{\kappa}2$.

(ii) For any $f \in \mathrm{Fn}(\kappa, 2, \omega)$ we have $F[U_f] = U_{\nu \circ f}$.

(iii) For any $X \subseteq {}^{\kappa}2$ we have $\theta(X) = \theta(F[X])$.

(iv) $\forall E \in A[F[E] \in A]$.

Proof. (i): Clearly F is one-one, and $F(F(f)) = f$ for any $f \in {}^{\kappa}2$. So (i) holds.

(ii): For any $g \in {}^{\kappa}2$,

$$\begin{aligned}
g \in F[U_f] \quad &\text{iff} \quad \exists h \in U_f[g = F(h)] \\
&\text{iff} \quad \exists h \in {}^{\kappa}2[f \subseteq h \text{ and } g = \nu \circ h] \\
&\text{iff} \quad \exists h \in {}^{\kappa}2[\nu \circ f \subseteq \nu \circ h \text{ and } g = \nu \circ h] \\
&\text{iff} \quad \nu \circ f \subseteq g \\
&\text{iff} \quad g \in U_{\nu \circ f}
\end{aligned}$$

(iii): Clearly $\theta_0(U_f) = \theta_0(F[U_f])$ for any $f \in \mathrm{Fn}(\kappa, 2, \omega)$. Also, $A \subseteq \bigcup_{n \in \omega C_n}$ iff $F[A] \subseteq \bigcup_{n \in \omega} F[C_n]$. So (iii) holds.

(iv): Suppose that $E \in A$. Let $X \subseteq {}^{\kappa}2$. Then

$$\begin{aligned}
\theta(X \cap F[E]) + \theta(X \backslash F[E]) &= \theta(F[F[X]] \cap F[E]) + \theta(F[F[X]] \backslash F[E]) \\
&= \theta(F[F[X] \cap E]) + \theta(F[F[X] \backslash E]) \\
&= \theta(F[X] \cap E) + \theta(F[X] \backslash E) \\
&= \theta(E) = \theta(F[E]).
\end{aligned}$$

□

Proposition 36.11. *If $\alpha < \kappa$ and $\varepsilon < 2$, then $\theta(U_{\{(\alpha,\varepsilon)\}}) = \frac{1}{2}$.*

Proof. By Proposition 36.10 we have $\theta(U_{\{(\alpha,\varepsilon)\}}) = \theta(U_{\{(\alpha,1-\varepsilon)\}})$, so the result follows from Proposition 36.9. □

Proposition 36.12. *For each $f \in \mathrm{Fn}(\kappa, 2, \omega)$ we have $U_f \in A$ and $\theta(U_f) = \frac{1}{2^{|\mathrm{dmn}(f)|}}$.*

Proof. We have $U_f = \bigcap_{\alpha \in \mathrm{dmn}(f)} U_{\{(\alpha, f(\alpha))\}}$, so $U_f \in A$ by Proposition 36.7. We prove that $\theta(U_f) = \frac{1}{2^{|\mathrm{dmn}(f)|}}$ by induction on $|\mathrm{dmn}(f)|$. For $|\mathrm{dmn}(f)| = 1$, this holds by Proposition 36.11. Now assume that it holds for $|\mathrm{dmn}(f)| = m$. For any f with $|\mathrm{dmn}(f)| =$

m and $\alpha \notin \text{dmn}(f)$ we have $2^{-|\text{dmn}(f)|} = \theta(U_f) = \theta(U_{f\cup\{(\alpha,0)\}}) + \theta(U_{f\cup\{(\alpha,1)\}})$. Since $\theta(U_{f\cup\{(\alpha,\varepsilon)\}}) \le \theta_0(U_{f\cup\{(\alpha,\varepsilon)\}}) = 2^{-|\text{dmn}(f)|-1}$ for each $\varepsilon \in 2$, it follows that $\theta(U_{f\cup\{(\alpha,\varepsilon)\}}) = 2^{-|\text{dmn}(f)|-1}$ for each $\varepsilon \in 2$. $\square$

Proposition 36.13. *If F is a finite subset of ${}^\kappa 2$, then $F \in A$ and $\theta(F) = 0$.*

Proof. This is obvious if $|F| \le 1$, and then the general case follows. $\square$

This finishes our development of measure theory. Now we start to see how a forcing order is obtained.

For any BA A, an *ideal* of A is a nonempty subset of A such that if $a, b \in A$, $a \le b$, and $b \in I$, then also $a \in I$; and if $a, b \in I$, then $a + b \in I$.

Proposition 36.14. *Let I be an ideal in a BA A. Define $\equiv_I = \{(a, b) : a, b \in A$ and $a \triangle b \in I\}$. Then $\equiv_I$ is an equivalence relation on A, and the collection of all equivalence classes can be made into a BA $(A/I, +, \cdot, -, [0]_I, [1]_I)$ such that the following conditions hold for all $a, b \in A$:*

(i) $[a]_I + [b]_I = [a + b]_I$.

(ii) $[a]_I \cdot [b]_I = [a \cdot b]_I$.

(iii) $-[a]_I = [-a]_I$.

Proof. Clearly $\equiv_I$ is reflexive on A and symmetric. Now suppose that $a \equiv_I b \equiv_I c$. Thus $a \triangle b \in I$ and $b \triangle c \in I$. Hence $a \cdot -c = a \cdot b \cdot -c + a \cdot -b \cdot -c \le b \triangle c + a \triangle b \in I$. Hence $a \cdot -c \in I$. Similarly $c \cdot -a \in I$, so $a \triangle c \in I$; thus $a \equiv_I c$.

Suppose that $a \equiv_I a'$ and $b \equiv_I b'$. Then

$$(a + b) \cdot -(a' + b') = a \cdot -a' \cdot -b' + b \cdot -a' \cdot -b' \le a \triangle a' + b \triangle b' \in I.$$

So $(a + b) \cdot -(a' + b') \in I$. Similarly $(a' + b') \cdot -(a + b) \in I$, so $(a + b) \triangle (a' + b') \in I$. Hence $(a + b) \equiv_I (a' + b')$. This shows that (i) is well-defined.

Similarly,

$$a \cdot b \cdot -(a' \cdot b') = a \cdot b \cdot -a' + a \cdot b \cdot -b' \le a \triangle a' + b \triangle b' \in I,$$

so $a \cdot b \cdot -(a' \cdot b') \in I$. Similarly $a' \cdot b' \cdot -(a \cdot b) \in I$, so $(a \cdot b) \triangle (a' \cdot b') \in I$, so $[a \cdot b]_I = [a' \cdot b']_I$, and (ii) is well-defined.

Also, $(-a) \triangle (-a') = a \triangle a' \in I$, so $[-a]_I = [-a']_I$, and (iii) holds.

Now it is straightforward to check that $(A/I, +, \cdot, -, [0]_I, [1]_I)$ is a BA. $\square$

Now the *random forcing order on κ* is $((A/I) \backslash \{[0]_I\}, \le, [1]_I)$, with A as in the above material on measure, and I is the ideal of members of A of measure 0. We denote it by ran_κ. For each $[a]_I$ in ran_κ we define $\theta([a]_I) = \theta(a)$. Clearly this definition is unambiguous.

Proposition 36.15. *ran_κ has ccc.*

Proof. Suppose to the contrary that $X \in [\text{ran}_\kappa]^{\omega_1}$ is pairwise disjoint. Then $X = \bigcup_{n\in\omega}\{x \in X : \theta(x) \ge \frac{1}{n+1}\}$, so we can choose $X' \in [X]^{\omega_1}$ and n such that $\theta(x) \ge \frac{1}{n+1}$

for all $x \in X'$. Write $x = [a_x]_I$ for each $x \in X'$. Let $y : n+2 \to X'$ be one-one. For each $i < n+2$ let $b_i = a_{y_i} \prod_{j<i} -a_{y_j}$. Then $\langle b_i : i < n+2 \rangle$ is a system of pairwise disjoint elements of A, and $\theta(b_i) = \theta(a_{j_i}) \geq \frac{1}{n+1}$ for all $i < n+2$. Hence $\theta(\sum_{i<n+2} b_i) = \sum_{i<n+2} \theta(b_i) \geq \frac{n+2}{n+1}$, contradiction. □

It follows that forcing with ran_κ preserves cofinalities and cardinals. If G is ran_κ-generic over a c.t.m. M, then for each $\alpha < \kappa$ one of the elements $[U_{\{(\alpha,0)\}}]_I$, $[U_{\{(\alpha,1)\}}]_I$ is in G since $\langle [U_{\{(\alpha,0)\}}]_I, [U_{\{(\alpha,1)\}}]_I \rangle$ is a maximal antichain. This gives a function $f : \kappa \to 2$. Its restriction to ω is a *random real*.

A BA A is *σ-complete* iff any countable subset of A has a sum.

Lemma 36.16. *If A is a σ-complete BA satisfying ccc, then A is complete.*

Proof. Let X be any subset of A; we want to show that it has a sum. By Zorn's lemma, let Y be a maximal set subject to the following conditions: Y consists of pairwise disjoint elements, and for any $y \in Y$ there is an $x \in X$ such that $y \leq x$. By ccc, Y is countable, and so $\sum Y$ exists. We claim that $\sum Y$ is the least upper bound of X.

Suppose that $x \in X$ and $x \not\leq \sum Y$. Then $x \cdot -\sum Y \neq 0$, and $Y \cup \{x \cdot -\sum Y\}$ properly contains Y and satisfies both of the conditions defining Y, contradiction. Hence $x \leq \sum Y$. So $\sum Y$ is an upper bound for X.

Suppose that z is any upper bound for X, but suppose that $\sum Y \not\leq z$. Thus $\sum Y \cdot -z \neq 0$, so by 2.2 there is a $y \in Y$ such that $y \cdot -z \neq 0$. Choose $x \in X$ such that $y \leq x$. Now $x \leq z$, so $y \cdot -z \leq z$, hence $y \cdot -z = 0$, contradiction. □

Lemma 36.17. *A/I is complete.*

Proof. By Proposition 36.15 and Lemma 36.16 it suffices to show that it is σ-complete. So, suppose that X is a countable subset of A/I. We can write $X = \{[y]_I : y \in Y\}$ for some countable subset Y of A. We claim that $[\bigcup Y]_I$ is the least upper bound for X. For, if $x \in X$, choose $y \in Y$ such that $x = [y]_I$. Then $y \subseteq \bigcup Y$, so $x \leq [\bigcup Y]_I$. Now suppose that $[z]_I$ is any upper bound of X. Then $[y]_I \leq [z]_I$ for any $y \in Y$, so $y \backslash z \in I$, i.e., $\theta(y \backslash z) = 0$, for any $y \in Y$. Hence

$$\theta\left(\bigcup Y \backslash z\right) \leq \sum_{y \in Y} \theta(y \backslash z) = 0;$$

so $[\bigcup Y]_I \leq [z]_I$, as desired. □

Theorem 36.18. *There is an isomorphism f of $\mathrm{RO}(\mathrm{ran}_\kappa)$ onto A/I such that $f(i(a)) = a$ for every $a \in \mathrm{ran}_\kappa$, where i is as in the definition of* RO.

Proof. Define $j : \mathrm{ran}_\kappa \to A/I$ by setting $j(a) = a$ for all $a \in \mathrm{ran}_\kappa$. Then the following conditions are clear:

(1) $j[\mathrm{ran}_\kappa]$ is dense in A/I. (In fact, $j[\mathrm{ran}_\kappa]$ consists of all nonzero elements of A/I.)

(2) If $a, b \in \mathrm{ran}_\kappa$ and $a \leq b$, then $j(a) \leq j(b)$.

(3) If $a, b \in \mathrm{ran}_\kappa$ and $a \perp b$, then $j(a) \cdot j(b) = 0$.

Hence our theorem follows from Theorem 13.22. $\square$

We now need the following general result about Boolean values.

Proposition 36.19. $[\![\exists x \in \check{A}\varphi(x)]\!] = \sum_{x\in A}[\![\varphi(\check{x})]\!]$.

Proof. Let $X = \{[\![\psi(\check{x})]\!] : x \in A\}$. First we show that $[\![\exists x \in \check{A}\varphi(x)]\!]$ is an upper bound for X. In fact, if $x \in A$ then $[\![\check{x} \in \check{A}]\!] = 1$ since $\check{x}_G = x \in A = \check{A}_G$ for any generic G, so that $1 \Vdash \check{x} \in \check{A}$. Hence

$$[\![\varphi(\check{x})]\!] = [\![\check{x} \in \check{A}]\!] \cdot [\![\varphi(\check{x})]\!] = [\![\check{x} \in \check{A} \wedge \varphi(\check{x})]\!] \leq [\![\exists x[x \in \check{A} \wedge \varphi(x)]]\!] = [\![\exists x \in \check{A}\varphi(x)]\!].$$

Now suppose that a is an upper bound for X, but $[\![\exists x \in \check{A}\varphi(x)]\!] \not\leq a$. Thus by definition,

$$\left(\sum_{\tau\in V^P}[\![\tau \in \check{A} \wedge \varphi(\tau)]\!]\right) \cdot -a \neq 0,$$

so there is a $\tau \in V^P$ such that $[\![\tau \in \check{A} \wedge \varphi(\tau)]\!] \cdot -a \neq 0$. Hence $\left(\sum_{b\in A}[\![\tau = \check{b} \wedge \varphi(\tau)]\!]\right) \cdot -a \neq 0$, so there is a $b \in A$ such that $[\![\tau = \check{b} \wedge \varphi(\tau)]\!] \cdot -a \neq 0$. But $[\![\tau = \check{b} \wedge \varphi(\tau)]\!] \leq [\![\varphi(\check{b})]\!]$, so this is a contradiction. $\square$

Theorem 36.20. *Suppose that M is a c.t.m. of ZFC, and $A, I, \mathrm{ran}_\kappa$ are as above, all in M. Suppose that G is ran_κ-generic over M, and $f \in {}^\omega\omega$ in $M[G]$. Then there is an $h \in M \cap {}^\omega\omega$ such that $f(n) < h(n)$ for all $n \in \omega$.*

Proof. Let σ be a ran_κ-name such that $\sigma_G = f$, and let $p \in \mathrm{ran}_\kappa$ be such that $p \Vdash \sigma : \omega \to \omega$. We claim that

$$E \stackrel{\text{def}}{=} \{q \in \mathrm{ran}_\kappa : \text{there is an } h \in {}^\omega\omega \text{ such that } q \Vdash \forall n \in \omega(\sigma(n) < \check{h}(n))\}$$

is dense below p. Clearly this gives the conclusion of the theorem.

To prove this, take any $r \leq p$; we want to find $q \in E$ such that $q \leq r$. Let k be the isomorphism from $\mathrm{RO}(\mathrm{ran}_\kappa)$ to A/I given by Theorem 36.18. Now temporarily fix $n \in \omega$. Let $i : \mathrm{ran}_\kappa \to \mathrm{RO}(\mathrm{ran}_\kappa)$ be the mapping from Chapter 9. Then by Proposition 36.19,

$$i(r) \leq [\![\exists m \in \omega(\sigma(\check{n}) < m)]\!] = \sum_{m\in\omega}[\![\sigma(\check{n}) < \check{m}]\!],$$

Applying k, we get

$$r \leq \sum_{m\in\omega} k([\![\sigma(\check{n}) < \check{m}]\!]). \tag{1}$$

Let $r \cdot k([\![\sigma(\check{n}) < \check{m}]\!]) = [a_m]$ for each $m \in \omega$. Now clearly if $m < p$, then $r \Vdash \sigma(\check{n}) < \check{m} \to \sigma(\check{n}) < \check{p}$, so $[a_m] \leq [a_p]$. Let $b_m = \bigcup_{p\leq m} a_p$ for each $m \in \omega$. Then $[a_m] = [b_m]$ for each m.

(2) $\sum_{m\in\omega}[b_m] = \left[\bigcup_{m\in\omega} b_m\right]$.

In fact, $\left[\bigcup_{m\in\omega} b_m\right]$ is clearly an upper bound for $\{[b_m] : m \in \omega\}$. If $[c]$ is any upper bound, then $\mu(b_m\backslash c) = 0$ for each m, and hence $\theta\left(\bigcup_{m\in\omega} b_m\backslash c\right) = 0$, so that $\left[\bigcup_{m\in\omega} b_m\right] \leq [c]$. So (2) holds.

Note that $r = \left[\bigcup_{m\in\omega} b_m\right]$; so $\theta(r) = \theta\left(\bigcup_{m\in\omega} b_m\right)$. By Proposition 36.5(iii) we get $\theta(r) = \sup\{\theta(b_m) : m \in \omega\}$. So we can choose $m \in \omega$ such that $\theta(b_m) \geq \theta(r) - \frac{1}{2^{n+2}}\theta(r)$. Let $h(n)$ be the least such m. Thus

$$\theta(r\backslash b_{h(n)}) = \theta(r) - \theta(b_{h(n)}) \leq \frac{1}{2^{n+2}}\theta(r). \tag{3}$$

Now

$$\begin{aligned}\theta\left(r\backslash \bigcap_{n\in\omega} b_{h(n)}\right) &= \theta\left(\bigcup_{n\in\omega}(r\backslash b_{h(n)})\right)\\ &\leq \sum_{n\in\omega}\frac{1}{2^{n+2}}\theta(r)\\ &= \frac{1}{2}\theta(r).\end{aligned}$$

It follows that

$$\theta\left(\bigcap_{n\in\omega} b_{h(n)}\right) > 0.$$

Let $q = \bigcap_{n\in\omega} b_{h(n)}$. So $[q] \in \mathrm{ran}_\kappa$. We claim that $[q] \leq r$ and $[q] \Vdash \forall n \in \omega(\sigma(n) < \check{h(n)})$. For, suppose that $[q] \in G$ with G ran_κ-generic over M, and suppose that $n \in \omega$. Then $[q] \leq [b_{h(n)} = [a_{h(n)} \leq r$. and also $[q] \leq k(\llbracket\sigma(checkn) < \check{h(n)}\rrbracket)$. Hence $i([q]) \leq \llbracket\sigma(checkn) < \check{h(n)}\rrbracket$, hence $[q] \Vdash \sigma(\check{n}) < \check{h(n)}$. Thus $[q] \in E$, as desired. □

Corollary 36.21. *Suppose that M is a c.t.m. of ZFC, and $\mathbb{P}_r$ is considered in M. Suppose that G is $\mathbb{P}_r$-generic over M. Then no $f \in {}^\omega\omega$ in $M[G]$ is a Cohen real.*

Proof. By Theorem 36.3 and Theorem 36.20. □

Thus we may say that adding a random real does not add a Cohen real.

Roitman, J. [79] *Adding a random or a Cohen real...* Fund. Math. 103 (1979), 47–60.

Sacks forcing

Let Seq be the set of all finite sequences of 0's and 1's. A *perfect tree* is a nonempty subset T of Seq with the following properies:

(1) If $t \in T$ and $m < \mathrm{dmn}(t)$, then $t \restriction m \in T$.

(2) For any $t \in T$ there is an $s \in T$ such that $t \subseteq s$ and $s^\frown\langle 0\rangle, s^\frown\langle 1\rangle \in T$.

Thus Seq itself is a perfect tree. *Sacks forcing* is the collection $\mathbb{Q}$ of all perfect trees, ordered by $\subseteq$ (not by $\supseteq$).

Note that an intersection of perfect trees does not have to be perfect. For example (with $\varepsilon_1, \varepsilon_2, \ldots$ any members of 2):

$$\begin{aligned} p &= \{\emptyset, \langle 0\rangle, \langle 0\varepsilon_1\rangle, \langle 0\varepsilon_1\varepsilon_2\rangle, \ldots\}; \\ q &= \{\emptyset, \langle 1\rangle, \langle 1\varepsilon_1\rangle, \langle 1\varepsilon_1\varepsilon_2\rangle, \ldots\}. \end{aligned}$$

Also, one can have p, q perfect, $p \cap q$ not perfect, but $r \subseteq p \cap q$ for some perfect r:

$$\begin{aligned} p = \{&\emptyset, \langle 1\rangle, \langle 1\varepsilon_1\rangle, \langle 1\varepsilon_1\varepsilon_2\rangle, \ldots \\ &\langle 0\rangle, \langle 01\rangle, \langle 01\varepsilon_2\rangle, \langle 01\varepsilon_2\varepsilon_3\rangle \ldots\}; \\ q = \{&\emptyset, \langle 1\rangle, \langle 1\varepsilon_1\rangle, \langle 1, \varepsilon_1\varepsilon_2\rangle, \ldots \\ &\langle 0\rangle, \langle 00\rangle, \langle 00\varepsilon_2\rangle, \langle 00\varepsilon_2\varepsilon_3\rangle \ldots\}; \\ r = \{&\emptyset, \langle 1\rangle, \langle 1\varepsilon_1\rangle, \langle 1, \varepsilon_1\varepsilon_2\rangle, \ldots\}. \end{aligned}$$

Theorem 36.22. *Suppose that M is a c.t.m. of ZFC. Consider $\mathbb{Q}$ within M, and let G be $\mathbb{Q}$-generic over M. Then the set*

$$\{s \in Seq : s \in p \text{ for all } p \in G\}$$

is a function from ω into 2.

Proof. For each $n \in \omega$ let

$$D_n = \{p \in \mathbb{Q} : \text{there is an } s \in \text{Seq such that } \mathrm{dmn}(s) = n \text{ and } s \subseteq t \text{ or } t \subseteq s \text{ for all } t \in p\}.$$

Then D_n is dense: if $q \in \mathbb{Q}$, choose any $s \in q$ such that $\mathrm{dmn}(s) = n$, and let $p = \{t \in q : s \subseteq t \text{ or } t \subseteq s\}$. Clearly $p \in D_n$ and $p \subseteq q$.

Now for each $n \in \omega$ let $p^{(n)}$ be a member of $G \cap D_n$, and choose $s^{(n)}$ accordingly.

(1) If $m < n$, then $s^{(m)} \subseteq s^{(n)}$.

In fact, choose $r \in G$ such that $r \subseteq p^{(m)} \cap p^{(n)}$. Then $s^{(m)} \subseteq t$ and $s^{(n)} \subseteq t$ for all $t \in r$ with $\mathrm{dmn}(t) \geq n$, so $s^{(m)} \subseteq s^{(n)}$.

(2) $s^{(m)} \in q$ for all $q \in G$.

In fact, let $q \in G$, and choose $r \in G$ such that $r \subseteq q$ and $r \subseteq p^{(m)}$. Take $t \in r$ with $\mathrm{dmn}(t) = m$. then $t = s^{(m)}$ since $r \subseteq p^{(m)}$. Thus $s^{(m)} \in q$ since $r \subseteq q$.

(3) If $t \in q$ for all $q \in G$, then $t = s^{(m)}$ for some m.

For, let $\mathrm{dmn}(t) = m$. Since $t \in p^{(m)}$, we have $t = s^{(m)}$.

From (1)–(3) the conclusion of the theorem follows. □

The function described in Theorem 36.19 is called a *Sacks real*.

If $p \in \mathbb{Q}$, a member f of p is a *branching point* iff $f^\frown\langle 0\rangle, f^\frown\langle 1\rangle \in p$.

Sacks forcing does not satisfy ccc:

Proposition 36.23. *There is a family of 2^ω pairwise incompatible members of $\mathbb{Q}$.*

Proof. Let $\mathscr{A}$ be a family of 2^ω infinite pairwise almost disjoint subsets of ω. With each $A \in \mathscr{A}$ we define a sequence $\langle P_{A,n} : n \in \omega \rangle$ of subsets of Seq, by recursion:

$$P_{A,0} = \{\emptyset\};$$
$$P_{A,n+1} = \begin{cases} \{f^\frown\langle 0\rangle : f \in P_{A,n}\} & \text{if } n \notin A, \\ \{f^\frown\langle 0\rangle : f \in P_{A,n}\} \cup \{f^\frown\langle 1\rangle : f \in P_{A,n}\} & \text{if } n \in A. \end{cases}$$

Note that all members of $P_{A,n}$ have domain n. We set $p_A = \bigcup_{n\in\omega} P_{A,n}$. We claim that p_A is a perfect tree. Condition (1) is clear. For (2), suppose that $f \in p_A$; say $f \in P_{A,n}$. Let m be the least member of A greater than n. If g extends f by adjoining 0's from n to $m-1$, then $g^\frown\langle 0\rangle, g^\frown\langle 1\rangle \in p_A$, as desired in (2).

We claim that if $A, B \in \mathscr{A}$ and $A \neq B$, then p_A and p_B are incompatible. For, suppose that q is a perfect tree and $q \subseteq p_A, p_B$. Now $A \cap B$ is finite. Let m be an integer greater than each member of $A \cap B$. Let f be a branching point of q with $\mathrm{dmn}(f) \geq m$; it exists by (2) in the definition of perfect tree. Let $\mathrm{dmn}(f) = n$. Then $f \in P_{A,n}$ and $f^\frown\langle 0\rangle, f^\frown\langle 1\rangle \in P_{A,n+1}$, so $n \in A$ by construction. Similarly, $n \in B$, contradiction. $\square$

Proposition 36.24. $\mathbb{Q}$ *is not ω_1-closed.*

Proof. For each $n \in \omega$ let

$$p_n = \{f \in \mathrm{Seq} : f(i) = 0 \text{ for all } i < n\}.$$

Clearly p_n is perfect, $p_n \subseteq p_m$ if $n > m$, and $\bigcap_{n\in\omega} P_n$ is $\{f\}$ with $f(i) = 0$ for all i, so that the descending sequence $\langle p_n : n \in \omega\rangle$ does not have any member of $\mathbb{Q}$ below it. $\square$

By 36.23 and 36.24, the methods of chapters 16 and 24 cannot be used to show that forcing with $\mathbb{Q}$ preserves cardinals, even if we assume CH in the ground model. Nevertheless, we will show that it does preserve cardinals. To do this we will prove a modified version of ω_1-closure.

If p is a perfect tree, an *n-th branching point* of p is a branching point f of p such that there are exactly n branching points g such that $g \subseteq f$. Thus $n > 0$. For perfect trees p, q and n a positive integer, we write $p \leq_n q$ iff $p \subseteq q$ and every n-th branching point of q is a branching point of p. Also we write $p \leq_0 q$ iff $p \subseteq q$.

Lemma 36.25. *Suppose that $p \subseteq q$ are perfect trees, and $n \in \omega$. Then:*

(i) If $p \leq_n q$, then $p \leq_i q$ for every $i < n$.

(ii) If $p \leq_n q$ and f is an n-th branching point of q, then f is an n-th branching point of p.

(iii) For each positive integer n there is an $f \in p$ such that f is an n-th branching point of q.

(iv) The following conditions are equivalent:

(a) $p \le_n q$.

(b) For every $f \in Seq$, *if* f *is an* n*-th branching point of* q, *then* $f^\frown\langle 0\rangle, f^\frown\langle 1\rangle \in p$.

(v) For each positive integer n *there are exactly* 2^{n-1} n*-th branching points of a perfect tree* p.

(vi) If p *and* q *are perfect trees, then so is* $p \cup q$.

(vii) If p *and* q *are perfect trees, then* $\{r : r$ *is a perfect tree and* $r \subseteq p$ *or* $r \subseteq q\}$ *is dense below* $p \cup q$.

Proof. (i): Assume that $p \le_n q$, $i < n$, and f is an i-th branching point of q. Then since q is perfect there are n-th branching points g, h of q such that $f^\frown\langle 0\rangle \subseteq g$ and $f^\frown\langle 1\rangle \subseteq h$. So $g, h \in p$, hence $f \in p$. This shows that $p \le_i q$.

(ii): Suppose that $p \le_n q$ and f is an n-th branching point of q. Let $r_0, \ldots, r_{n-1}$ be all of the branching points g of q such that $g \subseteq f$. Then by (i), $r_0, \ldots, r_{n-1}$ are all branching points of p. Hence f is an n-th branching point of p.

(iii): Let f be an n-th branching point of p. Then it is an m-th branching point of q for some $m \ge n$. Let r be an n-th branching point of q below f. Then $r \in p$, as desired. [But r might not be a branching point of p.]

(iv), (v), (vi): Immediate from the definitions.

(vii): Suppose that p, q, t are perfect trees and $t \subseteq p \cup q$; we want to find a perfect tree $r \subseteq t$ such that $r \subseteq p$ or $r \subseteq q$. If $t \subseteq p \cap q$, then $r = t$ works. Otherwise, there is some member f of t which is not in both p and q; say $f \in p \backslash q$. Then $r \stackrel{\text{def}}{=} \{g \in t : g \subseteq f$ or $f \subseteq g\}$ is a perfect tree with $r \subseteq t$ and $r \subseteq p$. □

Lemma 36.26. (Fusion lemma) *If* $\langle p_n : n \in \omega\rangle$ *is a sequence of perfect trees and* $\cdots \le_n p_n \le_{n-1} \cdots \le_2 p_2 \le_1 p_1 \le_0 p_0$, *then* $q \stackrel{\text{def}}{=} \bigcap_{n\in\omega} p_n$ *is a perfect tree, and* $q \le_n p_n$ *for all* $n \in \omega$.

Proof. Let n be a positive integer, and let s be an n-th branching point of p_n. If $n \le m$, then $p_m \le_n p_n$, so s is an n-th branching point of p_m; hence $s, s^\frown\langle 0\rangle, s^\frown\langle 1\rangle \in p_m$. It follows that $s, s^\frown\langle 0\rangle, s^\frown\langle 1\rangle \in q$, and s is a branching point of q. Thus we just need to show that q is a perfect tree.

Clearly if $t \in q$ and $n < \mathrm{dmn}(t)$, then $t \restriction n \in q$. Now suppose that $s \in q$; we want to find a $t \in q$ with $s \le t$ and t is a branching point of q. Let $n = \mathrm{dmn}(s)$. Now $s \in p_n$, and p_n has fewer than n elements less than s, so p_n has an n-th branching point $t \ge s$. By the first paragraph, $t \in q$. □

Let p be a perfect tree and $s \in p$. We define

$$p \restriction s = \{t \in p : t \subseteq s \text{ or } s \subseteq t\}.$$

Clearly $p \restriction s$ is still a perfect tree. Now for any positive integer n, let $t_0, \ldots, t_{2^n-1}$ be the collection of all immediate successors of n-th branching points of p. Suppose that for each $i < 2^n$ we have a perfect tree $q_i \le p \restriction t_i$. Then we define the *amalgamation of* $\{q_i : i < 2^n\}$ *into* p to be the set $\bigcup_{i<2^n} q_i$.

Lemma 36.27. *Under the above assumptions, the amalgamation* r *of* $\{q_i : i < 2^n\}$ *into* p *has the following properties:*

(i) r is a perfect tree.
(ii) $r \leq_n p$.

Proof. (i): Suppose that $f \in r$, $g \in \text{Seq}$, and $g \subseteq f$. Say $f \in q_i$ with $i < 2^n$. Then $g \in q_i$, so $g \in r$. Now suppose that $f \in r$; we want to find a branching point of r above f. Say $f \in q_i$. Let g be a branching point of q_i with $f \subseteq g$. Clearly g is a branching point of r.

(ii): Suppose that f is an n-th branching point of p. Then there exist $i, j < 2^n$ such that $f^\frown\langle 0\rangle = t_i$ and $f^\frown\langle 1\rangle = t_j$. So $f^\frown\langle 0\rangle \in q_i \subseteq r$ and $f^\frown\langle 1\rangle = t_j \in q_j \subseteq r$, and so f is a branching point of r. $\square$

Lemma 36.28. *Suppose that M is a c.t.m. of ZFC and we consider the Sacks partial order $\mathbb{Q}$ within M. Suppose that $B \in M$, $\tau \in M^{\mathbb{Q}}$, $p \in \mathbb{Q}$, and $p \Vdash \tau : \check{\omega} \to \check{B}$. Then there is a $q \leq p$ and a function $F : \omega \to [B]^{<\omega}$ in M such that $q \Vdash \tau(\check{n}) \in \check{F}_n$ for every $n \in \omega$.*

Proof. We work entirely within M, except as indicated. We construct two sequences $\langle q_n : n \in \omega\rangle$ and $\langle F_n : n \in \omega\rangle$ by recursion. Let $q_0 = p$. Suppose that q_n has been defined; we define F_n and q_{n+1}. Assume that $q_n \leq p$. Then $q_n \Vdash \tau : \check{\alpha} \to \check{B}$, so $q_n \Vdash \exists x \in \check{B}\tau(\check{n}) = x)$. Let $t_0, \ldots, t_{2^n-1}$ list all of the functions $f^\frown\langle 0\rangle$ and $f^\frown\langle 1\rangle$ such that f is an n-th branching point of q_n. Then for each $i < 2^n$ we have $q_n \restriction t_i \subseteq q_n$, and so $q_n \restriction t_i \Vdash \exists x \in \check{B}\tau(\check{n}) = x)$. Hence there exist an $r_i \subseteq q_n \restriction t_i$ and a $b_i \in B$ such that $r_i \Vdash \tau(\check{n}) = \check{b}_i$. Let q_{n+1} be the amalgamation of $\{r_i : i < 2^n\}$ into q_n, and let $F_n = \{b_i : i < 2^n\}$. Thus $q_{n+1} \leq_n q_n$ by 36.27. Moreover:

$$q_{n+1} \Vdash \tau(\check{n}) \in \check{F}_n. \tag{1}$$

In fact, let G be $\mathbb{Q}$-generic over M with $q_{n+1} \in G$. By 36.22(vii), there is an i such that $r_i \in G$. Since $r_i \Vdash \tau(\check{n}) = \check{b}_i$, it follows that $\tau_G(n) \in F_n$, as desired in (1).

Now with (1) the construction is complete.

By the fusion lemma 36.26 we get $s \leq_n q_n$ for each n. Hence the conclusion of the lemma follows. $\square$

Theorem 36.29. *If M is a c.t.m. of $ZFC + CH$ and $\mathbb{Q} \in M$ is the Sacks forcing partial order, and if G is $\mathbb{Q}$-generic over M, then cofinalities and cardinals are preserved in $M[G]$.*

Proof. Since $|\mathbb{Q}| \leq 2^\omega = \omega_1$ by CH, the poset $\mathbb{Q}$ satisfies the ω_2-chain condition, and so preserves cofinalities and cardinals $\geq \omega_2$. Hence it suffices to show that ω_1^M remains regular in $M[G]$. Suppose not: then there is a function $f : \omega \to \omega_1^M$ in $M[G]$ such that $\text{rng}(f)$ is cofinal in ω_1^M. Hence there is a name τ such that $f = \tau_G$, and hence there is a $p \in G$ such that $p \Vdash \tau : \check{\omega} \to \check{\omega}_1^M$. By Lemma 36.28, choose $q \leq p$ and $F : \omega \to [\omega_1^M]^{<\omega}$ in M such that $q \Vdash \tau(\check{n}) \in \check{F}_n$ for every $n \in \omega$. Take $\beta < \omega_1^M$ such that $\bigcup_{n\in\omega} F_n < \beta$. Now $q \Vdash \exists n \in \omega(\check{\beta} < \tau(\check{n})$, so there exist an $r \leq q$ and an $n \in \omega$ such that $r \Vdash \check{\beta} < \tau(\check{n})$. So we have:

(2) $r \Vdash \tau(\check{n}) \in \check{F}_n$;

(3) $\bigcup_{n\in\omega} F_n < \beta$;

(4) $r \Vdash \check{\beta} < \tau(\check{n})$.

These three conditions give the contradiction $r \Vdash \tau(\check{n}) < \tau(\check{n})$. $\square$

Baumgartner, J.; Laver, R. [79] *Iterated perfect-set forcing.* Ann. Math. Logic 17 (1979), 271–288.

Hechler MAD forcing

A family $\mathscr{A}$ of infinite subsets of ω is *maximal almost disjoint (MAD)* iff any two members of $\mathscr{A}$ are almost disjoint, and $\mathscr{A}$ is maximal with this property. By Theorem 20.1, there is a MAD family of size 2^ω. (Apply Zorn's lemma.)

Theorem 36.30. *Every infinite MAD family of infinite subsets of ω is uncountable.*

Proof. Suppose that $\mathscr{A}$ is a denumerable pairwise almost disjoint family of infinite subsets of ω; we want to extend it. Write $\mathscr{A} = \{A_n : n \in \omega\}$, the A_n's distinct. We define $\langle a_n : n \in \omega\rangle$ by recursion. Suppose that a_m has been defined for all $m < n$. Now $\bigcup_{m<n}(A_m \cap A_n)$ is finite, so we can choose

$$a_n \in A_n \backslash \left(\{a_m : m < n\} \cup \bigcup_{m<n} (A_m \cap A_n) \right).$$

Note that then $a_n \notin A_m$ for any $m < n$. Let $B = \{a_n : n \in \omega\}$. Then B is infinite, and $B \cap A_n \subseteq \{a_m : m \le n\}$. $\square$

Also recall that Martin's axiom implies that every MAD family has size 2^ω; see Theorem 21.7. We now want to introduce a forcing which will make a MAD family of size ω_1, with $\neg$CH.

The members of our partial order $\mathbb{H}$ will be certain pairs (p, q); we define $(p, q) \in \mathbb{H}$ iff the following conditions hold:

(1) p is a function from a finite subset of ω_1 into ${}^n2$ for some $n \in \omega$. We write $n = n_p$.

(2) q is a function with domain contained in $[\mathrm{dmn}(p)]^2$ and range contained in n_p.

(3) If $\{\alpha, \beta\} \in \mathrm{dmn}(q)$ and $q(\{\alpha, \beta\}) = m$, then for every i with $m \le i < n_p$ we have $(p(\alpha))(i) = 0$ or $(p(\beta))(i) = 0$.

Furthermore, for $(p_1, q_1), (p_2, q_2) \in \mathbb{H}$ we define $(p_1, q_1) \le (p_2, q_2)$ iff the following conditions hold:

(4) $\mathrm{dmn}(p_1) \supseteq \mathrm{dmn}(p_2)$.

(5) $p_1(\alpha) \supseteq p_2(\alpha)$ for all $\alpha \in \mathrm{dmn}(p_2)$.

(6) $q_1 \supseteq q_2$.

Note that (5) implies that $n_{p_2} \le n_{p_1}$.

The idea here is to produce almost disjoint sets a_α for $\alpha < \omega_1$; $p(\alpha)$ is the characteristic function of $a_\alpha \cap n_p$, and $a_\alpha \cap a_\beta \subseteq q(\{\alpha, \beta\})$.

Lemma 36.31. *Suppose that* $(p_3, q_3) \le (p_1, q_1), (p_2, q_2)$. *Then* $(p_3, q_1 \cup q_2) \in \mathbb{H}$, *and* $(p_3, q_3) \le (p_3, q_1 \cup q_2) \le (p_1, q_1), (p_2, q_2)$.

Proof. Condition (1) clearly holds for $(p_3, q_1 \cup q_2)$, since it only involves p_3. Clearly $q_1 \cup q_2$ is a relation with domain $\subseteq [\mathrm{dmn}(p_1)]^2 \cup [\mathrm{dmn}(p_2)]^2 \subseteq [\mathrm{dmn}(p_3)]^2$. To show that it is a function, suppose that $\{\alpha, \beta\} \in \mathrm{dmn}(q_1) \cap \mathrm{dmn}(q_2)$. Then $q_1(\{\alpha, \beta\}) = q_3(\{\alpha, \beta\}) = q_2(\{\alpha, \beta\})$. So $q_1 \cup q_2$ is a function, and it clearly maps into $\max(n_{p_1}, n_{p_2}) \le n_{p_3}$. Hence (2) holds for $(p_3, q_1 \cup q_2)$. Finally, suppose that $\{\alpha, \beta\} \in \mathrm{dmn}(q_1 \cup q_2)$. By symmetry, say $\{\alpha, \beta\} \in \mathrm{dmn}(q_1)$. Let $q_1(\{\alpha, \beta\}) = m$, and suppose that $m \le i < n_{p_3}$. Then $q_3(\{\alpha, \beta\}) = q_1(\{\alpha, \beta\}) = m$, so $(p_3(\alpha))(i) = 0$ or $(p_3(\beta))(i) = 0$. So (3) holds. The final inequalities are clear. □

Lemma 36.32. $\mathbb{H}$ *satisfies ccc.*

Proof. Suppose that N is an uncountable subset of $\mathbb{H}$; we want to find two compatible members of N. Now $\langle \mathrm{dmn}(p) : (p, q) \in N \rangle$ is an uncountable system of finite sets, so there exist an uncountable $N' \subseteq N$ and a finite subset H of ω_1 such that $\langle \mathrm{dmn}(p) : (p, q) \in N' \rangle$ is a Δ-system with root H. Next,

$$\begin{aligned} N' = \bigcup_{(f,g) \in J} &\{(p, q) \in N' : p \restriction H = f \text{ and } q \restriction [H]^2 = g\}, \quad \text{where} \\ J = \{&(f, g) : f : H \to \omega,\ g \text{ is a function}, \\ &\mathrm{dmn}(g) \subseteq [H]^2, \ \text{ and } \mathrm{rng}(g) \subseteq \omega\}. \end{aligned}$$

Since J is countable, let $(f, g) \in J$ be such that $N'' \stackrel{\text{def}}{=} \{(p, q) \in N' : p \restriction H = f$ and $q \restriction [H]^2 = g\}$ is uncountable. Now we claim that any two members (p_1, q_1) and (p_2, q_2) of N'' are compatible. Since $p_1 \restriction H = p_2 \restriction H$ and $\mathrm{dmn}(p_1) \cap \mathrm{dmn}(p_2) = H$, the relation $p_1 \cup p_2$ is a function. Say $n_{p_1} \le n_{p_2}$. We now define a function p_3 with domain $\mathrm{dmn}(p_1) \cup \mathrm{dmn}(p_2)$. Let $\alpha \in \mathrm{dmn}(p_1) \cup \mathrm{dmn}(p_2)$. Then we define $p_3(\alpha) : n_{p_2} \to 2$ by setting, for any $i < n_{p_2}$,

$$(p_3(\alpha))(i) = \begin{cases} (p_2(\alpha))(i) & \text{if } \alpha \in \mathrm{dmn}(p_2), \\ (p_1(\alpha))(i) & \text{if } \alpha \in \mathrm{dmn}(p_1) \backslash \mathrm{dmn}(p_2) \text{ and } i < n_{p_1}, \\ 0 & \text{otherwise.} \end{cases}$$

To check that $(p_3, q_1 \cup q_2) \in \mathbb{H}$, first note that (1) is clear. To show that $q_1 \cup q_2$ is a function, suppose that $\{\alpha, \beta\} \in \mathrm{dmn}(q_1) \cap \mathrm{dmn}(q_2)$. Then $\mathrm{dmn}(q_1) \cap \mathrm{dmn}(q_2) \subseteq [\mathrm{dmn}(p_1)]^2 \cap [\mathrm{dmn}(p_2)]^2 = [H]^2$, and it follows that $q_1(\{\alpha, \beta\}) = q_2(\{\alpha, \beta\})$. Thus $q_1 \cup q_2$ is a function. Furthermore,

$$\begin{aligned} \mathrm{dmn}(q_1 \cup q_2) &= \mathrm{dmn}(q_1) \cup \mathrm{dmn}(q_2) \\ &\subseteq [\mathrm{dmn}(p_1)]^2 \cup [\mathrm{dmn}(p_2)]^2 \\ &\subseteq [\mathrm{dmn}(p_1) \cup \mathrm{dmn}(p_2)]^2 \\ &= [\mathrm{dmn}(p_3)]^2. \end{aligned}$$

The range of $q_1 \cup q_2$ is clearly contained in n_{p_2}. So we have checked (2). For (3), suppose that $\{\alpha, \beta\} \in \mathrm{dmn}(q_1 \cup q_2)$, $(q_1 \cup q_2)(\{\alpha, \beta\}) = m$, and $m \le i < n_{p_2}$. We consider some cases:

Case 1. $\{\alpha, \beta\} \in \mathrm{dmn}(q_2)$. Then $\alpha, \beta \in \mathrm{dmn}(p_2)$, so $p_3(\alpha) = p_2(\alpha)$ and $p_3(\beta) = p_2(\beta)$. Hence $(p_3(\alpha))(i) = 0$ or $(p_3(\beta))(i) = 0$, as desired.

Case 2. $\{\alpha, \beta\} \in \mathrm{dmn}(q_1)\backslash\mathrm{dmn}(q_2)$ and $i < n_{p_1}$. Thus $\alpha, \beta \in \mathrm{dmn}(p_1)$. If $\alpha \in \mathrm{dmn}(p_2)$, then $p_1(\alpha) = p_2(\alpha)$, and so $(p_3(\alpha))(i) = (p_1(\alpha))(i)$. If $\alpha \notin \mathrm{dmn}(p_2)$, still $(p_3(\alpha))(i) = (p_1(\alpha))(i)$. Similarly for β, so the desired conclusion follows.

Case 3. $\{\alpha, \beta\} \in \mathrm{dmn}(q_1)\backslash\mathrm{dmn}(q_2)$ and $n_{p_1} \leq i$. Thus again $\alpha, \beta \in \mathrm{dmn}(p_1)$. If one of α, β is not in $\mathrm{dmn}(p_2)$, it follows that one of $(p_3(\alpha))(i)$ or $(p_3(\beta))(i)$ is 0, as desired. Suppose that both are in $\mathrm{dmn}(p_2)$. Then $\{\alpha, \beta\} \subseteq \mathrm{dmn}(p_1) \cap \mathrm{dmn}(p_2) = H$, and hence $\{\alpha, \beta\} \in \mathrm{dmn}(q_2)$, contradiction. □

Theorem 36.33. *Let M be a c.t.m. of ZFC, and consider $\mathbb{H}$ in M. Let G be $\mathbb{H}$-generic over M. Then cofinalities and cardinals are preserved in $M[G]$, and in $M[G]$ there is a MAD family of size ω_1.*

Proof. Cofinalities and cardinals are preserved by 36.32. For each $\alpha < \omega_1$, let

$$x_\alpha = \bigcup\{p(\alpha) : (p, q) \in G \text{ for some } q, \text{ and } \alpha \in \mathrm{dmn}(p)\}.$$

We claim that x_α is a function. For, suppose that $(a, b), (a, c) \in x_\alpha$. By the definition, choose $(p_1, q_1), (p_2, q_2) \in G$ such that $\alpha \in \mathrm{dmn}(p_1)$, $\alpha \in \mathrm{dmn}(p_2)$, $(a, b) \in p_1(\alpha)$, and $(a, c) \in p_2(\alpha)$. Then choose $(p_3, q_3) \in G$ such that $(p_3, q_3) \leq (p_1, q_1), (p_2, q_2)$. By (4) in the definition of $\mathbb{H}$ we have $\alpha \in \mathrm{dmn}(p_3)$, and by (5) we have $(a, b), (a, c) \in p_3(\alpha)$, so $a = c$.

Next we claim that in fact x_α has domain ω. (Its domain is clearly a subset of ω.) For, take any $m \in \omega$. It suffices to show that the set

$$D_{\alpha m} \stackrel{\text{def}}{=} \{(p, q) \in \mathbb{H} : \alpha \in \mathrm{dmn}(p) \text{ and } m \in \mathrm{dmn}(p(\alpha))\}$$

is dense. So, suppose that $(r, s) \in \mathbb{H}$. If $\alpha \in \mathrm{dmn}(r)$, let $t = r$. Suppose that $\alpha \notin \mathrm{dmn}(r)$. Extend r to t by adding the ordered pair $(\alpha, \langle 0 : i < n_r\rangle)$. Clearly $(t, s) \in \mathbb{H}$ and $(t, s) \leq (r, s)$. If $m < n_t$, then $(t, s) \in D_{\alpha m}$, as desired. Suppose that $n_t \leq m$. We now define

$$\begin{aligned} p = \{(\beta, g) : \beta \in \mathrm{dmn}(t),\ g \in {}^{m+1}2,\ t(\beta) \subseteq g,\ \text{and} \\ g(i) = 0 \text{ for all } i \in [n_t, m]\}. \end{aligned}$$

Clearly $(p, s) \in \mathscr{H}$, in fact $(p, s) \in D_{\alpha m}$, and $(p, s) \leq (t, s) \leq (r, s)$, as desired.

So $D_{\alpha m}$ is dense, and hence each x_α is a function mapping ω into 2. We define $a_\alpha = \{m \in \omega : x_\alpha(m) = 1\}$. We claim that $\langle a_\alpha : \alpha < \omega_1$ is our desired MAD family.

Now we show that each a_α is infinite. For each $m \in \omega$ let

$$\begin{aligned} E_m = \{(p, q) \in \mathbb{H} : \alpha \in \mathrm{dmn}(p),\ m < n_p, \text{ and there is} \\ \text{an } i \in [m, n_p) \text{ such that } (p(\alpha))(i) = 1\}. \end{aligned}$$

Clearly in order to show that a_α is infinite it suffices to show that each set E_m is dense. So, suppose that $(r, s) \in \mathbb{H}$. First choose $(t, u) \leq (r, s)$ with $(t, u) \in D_{\alpha 0}$. This is done just to make sure that α is in the domain of t. Let k be the maximum of n_t+1 and $m+1$. Define the function p as follows. $\mathrm{dmn}(p)=\mathrm{dmn}(t)$. For any $\gamma\in\mathrm{dmn}(t)$ and any $i<k$, let

$$(p(\gamma))(i)=\begin{cases}(t(\gamma))(i) & \text{if } i<n_t,\\ 0 & \text{if } n_t\le i \text{ and } \gamma\neq\alpha,\\ 1 & \text{if } n_t\le i \text{ and } \gamma=\alpha.\end{cases}$$

It is easy to check that $(p,u)\in\mathbb{H}$, in fact $(p,u)\in E_m$, and $(p,u)\le(r,s)$, as desired. So each a_α is infinite.

Next we show that distinct a_α,a_β are almost disjoint. Suppose that $\alpha,\beta<\omega_1$ with $\alpha\neq\beta$. Since $D_{\alpha 0}$ and $D_{\beta 0}$ are dense, there are $(p_1,q_1),(p_2,q_2)\in G$ with $\alpha\in\mathrm{dmn}(p_1)$ and $\beta\in\mathrm{dmn}(p_2)$. Choose $(p_3,q_3)\in G$ such that $(p_3,q_3)\le(p_1,q_1),(p_2,q_2)$. Thus $\alpha,\beta\in\mathrm{dmn}(p_3)$. Next we claim:

$$F\stackrel{\text{def}}{=}\{(r,s):\{\alpha,\beta\}\in\mathrm{dmn}(s)\}$$

is dense below (p_3,q_3). In fact, suppose that $(t,u)\le(p_3,q_3)$. We may assume that $\{\alpha,\beta\}\notin\mathrm{dmn}(u)$. Let $\mathrm{dmn}(r)=\mathrm{dmn}(t)$, and for any $\gamma\in\mathrm{dmn}(r)$ let $r(\gamma)$ be the function with domain n_t+1 such that $t(\gamma)\subseteq r(\gamma)$ and $(r(\gamma))(n_t)=0$. Let $\mathrm{dmn}(s)=\mathrm{dmn}(u)\cup\{\{\alpha,\beta\}\}$, with $u\subseteq s$ and $s(\{\alpha,\beta\})=n_t$. It is easily checked that $(r,s)\in\mathbb{H}$, in fact $(r,s)\in F$, and $(r,s)\le(t,u)$. So, as claimed, F is dense below (p_3,q_3). Choose $(p_4,q_4)\in F\cap G$.

We claim that $a_\alpha\cap a_\beta\subseteq q_4(\{\alpha,\beta\})$. To prove this, assume that $m\in a_\alpha\cap a_\beta$, but suppose that $q_4(\{\alpha,\beta\})\le m$. Thus $x_\alpha(m)=1=x_\beta(m)$, so there are $(e,b),(c,d)\in G$ such that $\alpha\in\mathrm{dmn}(e)$, $m\in\mathrm{dmn}(e(\alpha))$, $(e(\alpha))(m)=1$, and $\beta\in\mathrm{dmn}(c)$, $m\in\mathrm{dmn}(c(\beta))$, and $(c(\beta))(m)=1$. Choose $(p_5,q_5)\in G$ with $(p_5,q_5)\le(p_4,q_4),(a,b),(c,d)$. Then $\{\alpha,\beta\}\in\mathrm{dmn}(q_5)$, $q_5(\{\alpha,\beta\})=q_4(\{\alpha,\beta\})\le m<n_{p_5}$, $(p_5(\alpha))(m)=(e(\alpha))(m)=1$, and $(p_5(\beta))(m)=(c(\beta))(m)=1$, contradiction. So we have shown that $\langle a_\alpha:\alpha<\omega_1\rangle$ is an almost disjoint family.

To show that $\langle a_\alpha:\alpha<\omega_1\rangle$ is MAD, suppose to the contrary that b is an infinite subset of ω such that $b\cap a_\alpha$ is finite for all $\alpha<\omega_1$. Let σ be a name such that $\sigma_G=b$. For each $\alpha<\omega_1$ let

$$\begin{aligned}\tau_\alpha=\{(\check{i},(p,q)):i\in\omega,\ (p,q)\in\mathbb{H},\ \alpha\in\mathrm{dmn}(p),\\ i\in\mathrm{dmn}(p(\alpha)),\ \text{and } (p(\alpha))(i)=1\}.\end{aligned}$$

Cleary $\tau_{\alpha G}=a_\alpha$. For each $n\in\omega$ let $\mathscr{A}_n$ be maximal subject to the following conditions:

(1) $\mathscr{A}_n$ is a collection of pairwise incompatible members of $\mathbb{H}$.

(2) For each $(p,q)\in\mathscr{A}_n$, $(p,q)\Vdash\check{n}\in\sigma$ or $(p,q)\Vdash\check{n}\notin\sigma$.

Then

(3) $\mathscr{A}_n$ is maximal pairwise incompatible.

In fact, suppose that $(r,s)\perp(p,q)$ for all $(p,q)\in\mathscr{A}_n$. Now $(r,s)\Vdash\check{n}\in\sigma\vee\check{n}\notin\sigma$, so there is a $(t,u)\le(r,s)$ such that $(t,u)\Vdash\check{n}\in\sigma$ or $(t,u)\Vdash\check{n}\notin\sigma$. Then $\mathscr{A}_n\cup\{(t,u)\}$ still satisfies (1) and (2), and $(t,u)\notin\mathscr{A}_n$, contradiction.

Now choose

$$\alpha \in \omega_1 \backslash \bigcup_{\substack{n\in\omega,\\ (p,q)\in\mathscr{A}_n}} \operatorname{dmn}(p).$$

Let $m \in \omega$ be such that $b \cap a_\alpha \subseteq m$. Choose $(p_1, q_1) \in G$ such that $(p_1, q_1) \Vdash \sigma \cap \tau_\alpha \subseteq \check{m}$. Using $D_{\alpha 0}$, we may assume that $\alpha \in \operatorname{dmn}(p_1)$. Choose $n \in b$ with $n > m$ and $n \geq n_{p_1}$. Then take $(p_2, q_2) \in G \cap \mathscr{A}_n$. Then $(p_2, q_2) \Vdash \check{n} \in \sigma$, since $n \in b = \sigma_G$. Choose $(p_3, q_3) \in G$ with $(p_3, q_3) \leq (p_1, q_1), (p_2, q_2)$. Then by 36.32 we have $(p_3, q_1 \cup q_2) \in \mathbb{H}$ and $(p_3, q_1 \cup q_2) \leq (p_1, q_1), (p_2, q_2)$.

Now choose $k > \max(n_{p_3}, n)$, and define p_4 as follows. The domain of p_4 is $\operatorname{dmn}(p_3)$. For each $\beta \in \operatorname{dmn}(p_3)$ we define $p_4(\beta) : k \to 2$ by setting, for each $i < k$,

$$(p_4(\beta))(i) = \begin{cases} (p_3(\beta))(i) & \text{if } i < n_{p_3}, \\ 0 & \text{if } n_{p_3} \leq i \text{ and } \beta \neq \alpha, \\ 0 & \text{if } n_{p_3} \leq i,\ \beta = \alpha, \text{ and } i \neq n, \\ 1 & \text{if } \beta = \alpha \text{ and } i = n. \end{cases}$$

We check that $(p_4, q_1 \cup q_2) \in \mathbb{H}$. Conditions (1) and (2) are clear. For (3), suppose that $\{\beta, \gamma\} \in \operatorname{dmn}(q_1 \cup q_2)$, and $(q_1 \cup q_2)(\{\beta, \gamma\}) \leq i < n_{p_4}$. Remember that n_{p_4} is k. If $i < n_{p_3}$, then the desired conclusion follows since $(p_3, q_1 \cup q_2) \in \mathbb{H}$. If $n_{p_3} \leq i$, then the desired conclusion follows since at least one of β, γ is different from α. Hence, indeed, $(p_4, q_1 \cup q_2) \in \mathbb{H}$.

Clearly $(p_4, q_1 \cup q_2) \leq (p_2, q_2)$, and so $(p_4, q_1 \cup q_2) \Vdash \check{n} \in \sigma$. It is also clear that $(p_4, q_1 \cup q_2) \leq (p_1, q_1)$, so $(p_4, q_1 \cup q_2) \Vdash \sigma \cap \tau_\alpha \subseteq \check{m}$. Since $m < n$, it follows that $(p_4, q_1 \cup q_2) \Vdash \check{n} \notin \tau_\alpha$. But $(\check{n}, (p_4, q_1 \cup q_2))$ is clearly a member of τ_α, and hence $(p_4, q_1 \cup q_2) \Vdash \check{n} \in \tau_\alpha$, contradiction. $\square$

Miller, A. [03] *A MAD Q-set.* Fund. Math. 178 (2003), 271–281.

Collapsing to ω_1

Theorem 36.34. *Let M be a c.t.m. of ZFC, and in M let λ be an infinite cardinal, and set $\kappa = \lambda^+$ in M. Let P be $\operatorname{Fn}(\omega, \lambda, \omega)$, and let G be P-generic over M. Then cardinals $\geq \kappa$ are preserved in going to $M[G]$, but $\omega_1^{M[G]} = \kappa$.*

Thus we may say that all cardinals μ such that $\omega < \mu < \kappa$ become countable ordinals in $M[G]$.

Proof. Let $g = \bigcup G$. Clearly g is a function with domain contained in ω and range contained in λ. We claim that actually its domain is ω and its range is λ. For, let $m \in \omega$ and $\alpha \in \lambda$. Let

$$D = \{f \in P : m \in \operatorname{dmn}(f) \text{ and } \alpha \in \operatorname{rng}(f)\}.$$

Clearly D is dense. Hence $m \in \operatorname{dmn}(g)$ and $\alpha \in \operatorname{rng}(g)$, as desired. It follows that in $M[G]$, $|\lambda| = \omega$, and so the same is true for every ordinal α such that $\omega \leq \alpha \leq \lambda$.

Now we can finish the proof by showing in M that P has the κ-cc. Let $X \subseteq P$ with $|X| = \kappa$. Then $\langle \operatorname{dmn}(f) : f \in X \rangle$ is a system of κ many finite sets, so by the Δ-system lemma 10.1 with κ, λ replaced by ω, κ, there is a $N \in [X]^\kappa$ such that $\langle \mathrm{dmn}(f) : f \in M\rangle$ is a Δ-system, say with root r. Since $|{}^r\lambda| \leq \lambda < \kappa$, there are two members f, g of M such that $f \restriction r = g \restriction r$. So f and g are compatible, as desired. $\square$

We now want to do the same thing for regular limit cardinals κ. We introduce the *Lévy collapsing order*:

$$\begin{aligned}\mathrm{Lv}_\kappa = \{p : p &\text{ is a finite function, } \mathrm{dmn}(p) \subseteq \kappa \times \omega, \text{ and}\\ &\text{for all } (\alpha, n) \in \mathrm{dmn}(p),\ p(\alpha, n) \in \alpha\}.\end{aligned}$$

Again this set is ordered by $\supseteq$.

Lemma 36.35. *For κ regular uncountable,* Lv_κ *has the κ-cc.*

Proof. Very similar to part of the proof of 36.34. $\square$

Theorem 36.36. *Let M be a c.t.m. of ZFC, and suppose that in M κ is regular and uncountable. Let G be* Lv_κ*-generic over M. Then cardinals $\geq \kappa$ are preserved in $M[G]$, and $\omega_1^{M[G]} = \kappa$.*

Proof. Cardinals $\geq \kappa$ are preserved by 36.35. Suppose that $0 < \alpha < \kappa$; we will find a function mapping ω onto α in $M[G]$. Let $g = \bigcup G$. Clearly G is a function. Now for each $\alpha < \kappa$, and $m \in \omega$ let

$$D_{\alpha m} = \{p \in \mathrm{Lv}_\kappa : (\alpha, m) \in \mathrm{dmn}(p)\}.$$

Clearly $D_{\alpha m}$ is dense, so $(\alpha, m) \in \mathrm{dmn}(g)$. Thus $\mathrm{dmn}(g) = \kappa \times \omega$. Now suppose that $\alpha < \kappa$ and $\xi < \alpha$. Let

$$E_{\alpha\xi} = \{p \in \mathrm{Lv}_\kappa : \text{there is an } m \in \omega \text{ such that } (\alpha, m) \in \mathrm{dmn}(p) \text{ and } p(\alpha, m) = \xi\}.$$

We claim that $E_{\alpha\xi}$ is dense. For, suppose that $\alpha < \kappa$ and $\xi < \alpha$. Take any $q \in \mathrm{Lv}_\kappa$. Choose $m \in \omega$ such that $(\alpha, m) \notin \mathrm{dmn}(q)$, and let $p = q \cup \{((\alpha, m), \xi)\}$. Clearly $p \in E_{\alpha\xi}$, as desired.

It follows that $\langle g(\alpha, m) : m \in \omega\rangle$ maps ω onto α. $\square$

37. More examples of iterated forcing

We give some more examples of iterated forcing. These are concerned with a certain partial order of functions. For any regular cardinal κ we define

$$f <_\kappa g \quad \text{iff} \quad f, g \in {}^\kappa\kappa \text{ and there is an } \alpha < \kappa \text{ such that } f(\beta) < g(\beta) \text{ for all } \beta \in [\alpha, \kappa).$$

This is clearly a partial order on ${}^\kappa\kappa$. We say that $\mathscr{F} \subseteq {}^\kappa\kappa$ is *almost unbounded* iff there is no $g \in {}^\kappa\kappa$ such that $f <_\kappa g$ for all $f \in \mathscr{F}$. Clearly ${}^\kappa\kappa$ itself is almost unbounded; it has size 2^κ.

Theorem 37.1. *Let κ be a regular cardinal. Then any almost unbounded subset of ${}^\kappa\kappa$ has size at least κ^+.*

Proof. Let $\mathscr{F} \subseteq {}^\kappa\kappa$ have size $\leq \kappa$; we want to find an almost bound for it. We may assume that $\mathscr{F} \neq \emptyset$. Write $\mathscr{F} = \{f_\alpha : \alpha < \kappa\}$, possibly with repetitions. (Since maybe $|\mathscr{F}| < \kappa$.) Define $g \in {}^\kappa\kappa$ by setting, for each $\alpha < \kappa$,

$$g(\alpha) = \left(\sup_{\beta \leq \alpha} f_\beta(\alpha)\right) + 1.$$

If $\beta < \kappa$, then $\{\alpha < \kappa : g(\alpha) \leq f_\beta(\alpha)\} \subseteq \beta$, and so $f_\beta <_\kappa g$. □

Thus under GCH the size of almost unbounded sets has been determined. We are interested in what happens in the absence of GCH, more specifically, under $\neg$CH.

Theorem 37.2. *Suppose that κ is an infinite cardinal and* MA(κ) *holds. Suppose that $\mathscr{F} \subseteq {}^\omega\omega$ and $|\mathscr{F}| = \kappa$. Then there is a $g \in {}^\omega\omega$ such that $f <_\omega g$ for all $f \in \mathscr{F}$.*

Proof. Let $P = \{(p, F) : p \in \mathrm{Fn}(\omega, \omega, \omega) \text{ and } F \in [\mathscr{F}]^{<\omega}\}$. We partially order P by setting $(p, F) \leq (q, G)$ iff the following conditions hold:

(1) $p \supseteq q$.

(2) $F \supseteq G$.

(3) For all $f \in G$ and all $n \in (\mathrm{dmn}(p) \backslash \mathrm{dmn}(q))$, $p(n) > f(n)$.

To check that this really is a partial order, suppose that $(p, F) \leq (q, G) \leq (h, H)$. Obviously $p \supseteq h$ and $F \supseteq H$. Suppose that $f \in H$ and $n \in (\mathrm{dmn}(p) \backslash \mathrm{dmn}(h)$. If $n \in \mathrm{dmn}(q)$, then $p(n) = q(n) > f(n)$. If $n \notin \mathrm{dmn}(q)$, then $p(n) > f(n)$ since $f \in G$.

To show that $\mathscr{P}$ has ccc, suppose that $X \subseteq P$ is uncountable. Since $\mathrm{Fn}(\omega, \omega, \omega)$ is countable, there are $(p, F), (q, G) \in X$ with $p = q$. Then $(p, F \cup G) \in P$ and $(p, F \cup G) \leq (p, F), (p, G)$, as desired.

For each $h \in \mathscr{F}$ let $D_h = \{(p, F) \in P : h \in F\}$. Then D_h is dense. In fact, let $(q, G) \in P$ be given. Then $(q, G \cup \{h\}) \in P$ and $(q, G \cup \{h\}) \leq (q, G)$, as desired.

For each $n \in \omega$ let $E_n = \{(p, F) : n \in \mathrm{dmn}(f)\}$. Then E_n is dense. In fact, let $(q, G) \in P$ be given. We may assume that $n \notin \mathrm{dmn}(q)$. Choose $m > f(n)$ for each $f \in G$, and let $p = q \cup \{(n, m)\}$. Clearly $(p, G) \in E_n$ and $(p, G) \leq (q, G)$, as desired.

Now we apply MA(κ) to get a filter G on $\mathbb{P}$ intersecting all of these dense sets. Since G is a filter, the relation $g \stackrel{\text{def}}{=} \bigcup_{(p,F)\in G} p$ is a function. Since $G \cap E_n \neq \emptyset$ for each $n \in \omega$, g has domain ω. Let $f \in \mathscr{F}$. Choose $(p, F) \in G \cap D_f$. Let $m \in \omega$ be greater than each member of $\mathrm{dmn}(p)$. We claim that $f(n) < g(n)$ for all $n \geq m$. For, suppose that $n \geq m$. Choose $(q, H) \in G$ such that $n \in \mathrm{dmn}(q)$, and choose $(r, K) \in G$ such that $(r, K) \leq (p, F), (q, H)$. Then $f \in K$ since $F \subseteq K$. Also, $n \in \mathrm{dmn}(r)$ since $q \subseteq r$. So $n \in \mathrm{dmn}(r)\backslash\mathrm{dmn}(p)$. Hence from $(r, K) \leq (p, F)$ we get $g(n) = r(n) > f(n)$. □

As another illustration of iterated forcing, we now show that it is relatively consistent that every almost unbounded subset of ${}^\omega\omega$ has size 2^ω, while $\neg$ MA holds. This follows from the following theorem, using the fact that MA implies that $2^\kappa = 2^\omega$ for every infinite cardinal $\kappa < 2^\omega$.

Theorem 37.3. *There is a c.t.m. of ZFC with the following properties:*

(i) $2^\omega = \omega_2$.

(ii) $2^{\omega_1} = \omega_3$.

(iii) Every almost unbounded set of functions from ω to ω has size 2^ω.

Proof. Applying Theorem 24.15 to a model N of GCH, with $\lambda = \omega_1$ and $\kappa = \omega_3$, we get a c.t.m. M of ZFC such that in M, $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_3$. We are going to iterate within M, and iterate ω_2 times. At each successor step we will introduce a function almost greater than each member of ${}^\omega\omega$ at that stage. In the end, any subset of ${}^\omega\omega$ of size less than ω_2 appears at an earlier stage, and is almost bounded.

(1) If $\mathbb{Q}$ is a ccc forcing order in M of size $\leq \omega_1$, then there are at most ω_1 nice $\mathbb{Q}$-names for subsets of $(\omega \times \omega)\check{}$.

To prove (1), recall that a nice $\mathbb{Q}$-name for a subset of $(\omega \times \omega)\check{}$ is a set of the form

$$\bigcup\{\{\check{a}\} \times A_\alpha : a \in \omega \times \omega\}$$

where for each $a \in \omega \times \omega$, A_a is an antichain in $\mathbb{Q}$. Now by ccc the number of antichains in $\mathbb{Q}$ is at most $\sum_{\mu<\omega_1} |Q|^\mu \leq \omega_1$ by CH in M. So the number of sets of the indicated form is at most $\omega_1^\omega = \omega_1$. Hence (1) holds.

Now we are going to define by recursion functions $\mathbb{P}$, π, and σ with domain ω_2.

Let $\mathbb{P}_0$ be the trivial partial order $(\{0\}, 0, 0)$.

Now suppose that $\mathbb{P}_\alpha$ has been defined, so that it is a ccc forcing order in M of size at most ω_1. We now define π_α, σ^α, and $\mathbb{P}_{\alpha+1}$. By (1), the set of all nice $\mathbb{P}_\alpha$-names for subsets of $(\omega \times \omega)\check{}$ has size at most ω_1. We let $\{\tau^\alpha_\gamma : \gamma < \omega_1\}$ enumerate all of them.

(2) For every $\gamma < \omega_1$ there is a $\mathbb{P}_\alpha$-name σ^α_γ such that

$$1_{\mathbb{P}_\alpha} \Vdash_{\mathbb{P}_\alpha} \sigma^\alpha_\gamma : \check{\omega} \to \check{\omega} \text{ and } [\tau^\alpha_\gamma : \check{\omega} \to \check{\omega} \text{ implies that } \sigma^\alpha_\gamma = \tau^\alpha_\gamma].$$

In fact, clearly

$$1_{\mathbb{P}_\alpha} \Vdash_{\mathbb{P}_\alpha} \exists W[W : \check{\omega} \to \check{\omega} \text{ and } [\tau^\alpha_\gamma : \check{\omega} \to \check{\omega} \text{ implies that } W = \tau^\alpha_\gamma]],$$

and so (2) follows from the maximal principle.

This defines σ^α.

Now for each $H \in [\omega_1]^{<\omega}$ we define $\rho^\alpha_H = \{(\sigma^\alpha_\gamma, 1_{\mathbb{P}_\alpha}) : \gamma \in H\}$. So ρ^α_H is a $\mathbb{P}_\alpha$-name.

We now define

$$\pi^0_\alpha = \{(\mathrm{op}(\check{p}, \rho^\alpha_H), 1) : p \in \mathrm{Fn}(\omega,\omega,\omega) \text{ and } H \in [\omega_1]^{<\omega}\}.$$

Let G be $\mathbb{P}_\alpha$-generic over M. Then

$$(3) \qquad (\pi^0_\alpha)_G = \{(p, K) : p \in \mathrm{Fn}(\omega,\omega,\omega) \text{ and } K \in [{}^\omega\omega]^{<\omega}\}.$$

In fact, first suppose that $x \in (\pi^0_\alpha)_G$. Then there exist $p \in \mathrm{Fn}(\omega,\omega,\omega)$ and $H \in [\omega_1]^{<\omega}$ such that $x = (p, (\rho^\alpha_H)_G)$. Now $(\rho^\alpha_H)_G = \{(\sigma^\alpha_\gamma)_G : \gamma \in H\}$, and $(\sigma^\alpha_\gamma)_G \in {}^\omega\omega$ for each γ by (2). Thus x is in the right side of (3).

Second, suppose that $p \in \mathrm{Fn}(\omega,\omega,\omega)$ and $K \in [{}^\omega\omega]^{<\omega}$. For each $f \in K$ there is a $\gamma(f) < \omega_1$ such that $f = (\tau^\alpha_{\gamma(f)})_G$. Let $H = \{\gamma(f) : f \in K\}$. So H is a finite subset of ω_1, and hence is in N. By (1) we have $f = (\sigma^\alpha_{\gamma(f)})_G$ for each $f \in K$. Now $(\rho^\alpha_H)_G = K$, and so $(p, K) \in (\pi^0_\alpha)_G$, as desired. So (3) holds.

Next, we define

$$\begin{aligned}\pi^1_\alpha = \{(\mathrm{op}(\mathrm{op}(\check{p}, \rho^\alpha_H), \mathrm{op}(\check{p}', \rho^\alpha_{H'})), q) : {} & p, p' \in \mathrm{fin}(\omega,\omega),\\ & H, H' \in [\omega_1]^{<\omega},\ p' \subseteq p,\ H' \subseteq H,\ q \in \mathbb{P}_\alpha, \text{ and for all } \gamma \in H'\\ & \text{and all } n \in \mathrm{dmn}(p)\backslash\mathrm{dmn}(p'), q \Vdash_{\mathbb{P}_\alpha} \sigma^\alpha_\gamma(\check{n}) < (p(n))\check{}\ \}.\end{aligned}$$

Again, suppose that G is $\mathbb{P}_\alpha$-generic over M. Then

$$(4) \qquad \begin{aligned}(\pi^1_\alpha)_G = \{((p, K), (p', K')) : {} & (p, K), (p', K') \in (\pi^0_\alpha)_G,\ p' \subseteq p,\ K' \subseteq K,\\ & \text{and for all } f \in K' \text{ and all } n \in \mathrm{dmn}(p)\backslash\mathrm{dmn}(p'),\ f(n) < p(n)\}.\end{aligned}$$

To prove this, first suppose that $x \in (\pi^1_\alpha)_G$. Then there are $q \in G$, $p, p' \in \mathrm{Fn}(\omega,\omega,\omega)$ and $H, H' \in [\omega_1]^{<\omega}$ such that $x = ((p, (\rho^\alpha_H)_G), (p', (\rho^\alpha_{H'}))_G)$, $p' \subseteq p$, $H' \subseteq H$, and for all $\gamma \in H'$ and all $n \in \mathrm{dmn}(p)\backslash\mathrm{dmn}(p')$, $q \Vdash \sigma^\alpha_\gamma(\check{n}) < (p(n))\check{}$. Then with $K = (\rho^\alpha_H)_G$ and $K' = (\rho^\alpha_{H'}))_G$, the desired conditions clearly hold.

Second, suppose that p, p', K, K' exist as on the right side of (4). Then by the definition of π^0_α, there are $H, H' \in [\omega_1]^{<\omega}$ such that $K = (\rho^\alpha_H)_G$ and $K' = (\rho^\alpha_{H'})_G$. Then $K' = \{(\sigma^\alpha_\gamma)_G : \gamma \in H'\}$. Hence for every $\gamma \in H'$ and all $n \in \mathrm{dmn}(p)\backslash\mathrm{dmn}(p')$ we have $(\sigma^\alpha_\gamma)_G(n) < p(n)$. Since H' and $\mathrm{dmn}(p)\backslash\mathrm{dmn}(p')$ are finite, there is a $q \in G$ such that for every $\gamma \in H'$ and all $n \in \mathrm{dmn}(p)\backslash\mathrm{dmn}(p')$ we have $q \Vdash_{\mathbb{P}_\alpha} (\sigma^\alpha_\gamma)(\check{n}) < p(n)\check{}$. It follows now that $((p, K), (p', K')) \in (p^1_\alpha)_G$, as desired.

Next, we let $\pi^2_\alpha = \{(\mathrm{op}(0,0), 1_{\mathbb{P}_\alpha})\}$. Then for any generic G, $(\pi^2_\alpha)_G = (0,0)$. Finally, let $\pi_\alpha = \mathrm{op}(\mathrm{op}(\pi^0_\alpha, \pi^1_\alpha), \pi^2_\alpha)$. This finishes the definition of π_α.

By the argument in the proof of 37.2 we have

(5) $1_{\mathbb{P}_\alpha} \Vdash_{\mathbb{P}_\alpha} \pi_\alpha$ is $\check{\omega}_1 - cc$.

Now $\mathbb{P}_{\alpha+1}$ is determined by (I7) and (I8).

At limit stages we take direct limits, so that ccc is maintained. So the construction is finished, and $\mathbb{P}_\kappa$ is ccc.

Let G be $\mathbb{P}_\kappa$-generic over M.

(6) In $N[G]$, if $\mathscr{F} \subseteq {}^\omega\omega$ and $|\mathscr{F}| < \omega_2$, then there is a $g \in {}^\omega\omega$ such that $f <_\omega g$ for all $f \in \mathscr{F}$.

For, let $\mathscr{F} = \{f_\xi : \xi < \omega_1\}$, possibly with repetitions. Let

$$\mathscr{F}' = \{(\xi, i, j) : \xi < \omega_1,\ i, j \in \omega,\ \text{and } f_\xi(i) = j\}.$$

Now there is an $\alpha < \omega_2$ such that $\mathscr{F}' \in N[i^{-1}_{\alpha\omega_2}[G]]$, and hence also $\mathscr{F} \in N[i^{-1}_{\alpha\omega_2}[G]]$. For brevity write $G_\xi = i^{-1}_{\xi\omega_2}[G]$ for every $\xi < \omega_2$. Let

$$H_\alpha = \{\eta_{G_\alpha} : \eta \in \operatorname{dmn}(\pi^0_\alpha) \text{ and } p^\frown\langle\eta\rangle \in G_{\alpha+1} \text{ for some } p\}.$$

Let $\mathbb{Q}_\alpha = (\pi_\alpha)_{G_\alpha}$. Thus by (3) and (4),

$$\begin{aligned}
&(7) && Q_\alpha = \{(p, K) : p \in \operatorname{fin}(\omega, \omega) \text{ and } K \in [{}^\omega\omega]^{<\omega}\};\\
&(8) && \leq_{\mathbb{Q}_\alpha} = \{((p, K), (p', K')) : (p, K), (p', K') \in (\pi^0_\alpha)_G,\ p' \subseteq p,\ K' \subseteq K,\\
& && \qquad \text{and for all } f \in K' \text{ and all } n \in \operatorname{dmn}(p)\backslash\operatorname{dmn}(p'),\ f(n) < p(n)\}.
\end{aligned}$$

Hence G_α is $\mathbb{P}$-generic over N, $H_\alpha \in N[G_{\alpha+1}]$, and H_α is $\mathbb{Q}_\alpha$-generic over $N[G_\alpha]$. Let $g = \bigcup_{(p,F)\in H_\alpha} p$. Clearly g is a function. For each $m \in \omega$, let

$$E_m = \{(p, K) :\in Q_\alpha : m \in \operatorname{dmn}(p)\}.$$

Then E_m is dense. (See the proof of 37.2.) It follows that $g \in {}^\omega\omega$.

Now take any $f \in {}^\omega\omega$ (in $N[G_\alpha]$). The set $D \stackrel{\text{def}}{=} \{(p, K) \in Q_\alpha : f \in K\}$ is dense, by the proof of 37.2. Hence we can choose $(p, K) \in D \cap H_\alpha$. We claim that $f(m) < g(m)$ for all m such that $m > n$ for each $n \in \operatorname{dmn}(p)$. For, suppose that such an m is given. Choose $(p', K') \in E_m \cap H_\alpha$, and then choose $(p'', K'') \in H_\alpha$ with $(p'', K'') \leq (p, K), (p', K')$. Now $m \in \operatorname{dmn}(p') \subseteq \operatorname{dmn}(p'')$, and $f \in K$. so from $(p'', K'') \leq (p, K)$ and $m \notin \operatorname{dmn}(p)$ we get $f(m) < p''(m) = g(m)$, as desired. This finishes the proof of (6).

By (6) we have $\omega_2 \leq 2^\omega$.

(9) $|P_\alpha| \leq \omega_1$ for all $\alpha < \omega_2$.

We prove this by induction on α. It is clear for $\alpha = 0$. Assume that $|P_\alpha| \leq \omega_1$. Clearly $|\pi^0_\alpha| = \omega_1$, so by (I7), $|P_{\alpha+1}| \leq \omega_1$. Suppose that α is limit, and $|P_\beta| \leq \omega_1$ for all $\beta < \alpha$. Since $\mathbb{P}_\alpha$ is the direct limit of previous $\mathbb{P}_\beta$s, clearly $|P_\alpha| \leq \omega_1$.

(10) $|P_{\omega_2}| \leq \omega_2$.

This is clear from (8), since P_{ω_2} is the direct limit of earlier $\mathbb{P}_\beta$s.

Now by Proposition 24.3, replacing κ, λ, μ there by $\omega_2, \omega_1, \omega$, we get $2^\omega \le \omega_2$. So by the above, $2^\omega = \omega_2$ in $N[G]$. By Proposition 24.3, replacing κ, λ, μ there by $\omega_2, \omega_1, \omega_1$, we get $2^{\omega_1} \le \omega_3$. Since $2^{\omega_1} = \omega_3$ in N, it follows that $2^{\omega_1} = \omega_3$ in $N[G]$. □

We want to generalize 37.3 to higher cardinals. This requires some preparation.

Lemma 37.4. *Suppose that M is a c.t.m. of ZFC, and in M θ is a regular cardinal, $2^{<\theta} = \theta$, and $2^\theta = \theta^+$. We define a partial order $\mathbb{P}$ in M as follows:*

$$P = \{(p, F) : p \in \mathrm{Fn}(\theta, \theta, \theta), F \in \left[{}^\theta\theta\right]^{<\theta}\}$$
$$(p, F) \le (q, G) \quad \textit{iff} \quad q \subseteq p,\ G \subseteq F,\ \textit{and } \forall f \in G \forall \beta \in \mathrm{dmn}(p) \backslash \mathrm{dmn}(g)(p(\beta) > f(\beta));$$
$$1_\mathbb{P} = (0, 0).$$

Then the following conditions hold.

(i) $|P| \le \theta^+$.

(ii) $\mathbb{P}$ is θ-closed.

(iii) $\mathbb{P}$ has the θ^+-cc.

(iv) $\mathbb{P}$ preserves cofinalities and cardinals.

(v) If G is $\mathbb{P}$-generic over M, then there is a function $g \in {}^\theta\theta$ in $M[G]$ such that $f <_\theta g$ for all $f \in ({}^\theta\theta)^M$.

Proof. Clearly (i) holds.

$\mathbb{P}$ satisfies the θ^+-c.c.: Suppose that $B \subseteq P$ with $|B| \ge \theta^+$. Then since $|\mathrm{Fn}(\theta, \theta, \theta)| = \theta$, wlog there is a q such that $p = q$ for all $(p, F) \in B$, and so θ^+-c.c. is clear.

$\mathbb{P}$ is θ-closed: Suppose that $\langle (p_\alpha, F_\alpha) : \alpha < \beta \rangle$ is decreasing, with $\beta < \theta$. Let $q = \bigcup_{\alpha<\beta} p_\alpha$ and $G = \bigcup_{\alpha<\beta} F_\alpha$. Suppose that $\alpha < \beta$; we claim that $(q, G) \le (p_\alpha, F_\alpha)$. Suppose that $f \in F_\alpha$ and $\delta \in \mathrm{dmn}(q)\backslash\mathrm{dmn}(p_\alpha)$. Then there is a $\gamma < \beta$ such that $\delta \in \mathrm{dmn}(p_\gamma)$. We may assume that $\alpha < \gamma$. Hence $(p_\gamma, F_\gamma) \le (p_\alpha, F_\alpha)$, so $q(\delta) = p_\gamma(\delta) > f(\delta)$, as desired.

Now it follows that (iv) holds.

Now suppose that G is $\mathbb{P}$-generic over M. Define

$$g = \bigcup_{(p,F)\in G} p.$$

Clearly g is a function with domain and range included in θ. To show that g has domain θ, take any $\alpha < \theta$. Let $D = \{(p, F) : \alpha \in \mathrm{dmn}(p)\}$. Then D is dense. In fact, suppose that $(q, H) \in \mathbb{P}$. Wlog $\alpha \notin \mathrm{dmn}(q)$. Let p be the extension of g by adding α to its domain and defining $p(\alpha)$ to be any ordinal less than θ which is greater than each $f(\alpha)$ for $f \in H$. Clearly $(p, H) \le (q, H)$ and $(p, H) \in D$. So g has domain θ.

Finally, we claim that $f <^* g$ for all $f \in {}^\theta\theta \cap M$. In fact, clearly $E \overset{\text{def}}{=} \{(p, F) \in \mathbb{P} : f \in F\}$ is dense, and so we can choose $(p, F) \in E \cap G$. Take $\alpha < \theta$ such that $\sup(\mathrm{dmn}(p)) < \alpha$. Take any $\beta \in (\alpha, \theta)$. Choose (q, H) such that $\beta \in \mathrm{dmn}(q)$. Then choose $(r, K) \in G$ such that $(r, K) \le (p, F), (q, H)$. Then $\beta \in \mathrm{dmn}(r)\backslash\mathrm{dmn}(p)$, and $f \in F$, so $g(\beta) = r(\beta) > f(\beta)$. This shows that $f <^* g$. □

If π is a $\mathbb{P}$-name for a p.o., then we say that π is *full* for $\downarrow\theta$-sequences iff the following conditions (a)–(d) imply condition (e):

(a) $p \in \mathbb{P}$.

(b) $\alpha < \theta$.

(c) $\rho_\xi \in \mathrm{dmn}(\pi^0)$ for each $\xi < \alpha$.

(d) for all $\xi, \eta < \alpha$, if $\xi < \eta$, then $p \Vdash (\rho_\xi \in \pi^0) \wedge (\rho_\eta \in \pi^0) \wedge (\rho_\eta \leq \rho_\xi)$.

(e) There is a $\sigma \in \mathrm{dmn}(\pi^0)$ such that $p \Vdash \sigma \in \pi$ and $p \Vdash \sigma \leq \rho_\xi$ for each $\xi < \alpha$.

Lemma 37.5. *Let M be a c.t.m. of ZFC, and θ an infinite cardinal in M. Let $\mathscr{I}$ be the ideal in $\mathscr{P}(\theta)$ consisting of all sets of size less than θ. In M, let $(\mathbb{P}, \pi)$ be an α-stage iterated forcing construction with supports in $\mathscr{I}$ (Kunen's sense). Suppose that for each $\xi < \alpha$, the $\mathbb{P}_\xi$-name π_ξ is full for $\downarrow\theta$-sequences. Then $\mathbb{P}_\alpha$ is θ-closed.*

Proof. Let $\langle p^\nu : \nu < \sigma\rangle$ be a sequence of elements of $\mathbb{P}_\alpha$ such that $p^\nu \leq p^\mu$ if $\mu < \nu < \sigma$, and $\sigma < \theta$. We will define $p^\sigma = \langle p^\sigma_\xi : \xi < \alpha\rangle$ by recursion so that the following condition holds:

$$\text{For all } \xi < \alpha,\ p^\sigma \restriction \xi = \langle p^\sigma_\eta : \eta < \xi\rangle \in \mathbb{P}_\xi \text{ and } \forall\mu < \sigma(p^\sigma \restriction \xi \leq p^\mu \restriction \xi) \text{ and}$$
$$\mathrm{supp}(p^\sigma) = \bigcup_{\nu<\sigma} \mathrm{supp}(p^\nu).$$

The induction step to a limit ordinal ξ is clear, as is the case $\xi = 0$. Now we define p^σ_ξ, given $p^\sigma \restriction \xi$. By fullness we get $\rho^\sigma_\xi \in \mathrm{dmn}(\pi^0)$ such that

$$p^\sigma \restriction \xi \Vdash \rho^\sigma_\xi \in \pi \text{ and } p^\sigma \restriction \xi \Vdash \rho^\sigma_\xi \leq \rho^\sigma_\eta \text{ for each } \eta < \xi.$$

Clearly p^σ is as desired. □

Here is our generalization of 37.3.

Theorem 37.6. *Let M be a c.t.m. of GCH, and let θ be an uncountable regular cardinal in M. Then there is a generic extension N of M preserving cofinalities and cardinals such that in N the following hold:*

(i) $2^\theta = \theta^{++}$.

(ii) $2^{(\theta^+)} = \theta^{+++}$.

(iii) Every subset of ${}^\theta\theta$ of size less than 2^θ is almost unbounded.

Proof. First we apply Corollary 24.16 with $\lambda = \theta^+$ and $\kappa = \theta^{+++}$ to get a generic extension M' of M preserving cofinalities and cardinals in which $2^{<\theta} = \theta$, $2^\theta = \theta^+$, and $2^{\theta^+} = \theta^{+++}$.

We are going to iterate within M', and iterate θ^{++} times. At each successor step we will introduce a function almost greater than each member of ${}^\theta\theta$ at that stage. In the end, any subset of ${}^\theta\theta$ of size less than θ^{++} appears at an earlier stage, and is almost bounded.

(1) If $\mathbb{Q}$ is a θ^+-cc forcing order in M' of size less $\leq \theta^+$, then there are at most θ^+ nice $\mathbb{Q}$-names for subsets of $(\theta \times \theta)\check{}$.

To prove (1), recall that a nice $\mathbb{Q}$-name for a subset of $(\theta \times \theta)\check{}$ is a set of the form

$$\bigcup\{\{\check{a}\} \times A_\alpha : a \in \theta \times \theta\}$$

where for each $a \in \theta \times \theta$, A_a is an antichain in $\mathbb{Q}$. Now by θ^+-cc, the number of antichains in $\mathbb{Q}$ is at most $\sum_{\mu<\theta^+} |Q|^\mu \leq \theta^+$ by $2^\theta = \theta^+$. So the number of sets of the indicated form is at most $(\theta^+)^\theta = \theta^+$. Hence (1) holds.

Now we are going to define by recursion functions $\mathbb{P}$, π, and σ with domain θ^{++}.

Let $\mathbb{P}_0$ be the trivial partial order $(\{0\}, 0, 0)$.

Now suppose that $\mathbb{P}_\alpha$ has been defined, so that it is a θ^+-cc forcing order in M' of size at most θ^+, it is θ-closed, and every element has support of size less than θ. Also we assume that π_ξ has been defined for every $\xi < \alpha$ so that π_ξ is a $\mathbb{P}_\xi$-name for a forcing order, and it is full for $\downarrow\theta$-sequences. We now define π_α, σ^α, and $\mathbb{P}_{\alpha+1}$. By (1), the set of all nice $\mathbb{P}_\alpha$-names for subsets of $(\theta \times \theta)\check{}$ has size at most θ^+. We let $\{\tau^\alpha_\gamma : \gamma < \theta^+\}$ enumerate all of them.

(2) For every $\gamma < \theta_1$ there is a $\mathbb{P}_\alpha$-name σ^α_γ such that

$$1_{\mathbb{P}_\alpha} \Vdash_{\mathbb{P}_\alpha} \sigma^\alpha_\gamma : \check{\theta} \to \check{\theta} \text{ and } [\tau^\alpha_\gamma : \check{\theta} \to \check{\theta} \text{ implies that } \sigma^\alpha_\gamma = \tau^\alpha_\gamma].$$

In fact, clearly

$$1_{\mathbb{P}_\alpha} \Vdash_{\mathbb{P}_\alpha} \exists W[W : \check{\theta} \to \check{\theta} \text{ and } [\tau^\alpha_\gamma : \check{\theta} \to \check{\theta} \text{ implies that } W = \tau^\alpha_\gamma]],$$

and so (2) follows from the maximal principle.

This defines σ^α.

Now for each $H \in [\theta^+]^{<\theta}$ we define $\rho^\alpha_H = \{(\sigma^\alpha_\gamma, 1_{\mathbb{P}_\alpha}) : \gamma \in H\}$. So ρ^α_H is a $\mathbb{P}_\alpha$-name.

We now define

$$\pi^0_\alpha = \{(\mathrm{op}(\check{p}, \rho^\alpha_H), 1) : p \in \mathrm{Fn}(\theta, \theta, \theta) \text{ and } H \in [\theta^+]^{<\theta}\}.$$

Let G be $\mathbb{P}_\alpha$-generic over M'. Then

$$([\theta^+]^{<\theta})^{M'} = ([\theta^+]^{<\theta})^{M'[G]}. \tag{3}$$

In fact, $\subseteq$ is clear. Now suppose that $L \in ([\theta^+]^{<\theta})^{M'[G]}$. Then there exist an ordinal $\alpha < \theta$ and a bijection f from α onto L. Since $\mathbb{P}_\alpha$ is θ-closed, by 11.1 we have $f \in M'$, and hence $L \in M'$, as desired in (3). Similarly,

$$(\mathrm{Fn}(\theta, \theta, \theta))^{M'} = (\mathrm{Fn}(\theta, \theta, \theta))^{M'[G]}. \tag{4}$$

$$(\pi^0_\alpha)_G = \{(p, K) : p \in \mathrm{Fn}(\theta, \theta, \theta) \text{ and } K \in [{}^\theta\theta]^{<\theta}\}. \tag{5}$$

In fact, first suppose that $x \in (\pi^0_\alpha)_G$. Then there exist $p \in \mathrm{Fn}(\theta,\theta,\theta)$ and $H \in [\theta^+]^{<\theta}$ such that $x = (p, (\rho^\alpha_H)_G)$. Now $(\rho^\alpha_H)_G = \{(\sigma^\alpha_\gamma)_G : \gamma \in H\}$, and $(\sigma^\alpha_\gamma)_G \in {}^\theta\theta$ for each γ by (2). Thus x is in the right side of (3).

Second, suppose that $p \in \mathrm{Fn}(\theta,\theta,\theta)$ and $K \in [{}^\theta\theta]^{<\theta}$. For each $f \in K$ there is a $\gamma(f) < \theta^+$ such that $f = (\tau^\alpha_{\gamma(f)})_G$. Let $H = \{\gamma(f) : f \in K\}$. So H is a subset of θ^+ of size less than θ. By (2) we have $f = (\sigma^\alpha_{\gamma(f)})_G$ for each $f \in K$. Now $(\rho^\alpha_H)_G = K$, and so $(p, K) \in (\pi^0_\alpha)_G$, as desired. So (5) holds.

Next, we define

$$\begin{aligned}\pi^1_\alpha = \{(\mathrm{op}(\mathrm{op}(\check{p}, \rho^\alpha_H), \mathrm{op}(\check{p}', \rho^\alpha_{H'})), q) &: p, p' \in \mathrm{Fn}(\theta,\theta,\theta),\\ &H, H' \in [\theta^+]^{<\theta}, p' \subseteq p, H' \subseteq H, q \in \mathbb{P}_\alpha, \text{ and for all } \gamma \in H'\\ &\text{and all } \xi \in \mathrm{dmn}(p)\backslash\mathrm{dmn}(p'), q \Vdash_{\mathbb{P}_\alpha} \sigma^\alpha_\gamma(\check{\xi}) < (p(\xi))\check{}\,\}.\end{aligned}$$

Again, suppose that G is $\mathbb{P}_\alpha$-generic over M'. Then

$$\begin{aligned}(6)\qquad (\pi^1_\alpha)_G = \{((p, K), (p', K')) &: (p, K), (p', K') \in (\pi^0_\alpha)_G,\ p' \subseteq p,\ K' \subseteq K,\\ &\text{and for all } f \in K' \text{ and all } \xi \in \mathrm{dmn}(p)\backslash\mathrm{dmn}(p'),\ f(\xi) < p(\xi)\}.\end{aligned}$$

To prove this, first suppose that $x \in (\pi^1_\alpha)_G$. Then there are $q \in G$, $p, p' \in \mathrm{Fn}(\theta,\theta,\theta)$ and $H, H' \in [\theta^+]^{<\theta}$ such that $x = ((p, (\rho^\alpha_H)_G), (p', (\rho^\alpha_{H'}))_G)$, $p' \subseteq p$, $H' \subseteq H$, and for all $\gamma \in H'$ and all $n \in \mathrm{dmn}(p)\backslash\mathrm{dmn}(p')$, $q \Vdash \sigma^\alpha_\gamma(\check{\xi}) < (p(\xi))\check{}$. Then with $K = (\rho^\alpha_H)_G$ and $K' = (\rho^\alpha_{H'}))_G$, the desired conditions clearly hold.

Second, suppose that p, p', K, K' exist as on the right side of (4). Then by the definition of π^0_α, there are $H, H' \in [\theta^+]^{<\theta}$ such that $K = (\rho^\alpha_H)_G$ and $K' = (\rho^\alpha_{H'})_G$. Then $K' = \{(\sigma^\alpha_\gamma)_G : \gamma \in H'\}$. Hence for every $\gamma \in H'$ and all $\xi \in \mathrm{dmn}(p)\backslash\mathrm{dmn}(p')$ we have $(\sigma^\alpha_\gamma)_G(\xi) < p(\xi)$. Let $\langle(\xi_\nu, \psi_\nu) : \nu < \gamma\rangle$ enumerate all pairs (ξ, γ) such that $\xi \in \mathrm{dmn}(p)\backslash\mathrm{dmn}(p')$ and $\gamma \in H'$, with $\beta < \theta$, β limit. Now we define a system $\langle q_\nu : \nu \le \beta\rangle$ of members of $\mathbb{P}_\alpha$ by recursion. Let $q_0 = 1$. Suppose that q_ν has been defined so that $q_\nu \in G$. Now there is an $r \in G$ such that $r \Vdash \sigma^\alpha_{\gamma_\nu}(\hat{\xi}_\nu) < (p(\xi_\nu))\check{}$. Let $q_{\nu+1} \in G$ be such that $q_{\nu+1} \le r, q_\nu$. At limit stages $\le \beta$ we use that θ-closed property of $\mathbb{P}_\alpha$ to continue. Clearly q_β is as desired, showing that $((p, K), (p', K')) \in (p^1_\alpha)_G$.

Next, we let $\pi^2_\alpha = \{(\mathrm{op}(0, 0), 1_{\mathbb{P}_\alpha})\}$. Then for any generic G, $(\pi^2_\alpha)_G = (0, 0)$. Finally, let $\pi_\alpha = \mathrm{op}(\mathrm{op}(\pi^0_\alpha, \pi^1_\alpha), \pi^2_\alpha)$. This finishes the definition of π_α.

Using (5) and (6) it is clear that π_α is a $\mathbb{P}_\alpha$-name for a forcing order. To verify that it is full for $\downarrow\theta$-sequences, suppose that

(7) $\quad p \in P_\alpha,\ \beta < \theta,\ \varphi_\xi \in \mathrm{dmn}(\pi^0_\alpha)$ for each $\xi < \beta$,

(8) $\quad$ and if $\xi, \eta < \beta$ and $\xi < \eta$, then $p \Vdash_{\mathbb{P}_\alpha} (\varphi_\xi \in \pi^0_\alpha) \wedge (\varphi_\eta \in \pi^0_\alpha) \wedge (\varphi_\eta \le_{\pi_\alpha} \varphi_\xi)$.

We want to find $\psi \in \mathrm{dmn}(\pi^0_\alpha)$ such that

$$(9)\qquad p \Vdash \psi \in \pi^0_\alpha \text{ and } p \Vdash \psi \le_{\pi_\alpha} \varphi_\xi \text{ for each } \xi < \beta.$$

Since $\varphi_\xi \in \mathrm{dmn}(\pi^0_\alpha)$, there exist a $q_\xi \in \mathrm{Fn}(\theta,\theta,\theta)$ and an $H_\xi \in [\theta^+]^{<\theta}$ such that $\varphi_\xi = \mathrm{op}(\check{q}_\xi, \rho^\alpha_{H_\xi})$. Now if $\xi < \eta < \beta$, then $p \Vdash \varphi_\eta \le_{\pi_\alpha} \varphi_\xi$; hence $q_\xi \subseteq q_\eta$. Let $r = \bigcup_{\xi<\beta} \varphi_\xi$

and $K = \bigcup_{\xi<\beta} H_\xi$. Thus $r \in \mathrm{Fn}(\theta,\theta,\theta)$ and $K \in [\theta^+]^{<\theta}$. Let $\psi = \mathrm{op}(\check{r}, \rho^\alpha_K)$. Clearly $\psi \in \mathrm{dmn}(\pi^0_\alpha)$. Suppose that $\xi < \beta$. To show that $p \Vdash \psi \leq_{\pi_\alpha} q_\xi$, suppose that $p \in G$ with G $\mathbb{P}_\alpha$-generic over M'. Then $\psi_G = (r, (\rho^\alpha_K)_G)$, and clearly $(\rho^\alpha_K)_G) = K$. Suppose that $\gamma \in H_\xi$ and $\nu \in \mathrm{dmn}(r)\backslash\mathrm{dmn}(q_\xi)$. Say $\nu \in \mathrm{dmn}(q_\eta)$ with $\eta < \beta$. Clearly $\xi < \eta$. Since $(\varphi_\eta)_G \leq (\varphi_\xi)_G$ by (8), we have $r(\nu) = q_\eta(\nu) > (\sigma^\alpha_\gamma)_G(\nu)$. This proves that $\psi_G \leq (\varphi_\xi)_G$, and so (9) holds.

Now $\mathbb{P}_{\alpha+1}$ is defined by (I7) and (I8) in the definition of iteration. We now want to show that $\mathbb{P}_{\alpha+1}$ is θ^+-cc, and for this we will apply 15.10. we are assuming that $\mathbb{P}_\alpha$ is θ^+-cc, so it suffices to prove that $1 \Vdash_{\mathbb{P}_\alpha} \pi_\alpha - cc$. So, let G be $\mathbb{P}_\alpha$-generic over M'. As above, $2^{<\theta} = \theta$ in $M'[G]$. Now $|P_\alpha| \leq \theta^+$ by assumption. Hence $2^\theta = \theta^+$ in $M'[G]$ by 9.6 (with κ, λ, μ replaced by $\theta^+, \theta^+, \theta$ respectively). Hence $\pi_{\alpha G}$ is θ^+-cc by (5) and (6). So $\mathbb{P}_{\alpha+1}$ is θ^+-cc by 15.10.

$\mathbb{P}_{\alpha+1}$ is θ-closed by 37.5, since we have proved that π_α is full for $\downarrow\theta$-sequences. This finishes the recursion step from α to $\alpha+1$.

Now suppose that α is a limit ordinal $\leq \theta^{++}$. We let

$$\begin{aligned} P_\alpha = \{p : & p \text{ is a function with domain } \alpha,\ p_\xi \in P_\xi \text{ for all } \xi < \alpha \\ & \text{and } |\{\xi < \alpha : p_\xi \neq 1\}| < \theta\}. \end{aligned}$$

and for $p, q \in P_\alpha$, $p \leq q$ iff $p_\xi \leq q_\xi$ for all $\xi < \alpha$.

Now we show that $\mathbb{P}_\alpha$ has the θ^+-cc. Suppose that $\langle p^\gamma : \gamma < \theta^+\rangle$ is a system of members of P_α. Then we can apply the Δ-system theorem 10.1 to the system $\langle \mathrm{supp}(p^\gamma) : \gamma < \theta^+\rangle$, with κ, λ replaced by θ, θ^+ respectively. This gives us a set $L \in [\theta^+]^{\theta^+}$ and a set K such that for all distinct $\varphi, \gamma \in L$, $\mathrm{supp}(p^\gamma) \cap \mathrm{supp}(p^\delta) = K$. For $\gamma \in L$ and $\xi \in K$ we have $p^\gamma(\xi) \neq 1$, so we can write $p^\gamma(\xi) = \mathrm{op}(\check{q}^\gamma_\xi, \varphi_\xi)$ with $q^\gamma_\xi \in \mathrm{Fn}(\theta,\theta,\theta)$. Now for any $\gamma \in L$, the function $\langle a^\gamma_\xi : \xi \in K\rangle$ is a member of $\prod_{\xi\in K} \mathrm{Fn}(\theta,\theta,\theta)$, which has size at most θ. So there exist $L' \in [L]^{\theta+}$ and r such that $\langle q^\gamma_\xi : \xi \in K\rangle = \langle r_\xi : \xi \in K\rangle$ for all $\gamma \in L'$. Now it is clear that p^γ and p^δ are compatible for all $\gamma, \delta \in L'$, as desired.

By 37.5, $\mathbb{P}_\alpha$ is θ-closed. Clearly, for $\alpha < \theta^{++}$ $\mathbb{P}_\alpha$ has size at most θ^+.

This finishes the construction. For brevity let $\mathbb{R} = \mathbb{P}_{\theta^{++}}$.

Let G be $\mathbb{R}$-generic over M'.

(10) In $M'[G]$, if $\mathscr{F} \subseteq {}^\theta\theta$ and $|\mathscr{F}| < \theta^{++}$, then there is a $g \in {}^\theta\theta$ such that $f <_\theta g$ for all $f \in \mathscr{F}$.

For, let $\mathscr{F} = \{f_\xi : \xi < \theta^+\}$, possibly with repetitions. Let

$$\mathscr{F}' = \{(\xi, i, j) : \xi < \theta_1,\ i, j \in \theta, \text{ and } f_\xi(i) = j\}.$$

By 26.14 there is an $\alpha < \theta^{++}$ such that $\mathscr{F}' \in M'[i^{-1}_{\alpha\theta^{++}}[G]]$, and hence also $\mathscr{F} \in M'[i^{-1}_{\alpha\theta^{++}}[G]]$. For brevity write $G_\xi = i^{-1}_{\xi\theta^{++}}[G]$ for every $\xi < \theta^{++}$. Let

$$H_\alpha = \{\eta_{G_\alpha} : \eta \in \mathrm{dmn}(\pi^0_\alpha) \text{ and } p^\frown\langle\eta\rangle \in G_{\alpha+1} \text{ for some } p\}.$$

Let $\mathbb{Q}_\alpha = (\pi_\alpha)_{G_\alpha}$. Thus by (5) and (6),

$$Q_\alpha = \{(p, K) : p \in \text{fin}(\theta, \theta) \text{ and } K \in [{}^\theta\theta]^{<\theta}\}; \tag{11}$$

$$\begin{aligned} \leq_{\mathbb{Q}_\alpha} = \{((p, K), (p', K')) : (p, K), (p', K') \in (\pi^0_\alpha)_G,\ p' \subseteq p,\ K' \subseteq K, \\ \text{and for all } f \in K' \text{ and all } \xi \in \text{dmn}(p)\backslash\text{dmn}(p'),\ f(\xi) < p(\xi)\}. \end{aligned} \tag{12}$$

Now (10) follows from 37.4.

Replacing κ, λ, μ in 24.3 by $\theta^{++}, \theta^+, \theta$ respectively, we get $2^\theta \leq \theta^{++}$ in $M'[G]$. Hence by (10), $2^\theta = \theta^{++}$ in $M'[G]$.

Replacing κ, λ, μ in 24.3 by $\theta^{++}, \theta^+, \theta^+$ respectively, we get $2^{\theta^+} \leq \theta^{+++}$ in $M'[G]$. Since $2^{\theta^+} = \theta^{+++}$ in M', it follows that $2^{\theta^+} = \theta^{+++}$ in $M'[G]$. $\square$

38. Consistency results concerning $\mathscr{P}(\omega)/fin$

We give relative consistency theorems which show that consistently most of the functions described in the diagram in chapter 20 can be less than 2^ω. For the first consistency result, concerning $\mathfrak{a}$, we need to go into the theory of products of forcing orders.

If $\mathbb{P}_0$ and $\mathbb{P}_1$ are forcing orders, their *product* is the cartesian product $\mathbb{P}_0 \times \mathbb{P}_1$ with the order relation

$$(p_0, p_1) \le (q_0, q_1) \quad \text{iff} \quad p_0 \le q_0 \text{ and } p_1 \le q_1.$$

We define $i_0 : \mathbb{P}_0 \to \mathbb{P}_0 \times \mathbb{P}_1$ and $i_0 : \mathbb{P}_0 \to \mathbb{P}_0 \times \mathbb{P}_1$ by $i_0(p) = (p, 1)$ and $i_1(p) = (1, p)$.

Proposition 38.1. *i_0 and i_1 are complete embeddings.*

Proof. See the definition of complete embedding just before Proposition 26.2. Only (4) needs thought. For i_0, given $(p_0, p_1) \in \mathbb{P}_0 \times \mathbb{P}_1$ we take p_0 to be the reduction. Suppose that $p' \in \mathbb{P}_0$ and $p' \le p_0$. Then $i(p') = (p', 1)$ is compatible with (p_0, p_1); namely, (p', p_1) is below both of them. Similarly for i_1. □

Proposition 38.2. *Suppose that G is $(\mathbb{P}_0 \times \mathbb{P}_1)$-generic over M. Then $i_0^{-1}[G]$ is $\mathbb{P}_0$-generic over M, and $G = (i_0^{-1}[G] \times i_1^{-1}[G])$.*

Proof. The first assertion follows from Theorem 26.3. For the second assertion, $\subseteq$ is obvious. Now suppose that $(p_0, p_1) \in (i_0^{-1}[G] \times i_1^{-1}[G])$. Then $(p_0, 1) \in G$ and $(1, p_1) \in G$. Choose $(q_0, q_1) \in G$ below both of these. Then $(q_0, q_1) \le (p_0, p_1)$, so $(p_0, p_1) \in G$. □

Theorem 38.3. *Suppose that $G_0 \subseteq \mathbb{P}_0 \in M$ and $G_1 \subseteq \mathbb{P}_1 \in M$. Then the following conditions are equivalent:*

(i) $G_0 \times G_1$ is $(\mathbb{P}_0 \times \mathbb{P}_1)$-generic over M.

(ii) G_0 is $\mathbb{P}_0$-generic over M and G_1 is $\mathbb{P}_1$-generic over $M[G_0]$.

(iii) G_1 is $\mathbb{P}_1$-generic over M and G_0 is $\mathbb{P}_0$-generic over $M[G_1]$.

Proof. By symmetry it suffices to show that (i) and (ii) are equivalent. First suppose that $G_0 \times G_1$ is $(\mathbb{P}_0 \times \mathbb{P}_1)$-generic over M. Clearly $i_0^{-1}[G_0 \times G_1] = G_0$, so G_0 is $\mathbb{P}_0$-generic over M by Proposition 38.2. To show that G_1 is $\mathbb{P}_1$-generic over $M[G_0]$, take any dense $D \subseteq \mathbb{P}_1$, in $M[G_0]$. Let τ be a $\mathbb{P}_0$-name such that $D = \tau_{G_0}$. Choose $p_0 \in G_0$ such that

$$p_0 \Vdash (\tau \text{ is dense in } \mathbb{P}_1).$$

Let

$$D' = \{(q_0, q_1) \in (\mathbb{P}_0 \times \mathbb{P}_1) : q_0 \le p_0 \text{ and } q_0 \Vdash (\check{q}_1 \in \tau)\}.$$

(1) D' is dense below $(p_0, 1)$.

For, suppose that $(r_0, r_1) \le (p_0, 1)$. Since $r_0 \le p_0$ we have

$$r_0 \Vdash \exists x \in \check{\mathbb{P}}_1[x \in \tau \text{ and } x \le \check{r}_1].$$

Hence by Proposition 16.16 there exist $q_0 \le r_0$ and $q_1 \in \mathbb{P}_1$ such that

$$q_0 \Vdash (\check{q}_1 \in \tau \text{ and } \check{q}_1 \le \check{r}_1).$$

By Theorem 16.14 we then get $q_1 \in r_1$. Hence $(q_0, q_1) \leq (r_0, r_1)$ and $(q_0, q_1) \in D'$. So (1) holds.

By (1), choose $(q_0, q_1) \in (G_0 \times G_1) \cap D'$. Then $q_0 \Vdash \check{q}_1 \in \tau$, and $q_0 \in G_0$, so $q_1 \in \tau_{G_0} = D$. Also $q_1 \in G_1$. This proves (ii).

Conversely, assume (ii).

(2) $G_0 \times G_1$ is a filter on $\mathbb{P}_0 \times \mathbb{P}_1$.

For, clearly $G_0 \times G_1$ is closed upwards. Now suppose that $(p_0, p_1), (q_0, q_1) \in (G_0 \times G_1)$. Choose $s_0 \in G_0$ with $s_0 \leq p_0, q_0$, and choose $s_1 \in G_1$ so that $s_1 \leq p_1, q_1$. Then $(s_0, s_1) \in (G_0 \times G_1)$ and $(s_0, s_1) \leq (p_0, p_1), (q_0, q_1)$. so (2) holds.

To show that $G_0 \times G_1$ is generic, suppose that $D \in M$, $D \subseteq (\mathbb{P}_0 \times \mathbb{P}_1)$, D dense. Let

$$D^* = \{p_1 \in \mathbb{P}_1 : \exists p_0 \in G_0[(p_0, p_1) \in D]\}.$$

(3) D^* is dense in $\mathbb{P}_1$.

For, take $r_1 \in \mathbb{P}_1$. Let

$$D_0 = \{p_0 \in \mathbb{P}_0 : \exists p_1 \leq r_1[(p_0, p_1) \in D]\}.$$

Then D_0 is dense in $\mathbb{P}_0$, for if $s \in \mathbb{P}_0$ then there is a $(p_0, p_1) \in D$ with $(p_0, p_1) \leq (s, r_1)$, and then $p_0 \leq s$ and $p_0 \in D_0$. It follows that there is a $p_0 \in D_0 \cap G_0$. Take $p_1 \leq r_1$ such that $(p_0, p_1) \in D$. Then $p_1 \in D^*$ and $p_1 \leq r_1$. This proves (3).

Choose $r_1 \in D^* \cap G_1$; then take $p_0 \in G_0$ such that $(p_0, p_1) \in D$. So $(p_0, p_1) \in D \cap (G_0 \times G_1)$. □

Theorem 38.4. *Suppose that $G_0 \subseteq \mathbb{P}_0 \in M$ and $G_1 \subseteq \mathbb{P}_1 \in M$. Also suppose that $G_0 \times G_1$ is $(\mathbb{P}_0 \times \mathbb{P}_1)$-generic over M. (See Theorem 38.3.)*

Then $M[G_0 \times G_1] = M[G_0][G_1] = M[G_1][G_0]$.

Proof. We have $M \subseteq M[G_0][G_1]$ and $(G_0 \times G_1) \in M[G_0][G_1]$. Hence $M[G_0 \times G_1] \subseteq M[G_0][G_1]$ by Lemma 15.8. Also, $M \subseteq M[G_0 \times G_1]$ and $G_0 \in M[G_0 \times G_1]$, so $M[G_0] \subseteq M[G_0 \times G_1]$. Next, $G_1 \in M[G_0 \times G_1]$, so by Lemma 15.8, $M[G_0][G_1] \subseteq M[G_0 \times G_1]$. This proves that $M[G_0][G_1] = M[G_0 \times G_1]$. Similarly, $M[G_1][G_0] = M[G_0 \times G_1]$. □

Theorem 38.5. *Suppose that $I = I_0 \cup I_1$ with $I_0, I_1 \in M$. Let G be $\mathrm{Fn}(I, 2, \omega)$-generic over M. Let $G_0 = G \cap \mathrm{Fn}(I_0, 2, \omega)$ and $G_1 = G \cap \mathrm{Fn}(I_1, 2, \omega)$. Then:*

(i) G_0 is $\mathrm{Fn}(I_0, 2, \omega)$-generic over M.

(ii) G_1 is $\mathrm{Fn}(I_1, 2, \omega)$-generic over $M[G_0]$.

(iii) $M[G] = M[G_0][G_1]$.

Proof. Define $f : \mathrm{Fn}(I_0, 2, \omega) \times \mathrm{Fn}(I_1, 2, \omega)$ by setting $f(p, q) = p \cup q$ for any $p \in \mathrm{Fn}(I_0, 2, \omega)$ and $q \in \mathrm{Fn}(I_1, 2, \omega)$. Clearly f is an isomorphism. Note that $f^{-1}(r) = (r \cap \mathrm{Fn}(I_0, 2, \omega), r \cap \mathrm{Fn}(I_1, 2, \omega))$. By Lemma 25.9, $M[G] = M[f^{-1}[G]] = M[\mathrm{Fn}(I_0, 2, \omega) \times \mathrm{Fn}(I_1, 2, \omega)]$. Now (i) and (ii) hold by Theorem 38.3(ii). (iii) holds by Theorem 38.4. □

Lemma 38.6. *Let M be a c.t.m. and let $I, S \in M$. Let G be* $\mathrm{Fn}(I, 2, \omega)$*-generic over M. Suppose that $X \in M[G]$ and $X \subseteq S$. Then $X \in M[G \cap \mathrm{Fn}(I_0, 2, \omega)]$ for some $I_0 \subseteq I$ such that $I_0 \in M$ and $|I_0| \leq |S|)^M$.*

Proof. If S is finite, then $X \in M$; so assume that S is infinite. By Proposition 24.2 let τ be a nice name for a subset of $\check{S}$ such that $X = \tau_G$. Say $\tau = \bigcup_{s \in S}(\check{s} \times A_s)$, where each A_s is an antichain in $\mathrm{Fn}(I, 2, \omega)$. Let

$$I_0 = \bigcup\{\mathrm{dmn}(p) : \exists s \in S[p \in A_s]\}.$$

Let $G_0 = G \cap \mathrm{Fn}(I_0, 2, \omega)$. Thus $X \in M[G \cap \mathrm{Fn}(I_0, 2, \omega)]$. Now by ccc in M (see Lemma 16.7), each A_s is countable. Hence $|I_0| \leq |S|)^M$. $\square$

Theorem 38.7. *Suppose that M satisfies CH. and $I \in M$. Let G be* $\mathrm{Fn}(I, 2, \omega)$*-generic over M. Then in $M[G]$ there is a mad family of size ω_1.*

Proof. For a while we work with $\mathbb{P} \stackrel{\text{def}}{=} \mathrm{Fn}(\omega, 2, \omega)$. Note that if A is an antichain in $\mathbb{P}$ then $|A| \leq |\mathbb{P}| = \omega$. Hence there are at most ω_1 pairs (p, τ) such that $p \in \mathbb{P}$ and τ is a nice name for a subset of $\check{\omega}$. Let $\langle (p_\xi, \tau_\xi) : \xi < \omega_1 \rangle$ list all such pairs.

Now we define $A \in {}^{\omega_1}([\omega]^\omega)$ by recursion. Let $\langle A_n : n \in \omega \rangle$ be a system of infinite pairwise disjoint subsets of ω. Now suppose that $\xi \in [\omega, \omega_1)$ and A_η has been defined for all $\eta < \xi$. Then:

(1) There is a $B \in [\omega]^\omega$ such that the following conditions hold:
  (i) $\forall \eta < \xi[|A_\eta \cap B| < \omega]$.
  (ii) If
    (I) $p_\xi \Vdash (|\tau_\xi| = \omega)$ and $\forall \eta < \xi[p_\xi \Vdash |\tau_\xi \cap \check{A}_\eta| < \omega]$, then
    (II) $\forall n \in \omega \forall q \leq p_\xi \exists r \leq q \exists m \geq n[m \in B$ and $r \Vdash \check{m} \in \tau_\xi]$.

To prove (1), first note that if (1)(ii)(I) fails to hold, then we can use the proof described after Proposition 33.20 to construct B satisfying (i). So we may assume that (1)(ii)(I) holds. Now let $\langle C_i : i \in \omega \rangle$ enumerate $\{A_\eta : \eta < \xi\}$ without repetitions, and let $\langle (n_i, q_i) : i \in \omega \rangle$ enumerate $\omega \times \{q : q \leq p_\xi\}$. Clearly for all $i \in \omega$ we have $p_\xi \Vdash (|\tau_\xi \backslash (\check{C}_0 \cup \ldots \cup \check{C}_i)| = \check{\omega})$; hence each q_i also forces this. So

$$q_i \Vdash \exists m \geq \check{n}_i[m \in \tau_\xi \text{ and } m \notin (\check{C}_0 \cup \ldots \cup \check{C}_i)].$$

By Theorems 16.14 and 16.15 there exist $r_i \leq q_i$ and $m_i \geq n_i$ such that $m_i \notin (C_0 \cup \ldots \cup C_i)$ and $r_i \Vdash (\check{m}_i \in \tau_\xi)$. Let $B = \{m_i : i \in \omega\}$. Clearly (i) and (ii)(II) hold. Let $A_\xi = B$. This finishes the construction.

Let $\mathscr{A} = \{A_\xi : \xi < \omega_1\}$.

(2) If G is $\mathbb{P}$-generic over M, then $\mathscr{A}$ is mad in $M[G]$.

In fact, otherwise there is a $\xi < \omega_1$ such that $p_\xi \Vdash |\tau_\xi| = \omega$ and $p_\xi \Vdash \forall X \in \mathscr{A}[|\tau_\xi \cap X| < \omega]$. So (1)(i)(I) holds, and also $p_\xi \Vdash |\tau_\xi \cap \check{A}_\xi| < \omega$. So there exist a $q \leq p$ and an $n \in \omega$ such that $q \Vdash [\tau_\xi \cap A_\xi \subseteq \check{n}]$. This contradicts (1)(ii)(II). Thus (2) holds.

Now suppose that G os $\mathrm{Fn}(I, 2, \omega)$-generic over M, $X \in M[G]$, $|X| = \omega$, and $\forall Y \in \mathscr{A}[|X \cap Y| < \omega]$. By Lemma 38.6, $X \in M[G \cap \mathrm{Fn}(I_0, 2, \omega)]$for some $I_0 \subseteq I$ with $|I_0| = \omega$. Now $\mathrm{Fn}(I_0, 2, \omega) \cong \mathrm{Fn}(\omega, 2, \omega)$, so by Lemma 25.9 $M[G \cap \mathrm{Fn}(I_0, 2, \omega)] = M[H]$ for some H which is $\mathbb{P}$-generic over M. This contradicts (2). □

Corollary 38.8. *It is relatively consistent that $\mathfrak{a} < 2^\omega$.* □

Theorem 38.9. *It is relatively consistent that $\mathfrak{u} < 2^\omega$.*

Proof. Let M be a c.t.m. such that $2^\omega > \omega_1$ in M. Let U a nonprincipal ultrafilter on ω in M. Define

$$\begin{aligned} &\mathbb{P} = \{(F, H) : F \in [U]^{<\omega},\ H \in [\omega]^{<\omega}\};\\ &(F, H) \leq (F', H') \quad \text{iff} \quad F \supseteq F',\ H \supseteq H',\ \forall x \in F \forall m \in H \backslash H'[m \in x]. \end{aligned}$$

Clearly $\mathbb{P}$ is ccc, by considering second coordinates.

For each $x \in U$ let

$$D_x = \{(F, H) : x \in F\}.$$

Clearly D_x is dense. For each $m \in \omega$ let

$$E_m = \{(F, H) : \exists n \geq m[n \in H]\}.$$

This is dense too: given $(F, H) \in \mathbb{P}$, choose $n \in \bigcap F \backslash m$; then $(F, H \cup \{m\}) \leq (F, H)$.

Now let G be generic over M for $\mathbb{P}$. Define

$$a = \bigcup_{(F,H) \in G} H.$$

By the density of the E_m's, a is infinite. Now suppose that $x \in U$. Choose $(F, H) \in G$ such that $x \in F$. We claim that $a \backslash x \subseteq H$. For, suppose that $m \in a \backslash H$. Say $m \in H'$ with $(F', H') \in G$. Choose $(F'', H'') \in G$ such that $(F'', H'') \leq (F, H), (F', H')$. Then $m \in H' \subseteq H''$, $m \notin H$, and $x \in F$, so $m \in x$, as desired.

Now we do an iterated forcing, using the above construction at successor steps, obtaining:

(1) an increasing sequence $\langle M_\alpha : \alpha \leq \omega_1 \rangle$ of c.t.m., with $M_0 = M$;

(2) a sequence $\langle a_\alpha : \alpha < \omega_1 \rangle$ with each a_α an infinite subset of ω in M_α;

(3) an increasing sequence $\langle U_\alpha : \alpha < \omega_1 \rangle$ of ultrafilters on ω, each $U_\alpha \in M_\alpha$;

(4) for each $\alpha < \omega_1$ we have $\forall x \in U_\alpha[a_\alpha \leq x]$;

(5) $\{a_\beta : \beta < \alpha\} \subseteq U_\alpha$ for all $\alpha < \omega_1$.

Then we let U_{ω_1} be the filter generated by $\{a_\alpha : \alpha < \omega_1\}$ in M_{ω_1}. It is an ultrafilter, since each subset of ω in M_{ω_1} is in some M_α with $\alpha < \omega_1$, by Lemma 26.14. □

Lemma 38.10. *Let M be a c.t.m. of ZFC, and suppose that I is an ideal in $\mathscr{P}(\omega)^M$ containing all singletons. Define*

$$P = \{(b, y) : b \in I,\ y \in [\omega]^{<\omega}\};$$
$$(b, y) \leq (b', y') \quad iff \quad b \supseteq b',\ y \supseteq y',\ y \cap b' \subseteq y'.$$

Then P is ccc. Let G be P-generic over M, and define $d = \bigcup_{(b,y)\in G} y$. Then the following conditions hold:

(i) If $c \subseteq \omega$ and $c \notin I$, then $c \cap d$ is infinite.

(ii) If $c \subseteq \omega$ and $c \notin I$ then $c\backslash d$ is infinite.

(iii) If $b \in I$, then $b \cap d$ is finite.

Proof. Assume the hypotheses. Clearly P is ccc. For (i) and (ii), suppose that $c \subseteq \omega$ and $c \notin I$. For each $n \in \omega$ let

$$E_n = \{(b, y) : \exists m > n[m \in c \cap y\}.$$

To show that E_n is dense, let $(b, y) \in P$. Then $c\backslash b$ is infinite, as otherwise $c \subseteq b \cup (c\backslash b) \in I$. Choose $m > n$ with $m \in c\backslash b$. Then $(b, y \cup \{m\}) \in E_n$ and $(b, y \cup \{m\}) \leq (b, y)$, showing that E_n is dense.

The denseness of each set E_n clearly implies (i).

Next, define for any $n \in \omega$

$$H_n = \{(b, y) \in P : \exists m > n[m \in b \cap c\backslash y]\}.$$

To show that H_n is dense, let $(b, y) \in P$ be given. Since every finite subset of ω is in I, the set c is infinite. Choose $m \in c\backslash y$ with $m > n$. Then $(b \cup \{m\}, y) \in H_n$ and $(b \cup \{m\}, y) \leq (b, y)$. This shows that H_n is dense.

Now given $n \in \omega$, choose $(b, y) \in H_n \cap G$, and then choose $m > n$ such that $m \in b \cap c\backslash y$. We claim that $m \notin d$. For, suppose that $m \in d$; say $m \in y'$ with $(b', y') \in G$. Choose $(b'', y'') \in G$ such that $(b'', y'') \leq (b, y), (b', y')$. Thus $y'' \cap b \subseteq y$ and $y'' \cap b' \subseteq y'$. Now $m \in y'$, so $m \in y''$; also $m \in b$, so $m \in y$, contradiction. This finishes the proof of (ii).

For (iii), suppose that $b \in I$. Now the set $\{(c, y) \in P : b \subseteq c\}$ is clearly dense, so choose $(c, y) \in G$ such that $b \subseteq c$. We claim that $b \cap d \subseteq y$. In fact, suppose that $m \in b \cap d$. Say $(e, z) \in G$ with $m \in z$. Choose $(u, v) \in G$ such that $(u, v) \leq (c, y), (e, z)$. So $v \cap c \subseteq y$ and $v \cap e \subseteq z$. Now $m \in z \subseteq v$, and $m \in b \subseteq c$, so $m \in v \cap c \subseteq y$, as desired; (iii) holds. □

Lemma 38.11. *We work within a c.t.m. M. Suppose that κ is an infinite cardinal, and $\langle a_\xi : \xi < \kappa\rangle$ is a system of infinite subsets of ω which is an independent system. Let $\mathscr{A} = \{a_\xi : \xi < \kappa\}$. Thus $\langle [a_\xi] : \xi < \kappa\rangle$ is a system of independent elements of $\mathscr{P}(\omega)/\mathrm{fin}$. Let A be the completion of $Fr(\kappa)$, and let $\langle x_\xi : \xi < \kappa\rangle$ be the free generators of $Fr(\kappa)$. Then by Sikorski's extension theorem, there is a homomorphism f from $\mathscr{P}(\omega)/\mathrm{fin}$ into A such that $f([a_\xi]) = x_\xi$ for every $\xi < \kappa$. Let $h(b) = f([b])$ for any $b \subseteq \omega$. So h is a homomorphism from $\mathscr{P}(\omega)$ into A such that $h(a_\xi) = x_\xi$ for every $\xi < \kappa$. Also, $h(M) = 0$ for every finite $M \subseteq \omega$.*

Apply Lemma 38.10 to the ideal $\ker(h)$*, obtaining* P, G, d *as indicated there. Then:*

(i) If R *is a finite subset of* κ *and* $\varepsilon \in {}^R2$*, then* $\bigcap_{\alpha\in R} a_\alpha^{\varepsilon(\alpha)} \cap d$ *is infinite.*

(ii) If R *is a finite subset of* κ *and* $\varepsilon \in {}^R2$*, then* $\bigcap_{\alpha\in R} a_\alpha^{\varepsilon(\alpha)} \backslash d$ *is infinite.*

(iii) If $b \in ker(h)$*, then* $b \cap d$ *is finite.*

(iv) If $x \in \mathscr{P}(\omega) \cap M\backslash\mathscr{A}$*, then* $\mathscr{A} \cup \{x, d\}$ *is not independent.*

Proof.

(i) Let R and ε be as in (i). Then $\bigcap_{\alpha\in R} a_\alpha^{\varepsilon(\alpha)} \notin I$ by assumption, so the desired conclusion follows from (i) of Lemma 38.10.

(ii) is proved similarly, and (iii) follows from (iii) of Lemma 38.10.

Finally, for (iv), we show that if $x \in \mathscr{P}(\omega) \cap M\backslash\mathscr{A}$, then $\mathscr{A} \cup \{x, d\}$ is not an independent family.

Case 1. $h(x) = 0$. Then $x \cap d$ is finite by (iii).

Case 2. $h(x) \neq 0$. Then there is a finite subset R of κ and a $\varepsilon \in {}^R2$ such that $\bigcap_{\alpha\in R} x_\alpha^{\varepsilon(\alpha)} \leq h(x)$. It follows that $\bigcap_{\alpha\in R} a_\alpha^{\varepsilon(\alpha)}\backslash x$ is in the kernel of h, and so $\bigcap_{\alpha\in R} a_\alpha^{\varepsilon(\alpha)}\backslash x \cap d$ is finite. □

Theorem 38.12. *It is relatively consistent to have* $\mathfrak{i} < 2^\omega$*.*

Proof. We start with a c.t.m. M such that $2^\omega > \omega_1$ in M and with an independent family $\langle a_n : n \in \omega\rangle$ in $\mathscr{P}(\omega)$ in M. Then we do an iteration of length ω_1, applying Lemma 38.11 at successor steps, building an independent sequence $\langle a_\alpha : \alpha < \omega_1\rangle$. The final model is as desired, using Lemma 26.14. □

Lemma 38.13. *Let* M *be a c.t.m. of ZFC. Suppose that* κ *is an infinite cardinal and* $\langle a_i : i < \kappa\rangle$ *is a system of infinite subsets of* ω *such that* $\langle [a_i] : i < \kappa\rangle$ *is ideal independent, where* $[x]$ *denotes the equivalence class of* x *modulo the ideal fin of* $\mathscr{P}(\omega)$*. Then there is a generic extension* $M[G]$ *of* M *using a ccc partial order such that in* $M[G]$ *there is a* $d \subseteq \omega$ *with the following two properties:*

(i) $\langle [a_i] : i < \kappa\rangle^\frown\langle[\omega\backslash d]\rangle$ *is ideal independent.*

(ii) If $x \in (\mathscr{P}(\omega) \cap M)\backslash(\{a_i : i < \kappa\} \cup \{\omega\backslash d\})$*, then* $\langle [a_i] : i < \kappa\rangle^\frown\langle[\omega\backslash d], [x]\rangle$ *is not ideal independent.*

Proof. Let I be the ideal on $\mathscr{P}(\omega)$ generated by

$$\{\{m\} : m \in \omega\} \cup \{a_i \cap a_j : i, j < \kappa, i \neq j\},$$

and let f be the natural homomorphism from $\mathscr{P}(\omega)$ onto $\mathscr{P}(\omega)/I$. Note that $f(a_i) \neq 0$ for all $i < \kappa$, by ideal independence. Let B be the subalgebra of $\mathscr{P}(\omega)/I$ generated by $\{f(a_i) : i < \kappa\}$. Thus B is an atomic BA, with $\{f(a_i) : i < \kappa\}$ its set of atoms. Thus f is a homomorphism from $\mathscr{P}(\omega)$ onto B.

Now we apply Lemma 38.10 to the ideal $\ker(f)$, obtaining P, G, d as indicated there.

(1) If R is a finite subset of κ and $i \in \kappa\backslash R$, then $a_i \cap \bigcap_{j\in R}(\omega\backslash a_j) \cap d$ is infinite.

In fact, $a_i \cap \bigcap_{j\in R}(\omega\backslash a_j)$ is clearly not in the kernel of f, so (1) follows from (i) of Lemma 38.10.

(2) If R is a finite subset of κ, then $\omega\backslash\left(d\cup\bigcup_{i\in R}a_i\right)$ is infinite.

In fact, $\omega\backslash\bigcup_{i\in R}a_i$ is clearly not in the kernel of f, so (2) follows from (ii) of Lemma 38.10.

Now we can show that $\langle[a_i]:i<\kappa\rangle^\frown\langle[\omega\backslash d]\rangle$ is ideal independent. Suppose not. Then there are two possibilities.

Case 1. There exist a finite $R\subseteq\kappa$ and an $i\in\kappa\backslash R$ such that $[a_i]\le[\omega\backslash d]+\sum_{j\in R}[a_j]$. This contradicts (1).

Case 2. There is a finite $R\subseteq\kappa$ such that $[\omega\backslash d]\le\sum_{i\in R}[a_i]$. This contradicts (2).

This proves ideal independence.

It remains only to prove (ii). So, assume that $x\in(\mathscr{P}(\omega)\cap M)\backslash(\{a_i:i<\kappa\}\cup\{\omega\backslash d\})$.

Case 1. $x\in\ker(f)$. Then $[x]\le[\omega\backslash d]$ by (iii) of Lemma 1, as desired.

Case 2. $x\notin\ker(f)$. Choose $i<\kappa$ such that $f(a_i)\le f(x)$. Thus $a_i\backslash x\in\ker(f)$, Hence by (iii) of Lemma 1, $(a_i\backslash x)\cap d$ is finite. So $[a_i]\le[x]+[\omega\backslash d]$, as desired. □

Theorem 38.14. *It is relatively consistent that* $\mathrm{s_{mm}}<2^\omega$.

Lemma 38.15. *Let M be a c.t.m. of ZFC. Suppose that α is an infinite ordinal, and $\langle a_\xi:\xi<\alpha\rangle$ is a system of infinite subsets of ω such that $\langle[a_\xi]:\xi<\alpha\rangle$ is a free sequence in $\mathscr{P}(\omega)/\mathrm{fin}$, where $[a_\xi]$ denotes the equivalence class of a_ξ modulo the ideal fin.*

Then there is a generic extension $M[G]$ of M using a ccc partial order such that in $M[G]$ there exist infinite $d,e\subseteq\omega$ with the following properties:

(i) $\langle[a_\xi]:\xi<\alpha\rangle^\frown\langle[\omega\backslash d],[e]\rangle$ is a free sequence.

(ii) If $x\in(\mathscr{P}(\omega)\cap M)\backslash(\{a_\xi:\xi<\alpha\}\cup\{\omega\backslash d,e\})$, then $\langle[a_\xi]:\xi<\alpha\rangle^\frown\langle[\omega\backslash d],[e],[x]\rangle$ is not a free sequence.

Proof. For each $\xi\le\alpha$, the set $\{[a_\eta]:\eta<\xi\}\cup\{-[a_\eta]:\xi\le\eta<\alpha\}$ has the fip, by the free sequence property, and we let F_ξ be an ultrafilter on $\mathscr{P}(\omega)/\mathrm{fin}$ containing this set. Let $I=\{x:-[x]\in F_\xi$ for all $\xi\le\alpha\}$. Clearly I is an ideal on $\mathscr{P}(\omega)$ and $\{m\}\in I$ for all $m\in\omega$.

(1) If $\xi<\eta<\alpha$, then $[a_\eta]_I<[a_\xi]_I$.

In fact, suppose that $\xi<\eta<\alpha$. If $\nu\le\alpha$ and $[a_\eta]\cdot-[a_\xi]\in F_\nu$, then $\eta<\nu$, hence $\xi<\nu$ and so $[a_\xi]\in F_\nu$, contradiction. Hence $-([a_\eta]\cdot-[a_\xi])\in F_\nu$ for all $\nu\le\alpha$, and so $[a_\eta]_I\le[a_\xi]_I$. Now suppose that $[a_\eta]_I=[a_\xi]$. Then $a_\xi\cdot-a_\eta\in I$, so in particular $-[a_\xi]+[a_\eta]\in F_{\xi+1}$. Since also $[a_\xi]\in F_{\xi+1}$, it follows that $[a_\eta]\in F_{\xi+1}$. But $\xi<\eta$, contradiction. So (1) holds.

(2) $[a_0]_I\ne 1$.

This holds since $-[a_0]\in F_0$, and hence $(\omega\backslash a_0)\notin I$.

(3) If $\alpha=\beta+1$, then $[a_\beta]_I\ne 0$.

This is true since $[a_\beta]\in F_\alpha$, and hence $[a_\beta]\notin I$.

Now let J be an ideal in $\mathscr{P}(\omega)$ which is maximal subject to the following conditions:

(4) $I\subseteq J$.

(5) If $\xi<\eta<\alpha$, then $a_\xi\backslash a_\eta\notin J$.

(6) $\omega\backslash a_0 \notin J$.

(7) If $\alpha = \beta + 1$, then $a_\beta \notin J$.

Clearly then we have:

(8) For any $x \subseteq \omega$ one of the following conditions holds.

(a) $x \in J$.

(b) There exist $\xi < \eta < \alpha$ such that $a_\xi \cdot -a_\eta \cdot -x \in J$.

(c) $-a_0 \cdot -x \in J$.

(d) $\alpha = \beta + 1$ and $a_\beta \cdot -x \in J$.

Also we have

(9) If $F, K \in [\alpha]^{<\omega}$ and $F < K$, then $\bigcap_{\xi\in F} a_\xi \cap \bigcap_{\eta\in K} -a_\eta \notin J$.

Now we apply Lemma 1 to the ideal J to obtain a generic extension $M[G]$ such that, with $d = \bigcup_{(b,y)\in G} y$, the following conditions hold:

(10) If $c \subseteq \omega$ and $c \notin J$, then $c \cap d$ is infinite.

(11) If $c \subseteq \omega$ and $c \notin J$ then $c\backslash d$ is infinite.

(12) If $b \in J$, then $b \cap d$ is finite.

Hence by (9) we get

(13) If $F, K \in [\alpha]^{<\omega}$ and $F < K$, then $\bigcap_{\xi\in F} a_\xi \cap \bigcap_{\eta\in K} -a_\eta \cap d$ is infinite.

(14) If $F, K \in [\alpha]^{<\omega}$ and $F < K$, then $\bigcap_{\xi\in F} a_\xi \cap \bigcap_{\eta\in K} -a_\eta\backslash d$ is infinite.

Now let K be the ideal in $\mathscr{P}(\omega)^{M[G]}$ generated by J.

(15) If F is a finite subset of α, then $\bigcap_{\xi\in F} a_\xi \cap (\omega\backslash d) \notin K$.

In fact, otherwise we get a $c \in J$ such that $\bigcap_{\xi\in F} a_\xi \cap (\omega\backslash d) \subseteq c$, and so $\left(\bigcap_{\xi\in F} a_\xi\backslash c\right) \cap (\omega\backslash d) = \emptyset$. But clearly $\left(\bigcap_{\xi\in F} A_\xi\backslash c\right) \notin J$, so this contradicts (11). Similarly,

(16) If $F, L \in [\alpha]^{<\omega}$ and $F < L$, then $\bigcap_{\xi\in F} a_\xi \cap \bigcap_{\eta\in L} -a_\eta \cap d \notin K$.

Now we apply Lemma 1 with I replaced by K to obtain a generic extension $M[G][H]$ and an infinite subset e of ω such that $\langle [a_\xi : \xi < \alpha\rangle^\frown\langle[\omega\backslash d], [e]\rangle$ is a free sequence and the following condition holds:

(16) If $b \in K$, then $b \cap e$ is finite.

(17) If $x \in (\mathscr{P}(\omega) \cap M)\backslash(\{a_\xi : \xi < \alpha\} \cup \{\omega\backslash d, e\}$, then $\langle [a_\xi] : \xi < \alpha\rangle^\frown\langle[\omega\backslash d], [e], [x]\rangle$ is not a free sequence.

To prove this, we consider cases.

Case 1. $x \in K$. Then $x \cap e$ is finite by (16), as desired.

Case 2. $x \notin K$. Then $x \notin J$, and so by (8) we have three subcases.

Subcase 2.1. There exist $\xi < \eta < \alpha$ such that $a_\xi \cdot -a_\eta \cdot -x \in J$. Then by (12), $a_\xi \cdot -a_\eta \cdot -x \cdot d$ is finite, as desired.

Subcase 2.2. $-a_0 \cdot -x \in J$. Then by (12), $-a_0 \cdot -x \cdot d$ is finite, as desired.

Subcase 2.3. $\alpha = \beta + 1$ and $a_\beta \cdot -x \in J$. Then by (12), $a_\beta \cdot -x \cdot d$ is finite, as desired. $\square$

Theorem 38.16. *It is relatively consistent that* $\mathfrak{f} < 2^\omega$. $\square$

REAL NUMBERS

39. The integers

In this appendix we define and develop the main properties of the integers. The development is based upon Chapter 6, in which properties of natural numbers were given. At the end of that chapter a sketch of the construction of integers was given, and we now give full details.

Let $A = \omega \times \omega$. We define a relation $\sim$ on A by setting, for any $m, n, p, q \in \omega$,

$$(m, n) \sim (p, q) \quad \text{iff} \quad m + q = n + p.$$

This definition is motivated by thinking of (m, n) as representing, in some sense, $m - n$.

Lemma 39.1. *$\sim$ is an equivalence relation on A.*

Proof. For reflexivity, given $m, n \in \omega$ we want to show that $(m, n) \sim (m, n)$. By definition, this means that we want to show that $m+n = n+m$. This is given by 6.14(iv).

For symmetry, assume that $(m, n) \sim (p, q)$; we want to show that $(p, q) \sim (m, n)$. The assumption means, by definition, that $m + q = n + p$. Hence $p + n = q + m$ by 6.14(iv) again. Hence $(p, q) \sim (m, n)$. [In the definition, replace m, n, p, q by p, q, m, n respectively.]

For transitivity, assume that $(m, n) \sim (p, q) \sim (r, s)$. Thus $m + q = n + p$ and $p + s = q + r$. Hence $m + q + s = n + p + s = n + q + r$, so using 6.15(iii) we get $m + s = n + r$, so that $(m, n) \sim (r, s)$. □

We now let $\mathbb{Z}'$ be the collection of all equivalence classes under $\sim$. Elements of $\mathbb{Z}'$ are denoted by $[(m, n)]$ with $m, n \in \omega$.

For the purposes of this appendix, we treat binary operations on $\mathbb{Z}'$ as functions mapping $\mathbb{Z}' \times \mathbb{Z}'$ into $\mathbb{Z}'$.

Proposition 39.2. *There is a binary operation $+$ on $\mathbb{Z}'$ such that for any $m, n, p, q \in \omega$, $[(m, n)] + [(p, q)] = [(m + p, n + q)]$.*

Proof. Let

$$\begin{aligned} R = \{(x, y) :\ & \text{there exist } m, n, p, q \in \omega \text{ such that} \\ & x = ([(m, n)], [(p, q)]) \text{ and } y = [(m + p, n + q)]\}. \end{aligned}$$

We claim that R is a function. For, assume that $(x, y), (x, z) \in R$. Then we can choose $m, n, p, q, m', n', p', q' \in \omega$ such that the following conditions hold:

(1) $$x = ([(m, n)], [(p, q)]);$$
(2) $$y = [(m + p, n + q)];$$
(3) $$x = ([(m', n')], [(p', q')]);$$
(4) $$z = [(m' + p', n' + q')].$$

From (1) and (3) we get $[(m,n)] = [(m',n')]$ and $[(p,q)] = [(p',q')]$, hence $(m,n) \sim (m',n')$ and $(p,q) \sim (p',q')$, hence $m + n' = n + m'$ and $p + q' = q + p'$. Hence

$$m + p + n' + q' = m + n' + p + q' = n + m' + q + p' = n + q + m' + p',$$

from which it follows that $(m + p, n + q) \sim (m' + p', n' + q')$, hence $[(m + p, n + q)] = [(m' + p', n' + q')]$, hence $y = z$ by (2) and (4). This shows that R is a function.

Knowing that R is a function, the definition of R then says that for any $m, n, p, q \in \omega$, $([(m,n)], [(p,q)])$ is in the domain of R, and $R(([(m,n)], [(p,q)])) = [(m + p, n + q)]$. This is as desired in the proposition. □

Proposition 39.3. *The operation $+$ on $\mathbb{Z}'$ is associative and commutative. That is, if $x, y, z \in \mathbb{Z}'$, then $x + (y + z) = (x + y) + z$ and $x + y = y + x$.*

Proof. For any $a, b, c, d, e, f \in \omega$ we have

$$\begin{aligned}
[(a,b)] + ([(c,d)] + [(e,f)]) &= [(a,b)] + [(c + e, d + f)] \\
&= [(a + c + e, b + d + f)] \\
&= [(a + c, b + d)] + [(e,f)] \\
&= ([(a,b)] + [(c,d)]) + [(e,f)]; \\
[(a,b)] + [(c,d)] &= [(a + c, b + d)] \\
&= [(c + a, d + b)] \\
&= [(c,d)] + [(a,b)].
\end{aligned}$$

□

Now we define $0' = [(0,0)]$.

Proposition 39.4. *For any $a, b \in \omega$, $[(a,b)] + 0' = [(a,b)]$.* □

Proposition 39.5. *For any $x \in \mathbb{Z}'$ there is a $y \in \mathbb{Z}'$ such that $x + y = 0'$.*

Proof. Let $x \in \mathbb{Z}'$; hence there are $a, b \in \omega$ such that $x = [(a,b)]$. Let $y = [(b,a)]$. Then $x + y = [(a,b)] + [(b,a)] = [(a + b, b + a)] = [(0,0)] = 0'$. □

There are little group-theoretic facts that say that $0'$ and y above are unique:

Proposition 39.6. *If z is an element of $\mathbb{Z}'$ such that $x + z = x$ for all $x \in \mathbb{Z}'$, then $z = 0'$.*

Proof. $z = 0' + z$ (by 39.4) $= 0'$ (by assumption). □

Proposition 39.7. *If $x, y, z \in \mathbb{Z}'$ and $x + y = 0' = x + z$, then $y = z$.*

Proof. $y = 0' + y = x + z + y = z + x + y = z + 0' = z$. □

These are all of the properties of $+$ that we need.

Proposition 39.8. *There is a binary operation $\cdot$ on $\mathbb{Z}'$ such that for all $m, n, p, q \in \omega$, $[(m,n)] \cdot [(p,q)] = [(m \cdot p + n \cdot q, m \cdot q + n \cdot p)]$.*

Proof. Let

$$R = \{(x,y) : \text{there exist } m,n,p,q \in \omega \text{ such that} \\ x = ([(m,n)],[(p,q)]) \text{ and } y = [(m\cdot p + n\cdot q, m\cdot q + n\cdot p)]\}.$$

We claim that R is a function. For, assume that $(x,y),(x,z) \in R$. Then we can choose $m,n,p,q,m',n',p',q' \in \omega$ such that the following conditions hold:

$$x = ([(m,n)],[(p,q)]); \tag{1}$$

$$y = [(m\cdot p + n\cdot q, m\cdot q + n\cdot p)]; \tag{2}$$

$$x = ([(m',n')],[(p',q')]); \tag{3}$$

$$z = [(m'\cdot p' + n'\cdot q', m'\cdot q' + n'\cdot p')]. \tag{4}$$

From (1) and (3) we get $[(m,n)] = [(m',n')]$ and $[(p,q)] = [(p',q')]$, hence $(m,n) \sim (m',n')$ and $(p,q) \sim (p',q')$, hence $m + n' = n + m'$ and $p + q' = q + p'$. Hence

$$m\cdot p + m\cdot q' + n\cdot q + n\cdot p' = m\cdot q + m\cdot p' + n\cdot p + n\cdot q'. \tag{1}$$

Also,

$$m\cdot p' + n'\cdot p' + n\cdot q' + m'\cdot q' = n\cdot p' + m'\cdot p' + m\cdot q' + n'\cdot q'. \tag{2}$$

Hence

$$\begin{aligned} m\cdot p &+ n\cdot q + m'\cdot q' + n'\cdot p' + m\cdot p' + n\cdot q' \\ &= m\cdot p + n\cdot q + m\cdot p' + n'\cdot p' + n\cdot q' + m'\cdot q' \\ &= m\cdot p + n\cdot q + n\cdot p' + m'\cdot p' + m\cdot q' + n'\cdot q' \quad \text{by (2)} \\ &= m\cdot p + m\cdot q' + n\cdot q + n\cdot p' + m'\cdot p' + n'\cdot q' \\ &= m\cdot q + m\cdot p' + n\cdot p + n\cdot q' + m'\cdot p' + n'\cdot q' \quad \text{by (1)} \end{aligned}$$

Considering the first side of the top equation and the last part, we can cancel $m \cdot p'$ and $n \cdot q'$ by 6.15(iii), and we get

$$m\cdot p + n\cdot q + m'\cdot q' + n'\cdot p' = m\cdot q + n\cdot p + m'\cdot p' + n'\cdot q',$$

which easily yields $y = z$.

Thus R is a function, and this clearly proves the proposition. □

Proposition 39.9. *Let $x,y,z \in \mathbb{Z}'$. Then*

(i) $x \cdot y = y \cdot x$.
(ii) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.
(iii) $x \cdot (y + z) = x \cdot y + x \cdot z$.

Proof. Write $x = [(m, n)]$, $y = [(p, q)$, and $z = [(r, s)]$. Then

$$\begin{aligned}
x \cdot y &= [(m, n)] \cdot [(p, q)] \\
&= [(m \cdot p + n \cdot q, m \cdot q + n \cdot p)] \\
&= [(p \cdot m + q \cdot n, p \cdot n + q \cdot m)] \\
&= [(p, q)] \cdot [(m, n)] \\
&= y \cdot x; \\
x \cdot (y \cdot z) &= [(m, n)] \cdot ([(p, q)] \cdot [(r, s)]) \\
&= [(m, n)] \cdot [(p \cdot r + q \cdot s, p \cdot s + q \cdot r)] \\
&= [(m \cdot p \cdot r + m \cdot q \cdot s + n \cdot p \cdot s + n \cdot q \cdot r, \\
&\quad m \cdot p \cdot s + m \cdot q \cdot r + n \cdot p \cdot r + n \cdot q \cdot s)]; \\
(x \cdot y) \cdot z &= ([(m, n)] \cdot [(p, q)]) \cdot [(r, s)] \\
&= [(m \cdot p + n \cdot q, m \cdot q + n \cdot p)] \cdot [(r, s)] \\
&= [(m \cdot p \cdot r + n \cdot q \cdot r + m \cdot q \cdot s + n \cdot p \cdot s, \\
&\quad m \cdot p \cdot s + n \cdot q \cdot s + m \cdot q \cdot r + n \cdot p \cdot r)] \\
&= x \cdot (y \cdot z) \quad \text{by the above;} \\
x \cdot (y + z) &= [(m, n)] \cdot ([(p, q)] + [(r, s)]) \\
&= [(m, n)] \cdot [(p + r, q + s)] \\
&= [(m \cdot p + m \cdot r + n \cdot q + n \cdot s, m \cdot q + m \cdot s + n \cdot p + n \cdot r)]; \\
x \cdot y + x \cdot z &= [(m, n)] \cdot [(p, q)] + [(m, n)] \cdot [(r, s)] \\
&= [(m \cdot p + n \cdot q, m \cdot q + n \cdot p)] + [(m \cdot r + n \cdot s, m \cdot s + n \cdot r)] \\
&= [(m \cdot p + n \cdot q + m \cdot r + n \cdot s, m \cdot q + n \cdot p + m \cdot s + n \cdot r)] \\
&= x \cdot (y + z) \quad \text{by the above.}
\end{aligned}$$ □

Now we define $1' = [(1, 0)]$.

Proposition 39.10. *$1' \cdot x = x$ and $0' \cdot x = 0'$ for all $x \in \mathbb{Z}'$.*

Proof. Take any $x \in \mathbb{Z}$; say that $x = [(m, n)]$. Then

$$1' \cdot x = [(1, 0)] \cdot [(m, n)] = [(1 \cdot m + 0 \cdot m, 1 \cdot n + 0 \cdot m)] = [(m, n)] = x,$$

as desired.

For the second statement, note that $x \cdot 0' + x = x \cdot 0' + x \cdot 1' = x \cdot (0' + 1') = x \cdot 1' = x$, so $x \cdot 0' = 0'$. □

Proposition 39.11. *If $x, y \in \mathbb{Z}'$ and $x \cdot y = 0'$, then $x = 0'$ or $y = 0'$.*

Proof. Write $x = [(m, n)]$ and $y = [(p, q)]$. Now $x \cdot y = [(m, n)] \cdot [(p, q)] = [m \cdot p + n \cdot q, m \cdot q + n \cdot p)]$ and also $x \cdot y = 0' = [0, 0)]$, so $(m \cdot p + n \cdot q, m \cdot q + n \cdot p) \sim (0, 0)$, so

$$m \cdot p + n \cdot q = m \cdot q + n \cdot p. \tag{1}$$

Suppose that $x \neq 0'$; we will show that $y = 0'$, which will prove the proposition. Thus $[(m, n)] = x \neq 0' = [(0, 0)]$, so $m \neq n$. Hence $m < n$ or $n < m$.

Case 1. $m < n$. Then there is a nonzero natural number s such that $m + s = n$. Substituting this into (1) we get

$$\begin{aligned} m \cdot p + n \cdot q &= m \cdot p + (m + s) \cdot q \\ &= m \cdot p + m \cdot q + s \cdot q \quad \text{and} \\ m \cdot q + n \cdot p &= m \cdot q + (m + s) \cdot p \\ &= m \cdot q + m \cdot p + s \cdot p, \end{aligned}$$

and hence

$$m \cdot p + m \cdot q + s \cdot q = m \cdot q + m \cdot p + s \cdot p.$$

Then by 6.15(iii) we get $s \cdot q = s \cdot p$, and 6.20(viii) yields $q = p$. Hence $(p, q) \sim (0, 0)$, so $y = [(p, q)] = [(0, 0)] = 0'$.

Case 2. $n < m$. This is very similar to case 1. There is a nonzero natural number s such that $n + s = m$. Substituting this into (1) we get

$$\begin{aligned} m \cdot p + n \cdot q &= (n + s) \cdot p + n \cdot q \\ &= n \cdot p + s \cdot p + n \cdot q; \\ m \cdot q + n \cdot p &= (n + s) \cdot q + n \cdot p \\ &= n \cdot q + s \cdot q + n \cdot p, \end{aligned}$$

and hence

$$n \cdot p + s \cdot p + n \cdot q = n \cdot q + s \cdot q + n \cdot p.$$

Then by 6.15(iii) we get $s \cdot p = s \cdot q$, and 6.20(viii) yields $p = q$. Hence $(p, q) \sim (0, 0)$, so $y = [(p, q)] = [(0, 0)] = 0'$. □

This is all of the arithmetic properties of $\mathbb{Z}'$ that is needed. Now we introduce the order. First we only define the collection of positive elements:

$$P = \{[(m, n)] : m, n \in \omega \text{ and } m > n\}.$$

Note that this really means

$$P = \{x : \text{there exist } m, n \in \omega \text{ such that } x = [(m, n)] \text{ and } m > n\}.$$

Proposition 39.12. *For any $m, n \in \omega$, $[(m, n)] \in P$ iff $m > n$.*

Proof. $\Leftarrow$: true by definition. $\Rightarrow$: Suppose that $[(m, n)] \in P$. Choose $p, q \in \omega$ such that $p > q$ and $[(m, n)] = [(p, q)]$. Thus $(m, n) \sim (p, q)$, so $m + q = n + p$. If $m \leq n$, then by 6.17,

$$m + q \leq n + q < n + p = m + q,$$

contradiction. Hence $m < n$. □

Proposition 39.13. *For any $a, b \in \mathbb{Z}'$ we have:*

(i) If $a \neq 0'$, then $a \in P$ or $-a \in P$, but not both.

(ii) If $a, b \in P$, then $a + b \in P$.

(iii) If $a, b \in P$, then $a \cdot b \in P$.

Proof. Let $a = [(m, n)]$ and $b = [(p, q)]$. For (i), since $0' = [(0, 0)]$ we see that if $a \neq [(0, 0)]$ then $(m, n) \not\sim (0, 0)$ and so $m \neq n$. If $m < n$, then $-a = [(n, m)] \in P$. If $m > n$, then $a \in P$. If $a, -a \in P$, then by 39.12, $m < n$ and $n < m$, contradiction.

(ii): Assume that $a, b \in P$. Then by 39.12, $m > n$ and $p > q$. Clearly then $m+p > n+q$ by 6.17, so $a + b = [(m + p, n + q)] \in P$.

(iii): Assume that $a, b \in P$. Then by 39.12, $m > n$ and $p > q$. Write $n + s = m$ and $q + t = p$, with $s, t \neq 0$. Hence $s \cdot t \neq 0$. Now

$$(*) \qquad a \cdot b = [(m, n)] \cdot [(p, q)] = [(m \cdot p + n \cdot q, m \cdot q + n \cdot p)].$$

Now

$$\begin{aligned} m \cdot q + n \cdot p + s \cdot t &= m \cdot q + n \cdot (q + t) + s \cdot t \\ &= m \cdot q + n \cdot q + n \cdot t + s \cdot t \\ &= m \cdot q + n \cdot q + (n + s) \cdot t \\ &= m \cdot q + n \cdot q + m \cdot t \\ &= m \cdot (q + t) + n \cdot q \\ &= m \cdot p + n \cdot q, \end{aligned}$$

and so $m \cdot q + n \cdot p < m \cdot p + n \cdot q$, so that $a \cdot b \in P$ by $(*)$ and 39.12. □

Now we can define the order: $a < b$ iff $b - a \in P$. The main properties of $<$ are given in the following proposition.

Proposition 39.14. *Let $x, y, z \in \mathbb{Z}'$. Then*

(i) $x \not< x$.

(ii) If $x < y < z$, then $x < z$.

(iii) $x < y$, $x = y$, or $y < x$.

(iv) $x < y$ iff $x + z < y + z$.

(v) If $0' < x$ and $0' < y$, then $0' < x \cdot y$.

(vi) If $0' < z$, then $x < y$ implies that $x \cdot z < y \cdot z$.

Proof. (i): $x - x = 0'$, so (i) follows from 39.13(i).

(ii): Assume that $x < y < z$. So $y - x \in P$ and $z - y \in P$. Hence $z - x = (z - y) + y - x) \in P$ by 6.13(ii), so $x < z$.

(iii): We have $x = y$ or $x - y \in P$ or $y - x \in P$, so (iii) follows.

(iv): $x < y$ iff $y - x \in P$ iff $(y + z) - (x + z) \in P$ iff $x + z < y + z$.

(v): This is immediate from 39.13(iii).

(vi): Assume that $0' < z$ and $x < y$. So $z, y - x \in P$, so by 39.13(iii), $y \cdot z - x \cdot z = z \cdot (y - x) \in P$, and so $x \cdot z < y \cdot z$. □

This finishes our treatment of $\mathbb{Z}'$. Now we need to relate it to ω, and define our final version $\mathbb{Z}$ of the integers.

For any $m \in \omega$ let $f(m) = [(m, 0)]$.

Proposition 39.15. *f is a one-one function mapping ω into $\mathbb{Z}$. Moreover, for any $m, n \in \omega$ we have*

(i) $f(m + n) = f(m) + f(n)$.
(ii) $f(m \cdot n) = f(m) \cdot f(n)$.
(iii) $m < n$ iff $f(m) < f(n)$.

Proof. Suppose that $f(m) = f(n)$. Thus $[(m, 0)] = [(n, 0)]$, so $(m, 0) \sim (n, 0)$, hence $m + 0 = 0 + n$, hence $m = n$. So f is one-one. Next,

$$f(m + n) = [(m + n, 0)] = [(m, 0)] + [(n, 0)] = f(m) + f(n);$$

$$f(m \cdot n) = [(m \cdot n, 0)] = [(m \cdot n + 0 \cdot 0, m \cdot 0 + 0 \cdot n)] = [(m, 0)] \cdot [(n, 0)] = f(m) \cdot f(n)$$

$$\begin{aligned} f(m) < f(n) \quad &\text{iff} \quad [(m, 0)] < [(n, 0)] \\ &\text{iff} \quad m + 0 < 0 + n \\ &\text{iff} \quad m < n. \end{aligned}$$

□

We have now identified a part of $\mathbb{Z}'$ which acts like the natural numbers. We now want to apply the replacement process to officially define $\mathbb{Z}$.

Proposition 39.16. $\omega \cap \mathbb{Z}' = \emptyset$.

Proof. Suppose that $m \in \omega \cap \mathbb{Z}'$. Choose $n, p \in \omega$ such that $m = [(n, p)]$. But $[(n, p)]$ is an infinite set, since it contains all of the pairs $(n, p), (n + 1, p + 1), (n + 2, p + 2), \ldots$, contradiction. □

Now we define $\mathbb{Z} = (\mathbb{Z}' \backslash \mathrm{rng}(f)) \cup \omega$. There is a one-one function $g : \mathbb{Z} \to \mathbb{Z}'$, defined by $g([(m, n)]) = [(m, n)]$ if $[(m, n)] \in \mathbb{Z}' \backslash \mathrm{rng}(f)$, and $g(m) = f(m)$ for $m \in \omega$. Clearly g is a bijection. Now the operations $+'$ and $\cdot'$ are defined on $\mathbb{Z}$ as follows. For any $a, b \in \mathbb{Z}$,

$$a +' b = g^{-1}(g(a) + g(b));$$
$$a \cdot' b = g^{-1}(g(a) \cdot g(b)).$$

moreover, we define $a <' b$ iff $g(a) < g(b)$. With these definitions, g becomes an isomorphism of $\mathbb{Z}$ onto $\mathbb{Z}'$. Namely, if $a, b \in \mathbb{Z}$, then

$$g(a +' b) = g(g^{-1}(g(a) + g(b))) = g(a) + g(b);$$
$$g(a \cdot' b) = g(g^{-1}(g(a) \cdot g(b))) = g(a) \cdot g(b);$$
$$a <' b \quad \text{iff} \quad g(a) < g(b).$$

Moreover, the operations $+'$ and $\cdot'$ on ω coincide with the ones defined in Chapter 6, since if $m, n \in \omega$, then

$$m +' n = g^{-1}(g(m) + g(n)) = g^{-1}(f(m) + f(n)) = g^{-1}(f(m + n)) = m + n;$$
$$m \cdot' n = g^{-1}(g(m) \cdot g(n)) = g^{-1}(f(m) \cdot f(n)) = g^{-1}(f(m \cdot n)) = m \cdot n;$$

$$\begin{aligned} m <' n \quad &\text{iff} \quad g(m) < g(n) \\ &\text{iff} \quad f(m) < f(n) \\ &\text{iff} \quad m < n. \end{aligned}$$

All of the properties above, like the associative, commutative, and distributive laws, hold for $\mathbb{Z}$ since g is an isomorphism. Of course we use $+, \cdot, <$ now rather than $+', \cdot', <'$.

40. The rationals

Here we define the rational numbers and give their fundamental properties. For brevity we denote multiplication of integers by justaposition, as is usually done.

Let $A = \mathbb{Z} \times (\mathbb{Z}\backslash\{0\})$. We define a relation $\sim$ on A as follows:

$$(a,b) \sim (c,d) \quad \text{iff} \quad ad = bc$$

This definition and succeeding ones are well-motivated if you think of (a,b) as being $\frac{a}{b}$ intuitively.

Lemma 40.1. *$\sim$ is an equivalence relation on A.*

Proof. Reflexivity: If $(a,b) \in A$, then $ab = ba$, so $(a,b) \sim (a,b)$.

Symmetry: Assume that $(a,b) \sim (c,d)$. Thus $ad = bc$, so $cb = da$, and hence $(c,d) \sim (a,b)$.

Transitivity: Assume that $(a,b) \sim (c,d) \sim (e,f)$. Thus $ad = bc$ and $cf = de$. Hence $adf = bcf = bde$, so $0 = adf - bde = d(af - be)$. Since $d \neq 0$, it follows that $af - be = 0$, and hence $af = be$. This shows that $(a,b) = (e,f)$. $\square$

We let $\mathbb{Q}'$ be the set of all equivalence classes under $\sim$.

Proposition 40.2. *There is a binary operation $+$ on $\mathbb{Q}'$ such that for any $(a,b),(c,d) \in A$, $[(a,b)] + [(c,d)] = [(ad + bc, bd)]$.*

Proof. First note that if $(a,b),(c,d) \in A$, then $bd \neq 0$, so that at least the pair $(ad + bc, bd)$ is in A. Now let

$$\begin{aligned} R = \{(x,y) : & \text{ there exist } (a,b),(c,d) \in A \text{ such that} \\ & x = ([(a,b)],[(c,d)]) \text{ and } y = [(ad+bc,bd)]\}. \end{aligned}$$

We claim that R is a function. For, suppose that $(x,y),(x,z) \in R$. Then we can choose $(a,b),(c,d),(a',b'),(c',d') \in A$ such that $x = ([(a,b)],[(c,d)])$, $y = [(ad + bc, bd)]$, $x = ([(a',b')],[(c',d')])$, and $y = [(a'd' + b'c', b'd')]$. so $([(a,b)],[(c,d)]) = ([(a',b')],[(c',d')])$, hence $[(a,b)] = [(a',b')]$ and $[(c,d)] = [(c',d')]$, hence $(a,b) \sim (a',b')$ and $(c,d) \sim (c',d')$, hence

$$ab' = ba' \tag{1}$$

$$cd' = dc' \tag{2}$$

Hence

$$\begin{aligned} (ad+bc)b'd' &= adb'd' + bcb'd' \\ &= ab'dd' + cd'bb' \\ &= ba'dd' + dc'bb' \quad \text{by (1), (2)} \\ &= a'd'bd + b'c'bd \\ &= (a'd' + b'c')bd, \end{aligned}$$

and hence $(ad+bc, bd) \sim (a'd'+b'c', b'd')$. Thus $y = [(ad+bc, bd)] = [(a'd'+b'c', b'd')] = y'$. This proves that R is a function. The proposition is now clear. $\square$

Proposition 40.3. *If $x, y, z \in \mathbb{Q}'$, then*
(i) $x + (y + z) = (x + y) + z$.
(ii) $x + y = y + x$.

Proof. Let $x = [(a, b)]$, $y = [(c, d)]$, and $z = [(e, f)]$. Then

$$\begin{aligned}
x + (y + z) &= [(a, b)] + ([(c, d)] + [(e, f)]) \\
&= [(a, b)] + [(cf + de, df)] \\
&= [(adf + b(cf + de), bdf)]; \\
(x + y) + z &= ([(a, b)] + [(c, d)]) + [(e, f)] \\
&= [(ad + bc, bd)] + [(e, f)] \\
&= [((ad + bc)f + bde, bdf)] \\
&= [(adf + bcf + bde, bdf)] \\
&= x + (y + z); \\
x + y &= [(a, b)] + [(c, d)] \\
&= [(ad + bc, bd)] \\
&= [(cb + da, db)] \\
&= [(c, d)] + [(a, b)] \\
&= y + x.
\end{aligned}$$

$\square$

Now we define $0' = [(0, 1)]$.

Proposition 40.4. *$x + 0' = x$ for any $x \in \mathbb{Q}$. Moreover, for any $x \in \mathbb{Q}'$ there is a $y \in \mathbb{Q}'$ such that $x + y = 0'$.*

Proof. Let $x = [(a, b)]$. Then

$$\begin{aligned}
x + 0' &= [(a, b)] + [(0, 1)] \\
&= [(a \cdot 1 + b \cdot 0, b \cdot 1)] \\
&= [(a, b)] \\
&= x.
\end{aligned}$$

Next, let $y = [(-a, b)]$. Then

$$x + y = [(a, b)] + [(-a, b)] = [(ab + b(-a), bb)] = [(0, bb)] = [(0, 1)].$$

Here the last equality holds because $0 \cdot 1 = 0 = bb \cdot 0$. $\square$

The following two facts are proved as in appendix B, proof of B6 and B7.

Proposition 40.5. *If r is an element of $\mathbb{Q}'$ such that $x + r = x$ for all $x \in \mathbb{Q}'$, then $r = 0'$.*

Proposition 40.6. *If $x, y, z \in \mathbb{Q}'$ and $x + y = 0' = x + z$, then $y = z$.*

These are all of the properties of $+$ that we need.

Proposition 40.7. *There is a binary operation $\cdot$ on $\mathbb{Q}'$ such that for all $(a, b), (c, d) \in A$, $[(a, b)] \cdot [(c, d)] = [(ac, bd)]$.*

Proof. First note that if $(a, b), (c, d) \in A$, then $bd \neq 0$, so that $(ac, bd) \in A$. Now let

$$\begin{aligned} R = \{(x, y) :\ & \text{there exist } (a, b), (c, d) \in A \text{ such that} \\ & x = ([(a, b)], y = [(c, d)]), \text{ and } z = [(ac, bd)]\}. \end{aligned}$$

We claim that R is a function. For, suppose that $(x, y), (x, z) \in R$. Then we can choose $(a, b), (c, d), (a', b'), (c', d') \in A$ such that $x = ([(a, b)], [(c, d)])$, $y = [(ac, bd)]$, $x = ([(a', b')], [(c', d')])$, and $z = [(a'c', b'd')]$. So $([(a, b)], [(c, d)]) = ([(a', b')], [(c', d')])$, and hence $[(a, b)] = [(a', b')]$ and $[(c, d)] = [(c', d')]$, hence $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$, hence $ab' = ba'$ and $cd' = dc'$. Hence

$$\begin{aligned} & acb'd' = ab'cd' = ba'dc' = bda'c', \\ & \text{hence } (ac, bd) \sim (a'c', b'd'), \\ & \text{hence } y = [(ac, bd)] = [(a'c', b'd')] = z. \end{aligned}$$

So R is a function, and the conclusion is clear. □

Proposition 40.8. *For any $x, y, z \in \mathbb{Q}'$ we have*
(i) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.
(ii) $x \cdot y = y \cdot x$.
(iii) $x \cdot (y + z) = x \cdot y + x \cdot z$.

Proof. Write $x = [(a, b)]$, $y = [(c, d)$, and $z = [(e, f)]$. Then

$$\begin{aligned} x \cdot (y \cdot z) &= [(a, b)] \cdot ([(c, d)] \cdot [(e, f)]) \\ &= [(a, b)] \cdot [(ce, df)] \\ &= [(ace, bdf)] \\ &= [(ac, bd)] \cdot [(e, f)] \\ &= ([(a, b)] \cdot [(c, d)]) \cdot [(e, f)] \\ &= (x \cdot y) \cdot z; \\ x \cdot y &= [(a, b)] \cdot [(c, d)] \\ &= [(ac, bd)] \\ &= [(ca, db)] \\ &= [(c, d)] \cdot [(a, b)] \\ &= y \cdot x; \\ x \cdot (y + z) &= [(a, b)] \cdot ([(c, d)] + [(e, f)]) \end{aligned}$$

$$\begin{aligned}
&= [(a,b)]\cdot[(cf+de,df)]\\
&= [(a(cf+de),bdf)]\\
&= [(acf+ade,bdf)];\\
x\cdot y+x\cdot z &= [(a,b)]\cdot[(c,d)]+[(a,b)]\cdot[(e,f)]\\
&= [(ac,bd)]+[(ae,bf)]\\
&= [(acbf+bdae,bdbf)].
\end{aligned}$$

Thus for the distributive law (iii) we just need to show that $[(acf+ade,bdf)] = [(acbf+bdae,bdbf)]$, or equivalently that $(acf+ade,bdf)\sim(acbf+bdae,bdbf)$, or equivalently that $(acf+ade)bdbf = bdf(acbf+bdae)$. This last statement is proved as follows:

$$(acf+ade)bdbf = abbcdff+abbddef \text{ and } bdf(acbf+bdae) = abbcdff+abbddef. \quad \square$$

Next, we define $1' = [(1,1)]$.

Proposition 40.9. *Let $x\in\mathbb{Q}'$.*

(i) $x\cdot 1' = x$.

(ii) If $x\neq 0'$ then there is a unique $y\in\mathbb{Q}'$ such that $x\cdot y = 1'$.

Proof. Write $x=[(a,b)]$. Then $x\cdot 1' = [(a,b)]\cdot[(1,1)] = [(a,b)] = x$. For (ii), assume that $x\neq 0'$. Thus $[(a,b)]\neq[(0,1)]$, so $a\cdot 1\neq b\cdot 0$, i.e., $a\neq 0$. Let $y=[(b,a)]$. Then $x\cdot y = [(a,b)]\cdot[(b,a)] = [(ab,ba)]$, and this is equal to $[(1,1)] = 1'$ since $ab1 = ba1$. Suppose that also $x\cdot z = 1'$. Write $z = [(c,d)]$. then $[(1,1)] = x\cdot z = [(a,b)]\cdot[(c,d)] = [ac,bd)$, and so $ac = bd$, and hence $y = [(b,a)] = [(c,d)] = z$. $\square$

We turn to the order of the rationals. In general outline, we follow the procedure used for the integers.

First we define the set P of positive rationals:

$$P = \{[(a,b)]\in\mathbb{Q}' : ab>0\}.$$

As for the similar definition for integers, this definition says that if $ab>0$ then $[(a,b)]\in P$, but does not say anything about the converse, so we prove this converse:

Proposition 40.10. $[(a,b)]\in P$ *iff* $ab>0$.

Proof. As mentioned, $\Leftarrow$ holds by definition. Now assume that $[(a,b)]\in P$. This means that there is a $[(c,d)]\in\mathbb{Q}'$ such that $[(a,b)] = [(c,d)]$ and $cd>0$. So $(a,b)\sim(c,d)$, and hence $ad = bc$. Hence $adbd = bcbd$. Now we need the following little general fact:

(1) If $x\in\mathbb{Z}$ and $x\neq 0$, then $xx>0$.

In fact, we have $x>0$ or $-x>0$ by B13(i) and the definition of $<$ for integers, so by B14(v), $xx>0$ or $xx = (-x)(-x)>0$, as desired in (1).

Now by (1) and B14(v) we have $abdd = bcbd>0$. In particular, $ab\neq 0$. If $ab<0$, then $abdd<0dd=0$, contradiction. So $ab>0$. $\square$

Proposition 40.11. *Suppose that $r, s \in \mathbb{Q}'$.*

(i) If $r \neq 0'$, then $r \in P$ or $-r \in P$, but not both.

(ii) If $r, s \in P$, then $r + s \in P$.

(iii) If $r, s \in P$, then $r \cdot s \in P$.

Proof. Let $r = [(a, b)]$ and $s = [(c, d)]$.

(i): Assume that $r \neq 0'$. Then $ab \neq 0$, since $ab = 0$ would imply that $a = 0$ (since $b \neq 0$), and so $(a, b) = (0, b) \sim (0, 0)$ and hence $r = [(a, b)] = [(0, 0)] = 0'$, contradiction. If $ab > 0$, then $r \in P$, and if $-(ab) > 0$, then $(-a)b > 0$, so $-r = [(-a, b)] \in P$. Thus $r \in P$ or $-r \in P$. Suppose that $r \in P$ and $-r \in P$. Thus $ab > 0$ and $(-a)b > 0$, contradiction.

(ii): Suppose that $r, s \in P$. Then $ab > 0$ and $cd > 0$. Now $r + s = [(ad + bc, bd)]$, and $(ad + bc)bd = abdd + bbcd$. By (1) in the proof of 40.10, $dd > 0$ and $bb > 0$. Hence by properties of integers, $abdd + bbcd > 0$.

(iii): Suppose that $r, s \in P$. Then $ab > 0$ and $cd > 0$. Now $rs = [(ac, bd)]$ and $acbd = abcd > 0$. □

Now we can define the order: $a < b$ iff $b - a \in P$. The main properties of $<$ are given in the following proposition.

Proposition 40.12. *Let $x, y, z \in \mathbb{Q}'$. Then*

(i) $x \not< x$.

(ii) If $x < y < z$, then $x < z$.

(iii) $x < y$, $x = y$, or $y < x$.

(iv) $x < y$ iff $x + z < y + z$.

(v) If $0' < x$ and $0' < y$, then $0' < x \cdot y$.

(vi) If $0' < z$, then $x < y$ implies that $x \cdot z < y \cdot z$.

Proof. (i): $x - x = 0'$, so (i) follows from 40.11(i).

(ii): Assume that $x < y < z$. So $y - x \in P$ and $z - y \in P$. Hence $z - x = (z - y) + y - x) \in P$ by 40.11(ii), so $x < z$.

(iii): We have $x = y$ or $x - y \in P$ or $y - x \in P$, so (iii) follows.

(iv): $x < y$ iff $y - x \in P$ iff $(y + z) - (x + z) \in P$ iff $x + z < y + z$.

(v): This is immediate from 40.11(iii).

(vi): Assume that $0' < z$ and $x < y$. So $z, y - x \in P$, so by 40.11(iii), $y \cdot z - x \cdot z = z \cdot (y - x) \in P$, and so $x \cdot z < y \cdot z$. □

This finishes the main construction of the rational numbers. There are still two things to do, though: identify the integers among the rationals, and make a replacement so that the integers are a subset of the rationals.

For every integer a we define $f(a) = [(a, 1)]$.

Proposition 40.13. *f is an isomorphism of $\mathbb{Z}$ into $\mathbb{Q}'$. That is, f is an injection, and for any $a, b \in \mathbb{Z}$ we have $f(a + b) = f(a) + f(b)$ and $f(a \cdot b) = f(a) \cdot f(b)$.*

Proof. Suppose that $f(a) = f(b)$. Thus $[(a, 1)] = [(b, 1)]$, hence $(a, 1) \sim (b, 1)$, hence $a = a1 = 1b = b$. So f is an injection.

Now suppose that $a, b \in \mathbb{Z}$. Then

$$\begin{aligned} f(a) + f(b) &= [(a,1)] + [(b,1)] = [(a1 + 1b, 1)] = [(a+b, 1)] = f(a+b); \\ f(a) \cdot f(b) &= [(a,1)] \cdot [(b,1)] = [(ab, 1)] = f(ab). \end{aligned}$$ $\square$

Proposition 40.14. $\mathbb{Z} \cap \mathbb{Q}' = \emptyset$.

Proof. To show that $\omega \cap \mathbb{Q}' = \emptyset$ it suffices to show that each element of $\mathbb{Q}'$ is infinite. If $[(a,b)] \in \mathbb{Q}'$, then $(ca, cb) \in [(a,b)]$ for every $c \in \mathbb{Z}$, and $cb \neq db$ for $c \neq d$, and so $(ca, cb) \neq (da, db)$ for $c \neq d$; hence $[(a,b)]$ is infinite.

Now suppose that $x \in \mathbb{Z} \cap \mathbb{Q}'$ with $x \notin \omega$. Temporarily denote the equivalence relation used to define $\mathbb{Z}'$ by $\equiv$. Then there exist $m, n \in \omega$ such that $x = [(m,n)]_\equiv$, and there exists $(a,b) \in A$ such that $x = [(a,b)]_\sim$. Then $(a,b) \sim (2a, 2b)$, so also $[(2a,2b)]_\sim = [(a,b)]_\sim = x = [(m,n)]_\equiv$. Hence $(a,b), (2a,2b) \in [(m,n)]_\equiv$, and it follows that $(a,b) \equiv (2a,2b)$. So $a + 2b = b + 2a$, and hence $a = b$. Then $(0,0) \equiv (a,b)$, so $(0,0) \in [(a,b)]_\equiv = [(a,b)]_\sim$, and we infer that $(0,0) \in A$, contradiction. $\square$

We can now proceed very much like for $\mathbb{Z}$ and $\mathbb{Z}'$. We define $\mathbb{Q} = (\mathbb{Q}' \backslash \text{rng}(f)) \cup \mathbb{Z}$. There is a one-one function $g : \mathbb{Q} \to \mathbb{Q}'$, defined by $g([(a,b)]) = [(a,b)]$ if $[(a,b)] \in \mathbb{Q}' \backslash \text{rng}(f)$, and $g(a) = f(a)$ for $a \in \mathbb{Z}$. Clearly g is a bijection. Now the operations $+'$ and $\cdot'$ are defined on $\mathbb{Q}$ as follows. For any $a, b \in \mathbb{Q}$,

$$\begin{aligned} a +' b &= g^{-1}(g(a) + g(b)); \\ a \cdot' b &= g^{-1}(g(a) \cdot g(b)). \end{aligned}$$

moreover, we define $a <' b$ iff $g(a) < g(b)$. With these definitions, g becomes an isomorphism of $\mathbb{Q}$ onto $\mathbb{Q}'$. Namely, if $a, b \in \mathbb{Q}$, then

$$\begin{aligned} g(a +' b) &= g(g^{-1}(g(a) + g(b))) = g(a) + g(b); \\ g(a \cdot' b) &= g(g^{-1}(g(a) \cdot g(b))) = g(a) \cdot g(b); \\ a <' b \quad &\text{iff} \quad g(a) < g(b). \end{aligned}$$

Moreover, the operations $+'$ and $\cdot'$ on $\mathbb{Z}$ coincide with the ones defined previously, since if $a, b \in \mathbb{Z}$, then

$$\begin{aligned} a +' b &= g^{-1}(g(a) + g(b)) = g^{-1}(f(a) + f(b)) = g^{-1}(f(a+b)) = a + b; \\ a \cdot' b &= g^{-1}(g(a) \cdot g(b)) = g^{-1}(f(a) \cdot f(b)) = g^{-1}(f(a \cdot b)) = a \cdot b; \\ a <' b \quad &\text{iff} \quad g(a) < g(b) \\ &\text{iff} \quad f(a) < f(b) \\ &\text{iff} \quad a < b. \end{aligned}$$

All of the properties above, like the associative, commutative, and distributive laws, hold for $\mathbb{Z}$ since g is an isomorphism. Of course we use $+, \cdot, <$ now rather than $+', \cdot', <'$.

41. The reals

A subset A of $\mathbb{Q}$ is a *Dedekind cut* provided the following conditions hold:

(1) $\mathbb{Q} \neq A \neq \emptyset$;
(2) For all $r, s \in \mathbb{Q}$, if $r < s$ and $s \in A$, then $r \in A$.
(3) A has no largest element.

Let $\mathbb{R}'$ be the set of all Dedekind cuts.

If A and B are Dedekind cuts, then we define

$$A + B = \{x : \text{there are } a \in A \text{ and } b \in B \text{ such that } x = a + b\}.$$

Proposition 41.1. *If A and B are Dedekind cuts, then so is $A + B$.*

Proof. Since A and B are both nonempty, clearly $A + B$ is nonempty. Now take $r \in \mathbb{Q}\backslash A$ and $s \in \mathbb{Q}\backslash B$. So $t < r$ for all $t \in A$, and $u < s$ for all $u \in B$. Then $a + b < r + s$ for all $a \in A$ and $b \in B$, so that $x < r + s$ for all $x \in A + B$. In particular, $r + s \notin A + B$, by the irreflexivity of $<$. So we have shown that (1) holds for $A + B$.

Now suppose that $r < s \in A + B$. Write $s = a + b$ with $a \in A$ and $b \in B$. Then $r < s = a + b$, so $r - a < b$, and hence $r - a \in B$ by (2) for B. Hence $r = a + (r - a)$ shows that $r \in A + B$. So (2) holds for $A + B$.

Suppose that $x \in A + B$. Write $x = a + b$ with $a \in A$ and $b \in B$. Since a is not the greatest element of A, by (3) choose $a' \in A$ such that $a < a'$. Then $x = a + b < a' + b \in A + B$, proving (3) for $A + B$. $\square$

Proposition 41.2. *Let A, B, C be Dedekind cuts. Then*
(i) $A + B = B + A$.
(ii) $A + (B + C) = (A + B) + C$.

Proof. (i): obvious. (ii): Suppose that $x \in A + (B + C)$. Then there are $a \in A$ and $y \in (B + C)$ such that $x = a + y$; and there are $b \in B$ and $c \in C$ such that $y = b + c$. So $x = a + b + c$. Now $a + b \in (A + B)$, so $x \in ((A + B) + C)$. This shows that $A + (B + C) \subseteq (A + B) + C$. Since this is generally true for all Dedekind cuts A, B, C, we also have $(A + B) + C = C + (B + A) \subseteq (C + B) + A = A + (B + C)$. $\square$

Now we define, following Chapter 6,

$$Z = \{r \in \mathbb{Q} : r < 0\}.$$

Clearly Z is a Dedekind cut.

Proposition 41.3. *$A + Z = A$ for every Dedekind cut A.*

Proof. Let $a \in A$. Since A does not have a largest element, choose $b \in A$ such that $a < b$. Then $a - b < 0$, hence $a - b \in Z$, and so $a = b + (a - b)$ shows that $a \in A + Z$.

Conversely, suppose that $x \in A + Z$. Then there exist $a \in A$ and $b \in Z$ such that $x = a + b$. Since $b < 0$, we have $x < a$, and so $x \in A$, as desired. $\square$

It is easy to check that Z is the only element of $\mathbb{R}'$ such that $A + Z = A$ for all A.

Next, for any Dedekind cut A we define

$$-A = \{r \in \mathbb{Q} : \text{there is an } s \in \mathbb{Q} \text{ such that } r < s \text{ and } -s \notin A\}.$$

Proposition 41.4. *$A + -A = Z$ for any Dedekind cut A.*

Proof. First we show that $-A$ is itself a Dedekind cut. Since $A \neq \mathbb{Q}$, choose $r \in \mathbb{Q} \backslash A$. Then also $r + 1 \notin A$. so $-(r+1) < -r$ and $-(-r) = r \notin A$. It follows that $-(r+1) \in -A$. Hence $-A \neq \emptyset$. Next, choose $r \in A$. Then $-r \notin -A$, as otherwise there is an s such that $-r < s$ and $-s \notin A$; but $-s < r$, contradiction. So $-A \neq \mathbb{Q}$. Finally, suppose that $r \in -A$; we want to find a larger element in A. Choose s such that $r < s$ and $-s \notin A$. Take $t \in \mathbb{Q}$ such that $r < t < s$; for example, take $t = (r+s)/2$. Clearly then $t \in -A$, as desired. This checks that $-A$ is a Dedekind cut.

Now suppose that $x \in A + -A$. Then there are $a \in A$ and $b \in -A$ such that $x = a + b$. Choose $c \in \mathbb{Q}$ such that $b < c$ and $-c \notin A$. Suppose that $0 \leq x$. Then $x = a + b < a + c$, and so $-c < a + -x \leq a$, and hence $-c \in A$, contradiction. Hence $x < 0$, so that $x \in Z$.

Second suppose that $r \in Z$. Fix $b \notin A$.

(1) There is a positive integer p such that $b + \frac{pr}{2} \in A$.

In fact, to prove (1), also fix $a \in A$. Then $a < b$, as otherwise we would have $b \in A$. Hence there are positive integers s, t such that $b - a = \frac{s}{t}$. Since $\frac{r}{2} < 0$, there are also positive integers u, v such that $\frac{r}{2} = -\frac{u}{v}$. Then $b - a = \frac{s}{t} \leq s \leq su = sv(-\frac{r}{2})$. Hence $b + sv\frac{r}{2} \leq a$, and so $b + sv\frac{r}{2} \in A$, proving (1).

Let p be the smallest positive integer such that $b + p\frac{r}{2} \in A$. Recall that $b \notin A$, so that even if $p = 1$ we can assert that $b + (p-1)\frac{r}{2} \notin A$. Now

$$r = b + pr + (-b + (-p+1)\frac{r}{2} + \frac{r}{2}),$$

and $(-b + (-p+1)\frac{r}{2} + \frac{r}{2}) < (-b + (-p+1)\frac{r}{2}$, and $-(-b + (-p+1)\frac{r}{2}) = b + (p-1)\frac{r}{2} \notin A$. This shows that $r \in A + -A$. $\square$

The element $-A$ is unique: if $A + B = Z$, then $B = -A$. In particular, $-Z = Z$.

Next, we call a Dedekind cut A *positive* iff if has at least one positive member.

Proposition 41.5. *For any Dedekind cut A, exactly one of the following holds:*

(i) A is positive;

(ii) $A = Z$;

(iii) $-A$ is positive.

Proof. Suppose that A is not positive, and $A \neq Z$. Since A is not positive, all its members are negative or zero; since it has no largest element, $0 \notin A$. Thus $A \subseteq Z$. Since $A \neq Z$, we actually have $A \subset Z$. Choose $r \in Z \backslash A$. Now $r + r < 0 + r = r < 0$, and so $r < \frac{r}{2} < 0$. Hence $0 < -\frac{r}{2} < -r$. So $-\frac{r}{2} \in -A$, since $-(-r) = r \notin A$. This shows that $-A$ is positive.

So we have shown that one of (i)–(iii) holds.

Obviously (i) and (ii) do not simultaneously hold. Suppose that both A and $-A$ are positive. Hence there is a positive element $r \in A$, and a positive element $s \in -A$. By the definition of $-A$, choose t such that $s < t$ and $-t \notin A$. Then $-t < -s < 0 < r$, so $-t \in A$, contradiction. Thus (i) and (iii) do not simultaneously hold. Finally, suppose that $-Z$ is positive. Let r be a positive element of $-Z$. Then by definition there is an s such that $r < s$ and $-s \notin Z$. So $0 \le -s < -r$, contradicting r being positive. $\square$

On the basis of Proposition 41.5, the following definition makes sense. For any Dedekind cut A,

$$|A| = \begin{cases} A & \text{if } A = Z \text{ or } A \text{ is positive,} \\ -A & \text{if } -A \text{ is positive.} \end{cases}$$

Now we repeat the definition of product from Chapter 6. Let A and B be Dedekind cuts.

$$\begin{aligned} (a) \qquad & A \cdot B = \{r \in \mathbb{Q} : \text{there are } s \in A \text{ and } t \in B \text{ such that } 0 < s \\ & \qquad\qquad \text{and } 0 < t \text{ and } r < s \cdot t\} \text{ if } A \text{ and } B \text{ are positive,} \end{aligned}$$

$$(b) \qquad A \cdot B = Z \text{ if } A = Z \text{ or } B = Z,$$

$$(c) \qquad A \cdot B = -(|A| \cdot |B|) \text{ if } A \neq Z \neq B \text{ and exactly one of } A, B \text{ is positive}$$

$$(d) \qquad A \cdot B = (-A) \cdot (-B) \text{ if } -A \text{ and } -B \text{ are both positive.}$$

Proposition 41.6. *Let A, B, C be Dedekind cuts.*
(i) $A \cdot B = B \cdot A$.
(ii) $(-A) \cdot B = -(A \cdot B) = A \cdot (-B)$.
(iii) $A \cdot (B \cdot C) = (A \cdot B) \cdot C$.
(iv) $A \cdot (B + C) = A \cdot B + A \cdot C$.

Proof. (i): this is clear if both A and B are positive, or if one of them is Z. If both are different from Z and exactly one of them is positive, then $|A|$ and $|B|$ are both positive, and

$$A \cdot B = -(|A| \cdot |B|) = -(|B| \cdot |A|) = B \cdot A.$$

If $-A$ and $-B$ are both positive, then

$$A \cdot B = (-A) \cdot (-B) = (-B) \cdot (-A) = B \cdot A.$$

Thus (i) holds.

(ii): First we prove that $(-A) \cdot B = -(A \cdot B)$. This is true by (b) if one of A, B is Z, since $-Z = Z$. If A and B are positive, then

$$(-A) \cdot B = -(A \cdot B) \quad \text{by (c).}$$

If $-A$ and B are positive, then

$$\begin{aligned} -(A \cdot B) &= -(-((-A) \cdot B)) \quad \text{by (c)} \\ &= (-A) \cdot B. \end{aligned}$$

If A and $-B$ are positive, then

$$\begin{aligned} (-A) \cdot B &= B \cdot (-A) && \text{by (i)} \\ &= -(B \cdot A) && \text{by the previous case} \\ &= -(A \cdot B) && \text{by (i).} \end{aligned}$$

Finally, if $-A$ and $-B$ are positive, then

$$\begin{aligned} (-A) \cdot B &= -((-A) \cdot (-B)) \quad \text{by (c)} \\ &= -(A \cdot B) \quad \text{by (d).} \end{aligned}$$

Thus $(-A) \cdot B = -(A \cdot B)$ in general. The other part of (ii) follows from (i).

(iii):

(1) If A, B, C are all positive, then $A \cdot (B \cdot C) \subseteq (A \cdot B) \cdot C$.

For, assume that A, B, C are all positive. Clearly then $A \cdot B$ and $B \cdot C$ are positive. Now let $x \in A \cdot (B \cdot C)$. Then there exist s, t such that $x < s \cdot t$, $0 < s \in A$, and $0 < t \in B \cdot C$. Since $t \in B \cdot C$, there exist u, v such that $t < u \cdot v$, $0 < u \in B$, and $0 < v \in C$. Choose $s' \in A$ such that $s < s'$. Then $s \cdot u < s' \cdot u$, $0 < s' \in A$, and $0 < u \in B$, so $s \cdot u \in A \cdot B$. Then $x < s \cdot u \cdot v$, $0 < s \cdot u \in A \cdot B$, and $0 < v \in C$, so $x \in (A \cdot B) \cdot C$. This proves (1).

(2) If one of A, B, C is equal to Z, then $A \cdot (B \cdot C) = Z = (A \cdot B) \cdot C$.

This is clear.

(3) If A, B, C are all positive, then $A \cdot (B \cdot C) = (A \cdot B) \cdot C$.

In fact,

$$\begin{aligned} A \cdot (B \cdot C) &\subseteq (A \cdot B) \cdot C && \text{by (1)} \\ &= C \cdot (B \cdot A) && \text{by (i)} \\ &\subseteq (C \cdot B) \cdot A && \text{by (1)} \\ &= A \cdot (B \cdot C) && \text{by (i).} \end{aligned}$$

So (3) holds.

Now we can use (ii) to finish (iii):

$$\begin{aligned} A, B, -C \text{ positive: } A \cdot (B \cdot C) &= A \cdot -(B \cdot -C) \\ &= -(A \cdot (B \cdot -C) \\ &= -((A \cdot B) \cdot -C) \end{aligned}$$

$$
\begin{aligned}
&= (A\cdot B)\cdot C;\\
A,-B,C \text{ positive: } A\cdot(B\cdot C) &= A\cdot -(-B\cdot C)\\
&= -(A\cdot(-B\cdot C))\\
&= -((A\cdot -B)\cdot C)\\
&= (A\cdot B)\cdot C;\\
A,-B,-C \text{ positive: } A\cdot(B\cdot C) &= A\cdot((-B)\cdot(-C))\\
&= (A\cdot -B)\cdot -C\\
&= (A\cdot B)\cdot C;\\
C \text{ positive: } (A\cdot B)\cdot C &= C\cdot(B\cdot A)\\
&= (C\cdot B)\cdot A\\
&= A\cdot(B\cdot C);\\
-A,B,-C \text{ positive: } A\cdot(B\cdot C) &= A\cdot -(B\cdot -C)\\
&= -((-A)\cdot -(B\cdot -C))\\
&= (-A)\cdot(B\cdot -C)\\
&= ((-A)\cdot B)\cdot -C\\
&= (A\cdot B)\cdot C;\\
-A,-B,-C \text{ positive: } A\cdot(B\cdot C) &= A\cdot((-B)\cdot(-C))\\
&= -((-A)\cdot((-B)\cdot(-C)))\\
&= -(((-A)\cdot(-B))\cdot -C)\\
&= (A\cdot B)\cdot C.
\end{aligned}
$$

(iv): Clearly

(4) If one of A, B, C is Z, then $A\cdot(B+C) = A\cdot B + A\cdot C$.

(5) If A, B, C are positive, then $A\cdot(B+C) = A\cdot B + A\cdot C$.

For, first suppose that $x \in A\cdot(B+C)$. Then we can choose s, t so that $0 < s \in A$, $0 < t \in B+C$, and $x < s\cdot t$. Since $t \in B+C$, there are $b \in B$ and $c \in C$ such that $t = b+c$. Now choose $b' \in B$ with $b \le b'$ and $0 < b'$, and choose $c' \in C$ such that $c \le c'$ and $0 < c'$. Now $x = s\cdot b' + (x - s\cdot b')$, and clearly $s\cdot b' \in A\cdot B$, while

$$x - s\cdot b' < s\cdot(b'+c') - s\cdot b' = s\cdot c',$$

and clearly $s\cdot c' \in A\cdot C$. This proves $\subseteq$ in (5).

Now suppose that $y \in A\cdot B + A\cdot C$. Then we can write $y = u+v$ with $u \in A\cdot B$ and $v \in A\cdot C$. Say $u < s\cdot t$ with $0 < s \in A$ and $0 < t \in B$, and $v < a\cdot c$ with $0 < a \in A$ and $0 < c \in C$. Let s' be the maximum of s and a. Then $y < s'\cdot(t+c)$, $0 < s' \in A$, and $t + c \in B + C$. So $y \in A\cdot(B+C)$. This proves $\supseteq$ in (5).

(6) If $A, B, -C$ are positive, and also $B+C$ is positive, then $A\cdot(B+C) = A\cdot B + A\cdot C$.

For,

$$\begin{aligned}
A \cdot B &= A \cdot (B + C + -C) \\
&= A \cdot (B + C) + A \cdot (-C) \quad \text{by (5)} \\
&= A \cdot (B + C) + -(A \cdot C), \quad \text{by (ii)}
\end{aligned}$$

and (6) follows.

(7) If $A, B, -C$ are positive, and $B + C$ is negative, then $A \cdot (B + C) = A \cdot B + A \cdot C$.

For,

$$\begin{aligned}
-(A \cdot (B + C)) &= A \cdot (-(B + C)) \quad \text{by (ii)} \\
&= A \cdot (-B + -C) \\
&= A \cdot (-B) + A \cdot (-C) \quad \text{by (6)} \\
&= -(A \cdot B) + -(A \cdot C), \quad \text{by (ii)}
\end{aligned}$$

and (7) follows.

(8) If $A, B, -C$ are positive, and $B + C = Z$, then $A \cdot (B + C) = A \cdot B + A \cdot C$.

For, under these hypotheses, $C = -B$, and so

$$A \cdot (B + C) = A \cdot Z = Z = A \cdot B + -(A \cdot B) = A \cdot B + A \cdot (-B) = A \cdot B + A \cdot C.$$

(9) If $A, -B, C$ are positive, then $A \cdot (B + C) = A \cdot B + A \cdot C$.

This follows from (6)–(8) since $+$ is commutative.

(10) If $A, -B, -C$ are positive, then $A \cdot (B + C) = A \cdot B + A \cdot C$.

For,

$$\begin{aligned}
A \cdot (B + C) &= -(A \cdot (-B + -C)) \quad \text{by (ii)} \\
&= -(A \cdot (-B) + A \cdot (-C)) \quad \text{by (5)} \\
&= -(-(A \cdot B) + -(A \cdot C)) \quad \text{by (ii)} \\
&= A \cdot B + A \cdot C.
\end{aligned}$$

(11) If A is positive, then $A \cdot (B + C) = A \cdot B + A \cdot C$.

This is true by (6)–(10).

(12) If $-A$ is positive, then $A \cdot (B + C) = A \cdot B + A \cdot C$.

In fact, $(-A) \cdot (B + C) = (-A) \cdot B + (-A) \cdot C$ by (11), and (12) follows, using (ii). □

Now we define

$$I = \{r \in \mathbb{Q} : r < 1\}.$$

Clearly I is a Dedekind cut.

Proposition 41.7. *$A \cdot I = A$ for any Dedekind cut A.*

Proof. This is clear if $A = Z$. Now suppose that A is positive. Suppose that $r \in A \cdot I$. Then there are $s, t \in \mathbb{Q}$ such that $0 < s \in A$, $0 < t \in I$, and $r < s \cdot t$. Clearly then $r < s$, so $r \in A$ by the definition of Dedekind cut.

Conversely, suppose that $r \in A$. Choose $r', r'' \in A$ such that $r < r' < r''$ and $0 < r'$. Let $s = \frac{r'}{r''}$. Then $0 < s < 1$, so $s \in I$. Since $r < r' = r'' \cdot s$, it follows that $r \in A \cdot I$. Thus we have shown that $A \cdot I = A$ for A positive.

If $-A$ is positive, then $A \cdot I = -((-A) \cdot I) = -(-A) = A$, using D6(ii). $\square$

Proposition 41.8. *If A is a Dedekind cut and $A \neq Z$, then there is a Dedekind cut B such that $A \cdot B = I$.*

Proof. First suppose that A is positive. Let

$$B = \{r \in \mathbb{Q} : r < 0, \text{ or } 0 \leq r \text{ and } r \cdot s < 1 \text{ for every } s \in A \text{ for which } 0 < s\}.$$

Then $B \neq \emptyset$, since clearly $0 \in B$. Clearly if $r' < r \in B$, then also $r' \in B$. If $0 < s \in A$, then $\frac{1}{s} \notin B$. So B is a Dedekind cut.

We claim that $A \cdot B = I$. Suppose that $r \in A \cdot B$. Choose s, t so that $0 < s \in A$, $0 < t \in B$, and $r < s \cdot t$. Then by the definition of B, $s \cdot t < 1$, so $r < 1$. Hence $r \in I$.

Conversely, suppose that $r \in I$, so that $r < 1$. Choose r', r'', r''' so that $0, r < r' < r'' < r''' < 1$. Let $C = \{s \in \mathbb{Q} : s < r'''\}$. Clearly C is a Dedekind cut.

(1) $(A \cdot C) \subset A$.

In fact, clearly $(A \cdot C) \subseteq A$. Suppose that $A \cdot C = A$. Now

$$A = A \cdot I = (A \cdot C) + (A \cdot (I - C)) = A + (A \cdot (I - C)),$$

so $A \cdot (I - C) = Z$. Choose s, t so that $r''' < s < t < 1$. Then $-s < -r'''$ and $r''' \notin C$, so $-s \in -C$. Hence $0 < t - s \in (I - C)$. So $I - C$ is positive. Since A is also positive, it follows that $A \cdot (I - C)$ is positive, contradiction. Hence (1) holds.

By (1), choose $s \in A \backslash (A \cdot C)$. We may assume that $0 < s$. Thus

(2) For all a, c, if $0 < a \in A$ and $0 < c \in C$, then $a \cdot c \leq s$.

Now let $v = \frac{r'}{s}$. Thus $s \cdot v = r' > r$. Hence we will get $r \in A \cdot B$ as soon as we show that $v \in B$. Suppose that $0 < a \in A$. Now $0 < r'' \in C$, so by (2) we have $a \cdot r'' \leq s$. Hence

$$a \cdot v = a \cdot \frac{r'}{s} < a \cdot \frac{r''}{s} \leq 1,$$

so that $a \cdot v < 1$, as desired.

Thus we have finished the proof in the case that A is positive. If $-A$ is positive, then choose B so that $(-A) \cdot B = I$. Then $(A \cdot (-B)) = (-A) \cdot B = I$, using 41.7(ii). $\square$

This finishes the purely arithmetic part of the construction of the real numbers. Now we discuss ordering. We define $A < B$ iff $B - A$ is positive. Elementary properties of $<$ are given in the following proposition.

Proposition 41.9. *Let $A, B, C \in \mathbb{R}'$. Then*

(i) $A \not< A$.

(ii) If $A < B < C$, then $A < C$.

(iii) $A < B$, $A = B$, or $B < A$.

(iv) $A < B$ iff $A + C < B + C$.

(v) $Z < I$.

(vi) If $Z < A$ and $Z < B$, then $Z < A \cdot B$.

(vii) If $Z < C$, then $A < B$ implies that $A \cdot C < B \cdot C$.

(viii) $A < B$ iff $A \subset B$.

Proof. (i): $A - A = Z$, so $A \not< A$ by 41.5.

(ii) Suppose that $A < B < C$. Thus $B - A$ and $C - B$ are positive. Hence clearly also $C - A = C - B + B - A$ is positive.

(iii): Given A, B, by 41.5 we have $A - B$ positive, $A - B = Z$, or $-(A - B) = B - A$ is positive. By definition this gives $A < B$, $A = B$, or $B < A$.

(iv): First suppose that $A < B$. Thus $B - A$ is positive. Since $B + C - (A + C) = B - A$, it follows that $A + C < B + C$.

Second, suppose that $A + C < B + C$. Thus $B - A = B + C - (A + C)$ is positive, so $A < B$.

(v): Obviously I is positive.

(vi): Assume that $Z < A$ and $Z < B$. Thus A and B are positive. Clearly then $A \cdot B$ is positive. So $Z < A \cdot B$.

(vii): Assume that $Z < C$ and $A < B$. Then C and $B - A$ are positive, so also $C \cdot (B - A) = C \cdot B - (A \cdot C)$ is positive, and so $A \cdot C < B \cdot C$.

(viii): Suppose that $A < B$. Thus $B - A$ is positive. Choose x so that $0 < x \in B - A$. Then we can write $x = b + a$ with $b \in B$ and $a \in -A$. By the definition of $-A$, choose $s \in \mathbb{Q}$ so that $a < s$ and $-s \notin A$. Then $-s < -a$, so also $-a \notin A$. Also $b + a > 0$, so $b > -a$, and it follows that $b \notin A$. Now if $y \in A$, then $y < b$, as otherwise $b \leq y$ would imply that $b \in A$. But then $y \in B$. So $A \subseteq B$, and since $b \in B \backslash A$, even $A \subset B$.

Conversely, suppose that $A \subset B$. Choose $b \in B \backslash A$. Choose c, d such that $b < c < d \in B$. Now $-c < -b$ and $b \notin A$, so $-c \in -A$. Thus $d - c$ is a positive element of $B - A$, hence $B - A$ is positive and $A < B$. $\square$

The following theorem expresses the essential new property of the reals as opposed to the rationals.

Theorem 41.10. *Every nonempty subset of $\mathbb{R}'$ which is bounded above has a least upper bound. That is, if $\emptyset \neq \mathscr{X} \subseteq \mathbb{R}'$, and there is a Dedekind cut D such that $A \leq D$ for all $A \in \mathbb{R}'$, then there is a Dedekind cut B such that the following two conditions hold:*

(i) $A \leq B$ for every $A \in \mathscr{X}$.

(ii) For any Dedekind cut C, if $A \leq C$ for every $A \in \mathscr{X}$, then $B \leq C$.

Proof. Let $B = \bigcup_{A \in \mathscr{X}} A$. Since $\mathscr{X}$ is nonempty, and each Dedekind cut is nonempty, it follows that B is nonempty. To show that B does not consist of all rationals, we use the assumption that $\mathscr{X}$ has an upper bound. Let D be an upper bound for $\mathscr{X}$. Thus $A \leq D$ for all $A \in \mathscr{X}$. By 41.9(viii), $A \subseteq D$ for all $A \in \mathscr{X}$, and hence $B \subseteq D$. Since $D \neq \mathbb{Q}$, also

$B \neq \mathbb{Q}$. If $x < y \in B$, then $y \in A$ for some $A \in \mathscr{X}$, hence $x \in A$, hence $x \in B$. Thus B is a Dedekind cut.

For any $A \in \mathscr{X}$ we have $A \subseteq B$, and hence $A \leq B$ by D9(viii).

Now suppose that $A \subseteq C$ for all $A \in \mathscr{X}$, where C is a Dedekind cut. Then $B \subseteq C$, hence $B \leq C$ by 41.9(viii). □

Next we want to embed the rationals into $\mathbb{R}'$. For every rational r we define $f(r) = \{q \in \mathbb{Q} : q < r\}$. Clearly $f(r)$ is a Dedekind cut.

Proposition 41.11. *(i) f is one-one.*

(ii) $f(r+s) = f(r) + f(s)$ for any $r, s \in \mathbb{Q}$.

(iii) $f(r \cdot s) = f(r) \cdot f(s)$ for any $r, s \in \mathbb{Q}$.

Proof. (i): Suppose that $r, s \in \mathbb{Q}$; say $r < s$. Then $r \in f(s) \backslash f(r)$, so $f(r) \neq f(s)$.

(ii): First suppose that $x \in f(r+s)$. Thus $x < r+s$, so $x - s < r$. Let r' be a rational number such that $x - s < r' < r$. Then $x = r' + (x - r')$, and $x - r' < s$, so $x \in f(r) + f(s)$.

Conversely, suppose that $x \in f(r) + f(s)$. Choose $a \in f(r)$ and $b \in f(s)$ so that $x = a + b$. Then $a < r$ and $b < s$, so $x < r + s$, and so $x \in f(r+s)$.

(iii): Note that $f(0) = Z$; hence (iii) is clear if $r = 0$ or $s = 0$. Suppose that $r, s > 0$. Suppose that $x \in f(r \cdot s)$. So $x < r \cdot s$. Hence $\frac{x}{s} < r$. Choose $r' \in \mathbb{Q}$ such that $\frac{x}{s} < r' < r$ and $0 < r'$. Hence $\frac{x}{r'} < s$. Choose $s' \in \mathbb{Q}$ such that $\frac{x}{r'} < s' < s$ and $0 < s'$. Then $x < r' \cdot s'$, $0 < r' \in f(r)$, and $0 < s' \in f(s)$, so $x \in f(r) \cdot f(s)$.

Conversely, suppose that $x \in f(r) \cdot f(s)$. Then there are $r' \in f(r)$ and $s' \in f(s)$ such that $0 < r'$, $0 < s'$, and $x < r' \cdot s'$. Hence $x < r \cdot s$, so $x \in f(r \cdot s)$, as desired. This finishes the case in which $r, s > 0$.

To continue we need the following little fact:

(1) $-f(r) = \{q \in \mathbb{Q} : q < -r\}$ for any rational number r.

In fact, suppose that $q \in -f(r)$. Then there is a rational t such that $q < t$ and $-t \notin f(r)$. thus $-t \not< r$, so $r \leq -t$. Hence $t \leq -r$, so $q < -r$. Conversely, suppose that $q < -r$. Now $r \notin f(r)$, so $q \in -f(r)$. Thus (1) holds.

Now suppose that $r < 0 < s$. Then, using (1),

$$f(r) \cdot f(s) = -((-f(r)) \cdot f(s)) = -(f(-r) \cdot f(s)) = -f((-r) \cdot s) = f(r \cdot s).$$

Similarly if $s < 0 < r$. If $r, s < 0$, then

$$(f(r) \cdot f(s) = (-f(r)) \cdot (-f(s)) = f(-r) \cdot f(-s) = f((-r) \cdot (-s)) = f(r \cdot s).$$ □

Proposition 41.12. $\mathbb{Q} \cap \mathbb{R}' = \emptyset$.

Proof. First, $\omega \cap \mathbb{R}' = \emptyset$, since the members of ω are all finite, while the members of $\mathbb{R}'$ are all infinite.

Now suppose that $a \in \mathbb{Z} \cap \mathbb{R}'$. Then $a \notin \omega$ by the preceding paragraph, so $a = [(m, n)]$ for some $m, n \in \omega$. But also $a \in \mathbb{R}'$, so a is a set of rationals. In particular, (m, n) is a rational. Now (m,n) has either one or two elements; the only rationals with only one or two elements are 1 and 2. Since $\emptyset\in 1$ and $\emptyset\in 2$, we get $\emptyset\in(m,n)$, contradiction.

A similar argument shows that $a\in\mathbb{Q}\cap\mathbb{R}'$ leads to a contradiction. $\square$

We can now proceed very much like in previous appendices. We define $\mathbb{R}=(\mathbb{R}'\backslash\mathrm{rng}(f))\cup\mathbb{Q}$. There is a one-one function $g:\mathbb{R}\to\mathbb{R}'$, defined by $g(A)=A$ if $A\in\mathbb{R}'\backslash\mathrm{rng}(f)$, and $g(A)=f(A)$ for $A\in\mathbb{Q}$. Clearly g is a bijection. Now the operations $+'$ and $\cdot'$ are defined on $\mathbb{R}$ as follows. For any $a,b\in\mathbb{R}$,

$$\begin{aligned} a+'b&=g^{-1}(g(a)+g(b));\\ a\cdot' b&=g^{-1}(g(a)\cdot g(b)). \end{aligned}$$

moreover, we define $a<'b$ iff $g(a)<g(b)$. With these definitions, g becomes an isomorphism of $\mathbb{R}$ onto $\mathbb{R}'$. Namely, if $a,b\in\mathbb{R}$, then

$$\begin{aligned} g(a+'b)&=g(g^{-1}(g(a)+g(b)))=g(a)+g(b);\\ g(a\cdot' b)&=g(g^{-1}(g(a)\cdot g(b)))=g(a)\cdot g(b);\\ a<'b \quad &\text{iff}\quad g(a)<g(b). \end{aligned}$$

Moreover, the operations $+'$ and $\cdot'$ on $\mathbb{Q}$ coincide with the ones defined in appendix C, since if $a,b\in\mathbb{Q}$, then

$$\begin{aligned} a+'b&=g^{-1}(g(a)+g(b))=g^{-1}(f(a)+f(b))=g^{-1}(f(a+b))=a+b;\\ a\cdot' b&=g^{-1}(g(a)\cdot g(b))=g^{-1}(f(a)\cdot f(b))=g^{-1}(f(a\cdot b))=a\cdot b;\\ a<'b\quad &\text{iff}\quad g(a)<g(b)\\ &\text{iff}\quad f(a)<f(b)\\ &\text{iff}\quad a<b. \end{aligned}$$

All of the properties above, like the associative, commutative, and distributive laws, hold for $\mathbb{R}$ since g is an isomorphism. Of course we use $+,\cdot,<$ now rather than $+',\cdot',<'$.

Index of symbols

Index of words