SPACE-TIME–TIME

Homer G. Ellis

Abstract. Space-time–time is a natural hybrid of Kaluza's five-dimensional geometry and Weyl's conformal space-time geometry. Translations along the secondary time dimension produce the electromagnetic gauge transformations of Kaluza–Klein theory and the metric gauge transformations of Weyl theory, quantitatively related as Weyl postulated. Geometrically, this phenomenon resides in an exponential-expansion producing "conformality constraint", which replaces Kaluza's "cylinder condition" and is applicable to metrics of all dimensionalities and signatures. The de Sitter space-time metric is prototypically conformally constrained; its hyper-de Sitter analogs of signatures $+++-+$ and $+++--$ describe space-time–time vacua. The curvature tensors exhibit in space-time–time a wealth of "interactions" among geometrical entities with physical interpretations. Unique to the conformally constrained geometry is a sectionally isotropic, ultralocally determined "residual curvature", useful in construction of an action density for field equations. A space-time–time geodesic describes a test particle whose rest mass $\mathring{m}$ and electric charge q evolve according to definite laws. Its motion is governed by four apparent forces: the Einstein gravitational force proportional to $\mathring{m}$, the Lorentz electromagnetic force proportional to q, a force proportional to $\mathring{m}$ and to the electromagnetic four-potential, and a force proportional to $q^2/\mathring{m}$ and to the gradient of $\ln\phi$, where the scalar field ϕ is essentially the space-time–time residual radius of curvature. The particle appears suddenly at an event $\mathcal{E}_1$ with $q=-\phi(\mathcal{E}_1)$ and vanishes suddenly at an event $\mathcal{E}_2$ with $q=\phi(\mathcal{E}_2)$. At $\mathcal{E}_1$ and $\mathcal{E}_2$ the ϕ-force infinitely dominates the others, causing $\mathcal{E}_1$ and $\mathcal{E}_2$ to occur near where ϕ has an extreme value; application to the modeling of orbital transitions of atomic electrons suggests itself. The equivalence of a test particle's inertial mass and its passive gravitational mass is a consequence of the gravitational force's proportionality to $\mathring{m}$. No connection is apparent between $\mathring{m}$ and active gravitational mass or between q and active electric charge, nor does the theory seem to require any. Justification for applying the name "space-time–time" whether the signature be $+++-+$ or $+++--$ lies in a construction which, applied to Euclidean spheres, produces the de Sitter manifold and its time coordinate t ("space's time"), and, when applied to Minkowskian spheres, produces the hyper-de Sitter manifolds and their new coordinate ζ ("space-time's time"). If space-time–time metrics of the two signatures are placed on equal footing by complexification of ζ, the expanded geometry presents new elements which beg to be linked to quantum mechanical phase phenomena. The forging of such a link will allow one ultimately to say, not that geometry has been quantized, but that the quantum has been geometrized.

I. Introduction

Impelled by convictions about the nature of time [1], I have pursued the prospect that manifolds bearing "conformally constrained" metrics can serve as realistic models of physical systems in which gravity, electromagnetism, and other phenomena manifest themselves. This paper presents some of the products of that ongoing pursuit.

Roughly, a conformally constrained metric is one for which there is a vector field ξ such that the lengths of vectors Lie transported by ξ are conformally expanded if those vectors are orthogonal to ξ, but are left unchanged if those vectors are parallel to ξ [2]; the de Sitter metric is the prototype. The geometry of five-dimensional manifolds carrying such metrics is a natural hybrid of the five-dimensional Kaluza geometry, with its distinguished

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Killing vector field that "isometrically constrains" the metric [3], and the four-dimensional Weyl geometry, with its multiplicity of conformally related metrics and the associated gauge forms [4]. This Kaluza–Weyl offspring is an evolutionary improvement in that it retains and enhances the most useful characters of its progenitors while attenuating to benign and useful form those that have caused difficulty. Most notably, it retains both the Kaluza unification of gravity with electromagnetism and the Weyl association of metrical gauge changes (multiplications of the metric by conformal factors) with electromagnetic gauge changes (additions of gradients to the electromagnetic potential). Also, it converts the objectionable nonintegrability of length transference in the Weyl geometry to integrability without sacrificing the principle that length, because it is a comparative measure, depends on designation of a standard at each point, that is, on choice of a gauge. In the process it lends to the fifth dimension an essential significance that the Kaluza geometry fails to provide.

The picture that emerges from application of this hybrid geometry to the modeling of physical systems has in it some rather unexpected representations of elementary physical phenomena, quantum phenomena included. Because the models are clearly defined, with little room for ambiguity in their interpretations, these representations appear to be escapable only by denial of the whole enterprise. Taken on their own terms they will, I believe, add to our image of the world a certain coherency not present in existing representations. Whether they are accurate will be, of course, a matter for investigation.

In this paper I define and exemplify conformally constrained metrics and introduce the term "space-time–time" in Sec. II, exhibit canonical decompositions of such metrics in Sec. III, show in Sec. IV how they incorporate and relate metrical and electromagnetic gauge transformations, and exhibit in Secs. V and VI their connection forms and their geodesic equations in frame systems adapted to the vector field ξ of the constraint. In Sec. VII, acting on the assumption that the geodesics of space-time–time describe histories of test particles, I define the space-time–time momentum covector of such a particle and use it to make a physical interpretation of the space-time–time geometry, identifying certain scalars, vectors, and covectors along a geodesic as electric charge, rest mass, space-time proper time, and space-time momentum of the test particle in question, and certain geometrical fields as gravitational, electromagnetic potential, electromagnetic bivector, and scalar gradient fields exerting apparent forces on test particles in precisely determined ways. Section VIII examines how the space-time–time model distinguishes and to what extent it relates the concepts of inertial mass, passive gravitational mass, active gravitational mass, passive electric charge, and active electric charge. Section IX and the Appendix display the various curvature fields of a conformally constrained metric: curvature tensor, contracted curvature tensor, curvature scalar, and Einstein tensor. In Sec. X I define and compute "residual curvature", an important concept peculiar to conformally constrained metrics. Lastly, Sec. XI discusses the rationale for the term "space-time–time" and the need for extension of the conformally constrained geometry that consistent application of that rationale implies.

II. Conformally Constrained Metrics

Let $\mathcal{M}$ be a manifold and $\hat{G}$ a (symmetric and nondegenerate) metric on $\mathcal{M}$. That $\hat{G}$ is **conformally constrained** will mean that it meets the following condition, in which $\mathcal{L}_\xi$ denotes Lie differentiation along ξ.

Conformality Constraint. There exists on $\mathcal{M}$ a vector field ξ such that $\mathcal{L}_\xi \hat{G} = 2G$, where $G := \hat{G} - (\hat{G}\xi\xi)^{-1}(\hat{G}\xi \otimes \hat{G}\xi)$.

(The metric $\hat{G}$ is understood to be a "cocotensor" field: if P is a point of $\mathcal{M}$, then $\hat{G}(P)$ is an element of $T_P \otimes T_P$, that is, a linear mapping of the tangent space T^P of $\mathcal{M}$ at P into T_P, the cotangent space of $\mathcal{M}$ at P, regarded as the dual space of T^P. Thus $\hat{G}\xi$ is a covector field on $\mathcal{M}$, and $\hat{G}\xi\xi$ is a scalar field on $\mathcal{M}$, the "square length" of ξ under $\hat{G}$. Implicit in the conformality constraint is that $\hat{G}\xi\xi$ vanishes nowhere, that, to put it differently, ξ is nowhere null with respect to $\hat{G}$; a consequence is that ξ itself vanishes nowhere. The symmetric cocotensor field G is just the orthogonal projection of $\hat{G}$ along ξ, so the condition $\mathcal{L}_\xi \hat{G} = 2G$ causes the lengths of vectors orthogonal to and Lie transported by ξ to expand.)

The prototype of conformally constrained metrics is the de Sitter space-time metric, which in the Lemaître coordinate system takes the form

$$\hat{G} = e^{2t}(dx \otimes dx + dy \otimes dy + dz \otimes dz) - R^2(dt \otimes dt), \tag{1}$$

where R is the (uniform) space-time radius of curvature [1, 5]. Here $\xi = \partial/\partial t$, $\hat{G}\xi = -R^2 dt$, $\hat{G}\xi\xi = -R^2$, and $G = e^{2t}(dx \otimes dx + dy \otimes dy + dz \otimes dz)$. The manifold $\mathcal{M}$ covered by the Lemaître coordinate system is (together with $\hat{G}$) only half of the complete de Sitter space-time, which is a single-sheeted hyperboloidal "sphere" $\mathcal{H}$ of radius R in the Minkowski space $M(4,1)$. Though not geodesically complete, $\mathcal{M}$ is ξ-**complete** in that on every ξ-path (that is, on every maximally extended integral path of ξ) the integration parameter runs from $-\infty$ to ∞. Because $\mathcal{H}$ is homogeneous, it is a union of open "hemispheres" like $\mathcal{M}$, on each of which the metric of $\mathcal{H}$ is conformally constrained and ξ-complete.

Two additional examples of conformally constrained metrics are the hyper-de Sitter metrics $\hat{G}_\pm$ given by

$$\hat{G}_\pm = e^{2\zeta}(dx \otimes dx + dy \otimes dy + dz \otimes dz - dt \otimes dt) \pm R^2(d\zeta \otimes d\zeta), \tag{2}$$

defined on manifolds $\mathcal{M}_\pm$ that (with $\hat{G}_\pm$) are open halves of the two kinds of "spheres" of radius R found in $M(4,2)$. For both metrics $\xi = \partial/\partial\zeta$ and $G = e^{2\zeta}(dx \otimes dx + dy \otimes dy + dz \otimes dz - dt \otimes dt)$; but $\hat{G}_+\xi\xi = R^2$, whereas $\hat{G}_-\xi\xi = -R^2$, which of course reflects the fact that $\hat{G}_+$ has diagonal signature $+ + + - +$ and $\hat{G}_-$ has it $+ + + - -$. Both $\mathcal{M}_+$ and $\mathcal{M}_-$ are ξ-complete.

With these examples in mind let us agree to describe $\hat{G}$ as ξ-**completely conformally constrained** if $\hat{G}$ is conformally constrained and $\mathcal{M}$ is ξ-complete (with respect to the vector field ξ of the constraint), and as **locally** (ξ-**completely**) **conformally constrained** if $\mathcal{M}$ is a union of open submanifolds on each of which the restriction of $\hat{G}$ is (ξ-completely) conformally constrained. Then the metric of the de Sitter sphere $\mathcal{H}$ is locally, ξ-completely conformally constrained, as are the hyper-de Sitter sphere metrics that extend $\hat{G}_+$ and $\hat{G}_-$.

The five-dimensional Kaluza metrics are characterized by the "cylinder condition" $\mathcal{L}_\xi \hat{G} = 0$ [3], which makes ξ a Killing vector field of, hence "isometrically constrains", $\hat{G}$ [6]. Also, as readily follows, $\mathcal{L}_\xi G = 0$, so G is Lie-constant along every ξ-path. This projection G of the metric $\hat{G}$, defined on the five-dimensional manifold of $\hat{G}$, is essentially four-dimensional, being degenerate in the direction of ξ. It was intended (by Klein [3a] and by Einstein [3b], each of whom adopted it in preference to Kaluza's noninvariant alternative) to supplant the four-dimensional metric of space-time, and was therefore supposed to have diagonal signature $+++-$ for its nondegenerate part. Having to choose between $+++-+$ and $+++--$ for the signature of the full metric $\hat{G}$, Kaluza apparently opted for $+++-+$ [7]. As the first three $+$'s refer to spatial dimensions, one naturally is tempted to say (and many do say) that this causes Kaluza's extra dimension to be spatial also, and to call a Kaluza manifold a "space-time–space". But that is mere verbal analogy — it lacks any real justification in the form of a connection between the fifth coordinate, generated along ξ, and the three dimensions of physical space represented by the first three coordinates.

Rather than settle on one of these signatures for $\hat{G}$, I shall proceed as if either may be the case, and shall apply the descriptive term **space-time–time** to every five-dimensional manifold $\mathcal{M}$ bearing a locally, ξ-completely conformally constrained metric $\hat{G}$, of diagonal signature $+ + + - +$ (equivalently, $- - - + -$) or of signature $+ + + - -$ (equivalently, $- - - + +$), whose orthogonal projection along ξ has a space-time signature [8]. I intend in a subsequent paper to place the two kinds of space-time–time metric on equal footing as projections of a single, higher dimensional, conformally constrained metric. Physical interpretations aside, all the computations that follow will be valid whatever the dimensionality of $\mathcal{M}$ or the signature of $\hat{G}$.

III. Standard Forms of a Conformally Constrained Metric

Let $\hat{G}$ be a metric that is conformally constrained, and let $\mathcal{M}$ be its carrying manifold. One sees easily that

$$\hat{G} = G + \hat{\epsilon}\phi^2(A \otimes A), \tag{3}$$

where $\phi := |\hat{G}\xi\xi|^{\frac{1}{2}}$, $A := (\hat{G}\xi\xi)^{-1}\hat{G}\xi$, and $\hat{\epsilon} := 1$ if $\hat{G}\xi\xi > 0$, but -1 if $\hat{G}\xi\xi < 0$. The projected metric G, the scalar field ϕ, and the covector field A behave in the following ways under Lie differentiation along ξ: $\mathcal{L}_\xi\phi = 0$, $\mathcal{L}_\xi A = 0$, and $\mathcal{L}_\xi G = 2G$. This is demonstrable by a few simple calculations. First, $G\xi = \hat{G}\xi - (\hat{G}\xi\xi)^{-1}(\hat{G}\xi \otimes \hat{G}\xi)\xi = \hat{G}\xi - (\hat{G}\xi\xi)^{-1}(\hat{G}\xi\xi)\hat{G}\xi = 0$. Next, because $\mathcal{L}_\xi\xi = 0$, one has that $\mathcal{L}_\xi(\hat{G}\xi) = (\mathcal{L}_\xi\hat{G})\xi = 2G\xi = 0$, and $\mathcal{L}_\xi(\hat{G}\xi\xi) = (\mathcal{L}_\xi(\hat{G}\xi))\xi = 0$, so that clearly $\mathcal{L}_\xi\phi = 0$ and $\mathcal{L}_\xi A = 0$. From Eq. (3) it then follows that $\mathcal{L}_\xi G = \mathcal{L}_\xi\hat{G}$, whence $\mathcal{L}_\xi G = 2G$.

A decomposition of G comes about from solving the differential equation $\mathcal{L}_\xi G = 2G$. Specifically, if C is a scalar field on $\mathcal{M}$, then $\mathcal{L}_\xi(e^{-2C}G) = e^{-2C}[-2(\mathcal{L}_\xi C)G + \mathcal{L}_\xi G] = -2e^{-2C}(\mathcal{L}_\xi C - 1)G$. If $\mathcal{L}_\xi C = 1$, then $\mathcal{L}_\xi\mathring{G} = 0$, where $\mathring{G} := e^{-2C}G$. Thus $G = e^{2C}\mathring{G}$, and

$$\hat{G} = e^{2C}\mathring{G} + \hat{\epsilon}\phi^2(A \otimes A), \tag{4}$$

where C is a scalar field, $\mathcal{L}_\xi C = 1$, $\mathring{G}$ is a metric on $\mathcal{M}$ of the same signature and degeneracy as G, and $\mathcal{L}_\xi\mathring{G} = 0$. Application of $\mathcal{L}_\xi$ to both sides of Eq. (4) shows that this representation for $\hat{G}$, under the conditions that $\mathcal{L}_\xi C = 1$ and the Lie derivatives along ξ of $\mathring{G}$, ϕ, and A all vanish, is sufficient to make $\hat{G}$ satisfy the conformality constraint (with respect to ξ). With these conditions the representation therefore constitutes a characterization of conformally constrained metrics.

Continuing, let us introduce (by a standard construction) a coordinate system $[\![x^\mu, \zeta]\!]$ adapted to ξ so that $\xi = \partial/\partial\zeta$. (Here μ and other Greek letter indices will range, if $d > 1$, from 1 to $d - 1$, where $d := \dim \mathcal{M}$; if $d = 1$, then the only coordinate is ζ, so μ does not enter the game.) As a covector field, A has in $[\![x^\mu, \zeta]\!]$ the expansion $A = A_\mu dx^\mu + A_\zeta d\zeta$. But $A_\zeta = A(\partial/\partial\zeta) = A\xi = (\hat{G}\xi\xi)^{-1}\hat{G}\xi\xi = 1$, so $A = A_\mu dx^\mu + d\zeta$. Further, $0 = \mathcal{L}_\xi A = \mathcal{L}_{\partial/\partial\zeta}A = (\partial A_\mu/\partial\zeta)dx^\mu$, so $\partial A_\mu/\partial\zeta = 0$; thus the A_μ depend on the coordinates x^κ alone, and not on ζ. Also, $\partial\phi/\partial\zeta = \mathcal{L}_\xi\phi = 0$, so ϕ is a function of the x^κ only. The projected metric G has the expansion $G = dx^\mu \otimes g_{\mu\nu}dx^\nu + dx^\mu \otimes g_{\mu\zeta}d\zeta + d\zeta \otimes g_{\zeta\nu}dx^\nu + d\zeta \otimes g_{\zeta\zeta}d\zeta$. But $0 = G\xi = G(\partial/\partial\zeta) = g_{\zeta\nu}dx^\nu + g_{\zeta\zeta}d\zeta$, so $g_{\zeta\nu} = g_{\zeta\zeta} = 0$. Because G is symmetric, $g_{\mu\zeta}$ vanishes also, and therefore $G = dx^\mu \otimes g_{\mu\nu}dx^\nu$. The condition $\mathcal{L}_\xi G = 2G$ translates to $\partial g_{\mu\nu}/\partial\zeta = 2g_{\mu\nu}$. In this way we arrive at the adapted coordinates version of Eq. (3), viz.

$$\hat{G} = dx^\mu \otimes g_{\mu\nu}dx^\nu + \hat{\epsilon}\phi^2(A_\mu dx^\mu + d\zeta) \otimes (A_\nu dx^\nu + d\zeta), \tag{3'}$$

with $\partial\phi/\partial\zeta = \partial A_\mu/\partial\zeta = 0$ and $\partial g_{\mu\nu}/\partial\zeta = 2g_{\mu\nu}$.

To do the same for Eq. (4), let us now select the scalar field C. Because $\partial C/\partial\zeta = \mathcal{L}_\xi C = 1$, the possibilities are $C = \zeta + \theta$, thus $e^{2C} = e^{2\zeta}e^{2\theta}$, with $\partial\theta/\partial\zeta = 0$. But the factor $e^{2\theta}$ can be absorbed by redefining $\mathring{G}$, so let us take $C = \zeta$. Then $\mathring{G} = e^{-2\zeta}G = dx^\mu \otimes \mathring{g}_{\mu\nu}dx^\nu$, where $\mathring{g}_{\mu\nu} := e^{-2\zeta}g_{\mu\nu}$; consequently, $G = e^{2\zeta}\mathring{G}$, $g_{\mu\nu} = e^{2\zeta}\mathring{g}_{\mu\nu}$, and, because $\mathcal{L}_\xi\mathring{G} = 0$, $\partial\mathring{g}_{\mu\nu}/\partial\zeta = 0$. Let us also introduce the covector field $\mathring{A} := A - d\zeta$, for which $A = \mathring{A} + d\zeta$, $\mathring{A} = \mathring{A}_\mu dx^\mu$, $\mathring{A}_\mu = A_\mu$, $\mathcal{L}_\xi\mathring{A} = \mathcal{L}_\xi A = 0$, and $\partial\mathring{A}_\mu/\partial\zeta = \partial A_\mu/\partial\zeta = 0$. Then Eq. (4) takes the forms

$$\begin{aligned}\hat{G} &= e^{2\zeta}\mathring{G} + \hat{\epsilon}\phi^2\big(\mathring{A} + d\zeta\big) \otimes \big(\mathring{A} + d\zeta\big)\\ &= e^{2\zeta}(dx^\mu \otimes \mathring{g}_{\mu\nu}dx^\nu) + \hat{\epsilon}\phi^2\big(\mathring{A}_\mu dx^\mu + d\zeta\big) \otimes \big(\mathring{A}_\nu dx^\nu + d\zeta\big),\end{aligned} \tag{4'}$$

with $\partial\phi/\partial\zeta = \partial\mathring{A}_\mu/\partial\zeta = \partial\mathring{g}_{\mu\nu}/\partial\zeta = 0$.

The ability of a metric $\hat{G}$ to assume the standard forms (3′) and (4′) with the stated conditions on ϕ, A_μ, $\mathring{A}_\mu$, $g_{\mu\nu}$, and $\mathring{g}_{\mu\nu}$ satisfied is necessary and sufficient for the restrictions of $\hat{G}$ to the domains of all such adapted coordinate systems $[\![x^\mu, \zeta]\!]$ to be conformally constrained, thus for $\hat{G}$ to be locally conformally constrained.

IV. Gauge Transformations

Coordinate systems adapted to ξ such as $[\![x^\mu, \zeta]\!]$ of the preceding section may be constructed in the following well-known way. Pick a hypersurface $\mathcal{S}$ of $\mathcal{M}$ that is transverse to ξ, and a coordinate system $[\![y^\mu]\!]$ of $\mathcal{S}$, and suppose that no ξ-path crosses dom$[\![y^\mu]\!]$ twice [9]. For each point P of $\mathcal{M}$ that lies on a trajectory of ξ (that is, in some ξ-path's range) whose intersection with $\mathcal{S}$ is a point Q in the domain of $[\![y^\mu]\!]$, let $x^\mu(P) = y^\mu(Q)$ and let $\zeta(P)$ be the value attained at P by the integration parameter of ξ that starts with the value 0 at Q. Then $[\![x^\mu, \zeta]\!]$ is a coordinate system of $\mathcal{M}$ whose domain is the set of all such points P. It is adapted to ξ in the sense that $\xi = \partial/\partial\zeta$, and to $\mathcal{S}$ in that $\zeta|_{\mathcal{S}} = 0$.

The only arbitrary elements in this construction are the hypersurface $\mathcal{S}$ and the coordinate system $[\![y^\mu]\!]$ of $\mathcal{S}$. When one picks a different hypersurface $\mathcal{S}'$ transverse to ξ, and ξ-transfers $[\![y^\mu]\!]$ to $\mathcal{S}'$ to use as the coordinate system $[\![y^{\mu'}]\!]$ of $\mathcal{S}'$, so that $y^{\mu'}(Q') = y^\mu(Q)$ if Q in $\mathcal{S}$ and Q' in $\mathcal{S}'$ belong to the same trajectory of ξ, then the coordinate system $[\![x^{\mu'}, \zeta']\!]$ produced by the construction is related to $[\![x^\mu, \zeta]\!]$ by $x^{\mu'} = x^\mu$ and $\zeta' = \zeta - \lambda$, where $\lambda := \zeta - \zeta'$. The scalar field λ is constant on each trajectory of ξ traversing its domain, hence is independent of ζ, for if Q and Q' are the points where the ξ-trajectory intersects $\mathcal{S}$ and $\mathcal{S}'$, respectively, then $\lambda(P) = \zeta(Q') = -\zeta'(Q)$ for every point P on the trajectory.

From $\zeta' = \zeta - \lambda$ it follows that $d\zeta = d\lambda + d\zeta' = (\partial\lambda/\partial x^\mu)dx^\mu + d\zeta'$, hence that the covector field A, which has in $[\![x^\mu, \zeta]\!]$ the expansion $A = \mathring{A} + d\zeta = \mathring{A}_\mu dx^\mu + d\zeta$, has in $[\![x^\mu, \zeta']\!]$ the expansion $A = \mathring{A}' + d\zeta' = \mathring{A}'_\mu dx^\mu + d\zeta'$ with $\mathring{A}' = \mathring{A} + d\lambda$ and, consequently, $\mathring{A}_\mu + \partial\lambda/\partial x^\mu$. In the event that $\hat{G}$ is a space-time–time metric, the negative of twice the exterior differential of A will come to be identified as the electromagnetic field tensor F. We shall have then that $F = -2d_\wedge A = -2d_\wedge\big(\mathring{A} + d\zeta\big) = -2d_\wedge\mathring{A}$, hence that $\mathring{A}$ plays the role of electromagnetic four-vector potential. But we shall have also that $F = -2d_\wedge\mathring{A}'$, so that $\mathring{A}'$ plays the same role, but in a different gauge. This tells us that the transformation from the adapted coordinate system $[\![x^\mu, \zeta]\!]$ to the adapted coordinate system $[\![x^\mu, \zeta']\!]$ generates a gauge transformation $\mathring{A} \to \mathring{A} + d\lambda$ of the electromagnetic four-vector potential. The converse likewise is true: every gauge transformation $\mathring{A} \to \mathring{A} + d\lambda$ with λ a scalar field independent of ζ determines a transformation from the adapted coordinate system $[\![x^\mu, \zeta]\!]$

to an adapted coordinate system $[\![x^\mu, \zeta']\!]$ with $\zeta' = \zeta - \lambda$.

The discussion up to this point only recapitulates what Klein [3a] and Einstein [3b] worked out long ago for the Kaluza (–Klein) geometry. Their identification of electromagnetic four-potential gauge transformations with adapted-coordinates transformations in five dimensions was the first step on the road to the gauge theories that currently permeate theoretical physics. Missing from Kaluza–Klein theory and from these gauge theories, however, is any remembrance of Weyl's earlier association of electromagnetic gauge transformations with (conformal) gauge transformations of the metric of space-time [10]. In space-time–time this association is preserved, as we are now in position to see.

It is really quite simple. When $\hat{G}$ is a space-time–time metric, it is $\mathring{G}$ that takes the role of space-time metric. But there is not just one $\mathring{G}$, there are many, each corresponding to a particular choice of the hypersurface $\mathcal{S}$ in the construction of the adapted coordinates. If, as before, $\mathcal{S}$ and $\mathcal{S}'$ are two such choices, then $G = e^{2\zeta}\mathring{G} = e^{2\zeta'}\mathring{G}'$, where $\mathring{G}' = e^{2\lambda}\mathring{G}$. Thus the same coordinate transformation that generates the electromagnetic gauge transformation $\mathring{A} \to \mathring{A} + d\lambda$ generates Weyl's metrical gauge transformation $\mathring{G} \to e^{2\lambda}\mathring{G}$.

The coordinate transformations that generate the electromagnetic and the metrical gauge transformations, *being* coordinate transformations, alter only the representation of the space-time–time metric, not the metric itself. This is a principal advantage that the space-time–time geometry has over the Weyl geometry. Weyl, working before Kaluza first proposed using five dimensions to unify gravity and electromagnetism, impressed his infinitude of conformally related space-time metrics onto one four-dimensional manifold. That is very much like drawing all the maps of the world on a single sheet of paper, a practice that would conserve paper but confound navigators. In effect, the space-time–time geometry economizes on paper but avoids the confusion of maps on maps, by drawing a selection of the maps on individual sheets, then stacking the sheets so that each of the remaining maps can be generated on command by slicing through the stack in a particular way. Nothing is lost thereby, and much is gained, as we shall see.

V. Connection Forms and Covariant Differentiations

Further study of the geometry of the conformally constrained metric $\hat{G}$ will be facilitated if we work in a frame system that on the one hand takes full advantage of the orthogonality between G and $\hat{G} - G$ and on the other hand is Lie constant along ξ, but that elsewise is unrestricted. To accomplish this, let us back up a little and relabel the coordinate system $[\![x^\mu, \zeta]\!]$ adapted to ξ as $[\![x^{\mu'}, \zeta]\!]$. Then $A = A_{\mu'}dx^{\mu'} + d\zeta$, $G = dx^{\mu'} \otimes g_{\mu'\nu'}dx^{\nu'}$, $\mathring{A} = \mathring{A}_{\mu'}dx^{\mu'}$, and $\mathring{G} = dx^{\mu'} \otimes \mathring{g}_{\mu'\nu'}dx^{\nu'}$. Now let $\omega^d := \phi A$ (recall that $d := \dim \mathcal{M}$), and let $\{\omega^\mu\}$ be any pointwise linearly independent ordered set of $d-1$ covector fields that are smooth linear combinations of the $dx^{\mu'}$ with coefficients independent of ζ. Then $\omega^\mu = dx^{\mu'}J_{\mu'}{}^{\mu}$ and $dx^{\mu'} = \omega^\mu J_\mu{}^{\mu'}$, where $[J_{\mu'}{}^{\mu}]$ and $[J_\mu{}^{\mu'}]$ are reciprocal matrix fields and satisfy $\partial J_{\mu'}{}^{\mu}/\partial\zeta = \partial J_\mu{}^{\mu'}/\partial\zeta = 0$. The ordered set $\{\omega^\mu, \omega^d\}$ is also pointwise linearly independent; it therefore is a coframe system of $\mathcal{M}$, defined on the domain of the coordinate system $[\![x^{\mu'}, \zeta]\!]$. In this coframe system one has that $A = \phi^{-1}\omega^d$, $G = \omega^\mu \otimes g_{\mu\nu}\omega^\nu$, and $\mathring{G} = \omega^\mu \otimes \mathring{g}_{\mu\nu}\omega^\nu$, hence that

$$\hat{G} = \omega^\mu \otimes g_{\mu\nu}\omega^\nu + \hat{\epsilon}(\omega^d \otimes \omega^d) \tag{3''}$$

and

$$\hat{G} = e^{2\zeta}(\omega^\mu \otimes \mathring{g}_{\mu\nu}\omega^\nu) + \hat{\epsilon}(\omega^d \otimes \omega^d), \tag{4''}$$

where $g_{\mu\nu} = J_\mu{}^{\mu'} g_{\mu'\nu'} J_\nu{}^{\nu'}$ and $\mathring{g}_{\mu\nu} = J_\mu{}^{\mu'} \mathring{g}_{\mu'\nu'} J_\nu{}^{\nu'}$, with the consequences that $\partial g_{\mu\nu}/\partial\zeta = 2g_{\mu\nu}$ and $\partial \mathring{g}_{\mu\nu}/\partial\zeta = 0$. Also, $\mathring{A} = \mathring{A}_\mu \omega^\mu$, where $\mathring{A}_\mu = J_\mu{}^{\mu'} \mathring{A}_{\mu'}$, $\mathring{A}_{\mu'} = A_{\mu'}$, and, consequently, $\partial \mathring{A}_\mu/\partial\zeta = 0$ (note that in general $\mathring{A}_\mu \neq A_\mu$, even though A has the mixed expansion $A = \mathring{A}_\mu \omega^\mu + d\zeta$; in fact $A_\mu = 0$ and $A_d = \phi^{-1}$).

Upon identifying the frame system $\{e_\mu, e_d\}$ to which $\{\omega^\mu, \omega^d\}$ is dual, one has

$$e_\mu = J_\mu{}^{\mu'}(\partial/\partial x^{\mu'}) - \mathring{A}_\mu(\partial/\partial\zeta) \quad \text{and} \quad e_d = \phi^{-1}\xi = \phi^{-1}(\partial/\partial\zeta), \tag{5}$$

to go with

$$\omega^\mu = dx^{\mu'} J_{\mu'}{}^\mu \quad \text{and} \quad \omega^d = \phi A = \phi(A_{\mu'}dx^{\mu'} + d\zeta) = \phi(\mathring{A}_\mu\omega^\mu + d\zeta). \tag{6}$$

The vector field e_d is the unit normalization of ξ and is orthogonal to each of the vector fields e_μ. It is not difficult to see that $\mathcal{L}_\xi e_\mu = \mathcal{L}_\xi e_d = 0$ and $\mathcal{L}_\xi \omega^\mu = \mathcal{L}_\xi \omega^d = 0$. Thus we have a frame system and its dual coframe system that are Lie constant along ξ, but with the further property that e_d has length 1 and is orthogonal to each e_μ. Their constancy along ξ makes them gauge invariant: the adapted coordinates transformation $[\![x^{\mu'}, \zeta]\!] \to [\![x^{\mu'}, \zeta']\!]$ has no effect on them. This is what we sought. Borrowing terminology from fibre bundle theory we may call the e_μ and the tangent subspace they span at a point "horizontal", and e_d and the subspace it spans at a point "vertical", as determined with reference to the covector field A, standing in for a bundle connection 1-form.

To identify differentiations of a scalar field f by the various frame operators, let us adopt the abbreviations $f_{.\mu'} := \partial f/\partial x^{\mu'}$, $f_{.\zeta} := \partial f/\partial\zeta$, $f_{,\mu} := e_\mu f$, and $f_{,d} := e_d f$; also, let $f_{.\mu} := J_\mu{}^{\mu'} f_{.\mu'}$. Then from Eqs. (5) follow

$$f_{,\mu} = J_\mu{}^{\mu'} f_{.\mu'} - \mathring{A}_\mu f_{.\zeta} = f_{.\mu} - \mathring{A}_\mu f_{.\zeta} \quad \text{and} \quad f_{,d} = \phi^{-1} f_{.\zeta}. \tag{7}$$

When f is independent of ζ, then $f_{,d} = 0$ and $f_{,\mu} = f_{.\mu}$. In particular

$$\phi_{.\zeta} = \mathring{A}_{\mu.\zeta} = \mathring{g}_{\mu\nu.\zeta} = J_{\mu'}{}^{\mu}{}_{.\zeta} = J_\mu{}^{\mu'}{}_{.\zeta} = 0, \tag{8}$$

so

$$\mathring{A}_{\mu,d} = \mathring{g}_{\mu\nu,d} = J_{\mu'}{}^{\mu}{}_{,d} = J_\mu{}^{\mu'}{}_{,d} = 0 \tag{9}$$

and

$$\begin{gathered} \phi_{,\mu} = \phi_{.\mu}, \quad \mathring{A}_{\mu,\nu} = \mathring{A}_{\mu.\nu}, \quad \mathring{g}_{\mu\nu,\kappa} = \mathring{g}_{\mu\nu.\kappa}, \\ J_{\mu'}{}^{\mu}{}_{,\nu} = J_{\mu'}{}^{\mu}{}_{.\nu}, \quad \text{and} \quad J_\mu{}^{\mu'}{}_{,\nu} = J_\mu{}^{\mu'}{}_{.\nu}. \end{gathered} \tag{10}$$

On the other hand,

$$g_{\mu\nu.\zeta} = 2g_{\mu\nu}, \tag{11}$$

so

$$g_{\mu\nu,d} = 2\phi^{-1} g_{\mu\nu} \tag{12}$$

and

$$g_{\mu\nu,\kappa} = g_{\mu\nu.\kappa} - 2g_{\mu\nu}\mathring{A}_\kappa. \tag{13}$$

For the exterior differential of ω^μ we have $d_\wedge\omega^\mu = C_\kappa{}^\mu{}_\lambda\omega^\lambda\wedge\omega^\kappa$, with $C_\kappa{}^\mu{}_\lambda$ skew-symmetric in κ and λ and independent of ζ, so that

$$C_\kappa{}^\mu{}_{\lambda.\zeta} = 0, \quad C_\kappa{}^\mu{}_{\lambda,d} = 0, \quad \text{and} \quad C_\kappa{}^\mu{}_{\lambda,\nu} = C_\kappa{}^\mu{}_{\lambda.\nu} \tag{14}$$

(in terms of $J_{\mu'}{}^\mu$ and $J_\mu{}^{\mu'}$, $C_\kappa{}^\mu{}_\lambda = J_{[\kappa}{}^{\mu'}J_{\mu'}{}^\mu{}_{.\lambda]}$ [11]). From $\omega^d = \phi A$ it follows that $d_\wedge\omega^d = -(1/2)\phi F + d\phi\wedge A$, where

$$\begin{aligned} F &:= -2d_\wedge A = -2d_\wedge\mathring{A} \\ &= F_{\kappa\lambda}\omega^\lambda\wedge\omega^\kappa = F_{\kappa\lambda}\omega^\lambda\otimes\omega^\kappa \end{aligned} \tag{15}$$

with

$$\begin{aligned} F_{\kappa\lambda} &= -2\big(\mathring{A}_{[\kappa.\lambda]} + \mathring{A}_\mu C_\kappa{}^\mu{}_\lambda\big) \\ &= \mathring{A}_{\lambda.\kappa} - \mathring{A}_{\kappa.\lambda} - 2\mathring{A}_\mu C_\kappa{}^\mu{}_\lambda, \end{aligned} \tag{16}$$

in consequence of which

$$F_{\kappa\lambda.\zeta} = 0, \quad F_{\kappa\lambda,d} = 0, \quad \text{and} \quad F_{\kappa\lambda,\mu} = F_{\kappa\lambda.\mu}. \tag{17}$$

Thus

$$d_\wedge\omega^\mu = C_\kappa{}^\mu{}_\lambda\omega^\lambda\wedge\omega^\kappa, \tag{18}$$

and

$$d_\wedge\omega^d = -(1/2)\phi F_{\kappa\lambda}\omega^\lambda\wedge\omega^\kappa + \phi^{-1}\phi_{.\lambda}\omega^\lambda\wedge\omega^d. \tag{19}$$

By use of Eqs. (18) and (19), and the fact that, for $K, L, M = 1, \ldots, d$, $[e_K, e_L] = C_K{}^M{}_L\, e_M$ if $d_\wedge\omega^M = C_K{}^M{}_L\,\omega^L\otimes\omega^K$, the nonvanishing commutators of the frame system $\{e_\mu, e_d\}$ are readily expressed [12]:

$$[e_\kappa, e_\lambda] = C_\kappa{}^\mu{}_\lambda e_\mu - (1/2)\phi F_{\kappa\lambda}e_d \tag{20}$$

and

$$[e_\kappa, e_d] = -(1/2)\phi^{-1}\phi_{.\kappa}e_d = -[e_d, e_\kappa]. \tag{21}$$

Let us denote by $\hat{\mathbf{d}}$ the torsionless covariant differentiation on $\mathcal{M}$ that is compatible with $\hat{G}$, and by $\hat{\omega}_\kappa{}^\mu$, $\hat{\omega}_\kappa{}^d$, $\hat{\omega}_d{}^\mu$, and $\hat{\omega}_d{}^d$ the connection forms of $\hat{\mathbf{d}}$ in the frame system $\{e_\mu, e_d\}$, so that $\hat{\mathbf{d}}\hat{G} = 0$ and

$$\begin{aligned} \hat{\mathbf{d}}e_\kappa &= \hat{\omega}_\kappa{}^\mu\otimes e_\mu + \hat{\omega}_\kappa{}^d\otimes e_d, \\ \hat{\mathbf{d}}e_d &= \hat{\omega}_d{}^\mu\otimes e_\mu + \hat{\omega}_d{}^d\otimes e_d, \\ \hat{\mathbf{d}}\omega^\mu &= -\hat{\omega}_\kappa{}^\mu\otimes\omega^\kappa - \hat{\omega}_d{}^\mu\otimes\omega^d, \end{aligned} \tag{22}$$

and

$$\hat{\mathbf{d}}\omega^d = -\hat{\omega}_\kappa{}^d\otimes\omega^\kappa - \hat{\omega}_d{}^d\otimes\omega^d.$$

By standard methods these connection forms can be expressed in terms of the metric components in Eq. (3″) and the exterior differential coefficients in Eqs. (18) and (19). The result is that

$$\begin{aligned}\hat\omega_\kappa{}^\mu &= \omega_\kappa{}^\mu + \left[\phi^{-1}g_\kappa{}^\mu + \hat\epsilon(1/2)\phi F_\kappa{}^\mu\right]\omega^d,\\ \hat\omega_\kappa{}^d &= -\left[\hat\epsilon\phi^{-1}g_{\kappa\lambda} - (1/2)\phi F_{\kappa\lambda}\right]\omega^\lambda + \phi^{-1}\phi_{.\kappa}\omega^d,\\ \hat\omega_d{}^\mu &= \left[\phi^{-1}g^\mu{}_\lambda - \hat\epsilon(1/2)\phi F^\mu{}_\lambda\right]\omega^\lambda - \hat\epsilon\phi^{-1}\phi^{\cdot\mu}\omega^d\\ &= -\hat\epsilon g^{\mu\kappa}\hat\omega_\kappa{}^d,\end{aligned} \tag{23}$$

and

$$\hat\omega_d{}^d = 0;$$

in these equations

$$\begin{aligned}\omega_\kappa{}^\mu &:= \Gamma_\kappa{}^\mu{}_\lambda\omega^\lambda,\\ \Gamma_\kappa{}^\mu{}_\lambda &:= \{{}_\kappa{}^\mu{}_\lambda\} - (C_\kappa{}^\mu{}_\lambda + C_{\kappa\lambda}{}^\mu + C_{\lambda\kappa}{}^\mu),\\ \{{}_\kappa{}^\mu{}_\lambda\} &:= (1/2)(g_{\nu\lambda,\kappa} + g_{\kappa\nu,\lambda} - g_{\kappa\lambda,\nu})g^{\nu\mu},\\ C_{\kappa\lambda}{}^\mu &:= g_{\lambda\nu}C_\kappa{}^\nu{}_\pi g^{\pi\mu}, \quad \phi^{\cdot\mu} := \phi_{.\lambda}g^{\lambda\mu},\\ F_\kappa{}^\mu &:= F_{\kappa\lambda}g^{\mu\lambda}, \quad F^\mu{}_\lambda := g^{\kappa\mu}F_{\kappa\lambda},\\ g_\kappa{}^\mu &:= g_{\kappa\lambda}g^{\lambda\mu} = \delta_\kappa{}^\mu, \quad \text{and} \quad g^\mu{}_\lambda := g^{\mu\nu}g_{\nu\lambda} = \delta^\mu{}_\lambda,\end{aligned} \tag{24}$$

$[g^{\nu\mu}]$ being the matrix field inverse to $[g_{\mu\nu}]$.

An alternate covariant differentiation $\mathbf{d}$ on $\mathcal{M}$ is fixed by the stipulations that $\mathbf{d}e_\kappa = \omega_\kappa{}^\mu \otimes e_\mu$ and $\mathbf{d}e_d = 0$, or, equivalently, that $\mathbf{d}\omega^\mu = -\omega_\kappa{}^\mu \otimes \omega^\kappa$ and $\mathbf{d}\omega^d = 0$. It has the properties i) $\mathbf{d}G = 2A \otimes G$, ii) Tor $\mathbf{d} = d_\wedge\omega^d \otimes e_d = (d_\wedge A + \phi^{-1}d\phi \wedge A) \otimes \xi = [-(1/2)F + \phi^{-1}d\phi\wedge A]\otimes\xi$, and iii) $\mathbf{d}A = -\phi^{-1}d\phi\otimes A$. Because G is degenerate, properties (i) and (ii) do not alone determine $\mathbf{d}$; but properties (i), (ii), and (iii) do. These properties are gauge invariant, and so, therefore, is $\mathbf{d}$. Property (iii), a reformulation of $\mathbf{d}\omega^d = 0$, implies that $\mathbf{d}\hat G = \mathbf{d}G$, hence that $\mathbf{d}\hat G = 2A\otimes G$, in light of property (i). Although G has no inverse, it is useful to let $G^{-1} := e_\mu \otimes g^{\mu\nu}e_\nu$, and then one sees that $G^{-1}\hat G = G^{-1}G = \omega^\mu \otimes e_\mu$, $\hat G G^{-1} = GG^{-1} = e_\mu \otimes \omega^\mu$, and $\mathbf{d}\hat G^{-1} = \mathbf{d}G^{-1} = -2A \otimes G^{-1}$. All connection forms and coefficients of $\mathbf{d}$ other than the $\omega_\kappa{}^\mu$ and the $\Gamma_\kappa{}^\mu{}_\lambda$, that is, all with d as a suffix, vanish. This covariant differentiation is an analog in the conformally constrained geometry of the covariant differentiation (affine connection) in Weyl's geometry, the principal characteristic of which is that it satisfies the equation in property (i) above, properly interpreted.

Bringing into play Eq. (13) we can express the Christoffel symbols $\{{}_\kappa{}^\mu{}_\lambda\}$ in the more expanded form

$$\begin{aligned}\{{}_\kappa{}^\mu{}_\lambda\} = (1/2)&(g_{\nu\lambda.\kappa} + g_{\kappa\nu.\lambda} - g_{\kappa\lambda.\nu})g^{\nu\mu}\\ &- \left(g_{\nu\lambda}\mathring{A}_\kappa + g_{\kappa\nu}\mathring{A}_\lambda - g_{\kappa\lambda}\mathring{A}_\nu\right)g^{\nu\mu}.\end{aligned} \tag{25}$$

A further breaking out arises from replacing $g_{\mu\nu}$ by $e^{2\zeta}\mathring{g}_{\mu\nu}$ and, accordingly, $g^{\nu\mu}$ by $e^{-2\zeta}\mathring{g}^{\nu\mu}$, where $[\mathring{g}^{\nu\mu}]$ is the inverse of $[\mathring{g}_{\mu\nu}]$. That results in

$$\{{}_\kappa{}^\mu{}_\lambda\} = \{{}_\kappa{}^\mu{}_\lambda\}^\circ + \mathring{\Delta}_\kappa{}^\mu{}_\lambda, \tag{26}$$

in which

$$\begin{aligned}
\{{}_\kappa{}^\mu{}_\lambda\}^\circ &:= (1/2)(\mathring{g}_{\nu\lambda.\kappa} + \mathring{g}_{\kappa\nu.\lambda} - \mathring{g}_{\kappa\lambda.\nu})\mathring{g}^{\nu\mu}, \\
\mathring{\Delta}_\kappa{}^\mu{}_\lambda &:= -\big(\mathring{A}_\kappa \mathring{g}^\mu{}_\lambda + \mathring{g}_\kappa{}^\mu \mathring{A}_\lambda + \mathring{g}_{\kappa\lambda}\mathring{A}^\mu\big), \\
\mathring{A}^\mu &:= \mathring{A}_\lambda \mathring{g}^{\lambda\mu}, \\
\mathring{g}^\mu{}_\lambda &:= \mathring{g}^{\mu\nu}\mathring{g}_{\nu\lambda} = \delta^\mu{}_\lambda, \quad \text{and} \quad \mathring{g}_\kappa{}^\mu := \mathring{g}_{\kappa\lambda}\mathring{g}^{\lambda\mu} = \delta_\kappa{}^\mu.
\end{aligned} \tag{27}$$

This in turn gives

$$\omega_\kappa{}^\mu = \mathring{\omega}_\kappa{}^\mu + \mathring{\Delta}_\kappa{}^\mu$$

and (28)

$$\Gamma_\kappa{}^\mu{}_\lambda = \mathring{\Gamma}_\kappa{}^\mu{}_\lambda + \mathring{\Delta}_\kappa{}^\mu{}_\lambda,$$

where

$$\begin{aligned}
\mathring{\omega}_\kappa{}^\mu &:= \mathring{\Gamma}_\kappa{}^\mu{}_\lambda \omega^\lambda, \quad \mathring{\Delta}_\kappa{}^\mu := \mathring{\Delta}_\kappa{}^\mu{}_\lambda \omega^\lambda, \\
\mathring{\Gamma}_\kappa{}^\mu{}_\lambda &:= \{{}_\kappa{}^\mu{}_\lambda\}^\circ - \big(C_\kappa{}^\mu{}_\lambda + \mathring{C}_{\kappa\lambda}{}^\mu + \mathring{C}_{\lambda\kappa}{}^\mu\big),
\end{aligned} \tag{29}$$

and

$$\mathring{C}_{\kappa\lambda}{}^\mu := \mathring{g}_{\lambda\nu} C_\kappa{}^\nu{}_\pi \mathring{g}^{\pi\mu} \quad (= C_{\kappa\lambda}{}^\mu, \text{ as well}).$$

Yet another covariant differentiation $\mathring{\mathbf{d}}$ on $\mathcal{M}$ is fixed by the stipulations that $\mathring{\mathbf{d}}e_\kappa = \mathring{\omega}_\kappa{}^\mu \otimes e_\mu$ and $\mathring{\mathbf{d}}e_d = 0$, which are equivalent to $\mathring{\mathbf{d}}\omega^\mu = -\mathring{\omega}_\kappa{}^\mu \otimes \omega^\kappa$ and $\mathring{\mathbf{d}}\omega^d = 0$. It possesses and is determined by the properties i) $\mathring{\mathbf{d}}\mathring{G} = 0$, ii) Tor $\mathring{\mathbf{d}}$ = Tor $\mathbf{d}$, and iii) $\mathring{\mathbf{d}}A = \mathbf{d}A$, but like $\mathbf{d}$ it is not determined by (i) and (ii) alone. If $\mathring{G}^{-1} := e_\mu \otimes \mathring{g}^{\mu\nu} e_\nu$, then $\mathring{G}^{-1}\mathring{G} = \omega^\mu \otimes e_\mu$, $\mathring{G}\mathring{G}^{-1} = e_\mu \otimes \omega^\mu$, and $\mathring{\mathbf{d}}\mathring{G}^{-1} = 0$. All connection forms and coefficients of $\mathring{\mathbf{d}}$ other than the $\mathring{\omega}_\kappa{}^\mu$ and the $\mathring{\Gamma}_\kappa{}^\mu{}_\lambda$ vanish. Unlike $\mathbf{d}$, which, being determined by gauge invariant properties, is itself gauge invariant, $\mathring{\mathbf{d}}$ is not gauge invariant. That is to say, each new choice of a gauge brings with it a new $\mathring{G}$, and with that comes a (usually) new $\mathring{\mathbf{d}}$ compatible with the new $\mathring{G}$. This covariant differentiation is, in the space-time–time case, a generalized analog of the usual space-time covariant differentiation.

The formulas displayed above will enable us to write out in reasonably comprehensible form the geodesic equations and the various curvature tensor fields of the conformally constrained geometry. Some of their terms disappear in the corresponding formulas for the Kaluza geometry, which is described by the metric of Eq. (4′) with the factor $e^{2\zeta}$ removed; in the Kaluza–Klein geometry, which has in addition ϕ = constant, the terms involving derivatives of ϕ disappear as well. Thus in the conformally constrained geometry there are more hooks to hang physical interpretations on than in the Kaluza geometry, and even more yet than in the Kaluza–Klein geometry.

One aspect of the Kaluza and the Kaluza–Klein geometries that persists in the conformally constrained geometry is that the vanishing of the 2-form F is necessary (and sufficient) for the possibility of gauging away to zero the potential field $\mathring{A}$. Specifically, if $F = 0$, then $d_\wedge \mathring{A} = 0$, so (locally) there exists a scalar field λ such that $\mathring{A} = -d\lambda$, hence such that $\mathring{A}_\mu \omega^\mu = -\lambda_{,\mu}\omega^\mu - \lambda_{,d}\omega^d$. But then $\lambda_{,d} = 0$, so $\lambda_{.\zeta} = 0$, and if $\zeta' = \zeta - \lambda$, then $\mathring{A}' = \mathring{A} + d\lambda = 0$. An important distinction, however, is that, whereas in the Kaluza and the Kaluza–Klein geometries $\mathring{A}$ may be thus gauged away without disturbing the metric $\mathring{G}$, in the conformally constrained geometry the gauging away of $\mathring{A}$ is inevitably accompanied by a conformal alteration of $\mathring{G}$ ($\mathring{G}' = e^{2\lambda}\mathring{G}$). This foretells that in space-time–time physics a nonvanishing electromagnetic potential field will produce real effects even in regions where the electromagnetic field tensor vanishes, a phenomenon already predicted by quantum mechanics [13, 14].

VI. Geodesic Equations

Let $p\colon I \to \mathcal{M}$ be a path in $\mathcal{M}$, with parameter interval I, and let the components of the velocity of p in the adapted frame system $\{e_\mu, e_d\}$ be $\{\dot p^\mu, \dot p^d\}$, so that $\dot p = \dot p^\mu e_\mu(p) + \dot p^d e_d(p)$. Then the acceleration $\ddot p$ generated by the covariant differentiation $\hat{\mathbf{d}}$ is determined by the connection forms of $\hat{\mathbf{d}}$, through use of Eqs. (22), in the following way:

$$\begin{aligned}\ddot p &:= (\dot p^\mu)^{\cdot} e_\mu(p) + \dot p^\kappa \hat{\mathbf{d}} e_\kappa(p)\dot p + \big(\dot p^d\big)^{\cdot} e_d(p) + \dot p^d \hat{\mathbf{d}} e_d(p)\dot p \\ &= \ddot p^\mu e_\mu(p) + \ddot p^d e_d(p),\end{aligned} \tag{30}$$

where

$$\ddot p^\mu = (\dot p^\mu)^{\cdot} + \dot p^\kappa \hat\omega_\kappa{}^\mu(p)\dot p + \dot p^d \hat\omega_d{}^\mu(p)\dot p \tag{31}$$

and

$$\ddot p^d = \big(\dot p^d\big)^{\cdot} + \dot p^\kappa \hat\omega_\kappa{}^d(p)\dot p + \dot p^d \hat\omega_d{}^d(p)\dot p. \tag{32}$$

The condition that p be an affinely parametrized geodesic path of $\hat{\mathbf{d}}$ is that $\ddot p = 0$, which is equivalent to $\ddot p^\mu = \ddot p^d = 0$. From Eqs. (31), (32), (23), and (24), the fact that $\omega^\lambda(p)\dot p = \dot p^\lambda$ and $\omega^d(p)\dot p = \dot p^d$, and the skew-symmetry of $F_{\kappa\lambda}$ it follows that these geodesic equations are equivalent, respectively, to

$$(\dot p^\mu)^{\cdot} + \dot p^\kappa \Gamma_\kappa{}^\mu{}_\lambda \dot p^\lambda = \hat\epsilon \phi \dot p^d F^\mu{}_\lambda \dot p^\lambda - 2\phi^{-1}\dot p^d \dot p^\mu + \hat\epsilon \dot p^d \dot p^d \phi^{-1}\phi^{\cdot\mu} \tag{33}$$

and

$$\big(\dot p^d\big)^{\cdot} + \phi^{-1}\dot p^d \phi_{.\kappa}\dot p^\kappa = \hat\epsilon \phi^{-1}\dot p^\kappa g_{\kappa\lambda}\dot p^\lambda, \tag{34}$$

in which for brevity the compositions with p of the various scalar fields are implicit rather than express.

Utilizing Eqs. (28) to break up $\Gamma_\kappa{}^\mu{}_\lambda$, and remembering that $g_{\mu\nu} = e^{2\zeta}\mathring{g}_{\mu\nu}$ and $g^{\nu\mu} = e^{-2\zeta}\mathring{g}^{\nu\mu}$, we find that Eqs. (33) and (34) are equivalent, respectively, to

$$\begin{aligned}(\dot p^\mu)^{\cdot} + \dot p^\kappa \mathring\Gamma_\kappa{}^\mu{}_\lambda \dot p^\lambda = \hat\epsilon e^{-2\zeta}\phi \dot p^d \mathring F^\mu{}_\lambda \dot p^\lambda + 2\big(\mathring A_\kappa \dot p^\kappa - \phi^{-1}\dot p^d\big)\dot p^\mu \\ - \dot p^\kappa \mathring g_{\kappa\lambda}\dot p^\lambda \mathring A^\mu + \hat\epsilon e^{-2\zeta}\dot p^d \dot p^d \phi^{-1}\mathring\phi^{\cdot\mu}\end{aligned} \tag{33$'$}$$

and

$$\big(\dot p^d\big)^{\cdot} + \phi^{-1}\dot p^d \phi_{.\kappa}\dot p^\kappa = \hat\epsilon e^{2\zeta}\phi^{-1}\dot p^\kappa \mathring g_{\kappa\lambda}\dot p^\lambda, \tag{34$'$}$$

where $\mathring F^\mu{}_\lambda := \mathring g^{\kappa\mu}F_{\kappa\lambda}$ and $\mathring\phi^{\cdot\mu} := \phi_{.\lambda}\mathring g^{\lambda\mu}$. These equations display explicitly all occurrences of ζ except those implied by $\dot p^d = \omega^d(p)\dot p = \big[\phi(\mathring A_\kappa \omega^\kappa + d\zeta)\big](p)\dot p = \phi(\mathring A_\kappa \dot p^\kappa + \dot\zeta)$, where

$\dot{\zeta} := [\zeta(p)]^{\cdot} = d\zeta(p)\dot{p}$. If we take this decomposition of $\dot{p}^d$ partially into account, then we see that Eq. (33′) is equivalent to

$$\left(e^{2\zeta}\dot{p}^{\mu}\right)^{\cdot} + e^{2\zeta}\dot{p}^{\kappa}\mathring{\Gamma}_{\kappa}{}^{\mu}{}_{\lambda}\dot{p}^{\lambda} = \hat{\epsilon}\phi\dot{p}^d\mathring{F}^{\mu}{}_{\lambda}\dot{p}^{\lambda} - e^{2\zeta}\dot{p}^{\kappa}\mathring{g}_{\kappa\lambda}\dot{p}^{\lambda}\mathring{A}^{\mu} + \hat{\epsilon}\dot{p}^d\dot{p}^d\phi^{-1}\mathring{\phi}^{\cdot\mu}. \tag{33''}$$

Noting further that $\dot{\phi} = \phi_{.\kappa}\dot{p}^{\kappa}$, we find that Eq. (34′) is equivalent to

$$\left(\hat{\epsilon}\phi\dot{p}^d\right)^{\cdot} = e^{2\zeta}\dot{p}^{\kappa}\mathring{g}_{\kappa\lambda}\dot{p}^{\lambda}. \tag{34''}$$

As one knows, these geodesic equations entail that $\hat{G}(p)\dot{p}\dot{p}$ is constant. This integral takes either of the equivalent forms

$$\dot{p}^{\kappa}g_{\kappa\lambda}\dot{p}^{\lambda} + \hat{\epsilon}\dot{p}^d\dot{p}^d = \epsilon \tag{35}$$

and

$$e^{2\zeta}\dot{p}^{\kappa}\mathring{g}_{\kappa\lambda}\dot{p}^{\lambda} + \hat{\epsilon}\dot{p}^d\dot{p}^d = \epsilon, \tag{35'}$$

where $\epsilon := \operatorname{sgn}\left(\hat{G}(p)\dot{p}\dot{p}\right) = 1$, 0, or -1, provided that the affine parametrization of p is normal, that is, that arclength is the parameter when $\hat{G}(p)\dot{p}\dot{p} \neq 0$.

VII. Momentum, Rest Mass, Electric Charge, Proper Time, and Equations of Motion of a Test Particle in Space-Time–Time

Thus far it has been convenient to leave unspecified both the dimensionality d of the manifold $\mathcal{M}$ and the diagonal signature of the conformally constrained metric $\hat{G}$ carried by $\mathcal{M}$. Let us now restrict our attention to the case in which $d = 5$ and $\hat{G}$ is a space-time–time metric, with a view toward establishing a physical interpretation of the space-time–time geometry beyond that suggested by comparison of it with its Weyl and Kaluza antecedents. For this purpose it is advantageous to have the signature of the space-time part of the metric be $---+$; this causes the signature of $\hat{G}$ to be $---++$ if $\hat{\epsilon} = 1$, and to be $---+-$ if $\hat{\epsilon} = -1$.

The procedure to be used here to effect a physical interpretation of the geometry is a natural extension of the familiar space-time procedure. One assumes that an elementary test particle's journey through life is described, in whole or in part, by an affinely parametrized geodesic path p in space-time–time. One breaks the geodesic equation $\ddot{p} = 0$, or some equivalent thereof, into its component equations in a perspicuously appropriate frame system and compares these equations to the equations of motion of a test particle in the special theory of relativity, or, more closely, to the analogous equations of motion in the curved space-time of general relativity theory. Out of this comparison one identifies as far as possible the various geometric parameters of the path p with the classical physical parameters of the particle. In the same stroke one identifies terms in the geodesic component equations as representing forces due to classical physical fields, thus identifies the physical fields themselves with various of the geometrical fields derived from the space-time–time metric $\hat{G}$. As this amounts to solving a puzzle in which no piece is seen to fit until every piece is seen to do so, I shall dispense with many of the details and go as quickly as possible to the conclusions.

To begin, let us define the **space-time–time momentum covector** P of the test particle to be the metric dual of its space-time–time velocity, that is, $P := \hat{G}(p)\dot{p}$. Because

$\hat{G}$ is $\hat{\mathbf{d}}$-covariantly constant, $\dot{P} = \hat{G}(p)\ddot{p}$, and therefore the geodesic equation $\ddot{p} = 0$ is equivalent to $\dot{P} = 0$. This latter equation will provide the most immediate comparison to classical equations of motion. In the adapted coframe system $\{\omega^\mu, \omega^d\}$ the space-time–time momentum P has the expansion $P = P_\kappa \omega^\kappa(p) + P_d \omega^d(p)$, where

$$P_\kappa = \dot{p}^\mu g_{\mu\kappa} = e^{2\zeta} \dot{p}^\mu \mathring{g}_{\mu\kappa} \tag{36}$$

and

$$P_d = \hat{\epsilon} \dot{p}^d. \tag{37}$$

The covariant derivative of P has the expansion $\dot{P} = \dot{P}_\kappa \omega^\kappa(p) + \dot{P}_d \omega^d(p)$, where

$$\dot{P}_\kappa = (P_\kappa)^{\cdot} - P_\mu \Gamma_\kappa{}^\mu{}_\lambda \dot{p}^\lambda - \phi P_d F_{\kappa\lambda} \dot{p}^\lambda - \hat{\epsilon} P_d P_d \phi^{-1} \phi_{.\kappa} \tag{38}$$

and

$$\dot{P}_d = (P_d)^{\cdot} + P_d \phi^{-1} \phi_{.\mu} \dot{p}^\mu - \phi^{-1} P_\mu g^{\mu\nu} P_\nu \tag{39}$$

(compositions of scalar fields with p being suppressed in the notation), as follows from application of Eqs. (22) and (23) to $\dot{P} = (P_\kappa)^{\cdot} \omega^\kappa(p) + P_\mu \hat{\mathbf{d}}\omega^\mu(p)\dot{p} + (P_d)^{\cdot} \omega^d(p) + P_d \hat{\mathbf{d}}\omega^d(p)\dot{p}$. Let

$$\begin{aligned} \mathring{m} &:= \left(\mathring{G}^{-1}(p)PP\right)^{\frac{1}{2}} = \left(P_\mu \mathring{g}^{\mu\nu} P_\nu\right)^{\frac{1}{2}} \\ &= e^{2\zeta} \left(\mathring{G}(p)\dot{p}\dot{p}\right)^{\frac{1}{2}} = e^{2\zeta} \left(\dot{p}^\mu \mathring{g}_{\mu\nu} \dot{p}^\nu\right)^{\frac{1}{2}} \end{aligned} \tag{40}$$

and

$$\begin{aligned} q &:= P\xi(p) = \phi P_d \\ &= \hat{\epsilon}\phi \dot{p}^d = \hat{\epsilon}\phi^2 A(p)\dot{p} = \hat{\epsilon}\phi^2 \left(\mathring{A}_\mu \dot{p}^\mu + \dot{\zeta}\right). \end{aligned} \tag{41}$$

Then the equations $\dot{P}_\kappa = 0$ and $\dot{P}_d = 0$, equivalent jointly to $\dot{P} = 0$, are equivalent respectively to

$$\begin{aligned} (P_\kappa)^{\cdot} &= P_\mu \mathring{\Gamma}_\kappa{}^\mu{}_\lambda \dot{p}^\lambda + q F_{\kappa\lambda} \dot{p}^\lambda - e^{-2\zeta} \mathring{m}^2 \mathring{A}_\kappa + \hat{\epsilon}(q/\phi)^2 \phi^{-1} \phi_{.\kappa} \\ &= e^{-2\zeta} P_\mu \mathring{\Gamma}_\kappa{}^\mu{}_\lambda P_\nu \mathring{g}^{\nu\lambda} + e^{-2\zeta} q F_{\kappa\lambda} P_\nu \mathring{g}^{\nu\lambda} - e^{-2\zeta} \mathring{m}^2 \mathring{A}_\kappa + \hat{\epsilon}(q/\phi)^2 \phi^{-1} \phi_{.\kappa} \end{aligned} \tag{42}$$

and

$$\dot{q} = e^{-2\zeta} \mathring{m}^2. \tag{43}$$

These equations have, if the affine parametrization of p is normal, the integral $\hat{G}^{-1}(p)PP = \epsilon$. This is, of course, the same as $\hat{G}(p)\dot{p}\dot{p} = \epsilon$, and therefore the same as Eq. (35′), which is equivalent in terms of $\mathring{m}$ and q to

$$e^{-2\zeta} \mathring{m}^2 + \hat{\epsilon}(q/\phi)^2 = \epsilon. \tag{44}$$

Substitution of this integral into Eq. (43) yields

$$\dot{q} = \epsilon - \hat{\epsilon}(q/\phi)^2. \tag{45}$$

Equations (44), (41), and (43) imply that

$$\begin{aligned}(\mathring{m}^2)^{\cdot} &= 2\big[-\mathring{m}^2\mathring{A}_\kappa + \hat{\epsilon}e^{2\zeta}(q/\phi)^2\phi^{-1}\phi_{.\kappa}\big]\dot{p}^\kappa \\ &= 2\big[-e^{-2\zeta}\mathring{m}^2\mathring{A}_\kappa + \hat{\epsilon}(q/\phi)^2\phi^{-1}\phi_{.\kappa}\big]P_\lambda\mathring{g}^{\lambda\kappa}.\end{aligned} \tag{46}$$

The scalar $\mathring{G}(p)\dot{p}\dot{p}$, otherwise identifiable as $\dot{p}^\mu\mathring{g}_{\mu\nu}\dot{p}^\nu$ and as $e^{-4\zeta}\mathring{m}^2$, may be positive, zero, or negative on different geodesics and, generally, on different portions of the same geodesic. It is the square length of the "space-time part" $\dot{p}^\mu e_\mu(p)$ of the velocity $\dot{p}$, as measured by the degenerate metric $\mathring{G}$, whose space-time part has diagonal signature $---+$. Wherever on p this scalar is positive, that is, wherever the space-time part of $\dot{p}$ is timelike, we can introduce a real parameter $\mathring{\tau}$ such that

$$\begin{aligned}\mathring{\tau} &:= \int \big(\mathring{G}(p)\dot{p}\dot{p}\big)^{\frac{1}{2}}\,d\hat{\tau} = \int (\dot{p}^\mu\mathring{g}_{\mu\nu}\dot{p}^\nu)^{\frac{1}{2}}\,d\hat{\tau} \\ &= \int e^{-2\zeta}\mathring{m}\,d\hat{\tau} = \int \mathring{m}^{-1}\,dq\end{aligned} \tag{47}$$

($\hat{\tau}$ denoting the space-time–time affine parameter of p), and with it define space-time velocity components u^λ by $u^\lambda := dp^\lambda/d\mathring{\tau} := \dot{p}^\lambda/(\mathring{\tau})^{\cdot}$. Equations (42), (43), and (46) then are equivalent, wherever $\mathring{m}^2 > 0$, to

$$\frac{dP_\kappa}{d\mathring{\tau}} = P_\mu\mathring{\Gamma}_\kappa{}^\mu{}_\lambda u^\lambda + qF_{\kappa\lambda}u^\lambda - \mathring{m}\mathring{A}_\kappa + \hat{\epsilon}e^{2\zeta}\mathring{m}^{-1}(q/\phi)^2\phi^{-1}\phi_{.\kappa}, \tag{42$'$}$$

$$\frac{dq}{d\mathring{\tau}} = \mathring{m}, \tag{43$'$}$$

and

$$\frac{d(\mathring{m}^2)}{d\mathring{\tau}} = 2\big[-\mathring{m}^2\mathring{A}_\kappa + \hat{\epsilon}e^{2\zeta}(q/\phi)^2\phi^{-1}\phi_{.\kappa}\big]u^\kappa. \tag{46$'$}$$

Upon comparing these equations with the classical relativistic equations of motion for an electrically charged particle, and remembering the various definitions that have gone into them, one arrives at the following identifications and conclusions:

1. The scalar parameter $\mathring{\tau}$ is a (space-time) proper time parameter of the particle.
2. The u^λ are the components of the space-time proper velocity vector of the particle.
3. The P_κ are the components of the space-time momentum covector of the particle.
4. The scalar parameter $\mathring{m}$ is the rest mass of the particle.
5. The scalar parameter q is the electric charge of the particle.
6. The $F_{\kappa\lambda}$ are the components of the space-time electromagnetic field tensor.
7. The $\mathring{A}_\kappa$ are the components of a space-time covector potential field for the electromagnetic field.
8. The apparent forces to which the particle is subject, in that they contribute, according to Eq. (42$'$), additively to the space-time momentum rates $dP_\kappa/d\mathring{\tau}$, consist of
   a. the gravitational and other forces attributable to space-time geometry that are included in the term $P_\mu\mathring{\Gamma}_\kappa{}^\mu{}_\lambda u^\lambda$, familiar from general relativity theory;
   b. the Lorentz force of the electromagnetic field, expressed by the term $qF_{\kappa\lambda}u^\lambda$;
   c. a rest-mass proportional force in the direction of the electromagnetic potential, expressed by the term $-\mathring{m}\mathring{A}_\kappa$; and
   d. a force proportional to the square of the electric charge, inversely proportional to the rest mass, and in the direction of the gradient of the scalar field ϕ, expressed by the term $\hat{\epsilon}e^{2\zeta}\mathring{m}^{-1}(q/\phi)^2\phi^{-1}\phi_{.\kappa}$.

9. Neither the electric charge q nor the rest mass $\mathring{m}$ can be expected in general to remain constant, as they will evolve in accordance with Eqs. (43′) and (46′) while maintaining a kind of joint conservation, described by Eq. (44).

To go one step further, let $\mathring{P}^\mu := P_\nu \mathring{g}^{\nu\mu}$. Then $\mathring{P}^\mu = e^{2\zeta}\mathring{p}^\mu$, and $\mathring{P}^\mu = \mathring{m}u^\mu$ wherever $\mathring{m}^2 > 0$, in consequence of which we may identify the $\mathring{P}^\mu$ as the components of the space-time momentum vector of the particle. Consistent with this identification is the observation that $\mathring{m}^2 = \mathring{P}^\mu \mathring{g}_{\mu\nu}\mathring{P}^\nu$. In terms of $\mathring{P}^\mu$, q, and $\mathring{m}$, the geodesic equation (33″) reads

$$\big(\mathring{P}^\mu\big)^{\cdot} + e^{-2\zeta}\mathring{P}^\kappa \mathring{\Gamma}_\kappa{}^\mu{}_\lambda \mathring{P}^\lambda = e^{-2\zeta} q \mathring{F}^\mu{}_\lambda \mathring{P}^\lambda - e^{-2\zeta}\mathring{m}^2 \mathring{A}^\mu + \hat{\epsilon}(q/\phi)^2 \phi^{-1}\mathring{\phi}^{\cdot\mu}. \tag{48}$$

And this is equivalent, wherever $\mathring{m}^2 > 0$, to

$$\frac{d(\mathring{m}u^\mu)}{d\mathring{\tau}} + \mathring{m}u^\kappa \mathring{\Gamma}_\kappa{}^\mu{}_\lambda u^\lambda = q\mathring{F}^\mu{}_\lambda u^\lambda - \mathring{m}\mathring{A}^\mu + \hat{\epsilon}e^{2\zeta}\mathring{m}^{-1}(q/\phi)^2\phi^{-1}\mathring{\phi}^{\cdot\mu}, \tag{48′}$$

an equation which helps to cement the identifications and conclusions outlined above.

As one knows, the nonnull geodesic paths of $\hat{\mathbf{d}}$ are the paths that make stationary the arclength integral $\int_{\hat{\tau}_1}^{\hat{\tau}_2} |\dot{p}|\, d\hat{\tau}$, in which $|\dot{p}| := |\hat{G}(p)\dot{p}\dot{p}|^{\frac{1}{2}}$. The canonical momentum covector M whose components appear in the Euler equations for this variational problem can be expressed by

$$\begin{aligned} M &:= (\partial|\dot{p}|/\partial\dot{p}^\kappa)\omega^\kappa(p) + (\partial|\dot{p}|/\partial\dot{p}^d)\omega^d(p) \\ &= \operatorname{sgn}\big(\hat{G}(p)\dot{p}\dot{p}\big)|\dot{p}|^{-1}\hat{G}(p)\dot{p} \\ &= \epsilon|\dot{p}|^{-1}P = \epsilon|\dot{p}|^{-1}[P_\kappa\omega^\kappa(p) + P_d\omega^d(p)]. \end{aligned} \tag{49}$$

From this it follows that $P_\kappa = \epsilon|\dot{p}|M_\kappa = (1/2)(\partial L/\partial\dot{p}^\kappa)$ and $P_d = \epsilon|\dot{p}|M_d = (1/2)(\partial L/\partial\dot{p}^d)$, where $L := \epsilon|\dot{p}|^2 = \hat{G}(p)\dot{p}\dot{p}$, and that the equations of motion (42) and (43) (which, being equivalent to $\ddot{p} = 0$, hold only for affine parametrizations of p) can be derived from an action principle with L as the Lagrangian [15]. In terms of $\mathring{m}$ and q this Lagrangian can be formulated thus:

$$\begin{aligned} L &= e^{-2\zeta}\mathring{m}^2 + \hat{\epsilon}(q/\phi)^2 \\ &= \mathring{m}(v^\mu \mathring{g}_{\mu\nu}v^\nu)^{\frac{1}{2}} + q\mathring{A}_\mu v^\mu + q\dot{\zeta}; \end{aligned} \tag{50}$$

here $v^\mu := \dot{p}^\mu$ and Eqs. (40) and (41) have been invoked. But for the extra term $q\dot{\zeta}$, which refers to progression along the secondary time dimension and therefore has no space-time analog, this space-time–time Lagrangian would duplicate in appearance a standard space-time Lagrangian for the equations of motion of a charged particle in the special theory of relativity [16] and, by simple extension, in the general theory as well. In assessing this correspondence, however, one should bear in mind that in space-time–time $\mathring{m}$ and q are geometric parameters of the geodesic, not, as in space-time theories, mere handcrafted constants of no geometrical significance.

It is clear that the Lagrangian L, the geodesic equation $\ddot{p} = 0$, and its equivalent $\dot{P} = 0$ are all gauge invariant, inasmuch as gauge transformations are just coordinate transformations (of the type $[\![x^{\mu'}, \zeta]\!] \to [\![x^{\mu'}, \zeta - \lambda]\!]$), which do not affect $\hat{G}$, $\dot{p}$, $\ddot{p}$, P, or $\dot{P}$. What is not so apparent is that each component equation of motion is individually gauge invariant. This comes about because $\{\omega^\mu, \omega^d\}$, consequently $\{e_\mu, e_d\}$, and therefore $\dot{P}_\mu$, $\dot{P}_d$, $\ddot{p}^\mu$, and $\ddot{p}^d$, stay fixed when the gauge changes (as, likewise, do P_μ, P_d, $\dot{p}^\mu$, and $\dot{p}^d$). Thus, every one of Eqs. (42), (43), (42′), (43′), (48), and (48′) is individually gauge invariant (up to simple algebraic equivalence). Also gauge invariant is the electric charge q, as follows from the fact that $q = \hat{\epsilon}\phi^2 A(p)\dot{p}$, no part of which is altered by a change of gauge. Not gauge invariant, however, are the rest mass $\mathring{m}$ and the proper time $\mathring{\tau}$, which when $\zeta \to \zeta - \lambda$ behave so: $\mathring{m} \to e^{-\lambda}\mathring{m}$ and $d\mathring{\tau}/d\hat{\tau} \to e^{\lambda}(d\mathring{\tau}/d\hat{\tau})$. Nor are the components $\mathring{P}^\mu$ $(= \mathring{m}u^\mu)$ of the space-time momentum vector gauge invariant, for $\mathring{P}^\mu \to e^{-2\lambda}\mathring{P}^\mu$. The product $\mathring{m}(d\mathring{\tau}/d\hat{\tau})$, however, is gauge invariant, as is $\mathrm{sgn}(\mathring{m}^2)$. The lack of invariance for $\mathring{m}$, $\mathring{\tau}$, and $\mathring{P}^\mu$ of course reflects the fact that in the new gauge it is $e^{2\lambda}\mathring{G}$ instead of $\mathring{G}$ that is considered to be the metric of space-time.

Test particles obeying the equations of motion here detailed exhibit a complexity of behavior far beyond that of test particles in Einstein's space-time theory or in its extensions by Weyl, Kaluza, Klein, and others. This is owed in large measure to the unprecedented manner in which the electric charge q evolves and the equally unprecedented nature of the coupling of momentum rates to the gradient of ϕ. These have among their effects that a test particle can appear (seemingly out of nowhere) at a space-time event $\mathcal{E}_1$ with $q = -\phi(\mathcal{E}_1)$ and vanish at a later event $\mathcal{E}_2$ with $q = \phi(\mathcal{E}_2)$, and that *at* $\mathcal{E}_1$ and *at* $\mathcal{E}_2$ the ϕ-gradient force will, because of the growth of the coupling factor $e^{2\zeta}$ in Eq. (42′), *infinitely* dominate the other forces and thereby draw the particle irresistibly into the depths of one of the potential wells of $\hat{\epsilon}\phi$. These potential wells thus are the most probable locations for the occurrence of such "creation" and "annihilation" events. The thought that such behavior might be used to model orbital transitions ("quantum jumps") of electrons in atoms cannot be suppressed.

Because of its complexity I shall not here attempt further to describe space-time–time test particle behavior. Instead, I shall, in the next section, discuss subtleties in the concepts of mass and of charge that flow from these equations of motion, subtleties involving distinctions often unmade or neglected — to the detriment of science, for to fail to distinguish is to fail to know.

VIII. The Inertial-Passive Equivalence and the Passive-Active Distinction

In Newton's theory of gravity the assumption that a test particle's inertial mass m_i and its passive gravitational mass m_p are equal (and constant) reduces the equation of motion $(m_i\dot{\mathbf{r}})^{\cdot} + (m_p M/r^2)(\mathbf{r}/r) = \mathbf{0}$ to the equation $\ddot{\mathbf{r}} + (M/r^2)(\mathbf{r}/r) = \mathbf{0}$, in which neither of those masses appears. Einstein's theory of gravity incorporates that same equivalence by admitting only space-time geodesics as worldlines of test particles. It thereby adopts as its equation of motion a generalization of the *reduced* Newtonian equation, thus avoids even introducing m_i and m_p as concepts of significance for gravity. Because test particles in space-time–time must deal with the electromagnetic field alongside the gravitational field, this theory cannot exclude those concepts. It introduces them effortlessly, however, and in such a way as to maintain the numerical equivalence of m_i and m_p and to make them ignorable in the absence of nongravitational fields. Specifically, the same mass parameter $\mathring{m}$ that appears in the first term of Eq. (48′) in the role of inertial (rest) mass m_i appears also in the second term in the role of passive gravitational mass m_p; thus in space-time–time $m_i := \mathring{m} =: m_p$. And when the nongravitational fields $\phi_{.\kappa}$, $\mathring{A}_\kappa$, and $F_{\kappa\lambda}$ are all zero, then the horizontal subspaces are (space-time) hypersurface-forming, and $\mathring{m}$ has to be constant to satisfy Eq. (46′), whereupon Eq. (48′) reduces to $du^\mu/d\mathring{\tau} + u^\kappa \mathring{\Gamma}_\kappa{}^\mu{}_\lambda u^\lambda = 0$, which implies that the particle's space-time trajectory is geodesic, just as in Einstein's theory.

The constant M in the Newtonian equations of motion tells the strength of the gravitational field acting on the test particle; it is properly called the *active* gravitational mass of the particle considered to be producing that field, which of course is not the test particle. Newtonian theory treats every particle as both a test particle with $m_i = m_p$ and a field-generating particle with an active gravitational mass m_a. Although m_a and m_p refer to entirely different concepts, Newton's law of action and reaction, *applied instantaneously at a distance*, allows the inference that $m_a = m_p$.

In Einstein's theory the analog of M is the Schwarzschild mass parameter M_S, which also is properly called the active gravitational mass of the "particle" whose gravitational field the Schwarzschild metric represents. Although as noted that theory has no concept of passive gravitational mass, one can insert m_p and its equal m_i into the radial equation of geodesic motion for a Schwarzschild metric at the expected places to obtain an equivalent equation generalizing the unreduced Newtonian equation, with M_S in place of M. This done, however, one yet finds it impossible to establish by the Newtonian argument any equivalence between active and passive mass parameters. Even if the logically chimerical notion of a test particle with an active gravitational mass m_a as well as a passive gravitational mass m_p be entertained, the Newtonian argument that $m_a = m_p$ founders on the lack of any "instantaneous gravitational action and reaction at a distance" in Einstein's theory.

In space-time–time theory the situation is the same: there is no concept of an active gravitational mass of a test particle; an analog of the Newtonian M and the Schwarzschildean M_S can exist in a particular space-time–time, but it is a parameter of the gravitational field of that space-time–time, not of any test particle that the field acts upon; if particles with both active and passive gravitational masses be imagined, then the finiteness of the speed of propagation of gravitational effects precludes establishment of any relationship between the two masses. But in this theory a further, similar discrimination is unavoidable. The electric charge parameter q of a space-time–time test particle measures, in its initial appearance in Eq. (48′), the response of the particle to the electromagnetic field $F_{\kappa\lambda}$. Thus it plays there the role of a *passive* electric charge, just as $\mathring{m}$ takes the role of a passive gravitational mass in its second appearance in that equation. If $F_{\kappa\lambda}$ should have a form like that of a Coulomb field of strength Q, then Q would properly be called the *active* electric charge of the particle considered to be generating that field, but that particle could not strictly be treated as a test particle at all, still less as a test particle with passive electric charge Q. Between these concepts of active and of passive electric charge, just as between the concepts of active and of passive gravitational mass, lies a broad gulf, across which no bridge is apparent. Essentially the same gulf is present already in Maxwell–Lorentz electrodynamics. Attempts to bridge it there, by supposing test particles to have active (or at least semi-active) charge as well as passive charge, have produced among other oddities an equation of motion with an $\dddot{\mathbf{r}}$ term that lets in self-accelerated "runaway" solutions. The space-time–time equation of motion (48′) has no comparable term and no such solution. Space-time–time theory seems to require no bridge across the active-passive electric charge gulf, or for that matter across the active-passive gravitational mass gulf. It is conceivable, however, that some such connections lie hidden in the theory, to be exposed by future investigation [17].

In its third appearance in Eq. (48′) $\mathring{m}$ helps to measure the response of the test particle to the field $\mathring{A}_\mu$, and in the last term q and $\mathring{m}$ combine to help determine the particle's response to the field $\phi_{.\mu}$. The apparent forces involved are peculiar to space-time–time, so there are no names like "passive gravitational mass" and "passive electric charge" ready at hand to signify the roles played here by $\mathring{m}$ and $q^2/\mathring{m}$. This is perhaps fortunate, for such names tend to mislead by putting attention on the apparent forces themselves, rather than on the underlying geometry they spring from. It is this geometry that is presumed to model reality; the apparent forces and the test particles following geodesics are just convenient fictions to help us connect the geometry to our perceptions.

IX. Curvature

A full physical interpretation of the geometry of space-time–time must rest ultimately not only on delineation of the mechanics of test particles, but also on establishment of field equations for the evolution and interactions of ϕ, $\mathring{A}$, F, and $\mathring{G}$, analysis of the field dynamics those equations imply, and arrival at an understanding of the physical import of the unfamiliar scalar field ϕ. In preparation for a subsequent paper deriving such field equations I shall exhibit here and in the Appendix both concise and not so concise forms of the curvature tensor field $\hat{\Theta}$ of the conformally constrained metric $\hat{G}$, its contracted curvature tensor field $\hat{\Phi}$, its curvature scalar field $\hat{\Psi}$, and its Einstein tensor field $\hat{E}$. The adapted frame system $\{e_\mu, e_d\}$ and its dual $\{\omega^\mu, \omega^d\}$ are best suited to this purpose. As earlier, no restriction is placed on the dimensionality of $\mathcal{M}$ or the signature of $\hat{G}$.

If we adopt the convention that K, L, M, N, etc. range from 1 to d (retaining for κ, λ, μ, ν, etc. the range 1 to $d-1$), then we have that $\hat{\Theta} = \omega^K \otimes \hat{\Theta}_K{}^M \otimes e_M$, with the curvature 2-forms $\hat{\Theta}_K{}^M$ computed from the connection 1-forms of Eqs. (23) by means of the structural equation $\hat{\Theta}_K{}^M = 2(d_\wedge \hat{\omega}_K{}^M - \hat{\omega}_K{}^P \wedge \hat{\omega}_P{}^M)$. Upon performing the computations one finds that [11]

$$
\begin{aligned}
\hat{\Theta}_\kappa{}^\mu &= \Theta_\kappa{}^\mu \; - \big[\hat{\epsilon}\phi^2 F_\kappa{}^{(\mu} F_{\lambda)\nu} + \hat{\epsilon} 2\phi^{-2} g_{\kappa\lambda} g^\mu{}_\nu\big]\omega^\nu \wedge \omega^\lambda \\
&\qquad + \big[\hat{\epsilon}\big(\phi F_\kappa{}^\mu{}_{;\nu} + 2F_{(\kappa}{}^\mu \phi_{.\nu)} + 2F_\kappa{}^{(\mu}\phi_{.\nu)}\big) + 4\phi^{-2}\phi_{.[\kappa} g^{\mu]}{}_\nu\big]\omega^\nu \wedge \omega^d, \\
\hat{\Theta}_\kappa{}^d &= \big[\phi F_{\kappa\lambda;\nu} - 2\phi_{.(\kappa}F_{\lambda)\nu} + \hat{\epsilon}2\phi^{-2} g_{\kappa\lambda}\phi_{.\nu}\big]\omega^\nu \wedge \omega^\lambda \\
&\qquad - \big[2\phi^{-1}\phi_{.\kappa;\lambda} + \hat{\epsilon}(1/2)\phi^2 F_\kappa{}^\pi F_{\pi\lambda} + \hat{\epsilon}2\phi^{-2} g_{\kappa\lambda}\big]\,\omega^d \wedge \omega^\lambda, \\
\hat{\Theta}_d{}^\mu &= -\hat{\epsilon} g^{\mu\kappa}\hat{\Theta}_\kappa
\end{aligned}
\tag{51}
$$

and

$$\hat{\Theta}_d{}^d = 0.$$

In the first of these equations

$$
\begin{aligned}
\Theta_\kappa{}^\mu := {} & [2(\Gamma_\kappa{}^\mu{}_{\lambda,\nu} + \Gamma_\kappa{}^\pi{}_\lambda \Gamma_\pi{}^\mu{}_\nu + \Gamma_\kappa{}^\mu{}_\pi C_\lambda{}^\pi{}_\nu) \\
& - (g_\kappa{}^\mu F_{\lambda\nu} - F_{\kappa\lambda} g^\mu{}_\nu - g_{\kappa\lambda} F^\mu{}_\nu)]\,\omega^\nu \wedge \omega^\lambda.
\end{aligned}
\tag{52}
$$

The other abbreviations introduced in them are

$$
\begin{aligned}
\phi_{.\kappa;\lambda} &:= \phi_{.\kappa,\lambda} - \phi_{.\pi}\Gamma_\kappa{}^\pi{}_\lambda, \\
F_{\kappa\lambda;\nu} &:= F_{\kappa\lambda,\nu} - F_{\kappa\pi}\Gamma_\lambda{}^\pi{}_\nu - F_{\pi\lambda}\Gamma_\kappa{}^\pi{}_\nu,
\end{aligned}
\tag{53}
$$

and

$$F_\kappa{}^\mu{}_{;\nu} := F_\kappa{}^\mu{}_{,\nu} + F_\kappa{}^\pi \Gamma_\pi{}^\mu{}_\nu - F_\pi{}^\mu \Gamma_\kappa{}^\pi{}_\nu.$$

The ";" operation of Eqs. (53) harks back to the covariant differentiation $\mathbf{d}$ defined in Sec. V, for which $\mathbf{d}e_\kappa = \omega_\kappa{}^\mu \otimes e_\mu$, $\mathbf{d}e_d = 0$, $\mathbf{d}\omega^\mu = -\omega_\kappa{}^\mu \otimes \omega^\kappa$, and $\mathbf{d}\omega^d = 0$, with $\omega_\kappa{}^\mu = \Gamma_\kappa{}^\mu{}_\lambda \omega^\lambda$. Thus, ϕ being independent of ζ, $d\phi = \phi_{,\kappa}\omega^\kappa = \phi_{.\kappa}\omega^\kappa$ and $d\phi_{.\kappa} = \phi_{.\kappa,\lambda}\omega^\lambda$, so $\mathbf{d}(d\phi) = d\phi_{.\kappa} \otimes \omega^\kappa + \phi_{.\pi}\mathbf{d}\omega^\pi = (d\phi_{.\kappa} - \phi_{.\pi}\omega_\kappa{}^\pi) \otimes \omega^\kappa = \phi_{.\kappa;\lambda}\omega^\lambda \otimes \omega^\kappa$. A similar calculation finds that $\mathbf{d}F = F_{\kappa\lambda;\nu}\omega^\nu \otimes \omega^\lambda \otimes \omega^\kappa$. On the other hand $F_\kappa{}^\mu$ is not independent of ζ, so $\mathbf{d}(FG^{-1}) = \mathbf{d}(F_\kappa{}^\mu e_\mu \otimes \omega^\kappa) - (F_\kappa{}^\mu{}_{;\nu}\omega^\nu + F_\kappa{}^\mu{}_{;d}\omega^d) \otimes e_\mu \otimes \omega^\kappa$, where, because all of $\mathbf{d}$'s connection coefficients other than the $\Gamma_\kappa{}^\mu{}_\lambda$ vanish, $F_\kappa{}^\mu{}_{;d} := F_\kappa{}^\mu{}_{,d}$. It is easy to see that $g_{\mu\nu;\lambda} = g^{\mu\nu}{}_{;\lambda} = 0$ and that $F_\kappa{}^\mu{}_{;\nu} = g^{\mu\lambda}F_{\kappa\lambda;\nu}$. Should the need arise, Eqs. (53) can be refigured by use of the equivalences $\phi_{.\kappa,\lambda} = \phi_{.\kappa.\lambda} =: \phi_{.\kappa\lambda}$, $F_{\kappa\lambda,\nu} = F_{\kappa\lambda.\nu}$, and

$F_\kappa{}^\mu{}_{,\nu} = F_\kappa{}^\mu{}_{.\nu} + 2F_\kappa{}^\mu \mathring{A}_\nu$. The notation $\Theta_\kappa{}^\mu$ notwithstanding, the 2-forms of Eq. (52) are not curvature forms of $\mathbf{d}$, nor would they be if only the terms involving Γ were present.

By taking account of the skew-symmetries involved, one can extract from Eqs. (51) the curvature components $\hat{\Theta}_K{}^M{}_{LN}$ that appear in $\hat{\Theta}_K{}^M = \hat{\Theta}_K{}^M{}_{LN}\omega^N \wedge \omega^L = \hat{\Theta}_K{}^M{}_{LN}\omega^N \otimes \omega^L$. This is done in the Appendix.

The contracted curvature tensor field $\hat{\Phi}$ can be computed directly from Eqs. (51) by use of $\hat{\Phi} := \omega^R \hat{\Theta}(\cdot)e_R = \omega^K \otimes \omega^R(\hat{\Theta}_K{}^M \otimes e_M)e_R = \omega^K \otimes \hat{\Theta}_K{}^R e_R$, or, by referring to Eqs. (A.1) in the Appendix, from $\hat{\Phi} = \omega^K \otimes \hat{\Theta}_K{}^R{}_{LR}\omega^L$. The result is that $\hat{\Phi} = \omega^K \otimes \hat{\Phi}_{KL}\omega^L$, where

$$\begin{aligned}
\hat{\Phi}_{\kappa\lambda} &= \Phi_{\kappa\lambda} - \phi^{-1}\phi_{.\kappa;\lambda} + \hat{\epsilon}(1/2)\phi^2 F_\kappa{}^\rho F_{\rho\lambda} - \hat{\epsilon}(d-1)\phi^{-2}g_{\kappa\lambda},\\
\hat{\Phi}_{\kappa d} &= \hat{\epsilon}(1/2)(\phi F_\kappa{}^\rho{}_{;\rho} + 3F_\kappa{}^\rho \phi_{.\rho}) + (d-2)\phi^{-2}\phi_{.\kappa},\\
\hat{\Phi}_{d\lambda} &= \hat{\Phi}_{\lambda d},
\end{aligned} \tag{54}$$

and

$$\hat{\Phi}_{dd} = -\hat{\epsilon}\phi^{-1}\phi^{\cdot\rho}{}_{;\rho} - (1/4)\phi^2 F^\rho{}_\pi F^\pi{}_\rho - (d-1)\phi^{-2}.$$

In the first of these equations

$$\begin{aligned}
\Phi_{\kappa\lambda} &:= \Theta_\kappa{}^\rho e_\rho e_\lambda = \Theta_\kappa{}^\rho{}_{\lambda\rho}\\
&= 2\left(\Gamma_\kappa{}^\rho{}_{[\lambda,\rho]} + \Gamma_\kappa{}^\pi{}_{[\lambda}\Gamma_\pi{}^\rho{}_{\rho]} + \Gamma_\kappa{}^\rho{}_\pi C_\rho{}^\pi{}_\lambda\right) + (1/2)(d-1)F_{\kappa\lambda},
\end{aligned} \tag{55}$$

and in the last

$$\phi^{\cdot\rho}{}_{;\rho} := \phi^{\cdot\rho}{}_{,\rho} + \phi^{\cdot\pi}\Gamma_\pi{}^\rho{}_\rho = \phi_{.\kappa;\rho}g^{\kappa\rho}. \tag{56}$$

Because $\hat{G}^{-1} = e_\mu \otimes g^{\mu\nu}e_\nu + \hat{\epsilon}e_d \otimes e_d$, we have that $\hat{G}^{-1}\hat{\Phi} = \omega^\kappa \otimes \hat{\Phi}_\kappa{}^\nu e_\nu + \omega^d \otimes \hat{\epsilon}\hat{\Phi}_{dd}e_d$, with $\hat{\Phi}_\kappa{}^\nu := \hat{\Phi}_{\kappa\lambda}g^{\lambda\nu}$, hence that $\hat{\Psi} := \omega^P\left(\hat{G}^{-1}\hat{\Phi}\right)e_P = \hat{\Phi}_\pi{}^\pi + \hat{\epsilon}\hat{\Phi}_{dd}$. Applying this to Eqs. (54) one finds that

$$\hat{\Psi} = \Psi - 2\phi^{-1}\phi^{\cdot\rho}{}_{;\rho} + \hat{\epsilon}(1/4)\phi^2 F_\pi{}^\rho F_\rho{}^\pi - \hat{\epsilon}(d-1)d\,\phi^{-2}, \tag{57}$$

where

$$\Psi := \Phi_\pi{}^\pi \quad \text{and} \quad \Phi_\kappa{}^\nu := \Phi_{\kappa\lambda}g^{\lambda\nu}. \tag{58}$$

It then follows from $\hat{E} := \hat{\Phi} - (1/2)\hat{\Psi}\hat{G}$ that $\hat{E} = \omega^K \otimes \hat{E}_{KL}\omega^L$, where

$$\begin{aligned}
\hat{E}_{\kappa\lambda} = E_{\kappa\lambda} &- \phi^{-1}(\phi_{.\kappa;\lambda} - \phi^{\cdot\rho}{}_{;\rho}g_{\kappa\lambda})\\
&+ \hat{\epsilon}(1/2)\phi^2[F_\kappa{}^\rho F_{\rho\lambda} - (1/4)F_\pi{}^\rho F_\rho{}^\pi g_{\kappa\lambda}]\\
&+ \hat{\epsilon}(1/2)(d-2)(d-1)\phi^{-2}g_{\kappa\lambda},\\
\hat{E}_{\kappa d} = \hat{\epsilon}(1/2)&(\phi F_\kappa{}^\rho{}_{;\rho} + 3F_\kappa{}^\rho\phi_{.\rho}) + (d-2)\phi^{-2}\phi_{.\kappa},\\
\hat{E}_{d\lambda} = \hat{E}_{\lambda d}&,
\end{aligned} \tag{59}$$

and

$$\hat{E}_{dd} = -\hat{\epsilon}(1/2)\Psi - (3/8)\phi^2 F_\pi{}^\rho F_\rho{}^\pi + (1/2)(d-2)(d-1)\phi^{-2}.$$

The abbreviation

$$E_{\kappa\lambda} := \Phi_{\kappa\lambda} - (1/2)\Psi g_{\kappa\lambda} \tag{60}$$

is used in the first of Eqs. (59).

Hidden within these relatively concise expressions of $\hat{\Theta}$, $\hat{\Phi}$, $\hat{\Psi}$, and $\hat{E}$ is a wealth of "interactions" among the fields ϕ, $\mathring{A}_\mu$, $F_{\mu\nu}$, and $\mathring{g}_{\mu\nu}$. To bring them to visibility we shall have to "detelescope" the expressions with the aid of the expansions set out in Eqs. (25)–(29). The procedure is straightforward, but the product will occupy a considerable space. To reduce congestion the detelescoped expressions for $\hat{\Theta}$ and $\hat{\Phi}$ will be displayed in the Appendix, leaving here only those for $\hat{\Psi}$ and $\hat{E}$. In both places will appear additional abbreviations which can be described in the following way: The practice of inserting a "$^\circ$" to indicate raising of an index with the $\mathring{g}^{\mu\nu}$ rather than the $g^{\mu\nu}$ (as in $\mathring{\phi}^{\cdot\mu} := \phi_{.\lambda}\mathring{g}^{\lambda\mu}$) is continued, and is sharpened by the stipulations that if a "$^\circ$" is already present, then *only* the $\mathring{g}^{\mu\nu}$ can raise an index, and that $\mathring{A}^\mu := \mathring{A}_\lambda \mathring{g}^{\lambda\mu}$, *not* $A_\lambda \mathring{g}^{\lambda\mu}$). Further, it is understood that the "$^\circ$" travels with a raised index involved in a symmetrization or an antisymmetrization; thus $\mathring{F}_\kappa{}^{(\mu}\phi_{.\nu)} = (1/2)\big(\mathring{F}_\kappa{}^\mu\phi_{.\nu} + F_{\kappa\nu}\mathring{\phi}^{\cdot\mu}\big)$. Next, application of the covariant differentiation $\mathring{\mathbf{d}}$ defined in Sec. V, whose only nonvanishing connection coefficients are the $\mathring{\Gamma}_\kappa{}^\mu{}_\lambda$ of Eqs. (29), is signified by use of a "$_{:}$" *and* insertion of a "$^\circ$" if none is already present, provided that the field being differentiated is representable in terms of the e_μ, the ω^μ, and their tensor products alone, with coefficients independent of ζ (a representability that passes on to the differential field). As examples, $\mathring{\mathbf{d}}(d\phi) = \mathring{\mathbf{d}}(\phi_{.\kappa}\omega^\kappa) = \mathring{\phi}_{.\kappa:\lambda}\omega^\lambda \otimes \omega^\kappa$, $\mathring{\mathbf{d}}\mathring{A} = \mathring{\mathbf{d}}\big(\mathring{A}_\kappa\omega^\kappa\big) = \mathring{A}_{\kappa:\lambda}\omega^\lambda \otimes \omega^\kappa$, $\mathring{\mathbf{d}}\big(\mathring{G}^{-1}\mathring{A}\big) = \mathring{\mathbf{d}}\big(\mathring{A}^\mu e_\mu\big) = \mathring{A}^\mu{}_{:\lambda}\omega^\lambda \otimes e_\mu$, and $\mathring{\mathbf{d}}F = \mathring{\mathbf{d}}(F_{\kappa\lambda}\omega^\lambda \otimes \omega^\kappa) = \mathring{F}_{\kappa\lambda:\nu}\omega^\nu \otimes \omega^\lambda \otimes \omega^\kappa$, where

$$
\begin{aligned}
\mathring{\phi}_{.\kappa:\lambda} &= \phi_{.\kappa\lambda} - \phi_{.\pi}\mathring{\Gamma}_\kappa{}^\pi{}_\lambda, \\
\mathring{A}_{\kappa:\lambda} &= \mathring{A}_{\kappa.\lambda} - \mathring{A}_\pi\mathring{\Gamma}_\kappa{}^\pi{}_\lambda, \\
\mathring{A}^\mu{}_{:\lambda} &= \mathring{A}^\mu{}_{.\lambda} + \mathring{A}^\pi\mathring{\Gamma}_\pi{}^\mu{}_\lambda, \qquad\qquad (61)
\end{aligned}
$$

and

$$
\mathring{F}_{\kappa\lambda:\nu} = F_{\kappa\lambda.\nu} - F_{\kappa\pi}\mathring{\Gamma}_\lambda{}^\pi{}_\nu - F_{\pi\lambda}\mathring{\Gamma}_\kappa{}^\pi{}_\nu.
$$

Because $\mathring{\mathbf{d}}\mathring{G}^{-1} = 0$, the raising of an index with the $\mathring{g}^{\mu\nu}$ commutes with the "$^\circ_{:}$" operation; for example, $\mathring{A}^\mu{}_{:\lambda} = \mathring{A}_{\kappa:\lambda}\mathring{g}^{\kappa\mu}$ because $\mathring{\mathbf{d}}\big(\mathring{G}^{-1}\mathring{A}\big) = \mathring{G}^{-1}\mathring{\mathbf{d}}\mathring{A}$. Finally, $\mathring{\Theta}$, $\mathring{\Phi}$, $\mathring{\Psi}$, and $\mathring{E}$ stand for curvature fields built from $\mathring{\mathbf{d}}$ with the help of $\mathring{G}$ and $\mathring{G}^{-1}$. Specifically, $\mathring{\Theta} = \omega^\kappa \otimes \mathring{\Theta}_\kappa{}^\mu e_\mu$, $\mathring{\Theta}_\kappa{}^\mu = 2(d_\wedge\mathring{\omega}_\kappa{}^\mu - \mathring{\omega}_\kappa{}^\pi \wedge \mathring{\omega}_\pi{}^\mu) = \mathring{\Theta}_\kappa{}^\mu{}_{\lambda\nu}\omega^\nu \otimes \omega^\lambda$, $\mathring{\Psi} = \omega^\kappa \otimes \mathring{\Phi}_{\kappa\lambda}\omega^\lambda$, $\mathring{\Psi} = \mathring{\Psi}$, and $\mathring{E} = \omega^\kappa \otimes \mathring{E}_{\kappa\lambda}\omega^\lambda$, where

$$
\begin{aligned}
\mathring{\Theta}_\kappa{}^\mu{}_{\lambda\nu} &= 2\big(\mathring{\Gamma}_\kappa{}^\mu{}_{[\lambda.\nu]} + \mathring{\Gamma}_\kappa{}^\pi{}_{[\lambda}\mathring{\Gamma}_\pi{}^\mu{}_{\nu]} + \mathring{\Gamma}_\kappa{}^\mu{}_\pi C_\lambda{}^\pi{}_\nu\big), \\
\mathring{\Phi}_{\kappa\lambda} &= \mathring{\Theta}_\kappa{}^\rho{}_{\lambda\rho}, \\
\mathring{\Psi} &= \mathring{\Phi}_\pi{}^\pi, \qquad\qquad (62)
\end{aligned}
$$

and

$$
\mathring{E}_{\kappa\lambda} = \mathring{\Phi}_{\kappa\lambda} - (1/2)\mathring{\Psi}\mathring{g}_{\kappa\lambda}.
$$

Because $\mathring{\omega}_\kappa{}^d = \mathring{\omega}_d{}^\mu = \mathring{\omega}_d{}^d = 0$ and $\mathring{\Gamma}_\kappa{}^\mu{}_{\lambda,d} = 0$, the only nonvanishing curvature 2-forms of $\mathring{\mathbf{d}}$ are the $\mathring{\Theta}_\kappa{}^\mu$, so $\mathring{\Theta}$ as given is the curvature tensor field of $\mathring{\mathbf{d}}$. In space-time–time $\mathring{\Theta}$ is the curvature tensor field for the space-time metric $\mathring{G}$ picked out by the gauge selection of the hypersurface $\mathcal{S}$ on which to have $\zeta = 0$.

With these abbreviations all in place the detelescoped versions of $\hat{\Theta}$ and $\hat{\Phi}$ are as shown in the Appendix. From them one computes that

$$
\begin{aligned}
\hat{\Psi} = e^{-2\zeta}\mathring{\Psi} &+ (d-2)e^{-2\zeta}\big[2\mathring{A}^\rho{}_{:\rho} - (d-3)\mathring{A}^\rho\mathring{A}_\rho\big] \\
&- 2e^{-2\zeta}\phi^{-1}\big[\mathring{\phi}^{\cdot\rho}{}_{:\rho} - (d-3)\mathring{A}^\rho\phi_{.\rho}\big] \\
&+ \hat{\epsilon}(1/4)e^{-4\zeta}\phi^2\mathring{F}_\pi{}^\rho\mathring{F}_\rho{}^\pi - \hat{\epsilon}(d-1)d\,\phi^{-2}, \qquad\qquad (63)
\end{aligned}
$$

and then that

$$
\begin{aligned}
\hat{E}_{\kappa\lambda} &= \mathring{E}_{\kappa\lambda} \; + (d-3)\big[\mathring{A}_{(\kappa:\lambda)} - \mathring{A}^{\rho}{}_{:\rho}\mathring{g}_{\kappa\lambda}\big] \\
&\qquad + (d-3)\big[\mathring{A}_{\kappa}\mathring{A}_{\lambda} + (1/2)(d-4)\mathring{A}^{\rho}\mathring{A}_{\rho}\mathring{g}_{\kappa\lambda}\big] \\
&\qquad - \phi^{-1}\big[\mathring{\phi}_{.\kappa:\lambda} - \mathring{\phi}^{.\rho}{}_{:\rho}\mathring{g}_{\kappa\lambda}\big] \\
&\qquad - \phi^{-1}\big[2\mathring{A}_{(\kappa}\phi_{.\lambda)} + (d-4)\mathring{A}^{\rho}\phi_{.\rho}\mathring{g}_{\kappa\lambda}\big] \\
&\qquad + \hat{\epsilon}(1/2)e^{-2\zeta}\phi^{2}\big[\mathring{F}_{\kappa}{}^{\rho}F_{\rho\lambda} - (1/4)\mathring{F}_{\pi}{}^{\rho}\mathring{F}_{\rho}{}^{\pi}\mathring{g}_{\kappa\lambda}\big] \\
&\qquad + \hat{\epsilon}(1/2)(d-2)(d-1)e^{2\zeta}\phi^{-2}\mathring{g}_{\kappa\lambda}, \\
\hat{E}_{\kappa d} &= \hat{\epsilon}(1/2)e^{-2\zeta}\big[\phi\mathring{F}_{\kappa}{}^{\rho}{}_{:\rho} - (d-5)\phi\mathring{F}_{\kappa}{}^{\rho}\mathring{A}_{\rho} + 3\mathring{F}_{\kappa}{}^{\rho}\phi_{.\rho}\big] + (d-2)\phi^{-2}\phi_{.\kappa}, \\
\hat{E}_{d\lambda} &= \hat{E}_{\lambda d},
\end{aligned}
\tag{64}
$$

and

$$
\begin{aligned}
\hat{E}_{dd} &= -\hat{\epsilon}(1/2)e^{-2\zeta}\mathring{\Psi} - \hat{\epsilon}(d-2)e^{-2\zeta}\big[\mathring{A}^{\rho}{}_{:\rho} - (1/2)(d-3)\mathring{A}^{\rho}\mathring{A}_{\rho}\big] \\
&\qquad - (3/8)e^{-4\zeta}\phi^{2}\mathring{F}_{\pi}{}^{\rho}\mathring{F}_{\rho}{}^{\pi} + (1/2)(d-2)(d-1)\phi^{-2}.
\end{aligned}
$$

In the ancestral Kaluza geometry much of the complexity in these expressions goes away, taking with it many of the possibilities for interactions among the various fields. (For the sake of comparison the corresponding expressions for the Kaluza geometry are presented at the end of the Appendix.)

X. Residual Curvature

An important concept specific to the geometry of conformally constrained metrics is that of residual curvature. Loosely, the residual curvature is what remains of the usual curvature when the instruments used to measure it shrink to infinitesimal size — the curvature seen by a vanishingly small observer, so to speak. A little less loosely, it is the limiting curvature at the ends of the trajectories of ξ where the conformal factor in $G = e^{2\zeta}\mathring{G}$ becomes infinite. The notion of residual curvature does not apply to Kaluza metrics, which, being *isometrically* constrained, have no factor $e^{2\zeta}$ and therefore cannot have $e^{2\zeta} \to \infty$. It requires for its definition that translations along ξ generate actual expansion of the metric G. Moreover, $\mathcal{M}$ needs to be ξ-complete, in order that along each ξ-path the integration-parameter coordinate ζ might increase without bound [18].

Consider on $\mathcal{M}$ the frame system $\{e_{\bar{M}}\}$ for which $e_{\bar{\mu}} = e^{-\zeta}e_{\mu}$ and $e_{\bar{d}} = e_{d}$, with dual $\{\omega^{\bar{M}}\}$ given by $\omega^{\bar{\mu}} = e^{\zeta}\omega^{\mu}$ and $\omega^{\bar{d}} = \omega^{d}$. Referring to Eq. (4″) one sees that in this frame system

$$
\hat{G} = \omega^{\bar{\mu}} \otimes \mathring{g}_{\bar{\mu}\bar{\nu}}\omega^{\bar{\nu}} + \hat{\epsilon}\omega^{\bar{d}} \otimes \omega^{\bar{d}}, \tag{65}
$$

where $\mathring{g}_{\bar{\mu}\bar{\nu}} = \mathring{g}_{\mu\nu}$. From this it follows that $\mathcal{L}_{\xi}\big(\hat{G}e_{\bar{\mu}}e_{\bar{\nu}}\big) = \partial\mathring{g}_{\bar{\mu}\bar{\nu}}/\partial\zeta = 0$, $\mathcal{L}_{\xi}\big(\hat{G}e_{\bar{\mu}}e_{\bar{d}}\big) = \partial 0/\partial\zeta = 0$, and $\mathcal{L}_{\xi}\big(\hat{G}e_{\bar{d}}e_{\bar{d}}\big) = \partial\hat{\epsilon}/\partial\zeta = 0$, in other words that all metrical relationships determined by $\hat{G}$ among the vector fields $e_{\bar{M}}$ remain fixed under translation along ξ. The same of course holds true for the covector fields $\omega^{\bar{M}}$. Beyond the normality of e_{d} and the orthogonality between the e_{μ} and e_{d} the controlling fact here is that the e_{μ} are Lie constant along ξ, which entails that $\mathcal{L}_{\xi}e_{\bar{\mu}} = -e_{\bar{\mu}}$, hence that $\mathcal{L}_{\xi}\big(\hat{G}e_{\bar{\mu}}e_{\bar{\nu}}\big) = \big(\mathcal{L}_{\xi}\hat{G}\big)e_{\bar{\mu}}e_{\bar{\nu}} + \hat{G}\big(\mathcal{L}_{\xi}e_{\bar{\mu}}\big)e_{\bar{\nu}} + \hat{G}e_{\bar{\mu}}\big(\mathcal{L}_{\xi}e_{\bar{\nu}}\big) = 2Ge_{\bar{\mu}}e_{\bar{\nu}} - 2\hat{G}e_{\bar{\mu}}e_{\bar{\nu}} = 2e^{-2\zeta}\big(Ge_{\mu}e_{\nu} - \hat{G}e_{\mu}e_{\nu}\big) = 0$.

If T is a tensor field of $\mathcal{M}$, we can expand T in terms of the $e_{\bar{M}}$ and the $\omega^{\bar{M}}$, then can ask whether the components of T in this expansion have limits as $\zeta \to \infty$. If all do, then the tensor field T_∞ whose components in $\{e_{\bar M}\}$ are these limits is to be called the "residual" of T. More precisely, suppose that T is a tensor field of $\mathcal{M}$, that $\bar t$ is a component of T in $\{e_{\bar M}\}$ (and $\{\omega^{\bar M}\}$), and that P is a point in the domain of T. Let Q be a point lying on the trajectory of ξ through P and free to move along it. Let $\bar t_\infty(P) := \lim \bar t(Q)$ if this limit exists as Q moves along the trajectory so that $\zeta(Q) \to \infty$; if the limit does not exist, then assign no meaning to $\bar t_\infty(P)$. If $\bar t_\infty(P)$ thus defined exists for each such component $\bar t$ and point P, then the tensor field whose components in $\{e_{\bar M}\}$ are the corresponding scalar fields $\bar t_\infty$ is called the **residual of** T and is denoted by T_∞. Briefly put, if, for example, $T = \omega^K \otimes T_K{}^M{}_L \omega^L \otimes e_M = \omega^{\bar K} \otimes T_{\bar K}{}^{\bar M}{}_{\bar L} \omega^{\bar L} \otimes e_{\bar M}$, then $T_\infty := \omega^{\bar K} \otimes \bigl(\lim_{\zeta\to\infty} T_{\bar K}{}^{\bar M}{}_{\bar L}\bigr)\omega^{\bar L} \otimes e_{\bar M}$.

As an illustration, $\hat G = \omega^{\bar\mu} \otimes \hat g_{\bar\mu\bar\nu}\omega^{\bar\nu} + \hat\epsilon\omega^{\bar d} \otimes \omega^{\bar d}$, where $\hat g_{\bar\mu\bar\nu} = e^{-2\zeta} g_{\mu\nu} = \mathring{g}_{\mu\nu}$, and therefore $\hat G_\infty = \hat G$, inasmuch as $\lim_{\zeta\to\infty} \hat g_{\bar\mu\bar\nu} = \lim_{\zeta\to\infty} \mathring{g}_{\mu\nu} = \mathring{g}_{\mu\nu} = \hat g_{\bar\mu\bar\nu}$ and $\lim_{\zeta\to\infty}\hat\epsilon = \hat\epsilon$. Similarly, $\bigl(\hat G^{-1}\bigr)_\infty = \hat G^{-1} = \bigl(\hat G_\infty\bigr)^{-1}$ and $A_\infty = A$. On the other hand $\mathring{G} = \omega^{\bar\mu} \otimes \mathring{g}_{\bar\mu\bar\nu}\omega^{\bar\nu}$ with $\mathring{g}_{\bar\mu\bar\nu} = e^{-2\zeta}\mathring{g}_{\mu\nu}$, and $\lim_{\zeta\to\infty}\bigl(e^{-2\zeta}\mathring{g}_{\mu\nu}\bigr) = 0$, so $\mathring{G}_\infty = 0$. By the same token $\mathring{A}_\infty = 0$ and $F_\infty = 0$. But $\mathring{G}^{-1} = e_{\bar\mu} \otimes \mathring{g}^{\bar\mu\bar\nu} e_{\bar\nu}$, where $\mathring{g}^{\bar\mu\bar\nu} = e^{2\zeta}\mathring{g}^{\mu\nu}$, and if $\mathring{g}^{\mu\nu} \neq 0$, then $e^{2\zeta}\mathring{g}^{\mu\nu}$ has no limit as $\zeta \to \infty$, so $\bigl(\mathring{G}^{-1}\bigr)_\infty$ is not defined.

Several observations can be made: 1) The component $\bar t$ of T in $\{e_{\bar M}\}$ is related to the corresponding component t of T in $\{e_M\}$ by $\bar t = e^{(\alpha-\beta)\zeta} t$, where α is the number of contravariant, β the number of covariant indices of t that differ from d; thus $\lim_{\zeta\to\infty}\bar t$ can equally well be calculated as $\lim_{\zeta\to\infty}\bigl(e^{(\alpha-\beta)\zeta}t\bigr)$. 2) The residual of T, though defined in a particular gauge, is in fact gauge invariant: if $\zeta' = \zeta - \lambda$ with $\partial\lambda/\partial\zeta = 0$, $e_{\bar\mu'} = e^{-\zeta'}e_\mu$, $e_{\bar d'} = e_d$, $\omega^{\bar\mu'} = e^{\zeta'}\omega^\mu$, $\omega^{\bar d'} = \omega^d$, and, for example again, $T = \omega^K \otimes T_K{}^M{}_L \omega^L \otimes e_M$, then $T_{\bar\kappa'}{}^{\bar\mu'}{}_{\bar\lambda'} = e^{-\zeta'}T_\kappa{}^\mu{}_\lambda = e^\lambda e^{-\zeta} T_\kappa{}^\mu{}_\lambda = e^\lambda T_{\bar\kappa}{}^{\bar\mu}{}_{\bar\lambda}$, so $\omega^{\bar\kappa'} \otimes \bigl(\lim_{\zeta'\to\infty} T_{\bar\kappa'}{}^{\bar\mu'}{}_{\bar\lambda'}\bigr)\omega^{\bar\lambda'} \otimes e_{\bar\mu'} = \omega^{\bar\kappa} \otimes e^{-\lambda}\bigl(\lim_{\zeta'\to\infty} T_{\bar\kappa'}{}^{\bar\mu'}{}_{\bar\lambda'}\bigr)\omega^{\bar\lambda} \otimes e_{\bar\mu} = \omega^{\bar\kappa} \otimes \bigl(\lim_{\zeta\to\infty} T_{\bar\kappa}{}^{\bar\mu}{}_{\bar\lambda}\bigr)\omega^{\bar\lambda} \otimes e_{\bar\mu}$; this together with analogous results for the other components of T expresses the gauge invariance of T_∞. 3) The definition of T_∞ is also independent of the choice of the adapted frame system $\{e_\mu, e_d\}$, a consequence of the fact that if $\{e_{\mu''}, e_{d''}\}$ is another such system, then $e_{d''} = e_d$ and $e_{\mu''} = J_{\mu''}{}^\mu e_\mu$ with $J_{\mu''}{}^\mu = J_{\mu''}{}^{\mu'} J_{\mu'}{}^\mu$, which is independent of ζ (cf. Sec. V). 4) The components of T_∞ in the frame system $\{e_{\bar M}\}$ are constant on each trajectory of ξ. 5) If the metric is conformally constrained in two directions (two ξ's, with $\mathcal{L}_\xi \hat G = 2G$ for each), then each constraint produces a residual of T, and these can differ; strictly, then one should qualify *the* residual of T by citing the generating vector field ξ. 6) Algebraic symmetries of T are preserved in T_∞. 7) Residuals of algebraic derivates of tensor fields (sums, products, contractions, and the like) are the corresponding derivates of the residuals of the constituents.

Now let us see what the residual of the curvature tensor field $\hat\Theta$ is. Noting that $\hat\Theta_\kappa{}^\mu{}_{\lambda\nu}$ has one contravariant and three covariant indices distinct from d, we conclude in light of observation (1) above that $\hat\Theta_{\bar\kappa}{}^{\bar\mu}{}_{\bar\lambda\bar\nu} = e^{-2\zeta}\hat\Theta_\kappa{}^\mu{}_{\lambda\nu}$. Then, referring to the first of Eqs. (A.3) in the Appendix and multiplying both its members by $e^{-2\zeta}$, we see that as $\zeta \to \infty$ the only term on the right that is not extinguished by an exponential factor is the last, and from this it follows that

$$\bigl(\hat\Theta_\infty\bigr)_{\bar\kappa}{}^{\bar\mu}{}_{\bar\lambda\bar\nu} = -\hat\epsilon 2\phi^{-2}\mathring{g}_{\kappa[\lambda}\mathring{g}^\mu{}_{\nu]}. \tag{66}$$

Similar considerations show that

$$\bigl(\hat\Theta_\infty\bigr)_{\bar\kappa}{}^{\bar d}{}_{\bar\lambda\bar d} = -\hat\epsilon\phi^{-2}\mathring{g}_{\kappa\lambda} = -\bigl(\hat\Theta_\infty\bigr)_{\bar\kappa}{}^{\bar d}{}_{\bar d\bar\lambda} \tag{67}$$

and

$$\bigl(\hat\Theta_\infty\bigr)_{\bar d}{}^{\bar\mu}{}_{\bar d\bar\nu} = -\phi^{-2}\mathring{g}^\mu{}_\nu = -\bigl(\hat\Theta_\infty\bigr)_{\bar d}{}^{\bar\mu}{}_{\bar\nu\bar d}, \tag{68}$$

and that all remaining components of $\hat{\Theta}_\infty$ vanish. We find, therefore, that

$$\begin{aligned}\hat{\Theta}_\infty = 2\phi^{-2}\big[&\omega^\kappa \otimes \big(\hat{\epsilon} g_{\kappa\lambda}\omega^\lambda \wedge \omega^\mu\big) \otimes e_\mu \\ &+ \omega^\kappa \otimes \big(\hat{\epsilon} g_{\kappa\lambda}\omega^\lambda \wedge \omega^d\big) \otimes e_d + \omega^d \otimes \big(\omega^d \wedge \omega^\mu\big) \otimes e_\mu\big],\end{aligned} \tag{69}$$

which reduces to simply

$$\begin{aligned}\hat{\Theta}_\infty &= \hat{\epsilon} 2\phi^{-2}\big[\omega^K \otimes \big(\hat{g}_{KL}\omega^L \wedge \omega^M\big) \otimes e_M\big] \\ &= \hat{\epsilon}\phi^{-2}\big[\omega^K \otimes \hat{g}_{KL}\big(\omega^L \otimes \omega^M - \omega^M \otimes \omega^L\big) \otimes e_M\big].\end{aligned} \tag{70}$$

This, then, is the residual of the curvature tensor field of $\hat{G}$, or, a little more succinctly, the **residual curvature tensor field of** $\hat{G}$.

One computes easily the **residual** (of the) **contracted curvature tensor field of** $\hat{G}$ by computing the corresponding contraction of $\hat{\Theta}_\infty$:

$$\hat{\Phi}_\infty = -\hat{\epsilon}(d-1)\phi^{-2}\hat{G}; \tag{71}$$

the **residual** (of the) **curvature scalar field of** $\hat{G}$ by computing the trace of $\hat{G}^{-1}\hat{\Phi}_\infty$:

$$\hat{\Psi}_\infty = -\hat{\epsilon}(d-1)d\,\phi^{-2}; \tag{72}$$

and the **residual** (of the) **Einstein tensor field of** $\hat{G}$ by computing $\hat{\Phi}_\infty - (1/2)\hat{\Psi}_\infty\hat{G}_\infty$:

$$\hat{E}_\infty = \hat{\epsilon}(1/2)(d-2)(d-1)\phi^{-2}\hat{G}. \tag{73}$$

It is also easy to learn that the residuals $\mathring{\Theta}_\infty$ and $\mathring{\Phi}_\infty$ of the curvature fields $\mathring{\Theta}$ and $\mathring{\Phi}$ of $\mathring{\mathbf{d}}$ both vanish, that $\big(\mathring{G}^{-1}\mathring{\Phi}\big)_\infty = \mathring{G}^{-1}\mathring{\Phi}$ and $\mathring{\Psi}_\infty = \mathring{\Psi}$, and that $\mathring{E}_\infty$ vanishes.

Comparing Eq. (69) and Eqs. (A.3), we see that the components of $\hat{\Theta}_\infty$ in $\{e_\mu, e_d\}$ are just those additive contributions to the components of $\hat{\Theta}$ that do not depend on $\mathring{A}_\mu$ or on any derivative of ϕ, $\mathring{A}_\mu$, or $\mathring{g}_{\mu\nu}$. In the case of the prototypically conformally constrained de Sitter and hyper-de Sitter metrics of Eqs. (1) and (2) the latter quantities all vanish, leaving only the constants ϕ, and $\mathring{g}_{\mu\nu}$ to determine the curvature fields. Consequently, for these metrics $\hat{\Theta} = \hat{\Theta}_\infty$ as it is expressed in Eqs. (69) and (70), but particularized by the specialization of the $\mathring{g}_{\kappa\lambda}$ and by the fact that $\phi^{-2} = 1/R^2$. The manifolds with these metrics are, as previously remarked, open submanifolds of hyperboloidal "spheres" of radius R; they have, therefore, uniform sectional curvature of magnitude $1/R^2$, uniform at each point with respect to choice of section (isotropic, in other words), and uniform from point to point.

In the general case there is no such uniformity of ordinary sectional curvature. Residual sectional curvature, however, is always isotropic, and is uniform if ϕ is constant. If a and b are a pair of tangent vectors at the point P of $\mathcal{M}$, then from Eq. (70) it follows readily that, at P,

$$\big(\hat{G}\hat{\Theta}_\infty\big)abab = -\hat{\epsilon}\phi^{-2}\left[\big(\hat{G}aa\big)\big(\hat{G}bb\big) - \big(\hat{G}ab\big)^2\right]. \tag{74}$$

This equation implies that if the square $\big(\hat{G}(P)aa\big)\big(\hat{G}(P)bb\big) - \big(\hat{G}(P)ab\big)^2$ of the area of the bivector $a \wedge b$ is not 0, then the **residual sectional curvature of $\hat{G}$ at P in the direction of** $a \wedge b$, defined in complete analogy with the ordinary sectional curvature as the fraction of that square that the number $\big(\hat{G}\hat{\Theta}_\infty\big)(P)abab$ comes to, is $-\hat{\epsilon}\phi^{-2}(P)$. As this is independent of a and b, the residual sectional curvature is isotropic at P; clearly it is uniform from point to point only if ϕ is constant. Even when not uniform, however, it is constant on each trajectory of ξ, simply because $\mathcal{L}_\xi \phi = 0$. A shorter way of stating the facts is to say that $\hat{G}$ is **residually spherical** at each point of $\mathcal{M}$, with vertically uniform **residual radius of curvature** ϕ and **residual curvature** $-\hat{\epsilon}\phi^{-2}$.

We have seen that the $\hat{G}$ lengths of the horizontal vectors $e_{\bar{\mu}}$ in the reference frame $\{e_{\bar{M}}\}$ used in calculating residual curvature stay fixed as we push these vectors vertically along a trajectory of ξ. The other side of this is that their lengths as specified by the metric $\mathring{G}$ do not stay fixed as we push them along. In fact, $\mathring{G}e_{\bar{\mu}}e_{\bar{\mu}} = e^{-2\zeta}\mathring{G}e_\mu e_\mu = e^{-2\zeta}\mathring{g}_{\mu\mu}$, so $\mathring{G}e_{\bar{\mu}}e_{\bar{\mu}} \to 0$ as $\zeta \to \infty$. Also, $\mathring{G}e_{\bar{d}}e_{\bar{d}} = \mathring{G}e_d e_d = 0$. Thus the residual curvatures are limits of ordinary curvatures measured against frames of vanishingly small $\mathring{G}$ dimensions.

When $\hat{G}$ is a space-time–time metric, the $\mathring{G}$ lengths are the usual dimensions of space and time, and it is in terms of these familiar dimensions that the frame vectors $e_{\bar{\mu}}$ shrink to infinitesimal size as $\zeta \to \infty$. If we think of those frame vectors as abstract measuring instruments belonging to a family of increasingly microscopic observers stationed at a space-time event $\mathcal{E}$, then the effect of their shrinking is that the observers at the extreme microscopic end of the family are able to perceive and to measure the curvatures of only their most immediate surroundings, which to them are indistinguishable from a flat space-time region embedded in a hyper-de Sitter, space-time–time sphere of radius $\phi(\mathcal{E})$. Thus the residual curvatures, depending only on ϕ, represent an aspect of the geometry more infinitesimal in scale than that represented by the nonresidual portions of curvatures embodied in the tensor field $\hat{\Theta} - \hat{\Theta}_\infty$ and depending on $\mathring{A}_\mu$ and the derivatives of ϕ, $\mathring{A}_\mu$, and $\mathring{g}_{\mu\nu}$ as well as on ϕ and $\mathring{g}_{\mu\nu}$ — an ultralocal, as opposed to a merely local, aspect, one could say. This distinction between the local and the ultralocal aspects comes into play when field equations for space-time–time are to be derived from an action principle. By adopting for the action density the nonresidual portion $\hat{\Psi} - \hat{\Psi}_\infty$ of the curvature scalar field, one can favor space-time–times that extremize not total curvature, but the total deviation of curvature from the isotropic, ultralocal, vacuum emulating residual curvature [19].

XI. Space-Time–Time

The de Sitter metric $\hat{G}$ of Eq. (1) and the manifold $\mathcal{M}$ on which it is defined arise out of ordinary three-dimensional Euclidean space through the following construction [1]: the point of $\mathcal{M}$ whose address is $[\![x, y, z, t]\!]$ is (identified with) the Euclidean sphere of radius r centered at $[\![x, y, z]\!]$, where $t := -\ln(r/R)$; if this sphere and an infinitesimally neighboring sphere of radius $r + dr$ centered at $[\![x + dx, y + dy, z + dz]\!]$ miss being tangent to one another by the angular amount $d\alpha$ (the radian measure of their angle of intersection if the spheres meet), then the squared distance between the corresponding points of $\mathcal{M}$ is (and this defines $\hat{G}$) the number $R^2 d\alpha^2$.

The same construction applied to Minkowski space-time (which for present purposes is interchangeable with de Sitter space-time, being conformally equivalent to it and therefore having matching spheres and angles of intersection) yields both of the hyper-de Sitter metrics $\hat{G}_-$ and $\hat{G}_+$ of Eq. (2). The metric $\hat{G}_-$ results when the Minkowski spheres (three-dimensional hyper-hyperboloids of revolution, in the Euclidean sense) are of the one-sheeted variety, their points lying in spacelike directions from their centers. When the spheres are those of the two-sheeted variety, whose points lie in timelike directions from their centers, $\hat{G}_+$ results. In either case the sphere of radius s centered at $[\![x, y, z, t]\!]$ corresponds to the point (of $\mathcal{M}_-$ or of $\mathcal{M}_+$) with address $[\![x, y, z, t, \zeta]\!]$, where $\zeta := -\ln(s/R)$. And in either case the squared distance between neighboring points is $R^2 d\beta^2$, where $d\beta$ is the angular amount by which the corresponding neighboring spheres fall short of tangency [20].

It is because of this shared construction, which in the iterated application defines the new coordinate ζ in terms of the radius s precisely in the manner that when first applied it defines the new, temporal coordinate t in terms of the radius r, *and on no other ground*, that I have attached the label space-time–time to manifolds with conformally constrained metrics modeled on $\hat{G}_-$ or on $\hat{G}_+$. That the signature $+++--$ of $\hat{G}_-$ appears to fit the label and the signature $+++-+$ of $\hat{G}_+$ appears not to do so is of no consequence, for in either case the coordinate ζ represents a geometrical entity thoroughly comparable to the entity that the coordinate t represents, justifiably called a "time" — but a time of a higher order, of course. A fair description of the situation would be that t is "space's time" and ζ is "space-time's time" [21].

There being no geometrical reason to prefer the one kind of Minkowski sphere to the other, it seems a half-measure to model physical systems by use of conformally constrained metrics bearing either one of these signatures, to the exclusion of those bearing the other, or to use one today and the other tomorrow. Expansion of the geometry to include the two signatures on equal footing urges itself as an essential further step. One way to effect such an expansion is to complexify the secondary time coordinate ζ, and along with it the scalar field ϕ and the electromagnetic potential field $\mathring{A}$. When that is done, new elements become available for physical interpretation. Conspicuous among them are 1) a reciprocal coupling between pure imaginary gauge transformations of $\mathring{A}$ and complex phase shifts of ϕ, and 2) an investing of each geodesic with a varying complex phase rotation whose frequency parameter adjusts to the environs of the geodesic. These particular elements beg to be linked up with quantum mechanical phase phenomena, and that rather clearly demands the forging of a link between the geometrical field ϕ and the electron wave field (Schrödinger's ψ) of quantum theory. The forging of such a link will, I believe, allow one ultimately to say, not that geometry has been quantized, but that the quantum has been geometrized.

APPENDIX. Curvature Components

From the curvature 2-forms $\hat{\Theta}_K{}^M$ as expressed in Eqs. (51) one readily extracts the curvature components $\hat{\Theta}_K{}^M{}_{LN}$ that occur in $\hat{\Theta}_K{}^M = \hat{\Theta}_K{}^M{}_{LN}\,\omega^N\wedge\omega^L = \hat{\Theta}_K{}^M{}_{LN}\,\omega^N\otimes\omega^L$. Those that do not vanish identically are given by

$$\begin{aligned}
\hat{\Theta}_\kappa{}^\mu{}_{\lambda\nu} &= \Theta_\kappa{}^\mu{}_{\lambda\nu} - \hat{\epsilon}\phi^2 F_\kappa{}^{(\mu}F_{[\lambda)\nu]} - \hat{\epsilon}2\phi^{-2}g_{\kappa[\lambda}g^\mu{}_{\nu]},\\
\hat{\Theta}_\kappa{}^\mu{}_{d\nu} &= \hat{\epsilon}(1/2)\Big(\phi F_\kappa{}^\mu{}_{;\nu} + 2F_{(\kappa}{}^\mu\phi_{.\nu)} + 2F_\kappa{}^{(\mu}\phi_{.\nu)}\Big) + 2\phi^{-2}\phi_{.[\kappa}g^{\mu]}{}_\nu,\\
\hat{\Theta}_\kappa{}^d{}_{\lambda\nu} &= \phi F_{\kappa[\lambda;\nu]} - 2\phi_{.(\kappa}F_{[\lambda)\nu]} + \hat{\epsilon}2\phi^{-2}g_{\kappa[\lambda}\phi_{.\nu]},\\
\hat{\Theta}_d{}^\mu{}_{\lambda\nu} &= -\hat{\epsilon}g^{\mu\kappa}\hat{\Theta}_\kappa{}^d{}_{\lambda\nu},\\
\hat{\Theta}_\kappa{}^d{}_{\lambda d} &= -\phi^{-1}\phi_{,\kappa;\lambda} - \hat{\epsilon}(1/4)\phi^2 F_\kappa{}^\pi F_{\pi\lambda} - \hat{\epsilon}\phi^{-2}g_{\kappa\lambda},
\end{aligned}\tag{A.1}$$

and

$$\hat{\Theta}_d{}^\mu{}_{d\nu} = \hat{\epsilon}g^{\mu\kappa}\hat{\Theta}_\kappa{}^d{}_{\nu d},$$

and the antisymmetry $\hat{\Theta}_K{}^M{}_{NL} = -\hat{\Theta}_K{}^M{}_{LN}$. Here, in accordance with Eq. (52),

$$\begin{aligned}
\Theta_\kappa{}^\mu{}_{\lambda\nu} := {}& 2\left(\Gamma_\kappa{}^\mu{}_{[\lambda,\nu]} + \Gamma_\kappa{}^\pi{}_{[\lambda}\Gamma_\pi{}^\mu{}_{\nu]} + \Gamma_\kappa{}^\mu{}_\pi C_\lambda{}^\pi{}_\nu\right)\\
&- \left(g_\kappa{}^\mu F_{\lambda\nu} - F_{\kappa[\lambda}g^\mu{}_{\nu]} - g_{\kappa[\lambda}F^\mu{}_{\nu]}\right).
\end{aligned}\tag{A.2}$$

When the righthand members of Eqs. (A.1) are "detelescoped" by use of Eqs. (25)–(29), there results that

$$
\begin{aligned}
\hat{\Theta}_\kappa{}^\mu{}_{\lambda\nu} &= \mathring{\Theta}_\kappa{}^\mu{}_{\lambda\nu} + 2\Big(\mathring{A}_{(\kappa:[\lambda)}\mathring{g}^\mu{}_{\nu]} + \mathring{g}_{\kappa[\lambda}\mathring{A}^{(\mu}{}_{:\nu)]}\Big) \\
&\qquad + 2\Big(\mathring{A}_\kappa\mathring{A}_{[\lambda}\mathring{g}^\mu{}_{\nu]} + \mathring{g}_{\kappa[\lambda}\mathring{A}^\mu\mathring{A}_{\nu]} - \mathring{A}^\pi\mathring{A}_\pi\mathring{g}_{\kappa[\lambda}\mathring{g}^\mu{}_{\nu]}\Big) \\
&\qquad - \hat{\epsilon}e^{-2\zeta}\phi^2\mathring{F}_\kappa{}^{(\mu}F_{[\lambda)\nu]} - \hat{\epsilon}2e^{2\zeta}\phi^{-2}\mathring{g}_{\kappa[\lambda}\mathring{g}^\mu{}_{\nu]}, \\
\hat{\Theta}_\kappa{}^\mu{}_{d\nu} &= \hat{\epsilon}(1/2)e^{-2\zeta}\Big[\phi\mathring{F}_\kappa{}^\mu{}_{:\nu} + 2\mathring{F}_{(\kappa}{}^\mu\phi_{.\nu)} + 2\mathring{F}_\kappa{}^{(\mu}\phi_{.\nu)} \\
&\qquad\qquad + 2\phi\Big(\mathring{F}_{(\kappa}{}^\mu\mathring{A}_{\nu)} + \mathring{F}_\kappa{}^{(\mu}\mathring{A}_{\nu)} - \mathring{F}_{[\kappa}{}^\pi\mathring{A}_\pi\mathring{g}^{\mu]}{}_\nu\Big)\Big] \\
&\quad + 2\phi^{-2}\phi_{.[\kappa}\mathring{g}^{\mu]}{}_\nu, \\
\hat{\Theta}_\kappa{}^d{}_{\lambda\nu} &= \phi\mathring{F}_{\kappa[\lambda:\nu]} - 2\phi_{.(\kappa}F_{[\lambda)\nu]} - 2\phi\Big(\mathring{A}_{(\kappa}F_{[\lambda)\nu]} - \mathring{g}_{\kappa[\lambda}\mathring{A}^\pi F_{\pi\nu]}\Big) \\
&\quad + \hat{\epsilon}2e^{2\zeta}\phi^{-2}\mathring{g}_{\kappa[\lambda}\phi_{.\nu]}, \\
\hat{\Theta}_d{}^\mu{}_{\lambda\nu} &= -\hat{\epsilon}e^{-2\zeta}\mathring{g}^{\mu\kappa}\hat{\Theta}_\kappa{}^d{}_{\lambda\nu}, \\
\hat{\Theta}_\kappa{}^d{}_{\lambda d} &= -\phi^{-1}\Big[\mathring{\phi}_{.\kappa:\lambda} + \Big(2\mathring{A}_{(\kappa}\phi_{.\lambda)} - \mathring{g}_{\kappa\lambda}\mathring{A}^\pi\phi_{.\pi}\Big)\Big] \\
&\quad - \hat{\epsilon}(1/4)e^{-2\zeta}\phi^2\mathring{F}_\kappa{}^\pi F_{\pi\lambda} - \hat{\epsilon}e^{2\zeta}\phi^{-2}\mathring{g}_{\kappa\lambda},
\end{aligned}
\tag{A.3}
$$

and

$$\hat{\Theta}_d{}^\mu{}_{d\nu} = \hat{\epsilon}e^{-2\zeta}\mathring{g}^{\mu\kappa}\hat{\Theta}_\kappa{}^d{}_{\nu d}.$$

The abbreviations that appear in these expressions are explained in Sec. IX.

From Eqs. (A.3) and the relation $\hat{\Phi}_{KL} = \hat{\Theta}_K{}^R{}_{LR}$ it now follows that

$$
\begin{aligned}
\hat{\Phi}_{\kappa\lambda} &= \mathring{\Phi}_{\kappa\lambda} + \Big[(d-3)\mathring{A}_{(\kappa:\lambda)} + \mathring{A}^\rho{}_{:\rho}\mathring{g}_{\kappa\lambda} + (d-3)\Big(\mathring{A}_\kappa\mathring{A}_\lambda + \mathring{A}^\rho\mathring{A}_\rho\mathring{g}_{\kappa\lambda}\Big)\Big] \\
&\quad - \phi^{-1}\mathring{\phi}_{.\kappa:\lambda} - 2\phi^{-1}\Big[\mathring{A}_{(\kappa}\phi_{.\lambda)} - (1/2)\mathring{A}^\rho\phi_{.\rho}\mathring{g}_{\kappa\lambda}\Big] \\
&\quad + \hat{\epsilon}(1/2)e^{-2\zeta}\phi^2\mathring{F}_\kappa{}^\rho F_{\rho\lambda} - \hat{\epsilon}(d-1)e^{2\zeta}\phi^{-2}\mathring{g}_{\kappa\lambda}, \\
\hat{\Phi}_{\kappa d} &= \hat{\epsilon}(1/2)e^{-2\zeta}\Big[\phi\mathring{F}_\kappa{}^\rho{}_{:\rho} - (d-5)\phi\mathring{F}_\kappa{}^\rho\mathring{A}_\rho + 3\mathring{F}_\kappa{}^\rho\phi_{.\rho}\Big] + (d-2)\phi^{-2}\phi_{.\kappa}, \\
\hat{\Phi}_{d\lambda} &= \hat{\Phi}_{\lambda d},
\end{aligned}
\tag{A.4}
$$

and

$$\hat{\Phi}_{dd} = -\hat{\epsilon}e^{-2\zeta}\phi^{-1}\Big[\mathring{\phi}^{\cdot\rho}{}_{:\rho} - (d-3)\mathring{A}^\rho\phi_{.\rho)}\Big] - (1/4)e^{-4\zeta}\phi^2\mathring{F}_\pi{}^\rho\mathring{F}_\rho{}^\pi - (d-1)\phi^{-2},$$

where $\mathring{\Phi}_{\kappa\lambda} = \mathring{\Theta}_\kappa{}^\rho{}_{\lambda\rho}$, as said in Sec. IX.

Here, for the sake of comparison with Eqs. (63) and (64), are the forms that $\hat{\Psi}$, $\hat{E}_{\kappa\lambda}$, $\hat{E}_{\kappa d}$, $\hat{E}_{d\lambda}$, and $\hat{E}_{dd}$ would take for the Kaluza geometry, whose metric is obtained from Eqs. (4′) by the replacement $e^{2\zeta} \to 1$:

$$\hat{\Psi} = \mathring{\Psi} - 2\phi^{-1}\mathring{\phi}^{\cdot\rho}{}_{:\rho} + \hat{\epsilon}(1/4)\phi^2\mathring{F}_\pi{}^\rho\mathring{F}_\rho{}^\pi, \tag{A.5}$$

$$\hat{E}_{\kappa\lambda} = \mathring{E}_{\kappa\lambda} - \phi^{-1}\left[\mathring{\phi}_{.\kappa:\lambda} - \mathring{\phi}^{.\rho}{}_{:\rho}\mathring{g}_{\kappa\lambda}\right] + \hat{\epsilon}(1/2)\phi^2\left[\mathring{F}_\kappa{}^\rho F_{\rho\lambda} - (1/4)\mathring{F}_\pi{}^\rho \mathring{F}_\rho{}^\pi \mathring{g}_{\kappa\lambda}\right],$$
$$\hat{E}_{\kappa d} = \hat{\epsilon}(1/2)\left[\phi \mathring{F}_\kappa{}^\rho{}_{:\rho} + 3\mathring{F}_\kappa{}^\rho \phi_{.\rho}\right],$$
$$\hat{E}_{d\lambda} = \hat{E}_{\lambda d}, \tag{A.6}$$

and

$$\hat{E}_{dd} = -\hat{\epsilon}(1/2)\mathring{\Psi} - (3/8)\phi^2\mathring{F}_\pi{}^\rho \mathring{F}_\rho{}^\pi.$$

The further replacement $\phi \to 1$ yields the forms of $\hat{\Psi}$, $\hat{E}_{\kappa\lambda}$, $\hat{E}_{\kappa d}$, $\hat{E}_{d\lambda}$, and $\hat{E}_{dd}$ for the Kaluza–Klein geometry.

References

1. H. G. Ellis, *Found. Phys.* **4** (1974), 311–319; Erratum: **5** (1975), p. 193; partially reprised in Sec. XI of the present paper. N.B. In several places the editor of the journal substituted "declaration" for the word "manifesto" used in the manuscript. I would have preferred "credo" as a compromise.
2. H. G. Ellis, *Abstracts of Contributed Papers*, 8th International Conference on General Relativity and Gravitation, Univ. of Waterloo, Waterloo, Ont., Canada, 1977, p. 138. In this abstract both a "weak form" and a "strong form" conformality constraint are defined. The present discussion refers to a yet stronger form (to be defined in Sec. II), which will be identified simply as *the* conformality constraint, the earlier versions being no longer in use.
3. Th. Kaluza, *S.–B. Preuss. Akad. Wiss.* **1921**, 966–972. The reference here is to the geometry with the "cylinder condition" as Kaluza intended it, without the additional restriction that the Killing vector field have constant length, appended later by Klein [a] and by Einstein [b]. This restriction conceivably lies implicit in Kaluza's stated and unstated assumptions; if so, he did not recognize it.
   a. O. Klein, *Z. Physik* **37** (1926), 895–906; **46** (1927), 188–208.
   b. A. Einstein, *S.–B. Preuss. Akad. Wiss., Phys.–math. Kl.* **1927**, 23–30.
   The Kaluza paper and the first Klein paper appear in English translation in *Modern Kaluza–Klein Theories* (vol. 65 in the series Frontiers in Physics), T. Appelquist, A. Chodos, and P. G. O. Freund, eds. (Addison–Wesley, Menlo Park, California, 1987); a facsimile of the Kaluza paper also is included.
4. H. Weyl, *S.–B. Preuss. Akad. Wiss.* **1918**, 465–480; *Ann. d. Physik* **59** (1919), 101–133.
5. E. Schrödinger, *Expanding Universes*, Cambridge Univ. Press, Cambridge, U.K., 1956, pp. 28–33.
6. Strict adherence to the meaning of "conformal" would classify metrics $\hat{G}$ for which $\mathcal{L}_\xi \hat{G} = 0$ as "isometrically conformally constrained" if $\mathcal{L}_\xi \hat{G} = 0$ and those for which $\mathcal{L}_\xi \hat{G} = 2G$ as "nonisometrically conformally constrained"; the less accurate "isometrically constrained" and "conformally constrained" have the advantage of brevity.
7. Th. Kaluza, *loc. cit.* Kaluza was not adamant about this choice, indeed seemed willing to let it go the other way if by so doing he could overcome a "very large" difficulty pointed out to him by Einstein (p. 971, last paragraph).
8. My reasons for calling $\mathcal{M}$ a space-time-*time* manifold can be found in Ref. 1; they are discussed here in Sec. XI.
9. This supposition, introduced here to simplify the construction, is not essentially restrictive, as cases in which it is false can be brought under it by judicious use of covering manifolds.
10. Weyl himself was mainly responsible for this amnesia, through his discovery of the tie between electromagnetic gauge transformations and electron wave field phase shifts (cf. *Space-Time-Matter* (Dover, New York, 1950), Preface to the First American Printing, p. v, and references cited there).
11. All skew-symmetrizations signified by [ ] and symmetrizations signified by ( ) are to be carried out on *two* indices only, namely the leftmost and the rightmost indices enclosed.
12. These commutators include the factor 1/2, viz., $[u,v] := (1/2)(uv - vu)$.
13. W. Ehrenberg and R. Siday, *Proc. Phys. Soc.* (London) **B62** (1949), 8–21.
14. Y. Aharonov and D. Bohm, *Phys. Rev.* **115** (1959), 485–491.
15. The derivation is not quite straightforward, because the coframe system $\{\omega^\mu, \omega^d\}$ is not required to be holonomic. The Euler equations turn out to be, after division by 2, $(\hat{g}_{KL}\dot{p}^L)^{\cdot} - (1/2)\dot{p}^M \hat{g}_{MN,K}\dot{p}^N + C_K{}^M{}_L \dot{p}^L \hat{g}_{MN}\dot{p}^N = 0$, where $K, L, M, N = 1, \ldots, d$ and $C_K{}^M{}_L = (d_\wedge \omega^M) e_L e_K = \omega^M [e_K, e_L]$.
16. See for example Eq. (2.41), p. 62, in *Electrodynamics and Classical Theory of Fields and Particles*, A. O. Barut (Macmillan, New York, 1964).
17. D. M. Chase, *Phys. Rev.* **95** (1954), 243–246, obtained results that seem to imply some connection in Einstein–Maxwell theory between the ratio of active charge to active mass and the ratio of passive charge to passive mass.
18. A weaker condition would suffice, merely that $\mathcal{M}$ be ξ*-forwardly complete*, in other words that on each ξ-path the integration parameter run to ∞.
19. The restricted field equations of Ref. 2 are derived from such an action principle. N.B. The equation there labeled $\delta\psi$) is wrong, and should be replaced by $\mathring{A}^\mu{}_{:\mu} - 3\mathring{A}^\mu A_\mu = (1/2)\mathring{R} - (3/4)K\psi^2 \mathring{F}^\mu{}_\nu \mathring{F}^\nu{}_\mu$. These equations, derived in a gauge chosen to simplify description of the integration region D^5, are not fully gauge invariant, nor are they intended to be.
20. See Ref. 1; Eq. (4) there needs a minus sign before the $d\beta^2$. See also R. L. Ingraham, *Nuovo Cimento* **46** B (1978), 16–32.
21. A point to be emphasized is that, irrespective of any label applied to it, the dimension coordinatized by ζ differs so radically in character from the three spatial dimensions that no attempt to make it "unobservable" is demanded. Making it periodic by rolling up the trajectories of ξ into circles (after the

practice of Kaluza–Klein theorists) would be in fact incompatible with the conformality constraint and therefore impossible. Cases in which the ξ-trajectories return repeatedly to the same space-time cross section, each return occurring at a new event, might usefully be contemplated, however.

First version: August, 1989
Revised: September, 2001
Revised: April, 2002

Homer G. Ellis
Department of Mathematics
University of Colorado at Boulder
395 UCB
Boulder, Colorado 80309-0395

Telephone: (303) 492-7754 (office); (303) 499-4027 (home)
Email: ellis@euclid.colorado.edu
Fax: (303) 492-7707