Problem 2. Let $X = [\![x, y, z]\!]$, a rectangular cartesian coordinate system for $\mathbb{E}^3$. Let $\Pi = [\![\rho, \vartheta, \varphi]\!]$, a spherical polar coordinate system for $\mathbb{E}^3$ related to X by

$$\begin{aligned} x &= \rho\,(\sin\vartheta)(\cos\varphi), & \quad & & \rho &= \sqrt{x^2+y^2+z^2}, \\ y &= \rho\,(\sin\vartheta)(\sin\varphi), & \quad & \text{and} & \cos\vartheta &= \frac{z}{\sqrt{x^2+y^2+z^2}}, \\ z &= \rho\,(\cos\vartheta), & & & \cos\varphi &= \frac{x}{\sqrt{x^2+y^2}} \quad \text{and} \quad \sin\varphi = \frac{y}{\sqrt{x^2+y^2}}, \end{aligned}$$

with $\operatorname{ran}\Pi = \{[\![\rho, \vartheta, \varphi]\!] \mid 0 < \rho,\ 0 < \vartheta < \pi, \text{ and } -\pi < \varphi < \pi\}$ and $\operatorname{dom}\Pi = \mathbb{E}^3 - \{P \mid y(P) = 0 \text{ and } x(P) \le 0\}$. One can show that Π and X are C^1-compatible (in fact, analytically compatible). Let $\mathcal{M}$ be the C^1 manifold for which a minimal C^1 atlas is $\{X\}$, and whose maximal C^1 atlas therefore has Π as well as X in it. Let $E := \{\partial_x, \partial_y, \partial_z\}$, $E' := \{\partial_\rho, \partial_\vartheta, \partial_\varphi\}$, and $E'' := \{\partial_\rho, (1/\rho)\partial_\vartheta, (1/\rho\sin\vartheta)\partial_\varphi\}$.

Define a frame systemization $\bar{E}$ of $\mathcal{M}$ by $\bar{E}^P := E$ for every point P of $\mathcal{M}$. Let $\bar{d}$ be the differentiation generated by $\bar{E}$, and let $\mathbf{d}$ be the covariant differentiation generated by $\bar{d}$. (Thus, if P is a point of $\mathcal{M}$ and T is a tensor field of $\mathcal{M}$ that is differentiable at P, then $\mathbf{d}T(P) := \bar{d}T(P) := d_{\bar{E}^P}T(P)$.)

Let $\omega_k{}^m$, $\omega_{k'}{}^{m'}$, and $\omega_{k''}{}^{m''}$ be the 1-forms of $\mathbf{d}$ in E, E', and E'', respectively. Then $\omega_k{}^m = 0$.

a. Compute $\omega_{k'}{}^{m'}$.

b. Compute $\omega_{k''}{}^{m''}$.

c. Write out each of the covariant derivatives $\mathbf{D}_{e_{k'}} e_{m'}$.

d. Write out each of the covariant derivatives $\mathbf{D}_{e_{k'}} \omega^{m'}$.

e. Write out each of the covariant derivatives $\mathbf{D}_{e_{k''}} e_{m''}$.

f. Write out each of the covariant derivatives $\mathbf{D}_{e_{k''}} \omega^{m''}$.