I. TOPOLOGICAL SPACES AND HOMEOMORPHISMS

If S is a nonempty set, then by a **topology for** S is meant a collection T of subsets of S such that (i) S is in T, (ii) the empty set $\varnothing$ is in T, (iii) if A and B are sets in T, then $A \cap B$ is in T, and (iv) if T' is a subcollection of T, then $\bigcup_{A \in T'} A$ is in T. By a **base for** T is meant a subcollection B of T such that every nonempty set in T is a union of one or more of the sets in B. By a **topological space** Σ is meant a set S together with a topology T for S. The elements of S are called the **points** of Σ. The sets in T are called the **open sets** of Σ; their complements (in S) are called the **closed sets** of Σ. Each of $\varnothing$ and S is both open and closed.

Example. S is a line, B is the collection of all intervals of S that include neither end point, and T is the collection of all unions of subcollections of B.

Example. S is a plane, B is the collection of all interiors of circles in S, and T is the collection of all unions of subcollections of B. Another base for the same topology T for S is the collection of all interiors of convex polygons in S. Yet another is the collection of all interiors of rectangles in S with one side horizontal. One other is the collection of all interiors of circles in S of radius less than ϵ, where ϵ is some positive number, the same for all the circles in question. Call this space Π.

Example. S is a sphere, B is the collection of all subsets of S "sliced off" by a plane that intersects S in a circle (a slice lying on one side of the plane only; the circle of intersection not included in the slice), and T is the collection of all unions of subcollections of B.

Example. S is a "punctured sphere" (a sphere with one point removed), B is the collection of all subsets of S "sliced off" by a plane that intersects S in a circle or a "punctured circle" (a slice lying one side of the plane only; the circle or punctured circle of intersection not included in the slice), and T is the collection of all unions of subcollections of B. Call this space $\Sigma^\star$.

If each of Σ and Σ' is a topological space, and F is a mapping of a subset of Σ into Σ', then F **is continuous at** P means that (i) P is a point of $\operatorname{dom} F$, and (ii) if U' is an open set of Σ' that has $F(P)$ in it, then there is an open set U of Σ that has P in it and is such that $F(U)$ is a subset of U'; F **is continuous** means that F is continuous at each point of its domain. By a **homeomorphism of** Σ **onto** Σ' is meant a one-to-one mapping F of Σ onto Σ' such that both F and F^{-1} are continuous. If such a homeomorphism exists, then Σ and Σ' are said to be **homeomorphic** and to be **topologically equivalent** (to one another). Topological equivalence is an equivalence relation (reflexive, symmetric, and transitive) on the set of all topological spaces.

Example. The plane Π and the punctured sphere $\Sigma^\star$ described above are topologically equivalent. Clearly Π is homeomorphic to the plane Π' tangent to $\Sigma^\star$ at the point opposite the puncture point. A homeomorphism F of Π' onto $\Sigma^\star$ is produced by "stereographic projection", described as follows: for each point P of Π', $F(P)$ is the point of $\Sigma^\star$ that lies on the line through P and the puncture point.

II. FINITE-DIMENSIONAL MANIFOLDS

By a **local coordinate system**, of **dimensionality** M, for the set S is meant a one-to-one mapping X of a subset of S onto an open set in $\mathbb{R}^M$; $\mathbb{R}^M$ is called the **coordinate space** of X, and $\operatorname{dom} X$ is called the **coordinate patch** of X. The mapping X^{-1} is called an **M-dimensional parametrization of** $\operatorname{dom} X$. The **coordinate functions** of X are denoted by x^m, $m = 1, \ldots, M$, so that $X = [\![\, x^1, \ldots, x^M \,]\!]$ (in other words, if $P \in \operatorname{dom} X$, then $X(P) = [\![\, x^1(P), \ldots, x^M(P) \,]\!]$, a vector in $\mathbb{R}^M$). If, to distinguish it from X, a local coordinate system of dimensionality M is labeled X', then its coordinate functions are denoted both by x'^m and by $x^{m'}$, $m = 1, \ldots, M$; X'', X''', and so on are treated similarly. If the coordinate patch of X is all of S, then X is called a **global coordinate system** for S. "Coordinate system" used alone means "local coordinate system". If $P \in \operatorname{dom} X$, then X is said to be a **coordinate system around** P. A synonym for "coordinate system" is "chart".

If f is a mapping of a subset of S into $\mathbb{R}$, then f **is** C^K (resp., C^∞) **with respect to** X means that $X(\operatorname{dom} f)$ is open in $\mathbb{R}^M$ and either $X(\operatorname{dom} f)$ is empty or else the function $f \circ X^{-1}\colon X(\operatorname{dom} f) \to \mathbb{R}$ is, if $K \geq 0$, continuous, and has, if $K \geq 1$, continuous partial derivatives of all mixtures of order K or less (resp., of all orders). These partial derivatives, composed with X, will be denoted by $\partial f/\partial x^m$, $\partial^2 f/\partial x^{m_2}\partial x^{m_1}$, $\partial^3 f/\partial x^{m_3}\partial x^{m_2}\partial x^{m_1}$, and so on. In a different notation these would read $\partial_m(f \circ X^{-1})(X)$, $\partial_{m_2}(\partial_{m_1}(f \circ X^{-1}))(X)$, and so on. That f **is analytic with respect to** X means that $X(\operatorname{dom} f)$ is open and either $X(\operatorname{dom} f)$ is empty or else the function $f \circ X^{-1}$ has, about each point of $X(\operatorname{dom} f)$, a real power series representation. If F is a mapping of a subset of S into $\mathbb{R}^N$, then F **is** C^K (C^∞, **analytic**) **with respect to** X means that for $n = 1, \ldots, N$ the n^{th} component function of the mapping $F \circ X^{-1}\colon X(\operatorname{dom} F) \to \mathbb{R}^N$ is C^K (C^∞, analytic). Clearly, if f or F is analytic with respect to X, then it is C^∞ with respect to X, and if it is C^L with respect to X, and $L \leq \infty$, then it is C^K with respect to X if $K \leq L$.

If each of X and X' is a coordinate system for the set S, then X' **is C^K-compatible** (resp., **C^∞-compatible, analytically compatible**) **with** X means that each of X and X' is C^K (resp., C^∞, analytic) with respect to the other. This relation is almost an equivalence relation on the set of all coordinate systems for S, as the following theorem indicates.

Theorem 1. C^K-compatibility, C^∞-compatibility, and analytic compatibility are reflexive and symmetric as relations among all local coordinate systems for the set S; if $S' \subset S$, then they are transitive among those local coordinate systems for S that are global for S'.

Henceforth, statements that are made about C^K-compatibility will be understood to generate by replacement analogous statements about C^∞-compatibility and analytic compatibility.

By a C^K **atlas** for the set S is meant a collection of one or more local coordinate systems for S whose coordinate patches collectively cover S, and each two of which are C^K-compatible. If each coordinate patch has in it a point of S not in any of the other patches, then this is a **minimal C^K atlas** for S. If the C^K atlas omits no coordinate system for S that is C^K-compatible with every other coordinate system in it, then it is called a **maximal C^K atlas** for S. Every C^K atlas for S that has a minimal C^K atlas for S as a subatlas (i. e., as a subset that is an atlas) is itself a subatlas of a maximal C^K atlas for S, viz. $\{ X \mid X$ is a local coordinate system for S that is C^K-compatible with every coordinate system in that minimal C^K atlas for $S\}$, and is a subatlas of no other maximal C^K atlas for S, however arrived at.

Definition. By a C^K (resp., C^∞, **analytic**) **manifold** is meant a set S, *distinct from* $\mathbb{R}$, *and from* $\mathbb{R}^M$ *for* $M = 1, 2, \ldots$, together with a maximal C^K (resp., C^∞, analytic) atlas for S. By a **manifold** is meant a C^K, a C^∞, or an analytic manifold. The elements of the underlying set S are called the **points** of the manifold. The elements of the maximal atlas of the manifold are called the **coordinate systems of the manifold** and their domains are called the **coordinate patches of the manifold**. If a manifold is C^K with $K \geq 1$, is C^∞, or is analytic, it is said to be K**-smooth** (**smooth** if $K \geq 1$, **doubly smooth** if $K \geq 2$, **triply smooth** if $K \geq 3$).

Theorem 2. If $\mathcal{M}$ is a manifold, then there is a unique topology for $\mathcal{M}$ with respect to which each coordinate system of $\mathcal{M}$ is a homeomorphism. This topology has as a base the set of all coordinate patches of $\mathcal{M}$.

If S' is an open subset of the C^K manifold $\mathcal{M}$, then S' together with the atlas whose coordinate systems are the restrictions to S' of the coordinate systems of $\mathcal{M}$ is a C^K manifold $\mathcal{M}'$. This manifold $\mathcal{M}'$ is referred to as the (open) **submanifold** $\mathcal{M}'$ of $\mathcal{M}$, and $\mathcal{M}$ is called an **extension of** $\mathcal{M}'$. The components (maximal connected subsets) of a manifold $\mathcal{M}$ are open, hence can be considered connected open submanifolds of $\mathcal{M}$.

Theorem 3. If the manifold $\mathcal{M}$ is connected, then all coordinate systems of $\mathcal{M}$ have the same dimensionality.

A proof of this theorem relies on the "invariance of domain" theorem that if an open set in $\mathbb{R}^M$ is homeomorphic to an open set in $\mathbb{R}^N$, then $M = N$. Henceforth every manifold considered will be supposed connected, and the common dimensionality of its coordinate systems will be called the **dimensionality of the manifold**.

III. DIFFERENTIABLE MAPPINGS BETWEEN SMOOTH MANIFOLDS

Henceforth $\mathcal{M}$ will denote an M-dimensional, and $\mathcal{N}$ an N-dimensional smooth manifold, unless otherwise indicated. For the present let F be a mapping of a subset of $\mathcal{M}$ into $\mathcal{N}$.

Definition. The statement that F **is Y-differentiable with respect to X at P** means that

i. X is a coordinate system of $\mathcal{M}$ around P,

ii. Y is a coordinate system of $\mathcal{N}$ around $F(P)$,

iii. P is an interior point of $\operatorname{dom} X \cap F^{-1}(\operatorname{dom} Y)$, and

iv. there exist a linear mapping $L\colon \mathbb{R}^M \to \mathbb{R}^N$ and a function $\eta\colon \operatorname{dom} X \cap F^{-1}(\operatorname{dom} Y) \to \mathbb{R}^N$ such that if $Q \in \operatorname{dom} X \cap F^{-1}(\operatorname{dom} Y)$, then

   a. $Y(F(Q)) - Y(F(P)) = L(X(Q) - X(P)) + \eta(Q)|X(Q) - X(P)|$, and

   b. $|\eta(Q)| \to 0$ as $|X(Q) - X(P)| \to 0$.

(Here the choice of norms for $\mathbb{R}^M$ and $\mathbb{R}^N$ is immaterial, inasmuch as all norms on $\mathbb{R}^K$ are uniformly equivalent to one another.)

Theorem 1. If F is Y-differentiable with respect to X at P, then the linear mapping L and the function η are uniquely determined by the conditions (iv.a) and (iv.b).

Theorem 2. If F is the identity mapping of $\mathcal{M}$ onto itself, P is a point of $\mathcal{M}$, and each of X and X' is a coordinate system of $\mathcal{M}$ around P, then F is X'-differentiable with respect to X at P; if $X' = X$, then $L = I^M$, the identity mapping of $\mathbb{R}^M$ onto itself.

If F is Y-differentiable with respect to X at P, then the linear mapping L is called the **Y-differential of F with respect to X at P**. The function whose domain is the set of all such points P and which assigns to each such point P the Y-differential of F with respect to X at P is called the **Y-differential of F with respect to X** and is denoted by $d(YF)/dX$. If F is the identity mapping of $\mathcal{M}$ onto itself, and X' is a coordinate system of $\mathcal{M}$ whose domain intersects that of X, then $d(X'F)/dX$ is called the **differential of X' with respect to X** and is denoted by dX'/dX.

Theorem 3. If F is Y-differentiable with respect to X at P, then each of the partial derivatives

$$\frac{\partial(y^n F)}{\partial x^m}(P) \quad (:= \partial_m (YFX^{-1})^n (X(P))) \tag{1}$$

exists, and the matrix that represents $(d(YF)/dX)(P)$ with respect to the standard bases in $\mathbb{R}^M$ and $\mathbb{R}^N$ is given by

$$\left[\frac{d(YF)}{dX}(P)\right] = {}^{m}\!\!\downarrow\!\! \overset{n\longrightarrow}{\left[\frac{\partial(y^n F)}{\partial x^m}(P)\right]} := \begin{bmatrix} \frac{\partial(y^1 F)}{\partial x^1}(P) & \cdots & \frac{\partial(y^N F)}{\partial x^1}(P) \\ \vdots & \ddots & \vdots \\ \frac{\partial(y^1 F)}{\partial x^M}(P) & \cdots & \frac{\partial(y^N F)}{\partial x^M}(P) \end{bmatrix}. \tag{2}$$

Thus if $u \in \mathbb{R}^M$, then

$$\left[\frac{d(YF)}{dX}(P)u\right] = [u]\left[\frac{d(YF)}{dX}(P)\right] = \overset{m\longrightarrow}{[u^m]}\left[\overset{n\longrightarrow}{\frac{\partial(y^nF)}{\partial x^m}}(P)\right]\overset{m}{\downarrow} = \left[\overset{n\longrightarrow}{u^m\frac{\partial(y^nF)}{\partial x^m}}(P)\right]. \tag{3}$$

Corollary. If each of X and X' is a coordinate system of $\mathcal{M}$ around P, then

$$\left[\frac{dX'}{dX}(P)\right] = \overset{m}{\downarrow}\left[\overset{m'\longrightarrow}{\frac{\partial x^{m'}}{\partial x^m}}(P)\right] := \begin{bmatrix} \frac{\partial x^{1'}}{\partial x^1}(P) & \cdots & \frac{\partial x^{M'}}{\partial x^1}(P) \\ \vdots & \ddots & \vdots \\ \frac{\partial x^{1'}}{\partial x^M}(P) & \cdots & \frac{\partial x^{M'}}{\partial x^M}(P) \end{bmatrix}, \tag{4}$$

where $(\partial x^{m'}/\partial x^m)(P) := \partial_m(X'FX^{-1})^{m'}(X(P))$, with F the identity mapping of $\mathcal{M}$ onto itself. In particular,

$$\left[\frac{dX}{dX}(P)\right] = \overset{m}{\downarrow}\left[\overset{n\longrightarrow}{\frac{\partial x^n}{\partial x^m}}(P)\right] = [\delta_m{}^n], \tag{5}$$

where $\delta_m{}^n := 0$ if $m \neq n$, 1 if $m = n$.

Theorem 4. If $P \in \operatorname{dom} F$, X is a coordinate system of $\mathcal{M}$ around P, Y is a coordinate system of $\mathcal{N}$ around $F(P)$, and YF is C^1 with respect to X, then F is Y-differentiable with respect to X at P.

Theorem 5. If $P \in \operatorname{dom} F$, each of X and X' is a coordinate system of $\mathcal{M}$ around P, and each of Y and Y' is a coordinate system of $\mathcal{N}$ around $F(P)$, then F is Y'-differentiable with respect to X' at P if and only if F is Y-differentiable with respect to X at P, and

$$\frac{d(Y'F)}{dX'}(P) = \frac{dY'}{dY}(F(P))\frac{d(YF)}{dX}(P)\frac{dX}{dX'}(P), \tag{6}$$

which in terms of the entries in the representing matrices is equivalent to

$$\frac{\partial(y^{n'}F)}{\partial x^{m'}}(P) = \frac{\partial x^m}{\partial x^{m'}}(P)\frac{\partial(y^nF)}{\partial x^m}(P)\frac{\partial y^{n'}}{\partial y^n}(F(P)). \tag{7}$$

Corollary. (Coordinate Chain Rule) If each of X, X', and X'' is a coordinate system of $\mathcal{M}$ around P, then

$$\frac{dX''}{dX'}(P)\frac{dX'}{dX}(P) = \frac{dX''}{dX}(P); \tag{8}$$

equivalently,

$$\frac{\partial x^{m'}}{\partial x^m}(P)\frac{\partial x^{m''}}{\partial x^{m'}}(P) = \frac{\partial x^{m''}}{\partial x^m}(P). \tag{9}$$

In particular

$$\frac{dX}{dX'}(P)\frac{dX'}{dX}(P) = \frac{dX}{dX}(P) = I^M; \tag{10}$$

equivalently,

$$\frac{\partial x^{m'}}{\partial x^m}(P)\frac{\partial x^n}{\partial x^{m'}}(P) = \frac{\partial x^n}{\partial x^m}(P) = \qquad (11)$$

Definition. The statement that F **is differentiable at** P means that F is Y-differentiable with respect to X at P for every coordinate system X of $\mathcal{M}$ around P and every coordinate system Y of $\mathcal{N}$ around $F(P)$; that F **is differentiable** means that if $P \in \operatorname{dom} F$, then F is differentiable at P.

According to this definition, differentiability of F is a property not of F alone, but of F and the maximal atlases of $\mathcal{M}$ and $\mathcal{N}$. If, for example, $\mathcal{N}'$ is a smooth manifold whose points are the same as those of $\mathcal{N}$, but whose maximal atlas is distinct from $\mathcal{N}$'s, then it might well be that F is differentiable as a mapping into $\mathcal{N}$, but not as a mapping into $\mathcal{N}'$. Strictly, then, each of the phrases "F is differentiable at P" and "F is differentiable" must be understood as followed by the qualifying phrase "as a mapping from the manifold $\mathcal{M}$ to the manifold $\mathcal{N}$". The same qualification must be applied to all other phrases describing a kind of differentiability or of continuity of F, such as "F is C^K" (to be defined), and "F is continuous at P" (to be defined).

Theorem 6. F is differentiable at P if and only if there exist a coordinate system X of $\mathcal{M}$ and a coordinate system Y of $\mathcal{N}$ such that F is Y-differentiable with respect to X at P.

Definition. F **is** C^K (C^∞, **analytic**) means that YF is C^K (C^∞, analytic) with respect to X for each coordinate system X of $\mathcal{M}$ and each coordinate system Y of $\mathcal{N}$ such that $\operatorname{dom} X \cap F^{-1}(\operatorname{dom} Y) \neq \varnothing$. (If $K = 0$, this definition applies also to the case where either $\mathcal{M}$ or $\mathcal{N}$ is C^0, as well as to the case where each is smooth.)

Theorem 7. When each of $\mathcal{M}$ and $\mathcal{N}$ is C^K (C^∞, analytic) or smoother, then F is C^K (C^∞, analytic) if and only if there exist a subatlas of $\mathcal{M}$'s maximal atlas and a subatlas of $\mathcal{N}$'s maximal atlas such that YF is C^K (C^∞, analytic) with respect to X whenever X is in the former subatlas, Y is in the latter subatlas, and $\operatorname{dom} X \cap F^{-1}(\operatorname{dom} Y) \neq \varnothing$.

Theorem 8. Each of these implications holds: F is analytic $\implies$ F is C^K for $K = 1, 2, \ldots \implies$ F is C^L for $L = 1, \ldots, K \implies$ F is differentiable $\implies$ F is C^0.

Definition. The statement that F **is Y-continuous with respect to X at** P means that

i. $P \in \operatorname{dom} F$,

ii. X is a coordinate system of $\mathcal{M}$ around P,

iii. Y is a coordinate system of $\mathcal{N}$ around $F(P)$, and

iv. if $Q \in \operatorname{dom} X \cap F^{-1}(\operatorname{dom} Y)$, then $|(YF)(Q) - (YF)(P)| \to 0$ as $|X(Q) - X(P)| \to 0$. That F **is continuous at** P means that F is Y-continuous with respect to X at P for every coordinate system X of $\mathcal{M}$ around P and every coordinate system Y of $\mathcal{N}$ around $F(P)$. That F **is continuous** means that if $P \in \operatorname{dom} F$, then F is continuous at P.

This definition and the next three propositions are sensible if either $\mathcal{M}$ or $\mathcal{N}$ is merely C^0, and the propositions are theorems in those cases as well as when $\mathcal{M}$ and $\mathcal{N}$ are both smooth.

Theorem 9. F is continuous at P if and only if there exist a coordinate system X of $\mathcal{M}$ and a coordinate system Y of $\mathcal{N}$ such that F is Y-continuous with respect to X at P.

Theorem 10. If $\operatorname{dom} F$ is open, then F is continuous if and only if F is C^0. F is continuous at P (resp., continuous) if and only if F is continuous at P (resp., continuous) as a mapping from the topological space $\mathcal{M}$ to the topological space $\mathcal{N}$.

Theorem 11. F is continuous at P if and only if F is continuous at P as a mapping of Σ into Σ', where Σ is $\mathcal{M}$ with the topology that has as a base the set of all coordinate patches of $\mathcal{M}$ and Σ' is $\mathcal{N}$ with the topology that has as a base the set of all coordinate patches of $\mathcal{N}$. F is continuous if and only if F is continuous as a mapping of Σ into Σ'.

Theorem 12. If F is differentiable at P, then F is continuous at P. If F is differentiable, then F is continuous.

That F is a **diffeomorphism of** $\mathcal{M}$ **onto** $\mathcal{N}$ means that F is a one-to-one mapping of $\mathcal{M}$ onto $\mathcal{N}$ and both F and F^{-1} are differentiable. If F is such a diffeomorphism, then $\mathcal{M}$ and $\mathcal{N}$ are said to be **diffeomorphic** and to be **differentiably equivalent** (to one another). Differentiable equivalence is an equivalence relation (reflexive, symmetric, and transitive) on the set of all differentiable manifolds.

Theorem 13. If $\mathcal{M}$ and $\mathcal{N}$ are diffeomorphic, then Σ and Σ' are homeomorphic, where Σ is $\mathcal{M}$ with the topology that has as a base the set of all coordinate patches of $\mathcal{M}$ and Σ' is $\mathcal{N}$ with the topology that has as a base the set of all coordinate patches of $\mathcal{N}$.

IV. DIFFERENTIABLE PATHS IN AND SCALAR FIELDS ON A SMOOTH MANIFOLD

By a **path in** $\mathcal{M}$ is meant a mapping $p\colon I \to \mathcal{M}$, where I is a nondegenerate interval of $\mathbb{R}$. By a **scalar field on** $\mathcal{M}$ is meant a mapping $f\colon U \to \mathbb{R}$, where U is a nonempty subset of $\mathcal{M}$. Let p be a path in $\mathcal{M}$, and f a scalar field on $\mathcal{M}$.

Definition. The statement that p **is X-differentiable at** t means that

i. t is a number in $\operatorname{dom} p$,

ii. X is a coordinate system of $\mathcal{M}$ around $p(t)$, and

iii. there exist a linear mapping $L\colon \mathbb{R} \to \mathbb{R}^M$ and a function $\eta\colon p^{-1}(\operatorname{dom} X) \to \mathbb{R}^M$ such that if $\bar{t} \in p^{-1}(\operatorname{dom} X)$, then

   a. $X(p(\bar{t})) - X(p(t)) = L(\bar{t} - t) + \eta(\bar{t})|\bar{t} - t)|$, and

   b. $|\eta(\bar{t})| \to 0$ as $|\bar{t} - t| \to 0$.

Theorem 1. If p is X-differentiable at t, then the linear mapping L and the function η are uniquely determined by the conditions (iii.a) and (iii.b).

If p is X-differentiable at t, then the linear mapping L is called the **X-differential of** p **at** t. The function whose domain is the set of all such numbers t and which assigns to each such number t the X-differential of p at t is called the **X-differential of** p and is denoted by $d(Xp)$. The function $d(Xp)(\cdot)(1)$ is called the **X-derivative of** p, and is denoted by $D(Xp)$; its value at t, $D(Xp)(t)$ (a vector in $\mathbb{R}^M$), is called the **X-derivative of** p **at** t.

Theorem 2. The path p is X-differentiable at t if and only if each of the functions $x^m p$ is differentiable at t, in which event $D(Xp)(t) = [\![\, (x^m p)^{\cdot}(t) \,]\!] = [\![\, (x^1 p)^{\cdot}(t), \cdots, (x^M p)^{\cdot}(t) \,]\!]$, and the row matrix that represents $d(Xp)(t)$ with respect to the standard bases of $\mathbb{R}$ and $\mathbb{R}^M$, as well as the vector $D(Xp)(t)$ with respect to the standard basis of $\mathbb{R}^M$, is given by

$$[D(Xp)(t)] = [d(Xp)(t)] = [\overset{m\longrightarrow}{(x^m p)^{\cdot}(t)}]. \tag{12}$$

Definition. The statement that f **is X-differentiable at** P means that

i. X is a coordinate system of $\mathcal{M}$ around P,

ii. P is an interior point of $\operatorname{dom} X \cap \operatorname{dom} f$, and

iii. there exist a linear mapping $L\colon \mathbb{R}^M \to \mathbb{R}$ and a function $\eta\colon \operatorname{dom} X \cap \operatorname{dom} f \to \mathbb{R}$ such that if $Q \in \operatorname{dom} X \cap \operatorname{dom} f$, then

   a. $f(Q) - f(P) = L(X(Q) - X(P)) + \eta(Q)|X(Q) - X(P)|$, and

   b. $|\eta(Q)| \to 0$ as $|X(Q) - X(P)| \to 0$.

Theorem 3. If f is X-differentiable at P, then the linear mapping L and the function η are uniquely determined by the conditions (iii.a) and (iii.b).

If f is X-differentiable at P, then the linear mapping L is called the **X-differential of f at P**. The function whose domain is the set of all such points P and which assigns to each such point P the X-differential of f at P is called the **X-differential of f** and is denoted by df/dX.

Theorem 4. If f is X-differentiable at P, then each of the partial derivatives

$$\frac{\partial f}{\partial x^m}(P) \quad (:= \partial_m(fX^{-1})(X(P))) \tag{13}$$

exists, and the column matrix that represents $(df/dX)(P)$ with respect to the standard bases in $\mathbb{R}^M$ and $\mathbb{R}$ is given by

$$\left[\frac{df}{dX}(P)\right] = \overset{m}{\downarrow}\left[\frac{\partial f}{\partial x^m}(P)\right]. \tag{14}$$

Thus if $u \in \mathbb{R}^M$, then

$$\left[\frac{df}{dX}(P)u\right] = [u]\left[\frac{df}{dX}(P)\right] = \overset{m\longrightarrow}{[u^m]}\left[\frac{\partial f}{\partial x^m}(P)\right]\overset{m}{\downarrow} = \left[u^m\frac{\partial f}{\partial x^m}(P)\right]. \tag{15}$$

Definition. The statement that p **is differentiable at** t means that p is X-differentiable at t for every coordinate system X of $\mathcal{M}$ around $p(t)$; that p is differentiable means that if $t \in \operatorname{dom} p$, then p is differentiable at t. The statement that f **is differentiable at** P means that f is X-differentiable at P for every coordinate system X of $\mathcal{M}$ around P; that f is differentiable means that if $P \in \operatorname{dom} f$, then f is differentiable at P.

Theorem 5. The path p is differentiable at t if and only if there is a coordinate system X of $\mathcal{M}$ such that p is X-differentiable at t. The scalar field f is differentiable at P if and only if there is a coordinate system X of $\mathcal{M}$ such that f is X-differentiable at P.

Theorem 6. (Contravariant Chain Rule) If p is differentiable at t, and each of X and X' is a coordinate system of $\mathcal{M}$ around $p(t)$, then

$$D(X'p)(t) = \frac{dX'}{dX}(p(t))D(Xp)(t); \tag{16}$$

equivalently,

$$D(x^{n'}p)(t) = D(x^n p)(t)\frac{\partial x^{n'}}{\partial x^n}(p(t)). \tag{17}$$

Theorem 7. (Covariant Chain Rule) If f is differentiable at P, and each of X and X' is a coordinate system of $\mathcal{M}$ around P, then

$$\frac{df}{dX'}(P) = \frac{df}{dX}(P)\frac{dX}{dX'}(P); \tag{18}$$

equivalently,

$$\frac{\partial f}{\partial x^{m'}}(P) = \frac{\partial x^m}{\partial x^{m'}}(P)\frac{\partial f}{\partial x^m}(P). \tag{19}$$

For p and f there are definitions and theorems analogous to those at the end of Chapter III having to do with F's being C^K, C^∞, analytic, or continuous.

V. DUAL VECTOR SPACES

Let U be a finite-dimensional vector space over $\mathbb{R}$, and let $M = \dim U$. By a **linear functional on** U is meant a (homogeneous) linear mapping of U into $\mathbb{R}$. With respect to the usual addition of and multiplication by real numbers of real-valued functions the set of all linear functionals on U is itself a vector space over $\mathbb{R}$; it is called the **dual space of** U and is denoted by U^*. In case $U = \mathbb{R}^M$ this dual space is also denoted by $\mathbb{R}_M$. It can be identified with $\mathbf{R}_M$ (the vector space of all M-rowed column matrices over $\mathbb{R}$, denoted alternatively by $\mathbf{R}^1_M$) by identifying the linear functional u^* in $\mathbb{R}_M$ with the column matrix $[u^*_m]$ in $\mathbf{R}_M$ if and only if $u^*_m = u^* e_m$, where $\{e_m\}$ is the standard basis for $\mathbb{R}^M$. That the numbers $u^* e_m$ determine u^* is a consequence of the linearity of u^*: if $u = u^m e_m$, then $u^* u = u^*(u^m e_m) = u^m(u^* e_m) = u^m u^*_m$ (in terms of matrix multiplication $[u^* u] = [u^m u^*_m] = [u^m][u^*_m] = [u][u^*]$, where $[u] = [u^m]$, the row matrix that represents u in the standard basis, and $[u^*] = [u^*_m] = [u^* e_m]$, the column matrix identified with u^*).

More generally, if $\{a_m\}$ is a basis for U, then to know an element u^* of U^* it is sufficient to know the numbers $u^* a_m$, for if $u = u^m a_m$, then $u^* u = u^m u^*_m$, where $u^*_m = u^* a_m$. In particular, for $n = 1, \ldots, M$ an element a^n of U^* is determined by the stipulation that $a^n a_m = \delta_m{}^n$, which is equivalent to the stipulation that if $u = u^m a_m$, then $a^n u = u^n$. The set $\{a^m\}$ is a basis for U^*, called the **basis dual to** $\{a_m\}$. From this it follows that $\dim U^* = M = \dim U$. If $u^* \in U^*$, and $u = u^m a_m$, then $u^* u = u^*(u^m a_m) = u^m(u^* a_m) = (a^m u)u^*_m = (u^*_m a^m)u$, so $u^* = u^*_m a^m = (u^* a_m)a^m$.

An isomorphism i of U onto U^* is determined by the formula $iu = u^m \delta_{mn} a^n$ if $u = u^m a_m$. Equivalent requirements that determine the same isomorphism are that $iuv = u^m \delta_{mn} v^n$ if $u = u^m a_m$ and $v = v^n a_n$, that $ia_m = \delta_{mn} a^n$, and that $ia_m a_n = \delta_{mn}$. This isomorphism, which maps the (ordered) basis $\{a_m\}$ onto the (ordered) dual basis $\{a^m\}$ (that is, $ia_m = a^m$) can change with a change of the basis $\{a_m\}$, consequently does not identify vectors in the dual space U^* with vectors in U in a "basis-free" manner. Likewise, the isomorphism j of U^* onto U^{**} (the dual of the dual of U) determined with regard to the basis $\{a^m\}$ of U^* in the same way that i is determined with regard to the basis $\{a_m\}$ of U is not independent of the choice of $\{a^m\}$, hence not independent of the choice of $\{a_m\}$. The isomorphism ji of U onto U^{**}, however, *is* basis-free. It assigns to each vector u in U the linear functional $l_u \colon U^* \to \mathbb{R}$ defined by the formula $l_u u^* = u^* u$ if $u^* \in U^*$, which makes no reference to any basis. This isomorphism is called the **natural** (also, the **canonical**) **isomorphism of** U **onto** U^{**} and is used to identify the vectors of U^{**} with those of U, whereupon j becomes identified with i^{-1}, inasmuch as $(ji)u$ is identified with u. When the basis vector a_m is thus identified with the element $(ji)a_m$, one has that $a_m u^* = u^* a_m = u^*_m$, in particular that $a_m a^n = a^n a_m = \delta_m{}^n$, so that $\{a_m\}$ is identified with the dual basis of its own dual basis. By inductive extension each of U^{**}, U^{****}, and so on is identified with U. Similarly, each of U^{***}, U^{*****}, and so on is identified with U^*.

If $U = \mathbb{R}^M$, so that $U^* = \mathbb{R}_M$, then in the same way that U^* is identified with $\mathbf{R}_M$, the second dual U^{**} can be identified with $\mathbf{R}^M$ (the vector space of all M-columned row matrices over $\mathbb{R}$, denoted alternatively by $\mathbf{R}^M_1$). Specifically, if $\{e^m\}$ is the basis of $\mathbb{R}_M$ dual to the standard basis $\{e_m\}$ of $\mathbb{R}^M$, and $l \in (\mathbb{R}_M)^*$, then l is identified with the row matrix $[l^m]$, where $l^m = le^m$. Combined with this identification, the identification of vectors in U with vectors in U^{**} becomes simply the correspondence between vectors in $\mathbb{R}^M$ and the row matrices that represent them in the standard basis.

If V is a finite-dimensional vector space over $\mathbb{R}$ (of dimensionality N, say), and L is a (homogeneous) linear mapping of U into V, then there is a (homogeneous) linear mapping $L^*\colon V^* \to U^*$ defined as follows: if $v^* \in V^*$, then, for each vector u in U, $(L^*v^*)u := v^*(Lu)$, which is to say, $L^*v^* := v^*L$. This mapping L^* is called the **dual of** L. If $\{a_m\}$ is a basis for U, $\{a^m\}$ is the basis for U^* dual to $\{a_m\}$, $\{b_n\}$ is a basis for V, and $\{b^n\}$ is the basis for V^* dual to $\{b_n\}$, then $La_m = L_m{}^n b_n$, where $L_m{}^n = b^n La_m$, and $L^*b^n = L^{*n}{}_m a^m$, where $L^{*n}{}_m = a_m L^* b^n = (L^*b^n)a_m = b^n(La_m) = L_m{}^n$. Thus if $u = u^m a_m$, then $Lu = (u^m L_m{}^n)b_n$, and if $v^* = v^*_n b^n$, then $L^*v^* = (v^*_n L^{*n}{}_m)a^m = (L_m{}^n v^*_n)a^m$.

When U^{**} and V^{**} are canonically identified with U and V, L^{**} becomes identified with a (homogeneous) linear mapping of U into V. If $u \in U$ and $v^* \in V^*$, then, under these identifications, $(L^{**}u)v^* := u(L^*v^*) = (L^*v^*)u := v^*(Lu) = (Lu)v^*$, so $L^{**} = L$.

Because $Lu = (u^m L_m{}^n)b_n$, the row matrix $[(Lu)^n]$ that represents Lu in the basis $\{b_n\}$ is $[u^m L_m{}^n]$, which factors into $[u^m][L_m{}^n]$, where $[u^m]$ is the row matrix that represents u in the basis $\{a_m\}$, and, consequently, $[L_m{}^n]$ is the matrix in $\mathbf{R}^N_M$ (the vector space of all M-rowed, N-columned matrices over $\mathbb{R}$) that represents L in the bases $\{a_m\}$ and $\{b_n\}$; thus $[(Lu)^n] = [u^m][L_m{}^n]$. Similarly, $[(L^*v^*)_m] = [L_m{}^n][v^*_n]$ where $[(L^*v^*)_m]$ is the column matrix that represents L^*v^* in the dual basis $\{a^m\}$, and $[v^*_n]$ is the column matrix that represents v^* in the dual basis $\{b^n\}$. Thus L^* is represented in the dual bases $\{a^m\}$ and $\{b^n\}$ by the same matrix, $[L_m{}^n]$, that represents L in the bases $\{a_m\}$ and $\{b_n\}$. For L^*, however, this matrix multiplies from the left, whereas for L it multiplies from the right.

VI. TANGENT VECTORS AND TANGENT COVECTORS

For each point P of the smooth manifold $\mathcal{M}$ let A_P denote the set of all coordinate systems of $\mathcal{M}$ around P.

Classical Definition of Tangent Vector. By a **contravariant tangent vector of** $\mathcal{M}$ (**tangent vector** or simply **vector** for short) **at** P is meant a mapping $u\colon A_P \to \mathbb{R}^M$ such that if $X \in A_P$ and $X' \in A_P$, then

$$u(X') = \frac{dX'}{dX}(P)u(X), \tag{20}$$

which in component form reads

$$u^{m'} = u^m \frac{\partial x^{m'}}{\partial x^m}(P), \tag{21}$$

provided $[u^m] = [u(X)]$ in $\mathbf{R}^M$ and $[u^{m'}] = [u(X')]$ in $\mathbf{R}^M$.

The coordinate chain rule ensures the consistency of this definition, for if also $X'' \in A_P$, then

$$u(X'') = \frac{dX''}{dX}(P)u(X) = \frac{dX''}{dX'}(P)\frac{dX'}{dX}(P)u(X) = \frac{dX''}{dX'}(P)u(X'). \tag{22}$$

Each tangent vector at P is determined by its representation in (i. e., its value at) a single coordinate system. This permits the following assertion.

Theorem 1. Under the usual addition and multiplication by scalars of mappings into a vector space the set of all tangent vectors at P is a real vector space. If X is any coordinate system in A_P, then an isomorphism of this space onto $\mathbb{R}^M$ is created by assigning to each tangent vector at P its representation in X.

Definition. The vector space of all tangent vectors at P is called the **tangent space (of** $\mathcal{M}$**) at** P and is denoted by $T^P(\mathcal{M})$ and by T^P (if $\mathcal{M}$ is implicit from the context). The isomorphism described in Theorem 1 is denoted by $\hat{X}^P$; thus if $u \in T^P$, then $\hat{X}^P u = u(X)$, so $[\hat{X}^P u] = [u^m]$ in $\mathbf{R}^M$.

Theorem and Definition. If p is a path in $\mathcal{M}$ that is differentiable at t, then there is a unique tangent vector at $p(t)$ whose representation in each coordinate system X around $p(t)$ is $D(Xp)(t)$. This tangent vector is called the **velocity of** p **at** t as well as the **derivative of** p **at** t. The function on $\{\, t \mid p \text{ is differentiable at } t \,\}$ whose value at each "time" t is the velocity of p at t is called the **velocity of** p and the **derivative of** p, and is denoted by $\dot{p}$, by p', and by Dp. If $p^m := x^m p$, then $(\dot{p}(\cdot)(X))^m = (D(Xp))^m = D(x^m p) = Dp^m$; this function is denoted by $\dot{p}^m$. Thus $\dot{p}^m := Dp^m = (p^m)' = (p^m)^{\cdot}$.

Classical Definition of Tangent Covector. By a **covariant tangent vector of** $\mathcal{M}$ (**tangent covector** or simply **covector** for short) **at** P is meant a mapping $v : A_P \to \mathbb{R}_M$ (the dual space of $\mathbb{R}^M$) such that if $X \in A_P$ and $X' \in A_P$, then

$$v(X') = v(X)\frac{dX}{dX'}(P), \tag{23}$$

which in component form reads

$$v_{m'} = \frac{\partial x^m}{\partial x^{m'}}(P) v_m, \tag{24}$$

where $[v_m] = [v(X)]$ in $\mathbf{R}_M$ and $[v_{m'}] = [v(X')]$ in $\mathbf{R}_M$.

As before, the coordinate chain rule ensures the consistency of the definition. Also as before, each tangent covector is determined by its representation in a single coordinate system, hence the following theorem holds.

Theorem 2. Under the usual addition and multiplication by scalars of mappings into a vector space the set of all tangent covectors at P is a real vector space. If X is any coordinate system around P, then an isomorphism of this space onto $\mathbb{R}_M$ is created by assigning to each tangent covector at P its representation in X.

Definition. The vector space of all tangent covectors at P is called the **cotangent space (of** $\mathcal{M}$**) at** P and is denoted by $T_P(\mathcal{M})$ and by T_P (if $\mathcal{M}$ is understood from the context). The isomorphism described in Theorem 2 is denoted by $\hat{X}_P$; thus if $v \in T_P$, then $\hat{X}_P v = v(X)$, so $[\hat{X}_P v] = [v_m]$ in $\mathbf{R}_M$.

Theorem and Definition. If f is a scalar field on a subset of $\mathcal{M}$, and f is differentiable at P, then there is a unique tangent covector at P whose representation in each coordinate system X around P is $(df/dX)(P)$. This tangent covector is called the **cogradient of** f **at** P as well as the **differential of** f **at** P. The function on $\{\, P \mid f$ is differentiable at $P \,\}$ whose value at each point P is the cogradient of f at P is called the **cogradient of** f and the **differential of** f, and is denoted by df. From the definition of $\partial f/\partial x^m$ it follows that $(df(\cdot)(X))_m = \partial f/\partial x^m$.

Classical Definition of (Tangent) Scalar. By a (**tangent**) **scalar of** $\mathcal{M}$ (**scalar** for short) **at** P is meant a mapping $\phi\colon A_P \to \mathbb{R}$ that is constant, in other words is such that if $X \in A_P$ and $X' \in A_P$, then $\phi(X') = \phi(X)$.

Theorem 3. If u is a tangent vector at P, and v is a tangent covector at P, then the relation $\phi(X) = v(X)u(X)$ if $X \in A_P$ determines a tangent scalar ϕ at P.

The tangent scalar ϕ at P is called the **contraction of** v **with** u (also the **inner product of** v **with** u) and is denoted by vu. (Note that, according to Theorem 3, $(vu)(X) = u^m v_m$ for each coordinate system X in A_P.)

Let $\mathcal{D}^1(P)$ denote the set of all scalar fields on $\mathcal{M}$ that are differentiable at P. Then with respect to the usual addition and multiplication by scalars $\mathcal{D}^1(P)$ is a vector space over $\mathbb{R}$. Further, if each of f and g is in $\mathcal{D}^1(P)$, then so is fg. Moreover, if $f \in \mathcal{D}^1(P)$, $f(P) = 0$, g is a scalar field on $\mathcal{M}$, P is an interior point of $\operatorname{dom} g$, and g is continuous at P, then $fg \in \mathcal{D}^1(P)$. These follow readily from the definition of fg and the identity

$$f(Q)g(Q) - f(P)g(P) = (f(Q) - f(P))g(P) + f(P)(g(Q) - g(P)) + (f(Q) - f(P))(g(Q) - g(P)). \tag{25}$$

Modern Definition of Tangent Vector. By a **tangent vector of** $\mathcal{M}$ **at** P is meant a mapping $L\colon \mathcal{D}^1(P) \to \mathbb{R}$ that is linear and treats products in the following way:

i. $L(fg) = (Lf)g(P) + f(P)(Lg)$ if $f \in \mathcal{D}^1(P)$ and $g \in \mathcal{D}^1(P)$;

ii. $L(fg) = (Lf)g(P)$ if $f \in \mathcal{D}^1(P)$, $f(P) = 0$, g is a scalar field on $\mathcal{M}$, P is an interior point of $\operatorname{dom} g$, and g is continuous at P.

Examples of tangent vectors at P in the modern sense are afforded by the operators $(\partial/\partial x^m)(P)$ on $\mathcal{D}^1(P)$ associated with the coordinate system X around P. In fact these constitute a basis for the tangent space at P according to the following theorem.

Theorem 4. Under the usual addition and multiplication by real numbers of real-valued mappings the set of all tangent vectors of $\mathcal{M}$ at P in the modern sense is a real vector space. If X is any coordinate system around P, then the set $\{(\partial/\partial x^m)(P)\}$ of operators on $\mathcal{D}^1(P)$ is a basis for this space, and an isomorphism is established, *via* $\mathbb{R}^M$ and $\mathbf{R}^M$, between the tangent space at P in the classical sense and the tangent space at P in the modern sense by the scheme

$$\begin{array}{lccccccc} & \text{Classical} & & \text{Cartesian} & & \text{Matrix} & & \text{Modern} \\ \text{Spaces:} & T^P & \longleftrightarrow & \mathbb{R}^M & \longleftrightarrow & \mathbf{R}^M & \longleftrightarrow & T^P \\ \text{Vectors:} & u & \longleftrightarrow & u(X) & \longleftrightarrow & [u^m] & \longleftrightarrow & u^m(\partial/\partial x^m)(P)\,. \end{array} \tag{26}$$

This isomorphism is the same for all choices of X.

Henceforth let the tangent spaces at P in the two senses be identified through the isomorphism described in this theorem. Then for each differentiable path p in $\mathcal{M}$ one has that $\dot{p} = \dot{p}^m(\partial/\partial x^m)(p)$ on $p^{-1}(\operatorname{dom} X)$.

Modern Definition of Tangent Covector. By a **tangent covector of** $\mathcal{M}$ **at** P is meant an element of the dual space of the tangent space at P.

According to this definition the cotangent space at P in the modern sense is simply the dual space of T^P. If X is a coordinate system of $\mathcal{M}$ around P, then the coordinate differentials $dx^m(P)$, defined as linear functionals on T^P by

$$dx^m(P)\left(u^n \frac{\partial}{\partial x^n}(P)\right) = u^m, \tag{27}$$

are tangent covectors in the modern sense. In fact $\{dx^m(P)\}$ is the basis dual to $\{(\partial/\partial x^m)(P)\}$, for $dx^m(P)(\partial/\partial x^n)(P) = \delta_n{}^m$.

Theorem 5. If X is any coordinate system of $\mathcal{M}$ around P, then an isomorphism is established, *via* $\mathbb{R}_M$ and $\mathbf{R}_M$, between the cotangent space at P in the classical sense and the cotangent space at P in the modern sense by the scheme

$$\begin{array}{lccccccc} & \text{Classical} & & \text{Cartesian} & & \text{Matrix} & & \text{Modern} \\ \text{Spaces:} & T_P & \longleftrightarrow & \mathbb{R}_M & \longleftrightarrow & \mathbf{R}_M & \longleftrightarrow & T_P \\ \text{Covectors:} & v & \longleftrightarrow & v(X) & \longleftrightarrow & [v_m] & \longleftrightarrow & v_m dx^m(P)\,. \end{array} \tag{28}$$

This isomorphism is the same for all choices of X; it is the dual of the isomorphism of Theorem 4, provided the classical cotangent space at P is identified with the dual of the classical tangent space at P in a way suggested by Theorem 3.

The two cotangent spaces at P will henceforth be identified by means of this isomorphism. It follows that if f is a differentiable scalar field on $\mathcal{M}$, then $df = (\partial f/\partial x^m)dx^m$ on $\operatorname{dom} X \cap \operatorname{dom} f$.

Definition. The basis $\{(\partial/\partial x^m)(P)\}$ and its dual basis $\{dx^m(P)\}$ are called, respectively, the (**coordinate**) **frame** (or **basis**) **at** P **determined by** X and the (**coordinate**) **coframe** (or **cobasis**) **at** P **determined by** X.

Finally, in the spirit of these modern definitions a **tangent scalar of** $\mathcal{M}$ **at** P is simply a real number (the constant value of the corresponding classical tangent scalar), and if $u = u^m(\partial/\partial x^m)(P)$ and $v = v_m dx^m(P)$, the contraction of v with u is the scalar (real number) $u^m v_m$, because

$$vu = (v_m dx^m(P))\left(u^n \frac{\partial}{\partial x^n}(P)\right) = u^n v_m \left(dx^m(P)\frac{\partial}{\partial x^n}(P)\right) = u^n \delta_n{}^m v_m = u^m v_m. \tag{29}$$

In particular, $df(P)u = u^m(\partial f/\partial x^m)(P) = uf$ if $f \in \mathcal{D}^1(P)$. This number is called the **derivative of** f **along** u, as well as the u-**derivative of** f, and is denoted by $D_u f$; thus, $D_u f = uf = df(P)u$.

Theorem and Definition. If $U \subset \mathcal{M}$ and $P \in U$, then

$$\{u \in T^P \mid D_u f = 0 \text{ if } f \in \mathcal{D}^1(P) \text{ and } f|_U \text{ is constant}\} \tag{30}$$

is a subspace of T^P. It is called the **subspace of** T^P **tangent to** U and is denoted by TU^P. Its vectors are said to be **tangent to** U **at** P.

Theorem and Definition. If $U \subset \mathcal{M}$ and $P \in U$, then

$$\{v \in T_P \mid vu = 0 \text{ if } u \text{ is tangent to } U \text{ at } P\} \tag{31}$$

is a subspace of T_P. It is called the **subspace of** T_P **cotangent to** U and is denoted by TU_P. Its covectors are said to be **cotangent to** U **at** P.

Theorem 6. If $U \subset \mathcal{M}$ and $P \in U$, then $\dim TU^P + \dim TU_P = M$. If $U = \{P\}$, then $TU^P = \{0\}$ and $TU_P = T_P$. If P is an interior point of U, then $TU^P = T^P$ and $TU_P = \{0\}$.

VII. TENSOR PRODUCTS

Let each of U and V be a finite-dimensional vector space over $\mathbb{R}$, and let $M = \dim U$ and $N = \dim V$. Let $\{a_m\}$ be a basis for U, and $\{b_n\}$ a basis for V, with $\{a^m\}$ and $\{b^n\}$ their dual bases in U^* and V^*. Let $\mathfrak{L}(U,V)$ denote the vector space of all (homogeneous) linear mappings of U into V. If $L \in \mathfrak{L}(U,V)$ and $u \in U$, then $Lu = u^m L_m{}^n b_n$, where $u^m = a^m u$ and $L_m{}^n = b^n L a_m$ (implying that $u = u^m a_m$ and $La_m = L_m{}^n b_n$). In terms of the following definition $u^m L_m{}^n b_n$ can be expressed as $(a^m \otimes L_m{}^n b_n)u$, hence $L = a^m \otimes L_m{}^n b_n$.

Definition. If $u^* \in U^*$ and $v \in V$, then by the **tensor product of** u^* **with** v is meant the mapping $u^* \otimes v \colon U \to V$ such that if $u \in U$, then $(u^* \otimes v)u = (u^* u)v$.

Theorem 1. If $u^*, \bar{u}^* \in U^*$, and $v, \bar{v} \in V$, and $\alpha \in \mathbb{R}$, then

i. $(u^* + \bar{u}^*) \otimes v = u^* \otimes v + \bar{u}^* \otimes v$,

ii. $u^* \otimes (v + \bar{v}) = u^* \otimes v + u^* \otimes \bar{v}$, and

iii. $(\alpha u^*) \otimes v = \alpha(u^* \otimes v) = u^* \otimes (\alpha v)$.

Theorem 2. The set S of all tensor products $u^* \otimes v$ with u^* in U^* and v in V is a spanning subset of $\mathfrak{L}(U,V)$, the subset $\{a^m \otimes b_n\}$ of S is a basis for $\mathfrak{L}(U,V)$, and $\dim \mathfrak{L}(U,V) = MN$. $S = \mathfrak{L}(U,V)$ if and only if $M = 1$ or $N = 1$. If $L \in \mathfrak{L}(U,V)$, then $L = a^m \otimes L_m{}^n b_n = a^m L_m{}^n \otimes b_n = L_m{}^n (a^m \otimes b_n)$, where $L_m{}^n = b^n L a_m$. If $u^* \in U^*$ and $v \in V$, then $b^n (u^* \otimes v) a_m = u^*_m v^n$, where $u^*_m = u^* a_m$ and $v^n = b^n v$.

Partly because of and partly in spite of this theorem the space $\mathfrak{L}(U,V)$ is called the **tensor product of** U^* **with** V and is denoted $U^* \otimes V$. In view of the previously adopted identification of the second duals U^{**} and V^{**} with U and V themselves the spaces $\mathfrak{L}(U^*,V)$, $\mathfrak{L}(U^*,V^*)$, and $\mathfrak{L}(U,V^*)$ are, respectively, $U \otimes V$, $U \otimes V^*$, and $U^* \otimes V^*$. In particular $U^* = U^* \otimes \mathbb{R}$ and $U = U \otimes \mathbb{R}$.

Theorem 3. The mapping $i \colon U^* \otimes V^* \to (U \otimes V)^*$ obtained by linear extension from the basic defining relations $i(a^m \otimes b^n)(a_k \otimes b_l) = (a^m a_k)(b^n b_l) = \delta_k{}^m \delta_l{}^n$ is an isomorphism of $U^* \otimes V^*$ onto $(U \otimes V)^*$ which maps the basis $\{a^m \otimes b^n\}$ of $U^* \otimes V^*$ onto the basis of $(U \otimes V)^*$ dual to the basis $\{a_m \otimes b_n\}$ of $U \otimes V$. This isomorphism is the same for all choices of the bases $\{a_m\}$ and $\{b_n\}$.

Henceforth the spaces $U^* \otimes V^*$ and $(U \otimes V)^*$ will be identified by means of the isomorphism of Theorem 3. Thus $(U \otimes V)^* = U^* \otimes V^*$, and, by implication, $(U^* \otimes V^*)^* = U \otimes V$, $(U^* \otimes V)^* = U \otimes V^*$, and $(U \otimes V^*)^* = U^* \otimes V$. Also, just as the bases $\{a_m \otimes b_n\}$ and $\{a^m \otimes b^n\}$ are dual to one another, so, by implication, are $\{a^m \otimes b_n\}$ and $\{a_m \otimes b^n\}$.

Let W be a finite-dimensional vector space over $\mathbb{R}$, with $\{c_p\}$ a basis for W. Then $U^* \otimes (V^* \otimes W) = \mathfrak{L}(U, \mathfrak{L}(V,W))$, whereas $(U^* \otimes V^*) \otimes W = (U \otimes V)^* \otimes W = \mathfrak{L}(\mathfrak{L}(U^*,V),W)$.

Theorem 4. For each element S of $U^* \otimes (V^* \otimes W)$ let iS be the mapping $T \colon U \otimes V \to W$ such that if $L = a_m \otimes L^{mn} b_n$, then $TL = L^{mn} S a_m b_n$. Then $i(a^m \otimes (b^n \otimes c_p)) = (a^m \otimes b^n) \otimes c_p$, and i is an isomorphism of $U^* \otimes (V^* \otimes W)$ onto $(U^* \otimes V^*) \otimes W$. Also, i is independent of the choice of the bases $\{a_m\}$, $\{b_n\}$, and $\{c_p\}$.

The spaces $U^* \otimes (V^* \otimes W)$ and $(U^* \otimes V^*) \otimes W$ will be identified from now on by agency of the isomorphism of Theorem 4. Under this identification the tensor product is associative, as a product of vectors as well as a product of spaces, for, as follows readily, if $u^* \in U^*$, $v^* \in V^*$, and $w \in W$, then $u^* \otimes (v^* \otimes w) = (u^* \otimes v^*) \otimes w$. Notations such as $u^* \otimes v^* \otimes w$ can now be used, being no longer ambiguous. Theorem 4 and these subsequent identifications apply equally well to other spaces: for example, $U \otimes (V^* \otimes W^*) = (U \otimes V^*) \otimes W^*$.

Although $\mathbb{R} \otimes V$ has been identified with $(\mathbb{R}^*)^* \otimes V$, it has not been defined as a space in its own right. It is useful to define it in the following way.

Definition and Theorem. If $r \in \mathbb{R}$ and $v \in V$, then $r \otimes v := rv$. The product $\mathbb{R} \otimes V$ is defined to be the vector space spanned by the set of all such products; consequently $\mathbb{R} \otimes V = V$.

This definition is reconcilable with the previous identification through use of the usual identification of a function f in $\mathbb{R}^*$ with the number $f(1)$ in $\mathbb{R}$.

VIII. DIFFERENTIALS OF MAPPINGS BETWEEN SMOOTH MANIFOLDS

As earlier, each of $\mathcal{M}$ and $\mathcal{N}$ is a smooth manifold, and F is a mapping of a subset of $\mathcal{M}$ into $\mathcal{N}$.

Theorem 1. Suppose that F is differentiable at P. If $h \in \mathcal{D}^1(F(P))$, then $hF \in \mathcal{D}^1(P)$. For each tangent vector u of $\mathcal{M}$ at P, let $Lu\colon \mathcal{D}^1(F(P)) \to \mathbb{R}$ be defined by $(Lu)h := u(hF)$. Then L is a linear mapping of $T^P(\mathcal{M})$, the tangent space of $\mathcal{M}$ at P, into $T^{F(P)}(\mathcal{N})$, the tangent space of $\mathcal{N}$ at $F(P)$; that is to say, $L \in T_P(\mathcal{M}) \otimes T^{F(P)}(\mathcal{N})$.

Definition. The linear mapping L is called the **differential of** F **at** P. The function whose domain is the set of all points at which F is differentiable, and which assigns to each such point P the differential of F at P, is called the **differential of** F and is denoted by dF. Thus $dF(P) \in T_P(\mathcal{M}) \otimes T^{F(P)}(\mathcal{N})$, and if $u \in T^P(\mathcal{M})$ and $h \in \mathcal{D}^1(F(P))$, then $(dF(P)u)h = u(hF)$. The function whose domain is the set of all points at which F is differentiable, and which assigns to each such point P the dual of the differential of F at P, is called the **dual of the differential of** F and is denoted by $(dF)^*$. Thus $(dF)^*(P) = dF(P)^*$, a linear mapping of $T_{F(P)}(\mathcal{N})$, the cotangent space of $\mathcal{N}$ at $F(P)$, into T_P, the cotangent space of $\mathcal{M}$ at P, that is to say, $(dF)^*(P) \in T^{F(P)}(\mathcal{N}) \otimes T_P(\mathcal{M})$ and if $v \in T_{F(P)}(\mathcal{N})$, then $(dF)^*(P)\, v := v\, dF(P) \in T_P(\mathcal{M})$.

Theorem 2. If F is differentiable at P, and U is a subset of $\mathcal{M}$ that has P in it, then $dF(P)(TU^P)$ is a subspace of $TF(U)^{F(P)}$, the subspace of $T^{F(P)}(\mathcal{N})$ tangent to $F(U)$ at $F(P)$. On the other hand $(dF)^*(P)(TF(U)_{F(P)})$ is a subspace of TU_P, the subspace of $T_P(\mathcal{M})$ cotangent to U at P.

Theorem 3. If F is constant on a set of which P is an interior point, then F is differentiable at P, and $dF(P) = 0$ (the zero element of $T_P(\mathcal{M}) \otimes T^{F(P)}(\mathcal{N})$).

Theorem 4. If F is differentiable at P, X is a coordinate system of $\mathcal{M}$ around P, and Y is a coordinate system of $\mathcal{N}$ around $F(P)$, then

$$dF(P) = dx^m(P) \otimes \frac{\partial(y^n F)}{\partial x^m}(P)\frac{\partial}{\partial y^n}(F(P)). \tag{32}$$

Let $\mathcal{O}$ be a smooth manifold and G a mapping of a subset of $\mathcal{N}$ into $\mathcal{O}$.

Theorem 5. (Chain Rule for Manifolds) If F is differentiable at P, and G is differentiable at $F(P)$, then GF is differentiable at P, and $d(GF)(P) = dG(F(P))dF(P)$.

Theorem 6. (Chain Rules for Velocities) If p is a path in $\mathcal{M}$, p is differentiable at t, F is differentiable at $p(t)$, and $q = F(p)$, then q is a path in $\mathcal{N}$ differentiable at t, and $\dot{q}(t) = dF(p(t))\dot{p}(t)$; if p is differentiable and F is differentiable, then q is differentiable, and $\dot{q} = dF(p)\dot{p}$. If p is differentiable, ϕ is a differentiable mapping of an interval of $\mathbb{R}$ into $\operatorname{dom} p$, and $q = p(\phi)$, then q is a differentiable path in $\mathcal{M}$, and $\dot{q} = \dot{p}(\phi)\dot{\phi}$.

Theorem 7. (Chain Rules for Cogradients) If g is a scalar field on $\mathcal{N}$, F is differentiable at P, g is differentiable at $F(P)$, and $h = gF$, then h is a scalar field on $\mathcal{M}$ differentiable at P, and

$dh(P) = dg(F(P))dF(P)$; if F is differentiable and g is differentiable, then h is differentiable, and $dh = dg(F)dF$. If g is differentiable, ϕ is a differentiable mapping of a subset of $\mathbb{R}$ into $\mathbb{R}$, $\operatorname{ran} g \cap \operatorname{dom} \phi \neq \varnothing$, and $h = \phi(g)$, then h is differentiable, and $dh = d\phi(g)dg$.

Theorem 8. If $P \in \mathcal{M}$, $u \in T^P$, and $f \in \mathcal{D}^1(P)$, then $uf = df(P)u = u^m(\partial f/\partial x^m)(P) = (ux^m)(\partial f/\partial x^m)(P)$ for each coordinate system X of $\mathcal{M}$ around P.

Theorem 9. If $P \in \mathcal{M}$, $f \in \mathcal{D}^1(P)$, $g \in \mathcal{D}^1(P)$, and $c \in \mathbb{R}$, then

i. $d(f+g)(P) = df(P) + dg(P)$,

ii. $d(cf)(P) = c\,df(P)$, and

iii. $d(fg)(P) = g(P)\,df(P) + f(P)\,dg(P)$.

IX. VECTOR FIELDS, COVECTOR FIELDS, AND TENSOR FIELDS

At each point P of the smooth manifold $\mathcal{M}$ there are the tangent space T^P, the cotangent space T_P, and the various tensor product spaces that can be built from them, such as $T_P \otimes T^P$, $T_P \otimes T_P$, $T_P \otimes T_P \otimes T^P$, $T_P \otimes T^P \otimes T_P$, and $T_P \otimes T_P \otimes T_P \otimes T^P$. The elements of the tensor product spaces are called **tangent tensors at** P and, for short, **tensors at** P. The type of such a tensor at P is determined by the space to which it belongs, and is indicated by addition of one or more prefixes co- or con- to the word tensor ("co-" from *co*variant vector, i. e., tangent covector, and "con-" from *con*travariant vector, i. e., tangent vector). The tensors in the product spaces listed above, for example, are called, respectively, (tangent) cocontensors, cocotensors, cococontensors, coconcotensors, and cocococontensors at P. The elements of T^P and T_P are also called tensors; contensor and cotensor are the applicable terms. The term "contravector" for an element of T^P is used also, to parallel "covector", but the shorter word "convector" is not, as it already has a meaning in ordinary discourse. Every tangent scalar at P is also called a tensor at P; no prefix is applied.

By a **vector** (or **contensor**) **field of** $\mathcal{M}$ is meant a function u on a subset U of $\mathcal{M}$ such that if P is a point of U, then $u(P)$ is a tangent vector at P; u is said to be a **vector** (or **contensor**) **field on** U. If X is a coordinate system of $\mathcal{M}$, then, for each m, $\partial/\partial x^m$ is a vector field (of $\mathcal{M}$) on $\operatorname{dom} X$; if U intersects $\operatorname{dom} X$, then on their intersection $u = u^m(\partial/\partial x^m)$, that is, $u(P) = u^m(P)(\partial/\partial x^m)(P)$ if $P \in U \cap \operatorname{dom} X$, where each **component** u^m **of** u **in** X is a scalar field on the intersection, given by $u^m(P) = u(P)x^m$ if $P \in U \cap \operatorname{dom} X$.

Definition. If u is a vector field of $\mathcal{M}$ on U, then u is said to be **differentiable** (resp., **continuous**) **at** P if and only if P is a point of U and, for every coordinate system X of $\mathcal{M}$ around P, each of the components u^m of u in X is differentiable (resp., continuous) at P; u is said to be **differentiable** (resp., **continuous**) if and only if u is differentiable (resp., continuous) at every point of U. To say that u **is** C^K (C^∞**, analytic**) is to say that, for every coordinate system X of $\mathcal{M}$ such that $U \cap \operatorname{dom} X \neq \varnothing$, each of the components u^m is C^K (C^∞, analytic).

If X' is a coordinate system of $\mathcal{M}$ whose domain intersects $U \cap \operatorname{dom} X$, then on $U \cap \operatorname{dom} X \cap \operatorname{dom} X'$ the components of u in X and of u in X' are related through the "contravariant transformation law" expressed by the equations

$$u^{m'} = u^m \frac{\partial x^{m'}}{\partial x^m} \quad \text{and} \quad u^m = u^{m'} \frac{\partial x^m}{\partial x^{m'}}. \tag{33}$$

If $\mathcal{M}$ is C^L (C^∞, analytic), then the partial derivatives $\partial x^{m'}/\partial x^m$ and $\partial x^m/\partial x^{m'}$ are all C^{L-1} (C^∞, analytic). Consequently, much as in the case of mappings between manifolds, to determine whether u is C^K, C^∞, or analytic it is sufficient (because products and sums of C^K, C^∞, or analytic scalar fields are themselves C^K, C^∞, or analytic) to make that determination for the components of u in all (relevant) coordinate systems of some subatlas of $\mathcal{M}$'s maximal atlas, so long as $\mathcal{M}$ is at least $(K+1)$-smooth. In particular, $\mathcal{M}$ being smooth, that is to say, 1-smooth, u will be C^0 if its components in all (relevant) coordinate systems of some minimal atlas of $\mathcal{M}$ are C^0. Also, if $\mathcal{M}$ is doubly smooth, then u will be differentiable at P if and only if its components in some coordinate system around P are so. It is clear that the smoothness hierarchy holds for vector fields — that if u is analytic, then it is C^∞, if u is C^∞, then it is C^K for $K = 1, 2, \dots$, and so on. Coordinate vector fields $\partial/\partial x^m$ are always C^{K-1} if $\mathcal{M}$ is C^K, C^∞ if $\mathcal{M}$ is C^∞, and analytic if $\mathcal{M}$ is analytic.

The definitions of **covector** (or **cotensor**) **field of** $\mathcal{M}$ and of what it means for a covector field of $\mathcal{M}$ to be differentiable at P, differentiable, C^K, C^∞, or analytic are entirely analogous to those for vector fields, as are the remarks one could make of the nature of those in the paragraph just above. The only differences are that the **components in X of the covector field v on U** are the scalar fields v_m, given by $v_m(P) = v(P)(\partial/\partial x^m)(P)$, such that $v = v_m dx^m$ on $U \cap \operatorname{dom} X$, and that the components in X' are related to those in X through the "covariant transformation law" expressed by

$$v_{m'} = \frac{\partial x^m}{\partial x^{m'}}\, v_m \quad \text{and} \quad v_m = \frac{\partial x^{m'}}{\partial x^m}\, v_{m'}. \tag{34}$$

It is clear that the coordinate covector fields dx^m will be C^{K-1} (C^∞, analytic) if $\mathcal{M}$ is C^K (C^∞, analytic). It is also clear that, for each vector field u whose domain intersects that of v, the scalar field vu, defined by $(vu)(P) := v(P)u(P)$ if $P \in \operatorname{dom} u \cap \operatorname{dom} v$, and having in a coordinate system X the representation $u^m v_m$, will be differentiable at P (differentiable, C^K, C^∞, analytic) if u and v individually are.

Tensor fields of $\mathcal{M}$ of other types are defined in the same manner as vector fields and covector fields of $\mathcal{M}$, and the definitions of the various degrees of differentiability of these tensor fields are analogous to those for vector fields and for covector fields. The values of the tensor field at different points are required to be all of the same type, and the tensor field itself is said to be of that type. For example, if at each point P in $\operatorname{dom} L$ the value of the tensor field L is in $T_P \otimes T^P$, then L is called a cocontensor field.

Negatives of tensor fields, sums of tensor fields of the same type, products of scalar fields with tensor fields, and tensor products of tensor fields with tensor fields are all defined in the obvious pointwise fashion. For example, if u is a vector field of $\mathcal{M}$, and v is a covector field of $\mathcal{M}$ whose domain intersects that of u, then the **tensor product $v \otimes u$ of v with u** is defined by $(v \otimes u)(P) := v(P) \otimes u(P)$ if $P \in \operatorname{dom} u \cap \operatorname{dom} v$. If f is a scalar field on $\mathcal{M}$ whose domain overlaps that of u, then $(f \otimes u)(P) := f(P) \otimes u(P) := f(P)u(P)$, so $f \otimes u = fu$. It is also the case that $u \otimes f = fu$, not to be confused with uf, which is defined by $(uf)(P) := u(P)f$.

In accordance with these definitions the cocontensor field L would have, in coordinate systems X and X', representations $L = dx^m \otimes L_m{}^n(\partial/\partial x^n)$ and $L = dx^{m'} \otimes L_{m'}{}^{n'}(\partial/\partial x^{n'})$. If the coordinate patches and $\operatorname{dom} L$ have an overlap, then on that overlap

$$L_{m'}{}^{n'} = \frac{\partial x^m}{\partial x^{m'}}\, L_m{}^n \frac{\partial x^{n'}}{\partial x^n} \quad \text{and} \quad L_m{}^n = \frac{\partial x^{m'}}{\partial x^m}\, L_{m'}{}^{n'} \frac{\partial x^n}{\partial x^{n'}}\,, \tag{35}$$

so that if the components $L_m{}^n$ of L in X are C^K (C^∞, analytic) on the overlap, then so are the components of L in X', and *vice versa*, provided that $\mathcal{M}$ is $(K+1)$-smooth.

If u is a vector field of $\mathcal{M}$, and v is a covector field of $\mathcal{M}$ whose domain intersects that of u, then, in each coordinate system X for which $\operatorname{dom} u \cap \operatorname{dom} v \cap \operatorname{dom} X \neq \varnothing$, the tensor field $v \otimes u$ has the representation

$$v \otimes u = (v_m dx^m) \otimes \left(u^n \frac{\partial}{\partial x^n}\right) = dx^m \otimes (v_m u^n) \frac{\partial}{\partial x^n} = (u^n v_m)\left(dx^m \otimes \frac{\partial}{\partial x^n}\right). \tag{36}$$

From this it follows that if each of u and v is C^K (C^∞, analytic) on $\operatorname{dom} u \cap \operatorname{dom} v$, then so is $v \otimes u$. For all other products of tensor fields, for sums and differences of tensor fields of the same type, and for products of scalar fields with tensor fields analogous conclusions can be drawn.

For suitable pairs of tensor fields u and v there are various “contractions” of the tensor product $v \otimes u$, analogous to the composition vu in the case where u is a vector field and v is a covector field; these contractions are themselves scalar fields, vector fields, covector fields, or tensor fields, and they, too, are C^K (C^∞, analytic) if each of u and v is. If $u = u^n(\partial/\partial x^n)$ and $v = v_m dx^m$, then $v \otimes u$ has just the one contraction mentioned:

$$(v \otimes u)\left(\frac{\partial}{\partial x^p}\right)(dx^p) = \left(v_m dx^m \otimes u^n \frac{\partial}{\partial x^n}\right)\left(\frac{\partial}{\partial x^p}\right)(dx^p) \tag{37}$$

$$= \left(v_m dx^m \frac{\partial}{\partial x^p}\right) \otimes \left(u^n \frac{\partial}{\partial x^n} dx^p\right) \tag{38}$$

$$= (v_m \delta^m{}_p) \otimes (u^n \delta_n{}^p) = (v_m \delta^m{}_p)(u^n \delta_n{}^p) \tag{39}$$

$$= v_p u^p = u^p v_p = vu. \tag{40}$$

If, however, $u = u^m \partial/\partial x^m$ and $v = dx^k \otimes v_{kl} dx^l$, then the product has the representation

$$v \otimes u = (dx^k \otimes v_{kl} dx^l) \otimes \left(u^m \frac{\partial}{\partial x^m}\right) \tag{41}$$

$$= (u^m v_{kl})\left(dx^k \otimes dx^l \otimes \frac{\partial}{\partial x^m}\right), \tag{42}$$

and it has two distinct contractions, represented in X by

$$(v \otimes u)\left(\frac{\partial}{\partial x^p}\right)(\cdot)(dx^p) = \left(dx^k \frac{\partial}{\partial x^p}\right) \otimes v_{kl} dx^l \otimes \left(u^m \frac{\partial}{\partial x^m} dx^p\right) \tag{43}$$

$$= \delta^k{}_p \otimes v_{kl} dx^l \otimes u^m \delta_m{}^p \tag{44}$$

$$= \delta^k{}_p v_{kl} \otimes dx^l \otimes u^m \delta_m{}^p \tag{45}$$

$$= v_{pl} \otimes dx^l \otimes u^p \tag{46}$$

$$= (u^p v_{pl}) dx^l \tag{47}$$

and

$$(v \otimes u)(\cdot)\left(\frac{\partial}{\partial x^p}\right)(dx^p) = dx^k \otimes \left(v_{kl} dx^l \frac{\partial}{\partial x^p}\right) \otimes \left(u^m \frac{\partial}{\partial x^m} dx^p\right) \tag{48}$$

$$= dx^k \otimes v_{kl} \delta^l{}_p \otimes u^m \delta_m{}^p \tag{49}$$

$$= dx^k \otimes v_{kp} \otimes u^p \tag{50}$$

$$= (u^p v_{kp}) dx^k. \tag{51}$$

X. FRAMES, COFRAMES, FRAME SYSTEMS, AND COFRAME SYSTEMS

By a **frame** (**coframe**) **at** the point P of the smooth manifold $\mathcal{M}$ is meant an ordered basis of the tangent (cotangent) space of $\mathcal{M}$ at P. By a **frame** (**coframe**) **system** (or **field**) **of** $\mathcal{M}$ is meant a function E (Ω) on a subset U of $\mathcal{M}$ such that if P is a point of U, then $E(P)$ $(\Omega(P))$ is a frame (coframe) at P; E (Ω) is said to be a **frame** (**coframe**) **system** (or **field**) **on** U and, if P is an interior point of U, **around** P. By the m^{th} **vector** (**covector**) **field of** E (Ω) is meant the vector (covector) field e_m (ω^m) of $\mathcal{M}$ on U whose value at P is the m^{th} vector (covector) in $E(P)$ $(\Omega(P))$. It is conventional to write $E = \{e_m\}$ and $\Omega = \{\omega^m\}$; then $E(P) = \{e_m\}(P) = \{e_m(P)\}$ and $\Omega(P) = \{\omega^m\}(P) = \{\omega^m(P)\}$, if $P \in U$. If $\operatorname{dom}\Omega$ and $\operatorname{dom}E$ intersect, and for each point P in their intersection $\Omega(P)$ is the basis of T_P dual to the basis $E(P)$ of T^P, then Ω is said to be **dual to** E **on** $\operatorname{dom}E \cap \operatorname{dom}\Omega$; in that event $\omega^m e_n = \delta^m{}_n$ (as a scalar field on $\operatorname{dom}E \cap \operatorname{dom}\Omega$). If in fact $\operatorname{dom}\Omega = \operatorname{dom}E$, then Ω is called the **dual of** E. For every frame system E there is just one coframe system Ω that is the dual of E; it is determined by the requirements that $\operatorname{dom}\Omega = \operatorname{dom}E$ and $\omega^m e_n = \delta^m{}_n$.

The frame (coframe) system E (Ω) is said to be **differentiable at** P if and only if each vector (covector) field e_m (ω^m) is differentiable at P, to be **differentiable** if and only each e_m (ω^m) is differentiable, and to be **smooth** if and only if each e_m (ω^m) is C^{K-1} (C^∞, analytic) if $\mathcal{M}$ is C^K (C^∞, analytic). These cannot occur unless $\mathcal{M}$ is doubly smooth, in which case if X is a coordinate system of $\mathcal{M}$, then $\{\partial/\partial x^m\}$ and $\{dx^m\}$ are, respectively, a smooth frame system and a smooth coframe system of $\mathcal{M}$ on $\operatorname{dom}X$, and $\{dx^m\}$ is the dual of $\{\partial/\partial x^m\}$; they are called the **frame system** and the **coframe system determined by** X. By a **coordinate frame** (**coframe**) **system of** $\mathcal{M}$ is meant a frame (coframe) system that for some coordinate system X of $\mathcal{M}$ is the frame (coframe) system determined by X.

Every vector field u whose domain intersects that of the frame system E and its dual Ω has on the intersection the representation $u = u^m e_m$, where the **components** u^m **of** u **in** E are the scalar fields $\omega^m u$. Similarly, every such covector field v has the representation $v = v_m \omega^m$, the **components** v_m **of** v **in** E being the scalar fields $v e_m$. In particular, if E' is a frame system of $\mathcal{M}$ whose domain intersects that of E, and Ω' is its dual, then each $e_{m'}$ has the components $\omega^m e_{m'}$ in E, each $\omega^{m'}$ has the components $\omega^{m'} e_m$ in Ω, each e_m has the components $\omega^{m'} e_m$ in E', and each ω^m has the components $\omega^m e_{m'}$ in Ω'. Thus

$$e_{m'} = (e_{m'})^m e_m = (\omega^m e_{m'}) e_m, \tag{52}$$

$$\omega^{m'} = (\omega^{m'})_m \omega^m = (\omega^{m'} e_m) \omega^m, \tag{53}$$

$$e_m = (e_m)^{m'} e_{m'} = (\omega^{m'} e_m) e_{m'}, \tag{54}$$

and

$$\omega^m = (\omega^m)_{m'} \omega^{m'} = (\omega^m e_{m'}) \omega^{m'}. \tag{55}$$

To make formulas such as these easier to read, the notations $A_{m'}{}^m$ and $A_m{}^{m'}$ are introduced to stand for the scalar fields $\omega^m e_{m'}$ and $\omega^{m'} e_m$, so that

$$e_{m'} = A_{m'}{}^{m} e_m, \quad \omega^{m'} = \omega^m A_m{}^{m'}, \quad e_m = A_m{}^{m'} e_{m'}, \quad \text{and} \quad \omega^m = \omega^{m'} A_{m'}{}^{m}. \tag{56}$$

If $P \in \operatorname{dom} E \cap \operatorname{dom} E'$, then the matrices $\overset{m'}{\downarrow}\overset{m\longrightarrow}{[A_{m'}{}^{m}(P)]}$ and $\overset{m}{\downarrow}\overset{m'\longrightarrow}{[A_m{}^{m'}(P)]}$ are, respectively, the transition matrix from the basis $\{e_m(P)\}$ of T^P to the basis $\{e_{m'}(P)\}$ of T^P and the transition matrix from the basis $\{\omega^m(P)\}$ of T_P to the basis $\{\omega^{m'}(P)\}$ of T_P; they are also, respectively, the transition matrix from $\{\omega^{m'}(P)\}$ to $\{\omega^m(P)\}$ and the transition matrix from $\{e_{m'}(P)\}$ to $\{e_m(P)\}$. As such, they are reciprocal matrices, and therefore $A_m{}^{m'} A_{m'}{}^{n} = \delta_m{}^{n}$ and $A_{m'}{}^{m} A_m{}^{n'} = \delta_{m'}{}^{n'}$. If each of E and E' is a coordinate frame system, say $E = \{\partial/\partial x^m\}$ and $E' = \{\partial/\partial x^{m'}\}$, then $A_m{}^{m'} = \partial x^{m'}/\partial x^m$ and $A_{m'}{}^{m} = \partial x^m/\partial x^{m'}$.

If E' is a *coordinate* frame system, and the frame system E is smooth, then the scalar fields $A_m{}^{m'}$, being the components of e_m in E', hence in X', are C^{K-1} (C^∞, analytic) if $\mathcal{M}$ is C^K (C^∞, analytic). Conversely, if the $A_m{}^{m'}$ satisfy the latter condition for every coordinate frame system E' whose domain intersects that of E, then E is smooth. Inasmuch as the entries of the reciprocal matrix $[A_{m'}{}^{m}(P)]$ are analytic functions of the entries of the matrix $[A_m{}^{m'}(P)]$, and the $A_{m'}{}^{m}$ are the components of ω^m in the coframe system dual to E', hence are the components of ω^m in X', the coframe system Ω dual to E is smooth whenever E itself is smooth. The converse also is true: E is smooth if Ω is smooth.

If u is a vector field of $\mathcal{M}$ whose domain has an overlap with the domain of the frame system E and the domain of the frame system E', then on the overlap $u = u^m e_m = u^m A_m{}^{m'} e_{m'}$, and therefore $u^{m'} = u^m A_m{}^{m'}$. From this it follows (by taking E' to be a coordinate frame system) that u will be differentiable at P, differentiable, or smooth if and only if in every smooth frame system E whose domain intersects that of u the components u^m are so. A similar propositon is true for covector fields v and their components v_m in E, and in both instances to know that the field is differentiable at P, differentiable, or smooth it sufffices to test the components of the field in every frame system of a set of smooth frame systems whose domains all contain P (for differentiability at P) or else overlap the domain of the field and together cover it. Application of these observations to the vector fields of a frame system (or to the covector fields of its dual coframe system) yields the conclusion that a frame system E and its dual are differentiable at P, differentiable, or smooth if and only if, for every frame system E' in a set of smooth frame systems whose domains contain P (for differentiability at P) or overlap and jointly cover $\operatorname{dom} E$, the scalar fields $A_m{}^{m'}$ (or, equally well, the scalar fields $A_{m'}{}^{m}$) are so.

For scalar fields f a useful notation is $f_{.m}$ to stand for $e_m f$, which makes $f_{.m} = \partial f/\partial x^m$ if $e_m = \partial/\partial x^m$. In this notation $f_{.m} = A_m{}^{m'} f_{.m'}$, and if $e_{m'} = \partial/\partial x^{m'}$, then $df = (\partial f/\partial x^{m'}) dx^{m'} = f_{.m'} dx^{m'} = f_{.m'} \omega^m A_m{}^{m'} = A_m{}^{m'} f_{.m'} \omega^m = f_{.m} \omega^m = (e_m f) \omega^m$. Also, if u is a vector field, then $uf = u^{m'} \partial f/\partial x^{m'} = u^{m'} f_{.m'} = u^m A_m{}^{m'} f_{.m'} = u^m f_{.m} = u^m e_m f = (df)u$.

Tensor fields of the various types all have their representations in frame systems, and their components in frame systems with overlapping domains are related by "transformation laws" involving the transition matrices $[A_m{}^{m'}]$ and $[A_{m'}{}^{m}]$. If, for example, T is a cococontensor field whose components in E and E' are $T^m{}_{kl}$ and $T^{m'}{}_{k'l'}$, then

$$T = T^m{}_{kl}(\omega^l \otimes \omega^k \otimes e_m) \tag{57}$$
$$= T^m{}_{kl}(\omega^{l'} A_{l'}{}^l \otimes \omega^{k'} A_{k'}{}^k \otimes A_m{}^{m'} e_{m'}) \tag{58}$$
$$= (A_{l'}{}^l A_{k'}{}^k T^m{}_{kl} A_m{}^{m'})(\omega^{l'} \otimes \omega^{k'} \otimes e_{m'}) \tag{59}$$
$$= T^{m'}{}_{k'l'}(\omega^{l'} \otimes \omega^{k'} \otimes e_{m'}), \tag{60}$$

so that $T^{m'}{}_{k'l'} = A_{l'}{}^l A_{k'}{}^k T^m{}_{kl} A_m{}^{m'}$. Specialization of E' to a coordinate frame system shows that, just as for vector fields and covector fields, differentiability at P, differentiability, and smoothness of T can be determined by examination of its components $T^m{}_{kl}$ in smooth frame systems E.

The notion of "contraction" is not confined to tensor fields that happen to be tensor products of other tensor fields. All that is necessary is that the tensor field to be contracted have a covariant slot and a contravariant slot, and then for each pair of its slots, one covariant, the other contravariant, there is a contraction in that pair of slots. Each such contraction is a tensor field with the same domain as, but with one fewer covariant slots and one fewer contravariant slots than the original field. If, for example, T is, as above, a cococontensor field, represented in E by $T = T^m{}_{kl}(\omega^l \otimes \omega^k \otimes e_m)$, then T has the two contractions given in E by

$$T(e_p)(\cdot)(\omega^p) = T^m{}_{kl}(\omega^l e_p \otimes \omega^k \otimes e_m \omega^p) \tag{61}$$
$$= T^m{}_{kl}(\delta^l{}_p \otimes \omega^k \otimes \delta_m{}^p) \tag{62}$$
$$= (\delta_m{}^p T^m{}_{kl} \delta^l{}_p)\omega^k \tag{63}$$
$$= (T^p{}_{kp})\omega^k \tag{64}$$

and

$$T(\cdot)(e_p)(\omega^p) = T^m{}_{kl}(\omega^l \otimes \omega^k e_p \otimes e_m \omega^p) \tag{65}$$
$$= T^m{}_{kl}(\omega^l \otimes \delta^k{}_p \otimes \delta_m{}^p) \tag{66}$$
$$= (\delta_m{}^p T^m{}_{kl} \delta^k{}_p)\omega^l \tag{67}$$
$$= (T^p{}_{pl})\omega^l. \tag{68}$$

These contractions of T, which ostensibly depend on the choice of the frame system E in which they are expressed, are in fact independent of it, and the same is true of all contractions of tensor fields of whatever (contractible) types. For example, starting from the frame system E' one has

$$T(e_{p'})(\cdot)(\omega^{p'}) = (A_{p'}{}^{l} A_{k'}{}^{k} T^{m}{}_{kl} A_{m}{}^{p'})\omega^{k'} \tag{69}$$

$$= T^{m}{}_{kl}(A_{m}{}^{p'} A_{p'}{}^{l})(\omega^{k'} A_{k'}{}^{k}) \tag{70}$$

$$= (T^{m}{}_{kl}\delta_{m}{}^{l})\omega^{k} \tag{71}$$

$$= (T^{m}{}_{km})\omega^{k} \tag{72}$$

$$= (T^{p}{}_{kp})\omega^{k} \tag{73}$$

$$= T(e_p)(\cdot)(\omega^p). \tag{74}$$

The frame-system–independent notation $T\,{}^{\mathrm{b}}_{\mathrm{a}}$ will be adopted to stand for the contraction of the tensor field T in the covariant slot numbered a and the contravariant slot numbered b (in the consecutive numbering of all of T's slots from left to right). Thus, in the example above, $T\,{}^{3}_{1}$ stands for the contraction represented in E by $T(e_p)(\cdot)(\omega^p)$, and $T\,{}^{3}_{2}$ is represented in E by $T(\cdot)(e_p)(\omega^p)$. One has, therefore, that

$$T\,{}^{3}_{1} = (T\,{}^{3}_{1})_k\, \omega^k = (T^{p}{}_{kp})\omega^k \tag{75}$$

and

$$T\,{}^{3}_{2} = (T\,{}^{3}_{2})_l\, \omega^l = (T^{p}{}_{pl})\omega^l. \tag{76}$$

When the tensor field to be contracted has only one contravariant slot, its contractions can be generated in a slightly different manner. In the case of T above, for example,

$$\omega^p T e_p = \omega^p (T^{m}{}_{kl}(\omega^l \otimes \omega^k \otimes e_m))e_p \tag{77}$$

$$= T^{m}{}_{kl}(\omega^l e_p \otimes \omega^k \otimes \omega^p e_m) \tag{78}$$

$$= T^{m}{}_{kl}(\delta^{l}{}_{p} \otimes \omega^k \otimes \delta^{p}{}_{m}) \tag{79}$$

$$= T^{p}{}_{kp}\omega^k \tag{80}$$

$$= T\,{}^{3}_{1} \tag{81}$$

and, similarly,

$$\omega^p T(\cdot)e_p = T\,{}^{3}_{2}\,. \tag{82}$$

The combinations $\omega^p T$, Te_p, $(\omega^p T)e_p$, $\omega^p(Te_p)$, $T(\cdot)e_p$, $(\omega^p T)(\cdot)e_p$, and $\omega^p(T(\cdot)e_p)$ that are implicit in these formulas are instances of more general compositions of tensor fields, as is the simplest composition of all, the contracted tensor product vu where u is a vector field and v is a covector field. To illustrate, consider in addition to the cococontensor field T a cocontensor field S. Then TS is to stand for a cococontensor field whose first covariant slot corresponds to the covariant slot of S, whose second covariant slot corresponds to the second covariant covariant slot of T, and whose contravariant slot corresponds to the contravariant slot of T; the two remaining slots, the first covariant slot of T and the contravariant slot of of S, are "composed" with one another and are thereby eliminated in TS. The field TS can be defined by telling, for each point P in its domain, each tangent vector u at P, and each tangent vector $\bar{u}$ at P, what the tangent vector $(TS)(P)u\bar{u}$ is. The rule is that $(TS)(P)u\bar{u} = (T(P)(S(P)u))\bar{u}$, the righthand member of this equation being interpreted as follows: the cocontensor $S(P)$ operates on the vector u to give the vector $S(P)u$; the cococontensor $T(P)$ operates on $S(P)u$ to give the cocontensor $T(P)(S(P)u)$, which then operates on the vector $\bar{u}$ to produce the vector $(T(P)(S(P)u))\bar{u}$. Linearity in u and in $\bar{u}$ being apparent, the operator $(TS)(P)$ clearly is in $\mathfrak{L}(T^P, \mathfrak{L}(T^P, T^P))$, i. e., is in $T_P \otimes T_P \otimes T^P$, so TS is a cococontensor field. Equally well, one could define TS by specifying that for each vector field u and each vector field $\bar{u}$ the vector field $(TS)u\bar{u}$ is to be the result of applying at each point P of $\operatorname{dom} T \cap \operatorname{dom} S \cap \operatorname{dom} u \cap \operatorname{dom} \bar{u}$ the rule given above, so that $(TS)u\bar{u} = (T(Su))\bar{u}$. This defines the composition TS in terms of elementary compositions of co... tensor fields with vector fields, which, pointwise, are just the operations of tangent co... tensors on tangent vectors.

The composition TS thus defined is nothing other than the contraction in the second and third slots (which come from the contravariant slot of S and the first covariant slot of T) of the tensor product $S \otimes T$ — for, on the one hand

$$(TS)u = T(Su) \tag{83}$$

$$= T((S^q{}_p\, \omega^p \otimes e_q)(u^r e_r)) \tag{84}$$

$$= T(S^q{}_p u^r (\omega^p e_r) \otimes e_q) \tag{85}$$

$$= (T^m{}_{kl}\, \omega^l \otimes \omega^k \otimes e_m)(S^q{}_p u^p e_q) \tag{86}$$

$$= T^m{}_{kl} S^q{}_p u^p (\omega^l e_q) \otimes \omega^k \otimes e_m \tag{87}$$

$$= T^m{}_{kl} S^l{}_p u^p\, \omega^k \otimes e_m \tag{88}$$

$$= (T^m{}_{kl} S^l{}_p\, \omega^p \otimes \omega^k \otimes e_m)u, \tag{89}$$

so that $TS = T^m{}_{kl} S^l{}_p\, \omega^p \otimes \omega^k \otimes e_m$, and on the other hand

$$S \otimes T = S^q{}_p T^m{}_{kl}\, \omega^p \otimes e_q \otimes \omega^l \otimes \omega^k \otimes e_m, \tag{90}$$

so that $(S \otimes T)^2_3 = S^n{}_p T^m{}_{kn}\, \omega^p \otimes \omega^k \otimes e_m = TS$.

A different composition of T with S is $T(\cdot)S$, in which the $(\cdot)$ indicates that the first covariant slot of T is held open and it is the second covariant slot of T that is "composed" with the contravariant slot of S. Thus $(T(\cdot)S)u\bar{u} = (T\bar{u})(Su)$, where now the cococontensor field T acts first on $\bar{u}$ and then on Su, whereas in $(TS)u\bar{u}$ it acts first on Su and then on $\bar{u}$. In both compositions the covariant slot of S is filled by the first vector acted upon, before the covariant slots of T come into play; if S had additional slots, these also would take precedence over those of T. Just as TS is the contraction of $S \otimes T$ in the second and third slots, $T(\cdot)S$ is the contraction of $S \otimes T$ in the second and fourth slots (those that arise from the contravariant slot of S and the second covariant slot of T); thus $T(\cdot)S = (S \otimes T)^2_4$.

In an analogous manner every contraction of a product $S \otimes T$ in a pair of slots of which one comes from S and the other from T gives rise to a composition of T with S. In more complex cases an unambiguous yet clean notation comparable to TS and $T(\cdot)S$ can be difficult to come by, but in every case the contraction notation $(S \otimes T)\,{}^{\mathrm{b}}_{\mathrm{a}}$ will be available.

In every frame system each component of a contraction of a tensor field is a sum of components of the uncontracted tensor field, hence the contraction is at least as smooth as the uncontracted field. In particular, every contraction of a product $S \otimes T$ is at least as smooth as the less smooth of S and T; the same is true, therefore, of the various compositions of T with S.

It is true, but not obviously so, that not every frame system is a coordinate frame system. A test that can be applied to a frame system to determine whether it is a coordinate frame system will be developed later.

XI. SYMMETRY, ANTISYMMETRY, AND EXTERIOR ALGEBRA

Let U be a finite-dimensional vector space over $\mathbb{R}$, let $\{a_m\}$ be a basis for U, and let $\{b^m\}$ be a basis for U^*. If p is a positive integer, then $(U^*)^p$ will denote the space $U^* \otimes \cdots \otimes U^*$ with p factors U^*.

Theorem 1. If $\varphi \in (U^*)^p$ (resp., $(U^*)^p \otimes U$), and π is a permutation of $\{1, \ldots, p\}$, then there is just one element φ_π of $(U^*)^p$ (resp., $(U^*)^p \otimes U$) such that if $u_1, \ldots, u_p \in U$, then

$$\varphi_\pi u_1 \ldots u_p = \varphi u_{\pi^{-1}(1)} \ldots u_{\pi^{-1}(p)}. \tag{91}$$

If σ is a permutation of $\{1, \ldots, p\}$, then $(\varphi_\pi)_\sigma = \varphi_{\sigma\pi}$. If $\varphi = b^{m_1} \otimes \cdots \otimes b^{m_p}$, then $\varphi_\pi = b^{m_{\pi(1)}} \otimes \cdots \otimes b^{m_{\pi(p)}}$; if $\varphi = b^{m_1} \otimes \cdots \otimes b^{m_p} \otimes a_m$, then $\varphi_\pi = b^{m_{\pi(1)}} \otimes \cdots \otimes b^{m_{\pi(p)}} \otimes a_m$.

Definition. If $\varphi \in (U^*)^p$ or $\varphi \in (U^*)^p \otimes U$, then

$$\text{Sym } \varphi := \frac{1}{p!} \sum_\pi \varphi_\pi \quad \text{and} \quad \text{Sk } \varphi := \frac{1}{p!} \sum_\pi (\operatorname{sgn} \pi) \varphi_\pi, \tag{92}$$

where the sums are over all permutations π of $\{1, \ldots, p\}$, and $\operatorname{sgn} \pi = 1$ or -1, according as π is even or odd.

Theorem 2. If $\varphi, \vartheta \in (U^*)^p$ or $\varphi, \vartheta \in (U^*)^p \otimes U$, and $\alpha \in \mathbb{R}$, then

i. $\text{Sym } (\varphi + \vartheta) = \text{Sym } \varphi + \text{Sym } \vartheta$,

ii. $\text{Sk } (\varphi + \vartheta) = \text{Sk } \varphi + \text{Sk } \vartheta$,

iii. $\text{Sym } (\alpha\varphi) = \alpha(\text{Sym } \varphi)$, and

iv. $\text{Sk } (\alpha\varphi) = \alpha(\text{Sk } \varphi)$.

Theorem 3. If $\varphi \in (U^*)^p$ (resp., $(U^*)^p \otimes U$), then Sym φ and Sk φ are in $(U^*)^p$ (resp., $(U^*)^p \otimes U$), and $\text{Sym } (\text{Sym } \varphi) = \text{Sym } \varphi$ and $\text{Sk } (\text{Sk } \varphi) = \text{Sk } \varphi$.

Definition. If $\varphi \in (U^*)^p$ or $\varphi \in (U^*)^p \otimes U$, then φ is **symmetric** means that $\varphi_\pi = \varphi$ for every transposition π of $\{1, \ldots, p\}$, and φ is **skew-symmetric** (**antisymmetric**, **alternating**) means that $\varphi_\pi = -\varphi$ for every transposition π of $\{1, \ldots, p\}$.

Theorem 4. Such a φ is symmetric if and only if $\varphi_\pi = \varphi$ for every permutation π of $\{1, \ldots, p\}$, and is skew-symmetric if and only if $\varphi_\pi = -\varphi$ for every odd permutation π of $\{1, \ldots, p\}$; also, such a φ is symmetric if and only if Sym $\varphi = \varphi$ and is skew-symmetric if and only if Sk $\varphi = \varphi$.

Sym φ is called the **symmetrization of** φ and the **symmetric part of** φ, and Sk φ is called the **skew-symmetrization of** φ and the **skew-symmetric** (**antisymmetric**, **alternating**) **part of** φ.

If, for each integer sequence $\{m_1, \ldots, m_p\}$ such that $1 \le m_i \le M$, $\varphi_{m_p \ldots m_1}$ is a real number, then for each such sequence $\{m_1, \ldots, m_p\}$ the real number $\varphi_{(m_p \ldots m_1)}$ and the real number

$\varphi_{[m_p \dots m_1]}$ are defined by

$$\varphi_{(m_p \dots m_1)} := \frac{1}{p!} \sum_{\pi} \varphi_{m_{\pi^{-1}(p)} \dots m_{\pi^{-1}(1)}} \quad \text{and} \quad \varphi_{[m_p \dots m_1]} := \frac{1}{p!} \sum_{\pi} (\operatorname{sgn} \pi) \varphi_{m_{\pi^{-1}(p)} \dots m_{\pi^{-1}(1)}}. \tag{93}$$

Theorem 5. If $\varphi \in (U^*)^p$, and $\varphi = \varphi_{m_p \dots m_1} b^{m_1} \otimes \cdots \otimes b^{m_p}$, then

$$\operatorname{Sym} \varphi = \varphi_{(m_p \dots m_1)} b^{m_1} \otimes \cdots \otimes b^{m_p} \tag{94}$$

and

$$\operatorname{Sk} \varphi = \varphi_{\,[m_p \dots m_1]} b^{m_1} \otimes \cdots \otimes b^{m_p}; \tag{95}$$

φ is symmetric if and only if $\varphi_{(m_p \dots m_1)} = \varphi_{m_p \dots m_1}$, and skew-symmetric if and only if $\varphi_{\,[m_p \dots m_1]} = \varphi_{m_p \dots m_1}$. If $\varphi \in (U^*)^p \otimes U$, and $\varphi = \varphi^m{}_{m_p \dots m_1} b^{m_1} \otimes \cdots \otimes b^{m_p} \otimes a_m$, then

$$\operatorname{Sym} \varphi = \varphi^m{}_{(m_p \dots m_1)} b^{m_1} \otimes \cdots \otimes b^{m_p} \otimes a_m \tag{96}$$

and

$$\operatorname{Sk} \varphi = \varphi^m{}_{[m_p \dots m_1]} b^{m_1} \otimes \cdots \otimes b^{m_p} \otimes a_m; \tag{97}$$

φ is symmetric if and only if $\varphi^m{}_{(m_p \dots m_1)} = \varphi^m{}_{m_p \dots m_1}$, and skew-symmetric if and only if $\varphi_{\,[m_p \dots m_1]} = \varphi^m{}_{m_p \dots m_1}$.

Theorem 6. If $\varphi \in (U^*)^2$ or $\varphi \in (U^*)^2 \otimes U$, then $\varphi = \operatorname{Sym} \varphi + \operatorname{Sk} \varphi$.

Definition. If $\varphi \in (U^*)^p$, and $\vartheta \in (U^*)^q$ or $\vartheta \in (U^*)^q \otimes U$, then $\varphi \wedge \vartheta := \operatorname{Sk} (\varphi \otimes \vartheta)$, and $\varphi \wedge \vartheta$ is called the **exterior**, the **alternating**, and the **wedge product of φ with ϑ**.

Theorem 7. If $\varphi, \bar{\varphi} \in (U^*)^p$, and $\vartheta, \bar{\vartheta} \in (U^*)^q$ or $\vartheta, \bar{\vartheta} \in (U^*)^q \otimes U$, and $\alpha \in \mathbb{R}$, then

i. $(\varphi + \bar{\varphi}) \wedge \vartheta = \varphi \wedge \vartheta + \bar{\varphi} \wedge \vartheta$,

ii. $\varphi \wedge (\vartheta + \bar{\vartheta}) = \varphi \wedge \vartheta + \varphi \wedge \bar{\vartheta}$, and

iii. $(\alpha\varphi) \wedge \vartheta = \alpha(\varphi \wedge \vartheta) = \varphi \wedge (\alpha\vartheta)$.

Theorem 8. If $\varphi \in (U^*)^p$, $\vartheta \in (U^*)^q$, and $\psi \in (U^*)^r$ or $\psi \in (U^*)^r \otimes U$, then

$$(\varphi \wedge \vartheta) \wedge \psi = (\varphi \otimes \vartheta) \wedge \psi = \varphi \wedge (\vartheta \otimes \psi) = \varphi \wedge (\vartheta \wedge \psi). \tag{98}$$

This theorem permits an unambiguous interpretation of $\varphi \wedge \vartheta \wedge \psi$ and, by extension, of such products as $\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_p$.

Definition. If p is a positive integer, then by a **p-form on U** is meant a skew-symmetric element of $(U^*)^p$, and by a **vector-valued p-form on U** is meant a skew-symmetric element of $(U^*)^p \otimes U$. By a **0-form on U** is meant an element of $\mathbb{R}$, and by a **vector-valued 0-form**

on U is meant an element of $\mathbb{R}\otimes U$. A p-form φ on U is called **simple** if there exist 1-form(s) $\varphi_1, \varphi_2, \ldots, \varphi_p$ such that $\varphi = \varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_p$, and **compound** if there do not. A vector-valued p-form φ on U is called **simple** if there exist 1-form(s) $\varphi_1, \varphi_2, \ldots, \varphi_p$, with φ_p vector-valued, such that $\varphi = \varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_p$, and **compound** if there do not.

Definition. The set of all p-forms on U is denoted by $\bigwedge^p U^*$, that is, $\bigwedge^p U^* := \{\varphi \in (U^*)^p \mid \varphi \text{ is skew-symmetric}\}$.

Theorem 9. If $p > 0$, $\bigwedge^p U^*$ is a subspace of $(U^*)^p$.

Theorem 10. The set of all vector-valued p-forms on U is $\bigwedge^p U^* \otimes U$, that is, $\bigwedge^p U^* \otimes U = \{\varphi \in (U^*)^p \otimes U \mid \varphi \text{ is skew-symmetric}\}$. If $p > 0$, $\bigwedge^p U^* \otimes U$ is a subspace of $(U^*)^p \otimes U$.

Theorem 11. If $\varphi \in (U^*)^p$, and $\varphi = \varphi_{m_p \ldots m_1} b^{m_1} \otimes \cdots \otimes b^{m_p}$, then

$$\mathrm{Sk}\ \varphi = \varphi_{[m_p \ldots m_1]} b^{m_1} \otimes \cdots \otimes b^{m_p} \tag{99}$$

$$= \varphi_{m_p \ldots m_1} b^{m_1} \wedge \cdots \wedge b^{m_p} \tag{100}$$

$$= \varphi_{[m_p \ldots m_1]} b^{m_1} \wedge \cdots \wedge b^{m_p}, \tag{101}$$

$$= \varphi \quad \text{if} \quad \varphi \in \textstyle\bigwedge^p U^*. \tag{102}$$

If $\varphi \in (U^*)^p \otimes U$, and $\varphi = \varphi^m{}_{m_p \ldots m_1} b^{m_1} \otimes \cdots \otimes b^{m_p} \otimes a_m$, then

$$\mathrm{Sk}\ \varphi = \varphi^m{}_{[m_p \ldots m_1]} b^{m_1} \otimes \cdots \otimes b^{m_p} \otimes a_m \tag{103}$$

$$= \varphi^m{}_{m_p \ldots m_1} b^{m_1} \wedge \cdots \wedge b^{m_p} \otimes a_m \tag{104}$$

$$= \varphi^m{}_{[m_p \ldots m_1]} b^{m_1} \wedge \cdots \wedge b^{m_p} \otimes a_m, \tag{105}$$

$$= \varphi \quad \text{if} \quad \varphi \in \textstyle\bigwedge^p U^* \otimes U. \tag{106}$$

Theorem 12. If $\varphi \in (U^*)^p$ and $\vartheta \in (U^*)^q$, then $\vartheta \wedge \varphi = (-1)^{pq}(\varphi \wedge \vartheta)$; also,

$$\varphi \wedge \vartheta = \mathrm{Sk}\ \varphi \wedge \vartheta = \varphi \wedge \mathrm{Sk}\ \vartheta = \mathrm{Sk}\ \varphi \wedge \mathrm{Sk}\ \vartheta. \tag{107}$$

If φ is simple, then $\varphi \wedge \varphi = 0$. If p is odd, then $\varphi \wedge \varphi = 0$.

Theorem 13. If $1 \leq p \leq \dim U$, then $\{b^{m_1} \wedge \cdots \wedge b^{m_p} \mid m_1 < \cdots < m_p\}$ is a basis for $\bigwedge^p U^*$, and $\{b^{m_1} \wedge \cdots \wedge b^{m_p} \otimes a_m \mid m_1 < \cdots < m_p\}$ is a basis for $\bigwedge^p U^* \otimes U$. If $\varphi \in (U^*)^p$, and $\varphi = \varphi_{m_p \ldots m_1} b^{m_1} \otimes \cdots \otimes b^{m_p}$, then

$$\varphi = \sum_{m_1 < \cdots < m_p} p!\, \varphi_{[m_p \ldots m_1]} b^{m_1} \wedge \cdots \wedge b^{m_p}, \tag{108}$$

$$= \sum_{m_1 < \cdots < m_p} p!\, \varphi_{m_p \ldots m_1} b^{m_1} \wedge \cdots \wedge b^{m_p} \quad \text{if} \quad \varphi \in \textstyle\bigwedge^p U^*. \tag{109}$$

If $\varphi \in (U^*)^p \otimes U$, and $\varphi = \varphi^m{}_{m_p \ldots m_1} b^{m_1} \otimes \cdots \otimes b^{m_p} \otimes a_m$, then

$$\varphi = \sum_{m_1 < \cdots < m_p} p!\, \varphi^m{}_{[m_p \ldots m_1]} b^{m_1} \wedge \cdots \wedge b^{m_p} \otimes a_m, \tag{110}$$

$$= \sum_{m_1 < \cdots < m_p} p!\, \varphi^m{}_{m_p \ldots m_1} b^{m_1} \wedge \cdots \wedge b^{m_p} \otimes a_m \quad \text{if} \quad \varphi \in {\textstyle\bigwedge^p} U^* \otimes U. \tag{111}$$

Theorem 14. If $M = \dim U$, then $\dim \bigwedge^p U^* = \binom{M}{p}$, and $\dim \bigwedge^p U^* \otimes U = \binom{M}{p} \cdot M$, where $\binom{M}{p}$ is the binomial coefficient $\frac{M!}{p!(M-p)!}$ if $p \leq M$, and is 0 if $p > M$.

If ϕ is a co...co- or a co...cocontensor field of the smooth manifold $\mathcal{M}$, then there is a positive integer p such that $\phi(P) \in (T_P)^p$ for every point P in $\operatorname{dom} \phi$ or $\phi(P) \in (T_P)^p \otimes T^P$ for every such point P. Because $T_P = (T^P)^*$, all the preceding definitions apply to $\phi(P)$ as an element of $(U^*)^p$ or of $(U^*)^p \otimes U$, with $U = T^P$. They can be applied to ϕ itself by applying them at each point P in $\operatorname{dom} \phi$ to $\phi(P)$. Specifically, the tensor fields ϕ_π (where π is a permutation of $\{1, \ldots, p\}$), Sym ϕ, and Sk ϕ are given by $\phi_\pi(P) = (\phi(P))_\pi$, (Sym $\phi)(P) =$ Sym $(\phi(P))$, and (Sk $\phi(P) =$ Sk $(\phi(P))$, and the latter two are called the **symmetrization of** ϕ and the **symmetric part of** ϕ, and the **skew-symmetrization of** ϕ and the **skew-symmetric** (**antisymmetric**, **alternating**) **part of** ϕ. Also, ϕ is **symmetric** means that $\phi(P)$ is symmetric if $P \in \operatorname{dom} \phi$, which is equivalent to $\phi_\pi = \phi$ for every transposition π of $\{1, \ldots, p\}$, and ϕ is **skew-symmetric** (**antisymmetric**, **alternating**) means that $\phi(P)$ is skew-symmetric if $P \in \operatorname{dom} \phi$, which is equivalent to $\phi_\pi = -\phi$ for every transposition π of $\{1, \ldots, p\}$.

In the same vein, if ϕ is a co...cotensor field of $\mathcal{M}$, and θ is a co...co- or a co...cocontensor field of $\mathcal{M}$, and $\operatorname{dom} \phi \cap \operatorname{dom} \theta \neq \varnothing$, then $\phi \wedge \theta$, the **exterior**, **alternating**, or **wedge product of** ϕ **with** θ, is defined by $(\phi \wedge \theta)(P) = \phi(P) \wedge \theta(P)$, and this is equivalent to $\phi \wedge \theta =$ Sk $(\phi \otimes \theta)$. The various special properties of this product carry over. For example, if ϕ is a p-cotensor field (that is, $\phi(P) \in (T_P)^p$ for each point P in $\operatorname{dom} \phi$), and θ is a q-cotensor field, then $\theta \wedge \phi = (-1)^{pq}(\phi \wedge \theta)$.

If ϕ is a skew-symmetric p-cotensor field of $\mathcal{M}$, then $\phi(P) \in \bigwedge^p T_P = \bigwedge^p (T^P)^*$, so $\phi(P)$ is a p-form on the tangent space of $\mathcal{M}$ at P; in this case ϕ itself is called a p**-form of** $\mathcal{M}$ and also a **differential** p**-form of** $\mathcal{M}$. If ϕ is a skew-symmetric p-cocontensor field of $\mathcal{M}$, then $\phi(P) \in \bigwedge^p T_P \otimes T^P$; in this case ϕ is referred to as a (**differential**) **vector-valued** p**-form of** $\mathcal{M}$. Thus the 1-forms are just the covector fields of $\mathcal{M}$, and the vector-valued 1-forms are the cocontensor fields. By a 0**-form of** $\mathcal{M}$ is meant a scalar field of $\mathcal{M}$, and by a **vector-valued** 0**-form of** $\mathcal{M}$ is meant a vector field of $\mathcal{M}$. A p-form ϕ of $\mathcal{M}$ is called **simple** if there exist 1-form(s) $\phi_1, \phi_2, \ldots, \phi_p$ such that $\phi = \phi_1 \wedge \phi_2 \wedge \cdots \wedge \phi_p$, and **compound** if there do not. A vector-valued p-form ϕ of $\mathcal{M}$ is called **simple** if there exist 1-form(s) $\phi_1, \phi_2, \ldots, \phi_p$, with ϕ_p vector-valued, such that $\phi = \phi_1 \wedge \phi_2 \wedge \cdots \wedge \phi_p$, and **compound** if there do not.

XII. DIFFERENTIATION IN FRAME SYSTEMS AND EXTERIOR DIFFERENTIATION

From this point onward the manifold $\mathcal{M}$ will be presumed to be doubly smooth unless the contrary is stated. If T is a tensor field of $\mathcal{M}$, then $\{P \mid T \text{ is differentiable at } P\}$ will be called the **domain of differentiability of** T and denoted by $\operatorname{ddom} T$. The same terminology and notation will be applied to scalar fields f. Clearly, $\operatorname{ddom} f \subset \operatorname{dom} f$, $\operatorname{ddom} T \subset \operatorname{dom} T$, and $(\operatorname{ddom} f \cap \operatorname{ddom} T) \subset \operatorname{ddom}(fT)$. If E is a frame system of $\mathcal{M}$, then $\operatorname{ddom} T \cap \operatorname{dom} E$ is the intersection of the domains of differentiability of the components of T in E. If each of S and T is a tensor field of $\mathcal{M}$, then $(\operatorname{ddom} S \cap \operatorname{ddom} T) \subset \operatorname{ddom}(S \otimes T)$ and, if S and T are of the same type, then $(\operatorname{ddom} S \cap \operatorname{ddom} T) \subset \operatorname{ddom}(S + T)$, and $\operatorname{ddom}(-T) = \operatorname{ddom} T$, so that $(\operatorname{ddom} S \cap \operatorname{ddom} T) \subset \operatorname{ddom}(S - T)$.

Definition. If E is a smooth frame system of $\mathcal{M}$, Ω is the dual of E, T is a tensor field of $\mathcal{M}$, and $\operatorname{ddom} T \cap \operatorname{dom} E \neq \varnothing$, then by the E-**differential of** T is meant

$$dT^m \otimes e_m \quad \text{if} \quad T = T^m e_m \quad \text{on } \operatorname{dom} T \cap \operatorname{dom} E, \tag{112}$$

$$dT_m \otimes \omega^m \quad \text{if} \quad T = T_m \omega^m \quad \text{on } \operatorname{dom} T \cap \operatorname{dom} E, \tag{113}$$

$$dT^m{}_n \otimes \omega^n \otimes e_m \quad \text{if} \quad T = T^m{}_n(\omega^n \otimes e_m) \quad \text{on } \operatorname{dom} T \cap \operatorname{dom} E, \tag{114}$$

$$dT_{mn} \otimes \omega^n \otimes \omega^m \quad \text{if} \quad T = T_{mn}(\omega^n \otimes \omega^m) \quad \text{on } \operatorname{dom} T \cap \operatorname{dom} E, \tag{115}$$

$$dT_m{}^n \otimes e_n \otimes \omega^m \quad \text{if} \quad T = T_m{}^n(e_n \otimes \omega^m) \quad \text{on } \operatorname{dom} T \cap \operatorname{dom} E, \tag{116}$$

$$dT^{mn} \otimes e_n \otimes e_m \quad \text{if} \quad T = T^{mn}(e_n \otimes e_m) \quad \text{on } \operatorname{dom} T \cap \operatorname{dom} E, \tag{117}$$

$$dT^m{}_{nk} \otimes \omega^k \otimes \omega^n \otimes e_m \quad \text{if} \quad T = T^m{}_{nk}(\omega^k \otimes \omega^n \otimes e_m) \quad \text{on } \operatorname{dom} T \cap \operatorname{dom} E, \tag{118}$$

and so on. The E-differential of T is denoted by $d_E T$. If $P \in (\operatorname{ddom} T \cap \operatorname{dom} E)$, then $d_E T(P)$ is called the E-**differential of** T **at** P. If u is a vector field whose domain intersects $\operatorname{ddom} T \cap \operatorname{dom} E$, then $(d_E T)u$ is called the E-**derivative of** T **along** u.

Thus $d_E T$ is a tensor field whose type is that of T augmented by an initial covariant slot, into which the vector field u is inserted to produce the E-derivative of T along u.

Theorem 1. If P is a point of $\mathcal{M}$, and each of E and E' is a frame system of $\mathcal{M}$ differentiable at P, then in order that $\operatorname{Sk}\, d_{E'}\phi(P) = \operatorname{Sk}\, d_E\phi(P)$ for every covector field ϕ of $\mathcal{M}$ differentiable at P it is necessary and sufficient that $A_{[m}{}^{m'}{}_{.n]}(P) = 0$, also that $A_{[m'}{}^{m}{}_{.n']}(P) = 0$, and also that $\big(\omega^{n'} \otimes \omega^{m'} \otimes [e_{m'}, e_{n'}]\big)(P) = \big(\omega^n \otimes \omega^m \otimes [e_m, e_n]\big)(P)$. If these conditions are satisfied, then $\operatorname{Sk}\, d_{E'}\phi(P) = \operatorname{Sk}\, d_E\phi(P)$ for every p-cotensor field ϕ of $\mathcal{M}$ differentiable at P.

(Here $A_m{}^{m'} := \omega^{m'} e_m$, $A_{[m}{}^{m'}{}_{.n]} := \frac{1}{2}(A_m{}^{m'}{}_{.n} - A_n{}^{m'}{}_{.m})$, and $[e_m, e_n] := \frac{1}{2}(e_m e_n - e_n e_m)$.)

Theorem 2. The relation "$\simeq$" defined by $E' \simeq E$ if and only if $A_{[m}{}^{m'}{}_{.n]}(P) = 0$ (hence if and only if $A_{[m'}{}^{m}{}_{.n']}(P) = 0$, hence, also, if and only if $\big(\omega^{n'} \otimes \omega^{m'} \otimes [e_{m'}, e_{n'}]\big)(P) =$

$\big(\omega^n \otimes \omega^m \otimes [e_m, e_n]\big)(P))$ is an equivalence relation on the set of all frame systems differentiable at P, one of whose equivalence classes includes all coordinate frame systems around P.

Corollary. If P is a point of $\mathcal{M}$, and each of E and E' is a coordinate frame system of $\mathcal{M}$ differentiable at P, then $\mathrm{Sk}\ d_{E'}\phi(P) = \mathrm{Sk}\ d_E\phi(P)$ for every p-cotensor field ϕ of $\mathcal{M}$ differentiable at P.

Definition. If ϕ is a p-cotensor field ϕ of $\mathcal{M}$, and $\mathrm{ddom}\,\phi \neq \varnothing$, then by the **exterior differential of** ϕ, denoted $d_\wedge\phi$ (and, on occasion, by $d\phi$), is meant the $(p+1)$-cotensor field whose domain is $\mathrm{ddom}\,\phi$ and whose value at each point P of $\mathrm{ddom}\,\phi$ is $\mathrm{Sk}\ d_E\phi(P)$, where E is any coordinate frame system around P; if ϕ is a scalar field of $\mathcal{M}$, and $\mathrm{ddom}\,\phi \neq \varnothing$, then $d_\wedge\phi := d\phi$. In either case, $d_\wedge\phi(P)$ is called the **exterior differential of** ϕ **at** P.

Theorem 3. If ϕ is a scalar field or a p-cotensor field of $\mathcal{M}$, θ is a scalar field or a q-cotensor field of $\mathcal{M}$, and $(\mathrm{ddom}\,\phi \cap \mathrm{ddom}\,\theta) \neq \varnothing$, then

i. $d_\wedge\phi$ is a $(p+1)$-form of $\mathcal{M}$ and $d_\wedge\theta$ is a $(q+1)$-form of $\mathcal{M}$,

ii. $d_\wedge(\phi+\theta) = d_\wedge\phi + d_\wedge\theta$ if $p = q$,

iii. $d_\wedge(\phi\otimes\theta) = d_\wedge(\phi\,\theta) = d_\wedge\phi\wedge\theta + \phi\, d_\wedge\theta$ if ϕ is a scalar field,

iv. $d_\wedge(\phi\otimes\theta) = d_\wedge(\phi\wedge\theta) = d_\wedge\phi\wedge\theta + (-1)^p(\phi\wedge d_\wedge\theta)$ if ϕ is a p-cotensor field,

v. $d_\wedge\phi = 0$ if $p \geq M$, and

vi. $d_\wedge(d_\wedge\phi) = 0$ if ϕ is C^2.

Definition. If ϕ is a p-form of $\mathcal{M}$, then ϕ is **closed** means that ϕ is differentiable and $d_\wedge\phi = 0$, and ϕ is **exact** means that $p > 0$ and $\phi = d_\wedge\theta$ for some $(p-1)$-form θ.

Theorem 4. If ϕ is a p-form of $\mathcal{M}$, and ϕ is exact, then ϕ is closed.

(This theorem has an approximate converse to the effect that if the domain of ϕ is "simply connected" (a topological requirement) then ϕ is exact if it is closed.)

Corollary. If E is a frame system of $\mathcal{M}$, and Ω is its dual coframe system, then in order that E be a coordinate frame system it is necessary that $d_\wedge\omega^m = 0$ and, equivalently, that $[e_m, e_n] = 0$.

(If the domain of E is "simply connected", then these necessary conditions are also sufficient to cause E to be a coordinate frame system.)

XIII. FRAME SYSTEMIZATIONS AND COVARIANT DIFFERENTIATIONS

Definition. By a **frame systemization of** $\mathcal{M}$ is meant a mapping $\bar{E}$ with domain $\mathcal{M}$ such that if $P \in \mathcal{M}$, then $\bar{E}\colon P \mapsto \bar{E}^P$, a smooth frame system of $\mathcal{M}$ around P. By a **coordinate frame systemization of** $\mathcal{M}$ is meant a frame systemizaton $\bar{E}$ of $\mathcal{M}$ such that if $P \in \mathcal{M}$, then $\bar{E}^P$ is a coordinate frame system.

Note that $\bar{E}^P$ is *not* a *frame* at P, rather is a frame *system* of $\mathcal{M}$ whose domain has P in it. If Q is a point in the domain of $\bar{E}^P$, then $\bar{E}^P(Q)$ *is* a frame at Q; in particular, $\bar{E}^P(P)$ is a frame at P.

Definition. If $\bar{E}$ is a frame systemization of $\mathcal{M}$, then by the **differentiation generated by** $\bar{E}$ is meant the mapping $\bar{d}$ such that

i. the domain of $\bar{d}$ is the set of all tensor fields T of $\mathcal{M}$ such that T is differentiable at some point, and

ii. if $T \in \operatorname{dom} \bar{d}$, then $\bar{d}T$ is a tensor field on $\operatorname{ddom} T$, and

iii. if $T \in \operatorname{dom} \bar{d}$ and $P \in \operatorname{ddom} T$, then $\bar{d}T(P) = d_{\bar{E}^P} T(P)$.

The essence of this definition is that for each point P of $\mathcal{M}$ there is a frame system $\bar{E}^P$ *around* P preferred for differentiating *at* P, in the sense that to differentiate a tensor field T at P it is sufficient to differentiate at P the components of T in the preferred frame system $\bar{E}^P$. Another point Q will have its own preferred frame system $\bar{E}^Q$ for differentiation at Q, even if Q happens to lie in $\operatorname{dom} \bar{E}^P$. An immediate consequence is that if $\bar{E}^P = \{e_{\bar{m}}\}$, with dual $\{\omega^{\bar{m}}\}$, then $\bar{d}e_{\bar{m}}(P) = 0$ and $\bar{d}\omega^{\bar{m}}(P) = 0$.

Definition. By a **covariant differentiation on** $\mathcal{M}$ is meant a mapping $\mathbf{d}$ such that

i. the domain of $\mathbf{d}$ is the set of all tensor fields T of $\mathcal{M}$ such that T is differentiable at some point,

ii. if $T \in \operatorname{dom} \mathbf{d}$, then $\mathbf{d}T$ is a tensor field of $\mathcal{M}$ whose domain is $\operatorname{ddom} T$, and if $P \in \operatorname{ddom} T$ and W is the tensor space that $T(P)$ is in, then $\mathbf{d}T(P)$ is in $T_P \otimes W$,

iii. if each of S, T, and $S + T$ is in $\operatorname{dom} \mathbf{d}$, then $\mathbf{d}(S + T) = \mathbf{d}S + \mathbf{d}T$,

iv. if f is a scalar field on $\mathcal{M}$ differentiable at some point, and each of T and fT is in $\operatorname{dom} \mathbf{d}$, then $\mathbf{d}(f \otimes T) = \mathbf{d}(fT) = df \otimes T + f\mathbf{d}T = df \otimes T + f \otimes \mathbf{d}T$,

v. if each of S, T, and $S \otimes T$ is in $\operatorname{dom} \mathbf{d}$, then $\mathbf{d}(S \otimes T)(\cdot) = \mathbf{d}S(\cdot) \otimes T + S \otimes \mathbf{d}T(\cdot)$, and

vi. If T is the cocontensor field whose value at each point P of $\mathcal{M}$ is the identity mapping of T^P onto T^P, then $\mathbf{d}T = 0$.

If $T \in \operatorname{dom} \mathbf{d}$, then the tensor field $\mathbf{d}T$ is called the **covariant differential of** T (**determined by** $\mathbf{d}$). If u is a vector field whose domain intersects $\operatorname{ddom} T$, then the tensor field $\mathbf{D}_u T$ on $\operatorname{ddom} T \cap \operatorname{dom} u$ defined by $\mathbf{D}_u T := (\mathbf{d}T)u$, that is, $(\mathbf{D}_u T)(P) := \mathbf{d}T(P)u(P)$ if $P \in \operatorname{ddom} T \cap \operatorname{dom} u$, is called the **covariant derivative of** T **along** u (**determined by** $\mathbf{d}$).

Theorem 1. If $\bar{E}$ is a frame systemization of $\mathcal{M}$, then the differentiation generated by $\bar{E}$ is a covariant differentation on $\mathcal{M}$.

Let $\mathbf{d}$ be a covariant differentiaton on $\mathcal{M}$. Let E be a smooth frame system of $\mathcal{M}$, with dual coframe system Ω.

Theorem 2. If u is a vector field of $\mathcal{M}$ that is differentiable at some point of $\operatorname{dom} E$, then on $\operatorname{ddom} u \cap \operatorname{dom} E$

$$\mathbf{d}u = (du^m + u^k \omega_k{}^m) \otimes e_m \tag{119}$$

$$= (u^m{}_{.l} + u^k \Gamma_k{}^m{}_l)\omega^l \otimes e_m, \tag{120}$$

$$= (u^m{}_{.l} + u^k \Gamma_k{}^m{}_l)(\omega^l \otimes e_m), \tag{121}$$

and in particular

$$\mathbf{d}e_k = \omega_k{}^m \otimes e_m \tag{122}$$

$$= \Gamma_k{}^m{}_l\, \omega^l \otimes e_m, \tag{123}$$

$$= \Gamma_k{}^m{}_l\, (\omega^l \otimes e_m), \tag{124}$$

where

$$\omega_k{}^m = \omega^m \mathbf{d}e_k \quad \text{and} \quad \Gamma_k{}^m{}_l = \omega_k{}^m e_l, \tag{125}$$

in consequence of which $\omega_k{}^m = \Gamma_k{}^m{}_l\, \omega^l$.

Definition. The covector fields $\omega_k{}^m$ and the scalar fields $\Gamma_k{}^m{}_l$ are called, respectively, the **1-forms** and the **coefficients in** E **of d** (and **of d in** E).

Theorem 3. If v is a covector field of $\mathcal{M}$ that is differentiable at some point of $\operatorname{dom} E$, then on $\operatorname{ddom} v \cap \operatorname{dom} E$

$$\mathbf{d}v = (dv_k - v_m \omega_k{}^m) \otimes \omega^k \tag{126}$$

$$= (v_{k.l} - v_m \Gamma_k{}^m{}_l)\omega^l \otimes \omega^k, \tag{127}$$

$$= (v_{k.l} - v_m \Gamma_k{}^m{}_l)(\omega^l \otimes \omega^k), \tag{128}$$

and in particular

$$\mathbf{d}\omega^m = -\omega_k{}^m \otimes \omega^k \tag{129}$$

$$= -\Gamma_k{}^m{}_l\, \omega^l \otimes \omega^k, \tag{130}$$

$$= -\Gamma_k{}^m{}_l\, (\omega^l \otimes \omega^k). \tag{131}$$

Theorem 4. If E' is a smooth frame system of $\mathcal{M}$ whose domain overlaps that of E, then on the overlap

$$\omega_{k'}{}^{m'} = (dA_{k'}{}^{m})A_{m}{}^{m'} + A_{k'}{}^{k}\omega_{k}{}^{m}A_{m}{}^{m'} \tag{132}$$

and

$$\Gamma_{k'}{}^{m'}{}_{l'} = A_{k'}{}^{m}{}_{.l'}A_{m}{}^{m'} + A_{k'}{}^{k}\Gamma_{k}{}^{m}{}_{l}A_{l'}{}^{l}A_{m}{}^{m'}, \tag{133}$$

where $[A_{m'}{}^{m}]$ is the transition matrix (field) from E to E'.

Theorem 5. If $\bar{E}$ is a frame systemization of $\mathcal{M}$ that generates $\mathbf{d}$, and P is a point of $\mathcal{M}$, then the 1-forms and the coefficients of $\mathbf{d}$ in $\bar{E}^{P}$ vanish at P, and if $P \in \operatorname{dom} E$, then

$$\omega_{k}{}^{m}(P) = dA_{k}{}^{\bar{m}}(P)A_{\bar{m}}{}^{m}(P) \quad \text{and} \quad \Gamma_{k}{}^{m}{}_{l}(P) = A_{k}{}^{\bar{m}}{}_{.l}(P)A_{\bar{m}}{}^{m}(P), \tag{134}$$

where $[A_{m}{}^{\bar{m}}]$ is the transition matrix from $\bar{E}^{P}$ to E.

Theorem 6. There is a frame systemization $\bar{E}$ of $\mathcal{M}$ that generates $\mathbf{d}$. In order that the frame systemization $\bar{\bar{E}}$ of $\mathcal{M}$ also generate $\mathbf{d}$ it is necessary and sufficient that, at each point P, $dA_{\bar{\bar{m}}}{}^{\bar{m}}(P) = 0$ and, equivalently, $dA_{\bar{m}}{}^{\bar{\bar{m}}}(P) = 0$, where $[A_{\bar{\bar{m}}}{}^{\bar{m}}]$ is the transition matrix from $\bar{E}^{P}$ to $\bar{\bar{E}}^{P}$.

Theorem 7. Let T be a tensor field of $\mathcal{M}$ that is differentiable at some point of $\operatorname{dom} E$.

If

$$T = T^{m}{}_{n}(\omega^{n} \otimes e_{m}) \quad \text{on } \operatorname{dom} T \cap \operatorname{dom} E, \tag{135}$$

then on $\operatorname{ddom} T \cap \operatorname{dom} E$

$$\mathbf{d}T = (dT^{m}{}_{n} - T^{m}{}_{p}\,\omega_{n}{}^{p} + T^{p}{}_{n}\,\omega_{p}{}^{m}) \otimes (\omega^{n} \otimes e_{m}) \tag{136}$$

$$= (T^{m}{}_{n.l} - T^{m}{}_{p}\Gamma_{n}{}^{p}{}_{l} + T^{p}{}_{n}\Gamma_{p}{}^{m}{}_{l})\,\omega^{l} \otimes (\omega^{n} \otimes e_{m}) \tag{137}$$

$$= (T^{m}{}_{n.l} - T^{m}{}_{p}\Gamma_{n}{}^{p}{}_{l} + T^{p}{}_{n}\Gamma_{p}{}^{m}{}_{l})(\omega^{l} \otimes \omega^{n} \otimes e_{m}). \tag{138}$$

If

$$T = T_{mn}(\omega^{n} \otimes \omega^{m}) \quad \text{on } \operatorname{dom} T \cap \operatorname{dom} E, \tag{139}$$

then on $\operatorname{ddom} T \cap \operatorname{dom} E$

$$\mathbf{d}T = (dT_{mn} - T_{mp}\,\omega_{n}{}^{p} - T_{pn}\,\omega_{m}{}^{p}) \otimes (\omega^{n} \otimes \omega^{m}) \tag{140}$$

$$= (T_{mn.l} - T_{mp}\Gamma_{n}{}^{p}{}_{l} - T_{pn}\Gamma_{m}{}^{p}{}_{l})\,\omega^{l} \otimes (\omega^{n} \otimes \omega^{m}) \tag{141}$$

$$= (T_{mn.l} - T_{mp}\Gamma_{n}{}^{p}{}_{l} - T_{pn}\Gamma_{m}{}^{p}{}_{l})(\omega^{l} \otimes \omega^{n} \otimes \omega^{m}). \tag{142}$$

If

$$T = T_m{}^n (e_n \otimes \omega^m) \quad \text{on } \operatorname{dom} T \cap \operatorname{dom} E, \tag{143}$$

then on ddom $T \cap \operatorname{dom} E$

$$\begin{aligned} \mathbf{d}T &= (dT_m{}^n + T_m{}^p\,\omega_p{}^n - T_p{}^n\,\omega_m{}^p) \otimes (e_n \otimes \omega^m) && (144)\\ &= (T_m{}^n{}_{.l} + T_m{}^p\Gamma_p{}^n{}_l - T_p{}^n\Gamma_m{}^p{}_l)\,\omega^l \otimes (e_n \otimes \omega^m) && (145)\\ &= (T_m{}^n{}_{.l} + T_m{}^p\Gamma_p{}^n{}_l - T_p{}^n\Gamma_m{}^p{}_l)(\omega^l \otimes e_n \otimes \omega^m). && (146) \end{aligned}$$

If

$$T = T^{mn}(e_n \otimes e_m) \quad \text{on } \operatorname{dom} T \cap \operatorname{dom} E, \tag{147}$$

then on ddom $T \cap \operatorname{dom} E$

$$\begin{aligned} \mathbf{d}T &= (dT^{mn} + T^{mp}\omega_p{}^n + T^{pn}\omega_p{}^m) \otimes (e_n \otimes e_m) && (148)\\ &= (T^{mn}{}_{.l} + T^{mp}\Gamma_p{}^n{}_l + T^{pn}\Gamma_p{}^m{}_l)\,\omega^l \otimes (e_n \otimes e_m) && (149)\\ &= (T^{mn}{}_{.l} + T^{mp}\Gamma_p{}^n{}_l + T^{pn}\Gamma_p{}^m{}_l). && (150) \end{aligned}$$

If

$$T = T^m{}_{nk}(\omega^k \otimes \omega^n \otimes e_m) \quad \text{on } \operatorname{dom} T \cap \operatorname{dom} E, \tag{151}$$

then on ddom $T \cap \operatorname{dom} E$

$$\begin{aligned} \mathbf{d}T &= (dT^m{}_{nk} - T^m{}_{np}\,\omega_k{}^p - T^m{}_{pk}\,\omega_n{}^p + T^p{}_{nk}\,\omega_p{}^m) \otimes (\omega^k \otimes \omega^n \otimes e_m) && (152)\\ &= (T^m{}_{nk.l} - T^m{}_{np}\Gamma_k{}^p{}_l - T^m{}_{pk}\Gamma_n{}^p{}_l + T^p{}_{nk}\Gamma_p{}^m{}_l)\,\omega^l \otimes (\omega^k \otimes \omega^n \otimes e_m) && (153)\\ &= (T^m{}_{nk.l} - T^m{}_{np}\Gamma_k{}^p{}_l - T^m{}_{pk}\Gamma_n{}^p{}_l + T^p{}_{nk}\Gamma_p{}^m{}_l)(\omega^l \otimes \omega^k \otimes \omega^n \otimes e_m). && (154) \end{aligned}$$

If

$$T = T_{mnk}(\omega^k \otimes \omega^n \otimes \omega^m) \quad \text{on } \operatorname{dom} T \cap \operatorname{dom} E, \tag{155}$$

then on ddom $T \cap \operatorname{dom} E$

$$\begin{aligned} \mathbf{d}T &= (dT_{mnk} - T_{mnp}\,\omega_k{}^p - T_{mpk}\,\omega_n{}^p - T_{pnk}\,\omega_m{}^p) \otimes (\omega^k \otimes \omega^n \otimes \omega^m) && (156)\\ &= (T_{mnk.l} - T_{mnp}\Gamma_k{}^p{}_l - T_{mpk}\Gamma_n{}^p{}_l - T_{pnk}\Gamma_m{}^p{}_l)\,\omega^l \otimes (\omega^k \otimes \omega^n \otimes \omega^m) && (157)\\ &= (T_{mnk.l} - T_{mnp}\Gamma_k{}^p{}_l - T_{mpk}\Gamma_n{}^p{}_l - T_{pnk}\Gamma_m{}^p{}_l)(\omega^l \otimes \omega^k \otimes \omega^n \otimes \omega^m). && (158) \end{aligned}$$

For each component t of T in E the corresponding component in E of $\mathbf{D}_{e_l}T$, that is, of $(\mathbf{d}T)e_l$, is denoted by $t_{:l}$. Thus, if $T = u = u^m e_m$, then $\mathbf{D}_{e_l}T = (\mathbf{d}u)e_l = (u^m{}_{.h} + u^k\Gamma_k{}^m{}_h)(\omega^h e_l)e_m = u^m{}_{:l}e_m$, where $u^m{}_{:l} = u^m{}_{.l} + u^k\Gamma_k{}^m{}_l$, and therefore $\mathbf{d}u = u^m{}_{:l}\,\omega^l \otimes e_m$. Similarly, if $v = v_k\omega^k$, then $\mathbf{d}v = v_{k:l}\,\omega^l\otimes\omega^k$, where $v_{k:l} = v_{k.l} - v_m\Gamma_k{}^m{}_l$. Likewise, if $T = T^m{}_n(\omega^n\otimes e_m)$, then $\mathbf{d}T = T^m{}_{n:l}(\omega^l \otimes \omega^n \otimes e_m)$, where $T^m{}_{n:l} = T^m{}_{n.l} - T^m{}_p\Gamma_n{}^p{}_l + T^p{}_n\Gamma_p{}^m{}_l$, and so on.

Covariant differentiation commutes with the operation of contraction, as the following calculation illustrates. If $T = T^m{}_{kl}(\omega^l \otimes \omega^k \otimes e_m)$, then $T\,{}^3_1 = v_k\,\omega^k$, where $v_k = T^p{}_{kp}$, so $\mathbf{d}(T\,{}^3_1) = v_{k:l}(\omega^l \otimes \omega^k)$, with

$$v_{k:l} = v_{k.l} - v_m\Gamma_k{}^m{}_l = (T^p{}_{kp})_{.l} - T^p{}_{mp}\Gamma_k{}^m{}_l. \tag{159}$$

On the other hand $\mathbf{d}T = T^m{}_{kl:n}\,\omega^n \otimes (\omega^l \otimes \omega^k \otimes e_m)$, where

$$T^m{}_{kl:n} = T^m{}_{kl.n} - T^m{}_{kp}\Gamma_l{}^p{}_n - T^m{}_{pl}\Gamma_k{}^p{}_n + T^p{}_{kl}\Gamma_p{}^m{}_n, \tag{160}$$

so $(\mathbf{d}T)\,{}^4_2 = T^m{}_{km:n}(\omega^n \otimes \omega^k) = T^m{}_{km:l}(\omega^l \otimes \omega^k)$, with

$$T^m{}_{km:l} = \delta^m{}_r(T^r{}_{ks.l})\delta^s{}_m - T^m{}_{kp}\Gamma_m{}^p{}_l - T^m{}_{pm}\Gamma_k{}^p{}_l + T^p{}_{km}\Gamma_p{}^m{}_l \tag{161}$$

$$= (\delta^m{}_r T^r{}_{ks}\delta^s{}_m)_{.l} - T^m{}_{pm}\Gamma_k{}^p{}_l \tag{162}$$

$$= (T^p{}_{kp})_{.l} - T^p{}_{mp}\Gamma_k{}^m{}_l \tag{163}$$

$$= v_{k:l}, \tag{164}$$

in consequence of which $(\mathbf{d}T)\,{}^4_2 = \mathbf{d}(T\,{}^3_1)$.

Divergence of a vector field, as defined in euclidean spaces, has a covariant analog. If u is a vector field of $\mathcal{M}$, differentiable at some point, then on ddom u

$$\mathbf{Div}\ u := \mathrm{Tr}\ \mathbf{d}u := (\mathbf{d}u)\,{}^2_1 = \omega^p(\mathbf{d}u)e_p; \tag{165}$$

if ddom u and dom E overlap, then on the overlap

$$\mathbf{Div}\ u = \omega^p(u^m{}_{:l}\,\omega^l \otimes e_m)e_p = u^m{}_{:l}\,(\omega^l e_p)(\omega^p e_m) = \delta^p{}_m u^m{}_{:l}\,\delta^l{}_p = u^p{}_{:p}. \tag{166}$$

More generally, if T is a tensor field with at least one contravariant slot, then for each of its contravariant slots T has a divergence $(\mathbf{d}T)\,{}^{\mathrm{b}+1}_1$, where b is the position number in T of the contravariant slot in question. If, for example, $T = T^{mn}{}_k(\omega^k \otimes e_n \otimes e_m)$ on dom $T \cap$ dom E, then, on ddom $T \cap$ dom E, $(\mathbf{d}T)\,{}^2_1 = T^{mp}{}_{k:p}(\omega^k \otimes e_m)$ and $(\mathbf{d}T)\,{}^3_1 = T^{pn}{}_{k:p}(\omega^k \otimes e_n)$.

Definition. If T is a tensor field of $\mathcal{M}$ differentiable at some point, then T is said to be **autoparallel** (**with respect to d**) if and only if there is a covector field λ on ddom T such that $\mathbf{d}T = \lambda \otimes T$; if λ is continuous, then T is said to be **continuously autoparallel** (**with respect to d**); if $\lambda = 0$, so that $\mathbf{d}T = 0$, then T is said to be **covariantly constant** (**with respect to d**).

An alternative approach to covariant differentiation is by way of the notion of a *connection*.

Definition. By a **connection on** $\mathcal{M}$ (called also an **affine connection on** $\mathcal{M}$) is meant a mapping Γ whose domain is the maximal atlas of $\mathcal{M}$ and which assigns to each coordinate system X a cococontensor field $\Gamma(X)$ on $\operatorname{dom} X$ in such a way that if $\operatorname{dom} X \cap \operatorname{dom} X' \neq \varnothing$, and $\Gamma(X) = \Gamma_k{}^m{}_l\,(dx^l \otimes dx^k \otimes \partial/\partial x^m)$ and $\Gamma(X') = \Gamma_{k'}{}^{m'}{}_{l'}(dx^{l'} \otimes dx^{k'} \otimes \partial/\partial x^{m'})$, then, on $\operatorname{dom} X \cap \operatorname{dom} X'$,

$$\Gamma_{k'}{}^{m'}{}_{l'} = \frac{\partial^2 x^m}{\partial x^{l'} \partial x^{k'}} \frac{\partial x^{m'}}{\partial x^m} + \frac{\partial x^k}{\partial x^{k'}} \Gamma_k{}^m{}_l \frac{\partial x^l}{\partial x^{l'}} \frac{\partial x^{m'}}{\partial x^m}. \tag{167}$$

The scalar fields $\Gamma_k{}^m{}_l$ are called the **coefficients in** X **of** Γ (and **of** Γ **in** X). The covector fields $\omega_k{}^m$ defined by $\omega_k{}^m := \Gamma_k{}^m{}_l\, dx^l$ are called the **1-forms in** X **of** Γ (and **of** Γ **in** X).

Definition. If Γ is a connection on $\mathcal{M}$, X is a coordinate system of $\mathcal{M}$, T is a tensor field of $\mathcal{M}$, and $\operatorname{ddom} T \cap \operatorname{dom} X \neq \varnothing$, then by the $\Gamma(X)$-**differential of** T is meant

$$(dT^m + T^p \omega_p{}^m) \otimes \frac{\partial}{\partial x^m} \qquad \text{if} \quad T = T^m \frac{\partial}{\partial x^m} \qquad \text{on } \operatorname{ddom} T \cap \operatorname{dom} X, \tag{168}$$

$$(dT_m - T_p\, \omega_m{}^p) \otimes dx^m \qquad \text{if} \quad T = T_m\, dx^m \qquad \text{on } \operatorname{ddom} T \cap \operatorname{dom} X, \tag{169}$$

$$(dT^m{}_n - T^m{}_p\, \omega_n{}^p + T^p{}_n\, \omega_p{}^m) \otimes \left(dx^n \otimes \frac{\partial}{\partial x^m}\right) \qquad \text{if} \quad T = T^m{}_n \left(dx^n \otimes \frac{\partial}{\partial x^m}\right) \qquad \text{on } \operatorname{ddom} T \cap \operatorname{dom} X, \tag{170}$$

$$(dT_{mn} - T_{mp}\, \omega_n{}^p - T_{pn}\, \omega_m{}^p) \otimes (dx^n \otimes dx^m) \qquad \text{if} \quad T = T_{mn}\,(dx^n \otimes dx^m) \qquad \text{on } \operatorname{ddom} T \cap \operatorname{dom} X, \tag{171}$$

$$(dT_m{}^n - T_m{}^p\, \omega_p{}^n + T_p{}^n\, \omega_m{}^p) \otimes \left(\frac{\partial}{\partial x^n} \otimes dx^m\right) \qquad \text{if} \quad T = T_m{}^n \left(\frac{\partial}{\partial x^n} \otimes dx^m\right) \qquad \text{on } \operatorname{ddom} T \cap \operatorname{dom} X, \tag{172}$$

$$(dT^{mn} + T^{mp} \omega_p{}^n + T^{pn} \omega_p{}^m) \otimes \left(\frac{\partial}{\partial x^n} \otimes \frac{\partial}{\partial x^m}\right) \qquad \text{if} \quad T = T^{mn} \left(\frac{\partial}{\partial x^n} \otimes \frac{\partial}{\partial x^m}\right) \qquad \text{on } \operatorname{ddom} T \cap \operatorname{dom} X, \tag{173}$$

and so on. The $\Gamma(X)$-differential of T is denoted by $d_{\Gamma(X)}T$.

Theorem 8. If Γ is a connection on $\mathcal{M}$, each of X and X' is a coordinate system of $\mathcal{M}$, T is a tensor field of $\mathcal{M}$, and $\operatorname{ddom} T \cap \operatorname{dom} X \cap \operatorname{dom} X' \neq \varnothing$, then $d_{\Gamma(X')}T = d_{\Gamma(X)}T$ on $\operatorname{ddom} T \cap \operatorname{dom} X \cap \operatorname{dom} X'$.

Definition. If Γ is a connection on $\mathcal{M}$, then by the **differentiation generated by** Γ is meant the mapping d_Γ such that

i. the domain of d_Γ is the set of all tensor fields T of $\mathcal{M}$ such that T is differentiable at some point,

ii. if $T \in \operatorname{dom} d_\Gamma$, then $d_\Gamma T$ is a tensor field on $\operatorname{ddom} T$, and

iii. if $T \in \operatorname{dom} d_\Gamma$, X is a coordinate system of $\mathcal{M}$, and $\operatorname{ddom} T \cap \operatorname{dom} X \neq \varnothing$, then $d_\Gamma T = d_{\Gamma(X)} T$ on $\operatorname{ddom} T \cap \operatorname{dom} X$.

Theorem 9. If Γ is a connection on $\mathcal{M}$, then d_Γ is a covariant differentiation on $\mathcal{M}$, and if X is a coordinate system of $\mathcal{M}$, then the 1-forms and the coefficients of this covariant differentiation in the frame system $\{\partial/\partial x^m\}$ are the 1-forms and coefficients in X of Γ. Conversely, if $\mathbf{d}$ is a covariant differentiation on $\mathcal{M}$, and, for each coordinate system X of $\mathcal{M}$, $\Gamma(X) = \Gamma_k{}^m{}_l\,(dx^l \otimes dx^k \otimes \partial/\partial x^m)$, where the $\Gamma_k{}^m{}_l$ are the coefficients of $\mathbf{d}$ in $\{\partial/\partial x^m\}$, then Γ is a connection on $\mathcal{M}$, and $\mathbf{d} = d_\Gamma$.

The import of this theorem is that every covariant differentiation determines and is determined by a connection, and every connection determines and is determined by a covariant differentiation. When $\mathbf{d}$ and Γ determine one another, they are said to be **associated**.

XIV. TORSION AND CURVATURE

Continue with $\mathbf{d}$ a covariant differentiation on $\mathcal{M}$, E a smooth frame system of $\mathcal{M}$, and Ω its dual.

Definition. By the **exterior differentiation of d** is meant the mapping $\mathbf{d}_\wedge$ whose domain is the set of all scalar, p-cotensor, or p-cocontensor fields ϕ of $\mathcal{M}$ that are differentiable at some point, given by $\mathbf{d}_\wedge\phi := d\phi$ if ϕ is a scalar field, and $\mathbf{d}_\wedge\phi := \mathrm{Sk}\,\mathbf{d}\phi$ if ϕ is a tensor field.

Theorem 1. There is a unique tensor field $\mathbf{T}$ on $\mathcal{M}$ with the property that if v is a covector field of $\mathcal{M}$ that is differentiable at P, then $\mathbf{d}_\wedge v(P) = d_\wedge v(P) - v(P)\mathbf{T}(P)$; $\mathbf{T}$ is a skew-symmetric cococontensor field, and, on $\mathrm{dom}\,E$, $\mathbf{T} = \mathbf{T}^m \otimes e_m$, where

$$\mathbf{T}^m = d_\wedge\omega^m - \mathbf{d}_\wedge\omega^m \tag{174}$$

$$= d_\wedge\omega^m - \omega^k \wedge {\omega_k}^m \tag{175}$$

$$= d_\wedge\omega^m + {\omega_k}^m \wedge \omega^k \tag{176}$$

$$= ({C_k}^m{}_l + {\Gamma_k}^m{}_l)(\omega^l \wedge \omega^k) \tag{177}$$

$$= ({C_k}^m{}_l + \Gamma_{[k}{}^m{}_{l]})(\omega^l \otimes \omega^k), \tag{178}$$

if $d_\wedge\omega^m = {C_k}^m{}_l(\omega^l \wedge \omega^k) = {C_k}^m{}_l(\omega^l \otimes \omega^k)$.

Definition. The tensor field $\mathbf{T}$ is called the **torsion of d**, and the skew-symmetric cocotensor fields $\mathbf{T}^m$ are called the **torsion 2-forms in** E **of d** (and **of d in** E).

Theorem 2. If $\mathbf{d}_\wedge v = d_\wedge v$ for every covector field v of $\mathcal{M}$ such that $\mathrm{ddom}\,v \neq \varnothing$, then $\mathbf{T} = 0$. If $\mathbf{T} = 0$, then $\mathbf{d}_\wedge\phi = d_\wedge\phi$ for every p-cotensor field ϕ of $\mathcal{M}$ such that $\mathrm{ddom}\,\phi \neq \varnothing$.

Theorem 3. If $P \in \mathcal{M}$, then $\mathbf{T}(P) = 0$ if and only if $\mathbf{d}$ is generated by a frame systemization $\bar{E}$ such that $\bar{E}^P$ is a coordinate frame system.

Theorem 4. There is a unique tensor field Θ on $\mathcal{M}$ with the property that if u is a vector field of $\mathcal{M}$ that is twice differentiable at P, then $\mathbf{d}_\wedge(\mathbf{d}u)(P) = \frac{1}{2}\Theta(P)u(P) - \mathbf{d}u(P)\mathbf{T}(P)$; Θ is a cocococontensor field, skew-symmetric in the second and third slots, and, on $\mathrm{dom}\,E$, $\Theta = \omega^k \otimes {\Theta_k}^m \otimes e_m = \omega^k \otimes {\Theta_k}^m{}_{ln}\,\omega^n \otimes \omega^l \otimes e_m$, where

$${\Theta_k}^m = 2(d_\wedge{\omega_k}^m - {\omega_k}^p \wedge {\omega_p}^m) \tag{179}$$

$$= {\Theta_k}^m{}_{ln}\,(\omega^n \wedge \omega^l) \tag{180}$$

$$= {\Theta_k}^m{}_{ln}\,(\omega^n \otimes \omega^l), \tag{181}$$

with

$${\Theta_k}^m{}_{ln} = 2({\Gamma_k}^m{}_{[l.n]} + {\Gamma_k}^p{}_{[l}\Gamma_{|p|}{}^m{}_{n]} + {\Gamma_k}^m{}_p {C_l}^p{}_n), \tag{182}$$

if $d_\wedge\omega^m = {C_k}^m{}_l(\omega^l \wedge \omega^k) = {C_k}^m{}_l(\omega^l \otimes \omega^k)$.

Definition. The tensor field Θ is called the **curvature of d**, and the skew-symmetric cocotensor fields $\Theta_k{}^m$ are called the **curvature 2-forms in** E **of d** (and **of d in** E). The contraction Θ_2^4, a cocotensor field on $\mathcal{M}$, is called the **contracted curvature of d** and is denoted by Φ.

Theorem 5. On $\operatorname{dom} E$, $\Phi = \omega^k \otimes \Theta_k{}^m e_m = \omega^k \otimes \Phi_{kl}\,\omega^l$, with $\Phi_{kl} = \Theta_k{}^m e_m e_l = \Theta_k{}^m{}_{lm}$.

On $\operatorname{dom} E$

$$d_\wedge \mathbf{T}^m = d_\wedge(d_\wedge\omega^m - \mathbf{d}_\wedge\omega^m) \tag{183}$$

$$= d_\wedge(\omega_k{}^m \wedge \omega^k) \tag{184}$$

$$= d_\wedge\omega_k{}^m \wedge \omega^k - \omega_k{}^m \wedge d_\wedge\omega^k \tag{185}$$

$$= (d_\wedge\omega_k{}^m - \omega_k{}^p \wedge \omega_p{}^m) \wedge \omega^k + \omega_k{}^p \wedge \omega_p{}^m \wedge \omega^k - \omega_k{}^m \wedge d_\wedge\omega^k \tag{186}$$

$$= \tfrac{1}{2}\Theta_k{}^m \wedge \omega^k - \omega_k{}^p \wedge \omega^k \wedge \omega_p{}^m - d_\wedge\omega^k \wedge \omega_k{}^m \tag{187}$$

$$= \tfrac{1}{2}(\omega^k \wedge \Theta_k{}^m) + (\mathbf{d}_\wedge\omega^k - d_\wedge\omega^k) \wedge \omega_k{}^m \tag{188}$$

$$= \tfrac{1}{2}(\omega^k \wedge \Theta_k{}^m) - \mathbf{T}^k \wedge \omega_k{}^m, \tag{189}$$

and therefore

$$\omega^k \wedge \Theta_k{}^m = 2(d_\wedge \mathbf{T}^m + \mathbf{T}^k \wedge \omega_k{}^m), \tag{190}$$

which is equivalent to

$$\operatorname{Sk}\,\Theta = 2\left((d_\wedge \mathbf{T}^m + \mathbf{T}^k \wedge \omega_k{}^m) \otimes e_m\right), \tag{191}$$

inasmuch as $\operatorname{Sk}\,\Theta = \omega^k \wedge \Theta_k{}^m \otimes e_m$. These reduce, when $d_\wedge \mathbf{T} + \mathbf{T}^k \wedge \omega_k{}^m = 0$, in particular when $\mathbf{T} = 0$, to $\omega^k \wedge \Theta_k{}^m = 0$ and $\operatorname{Sk}\,\Theta = 0$, which are equivalent to $\Theta_{[k}{}^m{}_{ln]} = 0$, in turn equivalent to

$$\Theta_k{}^m{}_{ln} + \Theta_l{}^m{}_{nk} + \Theta_n{}^m{}_{kl} = 0, \tag{192}$$

the **first Bianchi identity**.

Also, on $\operatorname{dom} E$

$$d_\wedge \Theta_k{}^m = 2 d_\wedge(d_\wedge\omega_k{}^m - \omega_k{}^p \wedge \omega_p{}^m) \tag{193}$$

$$= -2(d_\wedge\omega_k{}^p \wedge \omega_p{}^m - \omega_k{}^p \wedge d_\wedge\omega_p{}^m) \tag{194}$$

$$= -2\big((d_\wedge\omega_k{}^p - \omega_k{}^q \wedge \omega_q{}^p) \wedge \omega_p{}^m - \omega_k{}^p \wedge (d_\wedge\omega_p{}^m - \omega_p{}^q \wedge \omega_q{}^m)\big), \tag{195}$$

and therefore

$$d_\wedge \Theta_k{}^m = \omega_k{}^p \wedge \Theta_p{}^m - \Theta_k{}^p \wedge \omega_p{}^m. \tag{196}$$

From

$$\mathbf{d}\Theta_k{}^m(\cdot) = \mathbf{d}(\omega^m \Theta e_k)(\cdot) \tag{197}$$

$$= \mathbf{d}\omega^m(\cdot)\Theta e_k + \omega^m \mathbf{d}\Theta(\cdot)e_k + \omega^m\Theta\mathbf{d}e_k(\cdot) \tag{198}$$

$$= -\omega_p{}^m(\cdot) \otimes \omega^p\Theta e_k + \omega^m\mathbf{d}\Theta(\cdot)e_k + \omega^m\Theta(\omega_k{}^p(\cdot) \otimes e_p) \tag{199}$$

$$= -(\omega_p{}^m \otimes \Theta_k{}^p)(\cdot) + \omega^m\mathbf{d}\Theta(\cdot)e_k + (\omega_k{}^p \otimes \Theta_p{}^m)(\cdot) \tag{200}$$

there follows

$$\mathbf{d}_\wedge\Theta_k{}^m(\cdot) = -\omega_p{}^m \wedge \Theta_k{}^p + \mathrm{Sk}\,(\omega^m\mathbf{d}\Theta(\cdot)e_k) + \omega_k{}^p \wedge \Theta_p{}^m \tag{201}$$

$$= (\omega_k{}^p \wedge \Theta_p{}^m - \Theta_k{}^p \wedge \omega_p{}^m) + \mathrm{Sk}\,(\omega^m\mathbf{d}\Theta(\cdot)e_k), \tag{202}$$

in consequence of which

$$\mathrm{Sk}\,(\omega^m\mathbf{d}\Theta(\cdot)e_k) = \mathbf{d}_\wedge\Theta_k{}^m - d_\wedge\Theta_k{}^m. \tag{203}$$

If $\mathbf{d}_\wedge\Theta_k{}^m = d_\wedge\Theta_k{}^m$, which holds if $\mathbf{T} = 0$, then $\mathrm{Sk}\,(\omega^m\mathbf{d}\Theta(\cdot)e_k) = 0$. This is equivalent to $\Theta_k{}^m{}_{[ln:p]} = 0$, which in turn is equivalent to

$$\Theta_k{}^m{}_{ln:p} + \Theta_k{}^m{}_{np:l} + \Theta_k{}^m{}_{pl:n} = 0, \tag{204}$$

the **second Bianchi identity**.

Definition. If $P \in \mathcal{M}$, then $\mathbf{d}$ is **torsion free** (resp., **curvature free**) **at** P means that $\mathbf{T}(P) = 0$ (resp., $\Theta(P) = 0$). If $U \subset \mathcal{M}$, then $\mathbf{d}$ is **torsion free** (resp., **curvature free**) **on** U means that $\mathbf{T}|_U = 0$ (resp., $\Theta|_U = 0$); $\mathbf{d}$ is **torsion free** (resp., **curvature free**) means that $\mathbf{d}$ is torsion free (resp., curvature free) on $\mathcal{M}$.

Definition. If $U \subset \mathcal{M}$, then U is **featureless** (**with respect to d**) means that there is a frame systemization $\bar{E}$ that generates $\mathbf{d}$ and is such that if $P \in U$, then $\bar{E}^P$ is a coordinate frame system; that U is **flat** (**with respect to d**) means that there a frame systemization of $\mathcal{M}$ that generates $\mathbf{d}$ and is constant on U. That $\mathcal{M}$ is **flat** (**with respect to d**) means that there is a collection of one or more open flat subsets of $\mathcal{M}$ that covers $\mathcal{M}$.

Theorem 6. If U is a featureless subset of $\mathcal{M}$, then $\mathbf{d}$ is torsion free on U. If U is a flat subset of $\mathcal{M}$, then $\mathbf{d}$ is curvature free on U. If $\mathcal{M}$ is featureless, then $\mathbf{d}$ is torsion free. If $\mathcal{M}$ is flat, then $\mathbf{d}$ is curvature free.

XV. COVARIANT DERIVATIVES ON PATHS

Let $p : I \to \mathcal{M}$ be a smooth path in $\mathcal{M}$, and let u be a **vector field on** p, that is, $u : I \to \bigcup_{t\in I} T^{p(t)}$, and $u(t) \in T^{p(t)}$ if $t \in I$. Let u be differentiable in the sense that if E is a smooth frame system of $\mathcal{M}$ whose domain intersects the range of p, then each component of u in E is differentiable (as a mapping from I into $\mathbb{R}$). For each such frame system E let

$$\dot{u}_E = (u^m e_m(p))^{\boldsymbol{\cdot}}_E := (u^m)^{\cdot}\, e_m(p) + u^m \mathbf{d}e_m(p)\dot{p}. \tag{205}$$

Then

$$\dot{u}_E = \Big((u^m)^{\cdot} + u^k \omega_k{}^m(p)\dot{p}\Big)\, e_m(p) \tag{206}$$

$$= \Big((u^m)^{\cdot} + u^k \Gamma_k{}^m{}_l(p)\, \omega^l(p)\dot{p}\Big)\, e_m(p) \tag{207}$$

$$= \Big((u^m)^{\cdot} + u^k \Gamma_k{}^m{}_l(p)\dot{p}^l\Big)\, e_m(p). \tag{208}$$

With these it is straightforward to show that if E' is a smooth frame system of $\mathcal{M}$, and $(\operatorname{ran} p \cap \operatorname{dom} E \cap \operatorname{dom} E') \neq \varnothing$, then $\dot{u}_{E'} = \dot{u}_E$ on this intersection, thus that it is justified to define the **covariant** (or **absolute**) **derivative of** u (**determined by d**) as that vector field $\dot{u}$ on p such that $\dot{u}|_{p^{-1}(\operatorname{dom} E)} = \dot{u}_E$ for every smooth frame system E of $\mathcal{M}$ whose domain intersects the range of p.

In the event that there is a differentiable vector field w of $\mathcal{M}$ such that $u = w(p)$, it is easy to see that

$$\dot{u} = (w(p))^{\boldsymbol{\cdot}} = \mathbf{d}w(p)\dot{p}, \tag{209}$$

a chain rule.

Now let v be a **covector field on** p, that is, $v\colon I \to \bigcup_{t\in I} T_{p(t)}$, and $v(t) \in T_{p(t)}$ if $t \in I$. Let v be differentiable in the sense that if E is a smooth frame system of $\mathcal{M}$ whose domain intersects the range of p, then each component of v in E is differentiable (as a mapping from I into $\mathbb{R}$). For each such frame system E let

$$\dot{v}_E = (v_n \omega^n(p))^{\boldsymbol{\cdot}}_E := (v_n)^{\cdot} \omega^n(p) + v_n \mathbf{d}\omega^n(p)\dot{p}. \tag{210}$$

Then

$$\dot{v}_E = \Big((v_n)^{\cdot} - v_k\, \omega_n{}^k(p)\dot{p}\Big)\, \omega^n(p) \tag{211}$$

$$= \Big((v_n)^{\cdot} - v_k \Gamma_n{}^k{}_l(p)\, \omega^l(p)\dot{p}\Big)\, \omega^n(p) \tag{212}$$

$$= \Big((v_n)^{\cdot} - v_k \Gamma_n{}^k{}_l(p)\dot{p}^l\Big)\, \omega^n(p). \tag{213}$$

As in the case of a vector field, it is justified to define the **covariant** (or **absolute**) **derivative of** v (**determined by d**) as that vector field $\dot{v}$ on p such that $\dot{v}|_{p^{-1}(\operatorname{dom} E)} = \dot{v}_E$ for every smooth frame system E of $\mathcal{M}$ whose domain intersects the range of p.

In the event that there is a differentiable covector field w of $\mathcal{M}$ such that $v = w(p)$, then, as above, there is the chain rule

$$\dot{v} = (w(p))^{\boldsymbol{\cdot}} = \mathbf{d}w(p)\dot{p}. \tag{214}$$

In general, if T is any differentiable tensor field on p, and $\dot{T}_E$ is defined in the obvious way, then $\dot{T}_{E'}$ agrees with $\dot{T}_E$ where both are defined. This makes possible the definition of the **covariant** (or **absolute**) **derivative of** T (**determined by d**) as that tensor field $\dot{T}$ on p with the property that $\dot{T}|_{p^{-1}(\operatorname{dom} E)} = \dot{T}_E$ for every smooth frame system E of $\mathcal{M}$ whose domain intersects the range of p, and the chain rule

$$\dot{T} = (W(p))^{\boldsymbol{\cdot}} = \mathbf{d}W(p)\dot{p}, \tag{215}$$

if W is a differentiable tensor field of $\mathcal{M}$ and $T = W(p)$. To illustrate, if $T = T^m{}_n(\omega^n(p) \otimes e_m(p))$ on $p^{-1}(\operatorname{dom} E)$, then on $p^{-1}(\operatorname{dom} E)$

$$\dot{T} = \dot{T}_E = (T^m{}_n)^{\cdot}(\omega^n(p) \otimes e_m(p)) + T^m{}_n\,(\mathbf{d}\omega^n(p)\dot{p} \otimes e_m(p)) + T^m{}_n\,(\omega^n(p) \otimes \mathbf{d}e_m(p)\dot{p}) \tag{216}$$

$$= \Big((T^m{}_n)^{\cdot} - T^m{}_k\,\omega_n{}^k(p)\dot{p} + T^k{}_n\,\omega_k{}^m(p)\dot{p}\Big)\,(\omega^n(p) \otimes e_m(p)) \tag{217}$$

$$= \Big((T^m{}_n)^{\cdot} - T^m{}_k\,\Gamma_n{}^k{}_l(p)\,\omega^l(p)\dot{p} + T^k{}_n\,\Gamma_k{}^m{}_l(p)\,\omega^l(p)\dot{p}\Big)\,(\omega^n(p) \otimes e_m(p)) \tag{218}$$

$$= \Big((T^m{}_n)^{\cdot} - T^m{}_k\,\Gamma_n{}^k{}_l(p)\,\dot{p}^l + T^k{}_n\,\Gamma_k{}^m{}_l(p)\,\dot{p}^l\Big)\,(\omega^n(p) \otimes e_m(p)). \tag{219}$$

Definition. If T is a differentiable tensor field on p, then T is said to be **autoparallel** (**with respect to d**) if and only if there is a mapping $\lambda\colon I \to \mathbb{R}$ such that $\dot{T} = \lambda T$; if λ is continuous, then T is said to be **continuously autoparallel** (**with respect to d**); if $\lambda = 0$, so that $\dot{T} = 0$, then T is said to be **covariantly constant** (**with respect to d**).

Theorem 1. If T is a differentiable tensor field of $\mathcal{M}$, and $\operatorname{ran} p \subset \operatorname{ddom} T$, then $T(p)$ is a differentiable tensor field on p. If T is autoparallel (continuously autoparallel) with respect to $\mathbf{d}$, then $T(p)$ is autoparallel (continuously autoparallel) with respect to $\mathbf{d}$; if T is covariantly constant with respect to $\mathbf{d}$, then $T(p)$ is covariantly constant with respect to $\mathbf{d}$.

XVI. AUTOPARALLEL GEODESICS.

A particular vector field on the smooth path $p\colon I \to \mathcal{M}$ is the velocity $\dot{p}$. If $\dot{p}$ is differentiable, then the covariant derivative of $\dot{p}$ is called the **covariant** (or **absolute**) **acceleration of** p (**determined by d**), or, for short, just the **acceleration of** p, and is denoted by $\ddot{p}$.

Definition. The path p is said to be **autoparallel** (**with respect to d**), and is called an **autoparallel geodesic path** (**of d**) (an **autoparallel geodesic**, for short), if and only if p is doubly smooth and $\dot{p}$ is continuously autoparallel (with respect to **d**), that is, there is a continuous mapping $\lambda\colon I \to \mathbb{R}$ such that $\ddot{p} = \lambda\dot{p}$. If $\lambda = 0$, so that $\ddot{p} = 0$, and therefore $\dot{p}$ is constant (with respect to **d**), then p is said to be **affinely parametrized**. By an **autoparallel geodesic curve** (**of d**) is meant a subset C of $\mathcal{M}$ for which there is an autoparallel geodesic path p of **d** whose range is C and whose velocity $\dot{p}$ vanishes nowhere.

Definition. If ϕ is a C^1 mapping of an interval J onto I whose derivative vanishes nowhere, and $q := p(\phi)$, then q is said to be a **smooth regular reparametrization of** p (**by** ϕ), **sense-preserving** if $\dot{\phi} > 0$, **sense-reversing** if $\dot{\phi} < 0$. If both p and ϕ are C^2, then q is said to be a **doubly smooth regular reparametrization of** p (**by** ϕ).

Theorem 1. If q is a smooth regular reparametrization of p by ϕ, then q is a smooth path in $\mathcal{M}$, and $\dot{q} = \dot{p}(\phi)\,\dot{\phi}$. If q is a doubly smooth regular reparametrization of p by ϕ, then q is a doubly smooth path in $\mathcal{M}$, and $\ddot{q} = \ddot{p}(\phi)\,\dot{\phi}^2 + \dot{p}(\phi)\,\ddot{\phi}$.

Corollary. If p is autoparallel, then q is autoparallel. If p is affinely parametrized and is not a constant, then q is affinely parametrized if and only if $\ddot{\phi} = 0$, thus if and only if $\phi(s) = \alpha s + \beta$ for some nonzero number α and some number β.

Corollary. If p is autoparallel, with $\ddot{p} = \lambda\dot{p}$, and θ is any nonconstant solution on I of the linear differential equation $\ddot{\theta} = \lambda\dot{\theta}$, then $\dot{\theta} > 0$ or $\dot{\theta} < 0$, θ has an inverse ϕ, and if $q = p(\phi)$, then

i. q is a doubly smooth regular reparametrization of p, so q is autoparallel,

ii. q is affinely parametrized,

iii. q is sense-preserving if $\dot{\theta} > 0$,

iv. q is sense-reversing if $\dot{\theta} < 0$.

Every doubly smooth regular reparametrization of p that is affinely parametrized is obtained in this way.

Theorem 2. If p is an affinely parametrized autoparallel geodesic path, q is an affinely parametrized doubly smooth regular reparametrization of p by ϕ, s_1 and s_2 are two numbers in $\operatorname{dom}\phi$, and s_3 and s_4 are two numbers in $\operatorname{dom}\phi$, then

$$\frac{s_4 - s_3}{s_2 - s_1} = \frac{r_4 - r_3}{r_2 - r_1}, \tag{220}$$

where $r_i = \phi(s_i)$ for $i = 1, 2, 3, 4$.

The import of this theorem is that on each autoparallel geodesic curve C of $\mathbf{d}$ there is an invariant *relative* measure of the separation of two points P_3 and P_4 as compared to the separation of two points P_1 and P_2, given by the ratio $(r_4 - r_3)/(r_2 - r_1)$, where $P_i = p(r_i)$ for $i = 1, 2, 3, 4$, this ratio being the same for all affinely parametrized autoparallel geodesic paths p of $\mathbf{d}$ with range C.

Definition. That the affinely parametrized autoparallel geodesic path $p\colon I \to \mathcal{M}$ is **maximal** means that if $\bar{p}\colon \bar{I} \to \mathcal{M}$ is an affinely parametrized autoparallel geodesic path, and $I \cap \bar{I}$ is a nondegenerate interval, and $\bar{p}\,|_{I\cap\bar{I}} = p\,|_{I\cap\bar{I}}$, then $\bar{I} \subset I$.

Theorem 3. If P is a point of $\mathcal{M}$, u is a tangent vector at P, and r_0 is a number, then there is just one maximal affinely parametrized autoparallel geodesic path p of $\mathbf{d}$ such that $p(r_0) = P$ and $\dot{p}(r_0) = u$.

Corollary. If c is a nonzero number, and s_0 is a number, then the maximal affinely parametrized autoparallel geodesic path q such that $q(s_0) = P$ and $\dot{q}(s_0) = cu$ is a doubly smooth regular reparametrization of p, sense-preserving if $c > 0$, sense-reversing if $c < 0$. If each of r_1, r_2, s_1, and s_2 is a number, and $p(r_1) = q(s_1)$ and $p(r_2) = q(s_2)$, then $r_2 - r_1 = c(s_2 - s_1)$.

Theorem 4. The affinely parametrized autoparallel geodesic path p of $\mathbf{d}$ is maximal if and only every affinely parametrized doubly smooth regular reparametrization of p is maximal.

Definition. That the manifold $\mathcal{M}$ is **geodesically complete from** P (**with respect to** $\mathbf{d}$) means that P is a point of $\mathcal{M}$ and every maximal affinely parametrized autoparallel geodesic path of $\mathbf{d}$ that has P in its range has $\mathbb{R}$ for its domain. That $\mathcal{M}$ is **geodesically complete** (**with respect to d**) means that every maximal affinely parametrized autoparallel geodesic path of $\mathbf{d}$ has $\mathbb{R}$ for its domain.

Theorem 5. The manifold $\mathcal{M}$ is geodesically complete (with respect to $\mathbf{d}$) if and only it is geodesically complete from each of its points.

Definition. Let P be a point of $\mathcal{M}$. For each tangent vector u at P, let p_u denote the maximal affinely parametrized autoparallel geodesic path p of $\mathbf{d}$ such that $p(0) = P$ and $\dot{p}(0) = u$. Let U be the subset of T^P consisting of all vectors u for which $1 \in \operatorname{dom} p_u$. If $u \in U$, let $F(u) := p_u(1)$. The function F thus defined, whose domain is U and whose range is the set of all points of $\mathcal{M}$ reachable from P by autoparallel geodesics, is called the **exponential map of d at** P, and is denoted by Exp_P.

Theorem 6. The manifold $\mathcal{M}$ is geodesically complete from P if and only if $\operatorname{dom}\operatorname{Exp}_P = T^P$.

XVII. GEODESIC DEVIATION

Let p be a C^2 **two-parameter path net in** $\mathcal{M}$, that is, $p\colon I \times J \to \mathcal{M}$, where each of I and J is an interval of $\mathbb{R}$, and Xp is C^2 for every coordinate system X of $\mathcal{M}$. The range of p is a (possibly degenerate) surface Σ in $\mathcal{M}$, parametrized by p. For each number s in I and each number t in J let $\dot{p}(s,t)$ denote the velocity of $p|_{I\times\{t\}}$ at s, and let $\overset{*}{p}(s,t)$ denote the velocity of $p|_{\{s\}\times J}$ at t. If $(s,t) \in I \times J$, then $p(s,t)$ is a point of Σ, and each of $\dot{p}(s,t)$ and $\overset{*}{p}(s,t)$ is a tangent vector of $\mathcal{M}$ at $p(s,t)$ that is tangent to Σ, so each of $\dot{p}$ and $\overset{*}{p}$ is a vector field on p in the obvious sense. If $f\colon I \times J \to \mathbb{R}$, and f is differentiable, let $f^{\cdot}(s,t)$ denote the derivative at s of $f|_{I\times\{t\}}$, and let $f^{*}(s,t)$ denote the derivative at t of $f|_{\{s\}\times J}$. Then $f^{*\cdot} = f^{\cdot *}$ if f is a C^2 function.

If u is a differentiable vector field on p, let $u^{\bullet}(s,t)$ denote the covariant derivative at s of $u|_{I\times\{t\}}$, and let $u^{*}(s,t)$ denote the covariant derivative at t of $u|_{\{s\}\times J}$. Then each of $u^{\bullet}$ and u^{*} is a vector field on p.

If v is a differentiable covector field on p, let $v^{\bullet}(s,t)$ denote the covariant derivative at s of $v|_{I\times\{t\}}$, and let $v^{*}(s,t)$ denote the covariant derivative at t of $v|_{\{s\}\times J}$. Then each of $v^{\bullet}$ and v^{*} is a covector field on p.

Let X be a coordinate system of $\mathcal{M}$ whose domain intersects the range of p. Then, on $p^{-1}(\operatorname{dom} X)$, with $e_m = \partial/\partial x^m$ and $\omega^m = dx^m$,

$$\dot{p} = \dot{p}^m e_m(p) = (p^m)^{\cdot} e_m(p) \quad \text{and} \quad \overset{*}{p} = \overset{*}{p}{}^m e_m(p) = (p^m)^{*} e_m(p), \tag{221}$$

where $p^m := x^m p$, so

$$\overset{*}{p}{}^{\bullet} - \dot{p}^{*} = \left[(p^m)^{*\cdot} e_m(p) + (p^m)^{*}\mathbf{d}e_m(p)\dot{p}\right] - \left[(p^m)^{\cdot *} e_m(p) + (p^m)^{\cdot}\mathbf{d}e_m(p)\overset{*}{p}\right] \tag{222}$$

$$= \left(\omega^m(p) \otimes \mathbf{d}e_m(p)\dot{p}\right)\overset{*}{p} - \left(\omega^m(p) \otimes \mathbf{d}e_m(p)\overset{*}{p}\right)\dot{p}, \tag{223}$$

because $(p^m)^{*\cdot} = (p^m)^{\cdot *}$. From $\mathbf{d}(\omega^m \otimes e_m) = 0$ follows

$$\overset{*}{p}{}^{\bullet} - \dot{p}^{*} = \left(\mathbf{d}\omega^m(p)\overset{*}{p} \otimes e_m(p)\right)\dot{p} - \left(\mathbf{d}\omega^m(p)\dot{p} \otimes e_m(p)\right)\overset{*}{p} \tag{224}$$

$$= 2\left(\mathbf{d}_\wedge \omega^m \otimes e_m\right)(p)\overset{*}{p}\dot{p} \tag{225}$$

$$= 2\left((d_\wedge \omega^m - \mathbf{T}^m) \otimes e_m\right)(p)\overset{*}{p}\dot{p}, \tag{226}$$

consequently that

$$\overset{*}{p}{}^{\bullet} - \dot{p}^{*} = 2\,\mathbf{T}(p)\dot{p}\overset{*}{p}, \tag{227}$$

because $d_\wedge \omega^m = d_\wedge(dx^m) = 0$ and $\mathbf{T}$ is skew-symmetric.

Now let u be a C^2 vector field on p. On $p^{-1}(\operatorname{dom} X)$

$$u^{*} = (u^m)^{*} e_m(p) + u^m \mathbf{d}e_m(p)\overset{*}{p} \tag{228}$$

and

$$u^{\bullet} = (u^m)^{\cdot} e_m(p) + u^m \mathbf{d}e_m(p)\dot{p}, \tag{229}$$

from which follow, by a slight extension of previous calculations,

$$u^{*\bullet} = (u^m)^{*\bullet} e_m(p) + (u^m)^{*}\mathbf{d}e_m(p)\dot{p} + (u^m)^{\bullet}\mathbf{d}e_m(p)\overset{*}{p} + u^m\mathbf{d}(\mathbf{d}e_m)(p)\dot{p}\overset{*}{p} + u^m\mathbf{d}e_m(p)\overset{*}{p}^{\bullet}, \tag{230}$$

and

$$u^{\bullet *} = (u^m)^{\bullet *} e_m(p) + (u^m)^{\bullet}\mathbf{d}e_m(p)\overset{*}{p} + (u^m)^{*}\mathbf{d}e_m(p)\dot{p} + u^m\mathbf{d}(\mathbf{d}e_m)(p)\overset{*}{p}\dot{p} + u^m\mathbf{d}e_m(p)\dot{p}^{*}, \tag{231}$$

in consequence of which

$$u^{*\bullet} - u^{\bullet *} = \left((u^m)^{*\bullet} - (u^m)^{\bullet *}\right) e_m(p) + u^m \left(\mathbf{d}(\mathbf{d}e_m)(p)\dot{p}\overset{*}{p} - \mathbf{d}(\mathbf{d}e_m)(p)\overset{*}{p}\dot{p}\right) + u^m\mathbf{d}e_m(p)(\overset{*}{p}^{\bullet} - \dot{p}^{*}) \tag{232}$$

$$= 2u^m\mathbf{d}_{\wedge}(\mathbf{d}e_m)(p)\dot{p}\overset{*}{p} + u^m\mathbf{d}e_m(p)(\overset{*}{p}^{\bullet} - \dot{p}^{*}) \tag{233}$$

$$= 2u^m \left(\tfrac{1}{2}\Theta e_m - (\mathbf{d}e_m)\mathbf{T}\right)(p)\dot{p}\overset{*}{p} + u^m\mathbf{d}e_m(p)(2\,\mathbf{T}(p)\dot{p}\overset{*}{p}) \tag{234}$$

$$= \Theta(p)(u^m e_m(p))\dot{p}\overset{*}{p}, \tag{235}$$

so that

$$u^{*\bullet} - u^{\bullet *} = \Theta(p)u\dot{p}\overset{*}{p}. \tag{236}$$

This equation is classical. It is of interest that the torsion of $\mathbf{d}$ does not appear explicitly in it.

Finally, suppose that $\mathcal{M}$ is triply smooth, that p is C^3, and that the paths $p|_{I\times\{t\}}$ are affinely parametrized autoparallel geodesic paths of $\mathbf{d}$. Let $u = \dot{p}$ and $\eta = \overset{*}{p}$. Then

$$u^{\bullet} = \dot{p}^{\bullet} = \ddot{p} = 0 \tag{237}$$

and

$$\eta^{\bullet} = \overset{*}{p}^{\bullet} = \dot{p}^{*} + 2\,\mathbf{T}(p)\dot{p}\overset{*}{p} = u^{*} + 2\,\mathbf{T}(p)u\eta. \tag{238}$$

Therefore

$$\eta^{\bullet\bullet} = u^{*\bullet} + 2\left[(\mathbf{T}(p))^{\bullet}u\eta + \mathbf{T}(p)u^{\bullet}\eta + \mathbf{T}(p)u\eta^{\bullet}\right] \tag{239}$$

$$= u^{\bullet *} + \Theta(p)u\dot{p}\overset{*}{p} + 2\left[\mathbf{dT}(p)\dot{p}u\eta + \mathbf{T}(p)u^{\bullet}\eta + \mathbf{T}(p)u\eta^{\bullet}\right] \tag{240}$$

$$= 0 + \Theta(p)uu\eta + 2\left[\mathbf{dT}(p)uu\eta + 0 + \mathbf{T}(p)u\eta^{\bullet}\right]. \tag{241}$$

At last, then,

$$\ddot{\eta} - 2\,\mathbf{T}(p)u\dot{\eta} - (2\mathbf{dT} + \Theta)(p)uu\eta = 0, \tag{242}$$

and, equivalently,

$$\ddot{\eta} - 2\,\mathbf{T}(p)\dot{p}\dot{\eta} - (2\mathbf{dT} + \Theta)(p)\dot{p}\dot{p}\eta = 0. \tag{243}$$

This is the **Equation of Geodesic Deviation**, a second order, linear differential equation satisfied by η, which is a measure of the rate at which the autoparallel geodesics $p|_{I\times\{t\}}$ spread. When $\mathbf{T} = 0$ and $\Theta = 0$ it reduces to $\ddot{\eta} = 0$, which implies that η varies linearly with respect to the affine parameter along each autoparallel geodesic.

XVIII. SYMMETRIC INNER PRODUCTS ON VECTOR SPACES

Let U be a finite-dimensional vector space over $\mathbb{R}$, with dual space U^*, and let $M = \dim U = \dim U^*$. By a **symmetric inner product on** U is meant a mapping $\langle\cdot\,,\cdot\rangle\colon U \times U \to \mathbb{R}$ with the following properties:

i. if each of u, v, and w is a vector in U, then $\langle u, v+w\rangle = \langle u,v\rangle + \langle u,w\rangle$;

ii. if each of u and v is a vector in U, and c is a real number, then $\langle u, cv\rangle = c\langle u,v\rangle$; and

iii. if each of u and v is a vector in U, then $\langle u,v\rangle = \langle v,u\rangle$.

From (i) and (iii) follows

iv. if each of u, v, and w is a vector in U, then $\langle u+v, w\rangle = \langle u,w\rangle + \langle v,w\rangle$.

From (ii) and (iii) follows

v. if each of u and v is a vector in U, and c is a real number, then $\langle cu, v\rangle = c\langle u,v\rangle$.

That $\langle\cdot\,,\cdot\rangle$ is **positive** (resp., **negative**) **definite** means that

vi. if u is a vector in U other than 0_U (the zero vector of U), then $\langle u,u\rangle > 0$ (resp., < 0).

That $\langle\cdot\,,\cdot\rangle$ is **positive** (resp., **negative**) **semidefinite** means that

vii. if u is a vector in U other than 0_U (the zero vector of U), then $\langle u,u\rangle \geq 0$ (resp., ≤ 0).

That $\langle\cdot\,,\cdot\rangle$ is **definite** (resp., **semidefinite**) means that

viii. $\langle\cdot\,,\cdot\rangle$ is positive definite (resp., semidefinite) or negative definite (resp., semidefinite).

A consequence of (v) is that $\langle 0_U, v\rangle = 0$ for every vector v in U (in particular, that $\langle 0_U, 0_U\rangle = 0$). That $\langle\cdot\,,\cdot\rangle$ is **nondegenerate** means that, conversely,

ix. if $\langle u,v\rangle = 0$ for every vector v in U, then $u = 0_U$.

That $\langle\cdot\,,\cdot\rangle$ is **degenerate** means that $\langle\cdot\,,\cdot\rangle$ is not nondegenerate, thus that there is in U a nonzero vector u such that $\langle u,v\rangle = 0$ for every vector v in U. The set of all such vector(s) u is a subspace of U, called the **nullifying space** of $\langle\cdot\,,\cdot\rangle$; its dimensionality is positive if and only if $\langle\cdot\,,\cdot\rangle$ is degenerate, thus is 0 if and only if $\langle\cdot\,,\cdot\rangle$ is nondegenerate.

If $\langle\cdot\,,\cdot\rangle$ is a symmetric inner product on U, then u **is orthogonal to** v **(with respect to** $\langle\cdot\,,\cdot\rangle$**)** means that each of u and v is a vector in U, and $\langle u,v\rangle = 0$; if u is orthogonal to v (with respect to $\langle\cdot\,,\cdot\rangle$), then (iii) implies that v is orthogonal to u. In terms of orthogonality, $\langle\cdot\,,\cdot\rangle$ is degenerate if and only if some nonzero vector in U is orthogonal to every vector in U, and is nondegenerate if and only if the only vector in U that is orthogonal to every vector in U is 0_U.

With every symmetric inner product $\langle\cdot\,,\cdot\rangle$ on U is uniquely associated a linear transformation $L\colon U \to U^*$ (thus an element L of $U^* \otimes U^*$) such that if each of u and v is a vector in U, then $\langle u,v\rangle = (Lu)v$. The definition of L is simply that, for each vector u in U, Lu is the mapping $l\colon U \to \mathbb{R}$ given by $l\colon v \mapsto \langle u,v\rangle$; that l is linear, thus is in U^*, follows from (i) and (ii); that L is linear follows from (iv) and (v). The symmetry property (iii) implies that $(Lu)v = (Lv)u$. In terms of L, property (ix) and its converse say that $(Lu)v = 0$ for every vector v in U if and only if $u = 0_U$, in other words that $Lu = 0_{U^*}$ if and only if $u = 0_U$, another way of saying which is that the kernel of L is $\{0_U\}$. From this it follows that $\langle\cdot\,,\cdot\rangle$ is nondegenerate if and only if L is one-to-one (equivalently, is invertible, is nonsingular).

Conversely, with every linear transformation $L\colon U \to U^*$ such that $(Lu)v = (Lv)u$ for every vector u and every vector v in U is uniquely associated a symmetric inner product $\langle\cdot\,,\cdot\rangle$ on U, defined by $\langle u, v\rangle := (Lu)v$. As above, $\langle\cdot\,,\cdot\rangle$ is nondegenerate if and only if L is one-to-one.

Let $\langle\cdot\,,\cdot\rangle$ be a symmetric inner product on U, let L be the the associated linear transformation in $U^* \otimes U^*$, let $\{a_m\}$ be a basis for U, and let $\{a^m\}$ be the basis for U^* dual to $\{a_m\}$. If each of u and v is a vector in U, then

$$\langle u, v\rangle = \langle u^m a_m, v^n a_n\rangle = u^m \langle a_m, a_n\rangle v^n = u^m((La_m)a_n)v^n. \tag{244}$$

But the representation $L = a^k \otimes L_{kl}\, a^l$ implies that

$$La_m = (a^k a_m)L_{kl}\, a^l = \delta_m{}^k L_{kl}\, a^l = L_{ml}\, a^l, \tag{245}$$

hence that

$$(La_m)a_n = L_{ml}(a^l a_n) = L_{ml}\delta^l{}_n = L_{mn}, \tag{246}$$

thus that

$$L_{mn} = \langle a_m, a_n\rangle = \langle a_n, a_m\rangle = L_{nm} \tag{247}$$

and

$$\langle u, v\rangle = u^m L_{mn} v^n, \tag{248}$$

which tells that $\langle\cdot\,,\cdot\rangle$ is completely determined by the numbers $\langle a_m, a_n\rangle$, and is equivalent to

$$[\langle u, v\rangle] = [u][L][v]^T = \overset{m\longrightarrow}{[u^m]}\,[L_{mn}]\,[v^n]\overset{n}{\downarrow} = \overset{m\longrightarrow}{[u^m]}\,[\langle a_m, a_n\rangle]\,[v^n]\overset{n}{\downarrow}\,, \tag{249}$$

where $[u]$, $[v]$, and $[L]$ are the matrices that represent u, v, and L with respect to the bases $\{a_m\}$ and $\{a^m\}$, and $[\langle u, v\rangle]$ is the 1-by-1 matrix whose sole entry is $\langle u, v\rangle$. Briefly, $L = a^k \otimes \langle a_k, a_l\rangle\, a^l$.

If u is a vector in U, then the nonegative number $|\langle u, u\rangle|^{\frac{1}{2}}$ is denoted by $|u|$ and is called the **norm**, the **length**, the **magnitude**, and the **absolute value**, **of** u (**with respect to** $\langle\cdot\,,\cdot\rangle$). The number $\langle u, u\rangle$ is called the **square length of** u (**with respect to** $\langle\cdot\,,\cdot\rangle$). That u is **null** (resp., **positive**, **negative**) (**with respect to** $\langle\cdot\,,\cdot\rangle$) means that $u \neq 0_U$ and the square length $\langle u, u\rangle$ of u is 0 (resp., > 0, < 0). Clearly, $|u| = 0$ if u is null, and $|u| > 0$ if u is positive or negative. Moreover, if $u \neq 0_U$, then u is either positive or negative if $\langle\cdot\,,\cdot\rangle$ is definite.

Definition. If U' is a subspace of U, then

i. U' is $\langle\cdot\,,\cdot\rangle$-**positive** means that if u is a nonzero vector in U', then u is positive;

ii. U' is $\langle\cdot\,,\cdot\rangle$-**negative** means that if u is a nonzero vector in U', then u is negative.

iii. The nonnegative integer $\max\{\dim U' \mid U' \text{ is a } \langle\cdot\,,\cdot\rangle\text{-positive subspace of } U\}$ is called the **positivity index** of $\langle\cdot\,,\cdot\rangle$, and is denoted by $\langle +, +\rangle$.

iv. The nonnegative integer $\max\{\dim U' \mid U'$ is a $\langle\cdot\,,\cdot\rangle$-negative subspace of $U\}$ is called the **negativity index** of $\langle\cdot\,,\cdot\rangle$, and is denoted by $\langle -,-\rangle$.

v. The nonnegative integer that is the dimensionality of the nullifying space of U determined by $\langle\cdot\,,\cdot\rangle$ is called the **nullity** of $\langle\cdot\,,\cdot\rangle$, and is denoted by $\langle\circ,\circ\rangle$.

Theorem 1. If U^+ is a $\langle\cdot\,,\cdot\rangle$-positive subspace of U whose dimensionality is $\langle +,+\rangle$, U^- is a $\langle\cdot\,,\cdot\rangle$-negative subspace of U whose dimensionality is $\langle -,-\rangle$, and U° is the nullifying space of $\langle\cdot\,,\cdot\rangle$, then U is the direct sum of U^+, U^-, and U°, and therefore $\langle +,+\rangle + \langle -,-\rangle + \langle\circ,\circ\rangle = M$.

Theorem 2. The inner product $\langle\cdot\,,\cdot\rangle$ is

i. positive definite if and only if $\langle +,+\rangle \neq 0$, $\langle -,-\rangle = 0$, and $\langle\circ,\circ\rangle = 0$;

ii. negative definite if and only if $\langle +,+\rangle = 0$, $\langle -,-\rangle \neq 0$, and $\langle\circ,\circ\rangle = 0$;

iii. positive semidefinite if and only if $\langle -,-\rangle = 0$;

iv. negative semidefinite if and only if $\langle +,+\rangle = 0$;

v. nondegenerate if and only if $\langle\circ,\circ\rangle = 0$.

That the basis $\{a_m\}$ is **orthogonal** (with respect to $\langle\cdot\,,\cdot\rangle$) means that a_m is orthogonal to a_n if $m \neq n$; $\{a_m\}$ is **null** means that each of its vectors is null; $\{a_m\}$ is **normal** means that, for each m, $\langle a_m, a_m\rangle = 1$, 0, or -1 (equivalently, $|a_m| = 1$ if a_m is not null); and $\{a_m\}$ is **orthonormal** means that $\{a_m\}$ is both orthogonal and normal.

Theorem 3. There is a basis for U that is orthogonal. If $\{a_m\}$ is such a basis, then $\{\hat{a}_m\}$, where $\hat{a}_m = a_m$ if a_m is null, and $\hat{a}_m = a_m/|a_m|$ if a_m is not null, is a basis for U that is orthonormal.

Theorem 4. In every orthogonal basis for U the number of positive vector(s) is $\langle +,+\rangle$, the number of negative vector(s) is $\langle -,-\rangle$, and the number of null vector(s) is $\langle\circ,\circ\rangle$.

Theorem 5. The basis $\{a_m\}$ for U is null if and only if the matrix $[L]$ $(= [\langle a_m, a_n\rangle])$ that represents L with respect to $\{a_m\}$ and its dual is symmetric and has no nonzero diagonal entry; $\{a_m\}$ is orthogonal if and only if $[L]$ is diagonal with $\langle +,+\rangle$ positive diagonal entries, $\langle -,-\rangle$ negative diagonal entries, and $\langle\circ,\circ\rangle$ diagonal entries that are 0; $\{a_m\}$ is orthonormal if and only if $[L]$ is diagonal and each positive diagonal entry is 1 and each negative diagonal entry is -1; in every case, the nullity of L ($:= \dim(\text{kernel of } L)$) is $\langle\circ,\circ\rangle$.

Theorem 6. Let $\{a_m\}$ be any basis for U, and let each of $\{a_m\}^+$, $\{a_m\}^-$, and $\{a_m\}^\circ$ be a (perhaps empty) subset of $\{a_m\}$, with every vector of $\{a_m\}$ in one of them, and no vector of $\{a_m\}$ in two of them. There is on U just one symmetric inner product $\langle\cdot\,,\cdot\rangle^*$ with respect to which (i) $\{a_m\}$ is orthonormal, (ii) each vector (if any) in $\{a_m\}^+$ is positive, (iii) each vector (if any) in $\{a_m\}^-$ is negative, and (iv) each vector (if any) in $\{a_m\}^\circ$ is null. If, with respect to $\langle\cdot\,,\cdot\rangle$, $\{a_m\}$ is orthornormal, each vector (if any) in $\{a_m\}^+$ is positive, each vector (if any) in $\{a_m\}^-$ is negative, and each vector (if any) in $\{a_m\}^\circ$ is null, then $\langle\cdot\,,\cdot\rangle^* = \langle\cdot\,,\cdot\rangle$.

The import of this theorem is that to specify on U a symmetric inner product with given positivity index, negativity index, and nullity is tantamount to selecting some basis for U, decreeing that it should be orthonormal, and designating which of its vectors should be positive, which should be negative, and which should be null, in the prescribed numbers.

Suppose now that $\langle\cdot\,,\cdot\rangle$ is nondegenerate. Then L, which maps U linearly onto U^*, has an inverse L^{-1}, which maps U^* linearly onto U. Moreover, if each of u^* and v^* is in U^*, and $u = L^{-1}u^*$ and $v = L^{-1}v^*$, then $(L^{-1}u^*)v^* = (L^{-1}v^*)u^*$, because

$$(L^{-1}u^*)v^* = v^*(L^{-1}u^*) = (Lv)(L^{-1}(Lu)) = (Lv)u \tag{250}$$

and $(L^{-1}v^*)u^* = (Lu)v = (Lv)u$. Consequently, with L^{-1} is associated a nondegenerate symmetric inner product $\langle\cdot\,,\cdot\rangle^*$ on the dual space U^* of U such that, if each of u^* and v^* is in U^*, then

$$\langle u^*, v^*\rangle^* = (L^{-1}u^*)v^* = (Lv)u = \langle v, u\rangle = \langle u, v\rangle \tag{251}$$

where $u = L^{-1}u^*$ and $v = L^{-1}v^*$. Briefly put, $\langle u^*, v^*\rangle^* = \langle L^{-1}u^*, L^{-1}v^*\rangle$, and, equivalently, $\langle u, v\rangle = \langle Lu, Lv\rangle^*$. The inner product $\langle\cdot\,,\cdot\rangle^*$ is called the **dual of** $\langle\cdot\,,\cdot\rangle$.

As an element of $U^{**}\otimes U^{**}$, thus, by identification, of $U\otimes U$, L^{-1} has the representation $L^{-1} = a_m \otimes L^{mn}a_n$, where

$$L^{mn} := (L^{-1})^{mn} = (L^{-1}a^m)a^n = \langle a^m, a^n\rangle^* = \langle a^n, a^m\rangle^* = L^{nm}, \tag{252}$$

in terms of which $\langle u^*, v^*\rangle^* = u^*_m L^{mn} v^*_n$. The mapping $L^{-1}L$ has the representation

$$L^{-1}L = (a_l \otimes L^{ln}a_n)(a^m \otimes L_{mk}a^k) \tag{253}$$

$$= a^m \otimes L_{mk}(a_l a^k)L^{ln}a_n \tag{254}$$

$$= a^m \otimes L_{mk}(a^k a_l)L^{ln}a_n \tag{255}$$

$$= a^m \otimes L_{mk}\delta^k{}_l L^{ln}a_n \tag{256}$$

$$= a^m \otimes L_{mk}L^{kn}a_n. \tag{257}$$

But $L^{-1}L$ is the identity mapping of U onto U, which has the representation $a^m \otimes \delta_m{}^n a_n$, and therefore $L_{mk}L^{kn} = \delta_m{}^n$. Similarly, because LL^{-1} is the identity mapping of U^* onto U^*, $L^{mk}L_{kn} = \delta^m{}_n$ (which also follows from $L^{mk}L_{kn} = L^{km}L_{nk} = L_{nk}L^{km} = \delta_n{}^m = \delta^m{}_n$).

Theorem 7. The positivity index, the negativity index, and the nullity of $\langle\cdot\,,\cdot\rangle^*$ are the same as those of $\langle\cdot\,,\cdot\rangle$: $\langle +,+\rangle^* = \langle +,+\rangle$, $\langle -,-\rangle^* = \langle -,-\rangle$, and $\langle \circ,\circ\rangle^* = \langle \circ,\circ\rangle = 0$.

Let $b^m = \delta^{mn}La_n = La_m$, for $m = 1,\ldots,M$. Then $\{b^m\}$ is a basis for U^*, and $\langle b^m, b^n\rangle^* = \langle La_m, La_n\rangle^* = \langle a_m, a_n\rangle$.

Theorem 8. The basis $\{b^m\}$ is null with respect to $\langle\cdot\,,\cdot\rangle^*$ if and only if the basis $\{a_m\}$ is null with respect to $\langle\cdot\,,\cdot\rangle$; $\{b^m\}$ is orthogonal with respect to $\langle\cdot\,,\cdot\rangle^*$ if and only if $\{a_m\}$ is orthogonal with respect to $\langle\cdot\,,\cdot\rangle$; for each m, b^m is positive (resp., negative) with respect to $\langle\cdot\,,\cdot\rangle^*$ if and only if a_m is positive (resp., negative) with respect to $\langle\cdot\,,\cdot\rangle$. Also, $\{b^m\}$ is normal with respect to $\langle\cdot\,,\cdot\rangle^*$ if and only if $\{a_m\}$ is normal with respect to $\langle\cdot\,,\cdot\rangle$; for each m, $\langle b^m, b^m\rangle^* = 1$ (resp., -1) if and only if $\langle a_m, a_m\rangle = 1$ (resp., -1).

XIX. METRICS AND INNER PRODUCTS ON SMOOTH MANIFOLDS

In this chapter and the next $\mathcal{M}$ need be only smooth, not doubly smooth.

Definition. By a **metric on** $\mathcal{M}$ is meant a smooth, symmetric cocotensor field G of $\mathcal{M}$, globally defined on $\mathcal{M}$ ($\operatorname{dom} G = \mathcal{M}$), whose nullity is constant (the kernel of $G(P)$ has the same dimensionality at every point P). If G is a metric on $\mathcal{M}$, the **tangent inner product associated with** G is the mapping $\langle\cdot\,,\cdot\rangle_G : \bigcup_{P\in\mathcal{M}}(T^P \times T^P) \to \mathbb{R}$ defined by $\langle u, v\rangle_G := G(P)uv$ if each of u and v is in T^P; the mapping $|\cdot|_G : \bigcup_{P\in\mathcal{M}} T^P \to \mathbb{R}$ is defined by $|u|_G := |\langle u, u\rangle_G|^{\frac{1}{2}}$. By a **tangent inner product on** $\mathcal{M}$ is meant a mapping of $\bigcup_{P\in\mathcal{M}}(T^P \times T^P)$ into $\mathbb{R}$ that is $\langle\cdot\,,\cdot\rangle_G$ for some metric G on $\mathcal{M}$.

Let G be a metric on $\mathcal{M}$, and let $\langle\cdot\,,\cdot\rangle_G$ be the tangent inner product associated with G.

Theorem and Definition. At each point P of $\mathcal{M}$ the restriction of $\langle\cdot\,,\cdot\rangle_G$ to $T^P \times T^P$ is a symmetric inner product $\langle\cdot\,,\cdot\rangle^P$ on T^P whose nullity $\langle\circ,\circ\rangle^P$ is the nullity of $G(P)$. The metric G and the tangent inner product $\langle\cdot\,,\cdot\rangle_G$ are said to be **positive definite** (resp., **negative definite**, **definite**, **positive semidefinite**, **negative semidefinite**, **semidefinite**, **nondegenerate**, **degenerate**) if and only if, for each point P of $\mathcal{M}$, $\langle\cdot\,,\cdot\rangle^P$ is positive definite (resp., negative definite, definite, positive semidefinite, negative semidefinite, semidefinite, nondegenerate, degenerate).

Theorem 1. Each of $\langle +,+\rangle^P$, $\langle -,-\rangle^P$, and $\langle\circ,\circ\rangle^P$ is independent of the choice of P.

Definition. The nonnegative integers $\langle +,+\rangle^P$, $\langle -,-\rangle^P$, and $\langle\circ,\circ\rangle^P$ that are independent of P are called, respectively, the **positivity index**, the **negativity index**, and the **nullity**, **of** $\langle\cdot\,,\cdot\rangle_G$, and are denoted, respectively, by $\langle +,+\rangle_G$, $\langle -,-\rangle_G$, and $\langle\circ,\circ\rangle_G$.

Theorem 2. The metric G and the tangent inner product $\langle\cdot\,,\cdot\rangle_G$ are

i. positive definite if and only if $\langle +,+\rangle_G \neq 0$, $\langle -,-\rangle_G = 0$, and $\langle\circ,\circ\rangle_G = 0$;

ii. negative definite if and only if $\langle +,+\rangle_G = 0$, $\langle -,-\rangle_G \neq 0$, and $\langle\circ,\circ\rangle_G = 0$;

iii. positive semidefinite if and only if $\langle -,-\rangle_G = 0$;

iv. negative semidefinite if and only if $\langle +,+\rangle_G = 0$;

v. nondegenerate if and only if $\langle\circ,\circ\rangle_G = 0$.

Definition. If E is a smooth frame system of $\mathcal{M}$, then E is **orthogonal** (resp., **null**, **normal**, **orthonormal**) **with respect to** G **and** $\langle\cdot\,,\cdot\rangle_G$ means that if $P \in \operatorname{dom} E$, then $E(P)$ is orthogonal (resp., null, normal, orthonormal) with respect to $\langle\cdot\,,\cdot\rangle^P$.

Theorem 3. If the smooth frame system E is orthogonal, and $P \in \operatorname{dom} E$, then the number of positive vectors in $E(P)$ is $\langle +,+\rangle_G$, the number of negative vectors in $E(P)$ is $\langle -,-\rangle_G$, and the number of null vectors in $E(P)$ is $\langle\circ,\circ\rangle_G$.

Theorem 4. If X is a coordinate system of $\mathcal{M}$, then there is on $\operatorname{dom} X$ a smooth frame system E that is orthogonal. If E is such a frame system, then the frame system $\hat{E}$ defined by $\hat{e}^m = e^m$ if e_m is null, and $\hat{e}_m = e_m/|e_m|_G$ if e_m is not null, is orthonormal.

Let E be a smooth frame system of $\mathcal{M}$, with dual coframe system Ω; let $U = \operatorname{dom} E = \operatorname{dom} \Omega$. Then, on U,

$$G = \omega^m \otimes g_{mn} \omega^n, \tag{258}$$

where $g_{mn} = G e_m e_n$. Also, if each of u and v is a vector field of $\mathcal{M}$, and $U \cap \operatorname{dom} u \cap \operatorname{dom} v \neq \varnothing$, then, on the intersection,

$$\langle u, v\rangle_G = Guv = (\omega^m \otimes g_{mn}\omega^n)(u^k e_k)(v^l e_l) = u^k(\omega^m e_k) g_{mn} v^l (\omega^n e_l) = u^m g_{mn} v^n. \tag{259}$$

Suppose now that G is nondegenerate. At each point P of $\mathcal{M}$, $G(P)$ (an element of $\mathcal{L}(T^P, T_P)$, thus of $T_P \otimes T_P$) has an inverse $G(P)^{-1}$, which is an element of $\mathcal{L}(T_P, T^P)$, thus of $T^P \otimes T^P$. The concontensor field G^{-1} defined globally on $\mathcal{M}$ by $G^{-1}(P) := G(P)^{-1}$ is called the **inverse metric of** G. If u is a tangent vector at P, then the cotangent vector $G(P)u$ is called the **metric dual of** u (with respect to G); if u is a vector field of $\mathcal{M}$, then the covector field Gu is called the **the metric dual of** u. If u^* is a cotangent vector at P, then the tangent vector $G^{-1}(P)u^*$ is called the **inverse metric dual of** u^*; if u^* is a covector field of $\mathcal{M}$, then the vector field $G^{-1}u^*$ is called the **inverse metric dual of** u^*. Clearly, the inverse metric dual of the metric dual of u is u itself, and the metric dual of the inverse metric dual of u^* is u^*.

The **cotangent inner product associated with** G is the mapping $\langle\cdot\,,\cdot\rangle_G^*\colon \bigcup_{P\in\mathcal{M}}(T_P \times T_P) \to \mathbb{R}$ defined by $\langle\cdot\,,\cdot\rangle_G^* u^* v^* := G^{-1}(P)u^* v^*$ if each of u^* and v^* is in T_P. Clearly, at each point P of $\mathcal{M}$ the restriction of $\langle\cdot\,,\cdot\rangle_G^*$ to $T_P \times T_P$ is the symmetric inner product $\langle\cdot\,,\cdot\rangle_P$ on T_P that is the dual of the inner product $\langle\cdot\,,\cdot\rangle^P$ on T^P.

Theorem 5. Each of $\langle +,+\rangle_P$, $\langle -,-\rangle_P$, and $\langle \circ,\circ\rangle_P$ is independent of the choice of P.

Definition. The nonnegative integers $\langle +,+\rangle_P$, $\langle -,-\rangle_P$, and $\langle \circ,\circ\rangle_P$ that are independent of P are called, respectively, the **positivity index**, the **negativity index**, and the **nullity**, **of** $\langle\cdot\,,\cdot\rangle_G^*$, and are denoted, respectively, by $\langle +,+\rangle_G^*$, $\langle -,-\rangle_G^*$, and $\langle \circ,\circ\rangle_G^*$.

Theorem 6. The positivity index, the negativity index, and the nullity of $\langle\cdot\,,\cdot\rangle_G^*$ are the same as those of $\langle\cdot\,,\cdot\rangle_G$ ($\langle +,+\rangle_G^* = \langle +,+\rangle_G$, $\langle -,-\rangle_G^* = \langle -,-\rangle_G$, and $\langle \circ,\circ\rangle_G^* = \langle \circ,\circ\rangle_G = 0$).

Definition. If $\tilde{\Omega}$ is a smooth coframe system of $\mathcal{M}$, then $\tilde{\Omega}$ is **orthogonal** (resp., **null**, **normal**, **orthonormal**) **with respect to** G **and** $\langle\cdot\,,\cdot\rangle_G^*$ means that if $P \in \operatorname{dom}\tilde{\Omega}$, then $\tilde{\Omega}(P)$ is orthogonal (resp., null, normal, orthonormal) with respect to $\langle\cdot\,,\cdot\rangle_P$.

Let $\tilde{\Omega}$ be the **metric dual of** E, defined by $\tilde{\omega}^m := \delta^{mn} G e_n = G e_m$, for $m = 1, \ldots, M$. Then $\tilde{\Omega}$ is a smooth coframe system of $\mathcal{M}$ on U, and $\langle \tilde{\omega}^m, \tilde{\omega}^n\rangle_G^* = \langle G e_m, G e_n\rangle_G^* = G^{-1}(G e_m)(G e_n) = e_m(G e_n) = G e_m e_n = \langle e_m, e_n\rangle_G$.

Theorem 7. The coframe system $\tilde{\Omega}$ is null with respect to G and $\langle\cdot\,,\cdot\rangle_G^*$ if and only if the frame system E is null with respect to G and $\langle\cdot\,,\cdot\rangle_G$; $\tilde{\Omega}$ is orthogonal with respect to G and $\langle\cdot\,,\cdot\rangle_G^*$ if and only if E is orthogonal with respect to G and $\langle \cdot\,, \cdot \rangle_G$; $\tilde{\Omega}$ is normal with respect to G and $\langle \cdot\,, \cdot \rangle_G^*$ if and only if E is normal with respect to G and $\langle \cdot\,, \cdot \rangle_G$. For each m, $\langle \tilde{\omega}^m, \tilde{\omega}^m \rangle_G^* = 0$ (resp., < 0, > 0, $= 1$, $= -1$) on U if and only if $\langle e_m, e_m \rangle_G = 0$ (resp., < 0, > 0, $= 1$, $= -1$) on U.

If u is a vector field of $\mathcal{M}$, then

$$Gu = (\omega^m \otimes g_{mn}\omega^n)(u^k e_k) = u^k(\omega^m e_k) g_{mn}\omega^n = u^k g_{kn}\omega^n, \tag{260}$$

so the metric dual of u has the representation $Gu = u_m \omega^m$, where $u_m := u^k g_{km}$. The latter relation is often referred to as "lowering of the index of u by G". (Note the distinction between this representation and the representation $Gu = G(u^m e_m) = u^m G e_m = u^m \delta_{mn} \tilde{\omega}^n = \sum_m u^m \tilde{\omega}^m$.) In terms of the lowered index, one has that $\langle u, v \rangle_G = u_m v^m$.

On U, the inverse metric has the representation

$$G^{-1} = e_m \otimes g^{mn} e_n, \tag{261}$$

where $g^{mn} = G^{-1}\omega^m\omega^n$, related to g_{mn} by $g_{mk}g^{kn} = \delta_m{}^n$ and $g^{mk}g_{kn} = \delta^m{}_n$. From this follows that if each of u^* and v^* is a covector field of $\mathcal{M}$, and $U \cap \operatorname{dom} u^* \cap \operatorname{dom} v^* \neq \varnothing$, then, on the intersection,

$$\langle u^*, v^* \rangle^* = G^{-1}u^*v^* = (e_m \otimes g^{mn} e_n)(u^*_k \omega^k)(v^*_l \omega^l) = u^*_k (e_m \omega^k) g^{mn} v^*_l (e_n \omega^l) = u^*_m g^{mn} v^*_n. \tag{262}$$

Intermediately,

$$G^{-1}u^* = (e_m \otimes g^{mn} e_n)(u^*_k \omega^k) = u^*_k (e_m \omega^k) g^{mn} e_n = u^*_k g^{kn} e_n, \tag{263}$$

so the inverse metric dual of u^* has the representation $G^{-1}u^* = u^{*m} e_m$, where $u^{*m} := u^*_k g^{km}$. The latter relation is referred to as "raising of the index of u^* by G^{-1}". In terms of the raised index, $\langle u^*, v^* \rangle_G^* = u^{*m} v^*_m$.

The lowering and the raising of an index are inverse operations: if $u_m = u^k g_{km}$, then $u_l g^{lm} = u^k g_{kl} g^{lm} = u^k \delta_k{}^m = u^m$, which corresponds to $G^{-1}Gu = u$, and if $u^{*m} = u^*_k g^{km}$, then $u^{*l} g_{lm} = u^*_k g^{kl} g_{lm} = u^*_k \delta^k{}_m = u^*_m$, which corresponds to $GG^{-1}u^* = u^*$. Also, these operations are not confined to vector and covector fields. If, for example, T is a coconconcotensor field of $\mathcal{M}$, say $T = T_k{}^{mn}{}_l\,(\omega^l \otimes e_n \otimes e_m \otimes \omega^k)$, then, because $Ge_p = (e_p)^k g_{kq}\omega^q = \delta_p{}^k g_{kq}\omega^q = g_{pq}\omega^q$ and $G^{-1}\omega^p = (\omega^p)_k g^{kq} e_q = \delta^p{}_k g^{kq} e_q = g^{pq} e_q$,

$$GT = T_k{}^{pn}{}_l\,(\omega^l \otimes e_n \otimes Ge_p \otimes \omega^k) = T_k{}^{pn}{}_l\,(\omega^l \otimes e_n \otimes g_{pm}\omega^m \otimes \omega^k) \tag{264}$$

$$= T_k{}^{pn}{}_l\,g_{pm}\,(\omega^l \otimes e_n \otimes \omega^m \otimes \omega^k) = T_{km}{}^n{}_l\,(\omega^l \otimes e_n \otimes \omega^m \otimes \omega^k), \tag{265}$$

$$G(\cdot)T = T_k{}^{mp}{}_l\,(\omega^l \otimes Ge_p \otimes e_m \otimes \omega^k) = T_k{}^{mp}{}_l\,(\omega^l \otimes g_{pn}\omega^n \otimes e_m \otimes \omega^k) \tag{266}$$

$$= T_k{}^{mp}{}_l\,g_{pn}\,(\omega^l \otimes \omega^n \otimes e_m \otimes \omega^k) = T_k{}^m{}_{nl}\,(\omega^l \otimes \omega^n \otimes e_m \otimes \omega^k), \tag{267}$$

$$G^{-1}T = T_p{}^{mn}{}_l\,(\omega^l \otimes e_n \otimes e_m \otimes G^{-1}\omega^p) = T_p{}^{mn}{}_l\,(\omega^l \otimes e_n \otimes e_m \otimes g^{pk}e_k) \tag{268}$$

$$= T_p{}^{mn}{}_l\,g^{pk}\,(\omega^l \otimes e_n \otimes e_m \otimes e_k) = T^{kmn}{}_l\,(\omega^l \otimes e_n \otimes e_m \otimes e_k), \tag{269}$$

and

$$G^{-1}(\cdot)T = T_k{}^{mn}{}_p\,(G^{-1}\omega^p \otimes e_n \otimes e_m \otimes \omega^k) = T_k{}^{mn}{}_p\,(g^{pl}e_l \otimes e_n \otimes e_m \otimes \omega^k) \tag{270}$$

$$= T_k{}^{mn}{}_p\,g^{pl}\,(e_l \otimes e_n \otimes e_m \otimes \omega^k) = T_k{}^{mnl}\,(e_l \otimes e_n \otimes e_m \otimes \omega^k). \tag{271}$$

These various metric and inverse metric duals of T have contractions that T has no counterpart to, and lack counterparts to contractions that T has. For instance, there is no contraction $T\,{}^3_2$, but there is the contraction $(GT)\,{}^3_2$, namely,

$$(GT)\,{}^3_2 = \omega^p(GT)(\cdot)e_p = T_{km}{}^n{}_l\,(\omega^l \otimes \omega^p e_n \otimes \omega^m e_p \otimes \omega^k) \tag{272}$$

$$= T_{km}{}^n{}_l\,(\omega^l \otimes \delta^p{}_n \otimes \delta^m{}_p \otimes \omega^k) \tag{273}$$

$$= T_{kp}{}^p{}_l\,(\omega^l \otimes \omega^k). \tag{274}$$

On the other hand, there is the contraction $T\,{}^2_1 = T_k{}^{mp}{}_p\,(e_m \otimes \omega^k)$, but there is no contraction $(G^{-1}(\cdot)T)\,{}^2_1$. There is, however, the contraction $(G(\cdot)T)\,{}^2_3$, namely, $T_k{}^p{}_{pl}\,(\omega^l \otimes \omega^k)$, which, because $T_k{}^p{}_{pl} = T_{kr}{}^s{}_l g^{rp} g_{sp} = T_{kr}{}^s{}_l g^{rp} g_{ps} = T_{kr}{}^s{}_l \delta^r{}_s = T_{kr}{}^r{}_l = T_{kp}{}^p{}_l$, is the same as $(GT)\,{}^3_2$.

A case of interest occurs when the tensor field in question is the contracted curvature Φ of a covariant differentiation $\mathbf{d}$ on $\mathcal{M}$. As a cocotensor field, Φ itself has no contraction, but $G^{-1}\Phi$ does. This permits the following definition, which is not available in the absence of a nondegenerate metric.

Definition. If $\mathbf{d}$ is a covariant differentiation on $\mathcal{M}$, then by the **twice contracted curvature of d (with respect to** G) (the **curvature scalar of d**, for short) is meant the scalar field Ψ defined by $\Psi := (G^{-1}\Phi)\,{}^2_1$, where Φ is the contracted curvature of $\mathbf{d}$.

Theorem 8. In a frame system E,

$$\Psi := \mathrm{Tr}\ (G^{-1}\Phi) := (G^{-1}\Phi)\,{}^2_1 = \Phi_k{}^k = \Theta_k{}^{mk}{}_m, \tag{275}$$

and also

$$\Psi = \mathrm{Tr}\ (\Phi G^{-1}) = (\Phi G^{-1})\,{}^1_2 = \Phi^k{}_k = \Theta^{km}{}_{km}. \tag{276}$$

XX. LENGTHS OF PATHS AND CURVES IN METRIC MANIFOLDS

Let G be a metric on the smooth manifold $\mathcal{M}$. Call the pair $\{\mathcal{M}, G\}$ a **metric manifold**.

Definition. If $p\colon I \to \mathcal{M}$ is a smooth path in the metric manifold $\{\mathcal{M}, G\}$, and t is a number in I, then the nonnegative number $|\dot{p}(t)|_G$ is called the **speed of** p **at** t (**with respect to** G). The function on I whose value at each "time" t is the speed of p at t is called the **speed of** p (**with respect to** G) and is denoted by $|\dot{p}|_G$. If $I = [a, b]$, then by the **length of** p (**with respect to** G) is meant the nonnegative number $\ell(p)$ defined by

$$\ell(p) := \int_a^b |\dot{p}|_G = \int_a^b |\langle \dot{p}, \dot{p}\rangle_G|^{\frac{1}{2}} = \int_a^b |G(p)\dot{p}\dot{p}|^{\frac{1}{2}}. \tag{277}$$

Theorem 1. If p is constant, then $\ell(p) = 0$. If p is not constant, then $\ell(p) > 0$ if G is definite, and $\ell(p) = 0$ if and only if G is indefinite and, for each number t in $[a, b]$, the velocity of p at t is either the zero vector or a null vector.

Definition. That the path $p\colon I \to \mathcal{M}$ is **nonstop** means that p is smooth and the velocity $\dot{p}$ of p vanishes nowhere on I. If $I = [a, b]$, and $A = p(a)$ and $B = p(b)$, then p is said to be a **path from** A **to** B **in** $\mathcal{M}$. By a **smooth** (**doubly smooth**, **triply smooth**) **curve in** $\mathcal{M}$ is meant a subset C of $\mathcal{M}$ for which there is a (doubly smooth, triply smooth) nonstop path p in $\mathcal{M}$ whose range is C; every such path p is said to be a **parametrization of** C, and is said to **parametrize** C. If p is a nonstop path from A to B in $\mathcal{M}$, and p parametrizes C, then C is said to be a **smooth** (**doubly smooth**, **triply smooth**) **curve from** A **to** B **in** $\mathcal{M}$. By an **arc of** C is meant a smooth (doubly smooth, triply smooth) curve $\bar{C}$ in $\mathcal{M}$ that is parametrized by a path $\bar{p}$ such that $\bar{p} = p|_{[a,b]}$, where $p\colon I \in \mathcal{M}$ is a parametrization of C and $[a, b]$ is a nondegenerate subinterval of I; if $p(a) = A$ and $p(b) = B$, then $\bar{C}$ is called the **arc of** C **from** A **to** B, also the **arc** AB **of** C.

If $p\colon [a, b] \to \mathcal{M}$ is a smooth path in $\mathcal{M}$, and $q = p(\phi)$, a smooth regular reparametrization of p by $\phi\colon [c, d] \to [a, b]$, then $\dot{q} = \dot{p}(\phi)\dot{\phi}$. Because $\dot{\phi}$ vanishes nowhere, if either of p and q is nonstop, so is the other. Also,

$$\langle \dot{q}, \dot{q}\rangle_G = \langle \dot{p}(\phi)\dot{\phi}, \dot{p}(\phi)\dot{\phi}\rangle_G = \langle \dot{p}(\phi), \dot{p}(\phi)\rangle_G \dot{\phi}^2, \tag{278}$$

from which it follows that if p is nonstop, then the velocity of p and the velocity of q at corresponding times t and $\phi^{-1}(t)$ are both positive, both negative, or both null.

Definition. The nonstop path p in $\mathcal{M}$ is said to be **null** (resp., **positive**, **negative**) (**with respect to** G) if and only if at each "time" t in its interval the velocity of p at t is null (resp., positive, negative) with respect to G. The smooth curve C in $\mathcal{M}$ is said to be **null** (resp., **positive**, **negative**) (**with respect to** G) if and only if there is a nonstop null (resp., positive, negative) path p in $\mathcal{M}$ that parametrizes C.

Continuing, if, as above, $p\colon [a,b] \to \mathcal{M}$ is a smooth path in $\mathcal{M}$, and $q = p(\phi)$, a smooth regular reparametrization of p by $\phi\colon [c,d] \to [a,b]$, then

$$\ell(q) = \int_c^d |\langle \dot q, \dot q\rangle_G|^{\frac12} = \int_c^d |\langle \dot p(\phi), \dot p(\phi)\rangle_G \dot\phi^2|^{\frac12} \tag{279}$$

$$= \int_c^d |\langle \dot p(\phi), \dot p(\phi)\rangle_G|^{\frac12} |\dot\phi| = \int_c^d |\langle \dot p, \dot p\rangle_G|^{\frac12}(\phi)\,\mathrm{sgn}(\dot\phi)\dot\phi \tag{280}$$

$$= \mathrm{sgn}(\dot\phi(c)) \int_{\phi(c)}^{\phi(d)} |\langle \dot p, \dot p\rangle_G|^{\frac12} = \int_a^b |\langle \dot p, \dot p\rangle_G|^{\frac12} = \ell(p), \tag{281}$$

which justifies the following definition.

Definition. If each of A and B is a point of $\mathcal{M}$, and C is a smooth curve from A to B in $\mathcal{M}$, then by the **length of** C (**with respect to** G), denoted by $\ell(C)$, is meant $\ell(p)$, where p is any nonstop path from A to B in $\mathcal{M}$ that parametrizes C.

Theorem 2. If C is a smooth curve in $\mathcal{M}$, then $\ell(C) > 0$ if G is definite, and $\ell(C) = 0$ if and only if G is indefinite and C is null.

Definition. If C is a smooth curve in $\mathcal{M}$, and $p\colon I \to \mathcal{M}$ is a smooth path in $\mathcal{M}$ that parametrizes C, then p is said to be an **arclength-proportional parametrization of** C if and only if there is a positive number k such that if $[a,b]$ is a nondegenerate subinterval of I, then $b - a = k\,\ell\big(p|_{[a,b]}\big) = k\,\ell(\bar C)$, where $\bar C$ is the arc of C from $p(a)$ to $p(b)$; if $k = 1$, thus if $b - a = \ell\big(p|_{[a,b]}\big) = \ell(\bar C)$, then p is said to be an **arclength parametrization of** C, and is said to **parametrize** C **by arclength**. Also, p is said to be a **constant-speed parametrization of** C if and only if the speed $|\dot p|_G$ of p is constant, a **unit-speed parametrization of** C if and only if $|\dot p|_G = 1$.

Theorem 3. If C is a smooth curve in $\mathcal{M}$, then $p : I \to \mathcal{M}$ is an arclength-proportional parametrization of C if and only if p is a constant-speed parametrization of C whose speed is positive, in which event if $[a,b]$ is a nondegenerate subinterval of I, then $\ell\big(p|_{[a,b]}\big) = |\dot p(a)|_G\,(b-a)$, a consequence of which is that p parametrizes C by arclength if and only if p is a unit-speed parametrization of C.

Theorem 4. Let $p : I \to \mathcal{M}$ be a parametrization of the positive or negative smooth curve C in $\mathcal{M}$. Let a be any number in I, let c be any number, let k be any nonzero number, and let $\theta : I \to \mathbb{R}$ be defined by $\theta(t) := c + (1/k)\int_a^t |\dot p|_G$. Then θ has an inverse ϕ, and if $q = p(\phi)$, then q is an arclength-proportional parametrization of C with constant speed $|k|$, which is an arclength parametrization of C if and only if $|k| = 1$. If p itself is a constant-speed (thus arclength-proportional) parametrization of C, then $\theta(t) = c + (|\dot p(a)|_G/k)(t-a)$ and $\phi(s) = a + (k/|\dot p(a)|_G)(s-c)$. If p is a unit-speed parametrization, then $\theta(t) = c + (1/k)(t-a)$ and $\phi(s) = a + k\,(s-c)$.

XXI. METRIC GEODESICS

Let $\{\mathcal{M}, G\}$ be a doubly smooth metric manifold, and let C be a smooth curve in $\mathcal{M}$ that is either positive or negative.

Definition. The positive or negative curve C is said to be **geodesic** (**with respect to** G), and is called a **geodesic curve** (**of** G) (a **geodesic**, for short), if and only if C is doubly smooth and, for each arc $\bar{C}$ of C from a point A to a point B that lies entirely in a domain of some coordinate system of $\mathcal{M}$, the length $\ell(\bar{C})$ of $\bar{C}$ (with respect to G) is stationary under comparison with the lengths of all smooth curves from A to B in $\mathcal{M}$ that are "near" $\bar{C}$. (That the smooth curve $\bar{\bar{C}}$ from A to B in $\mathcal{M}$ is "near" $\bar{C}$ means here that there is a coordinate system X of $\mathcal{M}$ whose domain incorporates both $\bar{C}$ and $\bar{\bar{C}}$, and there are parametrizations $\bar{p}\colon [a,b] \to \mathcal{M}$ of $\bar{C}$ and $\bar{\bar{p}}\colon [a,b] \to \mathcal{M}$ of $\bar{\bar{C}}$ such that $\sup_{t\in[a,b]}(|X\,\bar{\bar{p}}(t) - X\bar{p}(t)| + |\hat{X}\,\dot{\bar{\bar{p}}}(t) - \hat{X}\,\dot{\bar{p}}(t)|)$ is small.)

Suppose C is doubly smooth. Let $p : I \to \mathcal{M}$ be a doubly smooth parametrization of C, let $\bar{C}$ be an arc of C from a point A to a point B, parametrized by a restriction of p to an interval $[a,b]$ such that $p(a) = A$ and $p(b) = B$. Suppose further that $\bar{C}$ lies entirely in the domain of the coordinate system X of $\mathcal{M}$. Then $\ell(\bar{C}) = \ell(p|_{[a,b]}) = \int_a^b |G(p)\dot{p}\dot{p}|^{\frac{1}{2}} = \int_a^b |\dot{p}^m g_{mn}(p)\dot{p}^n|^{\frac{1}{2}} = \int_a^b \mathcal{L}\left(Xp, (Xp)^{\cdot}\right)$, where, with $p^m = x^m p$ and $\dot{p}^m = (p^m)^{\cdot}$,

$$\mathcal{L}\left(Xp,(Xp)^{\cdot}\right) = \mathcal{L}\left(p^1,\cdots,p^M,\dot{p}^1,\cdots,\dot{p}^M\right) := \left|\dot{p}^k g_{kl}\left(X^{-1}\left([\![\, p^1,\cdots,p^M]\!]\right)\right)\dot{p}^l\right|^{\frac{1}{2}} = \left|\dot{p}^k g_{kl}(p)\dot{p}^l\right|^{\frac{1}{2}}. \tag{282}$$

The Euler–Lagrange equations for p that are necessary and sufficient for $\ell(\bar{C})$ to be stationary under comparison with the lengths of all smooth curves from A to B in $\mathcal{M}$ that are "near" $\bar{C}$ are, for $m = 1,\ldots,M$,

$$0 = \frac{\partial\mathcal{L}}{\partial p^m}\left(Xp,(Xp)^{\cdot}\right) - \left(\frac{\partial\mathcal{L}}{\partial \dot{p}^m}\left(Xp,(Xp)^{\cdot}\right)\right)^{\cdot} \tag{283}$$

$$= \frac{1}{2\,\mathcal{L}(Xp,(Xp)^{\cdot})}\frac{\partial\mathcal{L}^2}{\partial p^m}\left(Xp,(Xp)^{\cdot}\right) - \left(\frac{1}{2\,\mathcal{L}(Xp,(Xp)^{\cdot})}\frac{\partial\mathcal{L}^2}{\partial \dot{p}^m}\left(Xp,(Xp)^{\cdot}\right)\right)^{\cdot} \tag{284}$$

$$= \frac{1}{2\,\mathcal{L}(Xp,(Xp)^{\cdot})}\left[\frac{\partial\mathcal{L}^2}{\partial p^m}\left(Xp,(Xp)^{\cdot}\right) - \left(\frac{\partial\mathcal{L}^2}{\partial \dot{p}^m}\left(Xp,(Xp)^{\cdot}\right)\right)^{\cdot} + \frac{(\mathcal{L}(Xp,(Xp)^{\cdot}))^{\cdot}}{\mathcal{L}(Xp,(Xp)^{\cdot})}\frac{\partial\mathcal{L}^2}{\partial \dot{p}^m}\left(Xp,(Xp)^{\cdot}\right)\right], \tag{285}$$

which are equivalent to

$$\left(\frac{\partial\mathcal{L}^2}{\partial \dot{p}^m}\left(Xp,(Xp)^{\cdot}\right)\right)^{\cdot} - \frac{\partial\mathcal{L}^2}{\partial p^m}\left(Xp,(Xp)^{\cdot}\right) = \frac{(\mathcal{L}(Xp,(Xp)^{\cdot}))^{\cdot}}{\mathcal{L}(Xp,(Xp)^{\cdot})}\frac{\partial\mathcal{L}^2}{\partial \dot{p}^m}\left(Xp,(Xp)^{\cdot}\right). \tag{286}$$

Now

$$\mathcal{L}^2\left(Xp,(Xp)^{\cdot}\right) = \left|\dot{p}^k g_{kl}(p)\dot{p}^l\right| = \operatorname{sgn}\left(\dot{p}^k g_{kl}(p)\dot{p}^l\right)\left(\dot{p}^k g_{kl}(p)\dot{p}^l\right) \tag{287}$$

$$= \operatorname{sgn}\left(\dot{p}^k g_{kl}(p)\dot{p}^l\right)\left(\dot{p}^k g_{kl}\left(X^{-1}\left([\![p^1,\cdots,p^M]\!]\right)\right)\dot{p}^l\right) \tag{288}$$

$$= \operatorname{sgn}(C)\left(\dot{p}^k g_{kl}\left(X^{-1}\left([\![p^1,\cdots,p^M]\!]\right)\right)\dot{p}^l\right) = \operatorname{sgn}(C)\left(\dot{p}^k g_{kl}(p)\dot{p}^l\right), \tag{289}$$

where $\operatorname{sgn}(C) = 1$ or -1, according as C is positive or negative. Consequently,

$$\frac{\partial \mathcal{L}^2}{\partial p^m}\left(Xp,(Xp)^{\cdot}\right) = \operatorname{sgn}(C)\left(\dot{p}^k \frac{\partial}{\partial p^m}\left(g_{kl}\left(X^{-1}\left([\![p^1,\cdots,p^M]\!]\right)\right)\right)\dot{p}^l\right) \tag{290}$$

$$= \operatorname{sgn}(C)\left(\dot{p}^k \frac{\partial g_{kl}}{\partial x^m}(p)\dot{p}^l\right) = \operatorname{sgn}(C)\left(\dot{p}^k g_{kl.m}(p)\dot{p}^l\right), \tag{291}$$

and

$$\frac{\partial \mathcal{L}^2}{\partial \dot{p}^m}\left(Xp,(Xp)^{\cdot}\right) = \operatorname{sgn}(C)\left(\frac{\partial \dot{p}^k}{\partial \dot{p}^m}\, g_{kl}(p)\dot{p}^l + \dot{p}^k g_{kl}(p)\frac{\partial \dot{p}^l}{\partial \dot{p}^m}\right) \tag{292}$$

$$= \operatorname{sgn}(C)\left(\delta_m{}^k\, g_{kl}(p)\dot{p}^l + \dot{p}^k g_{kl}(p)\delta^l{}_m\right) \tag{293}$$

$$= \operatorname{sgn}(C)\left(g_{ml}(p)\dot{p}^l + \dot{p}^k g_{km}(p)\right) = \operatorname{sgn}(C)\left(2\,\dot{p}^k g_{km}(p)\right), \tag{294}$$

from which follows

$$\left(\frac{\partial \mathcal{L}^2}{\partial \dot{p}^m}\left(Xp,(Xp)^{\cdot}\right)\right)^{\cdot} = \operatorname{sgn}(C)\left(g_{ml.k}(p)\dot{p}^k\dot{p}^l + g_{ml}(p)(\dot{p}^l)^{\cdot} + (\dot{p}^k)^{\cdot} g_{km}(p) + \dot{p}^k g_{km.l}(p)\dot{p}^l\right) \tag{295}$$

$$= \operatorname{sgn}(C)\left(2\,(\dot{p}^k)^{\cdot} g_{km}(p) + \dot{p}^k\left(g_{ml.k}(p) + g_{km.l}(p)\right)\dot{p}^l\right). \tag{296}$$

The Euler–Lagrange equations are therefore equivalent to

$$(\dot{p}^k)^{\cdot} g_{km}(p) + \dot{p}^k\,[k\ \ l\,.\,m](p)\,\dot{p}^l = \lambda\,\dot{p}^k g_{km}(p), \tag{297}$$

where $\lambda = (\mathcal{L}(Xp,(Xp)^{\cdot}))^{\cdot}/\mathcal{L}(Xp,(Xp)^{\cdot})$ and

$$[k\ \ l\,.\,m] := \frac{1}{2}[g_{ml.k} + g_{km.l} - g_{kl.m}], \tag{298}$$

the **Christoffel symbol of the first kind**. If G is nondegenerate, then raising of the index m in these equations by G^{-1} produces the equivalent equations

$$(\dot{p}^m)^{\cdot} + \dot{p}^k\{{}_k{}^m{}_l\}(p)\,\dot{p}^l = \lambda\,\dot{p}^k, \tag{299}$$

where

$$\{{}_k{}^m{}_l\} := [k\ \ l\,.\,n]\,g^{nm} = \frac{1}{2}[g_{nl.k} + g_{kn.l} - g_{kl.n}]\,g^{nm}, \tag{300}$$

the **Christoffel symbol of the second kind**. These calculations establish the following theorem.

Theorem 1. In order that the positive or negative doubly smooth curve C in $\mathcal{M}$ be geodesic with respect to G it is necessary and sufficient that, for each doubly smooth parametrization

p of C, and each coordinate system X of $\mathcal{M}$ such that $\operatorname{dom} X \cap C \neq \varnothing$, the components in X of p and of $\dot{p}$ satisfy the differential equations

$$(\dot{p}^k)^{\cdot} g_{km}(p) + \dot{p}^k\,[k\ \ l\,.\,m](p)\,\dot{p}^l = \lambda\,\dot{p}^k g_{km}(p), \tag{301}$$

and, if G is nondegenerate, the equivalent differential equations

$$(\dot{p}^m)^{\cdot} + \dot{p}^k\{{}_k{}^m{}_l\}(p)\,\dot{p}^l = \lambda\,\dot{p}^k, \tag{302}$$

where $\lambda = (\ln|\dot{p}|_G)^{\cdot}$.

If p is an arclength-proportional parametrization of C, then $|\dot{p}|_G$ is constant, and vice versa, so this theorem has the following corollary.

Theorem 2. In order that the positive or negative doubly smooth curve C in $\mathcal{M}$ be geodesic with respect to G it is necessary and sufficient that, for each doubly smooth arclength-proportional parametrization p of C, and each coordinate system X of $\mathcal{M}$ such that $\operatorname{dom} X \cap C \neq \varnothing$, the components in X of p and of $\dot{p}$ satisfy the differential equations

$$(\dot{p}^k)^{\cdot} g_{km}(p) + \dot{p}^k\,[k\ \ l\,.\,m](p)\,\dot{p}^l = 0, \tag{303}$$

and, if G is nondegenerate, the equivalent differential equations

$$(\dot{p}^m)^{\cdot} + \dot{p}^k\{{}_k{}^m{}_l\}(p)\,\dot{p}^l = 0. \tag{304}$$

Definition. The smooth path p in $\mathcal{M}$ is said to be **geodesic** (**with respect to** G), and is called a **geodesic path** (**of** G) (a **geodesic**, for short), if and only if p is a doubly smooth parametrization of a geodesic curve of G.

Definition. That the geodesic curve C is **maximal** means that C is not a subset of any other geodesic curve.

Theorem 3. If P is a point of $\mathcal{M}$, u is a nonnull tangent vector at P, and r_0 is a number, then there is just one geodesic path p such that $p(r_0) = P$, $\dot{p}(r_0) = u$, and p is an arclength-proportional parametrization of a maximal geodesic curve C; if u is positive, then C is positive; if u is negative, then C is negative; if $|u|_G = 1$, then p is an arclength parametrization of C.

If the metric G is indefinite, then there are doubly smooth curves C in $\mathcal{M}$ that are null. If $\bar{C}$ is an arc of such a curve from a point A to a point B, then $\bar{C}$ is null and $\ell(\bar{C}) = 0$, so $\ell(\bar{C})$ is an absolute minimum with respect to the lengths of all smooth curves from A to B in $\mathcal{M}$. Because, however, the arclength function ℓ on the space of paths in $\mathcal{M}$ is not differentiable at any path that is null (for essentially the same reasons that the real function $|x^2|^{\frac{1}{2}}$, that is to say $|x|$, is not differentiable at 0), the preceding definition and theorems cannot be extended to encompass "null geodesic". If P is a point of $\mathcal{M}$, then according to the preceding theorem every nonnull tangent vector u at P generates by way of a parametrization p a geodesic curve that passes through P in the direction determined by u. The following definition causes this property to extend to the null tangent vectors at P.

Definition. The null curve C is said to be **geodesic** (**with respect to** G), and is called a **geodesic curve** (**of** G) (a **geodesic**, for short), if and only if C is doubly smooth and, for each doubly smooth parametrization p of C, and each coordinate system X of $\mathcal{M}$ such that $\operatorname{dom} X \cap C \neq \varnothing$, the components in X of p and of $\dot{p}$ satisfy the differential equations

$$(\dot{p}^k)^{\cdot} g_{km}(p) + \dot{p}^k\, [k\;\; l\,.\,m](p)\, \dot{p}^l = \lambda\, \dot{p}^k g_{km}(p), \tag{305}$$

and, if G is nondegenerate, the equivalent differential equations

$$(\dot{p}^m)^{\cdot} + \dot{p}^k \{{}_k{}^m{}_l\}(p)\, \dot{p}^l = \lambda\, \dot{p}^k, \tag{306}$$

for some continuous mapping $\lambda\colon \operatorname{dom} X \cap C \to \mathbb{R}$. If, for each such coordinate system X, $\lambda = 0$, then p is said to be a **pseudo–arclength-proportional parametrization of** C.

With appropriate modifications all the preceding definitions and theorems now extend to null geodesics.

XXII. METRIC CONNECTIONS

Continue with $\{\mathcal{M}, G\}$ a doubly smooth metric manifold, and with G nondegenerate.

Theorem 1. For each coordinate system X of $\mathcal{M}$ let $\Gamma(X) = \Gamma_k{}^m{}_l\,(dx^l \otimes dx^k \otimes \partial/\partial x^m)$, with

$$\Gamma_k{}^m{}_l := \{_k{}^m{}_l\} = \frac{1}{2}[g_{nl.k} + g_{kn.l} - g_{kl.n}]\,g^{nm}, \tag{307}$$

Then

i. Γ is a connection on $\mathcal{M}$,

ii. the covariant differentiation $\mathbf{d}$ associated with Γ is torsion free,

iii. $\mathbf{d}G = 0$,

iv. the autoparallel geodesic paths and curves of $\mathbf{d}$ are the geodesic paths and curves of the metric G, and

v. if the doubly smooth path p is a parametrization of the geodesic curve C of G, then p is an arclength-proportional or pseudo–arclength-proportional parametrization of C if and only if p is affinely parametrized as an autoparallel geodesic path of $\mathbf{d}$, in which case C is maximal if and only if p is maximal.

Definition. If $\mathbf{d}$ is a covariant differentiation on $\mathcal{M}$, then $\mathbf{d}$ is said to be **compatible** (also to be **consistent**) **with** G if and only if $\mathbf{d}G = 0$. By a **metric connection** is meant a connection on a doubly smooth manifold whose associated covariant differentiation is compatible with some metric on that manifold.

Theorem 2. The covariant differentiation $\mathbf{d}$ is compatible with G if and only if $\mathbf{d}G^{-1} = 0$.

Theorem 3. Suppose the covariant differentiation $\mathbf{d}$ is compatible with G. If each of u and v is a differentiable vector field that is covariantly constant with respect to $\mathbf{d}$, and $\operatorname{dom} u \cap \operatorname{dom} v \neq \varnothing$, then $\langle u, v\rangle_G$ is constant. If p is a differentiable path in $\mathcal{M}$, and each of u and v is a differentiable vector field on p that is covariantly constant with respect to $\mathbf{d}$, then $\langle u, v\rangle_G$ is constant.

The following proposition is referred to as the Fundamental Theorem of Metric Differential Geometry.

Theorem 4. If T is a skew-symmetric cococontensor field on $\mathcal{M}$, then there is just one covariant differentiation $\mathbf{d}$ on $\mathcal{M}$ that is compatible with G and has T for its torsion.

A proof of this theorem proceeds as follows: Suppose that $\mathbf{d}$ is such a covariant differentiation. Let E be a smooth frame system of $\mathcal{M}$, with dual coframe system Ω. Then $d_\wedge \omega^m = C_k{}^m{}_l(\omega^l \otimes \omega^k)$, with coefficients $C_k{}^m{}_l$ that are skew-symmetric in k and l. Also, if the torsion $\mathbf{T}$ of $\mathbf{d}$ is T, then $\mathbf{T} = \mathbf{T}^m \otimes e_m$, where $\mathbf{T}^m = T_k{}^m{}_l(\omega^l \otimes \omega^k)$, with coefficients $T_k{}^m{}_l$ that are skew-symmetric in k and l. Further, $\mathbf{d}_\wedge \omega^m = -\omega_k{}^m \wedge \omega^k = \omega^k \wedge \omega_k{}^m = -\Gamma_{[k}{}^m{}_{l]}(\omega^l \otimes \omega^k)$, where $\omega_k{}^m$ are the 1-forms and $\Gamma_k{}^m{}_l$ the coefficients of $\mathbf{d}$ in E. From $\mathbf{T}^m = d_\wedge \omega^m - \mathbf{d}_\wedge \omega^m$ follows that $\omega^k \wedge \omega_k{}^m = d_\wedge \omega^m - \mathbf{T}^m$, which is equivalent to $\Gamma_{[k}{}^{m}{}_{l]} = T_k{}^m{}_l - C_k{}^m{}_l$, in turn equivalent to

$$\Gamma_{kml} = \Gamma_{lmk} + 2(T_{kml} - C_{kml}), \tag{308}$$

the index m having been lowered by G. On the other hand, if $\mathbf{d}$ is compatible with G, then from $G = \omega^m \otimes g_{mn}\omega^n$ follows that

$$0 = \mathbf{d}G = \mathbf{d}\omega^m(\cdot) \otimes g_{mn}\omega^n + \omega^m \otimes dg_{mn}(\cdot) \otimes \omega^n + \omega^m \otimes g_{mn}\mathbf{d}\omega^n(\cdot) \tag{309}$$

$$= -\omega_k{}^m(\cdot) \otimes \omega^k \otimes g_{mn}\omega^n + \omega^m \otimes dg_{mn}(\cdot) \otimes \omega^n - \omega^m \otimes g_{mn}\omega_l{}^n(\cdot) \otimes \omega^l \tag{310}$$

$$= -\omega_k{}^m \otimes \omega^k \otimes g_{mn}\omega^n + dg_{mn} \otimes \omega^m \otimes \omega^n - \omega_l{}^n \otimes \omega^m \otimes g_{mn}\omega^l \tag{311}$$

$$= -\omega_k{}^m g_{mn} \otimes \omega^k \otimes \omega^n + dg_{mn} \otimes \omega^m \otimes \omega^n - \omega_l{}^n g_{mn} \otimes \omega^m \otimes \omega^l \tag{312}$$

$$= (-\omega_m{}^k g_{kn} + dg_{mn} - \omega_n{}^l g_{ml}) \otimes (\omega^m \otimes \omega^n) \tag{313}$$

$$= (-\omega_m{}^k g_{kn} + dg_{mn} - \omega_n{}^l g_{lm}) \otimes (\omega^m \otimes \omega^n), \tag{314}$$

which is equivalent to $dg_{mn} = \omega_{mn} + \omega_{nm}$, thus to $g_{mn.l}\omega^l = \Gamma_{mnl}\omega^l + \Gamma_{nml}\omega^l$, in turn equivalent to $\Gamma_{mnl} = -\Gamma_{nml} + g_{mn.l}$, and therefore to

$$\Gamma_{klm} = -\Gamma_{lkm} + g_{kl.m}. \tag{315}$$

It now follows that

$$\Gamma_{kml} = \Gamma_{lmk} + 2(T_{kml} - C_{kml}) \tag{316}$$

$$= -\Gamma_{mlk} + g_{lm.k} + 2(T_{kml} - C_{kml}) \tag{317}$$

$$= -\Gamma_{klm} - 2(T_{mlk} - C_{mlk}) + g_{lm.k} + 2(T_{kml} - C_{kml}) \tag{318}$$

$$= \Gamma_{lkm} - g_{kl.m} - 2(T_{mlk} - C_{mlk}) + g_{lm.k} + 2(T_{kml} - C_{kml}) \tag{319}$$

$$= \Gamma_{mkl} + 2(T_{lkm} - C_{lkm}) - g_{kl.m} - 2(T_{mlk} - C_{mlk}) + g_{lm.k} + 2(T_{kml} - C_{kml}) \tag{320}$$

$$= -\Gamma_{kml} + g_{mk.l} + 2(T_{lkm} - C_{lkm}) - g_{kl.m} - 2(T_{mlk} - C_{mlk}) + g_{lm.k} + 2(T_{kml} - C_{kml}), \tag{321}$$

thus that

$$2\Gamma_{kml} = g_{mk.l} + g_{lm.k} - g_{kl.m} - 2(T_{mlk} - C_{mlk}) + 2(T_{lkm} - C_{lkm}) + 2(T_{kml} - C_{kml}) \tag{322}$$

so that

$$\Gamma_{kml} = [k\ \ l\,.\,m] + 2(T_{klm} + T_{lkm} + T_{kml}) - 2(C_{klm} + C_{lkm} + C_{kml}), \tag{323}$$

and therefore that

$$\Gamma_k{}^m{}_l = \{_k{}^m{}_l\} + (T_{kl}{}^m + T_{lk}{}^m + T_k{}^m{}_l) - (C_{kl}{}^m + C_{lk}{}^m + C_k{}^m{}_l), \tag{324}$$

where

$$[k\ \ l\,.\,m] := \frac{1}{2}[g_{ml.k} + g_{km.l} - g_{kl.m}], \tag{325}$$

and

$$\{_k{}^m{}_l\} := [k\ \ l\,.\,n]\, g^{nm} = \frac{1}{2}[g_{nl.k} + g_{kn.l} - g_{kl.n}]\, g^{nm}. \tag{326}$$

It is straightforward to show that the $\Gamma_k{}^m{}_l$ given by this formula are the coefficients in E of a covariant differentiation **d** that is compatible with G and has the tensor field T for its torsion.

Note that

$$\Gamma_{(k}{}^m{}_{l)} = \{_k{}^m{}_l\} + (T_{kl}{}^m + T_{lk}{}^m) - (C_{kl}{}^m + C_{lk}{}^m) \tag{327}$$

$$\tag{328}$$

and

$$\Gamma_{[k}{}^m{}_{l]} = T_k{}^m{}_l - C_k{}^m{}_l, \tag{329}$$

thus that $\Gamma_k{}^m{}_l$ is symmetric in k and l if and only if $T_k{}^m{}_l = C_k{}^m{}_l$ and $\Gamma_k{}^m{}_l = \{_k{}^m{}_l\}$. If E is a coordinate frame system, then $C_k{}^m{}_l = 0$. Consequently, $\Gamma_k{}^m{}_l$ is symmetric in k and l in every coordinate frame system if and only if $T_k{}^m{}_l = 0$, thus if and only if $\mathbf{T} = 0$. For this reason metric connections associated with torsion free covariant differentiations are sometimes said to be **symmetric**.

The equations $dg_{mn} = \omega_{mn} + \omega_{nm}$ and $\omega^k \wedge \omega_k{}^m = d_\wedge \omega^m - \mathbf{T}^m$ provide an algorithm for calculating the 1-forms $\omega_k{}^m$ that is useful when the frame system E is chosen so that the g_{mn} are constants, particularly so when E is orthonormal. In that case the matrix $[\omega_k{}^m]$ is the solution of the matrix equation

$$[\omega^k] \wedge [\omega_k{}^m] = [d_\wedge \omega^m - \mathbf{T}^m] \quad (= [(C_k{}^m{}_l - T_k{}^m{}_l)(\omega^l \wedge \omega^k)]) \tag{330}$$

that has the symmetries implied by $[\omega_k{}^m] = S[G^{-1}] = [\omega_{kn}][g^{nm}]$, where, because $\omega_{mn} + \omega_{nm} = dg_{mn} = 0$, S is a skew-symmetric matrix. Linearity of the lefthand member allows the solution to be constructed as a sum of partial solutions, one for each of the entries $(C_k{}^m{}_l - T_k{}^m{}_l)(\omega^l \wedge \omega^k)$ of the righthand member.